A TWO STAGE PROCEDURE FOR THE CLASSIFICATION OF VECTOR BUNDLE MONOMORPHISMS WITH APPLICATIONS TO THE CLASSIFICATION OF IMMERSIONS HOMOTOPIC TO A MAP

by

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§0. Introduction

0.1. Let A be a subspace of a path connected space B. Let $* \in B$ be a base point and denote by $\pi_1(B,A,*)$ the set of homotopy classes of paths $c : [0,1] \to B$ with $c(1) = *$, $c(0) \in A$. Then $\pi_1(B,*)$ acts on $\pi_1(B,A,*)$ on the right with orbit space $\pi_0(A)$. Thus the problem of calculating $\pi_0(A)$ may be divided into two stages:

I Calculate $\pi_1(B,A,*)$

II Calculate the action of $\pi_1(B,*)$ on $\pi_1(B,A,*)$

0.2. If M^m, N^n are differentiable manifolds, the space $\mathrm{Imm}(M,N)$ of immersions of M in N is a subspace of the space N^M of all maps M to N (with the compact open topology). For fixed $f \in N^M$, let $N^M_{[f]}$ denote its path component. Applying 0.1 we have that $\pi_0(N^M_{[f]} \cap \mathrm{Imm}(M,N))$ (the set of regular homotopy classes of immersions homotopic to f, which we will denote by $[M \looparrowright N]_{[f]}$) is the orbit space of $\pi_1(N^M,\mathrm{Imm},f)$ (the set of regular homotopy classes of immersions with a homotopy to f given, denoted by $[M \looparrowright N]_f$) under an action of $\pi_1(N^M,f)$.

This work is an investigation into the second stage of the classification procedure. We were motivated to look closer at step two as we had observed in the literature several misstated results due to a failure to consider this step.

In §1 we recall the notion of affine structure and affine action.

In many situations, the sets encountered come equipped with an affine structure and the group actions are affine. This additional algebraic structure facilitates the expression of the final results.

In §2 we discuss general properties of π_1 actions in lifting problems. Here we give the general homotopy theoretic framework which is then applied in §3 to the lifting problem associated to the classification of monomorphisms of vector bundles. In §4 we give examples of trivial and non-trivial affine actions of immersion theory. In the appendix we give some calculations of $\pi_1(Y^X,f)$.

§1. Affine Structures

Definition 1.1. A set X is said to be affine (over a group G) if there is a map $\mu: X \times X \to G$ satisfying

a) $\mu(x,y) \cdot \mu(y,z) = \mu(x,z)$

b) for all $x \in X$, $\mu(x,\cdot) : X \to G$ is a bijection

Remark 1.1.1. μ determines (and is determined by) a simply transitive action $r : X \times G \to X$ by the equation $r(x,\mu(x,y)) = y$.

Definition 1.2. An affine map is a pair $(f,\bar{f})$ making the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\ \mu\ } & G \\ \big\downarrow{\scriptstyle f \times f} & & \big\downarrow{\scriptstyle \bar{f}} \\ X' \times X' & \longrightarrow & G' \end{array}$$

commute.

Definition 1.3. The group of affine transformations of X (over G, w.r.t. μ) will be denoted by $\mathrm{Aut}(X,G,\mu)$ (or just $\mathrm{Aut}(X)$).

Remark 1.3.1. There is a split exact sequence $1 \to G \to \mathrm{Aut}(X,G,\mu) \xrightarrow{\text{res}} \mathrm{Aut}(G) \to 1$, where $\text{res} : \mathrm{Aut}(X) \to \mathrm{Aut}(G)$ is given by $(f,\bar{f}) \to \bar{f}$. The action of G as automorphisms (on the left) of X is called translation.

§2. π_1 actions in lifting problems

2.1. π_1 actions on fibers

Given a (Serre) fibration $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ with B path connected and fiber F, one has a right action of $\pi_1(B)$ on $\pi_0(F)$ given by taking the end point of lifts of paths. The orbit space of this operation is $\pi_0(E)$.

The above situation is equivalent to that of 0.1, since if the inclusion $A \subset B$ is replaced by a fibration $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ then $\pi_1(B,A) \simeq \pi_0(F)$ and this bijection is compatible with the action of $\pi_1(B)$.

2.1.1. Naturality

If $\begin{matrix} E' & \to & E \\ \downarrow & & \downarrow \\ B' & \to & B \end{matrix}$ is a pullback (in the homotopy category)

then the action of $\pi_1(B')$ factors through $\pi_1(B)$.

2.2. Maps into fibrations

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ be a (Serre) fibration and X a complex. (These assumptions will be made throughout, although this is more restrictive than necessary).

The map $\begin{matrix} E^X \\ \rho\downarrow \\ B^X \end{matrix}$ is a Serre fibration. For $f \in B^X$, the fiber Γ_f over f (possibly empty) is the space of lifts over f. By 2.1 $\pi_1(B^X,f)$ acts on $\pi_0(\Gamma_f)$ with orbit space $\pi_0(\rho^{-1}B^X_{[f]})$ (denoted respectively by $[X,E]_f$ and $[X,E]_{[f]}$.) Thus the homotopy classification of liftings and of liftings "up to homotopy" differ by an action of π_1.

2.2.1. Naturality

If $\begin{matrix} E' & \to & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{g} & B \end{matrix}$ is a pullback, so is $\begin{matrix} E'^X & \to & E^X \\ \downarrow & & \downarrow \\ B'^X & \to & B^X \end{matrix}$ so, by 2.1.1, the action of $\pi_1(B'^X,f)$ factors through $\pi_1(B^X, gof)$.

2.3. Two stage lifting problems

Suppose $\begin{array}{c}E\\ \downarrow\\ B\end{array}$ and $\begin{array}{c}B\\ \downarrow\\ X\end{array}$ are fibrations. Let $\Gamma\begin{pmatrix}B\\ \downarrow\\ X\end{pmatrix} \subset B^X$ denote the space of sections.

The diagram

$$\begin{array}{ccc}\Gamma\begin{pmatrix}E\\ \downarrow\\ X\end{pmatrix} & \subset & E^X\\ \downarrow & & \downarrow \rho\\ \Gamma\begin{pmatrix}B\\ \downarrow\\ X\end{pmatrix} & \subset & B^X\end{array}$$

is a pullback.

For $f \in \Gamma\begin{pmatrix}B\\ \downarrow\\ X\end{pmatrix}$ denote by $\Gamma\begin{pmatrix}B\\ \downarrow\\ X\end{pmatrix}_{<f>}$ the path component of f. By 2.1 $\pi_1(\Gamma\begin{pmatrix}B\\ \downarrow\\ X\end{pmatrix}, f)$ acts on $\pi_0(\Gamma_f)$ ($= [X,E]_f$, see 2.2) with orbit space $\pi_0(\rho^{-1}\Gamma\begin{pmatrix}B\\ \downarrow\\ X\end{pmatrix}_{<f>})$ (denoted by $[X,E]_{<f>}$). Thus classification of liftings of a two stage fibration involves calculating an action of π_1.

Remark. By 2.1.1, the action of $\pi_1(\Gamma\begin{pmatrix}B\\ \downarrow\\ X\end{pmatrix})$ factors through $\pi_1(B^X)$ so little generality is lost by considering only the situation of 2.2.

2.4. Affine Structures

Proposition 2.4.1.

Let $\begin{array}{c}\tilde{G}\\ \downarrow\\ B\end{array}$ be a local coefficient system with fiber G and let $\begin{array}{c}\tilde{B}\\ \downarrow\\ B\end{array}$ be a covering space with fiber F. Suppose we are given a fiberwise action $\tilde{B} \times \tilde{G} \xrightarrow{r} \tilde{B}$ such that for $x \in F$, $r(x,\cdot) : G \to F$ is a bijection, i.e., so that F is affine over G. Then the action of $\pi_1(B)$ on F is affine.

Proof: Let $\alpha \in \pi_1(B)$ and let $f_\alpha : F \to F$, $g_\alpha : G \to G$ be the maps given by path lifting. Then $g_\alpha = \bar{f}_\alpha \in \mathrm{Aut}(G)$.

Example 2.4.2.

Suppose $\begin{array}{c}E\\ \downarrow\\ B\end{array}$, $\begin{array}{c}T\\ \downarrow\\ B\end{array}$ are fibrations with fibers F_b, Ω_b and suppose that the Ω_b are groups (H-spaces) and that we are given a fiberwise (H-space) action $E \times T \to E$ such that for $x \in F_b$ $r(x,\cdot) : \Omega_b \to F_b$ is a (homotopy) equivalence.

Then $\pi_0(\Omega_b)$ is a local coefficient system over B acting on the covering $\pi_0(F_b)$, so 2.4.1. applies.

Example 2.4.3.

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ be a fibration with fiber F. Let (SX,*) be a reduced suspension. Then the product $(B,*)^{(SX,*)} \times (F,*)^{(SX,*)}$ acts fiberwise on the fibration $\begin{matrix} (E,*)^{(SX,*)} \\ \downarrow \\ (B,*)^{(SX,*)} \end{matrix}$ as in 2.4.2. We get that $[(SX,*), (E,*)]_f$ is affine over the group $[(SX,*), (F,*)]$. Moreover the affine action of $\pi_1((B,*)^{(SX,*)})$ is via translations (since path lifting in products is trivial, see proof of 2.4.1).

Example 2.4.4.

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$, $\begin{matrix} T \\ \downarrow \\ B \end{matrix}$ be as in 2.4.2.

Then $\begin{matrix} E^X \\ \rho \downarrow \\ B^X \end{matrix}$ $\begin{matrix} T^X \\ \downarrow \\ B^X \end{matrix}$ are also as in 2.4.2 (except that $\rho^{-1}(f)$ may possibly be empty). Applying 2.4.1 and 2.4.2 we get that $[X,E]_f$ is affine over $[X,T]_f$ and the action of $\pi_1(B^X,f)$ is affine.

2.5. Affine structures for lifting problems in the stable range

2.5.1. Notation. For spaces we have the functor Σ, the unreduced suspension with distinguished points S and N, the south and north poles. For spaces with a base point, *, we have the functor Ω_*, loops at *, and P_*, paths ending at *. For spaces with two base points $*_0$ and $*_1$, we have the functor $P_{*_0,*_1}$ of paths beginning at $*_0$ and ending at $*_1$. Ω_* acts on P_* from the right. Ω_{*_1} acts on $P_{*_0,*_1}$ from the right and for $c \in P_{*_0,*_1}(X)$, $r(c,\cdot) : \Omega_{*_1}(X) \to P_{*_0,*_1}(X)$ is a homotopy equivalence.

If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a fibration (with section s, or sections s_0,s_1) then

$\Sigma E_B, \Omega_s E_B$, $P_s E_B$, $P_{s_0,s_1} E_B$ denote the fiberwise application of the functors Σ, Ω_*, P_*, $P_{*_0,*_1}$.

Theorem 2.5.2. (Becker [Be]).

Let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a fibration with $n-1$ connected fiber F. Let X be a $2n-1$ coconnected complex and $f : X \to B$ a map. Then $[X,E]_f$ is affine. Moreover, the action of $\pi_1(B^X,f)$ on $[X,E]_f$ is affine.

Lemma 2.5.3. For X $n-1$ connected, the inclusion $X \to P_{S,N}\Sigma X$ is $2n-1$ connected.

Lemma 2.5.4. Let $\begin{matrix} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & \underset{=}{\text{id}} & B \end{matrix}$ be a map of fibrations such that the map of fibers $F \to F'$ is m connected. Then if X is m coconnected, $[X,E]_f \to [X,E']_f$ is 1-1 and onto.

Proof of 2.5.2. By 2.5.3 and 2.5.4 we may replace $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ by $P_{S,N}\begin{smallmatrix} \Sigma E_B \\ \downarrow \\ B \end{smallmatrix}$. Now apply 2.4.4 to the fibrations $P_{S,N}\begin{smallmatrix} \Sigma E_B \\ \downarrow \\ B \end{smallmatrix}$, $\Omega_N \begin{smallmatrix} \Sigma E_B \\ \downarrow \\ B \end{smallmatrix}$.

2.6. Affine structures for lifting problems with fiber an Eilenberg MacLane space

Theorem 2.6.1.

Let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a fibration with fiber F a $K(G,n)$, $n > 1$. Then the set $[X,E]_f$ has an affine structure μ with group $H^n(X,\widetilde{G}_f)$ (where $\widetilde{G}_f$ is the local coefficient system on X induced by f from the local coefficient system $\pi_n(F_b)$). The action of $\pi_1(B^X,f)$ on $[X,E]_f$ is affine.

Moreover let $\psi: B \to K(\pi_1(B),1) = K$ be a map inducing the isomorphism $\pi_1(B) = \pi_1(K)$. Then the composite $\pi_1(B^X,f) \to \mathrm{Aut}([X,E]_f, H^n(X,\widetilde{G}_f),\mu) \to \mathrm{Aut}(H^n(X,\widetilde{G}_f))$ coincides with the composite

$$\pi_1(B^X,f) \to \pi_1(K^X,\psi f) \xrightarrow[\sim]{\text{cf } 5.1.3} \mathrm{Aut}\begin{pmatrix} \widetilde{G}_f \\ \downarrow \\ X \end{pmatrix} \to \mathrm{Aut}(H^n(X,\widetilde{G}_f)).$$

<u>Proof</u>: The affine structure is classical obstruction theory. Let $\widetilde{G}$ denote an operation of π on G (or a local coefficient system over $K = K(\pi,1)$. Let $\widetilde{K}$ denote the universal cover). Let Z be a pointed $K(G,n+1)$ with based π action inducing $\widetilde{G}$ on $\pi_n(Z) = G$. Set $L(\widetilde{G},n+1) = \widetilde{K} \times_\pi Z$, a $K(G,n+1)$ fibration over $K = K(\pi,1)$ with section u and projection p. Consider $E = P_u L(\widetilde{G},n+1)_K$ (see 2.5.1). The map $\begin{array}{c} E \\ \downarrow \\ L(\widetilde{G},n+1) \end{array}$, given by evaluating a path at its origin, is the universal (see [Ba], page 298) $K(G,n)$ fibration. (The fiber over $x \in Z \subset L(\widetilde{G},n+1)$ is $P_{x,*}Z$, a $K(G,n)$.)

By naturality, it will be enough to prove 2.6.1 in the universal case. Note that the fibers over $u(K)$ are H-spaces (the fiber over $* \in Z \subset L(\widetilde{G},n+1)$ is ΩZ) and $u^*E = \Omega_u L(\widetilde{G},n+1)_K$. The fibrations $\begin{array}{c} E \\ \downarrow \\ L(\widetilde{G},n+1) \end{array}$ and $\begin{array}{c} p^*u^*E \\ \downarrow \\ L(\widetilde{G},n+1) \end{array}$ satisfy 2.4.4. So $[X,E]_f$ is affine over $[X,p^*u^*E]_f = [X,\Omega_u L(\widetilde{G},n+1)_K]_{pof}$. This latter is isomorphic to $H^n(X,\widetilde{G}_f)$ as groups. By 2.4.2 and 2.1.1, the action of $\pi_1(B^X,f)$ is affine. Moreover, by the proof of 2.4.1, the map $\pi_1(L(\widetilde{G},n+1)^X,f) \to \mathrm{Aut}(H^n(X,\widetilde{G}_f))$ is given by path lifting in the fibration $\begin{array}{c} (\Omega_u L(\widetilde{G},n+1)_K)^X \\ \downarrow \\ K^X \end{array}$ (more precisely, by 2.4.2, by path lifting in the associated local coefficient system over K^X with fiber $[X,\Omega_u L(\widetilde{G},n+1)_K]_{pof} = H^n(X,\widetilde{G}_f)$.) This is easily seen to be given by the coefficient automorphism.

<u>Corollary 2.6.2</u>.

Let X have dimension n and let $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ be a fibration with $n-1$ connected fiber F. Then $[X,E]_f$ is affine over $H = H^n(X, \pi_n^f(F))$. The map $\pi_1(B^X,f) \to \mathrm{Aut}\ H$ factors through $\mathrm{Aut}(\pi_n^f(F))$.

<u>Proof</u>: Let $\begin{array}{c} E_n \\ \downarrow \\ B \end{array}$ be the first stage of a Postnikov tower for $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ with fiber $K(\pi_n(F),n)$. Then by 2.5.4 $[X,E]_f \to [X,E_n]_f$ is a bijection. Apply 2.6.1 to $\begin{array}{c} E_n \\ \downarrow \\ B \end{array}$.

Proposition 2.6.3.

Let $X = S^n$, the n-sphere, and let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a fibration with simply connected fiber F. Then $[X,E]_f$ has an affine structure with group $\pi_n(F)$. The composite $\pi_1(B^{S^n}) \to \mathrm{Aut}\big((\pi_n(F)\big)$ factors through $\pi_1(B)$. Furthermore, the map $\pi_{n+1}(B) \xrightarrow{\text{cf } 5.3} \ker(\pi_1(B^{S^n},f) \to \pi_1(B)) \to$ translation group of $[X,E]_f = \pi_n(F)$, is the boundary homomorphism.

Proof: Let $\to E_k \longrightarrow E_{k-1} \longrightarrow$ be a Postnikov decomposition for $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ where

$$p_k \searrow \ \swarrow$$
$$B$$

$\begin{smallmatrix} E_k \\ \downarrow \\ E_{k-1} \end{smallmatrix}$ has fiber an Eilenberg MacLane Space, $K(\pi_k(F),k)$. The fibers of

$\begin{smallmatrix} E^{S^n} \\ \downarrow \\ E^{S^n}_{k-1} \end{smallmatrix}$ are connected and simply connected for $k \le n-1$, hence by induction

$E^{S^n}_{n-1,f} = \{\tilde{f} : S^n \to E_{n-1}, p_{n-1} \circ \tilde{f} = f\}$ is connected and simply connected and thus

$E^{S^n}_{n-1,[f]} = \{g : S^n \to E_{n-1}, p_{n-1} \circ g \in [f]\}$ is connected and

$\pi_1(E^{S^n}_{n-1,[f]},\tilde{f}) \to \pi_1(B^{S^n},f)$ is an isomorphism, where $\tilde{f} : S^n \to E_{n-1}$ lifts f.

It follows that $[S^n,E]_{\tilde{f}} \to [S^n,E]_f$ is a bijection and one can apply 2.6.2.

The last assertion is an elementary verification.

§3. π_1 actions and monomorphisms of vector bundles

3.1. The fibration $\mathrm{Mono}(\xi,\eta) \to Y^X$

Let $\begin{smallmatrix} \xi^m \\ \downarrow \\ X \end{smallmatrix}$, $\begin{smallmatrix} \eta^n \\ \downarrow \\ Y \end{smallmatrix}$ be vector bundles and let $\mathrm{Mono}(\xi,\eta)$ be the space of all vector bundle maps ξ to η which are monomorphisms on each fiber. Each such map induced a map $X \to Y$. The map $\mathrm{Mono}(\xi,\eta) \to Y^X$ is a (Serre) fibration.

Let $E = \beta(\xi,\eta)$ be the fiber space over $X \times Y$ with fiber $\mathrm{Mono}(\xi_x, \eta_y)$ $\mathrm{Mono}(\xi_x, \xi_y) = V_{n,m}$ and structure group $O(m) \times O(n)$. Projecting onto X one has the two stage fibration $\begin{matrix} E \\ \downarrow \\ X \times Y \\ \downarrow \\ X \end{matrix}$. The space of sections $\Gamma\begin{pmatrix} E \\ \downarrow \\ X \end{pmatrix}$ is homeomorphic to the space $\mathrm{Mono}(\xi,\eta)$. The space of sections $\Gamma\begin{pmatrix} X \times Y \\ \downarrow \\ X \end{pmatrix}$ is homeomorphic to the space Y^X. Thus, for $f : X \to Y$ we have (see 2.3) $\pi_1(Y^X, f)$ acts on $\pi_0(\mathrm{Mono}_f(\xi,\eta))$ (denoted by $[\xi,\eta]_f$) with quotient $\pi_0(\mathrm{Mono}_{[f]}(\xi,\eta))$ (denoted by $[\xi,\eta]_f$). (Here $\mathrm{Mono}_f(\xi,\eta)$ is the space of monomorphisms covering f and $\mathrm{Mono}_{[f]}(\xi,\eta)$ is the space of monomorphisms covering maps homotopic to f). We remark that $\mathrm{Mono}_f(\xi,\eta)$ is homeomorphic to $\mathrm{Mono}_{id}(\xi, f^*(\eta))$.)

3.2. Naturality

Suppose $\begin{matrix} \eta & \longrightarrow & \eta' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{matrix}$ is a pullback of vector bundles.

Then the diagram

$$\begin{matrix} \beta(\xi,\eta) & \to & \beta(\xi,\eta') \\ \downarrow & & \downarrow \\ X \times Y & \to & X \times Y' \end{matrix} \quad \text{is also a pullback}$$

so the action of $\pi_1(Y^X, f)$ factors through $\pi_1(Y'^X, g \circ f)$.

Example 3.2.1.

Let $\psi : Y \to BO(m)$ classify η.
The action of $\pi_1(Y^X, f)$ factors through $\pi_1(BO(m)^X, \psi \circ f)$ (= $\pi_0(\mathrm{Aut}\, f^*(\eta))$, see 5.1.3.)

Example 3.2.2.

Let dimension $X = r$ and suppose η is trivial on the $r+1$ skeleton Y_{r+1} of Y. Then the action of $\pi_1(Y^X, f)$ is trivial.

Proof: We may suppose $f : X \to Y_{r+1}$. The action of $\pi_1(Y_{r+1}^X, f)$ is trivial since it factors through the trivial group $\pi_1(pt^X)$. But since

$\pi_1(Y^X_{r+1},f) \to \pi_1(Y^X,f)$ is surjective, $\pi_1(Y^X,f)$ also acts trivially.

3.3. Codimension zero monomorphisms

3.3.1. Theorem. Let $\begin{matrix}\xi & \eta\\ \downarrow & \downarrow \\ X & Y\end{matrix}$ be vector bundles of dimension n.

Let $g : Y \to BO(n)$ classify η. Then $[\xi,\eta]_f$ is affine over $\pi_1(BO(n)^X, g\circ f)$.

3.3.2. Corollary. $[\xi,\eta]_{[f]}$ corresponds bijectively with the coset space

$$\frac{\pi_1(BO(n)^X, g\circ f)}{\text{im } \pi_1(Y^X,f)}.$$

Proof of theorem. $Mono_f(\xi,\eta) = Mono_{id}(\xi, f^*(\eta))$ has $Aut(f^*(\eta))$, as action group. So $[\xi,\eta]_f$ has $\pi_0(Aut(f^*(\eta)))$ ($= \pi_1(BO(n)^X, g\circ f)$, see 5.1.3) as action group.

3.4. Codimension one monomorphisms

Let $\begin{matrix}\xi & \eta\\ \downarrow & \downarrow \\ X & Y\end{matrix}$ be vector bundles with $\dim \xi+1 = \dim \eta = n$.

Let $\begin{matrix}\omega\\ \downarrow \\ X\end{matrix}$ be the bundle of dimension 1 with first Stiefel Whitney class

class $W^1(\omega) = W^1(f^*(\eta))-W^1(\xi)$. The map $\begin{matrix}Mono_{[f]}(\xi\oplus\omega,\eta)\\ \downarrow \\ Mono_{[f]}(\xi,\eta)\end{matrix}$ is a 2 fold covering

which is split if η is orientable (by fixing orientations of $\xi \oplus \omega$ and η and requiring an extension to preserve orientations). The 2-fold covering $\begin{matrix}Mono_f(\xi \oplus \omega,\eta)\\ \downarrow \\ Mono_f(\xi,\eta)\end{matrix}$ is split (by requiring orientations to be preserved at the base

point). $Mono_f(\xi,\eta)$ has commuting action groups $Aut_+(\xi \oplus \omega)$, $Aut_+(F^*(\eta))$ where $Aut_+ \subset Aut$ is the normal subgroup of orientation perserving automorphisms. Hence with respect to the affine structure given by the action of $\pi_0(Aut_+(\xi\oplus\omega))$, $\pi_0(Aut_+(f^*(\eta))$ is the translation group and the action of $\pi_0(Aut(f^*(\eta))$ is affine. We have proven:

Theorem 3.4.1.

Let $\begin{matrix}\xi & \eta \\ \downarrow & \downarrow \\ X & Y\end{matrix}$ be vector bundles with $\dim \xi+1 = \dim \eta = n$.

Let $g : Y \to BO(n)$ classify η. Then $[\xi,\eta]_f$ has an affine structure with $\ker(\pi_1(B\theta^X(n), g\circ f) \to \pi_1(BO(n))$ acting as the translation group, and the action of $\pi_1(BO(n)^X, g\circ f)$ is affine. If η is orientable and $g : Y \to BSO(n)$ classifies η, then $[\xi,\eta]_f$ has an affine structure with $\pi_1(BSO(n)^X, g\circ f)$ acting as translation group.

Corollary 3.4.2.

If $g : Y \to BSO(n)$ classifies η, then $[\xi,\eta]_{[f]}$ corresponds bijectively with the coset space

$$\frac{\pi_1(BSO(n)^X, g\circ f)}{\operatorname{im} \pi_1(Y^X, f)} .$$

3.5. The case of a sphere

Let $\begin{matrix}\xi^m & \eta^n \\ \downarrow & \downarrow \\ S^k & Y\end{matrix}$ be a vector bundles, $m+2 \leq n$, and let $g : Y \to BO(n)$ classify η . The fiber $V_{n,m}$ of $B(\xi,\eta)$ is simply-connected so 2.6.3 applies. Combining 2.6.3 with 3.2.1 we obtain:

Theorem 3.5.1.

$[\xi,\eta]_f$ is affine over $\pi_k(V_{n,m})$. The action of $\pi_1(Y^{S^k}, f)$ is affine. Moreover the diagram

$$\begin{array}{ccccc}
\pi_{k+1}(Y) & \longrightarrow & \pi_{k+1}(BO(n)) & \xrightarrow{\partial} & \pi_k(V_{n,m}) = T([\xi,\eta],\pi_k(V_{n,m}),\gamma) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(Y^{S^k}, f) & \longrightarrow & \pi_1(BO(n)^{S^k}, g\circ f) & \longrightarrow & \operatorname{Aut}([\xi,\eta],\pi_k(V_{n,m}),\gamma) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(Y) & \longrightarrow & \pi_1(BO(n)) & \xrightarrow{\mu_*} & \operatorname{Aut} \pi_k(V_{n,m})
\end{array}$$

is commutative, where ∂ is the boundary homomorphism of the fibration $V_{n,m} \to BO(n-m) \to BO(n)$. μ_* is given by post multiplication by a non-rotation.

Corollary 3.5.2.

If η is orientable or μ_* is trivial, then $[\xi,\eta]_{[f]}$ is in bijection with a coset space of $\pi_k(V_{n,m})$.

Corollary 3.5.3.

If $\pi_1(Y^{S^k},f) \to \pi_1(BO(n)) = \mathbb{Z}/2\mathbb{Z}$ and μ_* are non-trivial then the action of $\pi_1(Y^{S^k},f)$ is non-trivial.

Corollary 3.5.4.

Suppose $\partial\colon \pi_{k+1}(BO(n)) \to \pi_k(V_{n,m})$ is the zero homomorphism. The action of $\pi_1(Y^{S^k},f)$ is trivial if η is orientable and factors through $\pi_1(BO(n)) = \mathbb{Z}/2\mathbb{Z}$ otherwise. Suppose in addition there is $\alpha \in \pi_1(Y^{S^k},f)$ with non-trivial image in $\pi_1(BO(n))$ (i.e. $res(\alpha) \in \pi_1(Y)$ reverses orientation) which fixes some element of $[\xi,\eta]_f$. Then the action of $\pi_1(Y^{S^k},f)$ is trivial if and only if μ_* is trivial and $[\xi,\eta]_{[f]}$ is in bijection with the set of orbits of $\pi_k(V_{n,m})$ under the operation of μ_*.

Proof: Follows from 3.5.1. and 1.3.1.

Example 3.5.4. a).

If $k+1 + 2m < 2n$, $13 \le n-m$ and $k+2 \le n$ then ∂ is zero (cf. [BM]). If $k+2 \le n$ and $k = 2,4,5$ or $6 \bmod 8$ then $\pi_{k+1}(BO(n)) = \pi_{k+1}(BO) = 0$.

Example 3.5.4. b).

Let $f : S^k \to Y$ be homotopic to a constant map. Let $\xi \oplus \xi' = f^*(\eta)$ and suppose ξ' admits an orientation reversing automorphism (e.g. if dim ξ' is odd or if ξ' admits a section). Then the map $S^n \times S^1 \to S^1 \overset{c}{\to} Y$, where c is an orientation reversing loop, fixes the monomorphism $\xi \subset \xi \oplus \xi' \simeq f^*(\eta) \to \eta$.

Example 3.5.4. c).

The map μ_* on $\pi_r(V_{n,k})$ has been calculated by James [J] to be $id - u_* s_* \Delta_* - \Delta_* s_* p_*$ where $\pi_r(V_{n,k}) \xrightarrow{\Delta^*} \pi_{r-1}(S^{n-k-1}) \xrightarrow{S_*} \pi_r(S^{n-k}) \xrightarrow{u_*} \pi_r(V_{n,k})$

and $\pi_r(V_{n,k}) \xrightarrow{p_*} \pi_r(S^{n-1}) \xrightarrow{S_*} \pi_{r+1}(S^n) \xrightarrow{\Delta_*} \pi_r(V_{n,k})$ p_* is projection onto base, u_* the inclusion of fiber and Δ_* are boundary homomorphisms of the obvious fibrations. S_* is suspension.

For example if $r=k=m$ with $n \geq m+2$, then $\Delta_* S_* p_* = 0$ since $\pi_m(S^n) = 0$. Hence $\mu_* = \text{id}-u_* S_* \Delta_* = \lambda_*$ (λ_* is premultiplication by a non-rotation). Since $\lambda_*^2 = 1 = \mu_*^2$ and $\lambda_*^m = \mu_*^n$ we get $\mu_* = 0$, if $n-m$ is odd.

From [P] $\pi_n(V_{2n-2,n}) = \mathbf{Z}_2 + \mathbf{Z}_2$ if $n \equiv 2 \bmod 4$. Moreover one can check $\Delta_* : \mathbf{Z}_2 + \mathbf{Z}_2 \to \pi_{m-1}(S^{m-3}) = \mathbf{Z}_2$ is surjective, S^* is an isomorphism and $u_* : \mathbf{Z}_2 \to \mathbf{Z}_2 + \mathbf{Z}_2$ is injective. Hence $\mu_* = \text{id}-u_* S_* \Delta_*$ fixes two elements and exchanges the other 2.

From [P], $\pi_8(V_{12,8}) = \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$ and one may check Δ is onto $\pi_7(S^3) = \mathbf{Z}_2$, $S^* : \pi_7(S^3) \to \pi_8(S^4) = \mathbf{Z}_2 + \mathbf{Z}_2$ is injective and $u_* : \pi_8(S^4) \to \pi_8(V_{12,8})$ is injective. Hence $\mu_* = \text{id}-u_* S_* \Delta_*$ fixes 4 elements and exchanges the other 4 in pairs.

One can also show the following:

If $n = 1 \bmod 4$ and $n \geq 5$ then $\pi_n(V_{2n-1,n}) = \mathbf{Z}_2 + \mathbf{Z}_2$ and there are 3 orbits.

If $n = 3 \bmod 4$, then $\pi_n(V_{2n-1,n}) = \mathbf{Z}/4\mathbf{Z}$ and $\mu_* = \text{id}$.

If $n = 1 \bmod 4$ and $n \geq 9$, then $\pi_n(V_{2n-3,n}) = \mathbf{Z}/12\mathbf{Z}$ and $\mu_*(X) = -x$ so there are 7 orbits.

If $k = n+2$ then $\pi_n(V_{n+2,n}) \simeq \pi_n(SO)$ and $\mu_* = \text{id}$.

3.6. The case of the first obstruction

Let $\begin{matrix} \xi^m & \eta^n \\ \downarrow & \downarrow \\ X & Y \end{matrix}$ and let $g : Y \to BO(n)$ classify η.

Suppose $n-m$ = dimension of $X = k$. The fiber $V_{n,m}$ of $B(\xi,\eta)$ is k-1 connected so 2.6.5. applies. Let $\widetilde{\pi_k(V_{n,m})}$ be the local coefficient system over X twisted by $\pi_1(X) \to \mathbf{Z}_2 \times \mathbf{Z}_2 \to \text{Aut}\, \pi_k(V_{n,m})$ where $\pi_1(X) \to \mathbf{Z}_2 \times \mathbf{Z}_2$ is the orientation homomorphism of ξ and η, and $\mathbf{Z}_2 \times \mathbf{Z}_2 \to \mathrm{Aut}\, \pi_k(V_{n,m})$ is induced by pre and post composition with non-rotations.

Proposition 3.6.1.

In the above $[\xi,\eta]_f$ is affine over $H^k(X, \widetilde{\pi_k(V_{n,m})}) = H$. The map $\pi_1(Y^X,f) \to \mathrm{Aut}(H)$ factors through $\pi_1(BO(n)) = \mathbf{Z}_2 \xrightarrow{\mu_*} \mathrm{Aut}(\pi_k(V_{n,m}))$.

Corollary 3.6.2.

If $\pi_1(Y^X,f) \to \pi_1(BO(n)) = \mathbf{Z}_2$ is non-trivial and the coefficient automorphism μ_* induces a non-trivial automorphism of $H^k(X,\widetilde{\pi_k(V_{n,m})})$, then the action of $\pi_1(Y^X,f)$ is non-trivial.

§4. Applications to Immersion Theory

4.1. Smale Hirsch Theorem

Let M^m, N^n be differentiable manifolds τ_M, τ_N their respective tangent bundles. One has a map $\mathrm{Imm}(M,N) \to \mathrm{Mono}(\tau_M, \tau_N)$ given by taking the differential. Smale-Hirsch theory says that this map is a weak homotopy equivalence provided either $m < n$ or (if $m=n$) M has no closed components.

One may think of this theorem as saying that the inclusion $\mathrm{Imm}(M,N) \subset N^M$ (see 0.1) may be replaced by the fibration $\mathrm{Mono}(\tau_M, \tau_N) \to N^M$ (see 2.1). Thus the sets $[M \looparrowright N]_f$ and $[M \looparrowright N]_{[f]}$ are equal to the sets $[\tau_M, \tau_N]_f$, respectively $[\tau_M, \tau_N]_{[f]}$.

4.2. Immersions of surfaces in orientable 3 manifolds (cf [Lil])

Let Σ^2 be any surface and N^3 any orientable 3-manifold. If f is any map, then $[\Sigma \looparrowright N]_{[f]}$ is in bijection with $H^1(\Sigma, \mathbf{Z}_2)$.

Proof: Any map $M^m \to N^{2m-1}$ is homotopic to an immersion (cf [LP]). N^3 orientable implies N^3 parallelizable (since $\pi_2(SO(3)) = 0$) so by 3.2.2. the action of $\pi_1(N^\Sigma,f)$ is trivial. Hence $[\Sigma^2 \looparrowright N^3]_{[f]} = [\tau_\Sigma, \tau_N]_f$. The fiber of $\beta(\tau_\Sigma, \tau_N)$ over $\Sigma \times N$ is $V_{3,2}$. Since $\pi_1(V_{3,2}) = \mathbf{Z}/2\mathbf{Z}$,

$\pi_2(V_{3,2}) = 0$ the set $[\tau_\Sigma, \tau_N]_f$ has an affine structure with group $H^1(\Sigma, \mathbb{Z}_2)$

4.3. Periodic Isotopy

A periodic isotopy is a map $S^1 \times N \to N$ which is the identify for $t = * \in S^1$ and an embedding for all $t \in S^1$. It is easy to see the following.

Proposition 4.3.1.

Let $\alpha \in \pi_1(N^M, f)$ be induced by a periodic isotopy on N. Then α acts trivially on $[M \looparrowright N]_f$.

Example 4.3.2.

Let L_m^{2n-1} denote a lens space. Then each element of $\pi_1(L) = \mathbb{Z}/m\mathbb{Z}$ is induced by a periodic isotopy. If $\dim M \leq 2n-3$, then $\pi_1(L^M, f) \to \pi_1(L)$ is an isomorphism (see 5.2.3). Hence by 4.3.1, $\pi_1(L^M, f)$ acts trivially.

4.4. Immersions of disks

Let $M = D^m$. If $m < n$ then $[D^m \looparrowright N^n]$ has one element. If $m = n$ then $[D^m \looparrowright N^m]_f$ is affine over $\mathbb{Z}/2\mathbb{Z}$ and the action of $\pi_1(N^D, f)$ is trivial if and only if N is orientable.

4.5. Immersions of M^m in S^{m+1} (cf [Li 2])

Proposition 4.5.1. $[M^m \looparrowright S^{m+1}]$ is in bijection with $[M, SO]$, provided $[M^m \looparrowright S^{m+1}]$ in non empty.

Proof: Let $SO(m+1) \to SO(m+2) \to S^{m+1}$ be the natural fibration. Then $\begin{array}{c} SO(m+2) \\ \downarrow \\ S^{m+1} \end{array}$ is the principal bundle associated to the tangent bundle (cf. [H]).
In particular, in the fibration $S^{m+1} \to BSO(m+1) \to BSO(m+2)$, the inclusion of the fiber classifies the tangent bundle of S^{m+1}. By 3.4.1, $[\tau_M, \tau_S]_f$ is affine over $\pi_1(BSO(m+1)^M)$ and by corollary 3.4.2, $[\tau_M, \tau_S]_{[f]}$ is in bijection with $\dfrac{\pi_1(BSO(m+1)^M)}{\operatorname{im} \pi_1({S^{m+1}}^M)} = \pi_1(BSO(m+2)^M) = \pi_1(BSO^M) = [M, SO]$ (cf.5.1.5).

Example 4.5.2.

$$[S^m \looparrowright R^{m+1}] = \pi_m(SO(m+1))$$

while $$[S^m \looparrowright S^{m+1}] = \pi_m(SO).$$

4.6. Immersions of spheres in manifolds

Applying 3.5.1, 3.5.2, 3.5.3, for $n \geq m+2$ we have

Proposition 4.6.1.

$[S^m \looparrowright N^n]_f$ is affine over $\pi_m(V_{n,m})$ and the action of $\pi_1(M^{S^m},f)$ is affine. If $\varepsilon : \pi_1(M) \to \mathbf{Z}_2$ is the orientation homomorphism and $\mu_* : \mathbf{Z}_2 \to \mathrm{Aut}\, \pi_n(V_{m,n})$ is given by postmultiplication by a non-rotation, then $\pi_1(N^{S^m},f) \to \mathrm{Aut}(\pi_m(V_{n,m}))$ is the composite $\mu_* \circ \varepsilon \circ res$ where $res : \pi_1(N^{S^m},f) \to \pi_1(N)$ is the restriction.

Corollary 4.6.2.

If μ_* or $\varepsilon o res$ is trivial then $[S^m \looparrowright N^n]_{[f]}$ is a coset space of $\pi_n(V_{n,m})$. If $\varepsilon o res$ and μ_* are non-trivial, the action of $\pi_1(N^{S^m}, f)$ is non-trivial.

Theorem 4.6.3. $(m+2 \leq n)$

a) Suppose the compositie $\pi_{m+1}(N) \to \pi_{m+1}(BO(n)) \xrightarrow{\partial} \pi_m(V_{n,m})$ is zero. The set of regular homotopy classes of immersion of S^m in N^n which are homotopic to a constant is in bijection with $\pi_m(V_{n,m})$ if N is orientable and with the set of orbits of the operation of μ_* on $\pi_m(V_{n,m})$ if N is non-orientable.

b) Suppose the map $\pi_{m+1}(BO(n)) \xrightarrow{\partial} \pi_m(V_{n,m})$ is zero. The action of $\pi_1(N^{S^m},f)$ on $[S^m \looparrowright N^n]_f$ is trivial if N is orientable and factors through $\mathbf{Z}/2\mathbf{Z}$ if N is non-orientable. If there is $\alpha \in [S^m \looparrowright N^n]_f$ which is fixed by $h \in \pi_1(N^{S^m},f)$ having non-zero image in $\pi_1(BO(m)) = \mathbf{Z}/2\mathbf{Z}$, then $[S^m \looparrowright N^n]_{[f]}$ corresponds bijectively with the set of orbits of the operation of μ_* on $\pi_m(V_{n,m})$.

Proof: b) follows from 4.6.1 and 1.3.1.

a) follows from 3.5.4 b) and 5.2.1.

4.7. Immersions of M^m in N^{2m}

Applying 3.6.1 and 3.6.2 we obtain

Proposition 4.7.1.

$[M^m \looparrowright N^{2m}]_f$ is affine over $H^m(M,\mathbb{Z}/2\mathbb{Z})$ if m is odd and $H^m(M,\tilde{\mathbb{Z}})$ if m is even ($\tilde{\mathbb{Z}}$ is the integers twisted by $f^*W^1(N) - W^1(M)$). For m odd and $f^*W^1(N) = 0$, μ_* induces multiplication by -1 on $H^n(M,\tilde{\mathbb{Z}}) = \mathbb{Z}$, hence if $\pi_1(N^M,f) \xrightarrow{res} \pi_1(N) \xrightarrow{W^1(N)} \mathbb{Z}/2\mathbb{Z}$ is non-trivial, the action of $\pi_1(N^M,f)$ is non-trivial.

Remark.

One can show [Li1] that the action is trivial if $[M^m \looparrowright N^{2m}]_f \simeq \mathbb{Z}/2\mathbb{Z}$ and factors through $\mathbb{Z}/2\mathbb{Z}$ if $[M^m \looparrowright N^{2m}]_f \simeq \mathbb{Z}$. Note that $\mathbb{Z}/2\mathbb{Z}$ can act on $\mathbb{Z}$, up to isomorphism, either by $x \to -x$ (one fixed point) or by $x \to 1-x$ (no fixed points).

§5. Appendix. Some calculations of $\pi_1(Y^X,f)$

5.1. The universal case

Let $\eta \downarrow Y$ be a vector bundle of dimension n (or a local coefficient system or other object) satisfying the following universal property: let $\xi \downarrow X$ be a n-dimensional vector bundle, $A \subset X$ and $\xi|_A \to \eta$ a bundle map which is an isomorphism on each fiber. Then there is an extension to a bundle map $\xi \to \eta$, which is an isomorphism on each fiber. (Technically, A is assumed to be a subcomplex of the complex X).

Let $\mathrm{Iso}(\xi,\eta)$ be the space of all bundle maps which are isomorphisms on each fiber.

Lemma 5.5.1.

If η is universal, then $\mathrm{Iso}(\xi,\eta)$ is contractible.

Proof: A map of the cone $C\ \mathrm{Iso}(\xi,\eta) \to \mathrm{Iso}(\xi,\eta)$ extending the identity of $\mathrm{Iso}(\xi,\eta)$ is produced as follows: Let $p : C\ \mathrm{Iso}(\xi,\eta) \times X \to X$ denote the projection and define $f : p^*\xi|_{\mathrm{Iso}(\xi,\eta) \times X} = \mathrm{Iso}(\xi,\eta) \times \xi \to \eta$ by $(\alpha,v) \to \alpha(v)$. By universality, f may be extended to all of $p^*\xi = C\ \mathrm{Iso}(\xi,\eta) \times \xi$.

If $\begin{matrix}\eta\\ \downarrow\\ Y\end{matrix}$ is universal and $F \in \mathrm{Iso}(\xi,\eta)$, the map $f : X \to Y$ induced by F is said to classify ξ.

Proposition 5.1.2.

Let $\begin{matrix}\eta\\ \downarrow\\ Y\end{matrix}$ be universal and let $f : X \to Y$ classify $\begin{matrix}\xi\\ \downarrow\\ X\end{matrix}$.

Then $Y^X_{[f]} = B\ \mathrm{Aut}(\xi)$.

Proof: Follows from 5.1.1 since $\mathrm{Aut}(\xi)$ acts effectively on the left of $\mathrm{Iso}(\xi,\eta)$ with orbit space $Y^X_{[f]}$.

Corollary 5.1.3. $\pi_1(Y^X,f) = \pi_0(\mathrm{Aut}(\xi))$.

Example 5.1.4.

If $f : X \to BO(n)$ is homotopically trivial then $\pi_1(BO(n)^X,f) = [X,O(n)]$.

Example 5.1.5.

Let X be a finite complex. The space BO^X is an H-space, so $\pi_1(BO^X,f) = \pi_1(BO^X,f)$ where $c : X \to BO$ is the constant map. So $\pi_1(BO^X,f) \simeq [X,O]$.

5.2. The map $\pi_1(Y^X,f) \to \pi_1(Y)$

Let X be a space with base point $*$. Restriction to $*$ yields a homomorphism $\pi_1(Y^X,f) \overset{\mathrm{Res}}{\to} \pi_1(Y,f(*))$.

Proposition 5.2.1.

Suppose f is null homotopic. Then Res is split.

Proof: We may assume f is the constant map. The projection $X \to *$ induces the splitting $\pi_1(Y) \to \pi_1(Y^X, f)$

Proposition 5.2.2.

$$\text{Image(Res)} \subset \text{centralizer of } f_*(\pi_1(X)).$$

Proof: $\pi_1(S^1)$ commutes with $\pi_1(X)$ in $\pi_1(S^1 \times X)$.

As a partial converse, we have

Proposition 5.2.3.

Suppose $\pi_i(Y) = 0$ for $2 \le i \le \dim X = m$. There is an exact sequence

$$1 \to H^m(X, \pi^f_{m+1}(Y)) \to \pi_1(Y^X, f) \to \text{centralizer of } f_*(\pi_1(X)) \to 1$$

where $\pi^f_{m+1}(Y)$ is the local coefficient system induced by f.

Proof: $\pi_1(Y^X, f)$ consists of homotopy classes of maps $S^1 \times X$ rel$* \times X$ to Y. By the assumption on $\pi_i(Y)$ any extension on the 2 skeleton of $S^1 \times X$ (rel$* \times X$) can be extended to all of $S^1 \times X$. An extension to the 2-skeleton exists if and only if there is θ making the diagram

$$\begin{array}{ccccc} \pi_1(Y) & \longrightarrow & \pi_1(Y \times X \times S') & \searrow & \\ f_* \uparrow & & \theta \, \vdots & & \\ \pi_1(X) & \longrightarrow & \pi_1(X \times S') & \xrightarrow{=} & \pi_1(X \times S') \end{array}$$

commute (cf. [Ba] page 265)

i.e. if and only if there is σ making

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{f_*} & \pi_1(Y) \\ \downarrow & \nearrow_{\sigma} & \\ \pi_1(X \times S^1) & & \end{array} \qquad \text{commute}$$

i.e. if and only if $\sigma(t)$ commutes with $f_*\pi_1(X)$ where t is the generator of

$\pi_1(S^1) = \mathbf{Z}$. This proves exactness at centralizer of $f_*(\pi_1(X))$.

Now let u be the composite $S^1 \times X \to X \xrightarrow{f} Y$. $\ker \pi_1(Y^X, f) \to \pi_1(Y)$ consists of homotopy classes of maps $S^1 \times X \to Y$ which are $(\mathrm{rel}* \times X)$ homotopic to u on $S^1 \vee X$. By our assumption on $\pi_i(Y)$, these correspond to homotopy classes of maps which are homotopic to u on the m skeleton of $S^1 \times X$. By the spectral sequence (cf [Ba] page 277) these are just $H^{m+1}(S^1 \times X, * \times X; \pi^u_{n+1}(Y)) = H^m(X, \pi^f_{n+1}(Y))$.

Example 5.2.4.

Let M^n be a connected manifold and $Y = S^{n+1}$.

Then $\pi_1(S^{n+1^M}) = \begin{cases} \mathbf{Z} & \text{if } M \text{ is orientable and closed} \\ \mathbf{Z}/2\mathbf{Z} & \text{if } M \text{ is non-orientable and closed} \\ 0 & \text{if } M \text{ is open.} \end{cases}$

Example 5.2.5.

Let $Y = \mathbb{R}P^{n+1}$ and $\dim X \le n$.

Then $1 \to H^n(X;Z_f) \to \pi_1(\mathbb{R}P^{n+1^X}, f) \to \mathbf{Z}/2\mathbf{Z} \to 1$ is exact where Z_f is the integers twisted by

$$\pi_1(X) \xrightarrow{f_*} \pi_1(\mathbb{R}P^{n+1}) = \mathbf{Z}_2 = \langle t \rangle \text{ and } t \text{ acts on } \mathbf{Z} \text{ by } (-1)^n.$$

5.3. The case $X = S^n$

Proposition 5.3.1.

There is an exact sequence

$$\pi_2(Y) \xrightarrow{d} \pi_{n+1}(Y) \to \pi_1(Y^{S^n}, f) \xrightarrow{\mathrm{Res}} \pi_1(Y)$$

where d is the Whitehead product with $[f] \in \pi_n(Y)$ and image(Res) = stabilizer of $[f]$.

Corollary 5.3.2.

If f is homotopic to a constant, then

$1 \to \pi_{n+1}(Y) \to \pi_1(Y^{S^n}, f) \to \pi_1(Y) \to 1$ is a split exact sequence.

<u>Proof of 5.3.1</u>. Since $\pi_1(S^1 \times S^n)$ stabilizes $\pi_n(S^1 \times S^n)$ image(Res) $\subset$ stabilizer of $[f]$. (In the following we use the decomposition $S^n \subset S^1 \vee S^n \subset S^1 \times S^n$). If an extension of f to $S^1 \vee S^n$ is given, the obstruction to extending to $S^1 \times S^n$ is just $\alpha[f]-[f] \in H^{n+1}(S^1 \times S^n, * \times S^n, \pi_n(Y)) = \pi_n(Y)$ where $\alpha \in \pi_1(Y)$ is given by $S^1 \to S^1 \vee S^n \to Y$. So image(Res) = stabilizer of $[f]$. Let u be the map $S^1 \times S^n \to S^n \xrightarrow{f} Y$. Then ker(res) is the set of maps $S^1 \times S^n \to Y$ homotopic, rel$* \times S^n$ to u on $S^1 \vee S^n$ (which is the n skeleton of $S^1 \times S^n$ rel$* \times S^n$ since there are no cells of dimension less than $n+1$). Thus by the spectral sequence ([Ba], page 277) we have $\ker(\text{res}) = \pi_{n+1}(Y)/d_n(\pi_2(Y))$. d_n is the Whitehead product ([Ba] page 285).

<u>Example 5.3.3</u>.

$1 \to \pi_{n+1}(BO) \to \pi_1(BO^{S^n}, f) \to \pi_1(BO) \to 1$ is exact since Whitehead products in $\pi_i(BO)$ are trivial, and the operation of π_1 is trivial (BO is an H-space).

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