THE STABLE SUSPENSION OF AN EILENBERG-MACLANE SPACE

BY
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Introduction. In Part I of this paper we set up a spectral sequence for the stable homotopy groups of a countable CW complex X, using the suspension triad sequences for the iterated suspensions S^rX. If X is $(n-1)$-connected, we calculate the differential operator d^1 for the first $n-1$ nontrivial dimensions in terms of the automorphism T_* of $\pi_m(X\times X)$ induced by the map which interchanges factors.

In Part II we continue the study of the behavior of Eilenberg-MacLane spaces under suspension, initiated in [2]. For $K(\pi, n)$, π finitely generated, we calculate the automorphism T_* on the p-primary component ($p\neq 2$) for a range of dimensions. The p-primary component ($p\neq 2$) of the first $n-1$ nontrivial stable homotopy groups can then be read off from the spectral sequence. The corresponding Postnikov invariants are zero, in contrast to those of the single suspension $SK(\pi, n)$; however, this is not true of the 2-primary component.

PART I. THE SPECTRAL SEQUENCE

1. **The suspension couple.** All spaces considered throughout Part I are to be 1-connected countable CW complexes with fixed base points. Let X be such. The suspension triad sequences for iterated suspensions S^rX (cf. [7] for definitions)

$$\cdots \xrightarrow{j_*} \pi_q(S^rX; C_+, C_-) \xrightarrow{\Delta} \pi_{q-2}(S^{r-1}X) \xrightarrow{E} \pi_{q-1}(S^rX) \xrightarrow{j_*} \cdots$$

give rise to an exact couple

$$\begin{tikzcd} A \arrow[rr, "E"] & & A \arrow[dl, "j_*"] \\ & C \arrow[ul, "\Delta"] & \end{tikzcd}$$

where

$$C_{r,s} = \begin{cases} \pi_{s+2r}(S^rX; C_+, C_-), & r \geqq 1 \\ \pi_s(X), & r = 0 \\ 0, & r < 0 \end{cases}$$

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$$A_{r,s} = \begin{cases} \pi_{s+2r}(S^r X), & r \geqq 0 \\ 0, & r < 0 \end{cases}$$

and we define $j_* = \text{identity}: A_{0,s} = \pi_s(X) \to \pi_s(X) = C_{0,s}$. Let $\Pi_m(X)$ denote the stable group $\pi_{m+q}(S^q X)$, q large, and $E_\infty: \pi_m(X) \to \Pi_m(X)$ the iterated suspension.

(1.1) PROPOSITION. *The spectral sequence associated with the above exact couple* [8, §6] *converges, and* $C^\infty_{r,s} \approx B_{r,s}/B_{r-1,s+1}$, *where* $B_{r,s}$ *is the image of* $E_\infty: \pi_{s+2r}(S^r X) \to \Pi_{s+r}(X)$ (*if* $r<0$, $B_{r,s}=0$).

The proof is of a standard type.

2. **The operator** d^1. We first give some definitions.

1. $r: X \times I \to SX$ identifies $X \times \dot{I} \cup x_0 \times I$ to a point, where x_0 is the base point. C_+ and C_- denote the images under r of $X \times [1/2, 1]$ and $X \times [0, 1/2]$ respectively.

2. $\phi: (SX, C_+, C_-) \to (S_1 X \vee S_2 X, S_1 X, S_2 X)$ is defined by

$$\phi(x, s) = \begin{cases} (x, 2s) \in S_2 X & \text{for } 0 \leqq s \leqq 1/2, \qquad x \in X, \\ (x, 2s-1) \in S_1 X & \text{for } 1/2 \leqq s \leqq 1 \end{cases}$$

where the subscripts merely distinguish between the two copies of SX.

3. ρ is the reflection of SX in the equator, $\rho(x, s) = (x, 1-s)$.

4. $X \boldsymbol{\times} Y = X \times Y / X \vee Y$ and $\theta: X \times Y \to X \boldsymbol{\times} Y$ is the identification map.

5. T maps $X \boldsymbol{\times} X$ onto itself by $T(x, x') = (x', x)$.

6. $\eta: S^2(X \boldsymbol{\times} Y) \to SX \boldsymbol{\times} SY$ is the homeomorphism defined by $\eta((x, y), s, t) = ((x, s), (y, t))$, for $x \in X$, $y \in Y$, and $s, t \in I$.

7. The composition $i'_* \partial: \pi_{m+1}(X \times Y, X \vee Y) \to \pi_m(X \vee Y) \to \pi_m(X \vee Y; X, Y)$ is an isomorphism for $m \geqq 2$ [3, Corollary 8.3]; Q is the inverse.

Toda [11] defined a generalized Hopf homomorphism for spheres; the analogue for general spaces is

8. $H: \pi_m(S^r X; C_+, C_-) \to \pi_{m+1}(S^r X \boldsymbol{\times} S^r X)$ is the composition $H = \theta_* Q \phi_*$.

Now the operator $d^1: C_{r+1,s} \to C_{r,s}$ in the spectral sequence is given by $d^1 = j_* \Delta$ for $r \geqq 1$. Consider the diagram

$$\begin{tikzcd}
\pi_{m+1}(S^{r+1}X; C_+, C_-) \arrow[r, "d^1"] \arrow[ddr, "H"'] & \pi_{m-1}(S^r X; C_+, C_-) \arrow[d, "H"] \\
& \pi_m(S^r X \boldsymbol{\times} S^r X) \arrow[d, "\eta_* E^2"] \\
& \pi_{m+2}(S^{r+1}X \boldsymbol{\times} S^{r+1}X).
\end{tikzcd}$$

(2.1) PROPOSITION. $(\rho \boldsymbol{\times} 1)_* \eta_* E^2 H d^1 = (1 - T_*) H$ *for* $r \geqq 1$.

Note that if X is $(n-1)$-connected, then ϕ_* and θ_* are isomorphisms at least for $m<3n+3r-2$ and epimorphisms for $m=3n+3r-2$ [9, Corollary 3.3, Corollary 3.5], and hence so is H. E^2 is an isomorphism for $m \leqq 4n+4r-2$, while $(\rho\times 1)_*$ and η_* are always such. Hence for $m<3n+3r-2$, d^1 is given up to isomorphism by the endomorphism $1-T_*$ of $\pi_{m+2}(S^{r+1}X\times S^{r+1}X)$.

If $X=S^{q-r-1}$, then using the facts that $T_*=(-1)^q\iota_{2q}\circ$ and $(\rho\times 1)_*=-\iota_{2q}\circ$, the proposition reduces to Theorem 3.7 of [11]. Proposition (2.1) can be proved by generalizing Toda's methods. Rather than give the proof, which is geometric, we shall prove an equivalent proposition ((2.8) below) for $r\geqq 2$ which is simpler and more enlightening.

The following shows that, since we remain in a range where E^2 is an isomorphism, we need only calculate T_* in the homotopy groups of $X\times X$.

(2.2) PROPOSITION.

$$T_*\eta_*E^2 = -\eta_*E^2T_*: \pi_m(S^rX\times S^rX)\to\pi_{m+2}(S^{r+1}X\times S^{r+1}X).$$

The proof follows at once from the definitions.

In order to apply the results of [7], we shall assume for the remainder of this section that X is an $(n-1)$-connected countable CW complex, $n\geqq 2$, with only a single vertex x_0. Consider the diagram

$$\text{(2.3)}\qquad \begin{tikzcd} \pi_{m+1}(S^{r+1}X;C_+,C_-) \arrow[r, "\Delta"] & \pi_{m-1}(S^rX) \arrow[r, "j_*"] \arrow[dr, "\phi_*'Q'"] & \pi_{m-1}(S^rX;C_+,C_-) \arrow[d, "Q\phi_*"] \\ \pi_m(S^rX\times S^rX, S^rX\vee S^rX) \arrow[u, "\lambda i_*p_*"] \arrow[ur, "p_*'\partial"] & & \pi_m(S^rX\times S^rX, S^rX\vee S^rX) \end{tikzcd}$$

where the homomorphisms are defined as follows: $\phi_*':\pi_{m-1}(S^rX)\to\pi_{m-1}(S^rX\vee S^rX)$ is induced by ϕ. Q' is the projection of $\pi_{m-1}(S^rX\vee S^rX)$ onto a direct summand, obtained by injection into the triad group followed by Q. Thus the right-hand triangle in (2.3) is commutative. The left-hand vertical homomorphisms are

$$\pi_m(S^rX\times S^rX, S^rX\vee S^rX)\xrightarrow{p_*}\pi_m((S^rX)_2,S^rX)\xrightarrow{i_*}\pi_m((S^rX)_\infty,S^rX)$$
$$\xrightarrow{\lambda}\pi_{m+1}(S^{r+1}X;C_+,C_-).$$

Here $(S^rX)_q$ denotes the qth space in the reduced product complex [6]; in particular, $(S^rX)_2$ is obtained from $S^rX\times S^rX$ by identifying $(x,x_0)=(x_0,x)$ for all $x\in S^rX$, and p is the identification map. $p':S^rX\vee S^rX\to S^rX$ is the restriction of p, i is the inclusion, and λ is the isomorphism denoted by ϕ in [7]. According to (10.1) of [7], $\Delta\lambda=\partial$; hence the left-hand triangle in (2.3) is also commutative. Now λ and Q are isomorphisms for all m, ϕ_* for $m<3n+3r-2$, and i_*, p_* for $m<3n+3r-1$, the latter by simple connectivity arguments. Hence in order to calculate $d^1=j_*\Delta$ for $m<3n+3r-2$ we need only find $Q'\phi_*' p_*' \partial$. To do this we use the generalized Whitehead products of [5]; we recall briefly their properties.

The join $A * B$ of two countable CW complexes A, B, each with a single vertex, is defined to be the space obtained from $A \times I \times B$ by making the identifications $(a, 0, b) = (a, 0, b')$, $(a, 1, b) = (a', 1, b)$, $(a_0, t, b_0) = (a_0, 1/2, b_0)$ for all $a, a' \in A$, $b, b' \in B$, $t \in I$.

The set of homotopy classes of mappings $X \to Y$, preserving base points, will be denoted by $\{X, Y\}$. If X is a suspension, this set is in a well-known way a group, Abelian if X is a double suspension.

If $\mu \in \{S^{p+1}A, Y\}$, $\nu \in \{S^{q+1}B, Y\}$ are represented by maps f, g respectively, then the *Whitehead product* $[\mu, \nu] \in \{S^pA * S^qB, Y\}$ is represented by the composition

$$S^pA * S^qB \xrightarrow{w} S^{p+1}A \vee S^{q+1}B \xrightarrow{f \vee g} Y \vee Y \xrightarrow{p'} Y$$

where

$$\begin{aligned} w(a, s, b) &= (b, 1 - 2s) \qquad \text{for } s \leqq \frac{1}{2}, \quad a \in S^pA, \ b \in S^qB \\ &= (a, 2s - 1) \qquad \text{for } s \geqq \frac{1}{2}. \end{aligned}$$

1. The product is natural: if $f\colon Y \to Y'$, then $f_*[\mu, \nu] = [f_*\mu, f_*\nu]$.

Let $A *' B$ denote the usual join, obtained from $A \times I \times B$ by identifying $(a, 0, b) = (a, 0, b')$, $(a, 1, b) = (a', 1, b)$ for all $a, a' \in A$, $b, b' \in B$. Then the (nonreduced) suspension is defined by $S_0A = A *' S^0$, where S^0 is a zero-sphere, and inductively by $S_0^pA = (S_0^{p-1}A) *' S^0$. There is a homeomorphism $S_0^{p+q}(A *' B) \to S_0^pA *' S_0^qB$, obtained by permuting B with the first p copies of S^0. Taking identification spaces, we obtain a homotopy equivalence $S^{p+q}(A * B) \to S^pA * S^qB$. Using this, we may suppose that the Whitehead product is represented by a map of a suspension. Then

2. The product is bilinear for $p, q \geqq 1$.

Let $\iota_i \in \{S^rX, S^rX \vee S^rX\}$ be the class of the inclusion onto the ith factor, $i = 1, 2$.

(2.4) LEMMA. *The image of*

$$\partial\colon \pi_m(S^rX \times S^rX, S^rX \vee S^rX) \to \pi_{m-1}(S^rX \vee S^rX), \quad r \geqq 1, m < 3n + 3r - 1,$$

consists of the elements $[\iota_1, \iota_2] \circ \alpha$, $\alpha \in \pi_{m-1}(S^{r-1}X * S^{r-1}X)$.

Proof. That $[\iota_1, \iota_2] \circ \alpha \in \operatorname{Image} \partial$ is shown in Theorem 7.13 of [5], where it is also shown that there is an isomorphism

$$\xi\colon \pi_m(S(S^{r-1}X * S^{r-1}X)) \to \pi_m(S^rX \times S^rX)$$

such that if α is as above, then there exists $\beta \in \pi_m(S^rX \times S^rX, S^rX \vee S^rX)$ with

$\partial\beta = [\iota_1, \iota_2] \circ \alpha$ and $\theta_*\beta = \xi E\alpha$. Now if θ_* and E are isomorphisms, then given any element β' there exists α such that $\theta_*\beta' = \xi E\alpha$. If β is an element corresponding to α as above, then since θ_* is an isomorphism, $\beta' = \beta$, and therefore $\partial\beta'$ has the desired form. Since θ_* and E are isomorphisms for $m < 3n + 3r - 1$, this proves (2.4).

To calculate $Q'\phi'_*p'_*\partial$: Let $r > 1$, and let $\gamma = [\iota_1, \iota_2] \circ \alpha$ be an element of Image ∂. If $m < 4n + 4r - 2$, then α is a suspension and composition is distributive. Since $p'_*\gamma = [\iota, \iota] \circ \alpha$,

$$\begin{aligned} \phi'_* p'_* \gamma &= [\iota_1 + \iota_2, \iota_1 + \iota_2] \circ \alpha \\ &= [\iota_1, \iota_1] \circ \alpha + [\iota_1, \iota_2] \circ \alpha + [\iota_2, \iota_1] \circ \alpha + [\iota_2, \iota_2] \circ \alpha \\ &\equiv [\iota_1, \iota_2] \circ \alpha + [\iota_2, \iota_1] \circ \alpha \mod \ker Q'. \end{aligned}$$

Hence $\phi'_* p'_*$: Image $\partial \to \pi_{m-1}(S^rX \vee S^rX)/\ker Q'$ is given by $1 + T_*$. Since T_* commutes with ∂ and Q', $Q'\phi'_*p'_*\partial = 1 + T_*$, and

$$Q\phi_* d^1 \lambda i_* p_* = (1 + T_*) \qquad \text{for } r > 1,\ m < 4n + 4r - 2. \tag{2.5}$$

In order to express this result in a better form, we introduce the Hopf homomorphism [7, §15]

$$h\colon \pi_{m+1}(S^{r+1}X; C_+, C_-) \to \pi_{m+1}(S^rX *_0 S^rX)$$

(here $A *_0 B$ denotes the join parametrized as $S(A \times B)$). A short calculation using the definition of h shows that

$$h = E\theta_*(\lambda i_* p_*)^{-1} \qquad \text{for } m < 3n + 3r - 1. \tag{2.6}$$

If we use T_* to denote the automorphism of $\pi_{m+1}(S^rX *_0 S^rX)$ induced by the homeomorphism T of $S^rX \times S^rX$, then (2.5) becomes

$$E\theta_* Q\phi_* d^1 = (1 + T_*)h. \tag{2.7}$$

Recalling that $H = \theta_* Q\phi_*$, we have

(2.8) Proposition.

$$EHd^1 = (1 + T_*)h\colon \pi_{m+1}(S^{r+1}X; C_+, C_-) \to \pi_{m+1}(S^rX *_0 S^rX),$$
$$\textit{for } r > 1,\ m < 3n + 3r - 1.$$

Within this range E and h are isomorphisms, while H is such for $m < 3n + 3r - 2$, so that d^1 is again expressed in terms of T_*, but with the loss of the case $r = 1$.

As a corollary to the above calculations, we have a generalization of G. W. Whitehead's EHP sequence [12]:

(2.9) Proposition. *Let X be $(n-1)$-connected, $n \geqq 2$. Then there is an exact sequence for $r \geqq 1$*

$$\pi_{3n+3r-2}(S^rX) \xrightarrow{E} \cdots \to \pi_m(S^rX) \xrightarrow{E} \pi_{m+1}(S^{r+1}X) \xrightarrow{E^{-2}H'}$$

$$\pi_{m-1}(S^{r-1}X * S^{r-1}X) \xrightarrow{P} \pi_{m-1}(S^rX) \to \cdots.$$

Here $H' = hj_*$ is James' generalized Hopf invariant [7, §15], $P(\alpha) = [\iota, \iota] \circ \alpha$, and we denote by E^{-2} the inverse of the composition $E\xi E$: $\pi_{m-1}(S^{r-1}X * S^{r-1}X) \to \pi_{m+1}(S^rX *_0 S^rX)$.

Proof. Let $m < 3n+3r-1$ throughout. It follows from the suspension triad sequence

$$\cdots \to \pi_m(S^rX) \xrightarrow{E} \pi_{m+1}(S^{r+1}X) \xrightarrow{j_*} \pi_{m+1}(S^{r+1}X; C_+, C_-) \xrightarrow{\Delta} \pi_{m-1}(S^rX) \to \cdots$$

that we need only prove that $PE^{-2}h = \Delta$; i.e. that $\Delta h^{-1}E\xi E = P$. According to Theorem 7.13 of [5], for $\alpha \in \pi_{m-1}(S^{r-1}X * S^{r-1}X)$, $\partial\theta_*^{-1}\xi E\alpha = [\iota_1, \iota_2] \circ \alpha$. Hence

$$\begin{aligned} \Delta h^{-1}E\xi E\alpha &= \Delta\lambda i_* p_* \theta_*^{-1}E^{-1}E\xi E\alpha && \text{by (2.6)} \\ &= p_*' \partial\theta_*^{-1}\xi E\alpha && \text{by (2.3)} \\ &= p_*' [\iota_1, \iota_2] \circ \alpha = [\iota, \iota] \circ \alpha = P(\alpha). \end{aligned}$$

This proves (2.9).

3. **An example.** As a simple example of the use of the spectral sequence we take $X = S^n$, $n > 2$. It is well-known that $T_* = (-1)^m$ in $\pi_q(S^m \times S^m)$, and that $S^m \times S^m$ is homeomorphic to S^{2m}. Hence it follows from (2.1) that for $s < 3n+r-3$,

$$C_{r,s} = \pi_{s+2r}(S^{n+r}; C_+, C_-) \approx \pi_{s+2r+1}(S^{2n+2r}) \qquad \text{for } r > 0,$$

while for $r > 1$ and $s < 3n+r-4$, $d^1_{r,s}$: $C_{r,s} \to C_{r-1,s}$ is given up to isomorphism by the endomorphism $1-(-1)^{n+r}$ of $\pi_{s+2r+1}(S^{2n+2r})$. Recall also that $C_{0,s} = \pi_s(S^n)$.

Let $\mathfrak{C}_2$ be the class of finite Abelian groups with order a power of 2. We distinguish two cases:

1. If n is odd, then $C^2_{r,s} \in \mathfrak{C}_2$ for $r > 0$. $d^1_{2,s}$: $C_{2,s} \to C_{1,s}$ is a $\mathfrak{C}_2$-epimorphism, and the fact that $d^1d^1 = 0$ implies that $d^1_{1,s} = 0 \bmod \mathfrak{C}_2$. Therefore $C^2_{0,s} \approx C_{0,s} = \pi_s(S^n) \bmod \mathfrak{C}_2$.

2. If n is even, then $C^2_{r,s} \in \mathfrak{C}_2$ for $r > 1$. If $s < 3n-3$, then we may calculate $d^1_{1,s}$ by noting that $j_* d^1_{1,s} = j_*\Delta$ is the operator $d^1_{2,s-2}$ in the spectral sequence for S^{n-1}. The latter is a $\mathfrak{C}_2$-isomorphism, and hence $d^1_{1,s}$ is a $\mathfrak{C}_2$-monomorphism. Therefore $C^2_{1,s} \in \mathfrak{C}_2$, and $C^2_{0,s} \approx C_{0,s}/d^1_{1,s}C_{1,s} \approx \pi_s(S^n)/G \bmod \mathfrak{C}_2$, where G is $\mathfrak{C}_2$-isomorphic to $\pi_{s+3}(S^{2n+2}) \approx \pi_s(S^{2n-1})$. Hence we have

(3.1) Proposition. *The stable suspension E_∞: $\pi_s(S^n) \to \Pi_s(S^n)$ is a $\mathfrak{C}_2$-isomorphism if n is odd and $s < 3n-2$, and a $\mathfrak{C}_2$-epimorphism with kernel $\mathfrak{C}_2$-isomorphic to $\pi_s(S^{2n-1})$ if n is even and $s < 3n-3$.*

This is a known result. The restrictions on s can be improved slightly by using less superficial information about H.

PART II. THE STABLE SUSPENSION OF AN EILENBERG-MACLANE SPACE

4. On Postnikov systems. All spaces in this section are to have the homotopy type of CW complexes.

Let E be 1-connected. Recall that a *Postnikov system* for E consists of fiberings $p_n: E_n \to E_{n-1}$, $n=3, 4, \cdots$, together with maps $h_n: E \to E_n$ such that h_n induces homotopy isomorphisms in dimensions $\leqq n$, $\pi_i(E_n)=0$ for $i>n$, and $p_n h_n = h_{n-1}$. Recall also that each fibering $E_n \to E_{n-1}$ may be taken as the fibre space induced from a contractible fibre space over $K(\pi_n, n+1)$, $\pi_n=\pi_n(E)$, by a map $k_n: E_{n-1} \to K(\pi_n, n+1)$ whose homotopy class is determined by the Postnikov invariant(1) $k_n \in H^{n+1}(E_{n-1}, \pi_n)$. If the homotopy groups of E are finite direct sums, $\pi_n \approx \pi_n^1 + \cdots + \pi_n^r$, then we may take $K(\pi_n, n+1) = \mathrm{X}_i\, K(\pi_n^i, n+1)$, and as a contractible fibre space we may take the product of contractible fibre spaces $\mathcal{P}_i \to K(\pi_n^i, n+1)$. One verifies immediately that, with these choices, the resulting fibering $p_n: E_n \to E_{n-1}$ is a composition

$$E_n = E_n^r \to E_n^{r-1} \to \cdots \to E_n^0 = E_{n-1}$$

in which the fibering $p_n^s: E_n^s \to E_n^{s-1}$ is induced from $\mathcal{P}_s \to K(\pi_n^s, n+1)$ by the map $k_n^s = \pi_n^s k_n p_n^1 \cdots p_n^{s-1}$, where $\pi_n^s: \mathrm{X}_i\, K(\pi_n^i, n+1) \to K(\pi_n^s, n+1)$ is the projection.

We shall again call the element $k_n^s \in H^{n+1}(E_n^{s-1}, \pi_n^s)$ determined by the map k_n^s a *Postnikov invariant*; it is equal to the transgression of the fundamental class in the fibering $E_n^s \to E_n^{s-1}$, and is related to k_n by

$$k_n^s = \pi_{n*}^s (p_n^1 \cdots p_n^{s-1})^* k_n \tag{4.1}$$

where $\pi_{n*}^s: H^{n+1}(E_n^{s-1}, \pi_n) \to H^{n+1}(E_n^{s-1}, \pi_n^s)$ is the coefficient homomorphism induced by projection. By reindexing the summands π_n^s for each fixed n, we may assume that for some $t=t(n)$

$$k_n^1, \cdots, k_n^t \text{ are } \neq 0, \quad \text{while } k_n^{t+1} = \cdots = k_n^r = 0. \tag{4.2}$$

We shall always assume that Postnikov systems are constructed in the above fashion.

The fibre F_n^s of $E_n^s \to E_n^{s-1}$ is a $K(\pi_n^s, n)$-space, and the group $\pi_n^s = \pi_n(F_n^s)$ will be identified with a summand of $\pi_n(E_n^s)$ under injection. Lemma (4.2) of [2] generalizes to

(1) In the classifying bundle for $K(\pi, n)$ we shall identify the groups of the fibre $\pi_n(K(\pi, n)) = H_n(K(\pi, n))$ with those of the base $\pi_{n+1}(K(\pi, n+1)) = H_{n+1}(K(\pi, n+1))$ under the Hurewicz homomorphism and the homology suspension. Then the above definition of k_n is the negative of the usual definition (as the obstruction to a cross-section).

(4.3) LEMMA. *$k_n^s=0$ if and only if the Hurewicz homomorphism $\mathcal{H}: \pi_n(E_n^s) \to H_n(E_n^s)$ maps π_n^s monomorphically onto a direct summand.*

Let $h_n^s = p_n^{s+1} \cdots p_n^r h_n: E \to E_n^s$.

(4.4) LEMMA. *In the above Postnikov system for E, if $k_n^{t+1} = \cdots = k_n^r = 0$, then for any m the m-fold suspension maps the summand* $\ker h_{n*}^t \subset \pi_n(E)$, *which is isomorphic to $\pi_n^{t+1} + \cdots + \pi_n^r$, monomorphically onto a direct summand G of $\pi_{n+m}(S^mE)$; and there is a Postnikov system for S^mE*

$$\cdots \to (S^mE)_{n+m} = (S^mE)_{n+m}^{r'} \to (S^mE)_{n+m}^{r'-1} \to \cdots$$

in which $G = \ker h_{n+m*}^{r'-1}$ and $k_{n+m}^{r'}(S^mE) = 0$.

The proof is straightforward, using the fact that

$$E_n = E_n^t \times K(\pi_n^{t+1} + \cdots + \pi_n^r, n).$$

5. **The stable homotopy groups.** Let π be a finitely generated Abelian group, written as $\pi = \sum_{1 \le i \le m} G_i$, where each G_i is cyclic of prime power or infinite order. Then $K(\pi, n)$ has the homotopy type of the product $\mathrm{X}_{1 \le i \le m} K_i$, where$K_i = K(G_i, n)$ is a countable CW complex. According to Theorem 19 of [10],

$$S(\mathrm{X}K_i) \equiv \bigvee S(K_{i_1} \times \cdots \times K_{i_r}), \tag{5.1}$$

where there is one term in the wedge for each index set $(i_1, \cdots, i_r)$ such that $1 \le i_1 < \cdots < i_r \le m$, and r ranges from 1 to m. Hence, using iterated suspension, we conclude that the Postnikov space

$$[S^r(\mathrm{X}K_i)]_{3n+r-1} \equiv [\bigvee S^rK_i \bigvee S^r(K_j \times K_k)]_{3n+r-1}, \tag{5.2}$$

where $1 \le i \le m$, $1 \le j < k \le m$. Since the stable homotopy group of a wedge of spaces is the direct sum of the stable homotopy groups of the spaces,

$$\Pi_q(K(\pi, n)) \approx \sum \Pi_q(K(G_i, n)) + \sum \Pi_q(K(G_j, n) \times K(G_k, n)) \tag{5.3}$$

for $q \le 3n-1$ in which i, j, k have the same range as before. Now

$$\Pi_q(K(G_i, n) \times K(G_j, n)) \approx \pi_q(K(G_i, n) \times K(G_j, n)) \approx \tilde{H}_{q-n}(G_i, n; G_j),$$

the first isomorphism because we are in the stable range, the second by Theorem 6.1 of [13]. Therefore in order to find $\Pi_q(K(\pi, n))$, $q \le 3n-1$, we need only find $\Pi_q(K(G_i, n))$.

For the remainder of the paper, G is to be a cyclic group of prime-power or infinite order. We shall use the fibering of [2] together with the previous lemmas on Postnikov systems to calculate T_* in the p-primary component, $p \ne 2$, of $\pi_q(K(G, n) \times K(G, n))$, $q \le 3n-1$, and shall then use the spectral sequence to find the p-primary component of $\Pi_q(K(G, n))$.

Let X be a countable CW complex which is a $K(G, n+1)$-space, and let

$K=\Omega X$, the space of loops on X. Then the fibering $F\rightarrow E\rightarrow X$ described in [2] is such that there are homotopy equivalences $q\colon E\rightarrow SK$, $h\colon K*K\rightarrow F$, and the following diagram commutes [2, (4.4)]:

(5.4)

$$
\begin{tikzcd}
\pi_i(F) \arrow[r, "i_*"] \arrow[d, "\mathcal{H}"'] & \pi_i(E) \arrow[d, "\mathcal{H}'"] \\
H_i(F) \arrow[d, "\mu"', "\approx"] & H_i(E) \arrow[d, "\approx"', "\nu"] \\
H_{i-1}(K\times K, K\vee K) \arrow[r, "\rho"] \arrow[d, "\lambda_p"'] & H_{i-1}(K) \arrow[d, "\lambda_p'"] \\
H_{i-1}(K\times K, K\vee K; Z_p) \arrow[r, "\rho"] \arrow[d, "\omega"'] & H_{i-1}(K, Z_p) \\
H_{i-n-1}(K, Z_p)\otimes H_n(K, Z_p) + H_n(K, Z_p)\otimes H_{i-n-1}(K, Z_p) \arrow[ur, "m_*"'] &
\end{tikzcd}
$$

Here $\mathcal{H}$, $\mathcal{H}'$ are Hurewicz homomorphisms; λ_p, λ_p' denote reduction mod p; ω is a projection onto a direct summand; and m_* is induced by multiplication in K. $\nu=E^{-1}q_*$, μ is the composition

$$
H_i(F)\xrightarrow{h_*^{-1}} H_i(K*K)\xrightarrow{\phi_*} H_i(S(K\times K))\xrightarrow{j_*} H_i(S(K\times K), S(K\vee K))
$$
$$
\xrightarrow{E^{-1}} H_{i-1}(K\times K, K\vee K)
$$

and ϕ_* and ρ are defined in [2]. Of these homomorphisms we need only the following information: for $2n+2<i\leqq 3n$, the p-primary component $\pi_i(F)_p$ is a Z_p-module which is mapped monomorphically by $\chi'=\omega\lambda_p\mu\mathcal{H}$, the image of χ' is invariant under T_* and i_* is an isomorphism for $i>n+1$. *For the remainder of this section let $2n+2<i\leqq 3n$ and let p be an odd prime.* Let $\mathcal{H}_p'=\lambda_p\mathcal{H}_i'\colon\pi_i(E)\rightarrow H_i(E, Z_p)$. We shall use (5.4) to prove:

(5.5) LEMMA. *A basis $g_1,\cdots,g_s$ for $\pi_i(E)_p$ can be chosen such that $g_1,\cdots,g_r$ is a basis for* ker $\mathcal{H}_p'$ *and $\chi(g_j)$ has the form $b\otimes\gamma_j\pm\gamma_j\otimes b$ for $j=1,\cdots,s$, where $\chi=\chi' i_*^{-1}$ and $b\in H_n(K, Z_p)$ is a generator.*

Proof. Suppose that we have chosen $g_1,\cdots,g_q$ satisfying the lemma. Let g be an element which is not in the subgroup $[g_1,\cdots,g_q]$ spanned by $g_1,\cdots,g_q$, with the further restriction that $g\in\ker\mathcal{H}_p'$ provided $\ker\mathcal{H}_p'\not\subset[g_1,\cdots,g_q]$. Then $\chi(g)=b\otimes\gamma_1+\gamma_2\otimes b$ for some γ_1,γ_2. Since $T_*\chi(g)=(-1)^{ni}(b\otimes\gamma_2+\gamma_1\otimes b)$, by invariance of $\chi(\pi_i(E))$ under T_*, there exists $g'\in\pi_i(E)_p$ such that $\chi(g')=b\otimes\gamma_2+\gamma_1\otimes b$. Then

$$\chi(g+g') = b \otimes (\gamma_1 + \gamma_2) + (\gamma_1 + \gamma_2) \otimes b,$$
$$\chi(g-g') = b \otimes (\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2) \otimes b.$$

Also, it cannot be the case that both $g+g'$ and $g-g'$ lie in $[g_1, \cdots, g_q]$; for then their sum $2g \in [g_1, \cdots, g_q]$ and since $p \neq 2$, g also lies in this subgroup, which violates the hypothesis. Hence we have found at least one new basis element with the desired image under χ, and it only remains to show that if $g \in \ker \mathcal{H}_p'$, then so are $g+g'$ and $g-g'$. But if $g \in \ker \mathcal{H}_p'$, then

$$(5.6) \qquad \lambda'_p \nu \mathcal{H}'(g') = m_* \chi(g') = \pm m_* \chi(g) = \pm \lambda_p' \nu \mathcal{H}'(g) = 0.$$

Hence $\mathcal{H}_p'(g') = 0$, and both $g+g'$ and $g-g'$ lie in $\ker \mathcal{H}_p'$.

We have thus shown that the partial basis $g_1, \cdots, g_q$ can be extended to satisfy the conditions of the lemma. Since $\pi_i(E)_p$ is finitely generated, this proves (5.5).

Let $g \in \pi_i(E)_p$ be one of the generators in Lemma (5.5), with $\chi(g) = b \otimes \gamma + (-1)^q \gamma \otimes b$. If $g \in \ker \mathcal{H}_p'$, then

$$0 = \lambda_p' \nu \mathcal{H}'(g) = m_* \chi(g) = b \cdot \gamma + (-1)^q \gamma \cdot b = (1 + (-1)^{q+ni}) b \cdot \gamma.$$

Now $b \cdot \gamma \neq 0$ [4, Theorem 3], so $(-1)^q = -(-1)^{ni}$. Thus

$$T_* \chi(g) = (-1)^{ni}(\gamma \otimes b - (-1)^{ni}(b \otimes \gamma)) = -\chi(g).$$

On the other hand, if $g \notin \ker \mathcal{H}_p'$, then

$$0 \neq \lambda_p' \nu \mathcal{H}'(g) = m_* \chi(g) = (1 + (-1)^{q+ni}) b \cdot \gamma,$$

so $(-1)^q = (-1)^{ni}$. Then $T_* \chi(g) = \chi(g)$.

We may express the condition $g \in \ker \mathcal{H}_p'$ in another way by considering a Postnikov system in the sense of §4 for SK, up to the term E_{3n}, in which the groups π_i^s are cyclic of prime order (recall that $2n+2 < i \leqq 3n$), and $k_i^s \neq 0$ for $s \leqq t$, $k_i^s = 0$ for $t < s \leqq r$. Suppose that $\pi_i^q = Z_p$ for some $q \leqq t$. Then by (4.3), since $k_i^q \neq 0$, $\mathcal{H}': \pi_i^q \to H_i(E_i^2)$ is not a monomorphism onto a direct summand, and hence $\mathcal{H}_p'$ maps π_i^q to zero.

Now let $p: E_i^r \to E_i^t$ be the projection, with fibre F, and let $g: E_i^t \to E_i^r$ be a cross-section. Then

$$\pi_i(E_i^r) = g_* \pi_i(E_i^t) + i_* \pi_i(F).$$

According to the above, $\mathcal{H}_p'$ is zero on π_i^q for $q \leqq t$, and hence on the p-primary component $g_* \pi_i(E_i^t)_p$. On the other hand, the Postnikov invariant of the fibering $E_i^r \to E_i^t$ is zero, and hence by (4.3) $\mathcal{H}'$ is a monomorphism of $\ker p_* = i_* \pi_i(F)_p$ onto a direct summand, and therefore so is $\mathcal{H}_p'$.

Thus d_i, which is the dimension of $\ker \mathcal{H}_p'$ in $\pi_i(E)_p$, or equivalently the dimension of $\ker \mathcal{H}_p'$ in $\pi_i(E_i^r)_p$, is equal to the number of cyclic invariants $k_i^s \neq 0 \bmod p$.

From the value of T_* on the image of χ we can obtain that of T_* on $\pi_{i-1}(K\times K)_p$ by noting that the image of the monomorphism

$$\pi_{i-1}(K\times K)_p \xrightarrow{\lambda_p\mathcal{H}} H_{i-1}(K\times K, Z_p) \xrightarrow{\approx} H_{i-1}(K\times K, K\vee K; Z_p)$$

followed by ω is equal to the image of χ, and that T_* commutes with this composition. The above calculations yield:

(5.7) Proposition. *Let $K=K(G, n)$, G cyclic of prime power or infinite order. Then $\pi_{i-1}(K\times K)_p$, $2n+2<i\leqq 3n$, admits a cyclic decomposition such that $T_*=-1$ on a submodule of dimension d_i and $T_*=1$ on a supplementary submodule. Here $d_i=\dim\ker\mathcal{H}_p'\colon\pi_i(SK)_p\to H_i(SK, Z_p)$, which is also equal to the number of (cyclic) Postnikov invariants $k_i^s(SK)\neq 0 \bmod p$.*

For the lowest two dimensions, $\mathcal{H}\colon\pi_{2n+q}(K\times K)\to H_{2n+q}(K\times K)$ is an isomorphism, $q=0, 1$; elementary calculation in the homology groups gives:
(5.8) Addendum. $T_*=(-1)^{n+q}$ in $\pi_{2n+q}(K\times K)$, $q=0, 1$.

(5.9) Proposition. *The stable suspension in the p-primary component $E_\infty\colon\pi_i(SK)_p\to\Pi_{i-1}(K)_p$ is an epimorphism for $i\leqq 3n$, p an odd prime. If in addition $i>2n+2$, then $\ker E_\infty$ has dimension d_i. If $i=2n+2$, then $\ker E_\infty=0$ if n is odd, and $\Pi_{i-1}(K)_p=0$ if n is even. If $i=2n+1$, then $\ker E_\infty=0$ if n is even, and $\Pi_{i-1}(K)_p=0$ if n is odd.*

Using (2.1) and the above values for T_*, the result for $i<3n$ can be read off from the spectral sequence. For $i=3n$, note that

$$\pi_{3n}(SK; C_+, C_-)\approx\pi_{3n}(SK)\approx\pi_{3n}(K*K)\approx\pi_{3n+1}(SK\times SK),$$

the first two isomorphisms coming from the suspension triad sequence and the fibering of [2] respectively. Hence $H\colon\pi_{3n}(SK; C_+, C_-)\to\pi_{3n+1}(SK\times SK)$ is an epimorphism of a finite group onto an isomorphic copy, and is therefore an isomorphism. This allows (2.1) to be applied in one higher dimension than would otherwise be possible.

We shall use a Postnikov system for $SK(G, n)$ as in §4, with $E_{n+1}=K(G, n+1)$, in order to determine which cyclic invariants are nonzero mod p. Recall from [2] that the invariants are known in the form $k_i^s=p_i^{s-1*}\bar{k}_i^s$, where $\bar{k}_i^s\in H^{i+1}(K(G, n+1), \pi_i^s)$ and $p_i^{s-1}\colon E_i^{s-1}\to K(G, n+1)$ is the composite fibering. There are two cases: 1. If $G=Z_{p^f}$ then $\pi_{2n+1}=Z_{p^f}=\pi_{2n+2}$, where $\pi_i=\pi_i(SK(G, n))$, while the higher homotopy groups within the range we consider are Z_p-modules. The classes $\bar{k}_{2n+1}$ and $\bar{k}_{2n+2}$ mod p are $b\cdot b$ and $b\cdot\beta(p^f)b$ respectively, where $b\in H^{n+1}(K(G, n+1), Z_p)$ is the basic class mod p, and $\beta(p^f)$ is the fth order Bockstein. The $\bar{k}_i^s$ for $2n+2<i\leqq 3n$ are of the form $b\cdot\theta_i^s$, where θ_i^s runs through a basis for $H^{i-n}(K(G, n+1), Z_p)$. 2. If $G=Z$, then $\pi_{2n+1}=Z$ and $\pi_{2n+2}=0$, while the higher homotopy groups within the range we consider are finite, with p-primary components which are Z_p-modules. $\bar{k}_{2n+1}$ mod p is equal to $b\cdot b$, while the $\bar{k}_i^s$ mod p for $2n+2<i\leqq 3n$ are of the

form $b\cdot \mathrm{St}_p^I b$, where $\mathrm{St}_p^I b$ runs through those elements of the Cartan basis [4] for $H^{i-n}(K(G, n+1), Z_p)$ such that the Steenrod operation St_p^I does not end with a Bockstein. Let $\overline{H}^{i-n}$ denote the subgroup with this basis.

Thus in order to determine which k_i^s are nonzero mod p, we must find the intersection of $H^{n+1}\cdot H^{i-n}$, or $H^{n+1}\cdot \overline{H}^{i-n}$, with the kernel of

$$\overset{*}{\bar{p}}_{i-1}\colon H^{i+1}(K(G, n+1), Z_p) \to H^{i+1}(E_{i-1}, Z_p).$$

Since $\bar{p}_{i-1} = p_{2n+1}^1 \cdots p_{i-1}^r$, we need only compute the kernel of each p_j^{s*} in turn. Now the sequence

$$H^q(K(\overset{s}{\pi}_j, j), Z_p) \xrightarrow{\tau} H^{q+1}(E_j^{s-1}, Z_p) \xrightarrow{\overset{s*}{p_j}} H^{q+1}(\overset{s}{E}_j, Z_p)$$

is exact within the range we are considering, and if $j > 2n+2$, then either π_j^s has order p, or has order prime to p, in which case we may disregard it. If $\pi_j^s = Z_p$, then $H^*(K(\pi_j^s, j), Z_p)$ is, in the stable range, just $A(b_j^s)$, where A is the mod p Steenrod algebra and b_j^s is the basic class. Thus

$$\ker \overset{s*}{p_j} = \tau A(\overset{s}{b}_j) = A\tau(\overset{s}{b}_j) = A(\overset{s}{k}_j) = A(\overset{s-1*}{\bar{p}_j}\overset{-s}{k_j}) = \overset{s-1*}{\bar{p}_j} A(\overset{-s}{k_j}).$$

For the lower dimensions there are two cases:

1. $G = Z_{p^f}$. $H^*(K(Z_{p^f}, j), Z_p)$ is, in the stable range, just $A(b_j) + A(\beta(p^f)b_j)$, where b_j is the basic class mod p (actually we need only use the subalgebra of A generated by those elements which do not have a Bockstein on the right). The kernel of p_{2n+1}^* is therefore $A(b\cdot b) + A(b\cdot\beta(p^f)b)$ if n is odd, and $A(b\cdot b)$ if n is even. Similarly the kernel of $(p_{2n+1}p_{2n+2})^*$ is

$$A(b\cdot b) + A(b\cdot\beta(p^f)b) + A\beta(p^f)(b\cdot\beta(p^f)b).$$

The last term has zero intersection with $H^{n+1}\cdot H^*$, and may therefore be omitted.

2. $G = Z$. Then $\ker p_{2n+1}^* = A(b\cdot b)$.

Putting together the above calculations, we have:

(5.10) Proposition. *For $2n+2 < i \leqq 3n$, p an odd prime, d_i is the dimension of the following Z_p-module, which is a factor module of a submodule of $H^{i+1}(K(G, n+1), Z_p)$:*

$$H^{n+1}\cdot H^{i-n}\Big/ A\left(H^{n+1}\cdot \sum_{n+1}^{i-n-1} H^i\right) \cap H^{n+1}\cdot H^{i-n}$$

where A is the mod p *Steenrod algebra, and*

1. *If $G = Z_{p^f}$, then $H^j = H^j(K(G, n+1), Z_p)$;*
2. *If $G = Z$, then H^j is the subgroup of $H^j(K(G, n+1), Z_p)$ with Cartan basis elements [4] $St_p^I b$ such that the Steenrod operation St_p^I does not end with a Bockstein on the left.*

The explicit calculation of d_i is purely mechanical.

(5.11) EXAMPLE. For m large, the first few stable groups $\Pi_r(K(Z_p, m))$ are:

r	m odd	m even
m	Z_p	Z_p
$2m$	0	Z_p
$2m+1$	Z_p	0
$2m+2(p-1)$	Z_p	0
$2m+2(p-1)+1$	Z_p	Z_p

(5.12) PROPOSITION. *The stable Postnikov invariants* $k_{i-1+r}(S^r(K(\pi, n)))$, $i \leqq 3n$, *r large, are all zero* mod $\mathcal{C}_2$, *the class of finite Abelian groups with order a power of* 2.

Proof. Let $2n+2<i\leqq 3n$, and let p be an odd prime. By (4.4), those Z_p-summands π_i^s of $\pi_i(SK(\pi, n))$ corresponding to zero invariants k_i^s suspend monomorphically, and with a choice of Postnikov system for $S^rK(\pi, n)$, the corresponding invariants $k_{i-1+r}^s(S^rK(\pi, n))$ are zero. But by (5.9), the stable suspension E_∞: $\pi_i(SK(\pi, n))_p \rightarrow \Pi_{i-1}(K(\pi, n))_p$ maps the remaining summands to zero, and is an epimorphism. This proves (5.12) for $i>2n+2$; in the bottom dimensions the proof is essentially the same.

The situation is no longer so simple for the 2-primary component. Firstly, E_∞ is not necessarily an epimorphism on $\pi_r(SK(\pi, n))_2$; for example, it can be seen from the spectral sequence that the cokernel of E_∞ acting on $\pi_{2n+2}(SK(Z_2, n))$ is Z_2. Secondly, the Postnikov invariants are not all zero. A short calculation shows that

$$k_{2n+2}(S^2K(Z_2, n)) = Sq^{n+1}b \in H^{2n+3}(K(Z_2, n + 2), Z_2).$$

This is already in the stable range.

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