linije plohe f odgovaraju time izotropnim linijama glavnih zakrivljenosti plohe F i obratno *izotropnim linijama glavnih zakrivljenosti* od f odgovaraju *asimptotske linije* od F.

Kao primjer uzmimo slučaj izotropnih *rotacijskih ploha* $z = f(x \cdot y)$ kojima pri *Euler*ovim transformacijama odgovaraju konoidi $Z = F(X/Y)$ i obrnuto. Napose *rotacijskoj kvadrici* $z = 2\sqrt{1 - xy}$ ili $4xy + z^2 = 4$ odgovara konoid $Z = X/Y + Y/X$ ili $XYZ = X^2 + Y^2$, dakle kubna pravčasta ploha sa Z-osi kao dvostrukom ravnalicom. Izotropne linije glavnih zakrivljenosti rotacijske plohe su njene paralele $z =$ const. i *meridijani* $X/Y =$ const., kojima na konoidu kao asimptotskim linijama odgovaraju izvodnice, i jedan sustav asimptotskih linija (koji nisu pravci). Te su krivulje *nulkrivulje* u nulsistemima koji odgovaraju simetrijama na izotropnim meridijanskim ravninama.

Kod *kubne pravčaste* plohe te su asimptotske linije racionalne krivulje 4. reda s infleksijskim točkama u kuspidalnim točkama plohe. Njene izotropne linije glavnih zakrivljenosti su *izotropne kružnice*, koje leže u dva sveska ravnina, čije su osi torzalni pravci plohe. Kako kvadrika ima dva sustava pravčastih izvodnica, to je *kubna pravčasta ploha omotaljka dvaju sustava izotropnih kugala*, čije su karakteristike izotropne kružnice i predstavljaju izotropne linije glavnih zakrivljenosti kubne pravčaste plohe.

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14. 4. 1981.

FRENET FRAMES AND THEOREMS OF JACOBI AND MILNOR FOR SPACE POLYGONS (GLOBAL GEOMETRY OF POLYGONS. III)

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In this paper we shall develop polygonal analogues of the tangent, normal, and binormal indicatrix curves for smooth space curves and we shall establish analogues of theorems involving the lengths of these curves and the areas they enclose. In particular, we shall prove an analogue of *Jacobi's* theorem which says that if the normal indicatrix of a space curve is simple, then it divides the sphere into two regions of equal area, and we prove a polygonal analogue of a theorem of *Milnor* [6] involving total absolute torsion of closed space curves. We finish by constructing analogues of *Frenet* Frames and tangent developables for space polygons.

1. Construction of the Indicatrix Polygons

For a smooth curve $X(t)$, $a \leqslant t \leqslant b$ with $X'(t) \times X''(t) \neq 0$ for every t, we may define a *tangent indicatrix* curve $T(t) = \dfrac{X'(t)}{\| X'(t) \|}$ describing the direction of the tangent line at each point and a *binormal indicatrix* $B(t) = \dfrac{X'(t) \times X''(t)}{\| X'(t) \times X''(t) \|}$ describing the position of the osculating plane at each point. We obtain a *normal indicatrix* by defining $N(t) = B(t) \times T(t)$.

We now consider a space polygon X determined by a cycle of vertices $(X_0, X_1, \ldots, X_{m-1})$, so X is a curve $X(t)$, $a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$ with $X_i = X(t_i)$ and $X(t)$ linear in any subinterval $t_i \leqslant t \leqslant t_{i+1}$. We will assume that the space polygon X is *general*, i. e. that no four consecutive vertices X_{i-1}, X_i, X_{i+1}, X_{i+2} are coplanar. (For a *closed* polygon with $X_0 = X_m$, we consider all subscripts to be reduced modulo m). This condition allows us to define the unit tangent vector $T_i = \dfrac{X_i - X_{i-1}}{\| X_i - X_{i-1} \|}$ at each edge of X', where the polygon has a well-defined tangent line and a unit binormal vector $B_i = \dfrac{T_i \times T_{i+1}}{\| T_i \times T_{i+1} \|}$ at every vertex, where the polygon has a well-defined osculating plane.

The condition that X is a general space polygon guarantees that the spherical polygon $T = (T_0, T_1, \ldots, T_{m-1})$ is a *general spherical polygon*, so that T_{i+1} will not lie on a great circle are from T_i through T_{i-1}. This polygon is the *tangent indicatrix* polygon of the curve X. Similarly the spherical polygon $B = (B_0, B_1, \ldots$ $\ldots, B_{m-1})$ is a general polygon, the *binormal indicatrix* of X.

We now may define a *normal indicatrix* polygon N with $2m$ vertices $N_{2i} = B_i \times T_i$, $N_{2i+1} = B_{i+1} \times T_i$, for $i = 0, 1, \dots, m-1$.

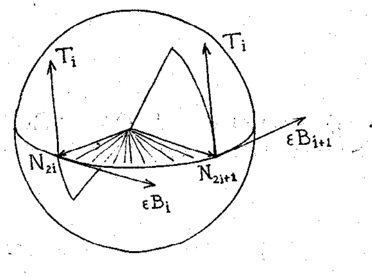

Fig. 1.

Remark. The definition used here is related to the one in [7], but there are some differences. The reader will find many references to related articles in the bibliography of [7].

To illustrate these definitions and justify them to some extent, we now prove some polygonal analogues of two classical theorems for smooth curves.

2. Jacobi's Theorem for Polygons

A general spherical polygon P determined by a cycle of vertices $(P_0, P_1, \dots, P_{n-1})$ is said to be *simple* if no two distinct geodesic arcs $\widehat{P_i P_{i+1}}$ and $\widehat{P_j P_{j+1}}$ intersect. A simple spherical polygon P separates the unit sphere into two regions R_1 and R_2, with R_1 lying to the left of the oriented polygon P. The *Gauss-Bonnet* Theorem for spherical regions states that $\mathscr{A}(R_1) = 2\pi - \Sigma(P)$, where $\Sigma(P)$ denotes the sum of the exterior angles of P. (The proof of this result may be established in an elementary way by induction on the number of edges of the polygon P.)

Jacobi's theorem states that if the normal indicatrix is a simple curve, then it bounds two regions on the sphere, each with area 2π. See for example [8],p. 407. To establish the polygonal analogue of this theorem, we need only show that the exterior angle sum for the polygon N is zero.

To prove this, we compute the exterior angles at N_{2i} and N_{2i+1}. The tangent vectors to the polygon N at N_{2i} are T_i and εB_i, and at N_{2i+1} the tangents are εB_{i+1} and T_i, where $\varepsilon = \pm 1$. Thus the exterior angles are both right angles and the algebraic signs are opposite. Specifically, $(T_i \times \varepsilon B_i) \cdot N_{2i} = (T_i \times \varepsilon B_i) \cdot (B_i \times T_i) = -\varepsilon$ and $(\varepsilon B_{i+1} \times T_i) \cdot N_{2i+1} = (\varepsilon B_{i+1} \times T_i) \cdot (B_{i+1} \times T_i) = \varepsilon$. It follows that the sum of the exterior angles is zero and that the normal indicatrix polygon divides the sphere into regions of equal area. (Figure 1)

3. Total Curvature and Total Torsion for Polygons

The *total curvature* of a space curve $X(t)$, $a \leqslant t \leqslant b$ is the length of the tangent indicatrix, $\int_a^b \| T'(t) \| \, dt = \int_a^b \varkappa(t) \| X'(t) \| \, dt$, where $\varkappa(t) \| X'(t) \|^3 = \| X'(t) \times X''(t) \|$. For a general polygon, the length of the tangent indicatrix

is the sum of the exterior angles Θ_i, where Θ_i is the angle from T_{i-1} to T_i, $0 \leqslant \Theta_i < \pi$.

The *total absolute torsion* of a space curve is the length of the binormal indicatrix $\int_a^b \| B'(t) \| \, dt = \int_a^b | \tau(t) | \| X'(t) \| \, dt$. For a general polygon, the length of the binormal indicatrix is the sum of the absolute values of the angles φ_i where φ_i is the angle from B_i to B_{i+1}, $-\pi < \varphi_i < \pi$. The angle φ_i is the dihedral angle between the »osculating planes« at the vertices X_i and X_{i+1}.

In [5], *Milnor* established the inequality $\varkappa(X) = \sum_{i=0}^{m-1} \Theta_i \geqslant 2\pi$ for any closed polygon, with equality only if the polygon is a convex (planar) polygon, and $\varkappa(X) > 4\pi$ if X is a knotted polygon. In [4], *Fáry* showed that $\varkappa(X)$ is the average of $\varkappa(P_{\xi\perp}(X))$, where $P_{\xi\perp} : E^3 \to E^2(\xi^\perp)$ is the orthogonal projection to the plane perpendicular to the unit vector ξ. In this section we prove a polyhedral analogue of a result of *Milnor* ([6], p. 290).

THEOREM. The total torsion $\tau(X)$ of a curve is π times the average number of inflections of the curve $P_{\xi\perp}(X)$ as ξ varies over the unit sphere.

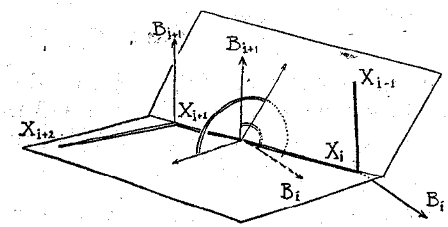

Fig. 2.

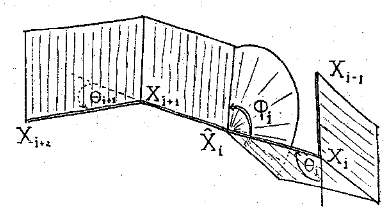

Fig. 3.

Proof. The edge $P_{\xi\perp}(X_i X_{i+1})$ will be an inflection edge if and only if the projection vector ξ lies in the sector of the sphere centered at $\frac{1}{2}(X_i + X_{i+1})$ cut out by the half planes at $X_i X_{i+1}$ containing X_{i-1} and X_{i+1} respectively. But the measure of this dihedral angle is $| \varphi_i |$, the absolute value of the angle between the binormal vectors of the two half-planes at $X_i X_{i+1}$ (Figures 2 and 3).

Remark. The concept of inflection edge of a polygon is a key idea in the Theorem of *Fabricius-Bjerre* [1] and in the theory of Self-Linking, as in [2].

4. Frenet Frame Polygons for Space Polygons

For a smooth space curve $X(t)$, $a \leqslant t \leqslant b$, with $X'(t) \times X''(t) \neq 0$ at each point, we define an orthonormal frame $F_X(t) = (T(t), N(t), B(t))$ called the *Frenet frame*. This gives a curve $F_X(t)$, $a \leqslant t \leqslant b$, in $SO(3)$, the space of orthonormal 3-frames with determinant $+1$, and the properties of this *Frenet frame curve* describe the geometry of the original curve up to a translation, i. e. if $F_X(t) = F_Y(t)$ for all t, then $Y(t) = X(t) + C$ for some constant vector C.

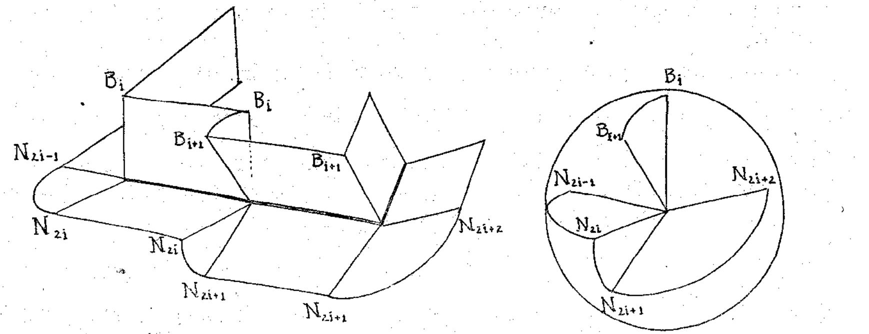

Fig. 4.

We may now define a polygonal analogue for the *Frenet* frame curve by defining a polygon F (Figure 4) with $2m$ vertices in the space or orthonormal frames with determinant 1, defined by

$$F_{2i} = (T_i, N_{2i}, B_i),$$

$$F_{2i+1} = (T_i, N_{2i+1}, B_{i+1}), \qquad (i = 0, 1, \dots, m-1).$$

In the smooth case we often deal with the pair consisting of the point on the curve and the frame at that point, $(X(t), F_X(t))$, (giving a curve in $E^3 \times SO(3)$). The corresponding geometric object in the polygonal case is a $4m$-gon defined by

$$G_{4i} = (X_i, F_{2i})$$

$$G_{4i+1} = (\hat{X}_i, F_{2i})$$

$$G_{4i+2} = (\hat{X}_i, F_{2i+1})$$

$$G_{4i+3} = (X_{i+1}, F_{2i+1})$$

where $\hat{X}_i = \frac{1}{2}(X_i + X_{i+1})$.

This polygon G, with $4m$ vertices, may be described by a parameter u with $0 \leqslant u \leqslant 4m$ such that $G(j) = G_j$ for $j = 0, 1, \dots, 4m-1$. We have

$$G(u) = (X(u), T(u), N(u), B(u)) = ((u - 4i) X_i + (4i + 1 - u) \hat{X}_i, F_{2i}),$$

$$4i \leqslant u \leqslant 4i + 1$$

$$\left(\hat{X}_i, T_i, \cos\left((u - (4i+1))\frac{\pi}{2}\right) N_{2i} + \sin\left((u - (4i+1))\frac{\pi}{2}\right) B_i,\right.$$

$$4i + 1 \leqslant u \leqslant 4i + 2$$

$$\left.- \sin\left((4i + 2 - u)\frac{\pi}{2}\right) N_{2i} + \cos\left((4i + 2 - u)\frac{\pi}{2}\right) B_i\right)$$

$$((u - (4i+2)) \hat{X}_i + (4i + 3 - u) X_{i+1}, F_{2i+1}), \qquad 4i + 2 \leqslant u \leqslant 4i + 3$$

$$\left(X_{i+1}, \cos\left((u - (4i+3))\frac{\pi}{2}\right) T_i + \sin\left((u - (4i+3))\frac{\pi}{2}\right) N_{2i+1},\right.$$

$$4i + 3 \leqslant u \leqslant 4i + 4$$

$$\left.- \sin\left((4i + 4 - u)\frac{\pi}{2}\right) T_i + \cos\left((4i + 4 - u)\frac{\pi}{2}\right) N_{2i+1}, B_{i+1}\right)$$

As in the smooth case, any *Frenet* frame curve $F(t)$, $a \leqslant t \leqslant b$ with weights (t_i) at each vertex corresponds to a unique curve $X^F(t)$ in E^3 such that $L(X^F_{i-1} X^F_i) = w(t_i)$, and the *Frenet* frame field for X^F is given by $F(t)$. In general such a curve $X(t)$ will not be *closed*, and the closure condition is precisely that $\Sigma\, w(t_i)\, T_i = 0$.

5. Tangent Developables and Related Surfaces

In the case of a smooth curve $X(t)$, the *tangent developable* is the surface $Y_T(t, v) = X(t) + vT(t)$. This surface is regular for all values $v \neq 0$, so the surface $Y_T(t, v)$, $v > 0$ is a regular surface called the *forward tangent developable*, which fits together with the *backward tangent developable* (with $v < 0$) along the curve $X(t)$ which is called the *edge of regression*.

For a general polygon X, we may again define $Y_T(u, v) = X(u) + vT(u)$ as defined in the p evious section. The forward tangent developable may then be described as a union of infinite planar wedges spanned by pairs of rays from X_i in the directions T_i and T_{i+1} for $i = 0, 1, \dots, m-1$.

This union of planar wedges may be cut along a ray X_0 in the direction of T_0 and then developed into the plane of the first wedge inductively, one wedge at a time, to get a possible multiple covering of portions of the plane. The image of X under this development will be a locally convex polygon in the sense that the line through any edge bounds a half plane containing both adjacent edges. The total curvature of the developed curve, given by the sum of angles θ_i, will be preserved under the deformation since these are precisely the angles between the rays of the wedges.

In a similar way we may define the *backward tangent developable* for the general polygon X, and the forward and backward developables then fit together along X to form the *tangent developable surface* of X, When the full tangent surface is developed into the plane, the image of X appears as the *fold curve*.

Similarly we may define the *normal surface* and *binormal surface* of a general polygon by $Y_N(u, v) = X(u) + vN(u)$ and $Y_B(u, v) = X(u) + vB(u)$.

We conclude this paper by giving the polygonal analogue of the classification of the forms of the tangential developable to a smooth space curve at points case, at a point of $X(t)$ where $\varkappa(t) \neq 0$ and $\tau(t) \neq 0$, the local form of the tangential developable is given by $(t, v) \to (t, v^2, v^3)$. In particular this is locally 1—1 with a cuspidal edge. In the case where $\varkappa(t_0) \neq 0$ but $\tau(t_0) = 0$ and $\tau'(t_0) \neq 0$, the local form is given by $(t, v) \to (t, v^2, tv^3)$, given implicitly by $z^2 = y^3 x^2$. This surface has a singularity at t_0 and the two sheets of the tangential developable intersect to produce a double curve emanating from the point. Topologically the singularity is a *Whitney* »pinch point« or »umbrella point«, so that a spherical neighborhood of the point in 3-space meet the tangential developable in a set which is topologically a cone over a figure eight.

The same thing happens in the polygonal case, i. e. there are two possible forms for the polyhedral tangential developable in the neighborhood of a vertex X_i of a general polygon. Since no four consecutive vertices are coplanar, the vertices X_{i-2} and X_{i+2} either lie on different sides of the plane through X_{i-1}, X_i and X_{i+1} or they lie on the same side. In the first case (Figure 5), the torsion angles φ_{i-1} and φ_i have the same algebraic sign and in the second they are different. It is in the sense that we say that the second situation represents a *torsion sign change* (Figure 6).

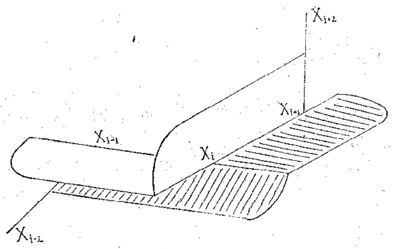

Fig. 5.

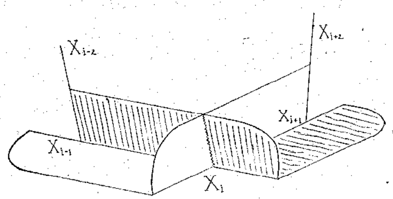

Fig. 6.

If we position the vertices so that X_{i-1}, X_i, and X_{i+1} lie in the horizontal plane, then two of the infinite wedges will lie in the plane. If X_{i+2} is above this plane the second wedge at $X_i X_{i+1}$ lies above this plane. In the case where X_{i-2} lies below the plane, the second wedge lies below the plane and the entire configuration is embedded in a neighborhood of X_i. If X_{i-2} and X_{i+2} both lie on the same side of the plane, the two wedges also lie on that same side of the plane and the configuration has a double line beginning at X_i where these two wedges intersect. Thus the two possible forms in the case of a general space polygon correspond precisely to the two cases for a general smooth space curve.

Remark. The developable surfaces associated with an embedded polygon figure in an essential way in the analysis of self-linking of space polygons, especially in [2], p. 1184.

Remark. This paper is labelled »Global Geometry of Polygons III« since the paper »Self-Linking Numbers of Space Polygons« [2] should be considered as the second paper in this series.

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