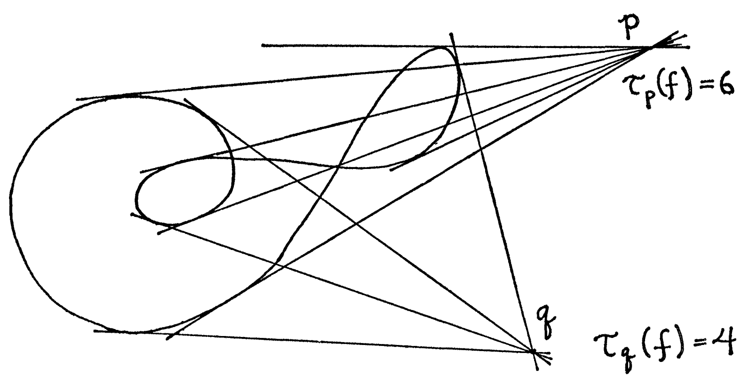
TOTAL CENTRAL CURVATURE OF CURVES

By Thomas F. Banchoff

Total central curvature refers to the measure of curvedness of a space curve contained in a ball (bounded by a sphere) obtained by averaging the total absolute curvatures of the image curves under central projection from all points on the sphere. The major object of this paper is to show that this total curvature coincides with the classical total absolute curvature of the original space curve. This result generalizes immediately to curves in n-space. As a corollary we show that a curve on S^3 in E^4 with total absolute curvature < 4 in E^4 can be unknotted in S^3.

We begin by studying, from an elementary standpoint, the specialization of this theorem to plane curves, and illustrate at the same time the methods to be used in the general case.

1. Total central curvature of plane curves. Let $f\colon S^1 \to E^2$ be a continuous map of the circle S^1 into the plane. A local support line to f at x is a line containing x and bounding a closed half-plane which contains the image of a neighborhood of x in S^1. Let $\tau_p(f)$ be the number of local support lines to f passing through the point p of E^2.



The *curvature of f with respect to a circle C* $= \tau_C(f)$ is defined to be the average value of $\tau_p(f)$ for points p on C, i.e.

$$\tau_C(f) = \frac{1}{l(C)} \int_{p \varepsilon C} \tau_p(f)\, ds_C$$

where ds_C denotes the element of arc of C so that $\int ds_C = l(C) =$ circumference of the circle C.

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The value of $\tau_C(f)$ does depend on the position of C relative to the image of f. For example, if f embeds S^1 as a convex curve, then $\tau_C(f) = 2$ if $C \supset f$ (i.e. if the image of f is contained in the region bounded by C) and $\tau_C(f) = 0$ if $f \supset C$. As the main result of this section we shall show that, for an arbitrary f, the value $\tau_C(f)$ is independent of the position of C if $C \supset f$.

We first consider the case where f is a polygon. A *polygon* $f : S^1 \to E^2$ is a continuous mapping determined by a finite number of vertex points $v_1, v_2, \cdots, v_m$ on S^1, with the condition that f be linear on each interval (v_i, v_{i+1}). We further assume that $f(v_i) \neq f(v_j)$ if $i \neq j$, and we follow the convention that $v_{i+1} = v_1$ if $i = m$ and $v_{i-1} = v_m$ if $i = 1$.

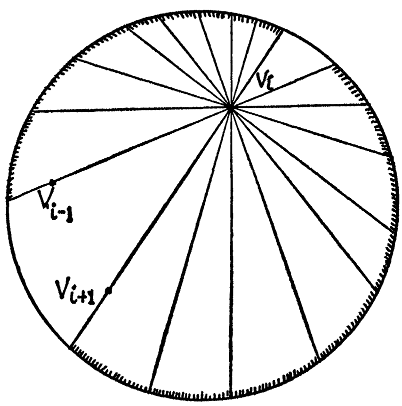
THEOREM 1. *If f is a polygon and $C \supset f$, then*

$$\tau_C(f) = 2 \sum_{i=1}^{m} \frac{1}{2\pi} \left| \textit{exterior angle of } f \textit{ at } v_i \right| .$$

Proof. Let $\tau_p(v_i) = 1$ if the line through p and $f(v_i)$ is a local support line to f at v_i, and let $\tau_p(v_i) = 0$ otherwise. If p does not lie on any line connecting the images of two distinct vertices, then $\tau_p(f) = \sum_{i=1}^{m} \tau_p(v_i)$. Since there are only finitely many lines determined in this way, this formula for $\tau_p(f)$ will hold for all but a finite number of points p on any circle C. Therefore

$$\begin{aligned} \tau_C(f) &= \frac{1}{l(C)} \int_{p \in C} \tau_p(f)\, ds_C \\ &= \frac{1}{l(C)} \int_{p \in C} \sum_{i=1}^{m} \tau_p(v_i)\, ds_C \\ &= \sum_{i=1}^{m} \frac{1}{l(C)} \int_{p \in C} \tau_p(v_i)\, ds_C . \end{aligned}$$

The expression $1/l(C) \int_{p \in C} \tau_p(v_i)\, ds_C$ just indicates the proportion of the circumference of C covered by support lines to the angle of f at v_i, and, by a theorem of Euclid, this number is $1/2\pi \cdot 2$ |exterior angle of f at v_i|.



This completes the proof of the theorem.

Remark. We have in fact shown that $\tau_C(f)$, for $C \supset f$, coincides with the classically defined total absolute curvature $\tau(f)$ of a polygon (up to a normalization constant $1/2\pi$), as given by Fáry [2] and Milnor [3]. The total curvature $\tau(f)$ for any continuous curve may be defined by

$$\tau(f) = \frac{1}{2\pi} \int_{\xi \in S^1} \tau(\xi \circ f)\, d\omega_1$$

where $d\omega_1$ is the element of arc on the unit circle, $\xi \circ f : S^1 \to E^1$ is the map $x \to \xi \cdot f(x)$ given by the ordinary inner product of $f(x)$ with the vector ξ, and $\tau(\xi \circ f) =$ number of local maxima or minima of $\xi \circ f$ on S^1. For a differentiable immersion f with continuously defined curvature $k(s)$, the value $\tau(f)$ coincides with $1/2\pi \int_C |k(s)|\, ds$, which explains the name "total absolute curvature."

A very complete treatment of the concept of total absolute curvature for continuous curves is given by Van Rooij [4], who shows explicitly that for any f which is not constant on any interval, we have $\lim_{n\to\infty} \tau(f_n) \to \tau(f)$, where $\{f_n\}$ is any sequence of polygons inscribed in f with mesh approaching zero. In precisely the same way we may show that $\lim_{n\to\infty} \tau_C(f_n) \to \tau_C(f)$, and since $\tau_C(f_n) = \tau(f_n)$ for planar polygons with all vertex images distinct, we may approximate f by a sequence of such polygons and we obtain the general result that $\tau_C(f) = \tau(f)$ for plane curves. The convergence results are also true for curves in E^3 (or in E^m), so we shall present proofs of subsequent theorems just for polygons, and the general results follow immediately.

Remark. In a sense the classical total absolute curvature $\tau(f)$ may be considered as the total central curvature $\tau_C(f)$ with respect to an "infinitely large circle," so that the family of support lines through a point p of C corresponds to local support lines perpendicular to a given direction ξ.

If $C \supset f$, we may view the function $\tau_p(f)$ as $\tau(\pi_p \circ f)$ where $\pi_p : E^2 - \{T_p\} \to E_p^1$ is the central projection from p to the line E_p^1 through the center of C parallel to the tangent line T_p to C at p, since we get a maximum or minimum of the projection function precisely whenever there is a local support line to f through p. It is this notion that we use in generalizing Theorem 1 to curves in higher-dimensional Euclidean spaces.

We may also view $\tau_p(f)$ as $\tau(\theta_p \circ f)$ where $\theta_p : E^2 - \{p\} \to S^1$ is the angular coordinate in a polar coordinate system with center p. The singularities of the radial coordinate function $\rho_p : E^2 \to E^1$ are studied in [1].

2. Total central curvature of curves in $\mathbf{E}^3$. Let $f: S^1 \to E^3$ be a continuous $1 - 1$ map of the circle S^1 and let S be a sphere such that $S \supseteq f$ (i.e. such that f lies in the closed ball bounded by the sphere S). Let $\pi_p : E^3 - \{T_p\} \to E_p^2$ be the *central projection* (where T_p is the tangent plane to S at p and E_p^2 is the plane parallel to T_p through the center of S), so for any x not in T_p, $\pi_p(x)$ is defined by the condition that $\pi_p(x)$ is in E_p^2 and $\pi_p(x)$ is collinear with x and p. (Note that π_p restricted to $S - \{p\}$ gives stereographic projection.)

For each p on S we obtain a number $\tau(\pi_p \circ f)$ which we average to obtain the *total central curvature* of f with respect to S,

$$\tau_S(f) = \frac{1}{V(S)} \int_{p \varepsilon S} \tau(\pi_p \circ f)\, dV_S \ ,$$

where dV_S is the element of area of S so that $V(S) = \int_{p \varepsilon S} dV_S$.

THEOREM 2. *Let f be a polygon embedded in E^3 and let $S \supseteq f$. Then $\tau_S(f) = \tau(f)$.*

Proof.

$$\begin{aligned}
\tau_S(f) &= \frac{1}{V(S)} \int_{p \varepsilon S} \tau(\pi_p \circ f)\, dV_S \\
&= \frac{1}{V(S)} \int_{p \varepsilon S} 2 \sum_{i=1}^{m} \frac{1}{2\pi} \left| \text{ext. angle of } \pi_p \circ f \text{ at } \pi_p \circ v_i \right| dV_S \\
&= 2 \sum_{i=1}^{m} \left(\frac{1}{V(S)} \int_{p \varepsilon S} \frac{1}{2\pi} \left| \text{ext. angle of } \pi_p \circ f \text{ at } \pi_p \circ v_i \right| dV_S \right) .
\end{aligned}$$

It remains to show that the expression in parenthesis represents $1/2\pi$ |ext. angle of f at v_i|, and we separate this off as the basic averaging lemma, which will also be used in the higher dimensional case.

Before proceeding to the proof of this lemma, we shall discuss the classical analogue (used by Fáry [2] in his proof of the knot theorem) and establish some notation for the central curvature case.

For a vector $\xi \varepsilon S^2$, let $P_\xi : E^3 \to E^2_\xi$ denote (orthogonal) projection into the plane orthogonal to ξ, so that an angle $\angle uvw$ of a polygon is projected into an angle $\angle P_\xi(u)P_\xi(v)P_\xi(w)$ whenever ξ is not parallel to either of the infinite rays $\overrightarrow{vu}$ or $\overrightarrow{vw}$. (Here $\angle uvw$ stands for the measure of the non-obtuse angle determined by the ordered triple of vertices.) Set

$$A(\angle\, uvw) = \frac{1}{4\pi} \int_{\xi \varepsilon S^2} \angle\, P_\xi(u)P_\xi(v)P_\xi(w)\, d\omega_2 \ ,$$

the average value of the projections of the angle into all planes through the origin. Since the integral $A(\angle uvw)$ remains unchanged under all rigid motions of E^3, the value $A(\angle uvw)$ depends only on the magnitude uvw. Moreover the function is additive in the following sense—if the vertex w' lies in a convex plane region bounded by the rays $\overrightarrow{vu}$ and $\overrightarrow{vw}$, then $\angle uvw' + \angle w'vw = \angle uvw$ and for every ξ not parallel to $\overrightarrow{vu}$, $\overrightarrow{vw'}$ or $\overrightarrow{vw}$, we have $\angle P_\xi(u)P_\xi(v)P_\xi(w') + \angle P_\xi(w')P_\xi(v)P_\xi(w) = \angle P_\xi(u)P_\xi(v)P_\xi(w)$, so $A(\angle uvw') + A(\angle w'vw) = A(\angle uvw)$. Also if $\angle uvw$ is a straight angle, then so is $\angle P_\xi(u)P_\xi(v)P_\xi(w)$ for almost all ξ, so $A(\angle uvw) = \angle uvw$ in this case. By additivity, $2A(\angle uvw') = A(\angle uvw') + A(\angle w'vw) = A(\angle uvw) = \angle uvw = 2\angle uvw'$ if $\angle uvw' = \pi/2$, and similarly $A(\angle uvw) = \angle uvw$ if $\angle uvw = \pi/2^r$ for any integer r. By additivity, and the continuity of the function A, we have $A(\angle uvw) = \angle uvw$ for every angle $\angle uvw$.

Note that we cannot use such direct argumentation in the case of central projection, since it is not clear *a priori* that the average value depends only on the magnitude of the angle. Set

$$A_S(\angle\ uvw) = \frac{1}{V(S)} \int_{p \in S} \angle\ \pi_p(u)\pi_p(v)\pi_p(w)\ dV_S\ .$$

This integral is invariant under rigid motions which keep the sphere fixed, but the group of such motions is not transitive on the sets of angles contained within S. The fact that $A_S(\angle uvw) = \angle uvw$ will follow however from the basic lemma.

AVERAGING LEMMA. *Let u, v, and w be distinct points contained within a sphere S and let C be a circle on S parallel to a plane containing u, v, and w. Let*

$$A_C(\angle\ uvw) = \frac{1}{l(C)} \int_{p \in C} \angle\ \pi_p(u)\pi_p(v)\pi_p(w)\ ds_C\ .$$

Then $A_C(\angle uvw) = \angle uvw$.

Proof. The proof proceeds in several steps, guided by an outline of the proof of the theorem of Euclid mentioned in Theorem 1.

The lemma is immediate if $\angle uvw = 0$ or π, so we assume that u, v, and w are not collinear.

a) If v lies on a diameter of S orthogonal to the plane of u, v, and w, then the value of $A_C(\angle uvw)$ depends only on the magnitude of $\angle uvw$, since any two such angles of equal magnitude in the same plane are congruent by a rotation of S about the diameter orthogonal to the plane. As in the classical case, we then show that $A_C(\angle uvw) = \angle uvw$.

b) If u, v, and w are on the surface of S and if the ray $\overrightarrow{vu}$ passes through the center o of the circle cut out on S by the plane of u, v, and w, we consider the reflection of S which fixes o and interchanges v and w to show that $A_C(\angle ovw)$

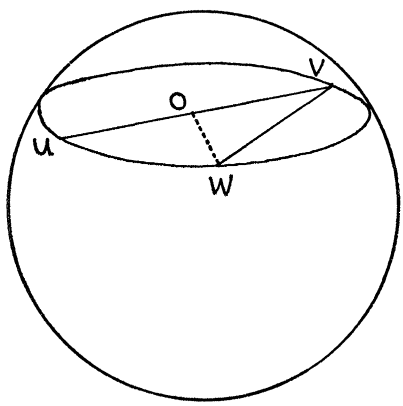


Figure for (b)

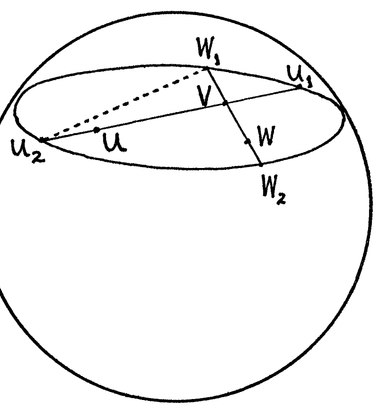


Figure for (d)

$= A_C(\angle owv)$. By part a), $\angle uow = A_C(\angle uow)$ so

$$2A_C(\angle ovw) = A_C(\angle ovw) + A_C(\angle owv) = A_C(\angle uow) = \angle uow$$
$$= \angle ovw + \angle owv = 2\angle ovw,$$

so $\angle uvw = \angle ovw = A_C(\angle ovw) = A_C(\angle uvw)$.

c) If u, v, and w are on S and if neither $\overrightarrow{vu}$ nor $\overrightarrow{vw}$ passes through o, we may introduce the auxiliary line $\overrightarrow{vo}$ and deduce $A_C(\angle uvw) = \angle uvw$ from the corresponding equalities for $\angle uvo$ and $\angle ovw$.

d) If u, v, and w are anywhere in S, extend $\overrightarrow{uv}$ to cut S in u_1 and $\overrightarrow{vu}$ to cut S in u_2 and obtain the points w_1 and w_2 similarly. Then $A_C(\angle uvw) = A_C(\angle vw_1u_2 + \angle vu_2w_1) = A_C(\angle vw_1u_2) + A_C(\angle vu_2w_1) = \angle vw_1u_2 + \angle vu_2w_1 = \angle uvw$. This concludes the proof of the averaging lemma.

Remark. If the circle C lies in the plane of u, v, and w, then this lemma reduces to Theorem 1.

We now use this lemma to show that $A_S(\angle uvw) = \angle uvw$, by arguing that since the average value of $\angle \pi_p(u)\pi_p(v)\pi_p(w)$ over all p in a circle C parallel to the plane of u, v, and w is $\angle uvw$, then the average over the whole sphere S must also be $\angle uvw$. Specifically, we choose a spherical coordinate system on S such that the circle in the plane of u, v, and w is given by $\phi =$ constant, and let $f(\theta, \phi, \angle uvw) = \angle \pi_p(u)\pi_p(v)\pi_p(w)$ where the coordinates of p are (θ, ϕ). For a circle C_0 given by $\phi = \phi_0$, we have $ds_{C_0} = r_0\, d\theta$ and

$$l(C_0) = 2\pi r_0 \quad \text{so} \quad A_{C_0}(\angle uvw) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta, \phi_0, \angle uvw)\, d\theta = \angle uvw$$

by the averaging lemma. Therefore

$$\begin{aligned} A_S(\angle uvw) &= \frac{1}{V(S)}\int_{p\in S} \angle \pi_p(u)\pi_p(v)\pi_p(w)\, dV_S \\ &= \frac{1}{4\pi r^2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} f(\theta, \phi, \angle uvw) r^2 \cos\phi\, d\theta\, d\phi \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2\pi}\int_0^{2\pi} f(\theta, \phi, \angle uvw)\, d\theta\right]\cos\phi\, d\phi \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \angle uvw \cos\phi\, d\phi \\ &= \angle uvw\,. \end{aligned}$$

Returning to the final step in the proof of Theorem 2, we have

$$\frac{1}{V(S)}\int_{p\in S}\frac{1}{2\pi}\left|\text{ext. angle of } \pi_p \circ f \text{ at } \pi_p \circ v_i\right| dV_S = \frac{1}{2\pi}\left|\text{ext. angle of } f \text{ at } v_i\right|$$

so the theorem is proved.

3. Total central curvature of curves in E^n. For a polygon $f: S^1 \to E^n$ contained in the region bounded by a hypersphere S we may again consider central projection $\pi_p : E^n - T_p \to E_p^{n-1}$ and define

$$\tau_S(f) = \frac{1}{V(S)} \int_{p \varepsilon S} \tau(\pi_p \circ f)\, dV_S$$

where dV_S is the element of volume of S so $V(S) = \int_{p\varepsilon S} dV_S$.

THEOREM 3. $\tau_S(f) = 2 \sum_{i=1}^{m} 1/2\pi$ $|$*ext. angle of f at* $v_i|$.

Proof. As in the proof of Theorem 2, we use the averaging lemma to show that $A_C(\angle uvw) = \angle uvw$ for any C cut out on S by a translate of the plane of u, v, and w, and deduce from this that the average value of $\angle \pi_p(u)\pi_p(v)\pi_p(w)$ over all of S is $\angle uvw$.

To make this explicit, we may parametrize the sphere S^{n-1} by $(\sin\theta_1$, $\cos\theta_1 \sin\theta_2$, $\cos\theta_1\cos\theta_2\sin\theta_3$, $\cdots$, $\cos\theta_1\cos\theta_2 \cdots \cos\theta_{n-2}\sin\theta_{n-1}$, $\cos\theta_1 \cdot \cos\theta_2 \cdots \cos\theta_{n-2}\cos\theta_{n-1})$ where the plane of u, v, and w is parallel to the plane spanned by x_{n-1} and x_n . The element of volume is then

$$d\omega^{n-1} = (\sin\theta_1)^{n-2}(\sin\theta_2)^{n-1} \cdots (\sin\theta_{n-2})$$

and

$$\begin{aligned} V(S^{n-1}) &= \int d\omega^{n-1} \\ &= \int_0^\pi \int_0^\pi \cdots \int_0^\pi \left(\int_0^{2\pi} (\sin\theta_1)^{n-2} \cdots (\sin\theta_{n-2})\, d\theta_{n-1}\right) d\theta_{n-2} \cdots d\theta_2\, d\theta_1 \\ &= \int_0^\pi \int_0^\pi \cdots \int_0^\pi 2\pi(\sin\theta_1)^{n-2} \cdots (\sin\theta_{n-2})\, d\theta_{n-2} \cdots d\theta_2\, d\theta_1 \ . \end{aligned}$$

If p is given by coordinates ρ, θ_1 , θ_2 , $\cdots$, θ_{n-2} , θ_{n-1} , we set $F(\rho, \theta_1, \theta_2, \cdots, \theta_{n-2}, \theta_{n-1}, f) = \tau(\pi_p \circ f)$. Then by the averaging lemma, for a given choice of θ_1^0 , θ_2^0 , $\cdots$, θ_{n-2}^0 , we have

$$\int_0^{2\pi} F(\rho, \theta_1^0, \theta_2^0, \cdots, \theta_{n-2}^0, \theta_{n-1}, f)\, d\theta_{n-1} = 2\pi\tau(f) \ .$$

Therefore

$$\begin{aligned} &\frac{1}{V(S)} \int_{p\varepsilon S} \tau(\pi_p \circ f)\, dV_S \\ &\qquad = \frac{1}{r^{n-1}V(S^{n-1})} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} F(\rho, \theta_1, \cdots, \theta_{n-2}, \theta_{n-1}) \\ &\qquad\quad \cdot (\sin\theta_1)^{n-2} \cdots (\sin\theta_{n-2}) r^{n-1}\, d\theta_{n-1}\, d\theta_{n-2} \cdots d\theta_1 \end{aligned}$$

$$= \frac{1}{V(S^{n-1})} \int_0^\pi \cdots \int_0^\pi 2\pi\tau(f)(\sin\theta_1)^{n-2} \cdots$$

$$\cdot(\sin\theta_{n-2})\, d\theta_{n-2} \cdots d\theta_1 = \tau(f) ,$$

and the theorem is proved.

COROLLARY 4. *If $f\colon S^1 \to S^3 \subset E^4$ and $\tau(f) < 4$ (as a curve in E^4), then f can be unknotted in S^3, i.e. there is a homeomorphism h of S^3 such that $h \circ f$ maps S^1 to a circle on S^3.*

Proof. By Theorem 3, $\tau(f) = 1/V(S^3) \int_{p\in S^3} \tau(\pi_p \circ f)\, dV_S$ so if $\tau(f) < 4$, then there is an open set U in S^3 such that $\tau(\pi_p \circ f) < 4$ for all p in U. It follows from the results of Milnor [3] that there is a homeomorphism g of E^3_p such that $g \circ \pi_p \circ f\colon S^1 \to E^3_p$ sends S^1 to a circle, and moreover such that g is the identity outside a compact set containing the image of $\pi_p \circ f$. The composed map $h = \pi_p^{-1} \circ g \circ \pi_p$ is then a homeomorphism and $h \circ f$ sends S^1 to the circle $\pi_p^{-1}(g \circ \pi_p \circ f(S^1))$ (the image of a circle under inverse stereographic projection).

Remark. Although the previous results have been global in character, it is possible to obtain a local version of Theorem 3 by using the global result. Specificially, if $f\colon S^1 \to E^n$ has image contained in the region bounded by a hypersphere S and if f has continuously defined curvature $|f''(x)|$ in a neighborhood of $x_0 \in S^1$, then $|f''(x_0)|$ equals the average of the curvatures $|(\pi_p \circ f)''(x_0))|$ of the curves $\pi_p \circ f$. For $\epsilon > 0$ is any sufficiently small number, the map $f\colon [x_0 - \epsilon, x_0 + \epsilon] \to E^n$ has continuous curvature on the interior and a well-defined tangent at each endpoint. We may then define a closed piecewise differentiable curve f_ϵ by going along f from $f(x_0 - \epsilon)$ to $f(x_0 + \epsilon)$ and then back along f to $f(x_0 - \epsilon)$. By Theorem 3,

$$\tau_S(f_\epsilon) = \tau(f_\epsilon) = 1 + \frac{1}{2\pi}\int_{x_0-\epsilon}^{x_0+\epsilon} |f''(s)|\, ds ,$$

where the constant 1 corresponds to the two zero angles at the singular points of f_ϵ, and

$$\tau_S(f_\epsilon) = \frac{1}{V(S)}\int_{p\in S} \tau(\pi_p \circ f)\, dV_S$$

$$= 1 + \frac{1}{V(S)}\int_{p\in S} \left(\frac{1}{2\pi}\int_{x_0-\epsilon}^{x_0+\epsilon} |(\pi_p \circ f)''(s)|\, ds\right) dV_S$$

$$= 1 + \frac{1}{2\pi}\int_{x_0-\epsilon}^{x_0+\epsilon} \left(\frac{1}{V(S)}\int_{p\in S} |(\pi_p \circ f)''(s)|\, dV_S\right) ds .$$

Since this is correct for any $\epsilon > 0$, it follows that

$$|f''(x_0)| = \frac{1}{V(S)}\int_{p\in S} |(\pi_p \circ f)''(x_0)|\, dV_S .$$

A similar local result has been obtained by other methods for smooth curves on the 2-sphere by P. Dombrowski (unpublished).

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BROWN UNIVERSITY