Self Linking Numbers of Space Polygons

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If a smooth closed space curve with nowhere vanishing curvature is pushed a small distance away from itself along its principal normal vectors to a new curve X_ϵ then the linking number of X with X_ϵ is independent of ϵ for ϵ sufficiently small and this number is called the *self-linking* number of X. This notion was first studied by Călugăreanu [3] using classical analytic techniques, and subsequently by Pohl [5] who used methods involving differential forms to establish a large number of properties of this self-linking number.

The purpose of this paper is to develop the theory of self-linking for polygons in 3-space in all of its ramifications. The proofs primarily rest upon a basic projection theorem which makes it possible to compute self-linking numbers simply and effectively by projecting to a plane and totalling numbers of apparent crossings and (signed) pairs of apparent inflection points.

The first section proves the basic theorem by deformation methods similar to those of [1]. Section 2 treats linking numbers of pairs of polygons in the same spirit and obtains a finite form of the *Gauss integral.* In section 3, we obtain integral formulas for the self-linking number of a polygon including a polygonal analogue of theorems involving total torsion of closed curves. Section 4 presents the connection with the definitions of Călugăreanu and Pohl in terms of normal variations and intersections with developable surfaces associated with the curve. Section 5 treats the behavior of self-linking under deformations, and the final section indicates how these concepts can be generalized to higher dimensional polyhedra.

James White and William Pohl have contributed helpful comments to this project and the author is grateful.

1. Definition of the self-linking number for a space polygon. Let $X = (X_0, X_1, \cdots, X_{m-1})$ be a space polygon determined by a cycle of vertices in general position. We set $T_i = (X_{i+1} - X_i)/|X_{i+1} - X_i||$, the unit *tangent* vector in the direction of the ith edge (where all subscripts are to be reduced modulo m). Set $B_i = (T_{i-1} \times T_i)/||T_{i-1} \times T_i||$, the unit *binormal* vector to the oriented plane containing both edges of X at X_i.

The vectors B_i and B_{i+1} are orthogonal to T_i so $B_i \times B_{i+1}$ is a multiple of T_i. We set $f_i(X) = \pm 1$ depending on the sign of $B_i \times B_{i+1} \cdot T_i$, indicating whether

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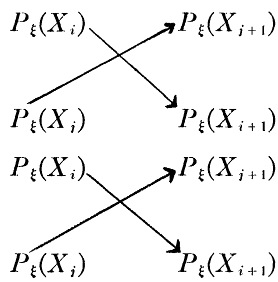
the ith edge has *positive* or *negative* torsion. The *torsion angle* $\varphi_i(X)$ is the signed angle from B_i to B_{i+1} in the oriented plane orthogonal to T_i.

For any point ξ of S^2, we let $P_\xi : E^3 \to E^2(\xi^\perp)$ denote the orthogonal projection $P_\xi(Y) = Y - (Y\cdot\xi)\xi$. Such a point ξ is said to be general for X if $P_\xi X$ is a polygon with vertices in general position in $E^2(\xi^\perp)$.

Set $c_{ij}(X)$ = algebraic sign of $(X_{i+1} - X_i) \times (X_{j+1} - X_j)\cdot(X_j - X_i)$. If $P_\xi(\bar{X}_i) = P_\xi(\bar{X}_j)$ for interior points $\bar{X}_i$ of the edge X_iX_{i+1} and $\bar{X}_j$ in X_jX_{j+1}, then we index this crossing by setting $C_{ij}(X, \xi) = \pm 1$ depending on the algebraic sign of $P_\xi(X_{i+1} - X_i) \times P_\xi(X_{j+1} - X_j)\cdot(\bar{X}_j - \bar{X}_i)$ (which will be the index $c_{ij}(X)$).

Thus

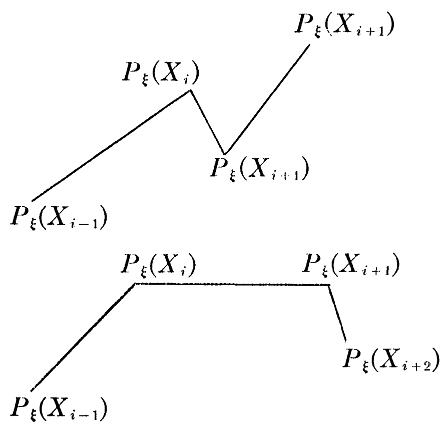
$$C_{ij}(X, \xi) = \begin{cases} -1 & \text{if} \\ +1 & \text{if} \end{cases}$$



Note that $C_{ij}(X, -\xi) = C_{ij}(X, \xi)$.

If $P_\xi(X_{i-1})$ and $P_\xi(X_{i+2})$ lie on opposite sides of the line through $P_\xi(X_i)$ and $P_\xi(X_{i+1})$, we set $F_i(X, \xi) = f_i(X)$ and set $F_i(X, \xi) = 0$ otherwise.

$$F_i(X, \xi) = \begin{cases} f_i(X) & \text{if} \\ 0 & \text{if} \end{cases}$$



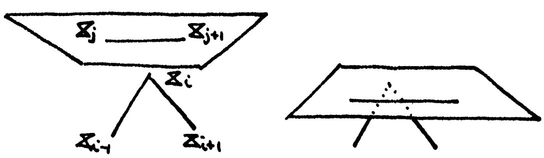
The sum $C(X, \xi) = \sum_{i<j} C_{ij}(X, \xi)$ is the algebraic number of crossings of X in the direction ξ and $F(X, \xi) = \sum_i F_i(X, \xi)$ is the algebraic number of inflection edges of X in the direction ξ. Note that $F(X, \xi)$ is even since any planar polygon has an even number of inflection edges.

Note that neither $C(X, \xi)$ nor $\frac{1}{2}F(X, \xi)$ is independent of the vector ξ. For example a quadrilateral formed from the vertices of a tetrahedron will have no crossings or inflections in some directions, no crossing and two inflections in others, and one crossing and two inflections in still others.

We set $SL(X, \xi) = C(X, \xi) + \frac{1}{2}F(X, \xi)$, *the self-linking number of* X *in the direction* ξ.

Theorem 1. *The sum* $SL(X, \xi)$ *is independent of the general vector* ξ *used to define it. This common value is what we call the self-linking number* $SL(X)$ *of* X.

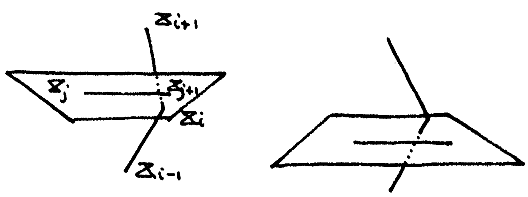
Proof. Given two vectors ξ_0 and ξ_1 which are both general for X, if we can find a spherical polygonal path $\xi(t)$, $0 \leqq t \leqq 1$ with $\xi(0) = \xi_0$ and $\xi(1) = \xi_1$ and $\xi(t)$ general for X for all t, then $C_{ij}(X, \xi(t))$ and $F_i(X, \xi(t))$ will be constant. Any two unit vectors general for X can be joined by a spherical polygon such that at only finitely many positions $\xi(t)$ in the interior of edges will the vector fail to be general for X, and such that exactly one triple of vertices of $P_{\xi(t)}X$ will be collinear at each exceptional position. We may enumerate all possible collineations for which any of the indices $C_{ij}(X, \xi(t))$ or $F_i(X, \xi(t))$ can change, and check the changes $\Delta C(X, \xi(t))$ and $\Delta F(X, \xi(t))$ in passing such an exceptional position. An exceptional position occurs precisely when $\xi(t)$ passes a great circle parallel to a plane containing three vertices of X, and we may illustrate the changes by exhibiting the views just below and just above this plane.



$\Delta C_{i-1,j}(X, \xi(t)) = -1$

$\Delta C_{ij}(X, \xi(t)) = 1$

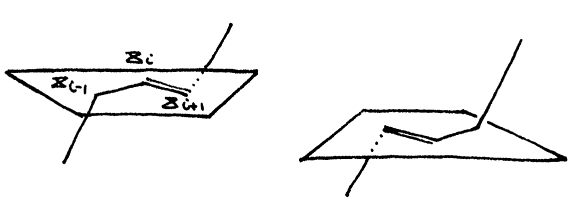
$\Delta SL(X, \xi(t)) = 0$



$\Delta C_{i-1,j}(X, \xi(t)) = 1$

$\Delta C_{ij}(X, \xi(t)) = -1$

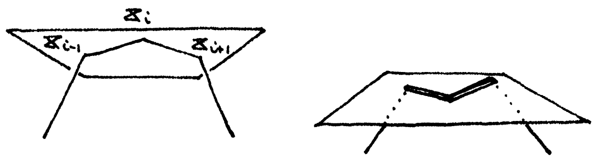
$\Delta SL(X, \xi(t)) = 0$



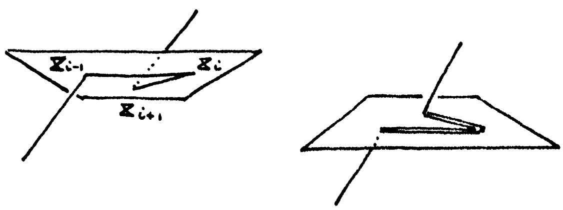
$\Delta F_{i-1}(X, \xi(t)) = -1$

$\Delta F_i(X, \xi(t)) = +1$

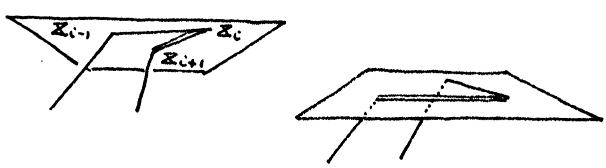
$\Delta SL(X, \xi(t)) = 0$



$$\Delta F_{i-1}(X, \xi(t)) = -1$$
$$\Delta F_i(X, \xi(t)) = +1$$
$$\Delta SL(X, \xi(t)) = 0$$



$$\Delta C_{i-1\ i+1}(X, \xi(t)) = +1$$
$$\Delta F_{i-1}(X, \xi(t)) = +1$$
$$\Delta F_i(X, \xi(t)) = +1$$
$$\Delta SL(X, \xi(t)) = 0$$



$$\Delta C_{i-1\ i+1}(X, \xi(t)) = +1$$
$$\Delta F_i(X, \xi(t)) = +1$$
$$\Delta F_{i-1}(X, \xi(t)) = +1$$
$$\Delta SL(X, \xi(t)) = 0$$

All other situations involving index changes either for C_{ij} or F_i can be obtained by reversing the orientations in the above pictures. Since $SL(X, \xi(t))$ remains constant in passing each of these exceptional positions, the values of $SL(X, \xi_0)$ and $SL(X, \xi_1)$ are the same, so the theorem is proved.

Remark. If $R : E^3 \to E^3$ is a reflection through a plane, then $C(RX, \xi) = -C(X, \xi)$ and $F(RX, \xi) = -F(X, \xi)$ so $SL(RX) = -SL(X)$.

Remark. We may also determine the self-linking number of a polygon by using central projections.

Given a general polygon X and a point Q of E^3 not on X, we may consider the projection of $E^3 - \{Q\}$ to the 2-sphere $S^2(Q)$ of radius 1 with center Q given by $R_Q : E^3 - \{Q\} \to S^2(Q)$, $R_Q(Y) = (Y - Q)/\|Y - Q\|$. The image of X under R_Q then has crossings and apparent inflections, and the algebraic numbers of these are denoted by

$$C(X, Q) = \sum_{i<j} C_{ij}(X, Q) \quad \text{and} \quad F(X, Q) = \sum_i F_i(X, Q)$$

respectively.

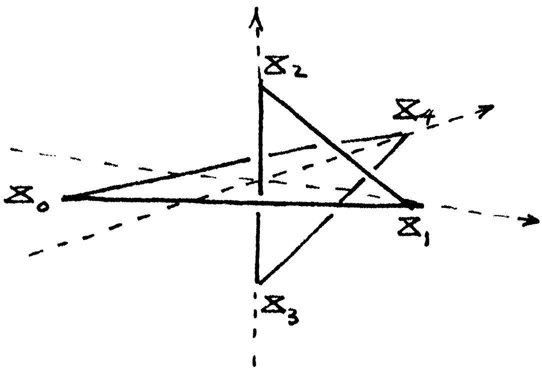
Theorem 2. *The number $SL(X, Q) = C(X, Q) + \frac{1}{2}F(X, Q)$ is independent of Q and it equals $SL(X)$.*

The proof of the independence is by a standard deformation argument, and the identification with $SL(X)$ is given by taking a sequence of Q's moving along a ray $\overline{Q\xi}$ towards infinity, so that ultimately $SL(X, Q) = SL(X, \xi) = SL(X)$.

Remark. Additional results concerning the absolute numbers of crossings and inflections proved by deformation arguments are found in the author's paper [1].

Examples. For any quadrilateral Q embedded in E^3, there is a direction for which the projection to the orthogonal plane is a convex polygon, so $SL(Q) = 0$. More generally, if a polygon X in E^3 has a projection which is locally convex then, the self-linking number is the algebraic number of crossing points. (This result in the smooth case is mentioned by Pohl [5]).

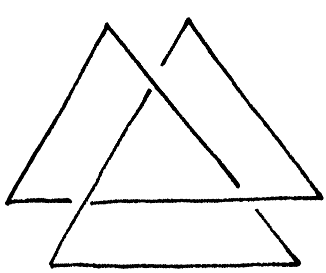
The simplest example of a polygon with non-zero self-linking number is a pentagon which does not lie on the boundary of its convex hull, for example, $X_0 = (-1, -1, 0)$, $X_1 = (1, 0, 0)$, $X_2 = (0, 0, 1)$, $X_3 = (0, 0, -1)$, $X_4 = (0, 1, 0)$.

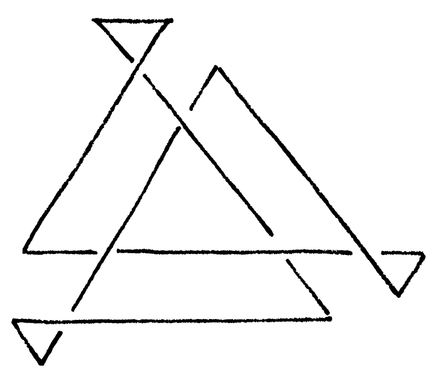


This polygon has a locally convex projection with a single crossing so $SL(X) = \pm 1$.

Note that this polygon represents a trivial knot, and indeed there is a projection for which the image is a simple curve, but with a pair of inflection edges both of the same algebraic sign.

A trefoil knot K has a locally convex projection with three crossings all of the same sign, so $SL(K) = \pm 3$. On the other hand, we may put in additional loops to alter the self-linking number without changing the knot type, and in particular, a polygon X with $SL(X) = 0$ may be knotted





2. Linking of pairs of polygons and the Gauss integral. For a non-intersecting pair of oriented smooth curves $X(u)$ and $Y(v)$, Gauss defined the linking number $L(X, Y)$ in terms of an integral expressing the degree of the secant mapping $(u, v) \to (X(u) - Y(v))/||X(u) - Y(v)||$ from the torus $\{(u, v) \mid 0 \leqq u \leqq 2\pi, 0 \leqq v \leqq 2\pi\}$ to the unit sphere S^2. We may compute this degree by choosing a point ξ in S^2 such that $P_\xi(X(u)) = P_\xi(Y(v))$ for only finitely many pairs (u, v). The degree of the mapping is obtained by adding the algebraic number of such crossings $C(X, Y, \xi)$, where the sign is the sign of

$$P_\xi(X(u)) \times P_\xi(Y(v)) \cdot (X(u) - Y(v)).$$

For polygons X and Y we consider a unit vector ξ such that $P_\xi(X)$ and $P_\xi(Y)$ are in general position. We set $C_{ij}(X, Y, \xi)$ equal to ± 1 depending on the sign $c_{ij}(X, Y)$ of $P_\xi(X_{i+1} - X_i) \times P_\xi(Y_{j+1} - Y_j) \cdot (\bar{X}_i - \bar{Y}_j)$ if $P_\xi(\bar{X}_i) = P_\xi(\bar{Y}_j)$ for interior points $\bar{X}_i$ of X_iX_{i+1} and $\bar{Y}_j$ of Y_jY_{j+1}, and 0 otherwise. Note that $C_{ij}(X, Y, -\xi) = C_{ij}(X, Y, \xi)$.

The integral

$$\frac{1}{4\pi} \int_{\xi \in S^2} C(X, Y, \xi)\, d\omega_2$$

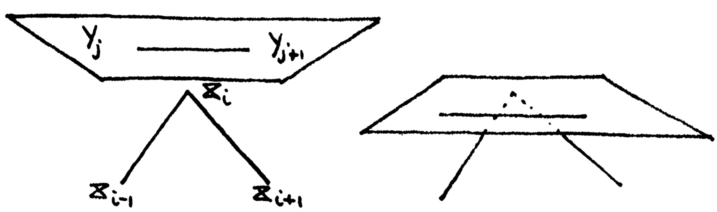
is called the *Gauss integral*, and for polygons we may express this as

$$= \frac{1}{4\pi} \int_{\xi \in S^2} \sum_{i,j} C_{ij}(X, Y, \xi)\, d\omega_2 = \sum_{i,j} \frac{1}{4\pi} \int_{\xi \in S^2} C_{ij}(X, Y, \xi)\, d\omega_2 .$$

As in the previous section, we may examine the dependence of $C(X, Y, \xi)$ on the general vector ξ.

Proposition. *$C(X, Y, \xi)$ is independent of the general vector ξ.*

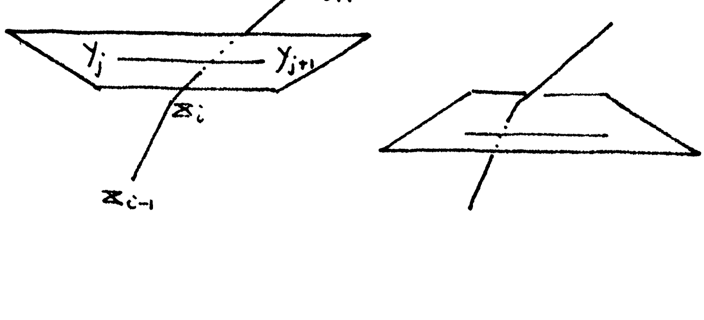
Proof. We find a path from ξ_0 to ξ_1 so that there are only finitely many collineations, involving a single vertex from one of the polygons and a pair of consecutive vertices from the other.



$\Delta C_{i-1\ j} = -1$

$\Delta C_{ij} = +1$

$\Delta \sum C_{ij} = 0$



$\Delta C_{i-1\ j} = -1$

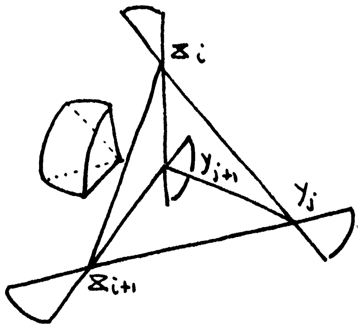
$\Delta C_{ij} = +1$

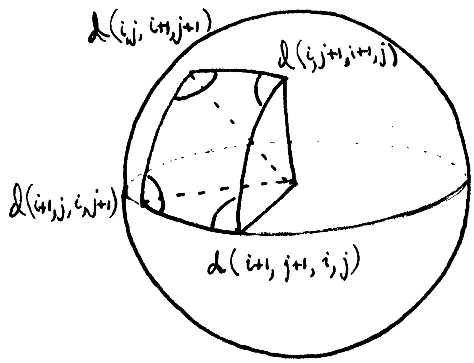
$\Delta \sum C_{ij} = 0$

This constant value $C(X, Y, \xi)$ is the linking number of X and Y and it can be expressed as the sum of the geometric quantities

$$C_{ij}(X, Y) = \frac{1}{4\pi} \int_{\xi \varepsilon S^2} C_{ij}(X, Y, \xi)\, d\omega_2 ,$$

representing the geometric probability that the ith edge of X crosses the jth edge of Y when projected along ξ.





This quantity represents the ratio of the oriented area of two antipodal quadrangles to the area of the whole sphere. The area of this quadrangle can be expressed in terms of the angles of the spherical polygon, and each such angle is a dihedral angle of the tetrahedron with vertices $X_iX_{i+1}Y_jY_{j+1}$. We set $d(i, j, k, l)$ = dihedral angle from the triangle $X_iX_kY_j$ to $X_iY_jY_l$ at the edge X_iY_j. We then have

$$\int_{\xi \varepsilon S^2} C_{ij}(X, Y, \xi)\, d\omega_2 = C_{ij}(X, Y) = -2\pi + d(i, j, i+1, j+1)$$
$$+ d(i+1, j, i, j+1) + d(i+1, j+1, i, j) + d(i, j+1, i+1, j).$$

Examples.

$X_0 = (-2, 0, 0)$ $\quad Y_0 = (2, 0, 0)$

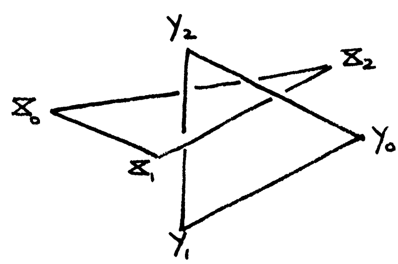
$X_1 = (1, -1, 0)$ $\quad Y_1 = (-1, 0, -1)$

$X_2 = (1, 1, 0)$ $\quad Y_2 = (-1, 0, 1)$

$c_{00}(X, Y) = +1$ $\quad c_{10} = -1$ $\quad c_{20} = +1$

$c_{01}(X, Y) = -1$ $\quad c_{11} = -1$ $\quad c_{21} = -1$

$c_{02}(X, Y) = +1$ $\quad c_{12} = -1$ $\quad c_{22} = +1$



(The indices c_{ij} are as defined in the previous section for a single polygon.) Thus

$$4\pi L(X, Y) = \sum_{i,j} \int_{\xi \epsilon S^2} C_{ij}(X, Y, \xi)\, d\omega_2 = \sum_{i,j} c_{ij} \text{ Area of } Q(i, i+1, j, j+1)$$

$$\begin{aligned}
&= (+1)\,(2\pi - d(0011) - d(1001) - d(1100) - d(0110))\\
&\quad(-1)\,(2\pi - d(0112) - d(1102) - d(1201) - d(0211))\\
&\quad(+1)\,(2\pi - d(0210) - d(1200) - d(1002) - d(0012))\\
&\quad(-1)\,(2\pi - d(1021) - d(2101) - d(2110) - d(1120))\\
&\quad(-1)\,(2\pi - d(1122) - d(2112) - d(2211) - d(1221))\\
&\quad(-1)\,(2\pi - d(1220) - d(2210) - d(2012) - d(1022))\\
&\quad(+1)\,(2\pi - d(2001) - d(0021) - d(0120) - d(2100))\\
&\quad(-1)\,(2\pi - d(2102) - d(0122) - d(0221) - d(2201))\\
&\quad(+1)\,(2\pi - d(2200) - d(0220) - d(0022) - d(2002)) = -2\pi - 2\pi = -4\pi
\end{aligned}$$

Since $d(0011) + d(0012) + d(0022) + d(0021) = 2\pi$ and since all other expressions $d(i, j, i+1, j+1) + d(i, j, i+1, j-1) + d(i, j, i-1, j-1) + d(i, j, i-1, j+1)$ are 0, thus $L(X, Y) = -1$. By reversing the orientation of X, we obtain $L(-X, Y) = 1$.

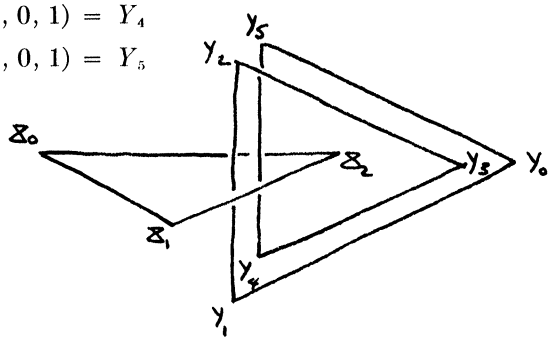
Example.

$X_0 = (-2, 0, 0) \qquad Y_0 = (2, 0, 0) = Y_3$

$X_1 = (1, -1, 0) \qquad Y_1 = (-1, 0, 1) = Y_4$

$X_2 = (1, 1, 0) \qquad Y_2 = (-1, 0, 1) = Y_5$

$$(c_{ij}) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$



note that

$$2\pi = d(0011) + d(0015) + d(0055) + d(0051)$$

$$2\pi = d(0314) + d(1304) + d(1403) + d(0413)$$

and all other expressions

$$d(i, j, i+1, j+1) + d(i, j, i+1, j-1) + d(i, j, i-1, j-1) + d(i, j, i-1, j+1) = 0.$$

Therefore,

$$4\pi L(X, Y) = \sum_{i,j} 2\pi c_{ij} + \sum_{i,j} (c_{ij}d(i, j, i+1, j+1) + c_{ij-1}d(i, j, i+1, j-1) + c_{i-1,j-1}d(i, j, i-1, j-1) + c_{i-1,j}d(i, j, i-1, j+1)$$
$$= -8\pi$$

Thus the linking number $L(X, Y) = -2$.

More generally, we may consider an r-fold covering $\tilde{Y}^r$ of the polygon Y, given by $\tilde{Y}_{ir} = Y_{i(\mathrm{mod}\ r)}$, $j = 0, 1, \cdots, rm - 1$ and we obtain $L(X, \tilde{Y}^r) = rL(X, Y)$. If we consider an s-fold covering $\tilde{X}^s$ of X at the same time, then $L(\tilde{X}^s, \tilde{Y}^r) = srL(X, Y)$.

As illustrated in the previous examples, the expressions for the dihedral angles combine by fours to form numbers

$$2\pi w_{ij}(X, Y) = c_{ij}d(i, j, i+1, j+1) + c_{i,j-1}d(i, j, i+1, j-1) + c_{i-1,j-1}d(i, j, i-1, j-1) + c_{i-1,j}d(i, j, i-1, j+1)$$

The expression $w_{ij}(X, Y)$ is called the *winding number* of the quadrilateral $(X_{i-1}, Y_{j-1}, X_{i+1}, Y_{j+1})$ about the oriented line in E^3 determined by the edge X_iY_j. This number will be 0 or ± 1, and this leads to a special expression for the linking number for a pair of polygons.

Theorem 3. *For any pair of polygons in X and Y general position, the linking number*

$$L(X, Y) = \sum_{i,j} c_{ij}(X, Y) + \sum_{i,j} w_{ij}(X, Y).$$

Comment. As we deform the polygons X and Y continuously in E^3, we may show that the expression

$$\sum c_{ij}(X, Y) + \sum w_{ij}(X, Y) = L(X, Y)$$

changes by ± 1 whenever an edge of X passes through an edge of Y. Small variations of position of the vertices leave all quantities unchanged, and we may therefore assume that the vertices of X and Y together form a set in general position in E^3. We may then move X parallel to itself through a family $X(t)$ of polygons so that an edge of $X(t)$ meets Y at only a finite number of positions $X(\bar{t})$. If the ith edge of $X(\bar{t})$ meets the jth edge of Y, then $c_{ij}(X(\bar{t} + \epsilon), Y) - c_{ij}(X(\bar{t} - \epsilon), Y) = \pm 1$ and $w_i(X(\bar{t} + \epsilon), Y) - w_i(X(\bar{t} - \epsilon), Y) = \mp 1$.

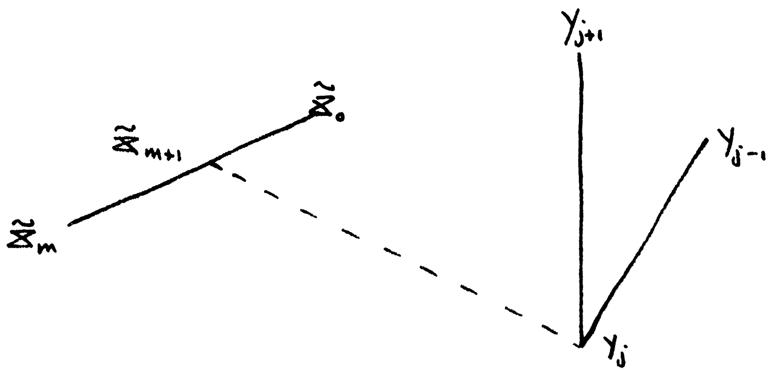
We may also investigate the behavior of the crossing indices for pairs of edges and the winding indices for diagonals as we introduce new vertices into the polygon X. Relabel the vertices cyclically if necessary so that the new vertex $\tilde{X}_{m+1}$ is the midpoint of the old edge X_mX_0. Then $c_{ij}(\tilde{X}, Y) = c_{ij}(X, Y)$

for $0 \leqq i < m$ and $c_{m+1,j}(\tilde{X}, Y) = c_{m,j}(X, Y) = c_{m,j}(\tilde{X}, Y)$ so

$$\sum_{i,j} c_{i,j}(\tilde{X}, Y) = \sum_{i,j} c_{ij}(X, Y) + \sum_{j=0}^{m-1} c_{mj}(X, Y).$$

Also $w_{ij}(\tilde{X}, Y) = w_{ij}(X, Y)$ for $0 \leqq i \leqq m$ and

$$\sum_{i,j} w_{ij}(\tilde{X}, Y) = \sum_{i,j} w_{ij}(X, Y) + \sum_{j=0}^{m-1} w_{m+1,j}(\tilde{X}, Y).$$



Note that $w_{m+1,j}(\tilde{X}, Y) = \pm 1$ if Y_{j-1} and Y_{j+1} lie on opposite sides of the plane through X_0, X_m, and Y_j, and 0 otherwise. If $w_{m+1,j}(\tilde{X}, Y) = \pm 1$, then $w_{m+1,j}(\tilde{X}, Y) + c_{m,j}(\tilde{X}, Y) = 0$. Thus $L(\tilde{X}, Y) - L(X, Y) =$ sum of indices $c_{m,j}(\tilde{X}, Y)$ for those vertices with $w_{m+1,j}(\tilde{X}, Y) = 0$. But these indices will alternate in sign and there will be an even number of them since the polygon is closed, so $L(\tilde{X}, Y) = L(X, Y)$, as expected.

3. The Gauss integral and total torsion for a polygon. In this section we use the results of the first two sections to obtain an integral formula for the self-linking number of a space polygon, and we then show how this interpretation gives a new result for the self-linking of a smooth curve.

As in the smooth case (as mentioned in [2] and [5]), if we take the Gauss integral of the previous section in the case where both polygons coincide, we no longer obtain an integer for

$$\frac{1}{4\pi} \int_{\xi \in S^2} \sum_{i,j} C_{ij}(X, X, \xi)\, d\omega_2 .$$

However

$$SL(X) = \frac{1}{4\pi} \int_{\xi \in S^2} SL(X, \xi)\, d\omega_2$$

$$= \frac{1}{4\pi} \int_{\xi \varepsilon S^2} (C(X, \xi) + \tfrac{1}{2}F(X, \xi))\, d\omega_2$$

$$= \frac{1}{4\pi} \int_{\xi \varepsilon S^2} \sum_{i,j} C_{ij}(X, \xi)\, d\omega_2 + \frac{1}{4\pi}\left(\frac{1}{2}\right) \int_{\xi \varepsilon S^2} F_i(X, \xi)\, d\omega_2 ,$$

so as in the smooth case, there is a "correction term" so that the sum of the Gauss integral and another integral over the curve is an integer, the self-linking number. In the smooth case, this "correction term" is shown by Pohl and by Călugăreanu to be the *total torsion* of the curve $\int_C \tau(s)\, ds$.

In the polygonal situation, the integral $\int_{\xi \varepsilon S^2} F_i(X, \xi)\, d\omega_2$ represents $F(X_i)$ multiplied by the geometric probability that $P_\xi(X)$ has an inflection edge at $P_\xi(X_i - X_{i+1})$. At the line through $X_i X_{i+1}$, the two half planes containing X_{i-1} and X_{i+1} respectively form a wedge, and the projection $P_\xi(X)$ has an inflection edge at $P_\xi(X_i - X_{i+1})$ if and only if the vector ξ or $-\xi$ centered at the midpoint of the edge lies in the wedge. It follows that $\int_{\xi \varepsilon S^2} F_i(X, \xi)\, d\omega_2 = 4F(X_i)$ (dihedral angle of the wedge at $X_i X_{i+1}$) $= 4F(X_i)\, d(i, i + 1, i - 1, i + 2)$. But $F(X_i)$ multiplied by this dihedral angle is precisely the signed angle $\varphi_i(X)$ between the binormal vectors B_i and B_{i+1}, and therefore

$$\frac{1}{4\pi}\left(\frac{1}{2}\right) \int_{\xi \varepsilon S^2} F_i(X, \xi)\, d\omega_2 = \frac{1}{2\pi} \sum_i \varphi_i(X).$$

This quantity represents the total turning of the binormal vector about the tangent vector, and this yields the *total torsion* of the polygon, $\tau(X) = \sum_i \varphi_i(X)$. We thus have

Theorem 4. *For a general space polygon X, $SL(X)$ is the sum of the Gauss integral*

$$\frac{1}{4\pi} \int_{\xi \varepsilon S^2} \sum_{i,j} C_{ij}(X, \xi)\, d\omega_2$$

and the normalized total torsion

$$\frac{1}{2\pi}\, \tau(X) = \frac{1}{2\pi} \sum_i \varphi_i(X).$$

Remark. In [4], Milnor established that the average (absolute) number of inflection points of projections of a smooth curve into planes is the total absolute torsion of the curve

$$\frac{1}{2\pi} \int_C |\tau(s)|\, ds = |\tau|\, (C).$$

The above theorem may be considered a polygonal analogue of a slightly refined version of this theorem which takes into account the sign of the torsion.

Even for smooth curves, the interpretation of the self-linking number in terms of averages of projections is new.

For a smooth curve X with curvature always positive and torsion changing sign a finite number of times, the self-linking number of X is given by $SL(X, \xi) = C(X, \xi) + \frac{1}{2}F(X, \xi)$ for any general direction, where $C(X, \xi)$ is the (finite) algebraic number of crossings of $P_\xi(X)$ and $F(X, \xi)$ is the algebraic number of inflection points of $P_\xi(X)$, each counted with the algebraic sign of the torsion of the point on the curve X.

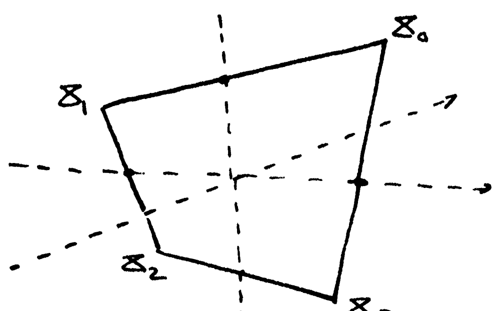
This makes it a relatively easy matter to calculate the self-linking number of any smooth or polygonal curve.

Remark. Polygons were also considered by Călugăreanu in [3], but using a much more complicated technique.

We may now evaluate the self-linking numbers of some polygons using the integral formula of Theorem 4:

$$\begin{aligned} 4\pi SL(X) &= \int_{\xi\varepsilon S^2} \sum_{i,j} C_{ij}(X, \xi)\, d\omega_2 + \int_{\xi\varepsilon S^2} \frac{1}{2} \sum_i F_i(X, \xi)\, d\omega_2 \\ &= \sum_{i,j} c_{ij}(X)(+2\pi - d(ij, i+1, j+1) - d(i+1, j, i, j+1) \\ &\qquad - d(i+1, j+1, I, j) - d(i, j+1, i+1, j)) \\ &\qquad + 2 \sum_i f_i(X) d(i, i+1, i-1, i+2) \end{aligned}$$

Example. For a quadrilateral with vertices $X_0 = (1, 1, 1)$, $X_1 = (-1, -1, 1)$, $X_2 = (-1, 1, -1)$, $X_3 = (1, -1, -1)$, we have $c_{02}(X) = 1 = c_{20}(X)$, $c_{13}(X) = -1 = c_{31}(X)$, $f_0(X) = -1 = f_2(X)$, and $f_1(X) = +1 = f_3(X)$. Thus



$$\begin{aligned} 4\pi SL(X) &= 2(2\pi - d(0213) - d(1203) - d(1302) - d(0321)) \\ &\quad - 2(2\pi - d(1320) - d(2310) - d(2013) - 3d(1023)) \\ &\quad - 2d(0123) + 2d(1203) - 2d(2301) + 2d(3012) \\ &= 0 \quad \text{since} \quad d(ijkl) = d(jikl) = d(jilk) = d(ijlk). \end{aligned}$$

For the pentagon $X_0 = (-1, -1, 0)$, $X_1 = (1, 0, 0)$, $X_2 = (0, 0, 1)$, $X_3 =$

$(0, 0, -1)$, $X_4 = (0, 1, 0)$, we have $c_{02} = -1$, $c_{03} = 1$, $c_{13} = -1$, $c_{14} = 1$, $c_{24} = -1$, $f_0 = 1$, $f_1 = -1$, $f_2 = -1$, $f_3 = -1$, $f_4 = 1$. Then

$$\begin{aligned}
4\pi SL(X) &= -2(2\pi - d(0213) - d(1203) - d(1302) - d(0312)) \\
&\quad + 2(2\pi - d(0314) - d(1304) - d(1403) - d(0413)) \\
&\quad - 2(2\pi - d(1324) - d(2314) - d(2413) - d(1423)) \\
&\quad + 2(2\pi - d(1420) - d(2410) - d(2014) - d(1024)) \\
&\quad - 2(2\pi - d(2430) - d(3420) - d(3024) - d(2034)) \\
&\quad + 2d(0124) - 2d(1203) - 2d(2314) - 2d(3420) + 2d(4013) \\
&= -4\pi + 2(+d(0213) - d(2014) + d(2034) \\
&\qquad + d(1302) - d(1304) + d(1324) \\
&\qquad + d(0312) - d(0314) + d(0324) \\
&\qquad + d(1403) + d(1423) - d(1420) \\
&\qquad + d(2413) - d(2410) + d(2430) \\
&= -4\pi
\end{aligned}$$

Similarly we have for *any* pentagon X,

$$SL(X) = \left(\sum_{i=0}^{4} c_{i,i+2}\right) + \sum_{i=0}^{4} w(i, i+2)$$

where

$$\begin{aligned}
w(i, i+2) = \frac{1}{2\pi} [(c_{i,i+1})\, d(i, i+2, i+1, i+3) \\
+ c_{i-1,i+1} d(i, i+2, i+1, i-1) + c_{i-1,i+2} d(i, i+2, i+3, i-1)]
\end{aligned}$$

represents the winding number of the triangle with vertices $X_{i-1}X_{i+1}X_{i+3}$ about the oriented line determined by the edge X_iX_{i+2}.

It follows that a pentagon has odd self-linking number if and only if there are an even number of diagonals X_iX_{i+2} which pass through the interior of the convex hull of the vertices. In particular if there are only four extreme vertices, so that some vertex, say X_2, is in the convex hull of the other four vertices, then there are precisely two interior diagonals, X_0X_2 and X_2X_4, so the self-linking number must be odd. Since there is a projection to a plane with non-self intersecting image and only two inflection edges, the self-linking number of such a pentagon will be ± 1. If the pentagon lies on the boundary of its convex hull, then there will be precisely one interior diagonal and the self-linking number must be even. But any such pentagon has a convex projection so the self-linking number is 0.

We may summarize this information about pentagons in a theorem:

Theorem 5. *A pentagon in general position in E^3 has self-linking number* 0 *or* ± 1 *depending on whether or not it lies in the boundary of its convex hull.*

For polygons of more than five vertices, we may carry out a similar analysis to show that

$$SL(X) = \sum_{i,j} c_{ij}(X) + \sum_{i,j} w_{ij}(X),$$

where now

$$2\pi w_{ij}(X) = c_{ij}d(i, j, i + 1, j + 1) + c_{i,j-1}d(i, j, i + 1, j - 1) + c_{i-1,j-1}d(i, j, i - 1, j - 1) + c_{i-1,j}(i, j, i - 1, j + 1),$$

as in the case of the linking number of a pair of polygons.

4. Linking with normal variation curves and cross-normal indices. In [3], Călugăreanu defined the self-linking number as the limit as $\epsilon \to 0$ of the linking number of $X(t)$ with the normal variation curve $X(t) + \epsilon N(t)$, where $N(t)$ is the principal normal. In this section we show how the same concept may be carried over to the polygonal case.

By the principal normal indicatrix polygon N of a space polygon X with vertices in general position, we mean a piecewise circular curve on S^2 with $2m$ vertices defined by setting $N_{2i} = T_i \times B_i$ and $N_{2i+1} = T_i \times B_{i+1}$ for $1 = 0, \cdots, m - 1$. This notion has been investigated in [2]. The polygon $X + \epsilon N$ will then have $2m$ segments, from $X_i + \epsilon N_{2i}$ to $\hat{X}_i + \epsilon N_{2i}$ and $\hat{X}_i + \epsilon N_{2i+1}$ to $X_{i+1} + \epsilon N_{2i+1}$ (where $\hat{X}_i = \frac{1}{2}(X_i + X_{i+1})$), and $2m$ circular arcs $X_i + \epsilon N_{2i-1}$ to $X_i + \epsilon N_{2i}$ and $\hat{X}_i + \epsilon N_{2i}$ to $\hat{X}_i + \epsilon N_{2i+1}$.

For sufficiently small ϵ, and for a projection P_ξ which is general for X, the edge $P_\xi(X_i X_{i+1})$ meets the edge $P_\xi(X_j, X_{j+1})$ if and only if exactly one of the image segments $P_\xi(X_i + \epsilon N_{2i}, \hat{X}_i + \epsilon N_{2i})$ or $P_\xi(\hat{X}_i + \epsilon N_{2i+1}, X_{i+1} + \epsilon N_{2i+1})$ meets $P_\xi(X_j, X_{j+1})$, and the crossing indices will be the same. The images of the arcs from $X_i + \epsilon N_{2i-1}$ to $X_i + \epsilon N_{2i}$ will not intersect $P_\xi X$ and the image of the arc from $\hat{X}_i + \epsilon N_{2i}$ to $\hat{X}_i + \epsilon N_{2i+1}$ meets $P_\xi(X_i, X_{i+1})$ if and only if $P_\xi(X_i, X_{i+1})$ is an inflection edge for $P_\xi(X)$, and the sign of the crossing is the sign of the inflection. It follows that $L(X, X + \epsilon N) = SL(X)$.

Since the normal polygon $X + \epsilon N$ may be deformed into the binormal polygon $X + \epsilon B$ with vertices $X_i + \epsilon B_i$, $i = 0, 1, \cdots, m - 1$ simply by rotating each sector from $\hat{X}_i + \epsilon N_{2i}$ to $\hat{X}_i + \epsilon N_{2i+1}$ ninety degrees and collapsing the sector from $X_i + \epsilon N_{2i-1}$ to $X_i + \epsilon N_{2i}$ to $X_i + \epsilon B_i$, the linking numbers $L(X, X + \epsilon N)$ and $L(X, X + \epsilon B)$ will be the same and each equals $SL(X)$.

Similarly we may deform $X + \epsilon N$ into a tangential polygon $X + \epsilon T$ with vertices $X_i + \epsilon T_i$, $i = 0, 1, \cdots, m - 1$, and again we obtain $L(X, X + \epsilon T) = SL(X)$.

To compute the linking number of X and $X + \epsilon N$, we may consider the algebraic number of times that X meets a disc bounded by $X + \epsilon N$. Such a disc is obtainable by taking $\{X + tN \mid t \geqq \epsilon\}$ in E^3 compactified by adding one point at infinity. The crossings of X with the "normal disc" are called *forward cross normals* by Pohl [5], and our result is that $SL(X) = L(X, X + \epsilon N) =$ algebraic number of intersections of X with $\{X + tN \mid t \geqq \epsilon\}$ = *algebraic* number of forward cross normals of X. Similarly we may show that $SL(X)$ equals the number of backward cross normals or forward or backward cross binormals or tangents. (This gives the seventh and eight theorems of [5].)

In [6] it is proved that the self-linking number of a smooth curve on a sphere is zero. We now prove a stronger proposition for polygons.

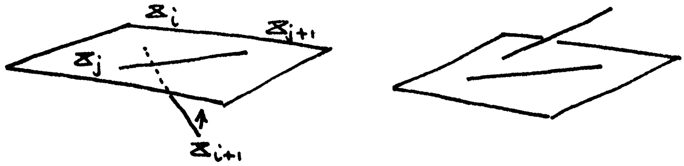
Proposition. *If a polygon X lies on the boundary of a convex body, then* $SL(X) = 0$.

Proof. We prove the stronger proposition that if X lies on the boundary of its own convex hull, then $SL(X) = 0$. If Q is a point within the convex hull of X, then we may compute $SL(X)$ by taking the intersection number of $X - \epsilon N$ with the cone over X from the point Q. But at every edge of X, the arc of $X - \epsilon N$ above the midpoint $\bar{X} = \frac{1}{2}(X_i + X_{i+1})$ lies in the exterior region of the wedge formed by the half-planes at X_iX_{i+1} containing X_{i+1} and X_{i+2} respectively, so this arc will not meet the cone which is contained entirely in the interior region. Since each arc at a vertex also will lie in the exterior region, the curve $X - \epsilon N$ will not meet the cone over X from Q at all.

Remark. It may be possible to find a point Q such that the cone from Q over X is embedded even when $SL(X) \neq 0$, but then some edge will have Q in the exterior of the wedge considered in the previous proposition. Compare the pentagonal example.

5. Behavior of self-linking under deformations. In this section we wish to examine the behavior of the self-linking number $SL(X(t))$ for a one-parameter family of polygons $X(t) = (X_0(t), X_1(t), \cdots, X_{m-1}(t))$. Such a one-parameter family is called a *general deformation* if $X(t)$ is a general polygon except for a finite number of positions $X(t_k)$, $k = 1, 2, \cdots, r$ such that at each exceptional position there is exactly one quadruple of vertices lying in a plane. We may study the changes in the indices $c_{ij}(X(t))$ and $f_i(X(t))$ as t passes one of these values t_k and thus describe the effect on $SL(X(t))$. In order to do this we must list all possible situations in which one vertex of $X(t)$ passes through the plane of three other vertices and describe the changes which affect the indices.

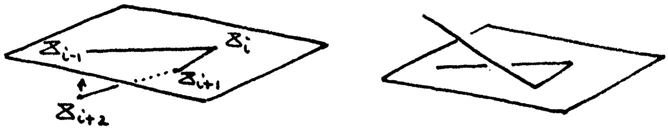
It is clear that $c_{ij}(X(t))$ can change only if the four vertices involved in the coplanarity are the vertices with indices i, $i + 1$, j, and $j + 1$; and similarly $f_i(X(t))$ can change only if the four vertices involved have indices $i - 1$, i, $i + 1$, $i + 2$.



$\Delta C_{ij} = \pm 2$

$\Delta SL = \pm 2$

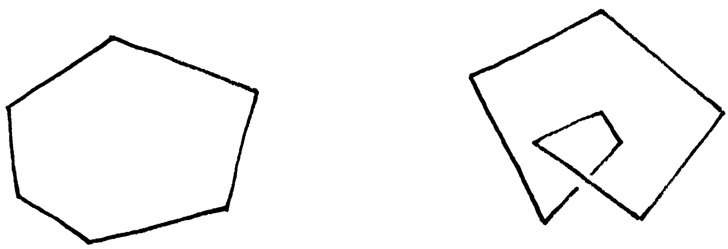
([5], Theorem 10)



$\Delta f_i = \pm 2$
$\Delta SL = \pm 1$
([5], Theorem 11).

The first coplanarity is called a *self-passage* and the second is called a *kink-pulling* in the smooth case treated in [5]. If there are no coplanarities of the second type during a deformation, then the binormal vectors $B_i(t)$ vary continuously and there are well-defined frames at each stage $X(t)$. Such a deformation is called *non-degenerate* (compare [5], p. 983). Such a general deformation is called an *isotopy* if there are no double points, i.e. no coplanarities of the first type. If the family $\{X(t)\}$ is an isotopy, then $SL(X(t))$ is unchanged ([5], Theorem 9).

There is no theorem in the literature which is a direct polygonal analogue of Feldman's Theorem, i.e. it is not known if every general polygon can be non-degenerately deformed into either (a) a convex polygon or (b) a polygon which is "almost" a locally convex planar curve with one crossing, where the number of vertices is fixed throughout the deformation.



(a) (b)

As in the smooth case it follows immediately that to reduce an arbitrary polygon to a convex polygon requires at least s self-passages and k kink-pullings so that $2s + k \geqq |SL(X)|$, and a non-degenerate deformation to (a) requires at least $\frac{1}{2}|SL(X)|$ self-passages and to (b) at least $\frac{1}{2}(|SL(X)| - 1)$. ([5], Theorems 14 and 13).

6. Self-linking of polygons on surfaces in E^4. Let M^2 be a triangulated 2-dimensional surfaces in E^4 and let X be a polygon in M^2 determined by a cycle of 2-simplexes $(\Delta_0^2, \Delta_1^2, \cdots, \Delta_{n-1}^2)$, where $\Delta_{i-1}^2 \cap \Delta_i^2 = \Delta_i^1$ is an edge of M^2 for each i. We may think of the polygon X as a $2n$-gon connecting the points $(\hat{\Delta}_0^2, \hat{\Delta}_0^1, \hat{\Delta}_1^2, \hat{\Delta}_1^1, \cdots, \hat{\Delta}_{n-1}^2, \hat{\Delta}_{n-1}^1)$ where $\hat{\Delta}_i^2$ is the barycenter of the ith 2-simplex of the cycle and $\hat{\Delta}_i^1$ is the barycenter of the ith edge. We assume that X is an embedded polygon, so that in particular no 2-simplex of M^2 appears twice in the cycle. We assume that the polygon X is in general position in E^4, so that in particular no open edge of X meets any 2-simplex which does not contain it.

For any direction ξ in S^3, we may consider the orthogonal projection $P_\xi : E^4 \to E^3(\xi^\perp)$. The direction ξ is said to be *general* for (M^2, X) if the vertices of $P_\xi(M^2)$ are in general position and the vertices of $P_\xi(X)$ are in general position in $E^3(\xi^\perp)$.

For each edge $e = (\hat{\Delta}_{i-1}{}^2, \hat{\Delta}_{i-1}{}^1)$ or $(\hat{\Delta}_{i-1}{}^1, \hat{\Delta}_i{}^2)$ of X and each 2-simplex Δ, we define a *crossing index* $c(e, \Delta, \xi)$ which is 0 if $P_\xi(e) \cap P_\xi(\Delta) = \varnothing$ and which equals $c(e, \Delta)$ if this intersection is non-empty, where $c(e, \Delta)$ is ± 1 depending on the sign of the oriented volume of the 4-simplex determined by the vertices $[\hat{\Delta}_{i-1}{}^2, V(\Delta^2), \hat{\Delta}_{i-1}{}^1]$. Similarly, for each face $\Delta_i{}^2$ in the cycle of X, we set $F(\Delta_i{}^2) =$ sign of the oriented volume of the 4-simplex spanned by $[\hat{\Delta}_{i-1}{}^2, V(\Delta_i{}^2), \hat{\Delta}_{i+1}{}^2]$, and we define $F(\Delta_i, \xi) = F(\Delta_i)$ if the faces $\Delta_{i-1}{}^2$ and $\Delta_{i+1}{}^2$ have images under P_ξ lying on opposite sides of the plane in $E^3(\xi^\perp)$ containing $P_\xi(\Delta_i{}^2)$ and 0 otherwise.

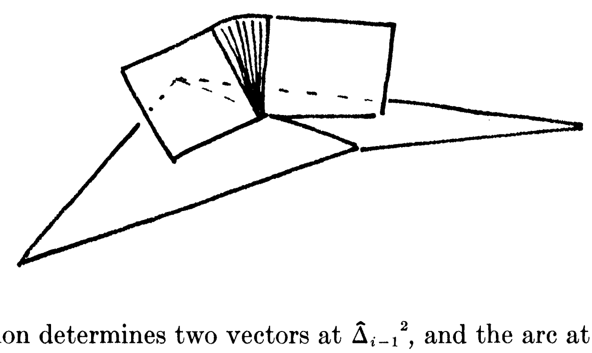
Theorem 1. *For any general vector ξ, the sum*

$$\sum_{e,\Delta} c(e, \Delta, \xi) + \frac{1}{2} \sum_{i=0}^{m-1} F(\Delta_i, \xi)$$

is independent of the direction used to define it. This expression is denoted by $SL(M^2, X)$.

Proof. By a standard deformation argument.

Theorem 2. *$SL(M^2, X) = L(M^2, X + \epsilon N)$ for sufficiently small ϵ, where N is the normal variation vector of X determined by M^2, i.e. N consists of $2n$ segments parallel to the segments of X together with $2m$ arcs, where the segments at $(\hat{\Delta}_{i-1}{}^2, \hat{\Delta}_{i-1}{}^1)$ and $(\hat{\Delta}_{i-1}{}^1, \hat{\Delta}_i{}^2)$ are in the 3-space containing $\Delta_{i-1}{}^2$ and $\Delta_i{}^2$ and the arc at $\hat{\Delta}_{i-1}{}^1$ connects the endpoints of these segments.*



This construction determines two vectors at $\hat{\Delta}_{i-1}{}^2$, and the arc at $\hat{\Delta}_{i-1}{}^2$ is the minor spherical arc joining these two points. Under projection, the parallel strips and the arcs at barycenters of edges are mapped to the convex side of the wedge formed by the images of the adjacent faces. The images of the arc at $\hat{\Delta}_i{}^2$ will intersect $P_\xi(\Delta_i{}^2)$ only if the images of the adjacent faces are on different sides of the plane determined by $P_\xi(\Delta_i^2)$, and the sign of the intersection is determined by the sign of the determinant in E^4.

Self-linking of Surfaces in E^5. For M^2 in general position in E^5, we have an index $c(\Delta_1^2, \Delta_2^2, \xi)$ for each ξ which is general for M^2 and an index $F(\Delta^2, \xi)$ indicating whether or not the image of the boundary of $St(\Delta^2)$ links $P_\xi(\Delta^2)$. The interpretation in terms of linking with a normal variation is that we may take the uniquely defined vectors at $\hat{\Delta}^2$ obtained by translating the normals to Δ^2 in the 3-spaces determined by Δ^2 and an adjacent 2-face. The image under P_ξ of the oriented spherical 2-cell determined by these three vectors will meet $P_\xi(\Delta^2)$ exactly when there is an *inflection face* in this direction. We get a normal variation by pushing each cell $(\Delta^0, \Delta^1, \Delta^2)$ off Δ^2 in the direction determined by Δ^2 and the cell meeting Δ^2 in Δ_1, then we adjust along $\hat{\Delta}^0\hat{\Delta}^2$. We do not have to indicate how the extension goes over vertices, since for sufficiently small ϵ, they will not be involved in the linking number of M^2 and $M^2 + \epsilon N$.

The same constructions yield a definition of self-linking of a k-manifolds M^k in E^{2k+1}, or a k-manifold with an l-dimensional submanifold in E^{k+l+1}, thus leading to polyhedral analogues for the constructions in [6].

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