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SINGULAR SPACES AND GENERALIZED POINCARÉ COMPLEXES

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Abstract. We introduce a method that associates to a singular space a CW complex whose ordinary rational homology satisfies Poincaré duality across complementary perversities as in intersection homology. The method is based on a homotopy theoretic process of spatial homology truncation, whose functoriality properties are investigated in detail. The resulting homology theory is not isomorphic to intersection homology and addresses certain questions in type II string theory related to massless D-branes. The two theories satisfy an interchange of third and second plus fourth Betti number for mirror symmetric conifold transitions. Further applications of the new theory to K-theory and symmetric L-theory are indicated.

1. Poincaré duality for pseudomanifolds

Stratified pseudomanifolds are finite dimensional spaces with a filtration by closed subsets such that the successive differences (the open strata) are manifolds, the top dimensional open stratum is dense, the complement of the top stratum (the "singular set") has codimension at least two, and every point has a distinguished neighborhood homeomorphic to the open cone on a compact pseudomanifold of smaller dimension. Examples include realizations of n-dimensional simplicial complexes such that every $(n-1)$-simplex is the face of precisely two n-simplices and every simplex is the face of some n-simplex, as well as irreducible complex algebraic or analytic varieties. The example of the suspension of a two-torus shows that the ordinary homology of a pseudomanifold does not in general satisfy Poincaré duality. Generalized Poincaré duality is obtained by using instead the intersection homology groups $IH^{\bar{p}}_*(-)$ of Goresky and MacPherson [11], [12], see also [1], [15], [3], or Cheeger's L^2-cohomology [9], [8], [10], in the case of a Riemannian pseudomanifold. It is shown in [9] and [12] that, for Riemannian pseudomanifolds that have conical singularities and no odd-codimensional strata, the linear dual of intersection homology for the middle perversity $\bar{p} = \bar{m}$ is isomorphic to L^2-cohomology. (The lower middle perversity $\bar{m}$ is the sequence

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$(\bar{m}(2), \bar{m}(3), \bar{m}(4), \ldots)$ given by $(0, 0, 1, 1, 2, 2, \ldots)$; the upper middle perversity $\bar{n}$ is the sequence $(\bar{n}(2), \bar{n}(3), \bar{n}(4), \ldots)$ given by $(0, 1, 1, 2, 2, 3, \ldots)$.)

Both theories are obtained as the (co)homology of a complex that is associated with the pseudomanifold X. In the case of intersection homology, this is a subcomplex of the ordinary chain complex of X, in the case of L^2-cohomology, it is the complex of those differential L^2-forms on the top stratum whose differential is an L^2-form as well. These complexes are not generally differential graded algebras, because in the former case, the intersection product of a $\bar{p}$-intersection chain and a $\bar{q}$-intersection chain is in general only a $(\bar{p}+\bar{q})$-intersection chain; in the latter case, the wedge product of two L^2-forms need not be an L^2-form. These complexes are thus already a certain distance removed from the homotopy type of X. We shall discuss here a method that associates to an oriented stratified pseudomanifold X spaces $I^{\bar{p}}X$, the *intersection spaces of* X, such that the ordinary reduced homology $\widetilde{H}_*(I^{\bar{p}}X; \mathbb{Q})$ satisfies generalized Poincaré duality. Thus the $I^{\bar{p}}X$ are generalized rational Poincaré complexes. The method constructs these spaces so that the given X is modified only near the singularities. The space away from a small neighborhood of the singular set is completely preserved. Roughly, the links are replaced with appropriate Moore sections of the links, depending on the perversity $\bar{p}$. The construction will be explained in more detail in the following sections. If X is a finite CW complex, then $I^{\bar{p}}X$ will be a finite CW complex as well, which is a desirable property for instance in fiberwise settings. Moreover, our construction is very explicit: if CW structures of the links are known, then the intersection spaces can be written down explicitly and concrete computations can be carried out readily.

The homology groups of $I^{\bar{p}}X$ are not isomorphic to $IH_*^{\bar{p}}(X)$, nor (a fortiori) to L^2-cohomology when $\bar{p} = \bar{m}$. The Calabi-Yau quintic

$$Q = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0\}$$

in $\mathbb{CP}^4$ has 125 isolated singular points, all of them nodes. The middle homology $H_3(I^{\bar{m}}Q)$ has rank 204. Almost all of these cycles remain invisible to intersection homology, as $\operatorname{rk} IH_3^{\bar{m}}(Q) = 2$. On the other hand, $\operatorname{rk} IH_2(Q) = \operatorname{rk} IH_4(Q) = 25$, while $\operatorname{rk} H_2(IQ) = \operatorname{rk} H_4(IQ) = 1$. (We will drop the perversity decoration when the perversity is understood to be the lower middle.) The two theories are, however, closely related. A good way to think of this relationship in the case of singular Calabi-Yau 3-folds is that $(IH_*(-), H_*(I-))$ is a mirror-pair in the sense of mirror symmetry. It turns out that the new homology theory $H_*(I-)$ solves certain problems in type II string theory that are not solved by either ordinary homology or by intersection homology. The role of intersection spaces in string theory is the topic of Section 5.

Let us mention some immediate advantages of this spatial approach to Poincaré duality for singular spaces. Contrary to the intersection chain complex or L^2-form complex, the ordinary cochain complex of $I^{\bar{p}}X$ is a differential graded algebra by simply using the ordinary cup product. In particular, $H^*(I^{\bar{p}}X)$ comes equipped with an *internal* cup product. Furthermore, cohomology operations are defined. If one wishes to define $\bar{p}$-intersection versions of generalized homology theories $E_*(-)$ in the sense of Eilenberg-Steenrod, then one cannot do this via chain theories. Indeed, it is proven in [4] that no nontrivial homology theory E_* is the homology of a chain theory. Here, a chain theory is a covariant functor L from the category of finite CW pairs (A, B) to the category of chain complexes such that

$$0 \to L_n(B) \longrightarrow L_n(A) \longrightarrow L_n(A, B) \to 0$$

is exact and $(A, B) \mapsto H_*(L(A, B))$ is a generalized homology theory. We regard E_* as trivial if there is a natural equivalence of homology theories

$$E_n(A, B) \cong \bigoplus_{p+q=n} H_p(A, B; E_q(\mathrm{pt})),$$

where H_* is ordinary singular homology. For example, oriented bordism cannot be derived from an underlying chain theory. Since the framework presented here is spatial, not just chain theoretic, it allows one to study $X \mapsto E_*(I^{\bar{p}}X)$, e.g. its duality properties, even when E_* is nontrivial. We will return to this point below for K-theory $E = \mathrm{KO}$ and symmetric L-theory $E = \mathbb{L}^\bullet$. The new theory also allows for cap products of the general form $\widetilde{H}^r(I^{\bar{m}}X; \mathbb{Q}) \otimes \widetilde{H}_i(X; \mathbb{Q}) \stackrel{\cap}{\longrightarrow} \widetilde{H}_{i-r}(I^{\bar{n}}X; \mathbb{Q})$. Products of this type are known not to exist for intersection homology. We will explain below why such products exist in the new theory but do not exist in intersection homology. Characteristic classes of topological pseudomanifolds, such as L-classes, lie generally in $H_*(X; \mathbb{Q})$, without possibility of lifting them to $IH_*(X; \mathbb{Q})$ or $H_*(IX; \mathbb{Q})$. Thus the value of the above product derives from the fact that it allows us to multiply Chern classes in $H^{\mathrm{even}}(I^{\bar{m}}X; \mathbb{Q})$ of some bundle with the characteristic classes of X to obtain a class in the homology of $I^{\bar{n}}X$. This may lead to extensions of the results of [5], [2] on characteristic class formulae for twisted signatures and twisted L-classes of pseudomanifolds.

2. Intersection spaces in the case of isolated singularities

Let us describe the construction of $I^{\bar{p}}X$ when X has only isolated singularities. In fact, the general method is already well illustrated by taking X to be of the form

$$X = M \cup_{\partial M} \mathrm{cone}(\partial M),$$

where M is an n-dimensional oriented compact manifold with boundary ∂M. The cone point is the only singularity and the link of this singularity is ∂M. Set $k = n - 1 - \bar{p}(n)$, a positive integer. The intersection homology groups of such an X are given by

$$IH_i^{\bar{p}}(X) = \begin{cases} H_i(M), & i < k \\ \mathrm{im}(H_i(M) \to H_i(M, \partial M)), & i = k \\ H_i(M, \partial M), & i > k. \end{cases} \tag{1}$$

Thus, intersection homology effectively truncates the chain complex of the link below the cut-off value k. Our method of construction will implement such a truncation spatially by using a process of spatial homology truncation. Eckmann-Hilton duality in homotopy theory dictates the direction of arrows for such a truncation, and this direction is such that one must truncate above k, not below. This fundamental point explains why the method yields a new kind of homology and not intersection homology.

Suppose, then, that $t_{<k}$ is a covariant assignment from spaces to spaces, assigning to L a Moore section $t_{<k}L$, that is, there is a natural transformation ϵ_k from $t_{<k}$ to the identity such that $\epsilon_{k*} : H_i(t_{<k}L) \to H_i(L)$ is an isomorphism when $i < k$, while $H_i(t_{<k}L) = 0$ when $i \geq k$. The existence and functoriality properties of such assignments are generally intricate and rather interesting in their own right.

Developing a homotopy theoretical method for spatial homology truncation forms a substantial part of the work presented here and constitutes the technical core of our method. We give more details on this in Section 4. Given $t_{<k}$, the intersection space $I^{\bar{p}}X$ is by definition the homotopy cofiber of the composite map

$$t_{<k}(\partial M) \xrightarrow{\epsilon_k} \partial M \hookrightarrow M.$$

(We should caution that this exposition of the construction of $I^{\bar{p}}X$ is somewhat simplified, since a naive truncation $(t_{<k}, \epsilon_k)$ as above does not actually exist, as we will see in Section 4. The more sophisticated truncation of that section is really used.) Duality then takes the following form.

Theorem 2.1. *Let $\bar{p}$ and $\bar{q}$ be complementary perversities. Then there is a generalized Poincaré duality isomorphism*

$$\widetilde{H}^i(I^{\bar{p}}X;\mathbb{Q}) \cong \widetilde{H}_{n-i}(I^{\bar{q}}X;\mathbb{Q}),$$

which is compatible with Poincaré-Lefschetz duality on $(M, \partial M)$.

While $IH_k^{\bar{p}}(X)$ is generally smaller than both $H_k(M)$ and $H_k(M,\partial M)$, the group $\widetilde{H}_k(I^{\bar{p}}X)$ is generally bigger than these two. In fact, $\widetilde{H}_k(I^{\bar{p}}X)$ is an extension of $H_k(M,\partial M)$ by $\ker(H_k(M) \to H_k(M,\partial M))$ and for $i \neq k$,

$$\widetilde{H}_i(I^{\bar{p}}X) = \begin{cases} H_i(M,\partial M), & i < k \\ H_i(M), & i > k, \end{cases}$$

which the reader may wish to compare to (1) above.

Suppose the dimension n of X is divisible by 4. Then the intersection form

$$\Phi_{HIX} : \widetilde{H}_{n/2}(IX;\mathbb{Q}) \otimes \widetilde{H}_{n/2}(IX;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

is symmetric. Let

$$\Phi_{IHX} : IH_{n/2}(X;\mathbb{Q}) \otimes IH_{n/2}(X;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

denote the Goresky-MacPherson intersection form. One can construct a complement to intersection homology in $\widetilde{H}_{n/2}(IX;\mathbb{Q})$, which contains a self-annihilating subspace of half the rank, and thus prove:

Theorem 2.2. *With $W(\mathbb{Q})$ the Witt group of the rationals, we have*

$$[\Phi_{HIX}] = [\Phi_{IHX}] \in W(\mathbb{Q}),$$

where $[-]$ denotes the Witt class of a symmetric nonsingular bilinear form.

If the dimension n of X is congruent to 2 mod 4, then there is a cap product

$$\widetilde{H}^{2l}(IX;\mathbb{Q}) \otimes \widetilde{H}_i(X;\mathbb{Q}) \xrightarrow{\cap} \widetilde{H}_{i-2l}(IX;\mathbb{Q})$$

such that

$$\begin{array}{ccc} \widetilde{H}^{2l}(IX;\mathbb{Q}) \otimes \widetilde{H}_i(X;\mathbb{Q}) & \xrightarrow{\cap} & \widetilde{H}_{i-2l}(IX;\mathbb{Q}) \\ \uparrow {\scriptstyle c^*\otimes \mathrm{id}} & & \downarrow {\scriptstyle c_*} \\ \widetilde{H}^{2l}(X;\mathbb{Q}) \otimes \widetilde{H}_i(X;\mathbb{Q}) & \xrightarrow{\cap} & \widetilde{H}_{i-2l}(X;\mathbb{Q}) \end{array}$$

commutes, where $c: IX \to X$ is the canonical map given by collapsing the mapping cone $\partial M \cup_{\epsilon_k} \mathrm{cone}\, t_{<k}(\partial M)$ in IX to a point, and the bottom horizontal arrow is the

ordinary cap product. To construct the desired product, one observes that since k is odd, either $2l > k$ or $2l < k$. If $2l > k$, use the cap product

$$\widetilde{H}^{2l}(M;\mathbb{Q}) \otimes H_i(M,\partial M;\mathbb{Q}) \xrightarrow{\cap} H_{i-2l}(M,\partial M;\mathbb{Q}),$$

observing that $i-2l<k$. If $2l<k$, use

$$H^{2l}(M,\partial M;\mathbb{Q}) \otimes H_i(M,\partial M;\mathbb{Q}) \xrightarrow{\cap} \widetilde{H}_{i-2l}(M;\mathbb{Q}) \xrightarrow{b_*} \widetilde{H}_{i-2l}(IX;\mathbb{Q}),$$

where $b : M \to IX$ is the canonical inclusion map from the target of a map to its mapping cone. Similar considerations lead to similar products in other cases vis-à-vis the residue class of n mod 4. Note that this method of obtaining the desired product is not available to intersection homology, since when $2l < k$ and $i-2l < k$, one would need a cap product

$$H^{2l}(M) \cap H_i(M,\partial M) \dashrightarrow H_{i-2l}(M).$$

Such a product does not exist, as the absolute chain-level cap product $C^{2l}(M) \otimes C_i(M) \longrightarrow C_{i-2l}(M)$ maps the subgroup $C^{2l}(M) \otimes C_i(\partial M)$ to $C_{i-2l}(\partial M)$, and these product chains may include cycles that are not null-homologous in M.

Let $\mathbb{L}^\bullet$ be the 0-connective symmetric L-spectrum, [17, §16, page 173]. Capping with the $\mathbb{L}^\bullet$-homology fundamental class of an n-dimensional oriented compact pseudomanifold X with isolated singularities induces a Poincaré duality isomorphism $\widetilde{H}^0(I^{\bar{m}}X;\mathbb{L}^\bullet) \otimes \mathbb{Q} \xrightarrow{\cong} \widetilde{H}_n(I^{\bar{n}}X;\mathbb{L}^\bullet) \otimes \mathbb{Q}$. A *$\bar{p}$-intersection vector bundle on X* may be defined as an actual vector bundle on $I^{\bar{p}}X$. Using simply connected surgery [7], one can show that there are infinitely many distinct 7-dimensional pseudomanifolds X, whose tangent bundle elements in the KO-theory $\widetilde{\mathrm{KO}}(X - \mathrm{Sing})$ of their nonsingular parts do not lift to $\widetilde{\mathrm{KO}}(X)$, but do lift to $\widetilde{\mathrm{KO}}(I^{\bar{n}}X)$, where $\bar{n}$ is the upper middle perversity. So this framework allows one to formulate the requirement that a pseudomanifold have a $\bar{p}$-intersection tangent bundle. In the isolated singularity case, the obstruction to lifting to $\widetilde{\mathrm{KO}}(I^{\bar{n}}X)$ lies in the KO-theory of the truncated link, which is roughly half as big as the KO-theory of the link itself.

3. Intersection spaces for more general singular sets

If the singular set is positive dimensional, then a process of fiberwise spatial homology truncation applied to the link bundle can sometimes be used. Let X be an n-dimensional, compact, stratified pseudomanifold with two strata $X = X_n \supset X_{n-c}$. The singular set $\Sigma = X_{n-c}$ is thus an $(n-c)$-dimensional closed manifold and the singularities are not isolated, unless $c = n$. Assume that X has a trivial link bundle, that is, a neighborhood of Σ in X looks like $\Sigma \times \mathring{\mathrm{cone}}(L)$, where L is a $(c-1)$-dimensional closed manifold, the link of Σ. With $k = c - 1 - \bar{p}(c)$, the intersection space $I^{\bar{p}}X$ is the homotopy cofiber of

$$\Sigma \times t_{<k}L \xrightarrow{\mathrm{id}_\Sigma \times \epsilon_k} \Sigma \times L = \partial M \overset{j}{\hookrightarrow} M.$$

The requisite Poincaré duality theorem can be established for these spaces. A piecewise linear X always has a stratification with trivial link bundles, namely the simplicial stratification. The behavior of the homotopy type of the intersection space under refinement of stratifications remains to be investigated. If there are more than two nested strata, then more elaborate homotopy colimit constructions involving iterated truncation techniques can be used. At present, intersection spaces have been explicitly constructed for the following classes of pseudomanifolds:

(1) Pseudomanifolds with isolated singularities,
(2) two-strata spaces with arbitrary bottom stratum but untwisted link bundle (as described above; of course we are referring to nested strata — the construction can also be applied to spaces with more than two strata, but then the singular strata must not be nested, but must have disjoint neighborhoods),
(3) three-strata spaces where the singular set is a disjoint union of circles each of which may contain isolated points as the bottom stratum (the link bundle around the circles may be twisted here), and
(4) two-strata spaces whose bottom stratum is a sphere of arbitrary dimension and whose link bundles may be twisted but have special structure groups.

We will illustrate how to deal with nested strata in the case of a three-strata space X^n whose singular set is a circle X_1, which contains a point stratum $X_0 = \{x_0\}$. Removing an open neighborhood $\mathring{\text{cone}}(L_0)$, where L_0 is the link of x_0, from X, we obtain a space X' which has one singular stratum Δ^1, a closed interval. Let L_1 be the link of this stratum, a closed manifold of dimension $n-2$. The link L_0 may be singular with singular stratum $L_0 \cap X_1 = \partial\Delta^1$ (two points). A regular neighborhood of $\partial\Delta^1$ in L_0 is isomorphic to two disjoint copies of $\text{cone}(L_1)$. If we remove the interiors of these two cones from L_0, we obtain a compact $(n-1)$-manifold W, which is a bordism between the two copies of L_1 on its boundary. A regular neighborhood of Δ_1 in X' is isomorphic to a product $\Delta^1 \times \text{cone}(L_1)$. Removing the interior of this neighborhood from X', we get a compact n-manifold M with boundary ∂M. The boundary is the pushout of a diagram Γ,

$$W \xleftarrow{\ f\ } \partial\Delta^1 \times L_1 \xhookrightarrow{\text{incl}\times\text{id}} \Delta^1 \times L_1,$$

for a suitable gluing map f. Given a perversity $\bar{p}$, set cut-off degrees

$$k_L = n-2-\bar{p}(n-1),\ k_W = n-1-\bar{p}(n).$$

Applying spatial homology truncation $t_{<k}$, one can construct a diagram $t_{<k}\Gamma$,

$$t_{<k_W}(W) \xleftarrow{\ t_{<k_W}(f)\ } \partial\Delta^1 \times t_{<k_L}(L_1) \xhookrightarrow{\text{incl}\times\text{id}} \Delta^1 \times t_{<k_L}(L_1).$$

A homotopy commutative diagram of the form

$$\begin{array}{ccccc} t_{<k_W}(W) & \longleftarrow & \partial\Delta^1 \times t_{<k_L}(L_1) & \hookrightarrow & \Delta^1 \times t_{<k_L}(L_1) \\ \downarrow & & \downarrow & & \downarrow \\ W & \longleftarrow & \partial\Delta^1 \times L_1 & \hookrightarrow & \Delta^1 \times L_1. \end{array}$$

gives rise to a map

$$\text{hocolim}(t_{<k}\Gamma) \longrightarrow \text{colim}(\Gamma) = \partial M.$$

Define $I^{\bar{p}}X$ to be the homotopy cofiber of the composition of this map with the inclusion of ∂M into M. Future research will have to determine the ultimate domain of pseudomanifolds for which an intersection space is definable.

Since spatial homology n-truncation of a space L in general requires choosing a complement of the n-cycle group, see Section 4, and since the construction of intersection spaces uses this truncation on the links L of singularities, the homotopy type of the intersection space $I^{\bar{p}}X$ may well depend, to some extent, on choices. For isolated singularities, we can show that the rational homology of $I^{\bar{p}}X$ is well-defined and independent of choices. Furthermore, if

$$\operatorname{Ext}(\operatorname{im}(H_k(M, \partial M) \to H_{k-1}(\partial M)), H_k(M)) = 0,$$

or

$$\operatorname{Ext}(H_k(M, \partial M), \operatorname{im}(H_k(\partial M) \to H_k(M))) = 0,$$

then the integral homology of $I^{\bar{p}}X$ in the cut-off degree k is independent of choices. Away from the cut-off degree, the integral homology is always independent of choices. The conditions are often satisfied in algebraic geometry for the middle perversity, for instance when X is a complex projective algebraic 3-fold with isolated hypersurface singularities that are weighted homogeneous and "well-formed" in the sense of [6]. In particular, this class of varieties includes conifolds, to be discussed in Section 5. When all links have every other homology group zero (simply connected 4-manifolds, smooth compact toric varieties, homogeneous spaces arising as the quotient of a complex simply connected semisimple Lie group by a parabolic subgroup such as flag manifolds, Grassmannians; smooth Schubert varieties), we can show that the homotopy type of $I^{\bar{p}}X$ is well-defined independent of choices.

4. Spatial homology truncation

Functoriality of spatial homology truncation is a subtler issue than functoriality for Postnikov sections, the Eckmann-Hilton dual problem. As we will see, in the former situation obstructions arise that have no analog in the latter situation. Since we wish to avail ourselves of the Whitehead and Hurewicz theorems, we shall place ourselves in the category of simply connected CW complexes K. Let k be a positive integer. On the object level, our truncation works roughly as follows.

Lemma 4.1. *Every simply connected k-dimensional CW complex L is homotopy equivalent rel $(k-1)$-skeleton to a complex L/k, whose k-th cellular cycle group $Z_k(L/k) \subset C_k(L/k)$ has a basis of cells.*

The conclusion of Lemma 4.1 is of course generally false, take for example $L = S^2 \cup_4 e^3 \cup_6 e^3$ and $k = 3$. This L is homotopy equivalent rel S^2 to $S^2 \cup_0 e^3 \cup_2 e^3$, which does have a cellular basis for Z_3.

Lemma 4.2. *Let P be a simply connected k-dimensional CW complex, whose group of k-cycles has a basis of cells. Then P contains a unique subcomplex $P_{<k}$, which has the same $(k-1)$-skeleton as P and is such that $H_i(P_{<k}) = 0$ for $i \geq k$ and the inclusion $P_{<k} \subset P$ induces an isomorphism $H_i(P_{<k}) \to H_i(P)$ for $i < k$.*

Note that the above example $S^2 \cup_4 e^3 \cup_6 e^3$ contains no such truncating subcomplex ($k = 3$), but $S^2 \cup_0 e^3 \cup_2 e^3$ does, namely $S^2 \cup_2 e^3$.

Let K be a simply connected CW complex of any dimension. Applying Lemma 4.1 to the k-skeleton $L = K^k$ of K and Lemma 4.2 to the output $P = K^k/k$ of Lemma 4.1, we get a complex $t_{<k}(K) = (K^k/k)_{<k}$ and a map

$$\epsilon_k : t_{<k}(K) \hookrightarrow K^k/k \simeq K^k \hookrightarrow K,$$

which is a homology isomorphism in degrees less than k. The truncation of maps $f : K \to L$ is a more delicate matter. Let $K = S^2 \cup_2 e^3$, a Moore space $M(\mathbb{Z}/_2, 2)$, and $L = K \vee S^3$. Let $f : K \to L$ be the map obtained by collapsing the 2-skeleton to a point and including the resulting 3-sphere in L in the obvious way. Then, no matter which maps $e : t_{<3}K \to K$ and $e' : t_{<3}L \to L$ with $e_* : H_i(t_{<3}K) \to H_i(K)$

and $e'_* : H_i(t_{<3}L) \to H_i(L)$ isomorphisms for $i < 3$ one takes, there exists no map $t_{<3}(f) : t_{<3}K \dashrightarrow t_{<3}L$ such that

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e'} \\ t_{<3}K & \overset{t_{<3}(f)}{\dashrightarrow} & t_{<3}L \end{array}$$

homotopy commutes. This phenomenon can be explained as follows. In the example, there is no way to choose complements Y_K, $Z_k(K) \oplus Y_K = C_k(K)$, and Y_L, $Z_k(L) \oplus Y_L = C_k(L)$, of the k-cycle groups such that $f_*(Y_K) \subset Y_L$. On the other hand, we can prove for general simply connected K, L:

Theorem 4.1 (Compression Theorem)**.** *Let $f : K \to L$ be a cellular map with $f_*(Y_K) \subset Y_L$. Then there exists a compression $t_{<k}(f) : t_{<k}K \to t_{<k}L$ of f, which agrees with f on the $(k-1)$-skeleton of K and makes the square*

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle \epsilon_k}\uparrow & & \uparrow{\scriptstyle \epsilon_k} \\ t_{<k}K & \xrightarrow{t_{<k}(f)} & t_{<k}L \end{array}$$

commutative up to homotopy rel $(k-1)$-skeleton.

We deduce that it is most natural to consider spatial homology truncation $t_{<k}$ as being defined on the category $\mathbf{CW}_{k\supset\partial}$ of pairs (K, Y) as above, with morphisms cellular maps that preserve the complements Y, and taking values in the rel $(k-1)$-skeleton homotopy category $\mathbf{HoCW}_{k-1}$ of CW complexes. Let $K = S^4 \cup_4 e^5$ and $L = S^3 \cup_2 e^4 \cup e^5$, where the 5-cell in L is attached to S^3 by an essential map. These two spaces have unique cycle-complements Y_K, Y_L in degree $k = 5$ and have unique truncation subcomplexes $t_{<5}(K, Y_K) \subset K$, $t_{<5}(L, Y_L) \subset L$. However, there is a map $f : K \to L$ and two nonhomotopic maps $g_1, g_2 : t_{<5}(K, Y_K) \to t_{<5}(L, Y_L)$, which agree with f on the 4-skeleton and have the property that

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle \epsilon_5}\uparrow & & \uparrow{\scriptstyle \epsilon_5} \\ t_{<5}(K, Y_K) & \xrightarrow{g_i} & t_{<5}(L, Y_L) \end{array}$$

homotopy commutes rel 4-skeleton, $i = 1, 2$. Hence, the homotopy class of the compression $t_{<k}(f)$ is generally not unique, even when the compression is pinned down on the $(k-1)$-skeleton. If it is unique, we shall refer to f as *compression rigid*. The need for a compression rigidity obstruction theory thus arises. Such an obstruction theory is provided by the next result.

Theorem 4.2. *Suppose that (K, Y_K) and (L, Y_L) are objects in $\mathbf{CW}_{k\supset\partial}$. Write $K_{<k} = t_{<k}(K, Y_K)$ and $L_{<k} = t_{<k}(L, Y_L)$.*
(1) Let $f : (K, Y_K) \to (L, Y_L)$ be a morphism in $\mathbf{CW}_{k\supset\partial}$ with a k-truncation $t_{<k}(f) : K_{<k} \to L_{<k}$. Then f is compression rigid if and only if for every map $g : K_{<k} \to L_{<k}$ with $j_*\omega(t_{<k}(f), g) = 0 \in C^{k+1}(K_{<k} \times I, \pi_k(L/k))$, one actually has $\omega(t_{<k}(f), g) = 0 \in C^{k+1}(K_{<k} \times I, \pi_k(L_{<k}))$. Here, $j : L_{<k} \subset L/k$ is the subcomplex inclusion and ω is the well-known obstruction cocycle for two maps

between k-dimensional simply connected CW complexes to be homotopic rel $(k-1)$-skeleton.
(2) A homotopy $H : K_{<k} \times I \to L/k$ can be compressed into $L_{<k}$ rel K^{k-1} if and only if a single obstruction

$$\omega_k(H) \in C^{k+1}(K_{<k} \times I, \pi_{k+1}(L/k, L_{<k}))$$

vanishes.
(3) If L is 2-connected, then $\pi_{k+1}(L/k, L_{<k}) \cong (\mathbb{Z}/_2)^b$, where $b = \operatorname{rk} H_k(L^k)$, assumed to be finite.

Consequently, f is compression rigid in any of the following cases:

- $\pi_k(L_{<k}) \to \pi_k(L/k)$ is injective, or
- $\partial_k = 0 : C_k(K) \to C_{k-1}(K)$, or
- $\partial_k : C_k(L) \to C_{k-1}(L)$ is injective, or
- $\operatorname{im}(\pi_k(L^k, L^{k-1}) \to \pi_{k-1}(L^{k-1})) \cap \ker(\pi_{k-1}(L^{k-1}) \to \pi_{k-1}(L^{k-1}, L^{k-2})) = 0$ $(k \geq 4)$, for instance, $\pi_{k-1}(L^{k-2}) = 0$.

Any morphism $f : K \to K$ is k-compression rigid if K has precisely one k-cell. Any morphism between closed simply connected 4-manifolds is compression rigid for any k. Any map between the links of isolated nodes in complex algebraic 3-folds is compression rigid for any k.

Theorem 4.3. *On any compression rigid subcategory $\mathbf{C} \subset \mathbf{CW}_{k\supset\partial}$, the covariant assignment $t_{<k} : \mathbf{CW}_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$ restricts to a functor, together with a natural transformation $\epsilon_k : t_{<k} \to \mathrm{id}$.*

If one inverts 2 and focuses on 2-connected spaces, then the compression rigidity obstruction vanishes and one receives an odd-primary spatial truncation functor $t_{<k}^{(\mathrm{odd})}$ defined on the full subcategory of $\mathbf{CW}_{k\supset\partial}$ whose objects are pairs (K, Y) with K 2-connected.

5. The role of intersection spaces in Type II string theory

The 10-dimensional target space of a nonlinear sigma model in string theory is often assumed to be of the form $M^4 \times X^6$, where M is a 4-manifold which we think of locally as the space-time of special relativity and X is a Calabi-Yau 3-fold. It is argued in [13], using the field equations $d^*d\omega = 0$ and $d^*\omega = 0$ for a differential form ω on $M \times X$, and observing that the Hodge-de Rham Laplacian Δ on $M \times X$ decomposes as $\Delta = \Delta_M + \Delta_X$, that Δ_X can be interpreted as a mass-operator for 4-dimensional fields, whose eigenvalues are masses as seen in four dimensions. In particular, harmonic forms on X correspond to massless particles in the low energy effective field theory. Thus a good cohomology theory for Calabi-Yau varieties should record all physically present massless particles.

The conifold transition is a valuable means to travel between Calabi-Yau 3-folds, as it covers a lot of terrain in the landscape of Calabi-Yau spaces, and may indeed connect all of them. It starts with a nonsingular Calabi-Yau 3-fold X_ϵ, whose complex structure depends on a complex parameter ϵ. In the limit as $\epsilon \to 0$, disjoint embedded 3-spheres with trivial normal bundle are collapsed to points, and one obtains a singular variety S (the *conifold*) with isolated nodes. By replacing the nodes with complex projective spaces $\mathbb{CP}^1$, one resolves the singularities and gets a nonsingular 3-fold Y, which is again Calabi-Yau. In type IIA string theory, there are twobranes present that wrap around the $\mathbb{CP}^1$ 2-cycles in Y, see [18], [14].

Their mass is proportional to the volume of the $\mathbb{CP}^1$ that they wrap around, so they become massless in S and ought to be recorded by a good homology theory. Let

$$b_A = \operatorname{rk}\operatorname{coker}(H_4X_\epsilon \to H_4S),\ b_B = \operatorname{rk}\ker(H_3X_\epsilon \to H_3S).$$

Note that $b_A + b_B$ equals the number of nodes in S. Since a good homology theory $\mathcal{H}^{\mathrm{IIA}}_*$ for type IIA string theory should also satisfy Poincaré duality (ideally the Kähler package), it should have the ranks

$$\operatorname{rk}\mathcal{H}^{\mathrm{IIA}}_2(S) = p + b_A,\ \operatorname{rk}\mathcal{H}^{\mathrm{IIA}}_3(S) = q,\ \operatorname{rk}\mathcal{H}^{\mathrm{IIA}}_4(S) = p + b_A,$$

where $q = \operatorname{rk}(H_3(S - \Sigma) \to H_3(S))$, $p = \operatorname{rk} H_2(S)$, because b_A is the number of massless twobranes present. Ordinary homology is obviously not a solution. A calculation shows that intersection homology $\mathcal{H}^{\mathrm{IIA}}_*(S) = IH_*(S)$ does have the desired ranks and thus is a suitable homology theory in the IIA regime. In type IIB string theory, there are threebranes present that wrap around the 3-spheres in X_ϵ and that become massless in the limit $\epsilon \to 0$. Since they should be accounted for by a good homology theory $\mathcal{H}^{\mathrm{IIB}}_*$, such a theory should have the ranks

$$\operatorname{rk}\mathcal{H}^{\mathrm{IIB}}_2(S) = p,\ \operatorname{rk}\mathcal{H}^{\mathrm{IIB}}_3(S) = q + 2b_B,\ \operatorname{rk}\mathcal{H}^{\mathrm{IIB}}_4(S) = p, \tag{2}$$

because b_B is the number of massless threebranes present and there are in addition $q + b_B$ elementary massless particles ($\operatorname{rk} H_3(S) = q + b_B$). Neither ordinary nor intersection homology have these ranks.

Theorem 5.1. *Let IS be the middle-perversity intersection space of the conifold S. Then $\mathcal{H}^{\mathrm{IIB}}_*(S) = H_*(IS)$ has the ranks (2).*

Since $Y \to S$ is a small resolution, $IH_*(S) \cong H_*(Y)$. The observation that $H_i(IS) \cong H_i(X_\epsilon)$, $0 < i < 6$, may be an indication that $H_i(IS)$ is even in more general contexts the homology of a certain class of smooth deformations.

To a Calabi-Yau 3-fold S, the mirror map associates another Calabi-Yau 3-fold S° such that type IIB string theory on $\mathbb{R}^4 \times S$ corresponds to type IIA string theory on $\mathbb{R}^4 \times S^\circ$. If S and S° are nonsingular, then $b_3(S^\circ) = (b_2 + b_4)(S)$ and $b_3(S) = (b_2 + b_4)(S^\circ)$ for the ordinary Betti numbers. It is conjectured in [16] that the mirror of a conifold transition is again a conifold transition, performed in the reverse direction.

Theorem 5.2. *Suppose that a singular Calabi-Yau 3-fold S sits in a conifold transition $X \rightsquigarrow S \rightsquigarrow Y$ and that its mirror S° sits in the mirror conifold transition $Y^\circ \rightsquigarrow S^\circ \rightsquigarrow X^\circ$. Then*

$$\begin{aligned} \operatorname{rk} IH_3(S) &= \operatorname{rk} H_2(IS^\circ) + \operatorname{rk} H_4(IS^\circ),\\ \operatorname{rk} IH_3(S^\circ) &= \operatorname{rk} H_2(IS) + \operatorname{rk} H_4(IS),\\ \operatorname{rk} H_3(IS) &= \operatorname{rk} IH_2(S^\circ) + \operatorname{rk} IH_4(S^\circ),\ \textit{and}\\ \operatorname{rk} H_3(IS^\circ) &= \operatorname{rk} IH_2(S) + \operatorname{rk} IH_4(S). \end{aligned}$$

Thus $(IH_*(-), H_*(I-))$ is a mirror-pair of homology theories for singular Calabi-Yau 3-folds.

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