COMPUTING TWISTED SIGNATURES AND L-CLASSES OF NON-WITT SPACES

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Abstract. In previous joint work with Cappell and Shaneson, we have established an Atiyah-Lusztig-Meyer-type multiplicative characteristic class formula for the twisted signature and, more generally, the twisted L-class, of a stratified Witt space. The present paper shows that these formulae hold even when the stratified space does not satisfy the Witt condition. It constitutes one of the first applications of signature homology.

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1. Introduction

Let X^n be a closed, oriented, Whitney stratified pseudomanifold. It is said to be a *Witt space*, if the middle-dimensional, middle-perversity intersection homology of the link of every stratum of odd codimension vanishes. In this case, intersection homology methods together with the Thom-Pontrjagin construction yield a homology L-class $L(X) \in H_*(X;\mathbb{Q})$, [GM80], [GM83], [Sie83]. If X is nonsingular, then $L(X)$ is the Poincaré-dual of the Hirzebruch L-class. Let $\mathcal{S}$ be a local coefficient system on X, equipped with a

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nondegenerate, symmetric, bilinear form $\mathcal{S} \otimes \mathcal{S} \to \mathbb{R}_X$. Using intersection chain sheaves twisted by $\mathcal{S}$, one constructs a twisted L-class $L(X;\mathcal{S})$. In [BCS03], we prove that for Witt spaces X,

$$L(X;\mathcal{S}) = \widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X),$$

where $[\mathcal{S}]_K \in KO(X)$ is the K-theory signature of $\mathcal{S}$ and $\widetilde{\mathrm{ch}} = \mathrm{ch} \circ \psi^2$ is a modified Chern character given by precomposing with the second Adams operation ψ^2.

The present paper removes the Witt hypothesis on X. We show that the above formula continues to hold for an arbitrary Whitney stratified pseudomanifold X, as long as it still possesses an L-class:

Theorem. *Let X^n be a closed, oriented, Whitney stratified pseudomanifold and let $\mathcal{S}$ be a nondegenerate, symmetric local system on X. If $L(X) \in H_*(X;\mathbb{Q})$ is defined, then*

$$L(X;\mathcal{S}) = \widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X). \tag{1}$$

(See section 7.) For the special case of the twisted signature $\sigma(X;\mathcal{S}) = L_0(X;\mathcal{S})$, one has therefore

$$\sigma(X;\mathcal{S}) = \langle \widetilde{\mathrm{ch}}[\mathcal{S}]_K, L(X)\rangle. \tag{2}$$

If X is not a Witt space, then the middle-perversity intersection chain sheaf $\mathbf{IC}^\bullet_{\bar{m}}(X)$ ceases to be Verdier-self-dual, and consequently an alternate construction of $L(X)$ is required. The obstruction theory of [Ban02] in terms of Lagrangian structures along strata of odd codimension can be used to decide which spaces X have an L-class. The construction of a well-defined L-class for spaces with vanishing obstructions is carried out in [Ban04]. Part of section 3 is devoted to a brief review of this theory.

For the situation where X is a smooth manifold, the signature formula (2) is the formula of [Mey72]. If in addition the local system is of geometric origin, that is, $\mathcal{S}$ is the higher direct image $\mathcal{S} = R^{2k}\pi_*\mathbb{R}_E$ for a smooth fiber bundle $F^{4k} \longrightarrow E \xrightarrow{\pi} X$, then (2) computes the signature of the total space, as $\sigma(E) = \sigma(X;\mathcal{S})$, and thus becomes the signature formula of [Ati69]. When $\pi_1(X)$ acts trivially on $H^{2k}(F;\mathbb{R})$, and thus $\mathcal{S}$ is constant, we have $(\widetilde{\mathrm{ch}}[\mathcal{S}]_K)_0 = \sigma(F)$ and all higher Chern character components vanish. The resulting relation is $\sigma(E) = \sigma(F)\sigma(X)$, due to [CHS57]. Further information on the signature of nonsingular fiber bundles can be found in [Lus71] and [LR92].

Similarly, twisted characteristic classes arise in geometric mapping situations involving singular spaces. In [CS91] it is shown that the signature $\sigma(Y)$ of the domain of a stratified map $f: Y^m \to X^n$, $m-n$ even, between Whitney stratified spaces with only even-codimensional strata is given by the twisted signature of X with coefficients in the middle cohomology of the "general fiber" (the fiber over the top stratum of X), plus a sum of twisted signatures which ranges over the components of the pure strata in the singular set of X. Analogous results for other characteristic classes are provided as well. If, on the other hand, X has only singular strata of odd codimension, then we prove in [Ban03] that

$$f_* L(Y) = L(X; \mathcal{S}), \tag{3}$$

where $\mathcal{S}$ is given on the top stratum of X by the middle cohomology of the "general fiber". Thus the singular strata of odd codimension do not contribute terms.

From the discussion of the above geometric situations we see in particular that the local system $\mathcal{S}$ is typically only given on the top stratum. If it extends, as a local system, over the entire space, then (1) can be used to compute the twisted L-classes. A necessary, and on normal spaces also sufficient, condition for the existence of a unique extension is that $\mathcal{S}$ has constant restrictions on all links. This holds automatically on the class of supernormal spaces, for which there exists a well understood classification theory, [CW91]. Our requirement that the local system be defined on the entire space cannot be eliminated without substitute, because formulae (1) and (2) will become false in general: One can construct examples of four-dimensional orbifolds with isolated singularities, together with local systems on their top stratum, such that these systems do not extend to the entire space, and the difference between the left and right hand side of (2) is given by a non-vanishing rho-invariant. These examples are particularly striking, since the underlying spaces have rather weak singularities, being rational homology manifolds.

Let us provide a compact outline of our strategy for proving the twisted L-class formula (1). By the Thom-Pontrjagin construction, the primary objective is to establish (2). To accomplish this, we use signature homology $S_*(-)$ introduced by Minatta [Min04] and Kreck. This homology theory has coefficients $S_{4k}(pt) \cong \mathbb{Z}$, given by the signature, and zero otherwise. The coefficient groups were already introduced in [Ban02]. In [Min04], signature homology is constructed as the bordism theory of topological stratifolds ([Kre]) equipped with self-dual sheaf-complexes satisfying intersection homology type axioms as introduced in [Ban02]. The pertinent feature for us is that the canonical map $\Omega_*^{SO}(X) \to S_*(X)$ is at odd primes surjective, allowing us to pull back signature calculations to smooth manifolds, starting with the identity map in $S_*(X)$. The following problem arises: If a stratified non-Witt space X carries perverse self-dual sheaves extending constant coefficients on the top stratum, then it is not automatically clear that it also carries perverse self-dual sheaves extending a non-constant local system

$\mathcal{S}$ on the top stratum[1]. We avoid these monodromy difficulties altogether by constructing a piecewise linear version $S_*^{PL}(-)$ of topological signature homology. As pointed out in, among other places, [Sul04] (section 2), the simplicial stratification of a PL space has the virtue of rendering all link bundles trivial. After defining the functor $S_*^{PL}(-)$, it must be verified that it is indeed a homology theory. This rather technical verification is relegated to the appendix, where it is carried out in complete detail.

The sections of this paper are organized as follows: Section 2 sets up terminology surrounding local coefficient systems and discusses prolongation questions for top stratum local systems. Section 3 reviews the results of [Ban02] and [Ban04], as well as the construction of the L-class of a non-Witt space with boundary. Proposition 3.1, which relates that L-class to the L-class of the boundary, is used in the proof of the twisted signature formula, theorem 6.1. Lemma 3.2 and lemma 3.3 are ingredients needed for the proof of the twisted L-class formula, theorem 7.1. Section 4 describes topological stratifolds and, using these as the geometric basis, the construction of topological signature homology as carried out in [Min04]. The coefficient groups are computed and the relation at odd primes to smooth oriented bordism, Witt space bordism, and KO-homology is summarized. In section 5, we construct PL signature homology as the bordism theory of compact, oriented PL-pseudomanifolds together with a triangulation from the PL-structure and a self-dual perverse sheaf constructible with respect to the simplicial stratification induced by the triangulation. In proving transitivity of the bordism relation, gluing is treated carefully (lemma 5.3). We show that the signature of a PL non-Witt space is a PL-invariant, and prove that topological and PL signature homology are isomorphic. Sections 6 and 7 establish the signature and L-class formula, respectively. An illustrative example involving non-Witt orbit spaces of Lie group actions is provided in section 8. The appendix, section 9, contains the proof that $S_*^{PL}(-)$ is a homology theory. The core of that proof is the construction of a Mayer-Vietoris sequence for PL signature homology, which uses simplicial codimension one transversality.

2. Local Coefficient Systems: Conventions and Terminology

Let $(X^n, \partial X)$ be a pseudomanifold with (possibly empty) boundary and filtration

$$X^n = X_n \supset X_{n-2} \supset X_{n-3} \supset \ldots \supset X_0 \supset \varnothing,$$

where the strata are indexed by dimension, the $X_i \cap \partial X$ stratify ∂X, and the $X_i - \partial X$ stratify $X - \partial X$; $\Sigma = X_{n-2}$ is the singular set.

[1] If X has only strata of even codimension, then this is true: The Deligne formula for the middle perversity yields a self-dual extension, both when applied to constant coefficients and when applied to $\mathcal{S}$.

Definition 2.1. A *Poincaré local system on* X is a locally constant sheaf $\mathcal{S}$ on X together with a nondegenerate, symmetric, bilinear pairing $\phi : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_X$. If X is disconnected, then we assume that both the rank of $\mathcal{S}_x$ and the signature of ϕ_x, $x \in X$, are constant functions on X.

Let $(\mathcal{S}, \phi)$ be a Poincaré local system on $X - \Sigma$ (the top stratum of X). The pairing ϕ, being nondegenerate, induces an isomorphism

$$\phi : \mathbf{Hom}(\mathcal{S}, \mathbb{R}_{X-\Sigma}) \xrightarrow{\cong} \mathcal{S}$$

Now assume $\partial X = \varnothing$. As $X - \Sigma$ is a manifold,

$$\mathcal{D}\mathcal{S}[-n] = \mathbf{Hom}(\mathcal{S}, \mathbb{R}_{X-\Sigma}) \otimes \mathcal{O}_{X-\Sigma},$$

where $\mathcal{O}_{X-\Sigma}$ is the orientation sheaf on $X - \Sigma$ and $\mathcal{D}$ the Borel-Moore-Verdier dualizing functor. Thus ϕ induces an isomorphism

$$\phi : \mathcal{D}\mathcal{S}[-n] \cong \mathcal{S} \otimes \mathcal{O}_{X-\Sigma}.$$

An orientation for X is an isomorphism $\mathcal{O}_{X-\Sigma} \cong \mathbb{R}_{X-\Sigma}$. Assuming X to be oriented, it follows that $\phi : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{X-\Sigma}$ induces a self-duality isomorphism

$$\phi : \mathcal{D}\mathcal{S}[-n] \cong \mathcal{S} \tag{4}$$

Let $(\mathcal{S}, \phi)$ be a Poincaré local system of stalk dimension m on the space X^n and let $\Pi_1(X)$ denote the fundamental groupoid of X. By $\mathfrak{Vect}_m$ denote the category whose objects are pairs (V, ψ), with V an m-dimensional real vector space and $\psi : V \times V \to \mathbb{R}$ a nondegenerate, symmetric, bilinear pairing whose morphisms are linear maps preserving the pairings:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{Vect}_m}((V_1, \psi_1), (V_2, \psi_2)) = \\ \{A : V_1 \to V_2 \text{ linear} \mid \psi_2(Av, Aw) = \psi_1(v, w), v, w \in V_1\}.\end{aligned}$$

The system $(\mathcal{S}, \phi)$ induces a covariant functor

$$\mu(\mathcal{S}) : \Pi_1(X) \longrightarrow \mathfrak{Vect}_m$$

as follows: For $x \in X$, let

$$\mu(\mathcal{S})(x) = (\mathcal{S}_x, \phi_x)$$

and for a path class $[\omega] \in \pi_1(X, x_1, x_2) = \mathrm{Hom}_{\Pi_1(X)}(x_2, x_1)$, $\omega : I \to X$, $\omega(0) = x_1$, $\omega(1) = x_2$, define the linear operator

$$\mu(\mathcal{S})[\omega] : \mu(\mathcal{S})(x_2) \longrightarrow \mu(\mathcal{S})(x_1)$$

to be the composition

$$\mu(\mathcal{S})(x_2) = \mathcal{S}_{\omega(1)} \cong (\omega^*\mathcal{S})_1 \underset{\text{restr}}{\xleftarrow{\cong}} \Gamma(I, \omega^*\mathcal{S}) \underset{\text{restr}}{\xrightarrow{\cong}} (\omega^*\mathcal{S})_0 \cong \mathcal{S}_{\omega(0)} = \mu(\mathcal{S})(x_1).$$

If we choose a base-point $x \in X$, then restricting $\mu(\mathcal{S})$ to the fundamental group $\pi_1(X, x) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$ gives an assignment of a linear automorphism on the stalk $\mathcal{S}_x$,

$$\mu(\mathcal{S})_x(g) : \mathcal{S}_x \longrightarrow \mathcal{S}_x,$$

preserving the pairing $\phi_x : \mathcal{S}_x \times \mathcal{S}_x \to \mathbb{R}$, to each $g \in \pi_1(X, x)$. Thus one obtains the monodromy representation

$$\mu(\mathcal{S})_x : \pi_1(X, x) \longrightarrow O(p, q; \mathbb{R})$$

($p + q = m$ is the rank of $\mathcal{S}_x$, $p - q$ the signature of ϕ_x). Conversely, a given functor $\mu : \Pi_1(X) \to \mathfrak{Vect}_m$ determines a Poincaré local system: Let X_0 be a path component of X, and $x_0 \in X_0$. Then $\pi_1(X_0, x_0)$ acts on $\mu(x_0) = (V, \phi)$ by the restriction μ_{x_0} and we have the associated local system

$$\mathcal{S}|_{X_0} = \widetilde{X_0} \times_{\pi_1(X_0, x_0)} V$$

over X_0 with an induced pairing ϕ, where $\widetilde{X_0}$ denotes the universal cover of X_0.

Definition 2.2. Let X be a stratified pseudomanifold with singular set Σ and let $\mathcal{X}$ denote the set of components of open strata of X of codimension at least 2. Each $Z \in \mathcal{X}$ has a link $Lk(Z)$. Call a Poincaré local system $\mathcal{S}$ on $X - \Sigma$ *strongly transverse to* Σ if the composite functor

$$\Pi_1(Lk(Z) - \Sigma) \xrightarrow{\text{incl}_*} \Pi_1(X - \Sigma) \xrightarrow{\mu(\mathcal{S})} \mathfrak{Vect}_m$$

is isomorphic to the trivial functor for all $Z \in \mathcal{X}$.

On normal spaces, strong transversality of local systems characterizes those systems that extend as local systems over the whole space:

Proposition 2.1. *Let X^n be normal. A Poincaré local system $\mathcal{S}$ on $X - \Sigma$ is strongly transverse to Σ if and only if it extends as a Poincaré local system over all of X. Such an extension is unique.*

The normality assumption is not necessary for the "if"-direction. Simple examples show that the normality assumption can not be omitted in the "only if"-direction and in the uniqueness statement. Proposition 2.1 allows us to state yet another useful characterization of strongly transverse local systems:

Corollary 2.1. *Let X^n be normal. A Poincaré local system $\mathcal{S}$ on $X - \Sigma$ is strongly transverse to Σ if and only if its monodromy functor $\mu(\mathcal{S}) : \Pi_1(X - \Sigma) \to \mathfrak{Vect}_m$ factors (up to isomorphism of functors) through $\Pi_1(X)$:*

$$\begin{array}{ccc} \Pi_1(X - \Sigma) & \xrightarrow{\text{incl}_*} & \Pi_1(X) \\ & \searrow^{\mu(\mathcal{S})} & \downarrow \\ & & \mathfrak{Vect}_m \end{array}$$

Corollary 2.2. *Let X^n be normal. A Poincaré local system $(\mathcal{S}, \phi)$ on $X^n - \Sigma$ strongly transverse to Σ has a K-theory signature*

$$[\mathcal{S}]_K \in KO(X).$$

3. L-Classes of Non-Witt Spaces

Let X be a stratified, oriented, topological pseudomanifold without boundary. If X has only strata of even codimension, then $\mathbf{IC}^\bullet_{\bar{m}}(X)$, the intersection chain sheaf with respect to the lower middle perversity $\bar{m}$, is Verdier-self-dual, since $\mathbf{IC}^\bullet_{\bar{m}}(X) = \mathbf{IC}^\bullet_{\bar{n}}(X)$, the intersection chain sheaf with respect to the upper middle perversity $\bar{n}$. More generally, $\mathbf{IC}^\bullet_{\bar{m}}(X)$ is still self-dual on X if X is a Witt space. If X is not a Witt space, then the canonical morphism $\mathbf{IC}^\bullet_{\bar{m}}(X) \to \mathbf{IC}^\bullet_{\bar{n}}(X)$ is not an isomorphism and $\mathbf{IC}^\bullet_{\bar{m}}(X)$ is not self-dual.

We will briefly review some relevant results of [Ban02], where a theory of intersection homology type invariants for non-Witt spaces is developed. Let X^n be an n-dimensional pseudomanifold with a fixed stratification

$$X = X_n \supset X_{n-2} \supset X_{n-3} \supset \ldots \supset X_0 \supset X_{-1} = \varnothing$$

such that X_j is closed in X and every non-empty $X_j - X_{j-1}$ is an open manifold of dimension j. Set $U_k = X - X_{n-k}$. Throughout this paper we will work with real coefficients and we will follow the indexing conventions of [GM83]. Thus self-duality will be understood with respect to $\mathcal{D}[n]$, rather than with respect to $\mathcal{D}$.

As shown in [Ban02], intersection homology type invariants on non-Witt spaces are given by objects of a certain full subcategory $SD(X) \subset D(X)$, where $D(X)$ denotes the bounded, constructible derived category. The objects of $SD(X)$ satisfy two properties: On the one hand, they are self-dual, on the other hand, they are as close to the middle perversity intersection chain sheaves as possible, that is, they interpolate between $\mathbf{IC}^\bullet_{\bar{m}}(X)$ and $\mathbf{IC}^\bullet_{\bar{n}}(X)$. The precise definition is as follows:

Definition 3.1. Let $SD(X)$ be the full subcategory of $D(X)$ whose objects $\mathbf{S}^\bullet$ satisfy the following axioms:

- **(SD1):** Top stratum: $\mathbf{S}^\bullet|_{U_2} \cong \mathbf{H}^{-n}(\mathbf{S}^\bullet)[n]|_{U_2}$
- **(SD2):** Lower bound: $\mathbf{H}^i(\mathbf{S}^\bullet) = 0$, for $i < -n$.
- **(SD3):** Stalk condition for the upper middle perversity $\bar{n}$:
  $\mathbf{H}^i(\mathbf{S}^\bullet|_{U_{k+1}}) = 0$, for $i > \bar{n}(k) - n, k \geq 2$.
- **(SD4):** Self-Duality: $\mathbf{S}^\bullet$ has an associated isomorphism
  $d : \mathcal{D}\mathbf{S}^\bullet[n] \xrightarrow{\cong} \mathbf{S}^\bullet$ such that $\mathcal{D}d[n] = \pm d$.

Here, $\mathbf{H}^i(\mathbf{S}^\bullet)$ denotes the cohomology sheaf of the complex $\mathbf{S}^\bullet$. Depending on X, the category $SD(X)$ may or may not be empty. If $\mathbf{S}^\bullet \in SD(X)$, then there exist morphisms $\mathbf{IC}^\bullet_{\bar{m}}(X;\mathcal{S}) \xrightarrow{\alpha} \mathbf{S}^\bullet \xrightarrow{\beta} \mathbf{IC}^\bullet_{\bar{n}}(X;\mathcal{S})$, where $\mathcal{S}$ is the self-dual local coefficient system $\mathcal{S} = \mathbf{H}^{-n}(\mathbf{S}^\bullet)|_{U_2}$, such that

$$\begin{array}{ccc} \mathbf{IC}^\bullet_{\bar{m}}(X;\mathcal{S}) & \xrightarrow{\alpha} & \mathbf{S}^\bullet \\ \uparrow \simeq & & \simeq \uparrow d \\ \mathcal{D}\mathbf{IC}^\bullet_{\bar{n}}(X;\mathcal{S})[n] & \xrightarrow{\mathcal{D}\beta[n]} & \mathcal{D}\mathbf{S}^\bullet[n] \end{array}$$

(where d is given by (**SD4**)) commutes, which clarifies the relation between intersection chain sheaves and objects of $SD(X)$. The main structure theorem on $SD(X)$ is a description as a Postnikov system with fibers given by categories of Lagrangian structures along the strata of odd codimension:

Theorem 3.1. *Let $n = \dim X$ be even. There is an equivalence of categories*

$$SD(X) \simeq \mathrm{Lag}(U_n - U_{n-1}) \rtimes \mathrm{Lag}(U_{n-2} - U_{n-3}) \rtimes \ldots \rtimes \mathrm{Lag}(U_4 - U_3) \rtimes \mathrm{Coeff}(U_2).$$

(Similarly for n odd.)

Assume k is odd and $\mathbf{A}^\bullet \in SD(U_k)$. Note that $\bar{n}(k) = \bar{m}(k) + 1$. We shall use the shorthand notation ${}^{\bar{m}}\mathbf{A}^\bullet = \tau_{\leq \bar{m}(k)-n} Ri_{k*}\mathbf{A}^\bullet$, ${}^{\bar{n}}\mathbf{A}^\bullet = \tau_{\leq \bar{n}(k)-n} Ri_{k*}\mathbf{A}^\bullet$, and $s = \bar{n}(k) - n$. The reason why ${}^{\bar{m}}\mathbf{A}^\bullet$ need not be self-dual is that the obstruction-sheaf

$$\mathcal{O}(\mathbf{A}^\bullet) = \mathbf{H}^s(Ri_{k*}\mathbf{A}^\bullet)[-s] \in D(U_{k+1})$$

need not be trivial. Its support is $U_{k+1} - U_k$, and it is isomorphic to the algebraic mapping cone of the canonical morphism ${}^{\bar{m}}\mathbf{A}^\bullet \to {}^{\bar{n}}\mathbf{A}^\bullet$. The obstruction-sheaf $\mathcal{O}(\mathbf{A}^\bullet)$ possesses an induced self-duality $\mathcal{D}\mathcal{O}(\mathbf{A}^\bullet)[n+1] \cong \mathcal{O}(\mathbf{A}^\bullet)$. A *Lagrangian structure* (along $U_{k+1} - U_k$) is a morphism $\mathcal{L} \longrightarrow \mathcal{O}(\mathbf{A}^\bullet)$, $\mathcal{L} \in D(U_{k+1})$, which induces injections on stalks and has the property that some distinguished triangle on $\mathcal{L} \longrightarrow \mathcal{O}(\mathbf{A}^\bullet)$ is an algebraic null-cobordism (in the sense of [CS91]) for $\mathcal{O}(\mathbf{A}^\bullet)$. If $\mathbf{B}^\bullet \in SD(U_k)$ and $\mathcal{L}_A \to \mathcal{O}(\mathbf{A}^\bullet)$, $\mathcal{L}_B \to \mathcal{O}(\mathbf{B}^\bullet)$ are two Lagrangian structures, then a *morphism of Lagrangian structures* is a commutative square in $D(U_{k+1})$:

$$\begin{array}{ccc} \mathcal{L}_A & \longrightarrow & \mathcal{O}(\mathbf{A}^\bullet) \\ \downarrow & & \downarrow \mathcal{O}(f) \\ \mathcal{L}_B & \longrightarrow & \mathcal{O}(\mathbf{B}^\bullet) \end{array}$$

where $f \in \mathrm{Hom}_{D(U_k)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$ and $\mathcal{O}(f) = \mathbf{H}^s(Ri_{k*}f)[-s]$. Thus Lagrangian structures form a category $\mathrm{Lag}(U_{k+1} - U_k)$. The expression $\mathrm{Lag}(U_{k+1} - U_k) \rtimes SD(U_k)$ denotes the twisted product of categories whose objects are pairs $(\mathcal{L} \xrightarrow{\lambda} \mathcal{O}(\mathbf{A}^\bullet), \mathbf{A}^\bullet)$, $\mathbf{A}^\bullet \in SD(U_k)$, $\lambda \in \mathrm{Lag}(U_{k+1} - U_k)$, and whose morphisms are pairs with first component a morphism $f \in \mathrm{Hom}_{D(U_k)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$ and second component a commutative square as above. One direction of the equivalence of categories in theorem 3.1 is induced by a covariant functor

$$\begin{array}{rccc} \boxplus: & \mathrm{Lag}(U_{k+1}-U_k) \rtimes SD(U_k) & \longrightarrow & SD(U_{k+1}), \\ & (\lambda, \mathbf{A}^\bullet) & \mapsto & \lambda \boxplus \mathbf{A}^\bullet, \end{array}$$

that is, a Lagrangian structure along $U_{k+1}-U_k$ naturally gives rise to a self-dual sheaf on U_{k+1}. For more details, we ask the reader to consult [Ban02]. Given a local coefficient system $\mathcal{S} \in \mathrm{Coeff}(U_2)$, we will write $SD(X;\mathcal{S}) \subset SD(X)$ for the full subcategory of objects $\mathbf{S}^\bullet$ such that $\mathbf{S}^\bullet|_{U_2} \cong \mathcal{S}[n]$.

Example 3.1. Let X^6 be the product of a circle with the (unreduced) suspension of complex projective space, $X^6 = S^1 \times \Sigma\mathbb{C}P^2$. This space has a stratum of odd codimension 5 consisting of the disjoint union of two circles. The link of this stratum is $\mathbb{C}P^2$ and there is no Lagrangian subspace in the middle cohomology $H^2(\mathbb{C}P^2)$ (e.g. the signature $\sigma(\mathbb{C}P^2) = 1 \neq 0$). Then the structure theorem implies $SD(X^6) = \varnothing$, so that there is no meaningful way to define intersection homology type invariants on X^6.

Let X^4 be the product of a circle with the suspension of a torus, $X^4 = S^1 \times \Sigma T^2$. The stratum of odd codimension 3 consists again of the disjoint union of two circles, but with link T^2. There are many Lagrangian subspaces $\mathcal{L}$ in the middle cohomology $H^1(T^2)$, and the structure theorem implies $SD(X^4) \neq \varnothing$. In fact, the functor $\boxplus$ constructs a self-dual sheaf on X^4 for every choice of $\mathcal{L}$.

The following lemma will be used in section 6.

Lemma 3.1. *Let X^n be a closed, topological, stratified, oriented pseudomanifold with singular set Σ. Let $(\mathcal{S},\phi)$ and $(\mathcal{T},\psi)$ be Poincaré local systems on $X-\Sigma$, and let $\gamma : \mathcal{S} \to \mathcal{T}$ be an isomorphism such that the diagram*

$$\begin{array}{ccc} \mathcal{S} \otimes \mathcal{S} & \xrightarrow{\phi} & \mathbb{R}_{X-\Sigma} \\ {\scriptstyle \gamma\otimes\gamma}\downarrow{\scriptstyle \cong} & \nearrow{\scriptstyle \psi} & \\ \mathcal{T} \otimes \mathcal{T} & & \end{array} \tag{5}$$

commutes. If there exists an extension $(\boldsymbol{S}^\bullet, d_S) \in SD(X;\mathcal{S})$ of $(\mathcal{S},\phi)$, $d_S : \mathcal{D}\boldsymbol{S}^\bullet[n] \xrightarrow{\simeq} \boldsymbol{S}^\bullet$, then $(\mathcal{T},\psi)$ also extends to $(\boldsymbol{T}^\bullet, d_T) \in SD(X;\mathcal{T})$, $d_T : \mathcal{D}\boldsymbol{T}^\bullet[n] \xrightarrow{\simeq} \boldsymbol{T}^\bullet$. Moreover,

$$\sigma(X;\mathcal{S}) = \sigma(X;\mathcal{T}).$$

Proof. We shall construct $(\mathbf{T}^\bullet, d_T) \in SD(X;\mathcal{T})$, together with an isomorphism $\overline{\gamma} : \mathbf{S}^\bullet \xrightarrow{\simeq} \mathbf{T}^\bullet$, in stages by induction on k. Set $\mathbf{T}^\bullet_2 = \mathcal{T}[n]$, $d_{T,2} = \psi$, $\overline{\gamma}_2 = \gamma[n]$, on $U_2 = X - \Sigma$. Here, we are reinterpreting ψ as a Verdier duality isomorphism

$$\psi : \mathcal{D}\mathbf{T}^\bullet_2[n] = (\mathbf{Hom}(\mathcal{T}, \mathbb{R}_{X-\Sigma}) \otimes \mathcal{O}_{X-\Sigma})[n] \xrightarrow{\simeq} (\mathcal{T} \otimes \mathcal{O}_{X-\Sigma})[n] \cong \mathbf{T}^\bullet_2,$$

where $\mathcal{O}_{X-\Sigma}$ denotes the orientation sheaf and the last isomorphism uses the orientation of $X - \Sigma$. Inductively, we will construct $\mathbf{T}^\bullet_k \in SD(U_k;\mathcal{T})$,

$d_{T,k} : \mathcal{D}\mathbf{T}^\bullet_k[n] \xrightarrow{\simeq} \mathbf{T}^\bullet_k$, and $\overline{\gamma}_k : \mathbf{S}^\bullet_k = \mathbf{S}^\bullet|_{U_k} \xrightarrow{\simeq} \mathbf{T}^\bullet_k$ so that $\mathbf{T}^\bullet = \mathbf{T}^\bullet_{n+1} \in SD(X;\mathcal{T})$, $d_T = d_{T,n+1}$, $\overline{\gamma} = \overline{\gamma}_{n+1}$, $X = U_{n+1}$. Assume $(\mathbf{T}^\bullet_k, d_{T,k}) \in SD(U_k;\mathcal{T})$ and $\overline{\gamma}_k$ have been constructed. If k is even, then $U_{k+1} - U_k$ is a stratum of even codimension, and thus the Goresky-MacPherson-Deligne extension

$$\mathbf{T}^\bullet_{k+1} = \tau_{\leq \bar{m}(k)-n} Ri_{k*}\mathbf{T}^\bullet_k$$

is a self-dual sheaf in $SD(U_{k+1};\mathcal{T})$. We set $\overline{\gamma}_{k+1} = \tau_{\leq \bar{m}(k)-n} Ri_{k*}\overline{\gamma}_k$. Let k be odd. According to theorem 3.1, there is an equivalence of categories

$$SD(U_{k+1};\mathcal{T}) \simeq \mathrm{Lag}(U_{k+1} - U_k) \rtimes SD(U_k;\mathcal{T}). \tag{6}$$

Since $\mathbf{S}^\bullet_{k+1} \in SD(U_k)$, it has an associated Lagrangian structure

$$\mathcal{L} \xrightarrow{\lambda_S} \mathcal{O}(\mathbf{S}^\bullet_k).$$

The isomorphism $\overline{\gamma}_k : \mathbf{S}^\bullet_k \to \mathbf{T}^\bullet_k$ induces an isomorphism

$$\mathcal{O}(\overline{\gamma}_k) : \mathcal{O}(\mathbf{S}^\bullet_k) \xrightarrow{\simeq} \mathcal{O}(\mathbf{T}^\bullet_k).$$

The composition

$$\lambda_T = \mathcal{O}(\overline{\gamma}_k) \circ \lambda_S : \mathcal{L} \longrightarrow \mathcal{O}(\mathbf{T}^\bullet_k)$$

is a Lagrangian structure for $\mathbf{T}^\bullet_k$. The pair $(\lambda_T, \mathbf{T}^\bullet_k)$ is thus an object in $\mathrm{Lag}(U_{k+1} - U_k) \rtimes SD(U_k;\mathcal{T})$. An application of the functor

$$\boxplus : \mathrm{Lag}(U_{k+1} - U_k) \rtimes SD(U_k) \longrightarrow SD(U_{k+1})$$

produces an object $\mathbf{T}^\bullet_{k+1} \in SD(U_{k+1})$ such that $\mathbf{T}^\bullet_{k+1}|_{U_k} = \mathbf{T}^\bullet_k$, $d_{T,k+1}|_{U_k} = d_{T,k}$. Application of $\boxplus$ to the isomorphism

$$\left(\begin{array}{ccc} \mathcal{L} & \xrightarrow{\lambda_S} & \mathcal{O}(\mathbf{S}^\bullet_k) \\ \Big\| & & \simeq \Big\downarrow \mathcal{O}(\overline{\gamma}_k) \\ \mathcal{L} & \xrightarrow{\lambda_T} & \mathcal{O}(\mathbf{T}^\bullet_k) \end{array}, \quad \mathbf{S}^\bullet_k \xrightarrow[\overline{\gamma}_k]{\simeq} \mathbf{T}^\bullet_k \right)$$

$$\in \mathrm{Hom}_{\mathrm{Lag}(U_{k+1}-U_k) \rtimes SD(U_k)}((\lambda_S, \mathbf{S}^\bullet_k), (\lambda_T, \mathbf{T}^\bullet_k))$$

yields an isomorphism

$$\overline{\gamma}_{k+1} : \mathbf{S}^\bullet_{k+1} = \lambda_S \boxplus \mathbf{S}^\bullet_k \xrightarrow{\simeq} \lambda_T \boxplus \mathbf{T}^\bullet_k = \mathbf{T}^\bullet_{k+1}$$

such that $\overline{\gamma}_{k+1}|_{U_k} = \overline{\gamma}_k$. Summarizing, we have constructed

$$\begin{aligned} &\mathbf{T}^\bullet = \mathbf{T}^\bullet_{n+1} \in SD(X), \\ &d_T = d_{T,n+1} : \mathcal{D}\mathbf{T}^\bullet[n] \cong \mathbf{T}^\bullet, \text{ and} \\ &\overline{\gamma} = \overline{\gamma}_{n+1} : \mathbf{S}^\bullet \xrightarrow{\simeq} \mathbf{T}^\bullet \end{aligned}$$

such that

$$\mathbf{T}^\bullet|_{X-\Sigma} = \mathcal{T}, \; \overline{\gamma}|_{X-\Sigma} = \gamma[n], \; d_T|_{X-\Sigma} = \psi.$$

We claim that the diagram

$$
\begin{array}{ccc}
\mathcal{D}\mathbf{S}^\bullet[n] & \xrightarrow[d_S]{\simeq} & \mathbf{S}^\bullet \\
{\scriptstyle \mathcal{D}\overline{\gamma}[n]}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \overline{\gamma}} \\
\mathcal{D}\mathbf{T}^\bullet[n] & \xrightarrow[d_T]{\simeq} & \mathbf{T}^\bullet
\end{array}
\tag{7}
$$

commutes. Indeed, over $X - \Sigma$ this diagram is

$$
\begin{array}{ccc}
\mathcal{D}\mathcal{S} & \xrightarrow[\phi]{\simeq} & \mathcal{S}[n] \\
{\scriptstyle \mathcal{D}\gamma}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \gamma[n]} \\
\mathcal{D}\mathcal{T} & \xrightarrow[\psi]{\simeq} & \mathcal{T}[n]
\end{array}
$$

i.e. (after removing the shift)

$$
\begin{array}{ccc}
\mathbf{Hom}(\mathcal{S}, \mathbb{R}_{X-\Sigma}) & \xrightarrow[\phi]{\simeq} & \mathcal{S} \\
{\scriptstyle \mathbf{Hom}(\gamma, \mathbb{R}_{X-\Sigma})}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \gamma} \\
\mathbf{Hom}(\mathcal{T}, \mathbb{R}_{X-\Sigma}) & \xrightarrow[\psi]{\simeq} & \mathcal{T}
\end{array}
$$

whose commutativity is provided by the commutativity of (5). Now by lemma 2.2 of [Ban02], restriction of morphisms induces an injection

$$\mathrm{Hom}_{D(X)}(\mathcal{D}\mathbf{T}^\bullet[n], \mathbf{T}^\bullet) \hookrightarrow \mathrm{Hom}_{D(X-\Sigma)}(\mathcal{D}\mathbf{T}^\bullet[n]|_{X-\Sigma}, \mathbf{T}^\bullet|_{X-\Sigma}),$$

whence diagram (7) commutes. If $n = 2m$, then applying hypercohomology to diagram (7), and using the compactness of X, yields the commutative square

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathcal{H}^{-m}(X; \mathbf{S}^\bullet), \mathbb{R}) & \xrightarrow{\simeq} & \mathcal{H}^{-m}(X; \mathbf{S}^\bullet) \\
\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow \\
\mathrm{Hom}(\mathcal{H}^{-m}(X; \mathbf{T}^\bullet), \mathbb{R}) & \xrightarrow{\simeq} & \mathcal{H}^{-m}(X; \mathbf{T}^\bullet)
\end{array}
$$

Rewriting this in terms of tensor products, we obtain the commutative diagram of bilinear forms

$$\begin{array}{ccc} \mathcal{H}^{-m}(X;\mathbf{S}^\bullet) \otimes \mathcal{H}^{-m}(X;\mathbf{S}^\bullet) & \xrightarrow{\mathcal{H}(d_S)} & \mathbb{R} \\ {\scriptstyle \mathcal{H}(\overline{\gamma})\otimes\mathcal{H}(\overline{\gamma})} \Big\downarrow \simeq & \nearrow {\scriptstyle \mathcal{H}(d_T)} & \\ \mathcal{H}^{-m}(X;\mathbf{T}^\bullet) \otimes \mathcal{H}^{-m}(X;\mathbf{T}^\bullet) & & \end{array}$$

which implies equality of the signatures,

$$\sigma(X;\mathcal{S}) = \sigma(\mathcal{H}(d_S)) = \sigma(\mathcal{H}(d_T)) = \sigma(X;\mathcal{T}).$$

□

Next, we review the construction of homology L-classes of stratified spaces with boundary, as well as their relation to the L-classes of the boundary. We adopt the Thom-Pontrjagin construction approach using global transversality as employed by Goresky-MacPherson [GM80] and Cappell-Shaneson [CS91]. Let $(X^n, \partial X)$ be a compact, oriented, Whitney stratified pseudomanifold-with-boundary endowed with a self-dual sheaf $\mathbf{S}^\bullet \in SD(\operatorname{int} X)$. Let S^k be the k-sphere with base-point $p \in S^k$. The cohomotopy set $\pi^k(X,\partial X) = [(X,\partial X),(S^k,p)]$ is a group for $2k > n+1$ and in that range the Hurewicz map is rationally an isomorphism

$$\pi^k(X,\partial X) \otimes \mathbb{Q} \cong H^k(X,\partial X;\mathbb{Q}) \tag{8}$$

Fix a point $q \in S^k$, $q \neq p$. A given continuous map $f : (X,\partial X) \to (S^k,p)$ is homotopic rel ∂X to a map $\tilde{f}$, the restriction of a smooth map on an open neighborhood of X in the ambient manifold implicit in the Whitney stratification, such that $\tilde{f}$ is transverse regular to q and $\tilde{f}^{-1}(q) \subset \operatorname{int} X$ is transverse to each stratum of X. Transversality implies the normal nonsingularity of the inclusion $i_f : \tilde{f}^{-1}(q) \hookrightarrow \operatorname{int} X$, so that the restriction $i_f^! \mathbf{S}^\bullet$ is a self-dual complex of sheaves, $i_f^! \mathbf{S}^\bullet \in SD(\tilde{f}^{-1}(q))$. If $f, g : (X,\partial X) \to (S^k,p)$ are homotopic transverse maps, then the preimage $H^{-1}(q) \subset \operatorname{int} X$ under a transverse homotopy rel ∂X, $H : X \times [0,1] \to S^k$, together with $i_H^! \mathbf{S}^\bullet \in SD(\operatorname{int} H^{-1}(q))$, $i_H : \operatorname{int} H^{-1}(q) \hookrightarrow \operatorname{int} X$, is a bordism between $(f^{-1}(q), i_f^! \mathbf{S}^\bullet)$ and $(g^{-1}(q), i_g^! \mathbf{S}^\bullet)$, so that the map

$$\begin{array}{rcl} \lambda_k(\mathbf{S}^\bullet) : \pi^k(X,\partial X) & \to & \mathbb{Z} \\ {[f]} & \mapsto & \sigma(i_f^! \mathbf{S}^\bullet) \end{array}$$

is a well-defined homomorphism. Here $\sigma(i_f^! \mathbf{S}^\bullet)$ denotes the signature of the form induced by self-duality on the middle-dimensional hypercohomology of $f^{-1}(q)$ with coefficients in $i_f^! \mathbf{S}^\bullet$. Under the identification (8), $\lambda_k(\mathbf{S}^\bullet)$ induces a map

$$\lambda_k(\mathbf{S}^\bullet) \otimes \mathbb{Q} : H^k(X,\partial X;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

defining an element $L_k(\mathbf{S}^\bullet) \in H_k(X, \partial X; \mathbb{Q}) \cong \mathrm{Hom}(H^k(X, \partial X; \mathbb{Q}), \mathbb{Q})$, the *$L$-class* of the space $(X, \partial X)$ with coefficients in $\mathbf{S}^\bullet$. The restriction $2k > n+1$ is removed by considering products of $(X, \partial X)$ with spheres.

The main result of [Ban04] is:

Theorem 3.2. *Let X^n be a closed, oriented, Whitney stratified pseudomanifold and $\mathcal{S}$ a Poincaré local system on the top stratum of X. If $SD(X; \mathcal{S}) \neq \varnothing$, then the L-classes $L_i(X; \mathcal{S}) = L_i(\boldsymbol{IC}^\bullet_{\mathcal{L}}(X; \mathcal{S})) \in H_i(X; \mathbb{Q})$, $\boldsymbol{IC}^\bullet_{\mathcal{L}}(X; \mathcal{S}) \in SD(X; \mathcal{S})$, are independent of the choice of Lagrangian structure $\mathcal{L}$.*

Thus a non-Witt space has a well-defined L-class $L(X)$, provided $SD(X; \mathbb{R}) \neq \varnothing$.

Proposition 3.1. *Let $(X, \partial X)$ be a Whitney stratified, compact, oriented pseudomanifold with boundary such that $SD(\operatorname{int} X; \mathbb{R}) \neq \varnothing$. Then*

$$\partial_* L_{k+1}(X) = L_k(\partial X),$$

where ∂_* is the connecting homomorphism $\partial_* : H_{k+1}(X, \partial X) \to H_k(\partial X)$.

Proof. Given $f : \partial X \to S^k$ transverse to $p \in S^k$, we shall describe how the cohomotopy coboundary operator

$$\delta^* : \pi^k(\partial X) \longrightarrow \pi^{k+1}(X, \partial X)$$

acts on $[f]$.
Write $c\partial X$ for the cone on ∂X and view $D^{k+1} \cong cS^k$. Then f extends over the cones as

$$cf : c\partial X \longrightarrow cS^k \cong D^{k+1}.$$

Let $q = (p, \frac{1}{2}) \in cS^k$ and N be an open collar neighborhood of ∂X in X. Consider the collapse maps

$$X \longrightarrow X/(X - N) \cong c\partial X$$

and

$$D^{k+1} \longrightarrow D^{k+1}/S^k \cong S^{k+1}.$$

Denote the images of p and q under the latter collapse again by $p, q \in S^{k+1}$. Then $\delta^*[f]$ is represented by the composition

$$g : (X, \partial X) \to (c\partial X, \partial X) \xrightarrow{cf} (D^{k+1}, S^k) \to (S^{k+1}, p).$$

Observe that g is transverse to q, since in fact $g^{-1}(q) = f^{-1}(p) \times \{\frac{1}{2}\}$ when regarded as a subvariety of the collar $N \cong \partial X \times [0, 1)$. Choose an object $\mathbf{S}^\bullet \in SD(\operatorname{int} X; \mathbb{R})$, so that $L(X) = L(\mathbf{S}^\bullet)$. Using the inclusions

$$\operatorname{int} X \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} \partial X,$$

we produce the self-dual sheaf $j^! i_! \mathbf{S}^\bullet \in SD(\partial X)$, and therefore $L(\partial X) = L(j^! i_! \mathbf{S}^\bullet)$. If

$$\lambda_k(j^! i_! \mathbf{S}^\bullet) : \pi^k(\partial X) \longrightarrow \mathbb{Z}$$

is the L-class of $j^! i_! \mathbf{S}^\bullet$ and

$$\lambda_{k+1}(\mathbf{S}^\bullet) : \pi^{k+1}(X, \partial X) \longrightarrow \mathbb{Z}$$

is the L-class of $\mathbf{S}^\bullet$, then

$$\begin{aligned} \lambda_{k+1}(\mathbf{S}^\bullet)(\delta^*[f]) &= \lambda_{k+1}(\mathbf{S}^\bullet)[g] \\ &= \sigma(g^{-1}(q); i_g^! \mathbf{S}^\bullet) \\ &= \sigma(f^{-1}(p); i_f^! j^! i_! \mathbf{S}^\bullet) \\ &= \lambda_k(j^! i_! \mathbf{S}^\bullet)[f], \end{aligned}$$

where $i_g : g^{-1}(q) \hookrightarrow \operatorname{int} X$, $i_f : f^{-1}(p) \hookrightarrow \partial X$, and so

$$\lambda_{k+1}(\mathbf{S}^\bullet) \circ \delta^* = \lambda_k(j^! i_! \mathbf{S}^\bullet).$$

Under the Hurewicz map and the universal coefficient isomorphism, this translates to

$$\partial_* L_{k+1}(X) = \partial_* L_{k+1}(\mathbf{S}^\bullet) = L_k(j^! i_! \mathbf{S}^\bullet) = L_k(\partial X).$$

□

Let X^n be a closed, oriented, stratified pseudomanifold and let

$$i : Y^m \hookrightarrow X^n$$

be a normally nonsingular inclusion of an oriented, stratified pseudomanifold Y^m. (Example: $i = i_f : f^{-1}(p) \hookrightarrow X$, where X is Whitney stratified and $f : X \to S^{n-m}$ is transverse regular to $p \in S^{n-m}$.) Consider an open neighborhood $E \subset X$ of Y, the total space of an $\mathbb{R}^{n-m}$- vector bundle over Y, and put $E_0 = E - Y$, the total space with the zero-section removed. Let $\tau \in H^{n-m}(E, E_0)$ denote the Thom class. If $\pi : E \to Y$ denotes the projection, then the composition

$$H_k(X) \xrightarrow{j_*} H_k(X, X-Y) \underset{\cong}{\xleftarrow{e_*}} H_k(E, E_0) \underset{\cong}{\xrightarrow{\tau\cap -}} H_{k-n+m}(E) \underset{\cong}{\xrightarrow{\pi_*}} H_{k-n+m}(Y)$$

defines a map

$$i^! : H_k(X) \longrightarrow H_{k-n+m}(Y).$$

In the following, let $u \in H^{n-m}(S^{n-m})$ denote the generator such that $\langle u, [S^{n-m}] \rangle = 1$.

Lemma 3.2. *Let X^n be a closed, oriented, Whitney stratified pseudomanifold, and let $f : X \to S^{n-m}$ be transverse regular to $p \in S^{n-m}$. Then*

$$f^*(u) \cap x = i_{f*} i_f^!(x)$$

for any $x \in H_*(X)$, where $i_f : f^{-1}(p) \hookrightarrow X$.

Proof. Set $F = f^{-1}(p)$, let $x \in H_k(X)$, and put $\tau' = (e^*)^{-1}(\tau)$, where $e^* : H^{n-m}(X, X-F) \xrightarrow{\cong} H^{n-m}(E, E_0)$ is the excision isomorphism. The commutative square

$$\begin{array}{ccc} H^{n-m}(X, X-F) \otimes H_k(X, X-F) & \xrightarrow{e^* \otimes e_*^{-1}} & H^{n-m}(E, E_0) \otimes H_k(E, E_0) \\ \Big\downarrow \cap & & \Big\downarrow \cap \\ H_{k-n+m}(X) & \xleftarrow{\quad i_{f*} \quad} & H_{k-n+m}(F) \end{array}$$

yields the relation

$$\tau' \cap j_*(x) = i_{f*}(\tau \cap e_*^{-1} j_*(x)) = i_{f*} i_f^! (x).$$

Viewing u as a class in $H^{n-m}(S^{n-m}, S^{n-m} - p)$, we can interpret it as the Thom class of $\{p\} \hookrightarrow S^{n-m}$. Thus, by naturality of the Thom class,

$$j^*(\tau') = f^*(u).$$

The commutativity of

$$\begin{array}{ccc} H^{n-m}(X, X-F) \otimes H_k(X) & \xrightarrow{1 \otimes j_*} & H^{n-m}(X, X-F) \otimes H_k(X, X-F) \\ \Big\downarrow j^* \otimes 1 & & \Big\downarrow \cap \\ H^{n-m}(X) \otimes H_k(X) & \xrightarrow{\quad \cap \quad} & H_{k-n+m}(X) \end{array}$$

implies

$$\tau' \cap j_*(x) = j^*(\tau') \cap x = f^*(u) \cap x.$$

□

Lemma 3.3. *Let X and f be as in lemma 3.2. If $L(X)$ is defined, then*

$$f^*(u) \cap L_k(X) = i_{f*} L_{k-n+m}(f^{-1}(p)).$$

Proof. We have

$$L_{k-n+m}(f^{-1}(p)) = L_{k-n+m}(i_f^! \mathbf{S}^\bullet) = i_f^! L_k(\mathbf{S}^\bullet) = i_f^! L_k(X),$$

$\mathbf{S}^\bullet \in SD(X)$, cf. [CS91]. The statement follows from lemma 3.2. □

4. Topological Signature Homology

We give a brief overview of signature homology, introduced by Augusto Minatta in his thesis [Min04].

The construction of signature homology is, on the geometric side, based on *stratifolds*. These are stratified topological objects introduced by Matthias Kreck in 1998, [Kre03], [Kre]. One of the motivations in introducing these objects was to create a reservoir of spaces generalizing manifolds by incorporating singularities which lends itself to the formation of homology theories whose cycles are given by suitably selected subclasses of stratifolds. In particular, the technical aspects of the definition of stratifolds are designed so as to make gluing along boundaries easy, as well as to entail codimension one transversality results.

In this paper, the term "stratifold" is used in the same way as in [Min04]. We recall the definitions that are relevant for us. Let X be a topological space. An i-dimensional *strat* for X is a pair $((W, \partial W), f)$, where $(W, \partial W)$ is a collared, topological, i-dimensional manifold-with-boundary and $f : W \to X$ a proper, continuous map whose restriction to the interior of W is a homeomorphism onto its image. An n-dimensional (topological) *stratifold* is a topological space X equipped with strats $((W_0, \partial W_0), f_0), \ldots, ((W_n, \partial W_n), f_n)$ satisfying the conditions:

1. the $f_i(\operatorname{int} W_i)$ are disjoint and $\bigcup_i f_i(\operatorname{int} W_i) = X$,
2. $\dim W_i = i$,
3. $f_i(\partial W_i) \subset \bigcup_{j \leq i-1} f_j(W_j) =: \Sigma_{i-1}$
4. a subset $U \subset X$ is open if and only if $f_i^{-1}(U)$ is open in W_i, for all i.

The subspace $X^i = \Sigma_i - \Sigma_{i-1}$ is called the i-th *pure stratum* of X (it is homeomorphic to the interior of W_i and thus an i-manifold). An *isomorphism* of stratifolds is a homeomorphism $\phi : X \to X'$ together with homeomorphisms $\phi_i : W_i \to W_i'$ respecting collars and such that $\phi f_i = f_i' \phi_i$. The stratifold X is *orientable* if X^{n-1} is empty and the top stratum X^n is orientable. An *orientation* of X is an orientation of the top stratum. The collars of the W_i define retractions $\pi_i : V_i \to X^i$ from an open neighborhood V_i of the pure stratum X^i in X to X^i. Let M be a manifold. A continuous map $g : X \to M$ is a *morphism* if every composition $g f_i : W_i \to M$ is constant on every $\{x\} \times [0, 1)$, where $x \in \partial W_i$ and we identify $\partial W_i \times [0, 1)$ with a neighborhood of ∂W_i in W_i using the collar. (Every continuous map $X \to M$ is homotopic to a morphism.) A morphism $g : X \to M$ is a *stratifold bundle* if there is a stratifold F so that g is a locally trivial fiber bundle projection with fiber F and structure group given by all stratifold isomorphisms $F \to F$. A stratifold X is *locally trivial* if each retraction $\pi_i : V_i \to X^i$ is a stratifold bundle; X is *locally conelike* if X is locally trivial and the fiber of π_i is the open cone on some compact stratifold.

An n-dimensional *stratifold-with-boundary* is a pair $(X, \partial X)$ with X a topological space, $\partial X \subset X$ a closed subset, such that $X - \partial X$ is a stratifold of dimension n, ∂X is a stratifold of dimension $n-1$, and ∂X has a collar in X. By collar we mean here the usual homeomorphism which now should be an isomorphism of stratifolds when restricted to the interior. The definition of a stratifold-with-boundary is designed so that gluing $(X, \partial X)$ and $(X', \partial X')$ along the boundary is straightforward, given an isomorphism of stratifolds $\partial X \cong \partial X'$. (The gluing is described in more detail in [Min04], see also lemma 5.2 of the present paper.)

In [Ban02, chapter 4], we introduced the bordism groups Ω_*^{SD} as follows: An element in Ω_n^{SD} is represented by a triple $(X^n, \mathbf{A}^\bullet, d)$, where X^n is an n-dimensional, closed, oriented, topological pseudomanifold and

$(\mathbf{A}^\bullet, d) \in SD(X)$ is a self-dual sheaf on X. The admissible nullbordisms are triples $(Y^{n+1}, \mathbf{B}^\bullet, \delta)$ where Y^{n+1} is an $(n+1)$-dimensional, compact, oriented pseudomanifold-with-boundary, $(\mathbf{B}^\bullet, \delta) \in SD(\operatorname{int} Y)$, $\delta : \mathcal{D}\mathbf{B}^\bullet[n+1] \xrightarrow{\cong} \mathbf{B}^\bullet$. Let i, j denote the inclusions $i : \operatorname{int} Y \hookrightarrow Y$, $j : \partial Y \hookrightarrow Y$. An application of the functor $j^* Ri_*$ produces an induced isomorphism

$$j^* Ri_*(\delta) : j^* Ri_* \mathcal{D}\mathbf{B}^\bullet[n+1] \xrightarrow{\cong} j^* Ri_* \mathbf{B}^\bullet.$$

We have the canonical isomorphisms

$$j^* Ri_* \mathcal{D}\mathbf{B}^\bullet[n+1] \cong \mathcal{D}(j^! Ri_! \mathbf{B}^\bullet)[n+1]$$

and

$$j^* Ri_* \mathbf{B}^\bullet \cong j^! Ri_! \mathbf{B}^\bullet[1].$$

Hence δ induces a self-duality isomorphism d for $j^! Ri_! \mathbf{B}^\bullet$:

$$d : \mathcal{D}(j^! Ri_! \mathbf{B}^\bullet)[n] \xrightarrow{\cong} j^! Ri_! \mathbf{B}^\bullet$$

We call d the *boundary of* δ and write $(\partial Y, j^! Ri_! \mathbf{B}^\bullet, d) = \partial(Y, \mathbf{B}^\bullet, \delta)$. The resulting bordism group is denoted by Ω_n^{SD}. The signature homomorphism

$$\sigma : \Omega_{4k}^{SD} \longrightarrow \mathbb{Z}$$

is onto, since e.g. $(\mathbb{C}P^{2k}, \mathbb{R}_{\mathbb{C}P^{2k}}[4k], d = \text{orient}) \in \Omega_{4k}^{SD}$. However, contrary for example to Witt bordism, σ is also injective: Suppose $\sigma(X, \mathbf{A}^\bullet, d) = 0$. Let Y^{4k+1} be the closed cone on X. Define a self-dual sheaf on the interior of the punctured cone by pulling back $\mathbf{A}^\bullet$ from X under the projection from the interior of the punctured cone, $X \times (0, 1)$, to X. According to the Postnikov system of theorem 3.1, the self-dual sheaf on the interior of the punctured cone will have a self-dual extension in $SD(\operatorname{int} Y)$ if and only if there exists a Lagrangian structure at the cone point (which has odd codimension $4k + 1$ in Y). That Lagrangian structure exists because $\sigma(X, \mathbf{A}^\bullet, d) = 0$. Let $(\mathbf{B}^\bullet, \delta) \in SD(\operatorname{int} Y)$ be any self-dual extension given by a choice of Lagrangian structure. Then $\partial(Y, \mathbf{B}^\bullet, \delta) = (X, \mathbf{A}^\bullet, d)$ and thus $[(X, \mathbf{A}^\bullet, d)] = 0 \in \Omega_{4k}^{SD}$. Clearly, $\Omega_n^{SD} = 0$ for $n \not\equiv 0(4)$. In summary, then, one has

$$\Omega_n^{SD} \cong \begin{cases} \mathbb{Z}, & n \equiv 0(4), \\ 0, & n \not\equiv 0(4). \end{cases}$$

Minatta [Min04] takes this as his starting point and constructs a bordism theory $S_*(-)$, called *signature homology*, whose coefficients are

$$S_n(pt) \cong \Omega_n^{SD} \cong \begin{cases} \mathbb{Z}, & n \equiv 0(4) \\ 0, & n \not\equiv 0(4) \end{cases}$$

which explains the nomenclature. Let us describe the construction of $S_*(-)$. Let X be a topological space. Elements of $S_n(X)$ are represented by quadruples

$$(S, \mathbf{S}^\bullet, d, S \xrightarrow{f} X),$$

where

- S is an n-dimensional, closed, oriented, locally conelike stratifold with dense top stratum and $S^{n-1} = \varnothing$ (i.e. no pure stratum of codimension 1),
- $\mathbf{S}^\bullet \in SD(S)$ is a constructible complex of sheaves on S with self-duality isomorphism $d : \mathcal{D}\mathbf{S}^\bullet[n] \cong \mathbf{S}^\bullet$, and
- $f : S \to X$ is a continuous map.

Two such quadruples $(S_1, \mathbf{S}^\bullet_1, d_1, f_1)$ and $(S_2, \mathbf{S}^\bullet_2, d_2, f_2)$ are *bordant* if there exists a quadruple

$$(T, \mathbf{T}^\bullet, \delta, T \xrightarrow{F} X),$$

where

- T is an $(n+1)$-dimensional, compact, oriented, locally conelike stratifold-with-boundary whose interior has dense top stratum and $(\operatorname{int} T)^{n-1} = \varnothing$,
- $\mathbf{T}^\bullet \in SD(\operatorname{int} T)$ is a constructible complex of sheaves on $\operatorname{int} T$ with self-duality isomorphism $\delta : \mathcal{D}\mathbf{T}^\bullet[n+1] \cong \mathbf{T}^\bullet$,
- $F : T \to X$ is a continuous map,
- $\partial(T, \mathbf{T}^\bullet, \delta) \cong (S_1, \mathbf{S}^\bullet_1, d_1) \sqcup (S_2, \mathbf{S}^\bullet_2, -d_2)$,
- near ∂T, $\mathbf{T}^\bullet$ is equipped with an isomorphism to the pullback of $\mathbf{S}^\bullet_1 \sqcup \mathbf{S}^\bullet_2$ under the retraction determined by the collar of ∂T in T.
- $F|_{\partial T} = f_1 \sqcup f_2$.

Let us describe signature homology at odd primes. Regard $\mathbb{Z}[\frac{1}{2}][t]$ as a graded ring with $\deg(t) = 4$, i.e.

$$(\mathbb{Z}[\tfrac{1}{2}][t])_n = \begin{cases} \mathbb{Z}[\frac{1}{2}]\langle t^k \rangle, & \text{if } n = 4k \\ 0, & \text{if } n \not\equiv 0(4). \end{cases}$$

Let $\Omega^{SO}_*(-)$ denote bordism of smooth, oriented manifolds. By $\otimes_{\Omega^{SO}_*(pt)}$ we mean tensor product in the category of $\Omega^{SO}_*(pt)$-modules. The ring homomorphism

$$\begin{array}{cccc} \tau: & \Omega^{SO}_*(pt) & \longrightarrow & \mathbb{Z}[\frac{1}{2}][t] \\ & [M^{4k}] & \mapsto & \sigma(M)t^k \end{array}$$

induces an $\Omega^{SO}_*(pt)$-module structure on $\mathbb{Z}[\frac{1}{2}][t]$. This homomorphism factors as

$$\begin{array}{cccc} \phi: & \Omega^{SO}_*(pt) & \longrightarrow & S_*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \\ & [M] & \mapsto & [(M, \mathbb{R}_M[\dim M], \text{orient}, M \to pt)] \otimes 1 \end{array}$$

followed by

$$\begin{array}{cccc} \eta: & S_*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] & \longrightarrow & \mathbb{Z}[\frac{1}{2}][t] \\ & [(S^{4k}, \mathbf{S}^\bullet, d, S \to pt)] \otimes x & \mapsto & x\sigma(S, \mathbf{S}^\bullet, d)t^k. \end{array}$$

As explained above, η is an isomorphism (already integrally). Tensoring with $\Omega^{SO}_*(X)$ over $\Omega^{SO}_*(pt)$, η induces an isomorphism

$$\eta(X) : \Omega^{SO}_*(X) \otimes_{\Omega^{SO}_*(pt)} S_*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\cong} \Omega^{SO}_*(X) \otimes_{\Omega^{SO}_*(pt)} \mathbb{Z}[\tfrac{1}{2}][t]. \tag{9}$$

Here, we consider $S_*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ as an $\Omega_*^{SO}(pt)$-module via ϕ. Define a natural homomorphism

$$\psi(X) : \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}(pt)} S_*(pt) \longrightarrow S_*(X)$$

by

$$[M \xrightarrow{f} X] \otimes [(S, \mathbf{S}^\bullet, d, S \to pt)] \mapsto [(M \times S, \pi_2^! \mathbf{S}^\bullet, \pi_2^! d, M \times S \xrightarrow{f \circ \pi_1} X)].$$

By the Landweber exact functor theorem (cf. [Lan76], example 3.4),

$$X \mapsto \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\tfrac{1}{2}][t]$$

is a homology theory. Thus (9) implies that

$$X \mapsto \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}(pt)} S_*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}]$$

is a homology theory. It follows that

$$\psi(X) \otimes \mathbb{Z}[\tfrac{1}{2}] : \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}(pt)} S_*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \longrightarrow S_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}]$$

is a natural transformation of homology theories. As $\psi(pt) \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism, $\psi \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism of homology theories. Composing the inverse of $\eta(X)$ with $\psi \otimes \mathbb{Z}[\frac{1}{2}]$ yields an isomorphism

$$\Omega_*^{SO}(-) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\tfrac{1}{2}][t] \xrightarrow{\cong} S_*(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}].$$

This proves

Theorem 4.1. *(Minatta.) There is an isomorphism of homology theories*

$$\Omega_*^{SO}(-) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\tfrac{1}{2}][t] \xrightarrow{\cong} S_*(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}]$$

Corollary 4.1. *The natural map*

$$\begin{array}{rcl} \kappa(X) : \Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}][t] & \longrightarrow & S_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \\ {[M \xrightarrow{f} X] \otimes at^l} & \mapsto & [(N, \mathbb{R}_N[d], \mathrm{orient}, N \to M \xrightarrow{f} X)] \otimes a, \end{array}$$

where $N^d = M \times \mathbb{C}P^{2l}$, is surjective. Moreover, given an element $x \otimes a \in S_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$, there exists an element of the form

$$[M^n \xrightarrow{f} X] \otimes \frac{1}{2^K} \in \Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}][t]$$

whose image under $\kappa(X)$ is $x \otimes a$.

Proof. The stated map $\kappa(X)$ makes the diagram

$$\begin{array}{ccc} \Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}][t] & \xrightarrow{\kappa(X)} & S_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \\ \downarrow \text{quotient} & \nearrow \simeq & \\ \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\frac{1}{2}][t] & & \end{array}$$

commutative, since for $[M \xrightarrow{f} X] \otimes at^l \in \Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}][t]$,

$$\eta(X)^{-1}([f] \otimes at^l) = \eta(X)^{-1}([f] \otimes a\sigma(\mathbb{C}P^{2l})t^l) = [f] \otimes [\mathbb{C}P^{2l}] \otimes a,$$

and

$$(\psi(X) \otimes \mathbb{Z}[\tfrac{1}{2}])([f] \otimes [\mathbb{C}P^{2l}] \otimes a) = [(N, \mathbb{R}_N[d], \text{orient}, N \to M \xrightarrow{f} X)] \otimes a$$

with $N = M \times \mathbb{C}P^{2l}$. The surjectivity of κ follows. Given $x \otimes a \in S_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$, there exists an element

$$y = \sum_i [N_i^{k_i} \xrightarrow{g_i} X] \otimes b_i t^{l_i}, \; b_i \in \mathbb{Z}[\tfrac{1}{2}], \; k_i + 4l_i = n,$$

such that $\kappa(X)(y) = x \otimes a$. Setting $M_i^n = N_i^{k_i} \times \mathbb{C}P^{2l_i}$, we have

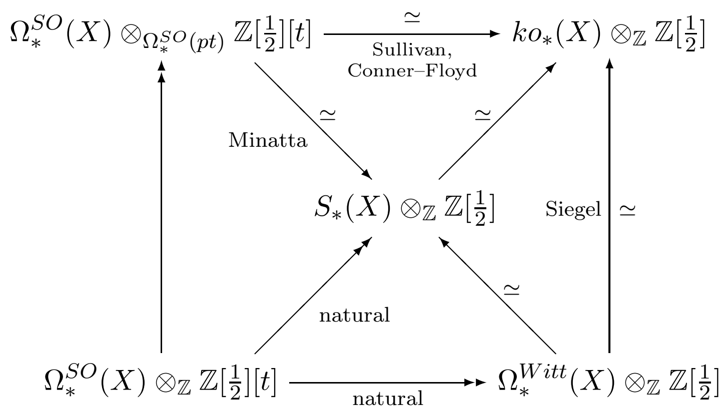
$$\kappa(X)(\sum_i [M_i^n \to N_i^{k_i} \xrightarrow{g_i} X] \otimes b_i) = \kappa(X)(y).$$

There exists a $K \in \mathbb{Z}$ such that $b_i = a_i/2^K$, $a_i \in \mathbb{Z}$, for all i. Then

$$\sum_i [M_i^n \to X] \otimes b_i = (\sum_i a_i [M_i^n \to X]) \otimes \frac{1}{2^K}.$$

□

This fact will be central to our proof of the twisted signature formula, as it will allow us to pull back calculations on singular spaces to calculations on nonsingular ones. The situation at odd primes is summarized in the following commutative diagram:



5. Piecewise Linear Signature Homology

Definition 5.1. Let X be an n-dimensional PL-pseudomanifold triangulated by a simplicial complex K. The *simplicial stratification on X induced by K* is given by the filtration

$$|K_n| = X \supset |K_{n-2}| \supset \cdots \supset |K_0|,$$

where $|K_i|$ is the underlying polyhedron of the i-skeleton of K, i.e. the union of all i-dimensional simplices of K. The notation X_K will make explicit that we wish to consider X as equipped with the simplicial stratification induced by K. Thus the top stratum $X - |K_{n-2}|$ is the union of the interiors of all n- and $(n-1)$-simplices of K. More generally, the stratification X_K can be defined similarly if K is only a cell complex (in the sense of [RS72], chapter 2; a cell is by definition a compact, convex polyhedron). In this paper, all simplicial complexes will be understood to be locally finite.

Lemma 5.1. *If X^n is a PL-pseudomanifold which is the underlying polyhedron of a cell complex K, then $SD(X_K) \subset SD(X_{K'})$ for any cellular subdivision K' of K (that is, each cell of K' is contained in a cell of K).*

Proof. Let $\mathbf{S}^\bullet \in SD(X_K)$. We have to check that $\mathbf{S}^\bullet$ is constructible with respect to the stratification $X_{K'}$ and that it satisfies **(SD1)** – **(SD4)** of definition 3.1 with respect to $X_{K'}$. Set

$$\begin{array}{ll} D_i = |K_i| - |K_{i-1}|, & U_k = |K| - |K_{n-k}|, \\ D'_i = |K'_i| - |K'_{i-1}|, & U'_k = |K'| - |K'_{n-k}|. \end{array}$$

As for constructibility, we have to show that the cohomology sheaves $\mathbf{H}^*(\mathbf{S}^\bullet)|_{D'_i}$ are locally constant. Since D'_i is a disjoint union of interiors of i-cells, it suffices to show that the cohomology sheaves are locally constant on such an interior $\operatorname{int} C'$, with C' an i-cell of K'. There exists a cell $C \in K$ such that $\operatorname{int} C' \subset \operatorname{int} C$. As $\mathbf{S}^\bullet$ is constructible with respect to K, we know that $\mathbf{H}^*(\mathbf{S}^\bullet)|_{\operatorname{int} C}$ is locally constant (hence constant since $\operatorname{int} C$ is contractible). In particular, $\mathbf{H}^*(\mathbf{S}^\bullet)|_{\operatorname{int} C'}$ is constant as the restriction of a constant sheaf.

(SD1): Let x be a point in the top stratum $U'_2 = D'_n \cup D'_{n-1}$ of X'_K. Then x lies in the interior of an n- or $(n-1)$-cell of K'. That interior is contained in the interior of some n- or $(n-1)$-cell of K. Thus $x \in D_n \cup D_{n-1}$, which is the top stratum U_2 of X_K. The axiom **(SD1)** for X_K furnishes an isomorphism

$$\mathbf{S}^\bullet|_{U_2} \cong \mathbf{H}^{-n}(\mathbf{S}^\bullet)[n]|_{U_2}$$

which yields the desired isomorphism by restriction from U_2 to U'_2.

(SD2): The statement $\mathbf{H}^j(\mathbf{S}^\bullet) = 0$ for $j < -n$ is independent of any choice of stratification.

(SD3): We must demonstrate that

$$\mathbf{H}^j(\mathbf{S}^\bullet|_{U'_{k+1}}) = 0, \quad \text{if } j > \bar{n}(k) - n,$$

for all $k \geq 1$ (where we set $\bar{n}(1) = 0$). We proceed by induction on k. For $k = 1$, the statement on U'_2 follows from **(SD1)**. Suppose we know that

$$\mathbf{H}^j(\mathbf{S}^\bullet|_{U'_k}) = 0, \quad \text{if } j > \bar{n}(k-1) - n.$$

Let $x \in U'_{k+1} = U'_k \cup D'_{n-k}$. If $x \in U'_k$ then $\mathbf{H}^j(\mathbf{S}^\bullet)_x = 0$ for $j > \bar{n}(k-1) - n$ by our induction hypothesis, so in particular the stalk vanishes for $j > \bar{n}(k) - n$, since $\bar{n}(k-1) \leq \bar{n}(k)$. Assume $x \in D'_{n-k}$, say $x \in \operatorname{int} C'_{n-k} \subset \operatorname{int} C_{n-c}$, where C'_{n-k} is an $(n-k)$-cell of K' and C_{n-c} is an $(n-c)$-cell of K, $c \leq k$. It follows that $x \in |K| - |K_{n-c-1}| = U_{c+1}$, so that $\mathbf{H}^j(\mathbf{S}^\bullet)_x = 0$ for $j > \bar{n}(c) - n$

by **(SD3)** for X_K. In particular $\mathbf{H}^j(\mathbf{S}^\bullet)_x = 0$ for $j > \bar{n}(k) - n$, as $c \leq k$ implies $\bar{n}(c) \leq \bar{n}(k)$.

(SD4): If Y is any topological pseudomanifold, then the dualizing complex $\mathbb{D}_Y^\bullet$ is constructible with respect to any topological stratification of Y. Thus $\mathbb{D}_X^\bullet$ is constructible with respect to X_K as well as $X_{K'}$. Consequently, the dual

$$\mathcal{D}\mathbf{S}^\bullet = \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet, \mathbb{D}_X^\bullet)$$

is constructible with respect to both X_K and $X_{K'}$, and the isomorphism $\mathcal{D}\mathbf{S}^\bullet[n] \cong \mathbf{S}^\bullet$ in $SD(X_K)$ given by axiom **(SD4)** may be regarded as an isomorphism in $SD(X_{K'})$. □

Corollary 5.1. *The signature is a PL-invariant for PL-pseudomanifolds X such that $SD(X_K) \neq \varnothing$ for some triangulating complex K in the PL-structure of X.*

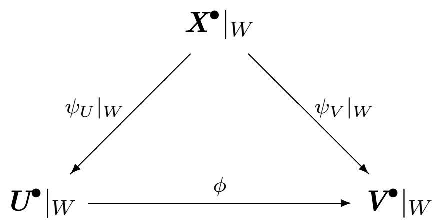
Proof. Suppose Y is a PL-pseudomanifold and $f : X \xrightarrow{\cong} Y$ is a PL-homeomorphism. As Y is PL, there exists a simplicial complex L with $Y = |L|$. Then there exist subdivisions K', L' of K, L, respectively, such that $f : |K'| \to |L'|$ is simplicial. Choose a self-dual sheaf $\mathbf{X}^\bullet \in SD(X_K)$. By lemma 5.1, $\mathbf{X}^\bullet \in SD(X_{K'})$. Since f is a simplicial isomorphism, we have $Rf_*\mathbf{X}^\bullet \in SD(Y_{L'})$. This shows that if $\sigma(X)$ is defined and X is PL-homeomorphic to Y, then $\sigma(Y)$ is defined also. Now let $\mathbf{Y}^\bullet \in SD(Y_L)$ be any sheaf; we have to show that $\sigma(\mathbf{X}^\bullet) = \sigma(\mathbf{Y}^\bullet)$. Again using lemma 5.1, $\mathbf{Y}^\bullet \in SD(Y_{L'})$, so that both $Rf_*\mathbf{X}^\bullet$ and $\mathbf{Y}^\bullet$ are self-dual sheaves constructible with respect to the same stratification of Y. Theorem 3.2 asserts that

$$\sigma(\mathbf{X}^\bullet) = \sigma(f_*\mathbf{X}^\bullet) = \sigma(\mathbf{Y}^\bullet).$$

□

The following is lemma 3.30 of [Min04], where a detailed proof can be found.

Lemma 5.2. *(Gluing Lemma.) Let X be a topological stratified pseudomanifold, and let $U, V \subset X$ be open subsets with $X = U \cup V$ and $W = U \cap V \neq \varnothing$. Given sheaves $\boldsymbol{U}^\bullet \in SD(U)$, $\boldsymbol{V}^\bullet \in SD(V)$ and an isomorphism $\phi : \boldsymbol{U}^\bullet|_W \xrightarrow{\cong} \boldsymbol{V}^\bullet|_W$, there exists, uniquely up to isomorphism, a sheaf $\boldsymbol{X}^\bullet \in SD(X)$ together with isomorphisms $\psi_U : \boldsymbol{X}^\bullet|_U \xrightarrow{\cong} \boldsymbol{U}^\bullet$, $\psi_V : \boldsymbol{X}^\bullet|_V \xrightarrow{\cong} \boldsymbol{V}^\bullet$, such that the diagram*



commutes.

Let X be a topological space. In order to define PL signature homology $S_n^{PL}(X)$, we describe first the class of objects that will represent elements in this theory.

Definition 5.2. An *admissible PL-representative* is a quadruple

$$(S, K, \mathbf{S}^\bullet, S \xrightarrow{f} X),$$

where:

- S is an n-dimensional, closed, oriented PL pseudomanifold.
- K is a simplicial complex triangulating S.
- $\mathbf{S}^\bullet \in SD(S_K)$ is a self-dual complex of sheaves on S (the self-duality isomorphism will be suppressed in the notation), constructible with respect to the simplicial stratification on S induced by K.
- $f : S \to X$ is a continuous map.

Definition 5.3. An *admissible PL-nullbordism* for an admissible PL-representative $(S, K, \mathbf{S}^\bullet, f)$ is a quadruple

$$(T, L, \mathbf{T}^\bullet, T \xrightarrow{F} X),$$

subject to the following requirements:

- T is a compact polyhedron containing S as a subpolyhedron.
- L is a simplicial complex triangulating T.
- $K \subset L$ is a subcomplex triangulating S.
- The complement $T - S$ is an oriented, $(n+1)$-dimensional pseudomanifold.
- S is collared in T, i.e. there exists a closed, polyhedral neighborhood N of S in T and an orientation preserving PL isomorphism
$$c : S \times I \xrightarrow{\cong} N$$
such that $c(s, 0) = s$ for all $s \in S$. We will use the notation $\partial T = S$, $\operatorname{int} T = T - S$.
- $\mathbf{T}^\bullet \in SD(\operatorname{int} T_L)$, i.e. $\mathbf{T}^\bullet$ is a self-dual sheaf on the interior of T, constructible with respect to the simplicial stratification induced on $\operatorname{int} T$ by L.
- Let U be the open subset $U = c(S \times (0, 1)) \subset T$ and let $c|$ denote the restriction $c| : S \times (0, 1) \xrightarrow{\cong} U \subset \operatorname{int} T$. The simplicial stratification on T induces by restriction a stratification of U. The simplicial stratification on S induces the product stratification on $S \times (0, 1)$. Then $c|$ is in general not stratum preserving. We require:
  1. $c|^*(\mathbf{T}^\bullet|_U)$ is constructible on $S \times (0, 1)$ with respect to the product stratification, and
  2. there is an isomorphism
  $$c|^*(\mathbf{T}^\bullet|_U) \cong \pi^! \mathbf{S}^\bullet,$$
  where $\pi : S \times (0, 1) \to S$ is the projection.
- $F : T \to X$ is a continuous map such that $F|_S = f$.

Two admissible PL-representatives $(S_1, K_1, \mathbf{S}_1^\bullet, f_1)$ and $(S_2, K_2, \mathbf{S}_2^\bullet, f_2)$ are *bordant* if there exists an admissible PL-nullbordism for $(S_1, K_1, \mathbf{S}_1^\bullet, f_1) \sqcup (-S_2, K_2, \mathbf{S}_2^\bullet, f_2)$. To see that this relation of bordism is an equivalence relation, we need only elaborate on transitivity.

Lemma 5.3. *The bordism relation on admissible PL-representatives is transitive.*

Proof. It suffices to describe how one can glue two admissible PL-nullbordisms for the same admissible PL-representative. Thus let $(T_i, L_i, \mathbf{T}_i^\bullet, T_i \xrightarrow{F_i} X)$, $i = 1, 2$, be two admissible PL-nullbordisms for the admissible PL-representative $(S, K, \mathbf{S}^\bullet, S \xrightarrow{f} X)$. Throughout this argument, the symbol π will always denote projection to S. Let $c_i : S \times I \xrightarrow{\cong} N_i$, $c_i(s, 0) = s$, be the associated collars such that

$$c_1|^*(\mathbf{T}_1^\bullet|_{U_1}) \cong \pi^! \mathbf{S}^\bullet, \tag{10}$$

$$c_2|^*(\mathbf{T}_2^\bullet|_{U_2}) \cong \pi^! \mathbf{S}^\bullet, \tag{11}$$

where the $c_i|$ are the restrictions $c_i| : S \times (0, 1) \xrightarrow{\cong} U_i = c_i(S \times (0, 1))$. On the disjoint union $T_1 \sqcup S \times [-1, 1]$, let $\sim_1$ denote the identification $(s, t) \sim_1 c_1(s, -t)$ for $t \in [-1, 0]$. Similarly, $\sim_2$ on $S \times [-1, 1] \sqcup T_2$ is the identification $(s, t) \sim_2 c_2(s, t)$ for $t \in [0, 1]$. Using these identifications, we form the polyhedron

$$W = T_1 \cup_{\sim_1} S \times [-1, 1] \cup_{\sim_2} T_2,$$

which is triangulated by the simplicial complex $L_1 \cup_K L_2$. From $\mathbf{T}_1^\bullet \in SD(\operatorname{int} T_1)$ and $\mathbf{T}_2^\bullet \in SD(\operatorname{int} T_2)$, we shall construct a sheaf $\mathbf{W}^\bullet \in SD(W)$ by applying the gluing lemma 5.2 twice. Let $U, V \subset W$ be the open subsets $U = \operatorname{int} T_1$, $V = S \times (-1, 1)$, so that $U \cap V = S \times (-1, 0) \subset W$. If we take $\mathbf{U}^\bullet = \mathbf{T}_1^\bullet \in SD(U)$ and $\mathbf{V}^\bullet = \pi^! \mathbf{S}^\bullet \in SD(V)$, then (10) induces an isomorphism $\mathbf{U}^\bullet|_{U \cap V} \cong \mathbf{V}^\bullet|_{U \cap V}$ and the gluing lemma implies that there exists a unique sheaf $\mathbf{W}_{<1}^\bullet \in SD(U \cup V)$, $U \cup V = T_1 \cup_{\sim_1} S \times (-1, 1)$, such that $\mathbf{W}_{<1}^\bullet|_{\operatorname{int} T_1} \cong \mathbf{T}_1^\bullet$, $\mathbf{W}_{<1}^\bullet|_{S \times (-1,1)} \cong \pi^! \mathbf{S}^\bullet$. Now consider the following open cover of W: $U = T_1 \cup_{\sim_1} S \times (-1, 1)$, $V = \operatorname{int} T_2$, so that $U \cap V = S \times (0, 1) \subset W$. Taking $\mathbf{U}^\bullet = \mathbf{W}_{<1}^\bullet \in SD(U)$ and $\mathbf{V}^\bullet = \mathbf{T}_2^\bullet \in SD(V)$, (11) induces an isomorphism $\mathbf{V}^\bullet|_{U \cap V} \cong \pi^! \mathbf{S}^\bullet \cong \mathbf{W}_{<1}^\bullet|_{U \cap V}$ and the gluing lemma implies that there exists a unique sheaf $\mathbf{W}^\bullet \in SD(U \cup V)$, $U \cup V = W$, such that $\mathbf{W}^\bullet|_U \cong \mathbf{W}_{<1}^\bullet$, $\mathbf{W}^\bullet|_V \cong \mathbf{T}_2^\bullet$. Let $F : W \to X$ be the map

$$F(w) = \begin{cases} F_1(w), & w \in T_1, \\ F_2(w), & w \in T_2. \end{cases}$$

(If $w \in T_1 \cap T_2 = S$, then $F_1(w) = f(w) = F_2(w)$.) The quadruple $(W, L_1 \cup_K L_2, \mathbf{W}^\bullet, F)$ is an admissible PL-representative obtained by gluing the $(T_i, L_i, \mathbf{T}_i^\bullet, T_i \xrightarrow{F_i} X)$, $i = 1, 2$, along their common boundary $(S, K, \mathbf{S}^\bullet, S \xrightarrow{f} X)$. □

Definition 5.4. *Piecewise linear signature homology* $S_n^{PL}(X)$ *is the set of equivalence classes*

$$S_n^{PL}(X) = \{[(S, K, \mathbf{S}^\bullet, f)] \mid (S, K, \mathbf{S}^\bullet, f) \text{ admissible PL-representative}\}$$

under the bordism relation[2]*. Disjoint union defines an abelian group structure on* $S_n^{PL}(X)$.

A continuous map $g : X \to X'$ induces a map

$$S_n^{PL}(g) : S_n^{PL}(X) \longrightarrow S_n^{PL}(X')$$

by

$$S_n^{PL}(g)[(S, K, \mathbf{S}^\bullet, f)] = [(S, K, \mathbf{S}^\bullet, gf)].$$

Clearly $S_n^{PL}(1_X) = 1_{S_n^{PL}(X)}$ and $S_n^{PL}(hg) = S_n^{PL}(h)S_n^{PL}(g)$.

Let $\mathcal{TOP}$ denote the category of topological spaces and continuous maps. Let $\mathcal{AB}$ denote the category of abelian groups and homomorphisms. Thus S_*^{PL} is a functor $S_*^{PL} : \mathcal{TOP} \to \mathcal{AB}$. The technical appendix (section 9) establishes that S_*^{PL} is a homology theory on compact PL pairs.

Proposition 5.1. *The coefficients of piecewise linear signature homology are*

$$S_n^{PL}(pt) \cong \begin{cases} \mathbb{Z}, & n \equiv 0(4) \\ 0, & n \not\equiv 0(4) \end{cases}$$

Proof. The argument given in section 4 can readily be restated in the PL category: The generators $\mathbb{C}P^{2k}$ have a PL structure, and the cone construction works piecewise linearly as well. □

Theorem 5.1. *There is a natural isomorphism*

$$\Theta : S_*^{PL}(X, A) \xrightarrow{\cong} S_*(X, A)$$

for compact, triangulable pairs (X, A).

Proof. Define a coefficient homomorphism

$$\Theta : S_n^{PL}(pt) \longrightarrow S_n(pt)$$

by

$$\Theta[(S, K, \mathbf{S}^\bullet, S \to pt)] = [(S_K, \mathbf{S}^\bullet, d, S_K \to pt)]$$

(recall that S_K denotes the space S stratified by the skeleta of K, cf. definition 5.1). This assignment is well defined, since S_K is indeed a locally conelike topological stratifold: as the i-dimensional strats we may take the closed i-simplices of K together with their attaching maps into S. When n

[2] As is done in [Min04], the construction of a small subclass $SD_0(S_K) \subset SD(S_K)$ shows that $S_n^{PL}(X)$ is indeed a set.

is not divisible by 4, both $S_n^{PL}(pt)$ and $S_n(pt)$ are the trivial group, and Θ is an isomorphism. When $n = 4k$, the diagram

$$\begin{array}{ccc} S_n^{PL}(pt) & \xrightarrow{\Theta} & S_n(pt) \\ & {\scriptstyle \sigma}\searrow\cong \quad \cong\swarrow{\scriptstyle \sigma} & \\ & \mathbb{Z} & \end{array}$$

commutes, whence Θ is an isomorphism for all n. By the Eilenberg-Steenrod uniqueness theorem, Θ has a unique extension to a natural isomorphism

$$\Theta : S_*^{PL}(X, A) \xrightarrow{\cong} S_*(X, A).$$

This extension is of course explicitly given by

$$\Theta[(S, K, \mathbf{S}^\bullet, S \to X \cup cA)] = [(S_K, \mathbf{S}^\bullet, d, S_K \to X \cup cA)].$$

□

Let us define a natural surjective map

$$\kappa_{PL}(X) : \Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}][t] \longrightarrow S_*^{PL}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}].$$

Given an element $[M \xrightarrow{f} X] \otimes at^l \in \Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}][t]$, set $N^d = M \times \mathbb{C}P^{2l}$. By J. H. C. Whitehead [Whi40], N can be smoothly triangulated by a simplicial complex K. Define

$$\kappa_{PL}(X)([M \xrightarrow{f} X] \otimes at^l) = [(N, K, \mathbb{R}_N[d], N \to M \xrightarrow{f} X)] \otimes a.$$

To see that $\kappa_{PL}(X)$ is well-defined, let K' be another simplicial complex smoothly triangulating N. Then the uniqueness result of [Whi40] asserts that the two triangulations are combinatorially equivalent, i.e. K has a subdivision K_* and K' has a subdivision K'_* such that K_* and K'_* are linearly isomorphic. The cylinder $N \times [0, 1]$ has a triangulation by a complex K_+ which restricts to K on $N \times \{1\}$ and K_* on $N \times \{0\}$. The cylinder $N \times [-1, 0]$ has a triangulation by a complex K_- which restricts to K' on $N \times \{-1\}$ and K'_* on $N \times \{0\}$. Then

$$N \times [-1, 1] = (N \times [-1, 0], K_-) \cup_{(N \times \{0\}, K_* \cong K'_*)} (N \times [0, 1], K_+),$$

together with the self-dual sheaf $\mathbb{R}_{N \times (-1,1)}[d + 1]$ and the composition $N \times [-1, 1] \to N \to X$, is an admissible PL-bordism from $(N, K, \mathbb{R}_N[d], N \to X)$ to $(N, K', \mathbb{R}_N[d], N \to X)$, and $\kappa_{PL}(X)$ is well-defined. The diagram

$$\begin{array}{ccc} \Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}][t] & \xrightarrow{\kappa(X)} & S_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \\ & {\scriptstyle \kappa_{PL}(X)}\searrow & \uparrow \cong \; {\scriptstyle \Theta \otimes \mathbb{Z}[\frac{1}{2}]} \\ & & S_*^{PL}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \end{array}$$

commutes, since in topological signature homology, N regarded as a stratifold with one stratum is bordant to N regarded as a stratifold with the simplicial stratification induced by a triangulation. Consequently, $\kappa_{PL}(X)$ is onto.

6. The Twisted Signature Formula

We start out by proving a very special case of the general twisted signature formula:

Lemma 6.1. *Let X^n be a closed, oriented, topological, stratified pseudomanifold such that $SD(X;\mathbb{R}) \neq \varnothing$. Let $(\mathbb{R}^r_X, \phi)$ be an untwisted Poincaré local system with pairing $\phi : \mathbb{R}^r_X \otimes \mathbb{R}^r_X \to \mathbb{R}_X$. Then*

$$\sigma(X; (\mathbb{R}^r_X, \phi)) = \sigma(\phi_{pt})\sigma(X).$$

Proof. The symmetric, bilinear, nondegenerate form ϕ can be diagonalized, i.e. there exists a linear isomorphism $\gamma : \mathbb{R}^r_X \to \mathbb{R}^r_X$ such that the composition ψ,

$$\begin{array}{ccc} \mathbb{R}^r_X \otimes \mathbb{R}^r_X & \xrightarrow{\psi} & \mathbb{R}_X \\ {\scriptstyle \gamma\otimes\gamma}\downarrow{\scriptstyle \cong} & \nearrow{\scriptstyle \phi} & \\ \mathbb{R}^r_X \otimes \mathbb{R}^r_X & & \end{array}$$

is given at any point by the matrix

$$\begin{pmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

in the standard basis of $\mathbb{R}^r$. In other words,

$$(\mathbb{R}^r_X, \psi) = \bigoplus^{P} (\mathbb{R}_X, 1) \oplus \bigoplus^{N} (\mathbb{R}_X, -1),$$

where P is the number of positive entries, N the number of negative entries, and $1 : \mathbb{R}_X \otimes \mathbb{R}_X \to \mathbb{R}_X$ denotes the form $v \otimes w \mapsto vw$. By assumption, $SD(X; (\mathbb{R}, 1))$ is not empty and we can choose an object $(\mathbf{S}^\bullet_0, d_0) \in SD(X; (\mathbb{R}, 1))$. Then the sheaf

$$(\mathbf{S}^\bullet, d_S) = \bigoplus^{P} (\mathbf{S}^\bullet_0, d_0) \oplus \bigoplus^{N} (\mathbf{S}^\bullet_0, -d_0) \in SD(X; (\mathbb{R}^r_X, \psi))$$

is a self-dual extension of $(\mathbb{R}^r_X, \psi)$. By lemma 3.1, there exists a self-dual extension $(\mathbf{T}^\bullet, d_T) \in SD(X; (\mathbb{R}^r_X, \phi))$ of $(\mathbb{R}^r_X, \phi)$ and

$$\begin{aligned}\sigma(X; (\mathbb{R}^r_X, \phi)) &= \sigma(X; (\mathbb{R}^r_X, \psi)) \\ &= \sigma(\mathbf{S}^\bullet, d_S) \\ &= \sum^P \sigma(\mathbf{S}^\bullet_0, d_0) - \sum^N \sigma(\mathbf{S}^\bullet_0, d_0) \\ &= (P - N)\sigma(\mathbf{S}^\bullet_0, d_0) \\ &= \sigma(\phi_{pt})\sigma(X).\end{aligned}$$

□

Proposition 6.1. *Let X^n be a triangulated, oriented pseudomanifold without boundary equipped with the simplicial stratification*[3]*. Let $(\mathcal{S}, \phi)$ be a Poincaré local system on $X - \Sigma$, strongly transverse to Σ. If $SD(X; \mathbb{R}) \neq \varnothing$, then $SD(X; \mathcal{S}) \neq \varnothing$.*

Proof. Choose any self-dual sheaf $\mathbf{T}^\bullet \in SD(X; \mathbb{R})$ with constant coefficients on $X - \Sigma$. The idea of the proof is to use the Lagrangian structures coming from $\mathbf{T}^\bullet$ (theorem 3.1) to build Lagrangian structures for an object $\mathbf{S}^\bullet \in SD(X; \mathcal{S})$. The two central provisions that will make this work are that $\mathcal{S}$ is constant on links, and that the Lagrangian sheaves are constant as well, as they are locally constant sheaves over contractible spaces (open simplices). Indeed, this is the very reason for our introduction of PL signature homology in this paper. If one attempts to use topological signature homology, then monodromy issues for the existence of Lagrangian structures will arise, whose discussion we can avoid completely by using PL signature homology.

Let $U_k = X - X_{n-k}$, $i_k : U_k \hookrightarrow U_{k+1}$, $k \geq 2$, where $X_n \supset X_{n-2} \supset X_{n-3} \supset \ldots \supset X_0 \supset X_{-1} = \varnothing$ is the simplicial stratification of X. The pure strata $X_{n-k} - X_{n-k-1} = U_{k+1} - U_k$ are a disjoint union of open $(n-k)$-simplices. We shall construct $\mathbf{S}^\bullet \in SD(X; \mathcal{S})$ in stages by induction on k. Set $\mathbf{S}^\bullet_2 = \mathcal{S}[n]$ on $U_2 = X - \Sigma$. Inductively, we will construct $\mathbf{S}^\bullet_k \in SD(U_k; \mathcal{S})$ so that $\mathbf{S}^\bullet = \mathbf{S}^\bullet_{n+1} \in SD(X; \mathcal{S})$, $X = U_{n+1}$. Assume $\mathbf{S}^\bullet_k \in SD(U_k; \mathcal{S})$ has been constructed. If k is even, then $U_{k+1} - U_k$ is a stratum of even codimension, and thus the Goresky-MacPherson-Deligne extension

$$\mathbf{S}^\bullet_{k+1} = \tau_{\leq \bar{m}(k)-n} Ri_{k*}\mathbf{S}^\bullet_k$$

is a self-dual sheaf in $SD(U_{k+1}; \mathcal{S})$. Let k be odd. According to theorem 3.1, there is an equivalence of categories

$$SD(U_{k+1}; \mathcal{S}) \simeq \mathrm{Lag}(U_{k+1} - U_k) \rtimes SD(U_k; \mathcal{S}). \tag{12}$$

Consequently, we must construct an object in $\mathrm{Lag}(U_{k+1} - U_k)$. Let $\overset{\circ}{\Delta} \subset U_{k+1} - U_k$ be an open $(n-k)$-simplex, and let $x \in \overset{\circ}{\Delta}$. We denote the

[3]If X is noncompact, then this is to be interpreted as follows: X is the interior of a compact, triangulated, oriented pseudomanifold-with-boundary Y and the stratification of X is the restriction of the simplicial stratification of Y.

inclusion of the link of $\mathring{\Delta}$ at x into U_k by $j : Lk(x) \hookrightarrow U_k$. As $\mathcal{S}$ is strongly transverse to Σ, we have

$$\mathbf{S}_k^\bullet|_{Lk(x)-\Sigma} \cong (\mathcal{S},\phi)[n]|_{Lk(x)-\Sigma} \cong (\mathbb{R}^r_{Lk(x)-\Sigma}[n], \phi|),$$

where r is the rank of $\mathcal{S}$. The signature $\sigma(Lk(x);\mathcal{S})$ can be evaluated with the help of lemma 6.1:

$$\begin{aligned} \sigma(Lk(x);\mathcal{S}) &= \sigma(Lk(x);(\mathbb{R}^r_{Lk(x)-\Sigma},\phi|)) \\ &= \sigma(\phi_{pt})\sigma(Lk(x);\mathbb{R}) \\ &= \sigma(\phi_{pt})\sigma(j^!\mathbf{T}^\bullet) \\ &= \sigma(\phi_{pt})\cdot 0 \\ &= 0. \end{aligned}$$

The term $\sigma(j^!\mathbf{T}^\bullet)$ vanishes, because $\mathbf{T}^\bullet$ possesses a Lagrangian structure along $U_{k+1} - U_k$. Letting $\mathbf{H}$ denote the derived sheaf

$$\mathbf{H} = \mathbf{H}^{\bar{n}(k)-n}(Ri_{k*}\mathbf{S}_k^\bullet),$$

we conclude that there exists a Lagrangian subspace in the stalk $\mathbf{H}_x$ (which is isomorphic to the middle dimensional hypercohomology of $Lk(x)$ with coefficients in $\mathbf{S}_k^\bullet$). Since the restriction of $\mathbf{H}$ to $\mathring{\Delta}$ is constant, the Lagrangian subspace extends uniquely to a Lagrangian subsheaf of $\mathbf{H}|\mathring{\Delta}$. Carrying this out for every connected component $\mathring{\Delta}$ of $U_{k+1} - U_k$ produces an object in $\mathrm{Lag}(U_{k+1} - U_k)$, which in turn, by (12), determines an object $\mathbf{S}_{k+1}^\bullet \in SD(U_{k+1})$. □

Theorem 6.1. *Let X^n be a closed, oriented, Whitney stratified, normal pseudomanifold of even dimension with singular set Σ, and let $(\mathcal{S},\phi)$ be a Poincaré local system on $X - \Sigma$, strongly transverse to Σ. Assume that $L(X) \in H_{2*}(X;\mathbb{Q})$ is defined. Then*

$$\sigma(X;\mathcal{S}) = \langle \widetilde{\mathrm{ch}}[\mathcal{S}]_K, L(X)\rangle.$$

Proof. First of all, the hypotheses of the theorem imply that the left hand side, $\sigma(X;\mathcal{S})$, is indeed defined: By Goresky [Gor78], X can be triangulated by a simplicial complex K such that the Whitney strata are triangulated by subcomplexes and the Whitney filtration of X defines a PL stratification of X. If X_{Whitney} denotes X equipped with the given Whitney stratification and X_K denotes X equipped with the simplicial stratification induced by K, then, by an obvious variant of lemma 5.1, we have $SD(X_{\mathrm{Whitney}}) \subset SD(X_K)$. The L-class $L(X)$ is the L-class of some self-dual sheaf $\mathbf{S}^\bullet \in SD(X_{\mathrm{Whitney}};\mathbb{R})$, $L(X) = L(\mathbf{S}^\bullet)$. Regarding this sheaf as an object $\mathbf{S}^\bullet \in SD(X_K;\mathbb{R})$, we can invoke proposition 6.1 to see that $SD(X_K;\mathcal{S}) \neq \varnothing$. Choose a sheaf $\mathbf{S}^\bullet_{\mathrm{twist}} \in SD(X_K;\mathcal{S})$. Then

$$\sigma(X;\mathcal{S}) = \sigma(\mathbf{S}^\bullet_{\mathrm{twist}}),$$

and this is independent of the choice of $\mathbf{S}^\bullet_{\mathrm{twist}}$ by theorem 3.2. As $\mathcal{S}$ is strongly transverse to Σ, there exists a unique extension $\overline{\mathcal{S}}$ of $\mathcal{S}$ to all of X

(proposition 2.1).

The argument that we will use to establish the equation $\sigma(X;\mathcal{S}) = \langle \widetilde{\mathrm{ch}}[\mathcal{S}]_K, L(X)\rangle$ consists of the following three steps:

1. The verification that assigning to a quadruple $(S, C, \mathbf{S}^\bullet, S \xrightarrow{f} X)$ (representing an element of $S_n^{PL}(X)$; in particular C is a simplicial complex triangulating S) the integer
$$\langle \widetilde{\mathrm{ch}}[f^*\overline{\mathcal{S}}]_K, L(S)\rangle$$
is a bordism invariant on the bordism group $S_n^{PL}(X)$.
2. The verification that assigning to a quadruple $(S, C, \mathbf{S}^\bullet, S \xrightarrow{f} X)$ the signature
$$\sigma(S; f^*\overline{\mathcal{S}})$$
is a bordism invariant on $S_n^{PL}(X)$. Again, $\sigma(S; f^*\overline{\mathcal{S}})$ is defined as the signature of any complex of sheaves in $SD(S; f^*\overline{\mathcal{S}})$, which is nonempty since $\mathbf{S}^\bullet \in SD(S;\mathbb{R})$, using proposition 6.1.
3. The facts that PL signature homology and topological signature homology are isomorphic (theorem 5.1) and that at odd primes, by corollary 4.1, every element of topological signature homology can be represented by a smooth manifold will allow us to pull the calculation back to manifolds, where the formula holds according to Atiyah [Ati69] and W. Meyer [Mey72].

Let us carry these steps out:

1. Let $(T^{n+1}, L, \mathbf{T}^\bullet, T \xrightarrow{F} X)$ be an admissible PL-nullbordism for $(S, C, \mathbf{S}^\bullet, S \xrightarrow{f} X)$, $\partial T = S$, $\mathbf{T}^\bullet \in SD(\mathrm{int}\, T_L;\mathbb{R})$, $\mathbf{S}^\bullet \in SD(S_C;\mathbb{R})$, $F|_{\partial T} = f$. Let $j : \partial T \hookrightarrow T$ be the inclusion, so that $F \circ j = f$. Using naturality of the Chern classes, we have

$$\begin{aligned}
\langle \widetilde{\mathrm{ch}}[f^*\overline{\mathcal{S}}]_K, L(S)\rangle &= \langle \widetilde{\mathrm{ch}}[j^*F^*\overline{\mathcal{S}}]_K, L(S)\rangle \\
&= \langle j^*\widetilde{\mathrm{ch}}[F^*\overline{\mathcal{S}}]_K, L(S)\rangle \\
&= \langle \widetilde{\mathrm{ch}}[F^*\overline{\mathcal{S}}]_K, j_*L(\partial T)\rangle \\
&= \langle \widetilde{\mathrm{ch}}[F^*\overline{\mathcal{S}}]_K, j_*\partial_*L(T)\rangle \\
&= 0.
\end{aligned}$$

Here, ∂_* is the connecting homomorphism in the long exact homology sequence of the pair $(T, \partial T)$. The identity $\partial_* L(T) = L(\partial T)$ was established in proposition 3.1.

2. Suppose again that $\partial(T^{n+1}, L, \mathbf{T}^\bullet, T \xrightarrow{F} X) = (S, C, \mathbf{S}^\bullet, S \xrightarrow{f} X)$. By proposition 6.1, $SD(\mathrm{int}\, T_L; (F^*\overline{\mathcal{S}})|_{\mathrm{int}\, T}) \neq \varnothing$, since $\mathbf{T}^\bullet \in SD(\mathrm{int}\, T_L;\mathbb{R})$. Choose a sheaf $\mathbf{T}^\bullet_{\mathrm{twist}} \in SD(\mathrm{int}\, T_L; (F^*\overline{\mathcal{S}})_{\mathrm{int}\, T})$. Setting

$$\mathbf{S}^\bullet_{\mathrm{twist}} = j^! Ri_! \mathbf{T}^\bullet_{\mathrm{twist}}$$

$j: \partial T \hookrightarrow T$, $i: \operatorname{int} T \hookrightarrow T$, defines an object in $SD(S; f^*\overline{\mathcal{S}})$. By theorem 3.2, this object can be used to calculate the twisted signature,

$$\sigma(S; f^*\overline{\mathcal{S}}) = \sigma(\mathbf{S}^\bullet_{\text{twist}}).$$

But

$$\sigma(\mathbf{S}^\bullet_{\text{twist}}) = \sigma(\partial \mathbf{T}^\bullet_{\text{twist}}) = 0$$

by [Ban02], corollary 4.1.

3. Consider the element

$$[(X, K, \mathbf{S}^\bullet, X \xrightarrow{\text{id}} X)] \otimes 1 \in S^{PL}_n(X) \otimes \mathbb{Z}[\tfrac{1}{2}],$$

where K is the simplicial complex provided by [Gor78]. In section 5, we constructed a surjection

$$\kappa_{PL}(X) : \Omega^{SO}_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}][t] \twoheadrightarrow S^{PL}_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}].$$

The surjectivity, together with the normal form of corollary 4.1, implies that there exists a smoothly triangulated manifold M^n, a continuous map $f: M \to X$, and a nonzero integer k such that

$$k \cdot [(X, K, \mathbf{S}^\bullet, X \xrightarrow{\text{id}} X)] = [(M, L, \mathbb{R}_M[n], M \xrightarrow{f} X)] \in S^{PL}_n(X).$$

Thus

$$\begin{aligned} k\sigma(X; \mathcal{S}) &= \sigma(M; f^*\overline{\mathcal{S}}) && \text{(bordism invariance, step 2.)} \\ &= \langle \widetilde{\text{ch}}[f^*\overline{\mathcal{S}}]_K, L(M) \rangle && \text{(by Atiyah/Meyer)} \\ &= k \langle \widetilde{\text{ch}}[\mathcal{S}]_K, L(X) \rangle && \text{(bordism invariance, step 1.)} \end{aligned}$$

□

7. The Twisted L-Class Formula

Theorem 7.1. *Let X^n be a closed, oriented, Whitney stratified, normal pseudomanifold of even dimension with singular set Σ, and let $(\mathcal{S}, \phi)$ be a Poincaré local system on $X - \Sigma$, strongly transverse to Σ. Assume that $L(X) \in H_{2*}(X; \mathbb{Q})$ is defined. Then*

$$L(X; \mathcal{S}) = \widetilde{\text{ch}}[\mathcal{S}]_K \cap L(X).$$

Proof. As in the proof of theorem 6.1, we see that $L(X; \mathcal{S})$ is in fact defined: $L(X; \mathcal{S}) = L(\mathbf{S}^\bullet_{\text{twist}})$, $\mathbf{S}^\bullet_{\text{twist}} \in SD(X_K; \mathcal{S})$, where K is a simplicial complex triangulating X compatibly with the Whitney stratification. By the Thom-Pontrjagin construction of the L-class, reviewed in section 3, it suffices to show that

$$\langle f^*(u), L_k(X; \mathcal{S}) \rangle = \langle f^*(u), (\widetilde{\text{ch}}[\mathcal{S}]_K \cap L(X))_k \rangle,$$

for any map $f: X \to S^k$ transverse regular to $p \in S^k$, where $u \in H^k(S^k)$ is the generator such that $\langle u, [S^k] \rangle = 1$. Set $F = f^{-1}(p)$ and let $i_f : F \hookrightarrow X$ be the normally nonsingular inclusion. The following calculation establishes the claim:

$$\begin{aligned}
\langle f^*(u), (\widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X))_k\rangle &= \textstyle\sum_{l\geq 0}\langle f^*(u), \widetilde{\mathrm{ch}}_{2l}[\mathcal{S}]_K \cap L_{2l+k}(X)\rangle \\
&= \textstyle\sum \langle \widetilde{\mathrm{ch}}_{2l}[\mathcal{S}]_K, f^*(u) \cap L_{2l+k}(X)\rangle \\
&= \textstyle\sum \langle \widetilde{\mathrm{ch}}_{2l}[\mathcal{S}]_K, i_{f*}L_{2l}(F)\rangle && \text{(by lemma 3.3)} \\
&= \textstyle\sum \langle \widetilde{\mathrm{ch}}_{2l}[\mathcal{S}|_F]_K, L_{2l}(F)\rangle && \text{(naturality of } \widetilde{\mathrm{ch}}) \\
&= \sigma(F;\mathcal{S}) && \text{(by theorem 6.1)} \\
&= \sigma(i_f^! \mathbf{S}^\bullet_{\mathrm{twist}}) && \text{(by definition)} \\
&= \langle f^*(u), L_k(X;\mathcal{S})\rangle && \text{(by construction of } L(X;\mathcal{S})).
\end{aligned}$$

□

8. An Example

We illustrate the use of theorem 6.1 with a simple example from the area of transformation groups. The space Y to be discussed will exhibit the following features:

- Y is a closed, oriented, Whitney stratified pseudomanifold which has only strata of even codimension.
- A compact Lie group G acts on Y.
- The action is compatible with the stratification in the sense that the orbit bundles coincide with the pure strata of Y. The action restricted to the pure strata is smooth.
- The orbit space $X = Y/G$ is a closed, oriented, Whitney stratified pseudomanifold which is *not* a Witt space.

Our goal is to calculate the signature $\sigma(Y)$ from data on the orbit space X. Let N be an even-dimensional, compact, oriented manifold whose boundary $\partial N = E$ fibers over an odd-dimensional manifold K with simply connected manifold fiber L, whose middle-dimensional cohomology is nontrivial but $\sigma(L) = 0$. Let $\pi_E : E \to K$ be the projection. Let G be a compact Lie group whose dimension is divisible by 4 and which fibers as $G_1 \longrightarrow G \xrightarrow{\pi_G} G_2$, where G_1, G_2 are compact Lie groups, both odd dimensional, and π_G is a group homomorphism. For instance, one could take $S^3 \cong SU(2) \longrightarrow U(2) \xrightarrow{\det} S^1$. Let M be the total space of a principal G-bundle $p : M \to N$ whose restriction to the boundary is a product, $\partial M = E \times G$. The manifold M has even dimension and its boundary $\partial M = E \times G$ fibers over $K \times G_2$ with projection $\pi_E \times \pi_G$. Setting

$$Y = M \cup_{\partial M} \mathrm{cyl}(\pi_E \times \pi_G),$$

where $\mathrm{cyl}(-)$ denotes the mapping cylinder, we obtain a Whitney stratified pseudomanifold with two strata and singular set $\Sigma_Y = K \times G_2$ of even dimension. The group G acts on Y as follows: The action on M is specified by the principal bundle structure. Restricted to ∂M, this action is $g\cdot(n,h) = (n, gh)$, $g \in G$. On Σ_Y let g act by $g \cdot (k, h_2) = (k, \pi_G(g)h_2)$. Then G acts on the cylinder $\partial M \times I$ by $g \cdot (x,t) = (g \cdot x, t)$ and this defines an action on the mapping cylinder, since $\pi_E \times \pi_G$ is a G-equivariant map. The action has precisely two orbit types. The orbit space is

$$X = Y/G = N \cup_{\partial N = E} \operatorname{cyl}(\pi_E),$$

a closed, oriented, Whitney stratified pseudomanifold of even dimension which is not a Witt space, since the singular stratum $\Sigma_X = K$ has odd codimension in X, and the link L has nonvanishing middle cohomology. For the characteristic class formulae to be applicable to the orbit space, it must possess (generalized) Poincaré duality, i.e. we need $SD(X) \neq \varnothing$. According to theorem 3.1, this is the case precisely when there exists a Lagrangian subspace in the middle-dimensional cohomology of L which is invariant under the monodromy action of $\pi_1(K)$ induced by π_E. Assume then that such an invariant Lagrangian subspace exists; this is of course automatic if K is simply connected, or more generally, if the monodromy action is trivial. Consequently, $L(X)$ is defined. The orbit projection $f : Y \to X$ is a stratified map. By the results of [Ban03], cf. equation (3) of the introduction,

$$f_* L(Y) = L(X; \mathcal{S}),$$

where the Poincaré local system $\mathcal{S}$ is given on the top stratum of X and has as stalks the middle-dimensional cohomology of G. In particular $\sigma(Y) = \sigma(X; \mathcal{S})$. Since the link L of Σ_X is simply connected, $\mathcal{S}$ is strongly transverse to Σ_X. Thus theorem 6.1 applies to yield

$$\sigma(Y) = \langle \widetilde{\operatorname{ch}}[\mathcal{S}]_K, L(X) \rangle.$$

As G is a parallelizable manifold, we have $\widetilde{\operatorname{ch}}_0[\mathcal{S}]_K = \sigma(G) = 0$ by the Hirzebruch signature theorem. Hence,

$$\sigma(Y) = \langle \widetilde{\operatorname{ch}}_2[\mathcal{S}]_K, L_2(X) \rangle + \langle \widetilde{\operatorname{ch}}_4[\mathcal{S}]_K, L_4(X) \rangle + \dots .$$

If the principal bundle $p : M \to N$ is trivial, then all Chern components vanish and it follows that $\sigma(Y) = 0$. In fact $\sigma(Y) = 0$ even if this principal bundle is not trivial, provided G is connected. Indeed, let $\gamma : N \to BG$ be the classifying map so that we have the pullback square

$$\begin{array}{ccc} M & \xrightarrow{\Gamma} & EG \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ N & \xrightarrow{\gamma} & BG \end{array}$$

For any complex of sheaves $\mathbf{A}^\bullet \in D(EG)$, the identity

$$Rp_* \Gamma^* \mathbf{A}^\bullet \cong \gamma^* Rq_* \mathbf{A}^\bullet$$

holds. The local system $\mathcal{S}$ can be written as the higher direct image $\mathcal{S} = \mathbf{H}^m(Rp_* \mathbb{R}_M)$, where $2m = \dim G$. Thus $\mathcal{S}$ is the pullback of a local system over BG:

$$\begin{array}{rcl} \mathcal{S} & = & \mathbf{H}^m(Rp_*\mathbb{R}_M) \\ & = & \mathbf{H}^m(Rp_*\Gamma^*\mathbb{R}_{EG}) \\ & \cong & \mathbf{H}^m(\gamma^* Rq_*\mathbb{R}_{EG}) \\ & \cong & \gamma^*\mathbf{H}^m(Rq_*\mathbb{R}_{EG}). \end{array}$$

If G is connected, then BG is simply connected. Hence the local system $\mathbf{H}^m(Rq_*\mathbb{R}_{EG})$ over BG is constant. Consequently $\mathcal{S}$ is constant as the pull-back of a constant coefficient system, and all $\widetilde{\mathrm{ch}}_i[\mathcal{S}]_K$ vanish.

9. Appendix: The Eilenberg-Steenrod Axioms for S_*^{PL}

We will show that S_*^{PL} is a homology theory on compact PL pairs, proceeding as follows: First, we verify that it is a homotopy functor (i.e. it satisfies the Eilenberg-Steenrod homotopy axiom), second, we construct a Mayer-Vietoris sequence (for certain triads), third, we prove that any homotopy functor with Mayer-Vietoris sequences is a homology theory. The interesting aspect is the construction of the Mayer-Vietoris boundary operator, where we have been guided by [Kre03]. Our proof of theorem 9.1 below is not only a piecewise linear version of the smooth constructions and arguments of [Kre03], but in addition we have to ensure that all constructions can be covered by constructible, self-dual complexes of sheaves. Thus the sheaf data is tracked rather carefully during the course of the proof.

Let $h_* : \mathcal{TOP} \to \mathcal{AB}$ be a functor. This is the formulation of the Mayer-Vietoris sequence that we will work with:

Axiom (MV). Let X be a space triangulated by a simplicial complex K. Let $U, V \subset X$ be open subsets which are both complements of subcomplexes of K and such that $U \cup V = X$. Then there exists a boundary operator

$$\partial_* : h_*(X) \longrightarrow h_{*-1}(U \cap V)$$

which is natural with respect to simplicial maps respecting the open covers and such that the sequence

$$\cdots \to h_i(U \cap V) \stackrel{i_* \oplus j_*}{\longrightarrow} h_i(U) \oplus h_i(V) \stackrel{k_* - l_*}{\longrightarrow} h_i(X) \stackrel{\partial_*}{\longrightarrow} h_{i-1}(U \cap V) \to \cdots$$

is exact, where

$$\begin{array}{ccc} U \cap V & \stackrel{i}{\longrightarrow} & U \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\ V & \stackrel{l}{\longrightarrow} & X \end{array}$$

are the inclusions.

Lemma 9.1. *(Cylinder lemma.) Let K and L be simplicial complexes such that*

1. $|L| = |K| \times [0, 1]$,
2. $L \cap (|K| \times \{0\}) = K = L \cap (|K| \times \{1\})$,
3. *L has no vertices in $|K| \times (0, 1)$, and*

4. *if* $\{v_1, \ldots, v_k\}$ *spans a simplex in* L, *then* $\{\mathrm{proj}(v_1), \ldots, \mathrm{proj}(v_k)\}$ *spans a simplex in* K, *where* $\mathrm{proj} : |K| \times [0,1] \to |K|$ *is the first factor projection.*

Suppose $\rho : L \to [-1,1]$ *is a simplicial map (where* $[-1,1]$ *has been triangulated arbitrarily) such that* $\rho(v,0) = \rho(v,1)$ *for all vertices* $v \in K$. *Then* $\rho^{-1}(0)$ *is a cylinder, i.e.*

$$\rho^{-1}(0) = \rho_0^{-1}(0) \times [0,1],$$

where $\rho_0 = \rho| : K = K \times \{0\} \to [-1,1]$.

Proof. A point $p \in |L|$ is of the form $p = (x,t)$, $x \in |K|$, $t \in [0,1]$. There exists a unique simplex $\Delta \in L$ such that $p \in \mathrm{int}\,\Delta$. That simplex is spanned by vertices $(v_1,0), \ldots, (v_k,0), (v_{k+1},1), \ldots, (v_l,1)$, where $v_1, \ldots, v_k, v_{k+1}, \ldots, v_l$ are vertices of K. With respect to Δ, p can be expressed as a linear combination

$$p = \sum_{i=1}^{k} \lambda_i (v_i, 0) + \sum_{j=k+1}^{l} \lambda_j (v_j, 1).$$

Consequently,

$$x = \sum_{i=1}^{k} \lambda_i v_i + \sum_{j=k+1}^{l} \lambda_j v_j, \; t = \sum_{j=k+1}^{l} \lambda_j.$$

By hypothesis, $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_l\}$ spans a simplex of K. Thus ρ_0 behaves linearly over that span,

$$\rho_0(x) = \sum \lambda_i \rho_0(v_i) + \sum \lambda_j \rho_0(v_j).$$

The calculation

$$\begin{aligned} \rho(p) &= \textstyle\sum \lambda_i \rho(v_i,0) + \sum \lambda_j \rho(v_j,1) \\ &= \textstyle\sum \lambda_i \rho(v_i,0) + \sum \lambda_j \rho(v_j,0) \\ &= \textstyle\sum \lambda_i \rho_0(v_i) + \sum \lambda_j \rho_0(v_j) \\ &= \rho_0(x) \end{aligned}$$

shows that $\rho(p) = 0$ if and only if $\rho_0(x) = 0$. □

Lemma 9.2. *(Shield lemma.) Let* K *and* L *be locally finite simplicial complexes,* $C \subset K$ *and* $A \subset L$ *subcomplexes with* $|C|$ *compact, and* $f : |K| \to |L|$ *a continuous map such that* $f(|C|) \subset |L| - |A|$. *Then there exist simplicial subdivisions* K' *of* K *and* L' *of* L, *and a simplicial approximation* $g : K' \to L'$ *to* f, *such that still* $g(|C|) \subset |L| - |A|$.

Proof. Fix a metric d on $|L|$. Since $|A|$ is closed in $|L|$, $d(y, A) > 0$ for every $y \in |L| - |A|$. By compactness of $|C|$, the continuous function $h : |C| \to (0, \infty)$, $h(x) = d(f(x), A)$ has a minimum at, say, $x_{\min}$. With $\epsilon = h(x_{\min}) > 0$, we conclude that $d(f(x), A) \geq \epsilon$ for all $x \in |C|$. Let $D \subset L$ be the subcomplex generated by $\{\Delta \in L \mid f(|C|) \cap \Delta \neq \varnothing\}$. As $f(|C|)$ is compact and L is locally finite, D is a finite complex. Then there exists a subdivision D' of D so that every simplex of D' has diameter $\leq \frac{\epsilon}{2}$. Let L' be any extension of D' to a subdivision of L, and let K' be any subdivision of K so that $f : |K| \to |L'|$ has a simplicial approximation $g : K' \to L'$. Given $x \in C$, there exists a simplex $\Delta \in D'$ such that both $f(x) \in \Delta$ and $g(x)$ in Δ. Thus for any $a \in A$,

$$\epsilon \leq d(f(x), a) \leq d(f(x), g(x)) + d(g(x), a) \leq \frac{\epsilon}{2} + d(g(x), a),$$

so that $d(g(x), a) \geq \frac{\epsilon}{2}$. It follows that $g(x) \notin A$ for any $x \in |C|$. □

Theorem 9.1. *$S_*^{PL} : \mathcal{TOP} \to \mathcal{AB}$ is a homotopy functor satisfying axiom (MV).*

Proof. To show that two homotopic maps $g, h : X \to Y$ induce equal maps $g_* = h_* : S_*^{PL}(X) \to S_*^{PL}(Y)$, one uses a verbatim reproduction of the usual proof of composing with a homotopy $X \times I \to Y$ to obtain a bordism between gf and hf, $f : S \to X$.

As for the Mayer-Vietoris boundary operator $\partial_* : S_n^{PL}(X) \longrightarrow S_{n-1}^{PL}(U \cap V)$: The PL space X is triangulated by a simplicial complex K. Let $(S, L, \mathbf{S}^\bullet, S \xrightarrow{f} X)$ be a quadruple representing an element in $S_n^{PL}(X)$. (So L is a simplicial complex which triangulates the PL pseudomanifold S.) Write $U = X - |A|$, $V = X - |B|$, where A and B are subcomplexes of K. By the simplicial approximation theorem, f is homotopic to $|g|$, with $g : L_0 \to K$ simplicial for some subdivision L_0 of L. The inverse images $P = g^{-1}(A)$ and $Q = g^{-1}(B)$ are subcomplexes of L_0. Separate P and Q by a simplicial map ρ as follows: The sets $|P|$ and $|Q|$ are disjoint closed subsets of S, so we can set $\rho(v) = -1$ for all vertices $v \in P$ and $\rho(v) = +1$ for all vertices $v \in Q$. To all other vertices (those in $L_0 - (P \cup Q)$) assign either -1 or $+1$ at will. Triangulate the interval $[-1, 1]$ with vertices -1 and $+1$. By abuse of notation, we shall denote both this complex and its underlying topological space by $[-1, 1]$. The vertex map ρ induces a simplicial map $\rho : L_0 \to [-1, 1]$. Consider the inverse image $T = \rho^{-1}(0)$. Triangulate the set T as follows: Subdivide $[-1, 1]$ so that 0 becomes a vertex. Then T is triangulated by a subcomplex of the corresponding subdivision L_1 of L_0. An open neighborhood of T in S is given by $\rho^{-1}(-1, 1) \cong T \times (-1, 1)$. Thus T is bicollared and has the structure of an $(n-1)$-dimensional, closed, triangulated pseudomanifold which is orientable. We assign a canonical orientation to T as follows: The space $\rho^{-1}[-1, 0]$ is a compact PL-pseudomanifold-with-boundary, whose orientation is induced by the orientation of S. Then the orientation of $\rho^{-1}[-1, 0]$ induces an orientation of its boundary $\partial\rho^{-1}[-1, 0] = T$. The quadruple $(T, L_1 \cap T, j^!(\mathbf{S}^\bullet|_{T \times (-1,1)}), |g||_T)$ represents an element in $S_{n-1}^{PL}(U \cap V)$, where $j : T = T \times \{0\} \hookrightarrow T \times (-1, 1)$. We define

$$\begin{array}{rcl} \partial_* : S_n^{PL}(X) & \longrightarrow & S_{n-1}^{PL}(U \cap V) \\ [(S, L, \mathbf{S}^\bullet, S \xrightarrow{f} X)] & \mapsto & [(T, L_1 \cap T, j^!(\mathbf{S}^\bullet|_{T \times (-1,1)}), |g||_T)]. \end{array}$$

To show that ∂_* is well-defined, we must discuss why the resulting bordism class is independent of the choice of simplicial approximation g to f, of the choice of ρ, and of the choice of representative $(S, L, \mathbf{S}^\bullet, f)$ in $[(S, L, \mathbf{S}^\bullet, f)]$. Suppose that f is homotopic to $|g'|$, with $g' : L_0' \to K$ simplicial for some subdivision L_0' of L. Set $P' = g'^{-1}(A)$ and $Q' = g'^{-1}(B)$. Let $\rho' : L_0' \to [-1, 1]$ be a simplicial map such that $\rho'(v) = -1$ for all vertices $v \in P'$, $\rho'(v) = +1$ for all vertices $v \in Q'$, and $\rho'(v) \in \{\pm 1\}$ for $v \in L_0' - (P' \cup Q')$. The polyhedral pseudomanifold $T' = \rho'^{-1}(0)$ is triangulated by a complex $L_1' \cap T'$, where L_1' is a subdivision of L_0'. We will show that $(T', L_1' \cap T', j'^!(\mathbf{S}^\bullet|_{T' \times (-1,1)}), |g'||_{T'})$ and $(T, L_1 \cap T, j^!(\mathbf{S}^\bullet|_{T \times (-1,1)}), |g||_T)$ are bordant. A continuous homotopy $S \times [-1, 1] \to X$ from g to g' has a simplicial approximation $H : L^\times \to K$ rel boundary, where $L^\times$ is a simplicial complex such that

(*i*) $|L^\times| = S \times [-1,1]$,
(*ii*) $L^\times$ is a subdivision of the cell complex $L \times [-1,1]$,
(*iii*) the subcomplex of $L^\times$ which triangulates $S \times \{+1\}$ equals L_0,
(*iv*) the subcomplex of $L^\times$ which triangulates $S \times \{-1\}$ equals L_0',
(*v*) $H|_{S\times\{+1\}} = g$,
(*vi*) $H|_{S\times\{-1\}} = g'$.

Set $P^\times = H^{-1}(A)$, $Q^\times = H^{-1}(B)$. Extend ρ and ρ' to a simplicial map $\rho^\times : L^\times \to [-1,1]$ such that $\rho^\times(v) = -1$ for all vertices $v \in P^\times$, $\rho^\times(v) = +1$ for all vertices $v \in Q^\times$. Now consider the polyhedron $S \times [-2,2]$ and triangulate it as follows: Triangulate the subspace $S\times[-1,1]$ using $L^\times$. Thus $S\times\{+1\}$ is triangulated by L_0 (cf. (*iii*) above). Triangulate $S\times\{+2\}$ by L_0 as well, and extend that triangulation of $S \times \{1,2\}$ to $S \times [1,2]$ without introducing any new vertices (use the standard algorithm for triangulating a prism). Similarly $S \times \{-1\}$ is triangulated by L_0' (cf. (*iv*) above). Triangulate $S \times \{-2\}$ by L_0' as well, and extend that triangulation of $S \times \{-2,-1\}$ to $S \times [-2,-1]$ without introducing any new vertices. Define a simplicial map $\overline{\rho} : S \times [-2,2] \to [-1,1]$ by

$$\begin{array}{ll} \overline{\rho}(v,2) = \rho(v), & \text{for vertices } (v,2) \in L_0 \times \{2\}, \\ \overline{\rho}(v) = \rho^\times(v), & \text{for vertices } v \in L^\times, \text{ and} \\ \overline{\rho}(v,-2) = \rho'(v), & \text{for vertices } (v,-2) \in L_0' \times \{-2\}, \end{array}$$

Set $W = \overline{\rho}^{-1}(0)$. Then W is transverse to the simplicial stratification of $S\times[-2,2]$ (an open neighborhood of W in $S \times [-2,2]$ is given by $\overline{\rho}^{-1}(-1,1) \cong W \times (-1,1)$), W is bicollared and has the structure of an n-dimensional, compact, oriented PL pseudomanifold-with-boundary, which is triangulated by the subcomplex $R\cap W$ of the subdivision R of the triangulation of $S\times[-2,2]$ which correponds to subdividing $[-1,1]$ so that 0 becomes a vertex. Note that the boundary of W is contained in the boundary of $S\times[-2,2]$, and in fact $\partial W = W\cap(S\times\{\pm 2\}) = T \sqcup -T'$, $R\cap\partial W = (L_1\cap T)\sqcup(L_1'\cap T')$. Furthermore, ∂W is collared in W: the required closed neighborhood N of ∂W in definition 5.3 may be taken to be $W\cap(S\times[1,2])\sqcup W\cap(S\times[-2,-1])$. Next, we shall cover W with a self-dual sheaf $\mathbf{W}^\bullet \in SD(W_{R\cap W})$. The sheaf $\mathbf{S}^\bullet$ on S is constructible with respect to the simplicial stratification induced by L. The pullback $\pi^!\mathbf{S}^\bullet$ under the projection $\pi : S \times (-2,2) \to S$ is self-dual on $S \times (-2,2)$ and constructible with respect to the product stratification whose i-dimensional closed stratum is $L^i \times (-2,2)$, with L^i the i-skeleton of L. The sheaf

$$\mathbf{W}^\bullet = j_W^! \pi^! \mathbf{S}^\bullet,$$

$j_W : \operatorname{int} W \hookrightarrow S \times (-2,2)$, is self-dual, since j_W can be factored as the normally nonsingular inclusion into the bicollar, $\operatorname{int} W = \operatorname{int} W \times \{0\} \hookrightarrow \operatorname{int} W \times (-1,1)$, followed by the open inclusion $\operatorname{int} W \times (-1,1) \cong \overline{\rho}^{-1}(-1,1) - (\partial W \times (-1,1)) \hookrightarrow S\times(-2,2)$. The cell complex $Z = L_0\times[1,2]\cup L^\times\cup L_0'\times[-2,-1]$ is a subdivision of the cell complex $L\times[-2,2]$, and R is a subdivision of Z, whence $\mathbf{W}^\bullet$ is constructible with respect to $R\cap W$ by lemma 5.1. By construction, $\mathbf{W}^\bullet$ is collared compatibly to $j^!(\mathbf{S}^\bullet|_{T\times(-1,1)})$ and $j'^!(\mathbf{S}^\bullet|_{T'\times(-1,1)})$, in the sense of definition 5.3. Finally, letting $\overline{H} : S \times [-2,2] \to X$ be the extension of H given by

$$\overline{H}(s,t) = \begin{cases} g(s), & 1 \le t \le 2 \\ H(s,t), & -1 \le t \le 1 \\ g'(s), & -2 \le t \le -1 \end{cases}$$

we have $\overline{H}|_T = |g||_T$, $\overline{H}|_{T'} = |g'||_{T'}$, and $\overline{H}(W) \subset U \cap V$. Consequently,

$$(W, R \cap W, \mathbf{W}^\bullet, \overline{H}|_W)$$

is an admissible PL-nullbordism for

$$(T, L_1 \cap T, j^!(\mathbf{S}^\bullet|_{T\times(-1,1)}), |g||_T) \sqcup (-T', L_1' \cap T', j'^!(\mathbf{S}^\bullet|_{T'\times(-1,1)}), |g'||_{T'}).$$

Our next goal is to show that the bordism class of $\partial_*(S, L, \mathbf{S}^\bullet, f)$ depends only on the bordism class of $(S, L, \mathbf{S}^\bullet, f)$. Suppose then that $(W, R, \mathbf{W}^\bullet, W \xrightarrow{F} X)$ is an admissible PL-nullbordism for the admissible PL-representative $(S, L, \mathbf{S}^\bullet, f)$. We will produce an admissible PL-nullbordism for $\partial_*(S, L, \mathbf{S}^\bullet, f) = (T, L_1 \cap T, \mathbf{T}^\bullet, |g||_T)$, where $\mathbf{T}^\bullet = j^!(\mathbf{S}^\bullet|_{T\times(-1,1)})$. Let $G : R_0 \to K$ be a simplicial approximation rel boundary to F, i.e. R_0 is a subdivision of R such that $R_0 \cap S = L_0$ and $G|_{L_0} = g$. We set $P_W = G^{-1}(A)$, $Q_W = G^{-1}(B)$ and note that $P_W \cap S = P$, $Q_W \cap S = Q$. Along the boundary S of W, let us attach a cylinder $S \times [1, 2]$ to obtain the polyhedron

$$\overline{W} = W \cup_{S=S\times\{1\}} S \times [1, 2].$$

Triangulate $\overline{W}$ in the following manner: On W use R_0. Triangulate $S \times \{2\}$ by L_0 and extend the triangulation to $S \times [1, 2]$ without introducing any new vertices and so that the resulting simplicial complex C, $|C| = S \times [1, 2]$, satisfies the condition: if a set of vertives $\{v_1, \ldots, v_k\}$ spans a simplex in C, then $\{\mathrm{proj}(v_1), \ldots, \mathrm{proj}(v_k)\}$ (which is a set of vertices of L_0) spans a simplex in L_0, with $\mathrm{proj} : S \times [1, 2] \to S$. Call the complex thus obtained $\overline{R}_0 = R_0 \cup_{L_0=L_0\times\{1\}} C$, $|\overline{R}_0| = \overline{W}$. Next, extend the simplicial map $\rho : L_0 \to [-1, 1]$ to a simplicial map $\overline{\rho} : \overline{R}_0 \to [-1, 1]$ by requiring

$$\begin{array}{ll} \overline{\rho}(v, i) = \rho(v), & \text{for vertices } (v, i) \in L_0 \times \{i\},\ i = 1, 2, \\ \overline{\rho}(v) = -1, & \text{for vertices } v \in P_W, \\ \overline{\rho}(v) = +1, & \text{for vertices } v \in Q_W, \text{ and} \\ \overline{\rho}(v) \in \{\pm 1\}, & \text{for vertices } v \in R_0 - (L_0 \times \{1\} \cup P_W \cup Q_W). \end{array}$$

Set $\overline{T} = \overline{\rho}^{-1}(0)$. Then $\overline{T}$ is transverse to the simplicial stratification of $\overline{W}$ induced by $\overline{R}_0$, has a neighborhood in $\overline{W}$ homeomorphic to $\overline{T} \times (-1, 1)$, and thus is a compact, oriented, PL-pseudomanifold with boundary $\partial\overline{T} = \overline{T} \cap S \times \{2\} = T \times \{2\}$. The triangulation of $\overline{T}$ is given by the simplicial complex $\overline{R}_1 \cap \overline{T}$, where $\overline{R}_1$ is a simplicial subdivision of $\overline{R}_0$ corresponding to regarding $\overline{\rho}$ as a simplicial map into $[-1, 1]$ with $0 \in [-1, 1]$ a vertex. Note that $\overline{R}_1$ can be constructed so that $\overline{R}_1 \cap S \times \{2\} = L_1$. Define a simplicial map $\overline{G} : \overline{R}_0 \to K$ on vertices $v \in \overline{R}_0$ by

$$\overline{G}(v) = \begin{cases} G(v), & v \in R_0 \\ g(v), & v \in L_0 \times \{2\} \end{cases}$$

Then $\overline{G}(\overline{T}) \subset U \cap V$ and $\overline{G}|_{T\times\{2\}} = g|_T$.

Lastly, we shall construct a self-dual sheaf $\overline{\mathbf{T}}^\bullet \in SD(\overline{T}_{\overline{R}_1\cap\overline{T}})$. Let $c : S \times [0, 1] \xrightarrow{\cong} N \subset W$ be a collar for S in W, with N a closed neighborhood of S in W. We assume here that this collar has been parametrized so that $c(s, 1) = s$ for all $s \in S$. With $c|$ the restriction $c| : S \times (0, 1) \xrightarrow{\cong} \mathrm{int}\, N$ and $\pi_{01} : S \times (0, 1) \to S$ the projection, we have an isomorphism

$$c|^*(\mathbf{W}^\bullet|_{\mathrm{int}\, N}) \cong \pi_{01}^! \mathbf{S}^\bullet \qquad (13)$$

(definition 5.3). We will apply the gluing lemma 5.2 to construct a self-dual sheaf $\overline{\mathbf{W}}^\bullet$ on the interior of $\overline{W}$. With

$$\begin{array}{l} U = \operatorname{int}(N \cup_{S = S\times\{1\}} S \times [1,2]), \\ V = \operatorname{int} W, \end{array}$$

we have $U \cap V = \operatorname{int} N$, $U \cup V = \operatorname{int} \overline{W}$. Define

$$c_U : S \times (0,2) \xrightarrow{\cong} U$$

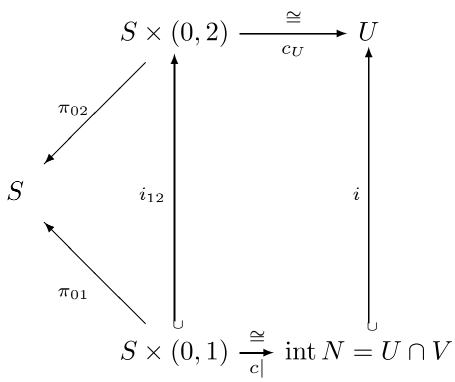
by

$$c_U(s,t) = \begin{cases} c(s,t), & t < 1, \\ (s,t), & t \geq 1, \end{cases}$$

and let

$$\begin{array}{l} \mathbf{U}^\bullet = (c_U^{-1})^* \pi_{02}^! \mathbf{S}^\bullet, \;\; \pi_{02} : S \times (0,2) \to S, \\ \mathbf{V}^\bullet = \mathbf{W}^\bullet, \end{array}$$

$\mathbf{U}^\bullet \in SD(U)$, $\mathbf{V}^\bullet \in SD(V)$. In order to verify $\mathbf{U}^\bullet|_{U\cap V} \cong \mathbf{V}^\bullet_{U\cap V}$, consider the diagram



which commutes by construction of c_U. We calculate, using (13),

$$\begin{array}{rcl} c|^*(\mathbf{V}^\bullet|_{U\cap V}) & = & c|^*(\mathbf{W}^\bullet|_{\operatorname{int} N}) \cong \pi_{01}^! \mathbf{S}^\bullet \cong i_{12}^* \pi_{02}^! \mathbf{S}^\bullet \\ & = & c|^* i^* (c_U^{-1})^* \pi_{02}^! \mathbf{S}^\bullet = c|^* i^* \mathbf{U}^\bullet \\ & = & c|^*(\mathbf{U}^\bullet|_{U\cap V}). \end{array}$$

This isomorphism induces an isomorphism $\mathbf{U}^\bullet|_{U\cap V} \cong \mathbf{V}^\bullet_{U\cap V}$, since $c|$ is a PL-homeomorphism. By the gluing lemma, there exists a unique $\overline{\mathbf{W}}^\bullet \in SD(\operatorname{int} \overline{W})$ such that $\overline{\mathbf{W}}^\bullet|_U \cong (c_U^{-1})^* \pi_{02}^! \mathbf{S}^\bullet$, $\overline{\mathbf{W}}^\bullet|_{\operatorname{int} W} \cong \mathbf{W}^\bullet$. The sheaf $\overline{\mathbf{W}}^\bullet$ is constructible with respect to the simplicial stratification on $\operatorname{int} \overline{W}$ induced by $\overline{R}_0 - L_0 \times \{2\}$. Using the inclusions

$$\begin{array}{ccc} \operatorname{int} \overline{T} = \operatorname{int} \overline{\rho}^{-1}(0) & \overset{\jmath}{\hookrightarrow} & \overline{\rho}^{-1}(-1,1) \cap \operatorname{int} \overline{W} \cong \operatorname{int} \overline{T} \times (-1,1) \\ & & \big\downarrow \iota \\ & & \operatorname{int} \overline{W} \end{array}$$

we define

$$\overline{\mathbf{T}}^\bullet = \jmath^! \iota^* \overline{\mathbf{W}}^\bullet$$

on $\operatorname{int}\overline{T}$. The self-duality of $\overline{\mathbf{T}}^\bullet$ follows from the facts that ι is an open inclusion and $\bar{\jmath}$ is a normally nonsingular inclusion. The interior of $\overline{W}$ is simplicially stratified by $\overline{R}_0 - L_0 \times \{2\}$. The open inclusion ι induces a stratification on $\overline{\rho}^{-1}(-1,1) \cap \operatorname{int}\overline{W}$, which induces a stratification on $\operatorname{int}\overline{T} \times (-1,1)$. This stratification agrees with the product stratification on $\operatorname{int}\overline{T} \times (-1,1)$ where $\operatorname{int}\overline{T}$ is simplicially stratified by $\overline{R}_1 \cap \operatorname{int}\overline{T}$. Hence $\overline{\mathbf{T}}^\bullet$ is constructible with respect to $\overline{R}_1 \cap \operatorname{int}\overline{T}$. In summary, $\overline{\mathbf{T}}^\bullet \in SD(\operatorname{int}\overline{T}_{\overline{R}_1 \cap \operatorname{int}\overline{T}})$. It remains to be shown that $(T, \mathbf{T}^\bullet)$ is collared in $(\overline{T}, \overline{\mathbf{T}}^\bullet)$ (in the sense of definition 5.3). The cylinder lemma 9.1 implies that $\overline{T} \cap S \times [1,2] = T \times [1,2]$. The set $\overline{N} = \overline{T} \cap S \times [1,2]$ is a closed neighborhood of $T = T \times \{2\}$ in $\overline{T}$. Define a PL-homeomorphism

$$c_T : T \times [0,1] \xrightarrow{\cong} T \times [1,2] = \overline{N} \subset \overline{T}$$

by

$$c_T(x,y) = (x, y+1),\ x \in T,\ y \in [0,1].$$

We claim that

$$c_T|^*(\overline{\mathbf{T}}^\bullet|_{\operatorname{int}\overline{N}}) \cong \pi^! \mathbf{T}^\bullet,$$

where $c_T| : T \times (0,1) \xrightarrow{\cong} \operatorname{int}\overline{N}$ is the restriction of c_T and $\pi : T \times (0,1) \to T$ is the projection. To establish this claim, consider the commutative diagram

$$\begin{array}{ccccccc}
\operatorname{int}\overline{N} = T \times (1,2) & \overset{\beta}{\hookrightarrow} & S \times (1,2) & \overset{\alpha}{\hookrightarrow} & S \times (0,2) & \xrightarrow{\pi_{02}} & S \\
\Big\uparrow{\scriptstyle c_T| \,\cong} & & & & & & \Big\uparrow{\scriptstyle i_{-11}} \\
T \times (0,1) & \xrightarrow{\pi} & T = T \times \{0\} & & \overset{j}{\hookrightarrow} & & T \times (-1,1)
\end{array}$$

(with α, β the obvious inclusions), which shows that

$$\begin{array}{rcl}
c_T|^*\beta^!\alpha^*\pi_{02}^!\mathbf{S}^\bullet & \cong & c_T|^!\beta^!\alpha^!\pi_{02}^!\mathbf{S}^\bullet \\
& = & \pi^! j^! i_{-11}^! \mathbf{S}^\bullet \\
& \cong & \pi^! j^! (\mathbf{S}^\bullet|_{T \times (-1,1)}).
\end{array} \tag{14}$$

The diagram

$$\begin{array}{ccc}
\operatorname{int}\overline{W} & \overset{\delta}{\hookleftarrow} & U \\
\Big\uparrow{\scriptstyle \gamma} & & \Big\uparrow{\scriptstyle c_U \,\cong} \\
S \times (1,2) & \overset{\alpha}{\hookrightarrow} & S \times (0,2)
\end{array}$$

commutes, as $c_U(s,t) = (s,t)$ for $t > 1$ (γ and δ are again the obvious inclusions). Thus

$$\begin{array}{rcl}
\gamma^*\overline{\mathbf{W}}^\bullet & \cong & \alpha^* c_U^* \delta^* \overline{\mathbf{W}}^\bullet = \alpha^* c_U^*(\overline{\mathbf{W}}^\bullet|_U) \\
& \cong & \alpha^* c_U^* (c_U^{-1})^* \pi_{02}^! \mathbf{S}^\bullet \\
& \cong & \alpha^* \pi_{02}^! \mathbf{S}^\bullet.
\end{array} \tag{15}$$

Employing the commutative diagram

$$\begin{array}{ccc}
\operatorname{int}\overline{T}\times(-1,1)\cong\overline{\rho}^{-1}(-1,1)\cap\operatorname{int}\overline{W} & \xrightarrow{\ \iota\ } & \operatorname{int}\overline{W} \\
\uparrow{\scriptstyle \overline{\jmath}} & & \\
\operatorname{int}\overline{T}=\operatorname{int}\overline{T}\times\{0\} & & \uparrow{\scriptstyle \gamma} \\
\uparrow{\scriptstyle \kappa} & & \\
\operatorname{int}\overline{N}\ =T\times(1,2) & \xrightarrow{\ \beta\ } & S\times(1,2)
\end{array}$$

we see that

$$\begin{aligned}
\beta^!\gamma^*\overline{\mathbf{W}}^\bullet &\cong \beta^!\gamma^!\overline{\mathbf{W}}^\bullet \cong \kappa^!\overline{\jmath}^!\iota^!\overline{\mathbf{W}}^\bullet \\
&\cong \kappa^*\overline{\jmath}^!\iota^*\overline{\mathbf{W}}^\bullet.
\end{aligned} \tag{16}$$

Combining (14), (15) and (16) yields the desired isomorphism:

$$\begin{aligned}
c_T|^*(\overline{\mathbf{T}}^\bullet|_{\operatorname{int}\overline{N}}) &= c_T|^*\kappa^*\overline{\mathbf{T}}^\bullet = c_T|^*\kappa^*\overline{\jmath}^!\iota^*\overline{\mathbf{W}}^\bullet \\
&\cong c_T|^*\beta^!\gamma^*\overline{\mathbf{W}}^\bullet \cong c_T|^*\beta^!\alpha^*\pi_{02}^!\mathbf{S}^\bullet \\
&\cong \pi^!j^!(\mathbf{S}^\bullet|_{T\times(-1,1)}) \\
&= \pi^!\mathbf{T}^\bullet.
\end{aligned}$$

We have shown that $(\overline{T},\overline{R}_1\cap\overline{T},\overline{\mathbf{T}}^\bullet,|\overline{G}||_{\overline{T}})$ is an admissible PL-nullbordism for $(T,L_1\cap T,\mathbf{T}^\bullet,|g||_T)$. This concludes the proof that ∂_* is well-defined.

Naturality: Let X be a space triangulated by the simplicial complex K and X' a space triangulated by the complex K'. Let $U\cup V=X$ and $U'\cup V'=X'$ be open covers so that each of U,V,U',V' is the complement of a subcomplex. Given a simplicial map $h:K\to K'$ such that $|h|(U)\subset U'$ and $|h|(V)\subset V'$, we have to show that the square

$$\begin{array}{ccc}
S_n^{PL}(X) & \xrightarrow{\ \partial_*\ } & S_{n-1}^{PL}(U\cap V) \\
\downarrow{\scriptstyle S_n^{PL}(|h|)} & & \downarrow{\scriptstyle S_{n-1}^{PL}(|h||_{U\cap V})} \\
S_n^{PL}(X') & \xrightarrow{\ \partial_*\ } & S_{n-1}^{PL}(U'\cap V')
\end{array}$$

commutes. Write $U'=X'-|A'|$, $V'=X'-|B'|$, with A',B' subcomplexes of K'. The map $hg:L_0\to K'$ is a simplicial approximation to $|h|f$. With $P'=(hg)^{-1}(A')$, $Q'=(hg)^{-1}(B')$, we have $P'\subset P$ and $Q'\subset Q$. Consequently, the very same ρ that determines $T=\rho^{-1}(0)$, $\partial_*[(S,L,\mathbf{S}^\bullet,f)]=[(T,L_1\cap T,\mathbf{T}^\bullet,|g||_T)]$, also works for P',Q', and

$$\begin{aligned}
\partial_*S_n^{PL}(|h|)[(S,L,\mathbf{S}^\bullet,f)] &= \partial_*[(S,L,\mathbf{S}^\bullet,|h|f)] \\
&= [(T,L_1\cap T,\mathbf{T}^\bullet,|hg||_T)] \\
&= S_{n-1}^{PL}(|h||)[(T,L_1\cap T,\mathbf{T}^\bullet,|g||_T)] \\
&= S_{n-1}^{PL}(|h||)\partial_*[(S,L,\mathbf{S}^\bullet,f)].
\end{aligned}$$

Exactness of

$$S_n^{PL}(U \cap V) \xrightarrow{i_* \oplus j_*} S_n^{PL}(U) \oplus S_n^{PL}(V) \xrightarrow{k_* - l_*} S_n^{PL}(X):$$

The equality $ki = lj$ implies $(k_* - l_*) \circ (i_* \oplus j_*) = 0$. Suppose that

$$[(S_U, L_U, \mathbf{S}_U^\bullet, S_U \xrightarrow{f_U} U)] \in S_n^{PL}(U)$$

and

$$[(S_V, L_V, \mathbf{S}_V^\bullet, S_V \xrightarrow{f_V} V)] \in S_n^{PL}(V)$$

are elements such that

$$k_*[(S_U, L_U, \mathbf{S}_U^\bullet, f_U)] = l_*[(S_V, L_V, \mathbf{S}_V^\bullet, f_V)].$$

Let $(W^{n+1}, C, \mathbf{W}^\bullet, W \xrightarrow{F} X)$ be an admissible PL-bordism between $(S_U, L_U, \mathbf{S}_U^\bullet, kf_U)$ and $(S_V, L_V, \mathbf{S}_V^\bullet, lf_V)$. By the shield lemma 9.2, there exists a subdivision C_0 of C, a subdivision K_0 of K, and a simplicial approximation $G: C_0 \to K_0$ of F such that $|G|(S_U) \subset U$ and $|G|(S_V) \subset V$. The sets $P = G^{-1}(A) \cup S_V$, $Q = G^{-1}(B) \cup S_U$ are disjoint. Let $\rho: C_0 \to [-1,1]$ be any simplicial map so that $\rho(v) = -1$ for vertices $v \in P$ and $\rho(v) = +1$ for vertices $v \in Q$. The preimage $W_0 = \rho^{-1}(0)$ is a closed, oriented, bicollared (as $\rho^{-1}(-1,1) \cong W_0 \times (-1,1)$) PL-pseudomanifold triangulated by a complex $C_1 \cap W_0$, with C_1 a suitable subdivision of C_0 as described above. The sheaf $\mathbf{W}_0^\bullet = j^!(\mathbf{W}^\bullet|_{\rho^{-1}(-1,1)})$ is self-dual on W_0, where j is the normally nonsingular inclusion $j: \rho^{-1}(0) \hookrightarrow \rho^{-1}(-1,1)$. Note that $|G|(W_0) \subset U \cap V$, whence

$$[(W_0, C_1 \cap W_0, \mathbf{W}_0^\bullet, |G||_{W_0})] \in S_n^{PL}(U \cap V).$$

We claim that

$$i_*[(W_0, C_1 \cap W_0, \mathbf{W}_0^\bullet, |G||_{W_0})] = [(S_U, L_U, \mathbf{S}_U^\bullet, |G||_{S_U})]:$$

With $W_{\geq 0} = \rho^{-1}[0,1]$, the quadruple $(W_{\geq 0}, C_1 \cap W_{\geq 0}, \mathbf{W}^\bullet|_{\operatorname{int} W_{\geq 0}}, |G||_{W_{\geq 0}})$ is an admissible PL-bordism between $(W_0, C_1 \cap W_0, \mathbf{W}_0^\bullet, |G||_{W_0})$ and $(S_U, L_U, \mathbf{S}_U^\bullet, |G||_{S_U})$, and $|G|(W_{\geq 0}) \subset U$. Now $(S_U, L_U, \mathbf{S}_U^\bullet, |G||_{S_U})$ is of course bordant to $(S_U, L_U, \mathbf{S}_U^\bullet, f_U)$. Using $W_{\leq 0} = \rho^{-1}[-1,0]$, one sees similarly that

$$j_*[(W_0, C_1 \cap W_0, \mathbf{W}_0^\bullet, |G||_{W_0})] = [(S_V, L_V, \mathbf{S}_V^\bullet, f_V)].$$

Exactness of

$$S_n^{PL}(X) \xrightarrow{\partial_*} S_{n-1}^{PL}(U \cap V) \xrightarrow{i_* \oplus j_*} S_{n-1}^{PL}(U) \oplus S_{n-1}^{PL}(V):$$

To see that $(i_* \oplus j_*) \circ \partial_* = 0$, let $[(S, L, \mathbf{S}^\bullet, S \xrightarrow{f} X)] \in S_n^{PL}(X)$. Then, without repeating all the details, $W_{\leq 0} = \rho^{-1}[-1,0]$ can be used to construct an admissible PL-nullbordism for $\partial_*(S, L, \mathbf{S}^\bullet, f)$ in V, and $W_{\geq 0} = \rho^{-1}[0,1]$ is used for U.

Let $[(T^{n-1}, C, \mathbf{T}^\bullet, T \xrightarrow{h} U \cap V)] \in S_{n-1}^{PL}(U \cap V)$ be an element such that

$$i_*[(T^{n-1}, C, \mathbf{T}^\bullet, h)] = 0 \text{ and } j_*[(T^{n-1}, C, \mathbf{T}^\bullet, h)] = 0.$$

Let $(S_U, L_U, \mathbf{S}_U^\bullet, S_U \xrightarrow{f_U} U)$ be an admissible PL-nullbordism for $(T^{n-1}, C, \mathbf{T}^\bullet, ih)$, and let $(S_V, L_V, \mathbf{S}_V^\bullet, S_V \xrightarrow{f_V} V)$ be an admissible PL-nullbordism for $(T^{n-1}, C, \mathbf{T}^\bullet, jh)$. As described in the proof of transitivity of the bordism relation, lemma 5.3, these two nullbordisms can be glued along their common boundary, yielding an admissible PL-representative. However, in order to show that this representative has a ρ such that $\rho^{-1}(0) = T$, it is technically advantageous to glue S_U and S_V to the two ends of a cylinder on T. Thus triangulate $T \times [-1,1]$ by a simplicial complex R so that $R \cap T \times \{1\} = C = R \cap T \times \{-1\}$ and R has no vertices in the interior $T \times (-1,1)$. Then apply the gluing process of the proof of lemma 5.3 twice: First, to glue $(S_U, L_U, \mathbf{S}^\bullet_U, f_U)$ and

$$(T \times [-1,1], R, \pi^! \mathbf{T}^\bullet, T \times [-1,1] \xrightarrow{\text{proj}} T \xrightarrow{h} U \cap V),$$

$\pi : T \times (-1,1) \to T$, along their common boundary $\partial S_U = T = T \times \{+1\}$, and then to glue the resulting object along $T \times \{-1\}$ to $(S_V, L_V, \mathbf{S}^\bullet_V, f_V)$. The end-product is an admissible PL-representative $(S, L, \mathbf{S}^\bullet, S \xrightarrow{f} X)$ with

$$S = S_U \cup_{\partial S_U = T \times \{+1\}} T \times [-1,1] \cup_{T \times \{-1\} = \partial S_V} S_V, \; L = L_U \cup_C R \cup_C L_V.$$

According to lemma 9.2, there exist subdivisions L_0 of L and K_0 of K, and a simplicial approximation $g : L_0 \to K_0$ of f such that still $|g|(S_U) \subset U$ and $|g|(S_V) \subset V$. This can be done in such a way that $L_0 \cap T \times \{1\} = L_0 \cap T \times \{-1\}$ — let us call this simplicial complex C_0 — and so that L_0 has no vertices in $T \times (-1,1)$. With $P = g^{-1}(A)$, $Q = g^{-1}(B)$, we have $P \subset \operatorname{int} S_V$, $Q \subset \operatorname{int} S_U$. Define the simplicial map $\rho : L_0 \to [-1,1]$ on vertices v by

$$\rho(v) = \begin{cases} +1, & v \in S_U, \\ -1, & v \in S_V. \end{cases}$$

This determines ρ completely, since there are no vertices in the interior of the middle cylinder. In particular, every vertex $v \in C_0 \times \{+1\}$ has $\rho(v) = +1$ and every vertex $v \in C_0 \times \{-1\}$ has $\rho(v) = -1$. Using this ρ to calculate ∂_*, we have

$$\partial_*[(S, L, \mathbf{S}^\bullet, f)] = [(T, C, \mathbf{T}^\bullet, h)].$$

Exactness of

$$S_n^{PL}(U) \oplus S_n^{PL}(V) \xrightarrow{k_* - l_*} S_n^{PL}(X) \xrightarrow{\partial_*} S_{n-1}^{PL}(U \cap V):$$

Let us first discuss $\partial_* \circ (k_* - l_*) = 0$. Suppose $[(S_U,\, L_U,\, \mathbf{S}^\bullet_U,\, S_U \xrightarrow{f_U} U)] \in S_n^{PL}(U)$ and $[(S_V,\, L_V,\, \mathbf{S}^\bullet_V,\, S_V \xrightarrow{f_V} V)] \in S_n^{PL}(V)$. According to lemma 9.2, there exist subdivisions L_{U0} of L_U, L_{V0} of L_V, K_0 of K, and simplicial approximations $g_U : L_{U0} \to K_0$ of f_U and $g_V : L_{V0} \to K_0$ of f_V such that still $|g_U|(S_U) \subset U$ and $|g_V|(S_V) \subset V$. Set $P = (g_U \sqcup g_V)^{-1}(A)$ and $Q = (g_U \sqcup g_V)^{-1}(B)$. From $g_U^{-1}(A) = \varnothing$ it follows that in fact $P = g_V^{-1}(A)$, and so $P \subset S_V$. Similarly $Q \subset S_U$. Thus we can set $\rho : L_{U0} \sqcup L_{V0} \to [-1,1]$ to be identically $+1$ on L_{U0} and identically -1 on L_{V0}. The resulting simplicial map does not assume any value other than ± 1 since S_U and S_V are disjoint. Hence $T = \rho^{-1}(0) = \varnothing$ and

$$\partial_*(k_*[(S_U, L_U, \mathbf{S}^\bullet_U, f_U)] - l_*[(S_V, L_V, \mathbf{S}^\bullet_V, f_V)]) = 0.$$

Suppose $[(S, L, \mathbf{S}^\bullet, S \xrightarrow{f} X)] \in S_n^{PL}(X)$ is an element such that

$$\partial_*[(S, L, \mathbf{S}^\bullet, f)] = [(T, L_1 \cap T, \mathbf{T}^\bullet, |g||_T)] = 0 \in S_{n-1}^{PL}(U \cap V),$$

$g : L_0 \to K$ a simplicial approximation of f, $T = \rho^{-1}(0)$, $\rho : L_0 \to [-1,1]$ simplicial. The spaces $W_{\leq 0} = \rho^{-1}[-1,0]$ and $W_{\geq 0} = \rho^{-1}[0,1]$ are compact, oriented PL-pseudomanifolds with $\partial W_{\leq 0} = T$, $\partial W_{\geq 0} = -T$, and are triangulated by $L_1 \cap W_{\leq 0}$ and $L_1 \cap W_{\geq 0}$, respectively. As ρ has been constructed to be identically -1 on $P = g^{-1}(A)$ and identically $+1$ on $Q = g^{-1}(B)$, we have $|g|(W_{\leq 0}) \subset V$ and

$|g|(W_{\geq 0}) \subset U$. Let $(W, R, \mathbf{W}^\bullet, W \xrightarrow{F} U \cap V)$ be an admissible PL-nullbordism for $(T, L_1 \cap T, \mathbf{T}^\bullet, |g||_T)$. Let

$$\begin{aligned}(S_U, L_U, \mathbf{S}^\bullet_U, S_U \xrightarrow{f_U} U) &= (W_{\geq 0}, L_1 \cap W_{\geq 0}, \mathbf{S}^\bullet|_{\operatorname{int} W_{\geq 0}}, |g||_{W_{\geq 0}}) \\ &\quad \cup_{(T, L_1 \cap T, \mathbf{T}^\bullet, |g||_T)} (W, R, \mathbf{W}^\bullet, F), \\ (S_V, L_V, \mathbf{S}^\bullet_V, S_V \xrightarrow{f_V} V) &= (-W_{\leq 0}, L_1 \cap W_{\leq 0}, \mathbf{S}^\bullet|_{\operatorname{int} W_{\leq 0}}, |g||_{W_{\leq 0}}) \\ &\quad \cup_{(T, L_1 \cap T, \mathbf{T}^\bullet, |g||_T)} (W, R, \mathbf{W}^\bullet, F)\end{aligned}$$

be obtained by gluing as described in the proof of lemma 5.3. We claim that

$$k_*[(S_U, L_U, \mathbf{S}^\bullet_U, f_U)] - l_*[(S_V, L_V, \mathbf{S}^\bullet_V, f_V)] = [(S, L_1, \mathbf{S}^\bullet, |g|)].$$

As the underlying space for the bordism, we take the polyhedron

$$Z = (S_V \times [-1, 0]) \cup_{W \times \{0\}} (S_U \times [0, 1]),$$

whose boundary is

$$\partial Z = S_U \sqcup -S_V \sqcup -S.$$

This bordism is to be triangulated as follows: Triangulate both $S_V \times \{-1\}$ and $S_V \times \{0\}$ by L_V, and extend this to a triangulation of $S_V \times [-1, 0]$ without introducing further vertices. Similarly, triangulate both $S_U \times \{1\}$ and $S_U \times \{0\}$ by L_U, and extend to $S_U \times [0, 1]$ without introducing further vertices. These triangulations give rise to a triangulation C of Z, since $L_U \cap W = R = L_V \cap W$. Up to isomorphism, there exists a unique sheaf $\mathbf{Z}^\bullet \in SD(\operatorname{int} Z)$ such that

- $\mathbf{Z}^\bullet|_{S_U \times (0,1)} \cong \pi^!_{01} \mathbf{S}^\bullet_U$, $\pi_{01} : S_U \times (0, 1) \to S_U$,
- $\mathbf{Z}^\bullet|_{S_V \times (-1,0)} \cong \pi^!_{-10} \mathbf{S}^\bullet_V$, $\pi_{-10} : S_V \times (-1, 0) \to S_V$,
- $\mathbf{Z}^\bullet|_{\operatorname{int} W \times (-1,1)} \cong \pi^!_{-11} \mathbf{W}^\bullet$, $\pi_{-11} : \operatorname{int} W \times (-1, 1) \to \operatorname{int} W$

(use the gluing lemma 5.2). Define a continuous map $G : Z \to X$ by

$$G(x, t) = \begin{cases} f_U(x), & (x, t) \in S_U \times [0, 1], \\ f_V(x), & (x, t) \in S_V \times [-1, 0]. \end{cases}$$

That this is well-defined follows from $f_U(x) = F(x) = f_V(x)$ for $(x, 0) \in W \times \{0\}$. On the boundary it restricts to

$$\begin{aligned} G|_{\partial Z} &= f_U \sqcup f_V \sqcup (f_V|_{W_{\leq 0}} \cup f_U|_{W_{\geq 0}}) \\ &= f_U \sqcup f_V \sqcup (|g||_{W_{\leq 0}} \cup |g||_{W_{\geq 0}}) \\ &= f_U \sqcup f_V \sqcup |g|. \end{aligned}$$

Summarizing, the quadruple

$$(Z, C, \mathbf{Z}^\bullet, G)$$

is an admissible PL-nullbordism for

$$(S_U, L_U, \mathbf{S}^\bullet_U, k f_U) \sqcup (-S_V, L_V, \mathbf{S}^\bullet_V, l f_V) \sqcup (-S, L_1, \mathbf{S}^\bullet, |g|),$$

establishing the claim. Now of course $[(S, L_1, \mathbf{S}^\bullet, |g|)] = [(S, L, \mathbf{S}^\bullet, f)] \in S^{PL}_n(X)$. □

Proposition 9.1. *Any homotopy functor $h_* : \mathcal{TOP} \to \mathcal{AB}$ satisfying axiom (MV) is a homology theory on the admissible category of compact polyhedra.*

Proof. The map $X \to pt$ induces a map $h_i(X) \to h_i(pt)$. Define $\widetilde{h}_i(X) = \ker(h_i(X) \to h_i(pt))$. For a PL pair (K, L) with $L \neq \varnothing$, we set

$$h_i(K, L) = \widetilde{h}_i(K \cup_L cL),$$

where $K \cup_L cL$ denotes the PL space obtained by attaching the cone on L to K, along $L \subset K$. When $L = \varnothing$, set $h_i(K, L) = h_i(K)$. We prove the exactness axiom. Let $U \subset K \cup cL$ be the complement of the cone point. Then U is an open subset homotopy equivalent to K. Let V be the complement of K, i.e. the open cone on L. Then V is a contractible open subset and $U \cup V = K \cup cL$. The intersection $U \cap V$ is homotopy equivalent to L. For $i > 0$, consider the diagram

$$\begin{array}{ccccccc}
h_i(U \cap V) & \longrightarrow & h_i(U) \oplus h_i(V) & \longrightarrow & h_i(U \cup V) & \xrightarrow{\partial_*} & h_{i-1}(U \cap V) \\
\downarrow \cong & & \downarrow \cong & & \| & & \downarrow \cong \\
h_i(L) & \longrightarrow & h_i(K) & \longrightarrow & h_i(K, L) & \overset{\partial_*}{\dashrightarrow} & h_{i-1}(L)
\end{array}$$

whose first exact row is provided by axiom (MV). This defines the (natural) boundary operator $\partial_* : h_i(K, L) \to h_{i-1}(L)$ and establishes the exactness of the sequence of the pair (K, L). (We leave $i = 0$ to the reader.) Let us move on to the excision axiom. Given an open subset $U \subset |K|$ whose closure is contained in the interior of $|L|$ and which is of the form $U = |K| - |P|$ for some subcomplex $P \subset K$, let $Q \subset L$ be the subcomplex such that $|L| - U = |Q|$. We have to prove that the inclusion $(P, Q) \to (K, L)$ induces an isomorphism

$$h_i(P, Q) \xrightarrow{\cong} h_i(K, L).$$

Setting

$$X = K \cup cL, \; Y = P \cup cQ,$$

it suffices, by the exact sequence of the pair (X, Y), to show that $h_i(X, Y) = \widetilde{h}_i(X \cup cY) = 0$. Let $V \subset X \cup cY$ be an open regular neighborhood of cL ($\subset X \subset X \cup cY$), so that cL is a deformation retract of V. Let $W \subset X \cup cY$ be an open regular neighborhood of cY, so that cY is a deformation retract of W. Then

1. V is contractible, $V \simeq cL \simeq pt$,
2. W is contractible, $W \simeq cY \simeq pt$,
3. $V \cap W$ is contractible, $V \cap W \simeq cQ \simeq pt$, and
4. $V \cup W = X \cup cY$.

Thus axiom (MV) implies that $\widetilde{h}_i(X \cup cY) = 0$ for all i. Consequently, h_* satisfies all of the Eilenberg-Steenrod axioms for a homology theory. □

By theorem 9.1, we have in particular:

Corollary 9.1. *$S_*^{PL}(-)$ is a homology theory.*

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