SPACES OF FINITE CHARACTERISTIC

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[Received 2 February 1959]

Introduction

It is well known that, if the singular homology groups of a simply-connected space are torsion groups in dimensions greater than 1, then so are the homotopy groups. In fact, the orders of the elements of the qth homotopy group will be bounded if the orders of the homology classes in dimensions not exceeding q are bounded: various conjectures have been made about the relation between the bounds. In this paper a crude upper limit for the bound of the orders of homotopy classes will be found, with more reasonable results for suspensions. It suffices to consider only CW complexes since any space may be replaced by its singular complex (**12**) for these purposes.

Particular interest attaches to spaces with one non-trivial homology group of positive dimension (Moore spaces) since these are the bricks with which a complex of given homotopy type can be constructed from the homology groups, and certain invariants (akin to the Postnikov invariants). These can be represented as suspensions in such a way that the homotopy class of the identity map has finite order under track addition (**1**).

A complex A will be said to have *characteristic* p ($\leqslant \infty$) if the homotopy class of the identity map of its suspension has order p under track addition. The *homology characteristic* of a space will mean the least common multiple of the orders of the (singular) homology classes of positive dimension. Thus the real projective plane has homology characteristic 2 and characteristic 4 [(**1**) Part II]. A connected complex of finite (homological) dimension will be shown to have finite characteristic if and only if it has finite homology characteristic; the latter divides the former, and the former a power of the latter.

A simply-connected complex of finite homology characteristic has the homotopy type of a wedge, or, equally, a direct product, of spaces of mutually prime prime-power homology characteristics; in examining the qth homotopy group it suffices to consider a complex with the same homology groups in dimension up to $q+1$, and no others. Therefore it is sufficient to consider complexes of prime-power characteristic.

Quart. J. Math. Oxford (2), 11 (1960), 124–36.

Let p be a prime, and suppose that A is of characteristic p^m, and $(n-1)$-connected $(n \geqslant 1)$; let E denote suspension. We have the proposition:

PROPOSITION. $p^{mk}\pi_q(EA) = 0$ *if* $q \leqslant 2^k n$. *If A is itself a suspension, then*

$$p^{m+k}\pi_q(EA) = 0 \text{ if } q \leqslant p^{k+1}n.$$

The last result may be best-possible: $\pi_{10}(EA)$ is cyclic of order 8 when A is the 3-fold suspension of the real projective plane. Various crude results can be deduced: for example, if $p^r H_t(A) = 0$ for $1 \leqslant t \leqslant q$, then a space B can be found of characteristic at most $p^{r(q+1)}$, with $\pi_q(EA)$ as a homomorphic image of $\pi_q(EB)$. Again, if A is at least simply connected in the proposition, the first result yields a bound of the order of p^{2mk} for $\pi_q(A)$ if $q \leqslant 2^k(n-1)$; for $q \leqslant 2(n-1)$, A can be replaced by a suspension, and a sharper result can be obtained.

Though spaces of any odd characteristic can be constructed, no space of characteristic 2 is known at present, and it is reasonable to conjecture that none exist. An existence problem of a different kind is this: is there a compactum of finite (covering) dimension and finite homology characteristic which has infinitely many non-zero singular homology groups? Such spaces are known (3) with infinite homology characteristic.

1. Characteristic and homology characteristic

The first propositions relate the two types of characteristic; here 'spaces' mean CW complexes with base points, E denotes suspension, and $\pi_1^A(x)$ the track group of base-point preserving homotopy classes of maps $EA \to X$ (as defined in the Appendix). The homotopy class of the identity map $EA \to EA$ is written $\mathbf{1}_A$. We have the lemmas:

LEMMA 1.1. *If $\mathbf{1}_A$ has order $p < \infty$, then A is connected, and $pH_q(A) = 0$ (all $q > 0$).*

For, if α, β: $EA \to X$, the track-sum $\alpha+\beta$: $EA \to X$ is defined by composing a map ρ: $EA \to EA \vee EA$ with a map $(\alpha \vee \beta)$: $EA \vee EA \to X$ (see Appendix); ρ pinches a middle section $A \subset EA$ to a point. It is easily verified that

$$(\alpha+\beta)_* = \alpha_*+\beta_*\colon H_q(EA) \to H_q(X), \quad \text{for } q > 0.$$

Hence, if $z \in H_q(EA)$,

$$pz = p(\mathbf{1}_{A*}z) = (p\mathbf{1}_A)_*z = 0.$$

Also,

$$H_q(EA) \approx H_{q-1}(A) \quad (q > 1),$$

and $H_0(A) \approx H_1(EA)+Z$ must be free abelian.

LEMMA 1.2. *If A is connected, and has homology characteristic p, the characteristic of A divides the product of the maximum of the orders of the elements of the cohomology groups $H^q(A; Z_p)$ $(q = 1, 2, \ldots)$.*

The lemma has content only when all but a finite number of the cohomology groups are 0. It is shown in (**5**) and [(**1**) Part II] that there is a sequence of groups

$$0 = Q_0, Q_1, \ldots, Q_r, \ldots, Q_\infty = \pi_1^A(X),$$

such that Q_r is a central extension of a subgroup of Q_{r-1} by a factor group of $H^r(A; \pi_{r+1}(X))$ for each $r \geqslant 1$; Q_∞ is the inverse limit of the system of groups Q_r and projections $Q_r \to Q_{r-1}$. The universal-coefficient theorem and the conditions imply that the maximum order of elements of $H^r(A; \pi_{r+1}(X))$ divides that of $H^r(A; Z_p)$.† The lemma follows by induction on r.

For simply-connected complexes of finite dimension, (1.1) and (1.2) imply that finite characteristic and finite homology characteristic are equivalent properties.

LEMMA 1.3. *If A is 1-connected and has homology characteristic p, then for any $q > 1$, $p^s \pi_q(A) = 0$ for some power s (depending on q).*

This follows from Serre's C-theory form of the Hurewicz theorem since the abelian groups whose elements have bounded orders dividing powers of p form a class (**10**).

THEOREM 1.4. *If A is 1-connected and of homology characteristic $p = p_1 \ldots p_k$, where $p_1, \ldots, p_k$ are powers of distinct primes, then there are complexes $A_1, \ldots, A_k$ of homology characteristics $p_1, \ldots, p_k$, respectively, such that A, $\bigvee_{1 \leqslant i \leqslant k} A_i$, $\prod_{1 \leqslant i \leqslant k} A_i$ have the same homotopy type.*

This is a special case of a theorem concerning spaces whose homology groups are torsion groups. If A, B are 1-connected and of mutually prime homology characteristics, the inclusion $A \vee B \subset A \times B$ induces an homology isomorphism, and hence (by a theorem of J. H. C. Whitehead's) is a homotopy equivalence. Therefore $\bigvee A_i$ and $\prod A_i$ must have the same homotopy type.

† Let $M(Q)$ denote the maximum order of the elements of Q. If $pQ = 0$, both $\mathrm{Hom}(Q, G)$ and $\mathrm{Ext}(Q, G)$ are Z_p modules in an obvious way, and it is easily proved that the maximum orders of their elements divide $M(Q)$, and are $M(Q)$ when $G = Z_p$. Since

$$H^r(A; G) \approx \mathrm{Hom}(H_r(A), G) + \mathrm{Ext}(H_{r-1}(A), G),$$

it follows that $M(H^r(A; G))$ divides $M(H^r(A; Z_p))$ for all G when A has finite homology characteristic p.

An equivalence $A \equiv \prod A_t$ can be constructed by considering the Postnikov system of A; the dual method, described in (9), is to construct a sequence of spaces $B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset B_\infty$, and maps $f_r \colon B_r \to A$ such that

(i) $B_t =$ point,

(ii) $f_n \mid B_{n-1} = f_{n-1}$,

(iii) $f_{n*} \colon H_i(B_n) \approx H_i(A) \quad (i \leqslant n)$,†

(iv) $H_i(B_n) = 0 \quad (i > n)$.

B_n is obtained from B_{n-1} by attaching a cone on a Moore space Y_n by means of a map $Y_n \to B_{n-1}$, where the only non-trivial homology groups of Y_n are
$$H_0(Y_n) = Z, \qquad H_{n-1}(Y_n) = H_n(A).$$

Assume, as an inductive hypothesis, that
$$B_{n-1} = \bigvee B_{n-1,t} \quad (1 \leqslant t \leqslant k),$$
where the homology characteristic of $B_{n-1,t}$ divides p_t. We can also choose Y_n in similar form $\bigvee Y_{n,t}$. By the remark above,
$$\pi_q(B_{n-1}) = \pi_q(\bigvee B_{n-1,t}) \approx \pi_q(\prod B_{n-1,t}) = \sum \pi_q(B_{n-1,t}).$$
By (1.3), all maps $Y_{n,t} \to B_{n-1}$ can be deformed into $B_{n-1,t}$ $(1 \leqslant t \leqslant k)$.‡ Therefore B_n can be constructed so as to be a similar wedge $\bigvee B_{n,t}$. Since B_1 is a point, by the principle of induction
$$B_\infty = \bigcup B_n = \bigvee_{1 \leqslant t \leqslant k} (\bigcup B_{n,t}) \equiv A.$$
This completes the proof.

2. Theorems on suspensions

We first prove the lemma:

LEMMA 2.1. *If A has characteristic p, then*
$$pE\pi_q(A) = 0 \quad (E \colon \pi_q(A) \to \pi_{q+1}(EA)).$$

Proof. The track-group functor $\pi_1^A(X)$ is contravariant in A; a homotopy class $\phi \colon A \to B$ induces a homomorphism $\phi^* \colon \pi_1^B(X) \to \pi_1^A(X)$ by $\phi^*(\theta) = \theta \circ E\phi$. Now $\pi_1^{S^q}(X)$ can be identified with $\pi_{q+1}(X)$, so that $\phi^*(1_A) = E\phi$ if $\phi \in \pi_q(A)$. Therefore $pE\phi = (p1_A) \circ E\phi = 0$, which proves the lemma.

† The epimorphism when $i = n$ in (9) can be replaced by an isomorphism.

‡ Let C_t be the class of torsion groups whose orders are prime to p_t. Then $H_q(B_{n-1}, B_{n-1,t})$ are in C_t, and so therefore are $\pi_q(B_{n-1}, B_{n-1,t})$. The inclusion $B_{n-1,t} \subset B_n$ therefore induces an isomorphism between $H^r(Y_{n,t}, \pi_q(B_{n-1,t}))$ and $H^r(Y_{n,t}, \pi_q(B_{n-1}))$, and therefore between the homotopy classes of maps of $Y_{n,t}$ into these spaces.

Information about the kernel of E can be obtained from the exact couple of the suspension, described in (2), and the following theorems (p denotes an arbitrary prime):

THEOREM 2.2. *If A is $(n-1)$-connected and has characteristic p^m, then*

$$p^{mk}\pi_q(EA) = 0 \text{ for } q \leqslant 2^k n \quad (n \geqslant 1).$$

THEOREM 2.3. *If A in (2.2) is itself a suspension, then*

$$p^{m+k}\pi_q(EA) = 0 \text{ for } q \leqslant p^{k+1}n.$$

The proofs will be given in § 5 and § 7 respectively.

3. Collapsed products and Whitehead products

The collapsed product $A \mathbin{⋇} B$ of two spaces A, B with base points is the result of identifying $A \vee B$ in $A \times B$ with the base point. This is a functor: maps $\alpha\colon A \to A'$, $\beta\colon B \to B'$ determine $\alpha \mathbin{⋇} \beta\colon A \mathbin{⋇} B \to A' \mathbin{⋇} B'$. Notice that $⋇$ is associative, and distributive over $\vee$; S^0 acts as a two-sided unit, and the suspension EA can be defined as $A \mathbin{⋇} S^1$. The k-fold collapsed product $A^{(k)}$ is defined inductively by

$$A^{(0)} = S^0, \quad A^{(1)} = A, \quad A^{(k+1)} = A \mathbin{⋇} A^{(k)}.$$

We consider the lemmas:

LEMMA 3.1. *The characteristic of $A \mathbin{⋇} B$ divides the characteristics of A and B.*

In particular, $A \mathbin{⋇} B$ has finite characteristic if either A or B has. This will be deduced from the next lemma, which is proved in the Appendix. Let $\iota_A\colon A \to A$ denote the identity map or its homotopy class indifferently. Then every map (or homotopy class) $\beta\colon EB \to X$ determines $\iota_A \mathbin{⋇} \beta\colon A \mathbin{⋇} EB = E(A \mathbin{⋇} B) \to A \mathbin{⋇} X$.

LEMMA 3.2. *The transformation $\beta \to \iota_A \mathbin{⋇} \beta$ is a homomorphism*

$$\pi_1^B(X) \to \pi_1^{A⋇B}(A \mathbin{⋇} X).$$

Proof of Lemma 3.1. $\qquad \mathbf{1}_A = E\iota_A,$

and $\qquad \mathbf{1}_{A⋇B} = E(\iota_A \mathbin{⋇} \iota_B) = \iota_A \mathbin{⋇} E\iota_B = \iota_A \mathbin{⋇} \mathbf{1}_B.$

By the lemma, $\qquad p(\iota_A \mathbin{⋇} \mathbf{1}_B) = \iota_A \mathbin{⋇} (p\mathbf{1}_B) = 0;$

this completes the proof.

It is convenient to define the Whitehead product

$$[\pi_1^A(X), \pi_1^B(X)] \subset \pi_1^{A⋇B}(X)$$

by means of commutators [cf. (1) § 8.2], generalizing Fox's characterization (7), rather than by Cohen's method (4); the definitions are equivalent for suspensions. The natural maps of $A \times B$ to $A \divideontimes B$, A, B induce monomorphisms by which the groups

$$\pi_1^{A \divideontimes B}(X), \quad \pi_1^A(X), \quad \pi_1^B(X)$$

can be embedded in $\pi_1^{A \times B}(X)$. It is shown in (1) (loc. cit.) that there is a short exact sequence

$$\pi_1^{A \divideontimes B}(X) \to \pi_1^{A \times B}(X) \xrightarrow{i^*} \pi_1^A(X) + \pi_1^B(X),$$

where i^* is induced by the inclusion $A \vee B \subset A \times B$. Then, if

$$\alpha \in \pi_1^A(X), \qquad \beta \in \pi_1^B(X),$$

the product $[\alpha, \beta]$ is to be the commutator

$$[\alpha, \beta] = \alpha + \beta - \alpha - \beta$$

in $\pi_1^{A \times B}(X)$; it lies in the subgroup $\pi_1^{A \divideontimes B}(X) = \mathrm{Ker}(i^*)$.

Remark. If $\theta \colon X \to Y$, then $\theta \circ [\alpha, \beta] = [\theta \circ \alpha, \theta \circ \beta]$.

This product has the important property that, if $\theta \colon X \to Y$ is a bundle mapping of a principal fibre bundle with group G, and

$$\Delta \colon \pi_1^A(X) \to \pi_0^A(G)$$

is the transgression homomorphism (always defined), then

$$\Delta[\alpha, \beta] = \langle \alpha, \beta \rangle,$$

where $\langle \alpha, \beta \rangle$ is the Samelson product in $\pi_0^{A \times B}(G)$: for both products are (or can be) defined as commutators. This leads to the statement of the Milnor–Hilton theorem given in § 4.

Another property that will be used in the proof of (2.3) is that of the lemma:

LEMMA 3.3. *If A is a suspension, so is $A \divideontimes B$, and $\pi_1^A(X)$, $\pi_1^{A \divideontimes B}(X)$ are abelian, while $[\pi_1^A(X), \pi_1^B(X)]$ is linear in $\pi_1^A(X)$. An analogous result applies if B is a suspension.*

The first part is obvious: to prove the linearity, observe that it suffices to show that the subgroup $Q \subset \pi_1^{A \times B}(X)$ generated by $\pi_1^A(X)$, $\pi_1^{A \divideontimes B}(X)$ is abelian since

$$[\alpha_1 + \alpha_2, \beta] = \alpha_1 + [\alpha_2, \beta] - \alpha_1 + [\alpha_1, \beta].$$

Let $\xi \colon A \times B \to Y$ be the identification map pinching B to a point; then $\xi^* \colon \pi_1^Y(X) \approx Q \subset \pi_1^{A \times B}(X)$, and so $\pi_1^Y(X)$ is to be proved abelian. This will follow if Y is shown to have the homotopy type of some suspension.

Now Y, A, $A \divideontimes B$ are 1-connected, and the last two are suspensions: it suffices to construct a map $\eta\colon Y \to A \vee A \divideontimes B$ that induces a homology isomorphism. Let $\rho\colon A \to A \vee A$ be the *canonical pinch* as defined in the Appendix: ρ induces $\rho \times 1\colon A \times B \to (A \vee A) \times B$, and so

$$\rho'\colon Y \to Y \vee Y.$$

The maps $A \times B \to A$, $A \divideontimes B$ pinch B to a point, and so induce maps $Y \to A$, $A \divideontimes B$, and hence a map $\lambda\colon Y \vee Y \to A \vee A \divideontimes B$; it is easily verified that $\eta = \lambda \circ \rho\colon Y \to A \vee A \divideontimes B$ is a suitable map.

Example. Take $X = EA$; then $[\mathbf{1}_A, \mathbf{1}_A] \in \pi_1^{A^{(2)}}(EA)$, and

$$(p\mathbf{1}_A) \circ [\mathbf{1}_A, \mathbf{1}_A] = [p\mathbf{1}_A, p\mathbf{1}_A].$$

When A is a suspension, this is

$$p^2[\mathbf{1}_A, \mathbf{1}_A] = [\mathbf{1}_A, \mathbf{1}_A] \circ (p^2\mathbf{1}_{A^{(2)}}).$$

4. The Milnor-Hilton theorem

In (8) Milnor generalized to the loop space of a wedge of suspensions of connected complexes Hilton's theorem (6) on the loop space of a wedge of simply-connected spheres. Since the statement of the theorem is not readily available, it is stated here in the form in which it will be used. Milnor's construction is more explicit and can yield more information than the bare statement below.

The theorem concerns the homotopy groups of a wedge of suspensions $X = EA_1 \vee \ldots \vee EA_s$, of connected complexes $A_1, \ldots, A_s$. A countable set $\{A_t\}$ $(t = 1, 2, \ldots)$ of complexes, together with homotopy classes of maps $u_t\colon EA_t \to X$ will be constructed so that the following theorem is true:

THE MILNOR–HILTON THEOREM. *$u_{t*}\colon \pi_q(EA_t) \to \pi_q(X)$ is a monomorphism for each $t \geqslant 1$, and*

$$\pi_q(X) = \sum_t u_{t*}\,\pi_q(EA_t).$$

In fact, π_q can be replaced by the track group π_q^B of a *suspension* B.

Construction of A_t, u_t. Construct first symbolic basic products $\omega_1, \omega_2, \ldots$ in symbols $\mathbf{1}, \ldots, \mathbf{s}$; each ω_t will be a certain number $w_t > 0$ (called the *weight* of ω_t) of symbols, bracketed in some way, and will have a non-negative integer c_t (called the *class* of ω_t) associated with it. The construction is not unique, and proceeds inductively.

There are to be s symbolic basic products $\omega_t = \mathbf{t}$ $(t \leqslant s)$ of weight 1; all have class 0. Assume that the symbolic basic products of weight less than m have been defined and are $\omega_1,..., \omega_n$, such that

$$\text{(i)} \quad r < t \text{ implies } w_r \leqslant w_t, \qquad \text{(ii)} \quad c_t < t.$$

Take all symbols $[\omega_p, \omega_q]$ such that

$$\text{(i)} \quad w_p + w_q = m, \qquad \text{(ii)} \quad c_q \leqslant p < q;$$

order these arbitrarily and label them $\omega_{n+1}, \omega_{n+2},\ldots$. These are to be the symbolic basic products of weight m; the class of $[\omega_p, \omega_q]$ is to be p.

Define $A_t = \omega_t(A_1,..., A_s)$ by induction, so that

$$\text{if } t \leqslant s,\ A_t = A_t; \quad \text{if } \omega_t = [\omega_p, \omega_q],\ A_t = A_p \divideontimes A_q.$$

Define maps $u_t = \omega_t(\mathbf{1}_{A_1},..., \mathbf{1}_{A_s})$ by

$$\text{if } t \leqslant s,\ u_t = \mathbf{1}_{A_t}; \quad \text{if } \omega_t = [\omega_p, \omega_q],\ u_t = [u_p, u_q].$$

In general, if α_t: $EA_t \to X$ $(1 \leqslant t \leqslant s)$ are homotopy classes of maps, $\omega_t(\alpha_1,..., \alpha_s)$ can be similarly defined as a repeated Whitehead product.

The theorem leads to the following generalization of a construction of Hilton's (6). Let $A = A_1 = ... = A_s$, and $X = \bigvee_{t \leqslant s} EA_t$ as before. Construct homotopy classes of maps

$$EA \xrightarrow{\rho_s} X \xrightarrow{\mu} EA,$$

so that ρ_s is the track-sum $u_1 + u_2 + ... + u_s$ of the s identity maps $EA \to EA_t$, and $\mu \mid EA_t$ is also the identity map $EA_t \to EA$. Thus $\mu \circ \rho_s = (s\mathbf{1}_A)$. Let ρ_* be the homomorphism induced by ρ_s, i.e.

$$\rho_*\colon \pi_q(EA) \to \pi_q(X) = \sum u_{t*} \pi_q(EA_t).$$

By composing ρ_* with the natural projections, natural homomorphisms M_t are obtained:

$$M_t\colon \pi_q(EA) \to \pi_q(EA_t) \quad (t = 1, 2,...) \tag{4.1}$$

such that $\rho_*(\phi) = \sum u_{t*} M_t(\phi)$, and hence

$$(s\mathbf{1}_A) \circ \phi = \mu_* \rho_*(\phi) = \sum \mu \circ u_t \circ M_t(\phi). \tag{4.2}$$

Notice that, when $t \leqslant s$,

$$\mu \circ u_t \circ M_t(\phi) = \phi.$$

5. Proof of Theorem 2.2

In (4.2), take $s = p^m$; for any $\phi \in \pi_q(EA)$,

$$0 = (p^m \mathbf{1}_A) \circ \phi = p^m \phi + \sum_{t > p^m} \mu \circ u_t \circ M_t(\phi).$$

Now, for $t > p^m$, A_t is at least a 2-fold collapsed product of A, and so at least $(2n-1)$-connected and, by Lemma 3.1, of characteristic dividing p^m. Assume the theorem true for all spaces of all prime-power characteristics, all connectivities n, and all $q \leqslant 2^{k-1}n$. Then, if $q \leqslant 2^k n$, $p^{m(k-1)}\pi_q(EA_t) = 0$ for all $t > p^m$. Thus

$$0 = p^{mk}\phi + \sum_{t>p^m} \mu \circ u_t \circ (p^{m(k-1)}M_t(\phi)) = p^{mk}\phi.$$

Therefore the theorem is also true for $q \leqslant 2^k n$; it is trivially true for $q < n$, and so always true. This proves the theorem.

This procedure will be used later, and called 'induction on k'. The next lemma will also be used:

LEMMA 5.1. *If A is a suspension of characteristic p^m, then for each $1 \leqslant r \leqslant m$, $(p^r\mathbf{1}_A) \circ (p^{m-r}\phi) = 0$ for any ϕ in $\pi_q(EA)$ and all q.*

Notice that this gives Theorem 2.3 when $p = 2$, by induction on k: for

$$0 = (p\mathbf{1}_A) \circ (p^{m-1}\phi) = p^m\phi + \sum \mu \circ u_t \circ (p^{m-1}M_t(\phi));$$

on multiplying by p^k and using the appropriate inductive hypothesis, it follows that $p^{m+k}\pi_q(EA) = 0$ for $q \leqslant 2^{k+1}n$. A more delicate argument is required for the other primes.

Proof of Lemma 5.1. The theorem is true when $r = m$. Notice that, if true for any particular r, then

$$(p^{r+t}\mathbf{1}_A) \circ (p^{m-r}\phi) = 0 \quad \text{for all } t \geqslant 0.$$

Assume that, for some $r \geqslant 1$ and all $1 \leqslant s \leqslant m-1$,

$$0 = (p^{r+s}\mathbf{1}_A) \circ (p^{m-r-s}\phi) = 0.$$

Put, for convenience, $\psi = p^{m-r-1}\phi$, and expand $(p^{r+1}\mathbf{1}_A) \circ \psi$ by (4.2) with $s = p$ as follows:

$$\begin{aligned} 0 &= (p^r\mathbf{1}_A) \circ (p\mathbf{1}_A) \circ \psi \\ &= (p^r\mathbf{1}_A) \circ \Big\{p\psi + \sum_{t>p} \mu \circ u_t \circ M_t(\psi)\Big\} \\ &= (p^r\mathbf{1}_A) \circ p^{m-r}\phi + \Big\{\sum_{t>p} (p^r\mathbf{1}_A) \circ \mu \circ u_t \circ M_t(\psi)\Big\}. \end{aligned} \tag{5.2}$$

Now

$$(p^r\mathbf{1}_A) \circ \mu \circ u_t = \mu \circ \Big(\bigvee_{i\leqslant p} (p^r\mathbf{1}_{A_i}) \circ u_t\Big) = \mu \circ u_t \circ (p^{rw_t}\mathbf{1}_{A_t}),$$

by Lemma 3.4 (cf. example), since A is a suspension.

By the inductive hypothesis (since $w_t > 1$)

$$(p^{rw_t}\mathbf{1}_A) \circ M_t(p^{m-r-1}\phi) = 0 \quad (\text{all } t > p).$$

Therefore (5.2) reduces to $0 = (p^r\mathbf{1}_A) \circ (p^{m-r}\phi)$, and Lemma 5.1 follows by induction on $m-r$.

6. The distributive law

Following Hilton, a distributive law for composition is obtained by taking $s = 2$ in (4.2); in this case, write H_t instead of M_t (conforming with Hilton's notation). If $\alpha, \beta\colon EA \to Y$, $\alpha \vee \beta\colon EA \vee EA = X \to Y \vee Y$; let $\theta\colon X \to Y$ be obtained by collapsing $Y \vee Y$ on Y. Then (4.2) implies

$$(\alpha+\beta) \circ \phi = \theta \circ \rho_2 \circ \phi = \alpha \circ \phi + \beta \circ \phi + \sum_{t>2} \theta \circ u_t \circ H_t(\phi). \quad (6.1)$$

This is the distributive law referred to; the other,

$$\alpha \circ (\phi + \phi') = \alpha \circ \phi + \alpha \circ \phi'$$

is always true and has been used already.

The intention is to expand $(s\mathbf{1}_A) \circ \phi$ by the distributive law; it is necessary to examine $\{(k\mathbf{1}_{A_1}) \vee \mathbf{1}_{A_2}\} \circ u_t\colon EA_t \to EA_1 \vee EA_2$ when A is a suspension.

A symbolic basic product ω_t of weight w_t in symbols $\mathbf{1}$, $\mathbf{2}$ will be said to be of *type* a if there are a $\mathbf{1}$'s and $(w_t - a)\,\mathbf{2}$'s in it. Let $\alpha = k\mathbf{1}_A$, $\beta = \mathbf{1}_A\colon EA \to EA$ define $\theta\colon X = EA \vee EA \to EA$ as before. We have

LEMMA 6.2. $\theta \circ u_t = \mu \circ u_t \circ (k^{a_t}\mathbf{1}_{A_t})$, *where ω_t is of type a_t.*

This follows from the definition and Lemma 3.3 (cf. example), by induction on t. Then (6.1) yields, with Lemma 6.2,

$$\{(k+1)\mathbf{1}_A\} \circ \phi - (k\mathbf{1}_A) \circ \phi - \phi = \sum_{t>2} \mu \circ u_t \circ (k^{a_t}\mathbf{1}_{A_t}) \circ H_t(\phi). \quad (6.3)$$

Sum this for $1 \leqslant k < s$ to obtain the lemma:

LEMMA 6.4. $(s\mathbf{1}_A) \circ \phi - s\phi = \sum_{t>2} \mu \circ u_t \circ \left\{\sum_{k=1}^{s-1} [(k^{a_t}\mathbf{1}_{A_t}) \circ H_t(\phi)]\right\}.$

This is the first step in the reduction of the left-hand side. On expanding the terms in the braces, we obtain invariants of type $H_s H_t(\phi)$, and leave more terms to be expanded. It is convenient to extend the notation temporarily to cover the general case.

Let τ denote an ordered sequence $(t_1, \dots, t_r)$, where $t_i > 2$, and let T be the set of all such sequences ($r = 1, 2, 3, \dots$). Define, by induction on r, spaces A_τ, maps $\bar{u}_\tau\colon EA_\tau \to EA$ and homomorphisms

$$H_\tau\colon \pi_Q(EA) \to \pi_Q(EA_\tau)$$

by setting

$$\left.\begin{array}{llll} A_\tau = A_t, & \bar{u}_\tau = \mu \circ u_t, & H_\tau = H_t, & \text{if } \tau = (t) \\ A_\tau = (A_t)_\nu, & \bar{u}_\tau = \bar{u}_t \circ \bar{u}_\nu, & H_\tau = H_\nu \circ H_t, & \text{if } \tau = (t, \nu) \end{array}\right\}. \quad (6.5)$$

Remark. These constructions are natural; $\bar{u}_t \circ \bar{u}_\nu$ denotes a composition

$$E(A_t)_\nu \to EA_t \to EA.$$

Repetition of Lemma 6.4 yields a formula of type

$$(s\mathbf{1}_A)\circ\phi - s\phi = \sum \bar{u}_\tau \circ \{\sigma_\tau(s) H_\tau(\phi)\},$$

where $\sigma_\tau(s)$ is an integer; it is important to know what these integers are.

Define $\sigma(s;\, \alpha_1,\ldots,\alpha_r)$ for integers $\alpha_1,\ldots,\, \alpha_r$ inductively by

$$\sigma(s;\,\alpha) = \sum_{t=0}^{s-1} t^{\alpha}, \quad \ldots, \quad \sigma(s;\,\alpha_1,\ldots,\alpha_r) = \sum_{t=0}^{s-1} \sigma(t^{\alpha_1};\, \alpha_2,\ldots,\alpha_r). \tag{6.6}$$

If $\tau = (t_1,\ldots,t_r)$, and ω_t is of type a_t, set

$$\sigma_\tau(s) = \sigma(s;\, a_{t_1},\ldots,a_{t_r}). \tag{6.7}$$

Lemma 6.8. *If A is a suspension, $\phi\in\pi_q(EA)$, and A_τ, $\bar{u}_\tau$, H_τ, $\sigma_\tau(s)$ are defined by* (6.5), (6.7), *then*

$$(s\mathbf{1}_A)\circ\phi - s\phi = \sum \bar{u}_\tau \circ \{\sigma_\tau(s) H_\tau(\phi)\},$$

the summation being over all τ in T.

Proof. Assume the lemma true for n-connected suspensions, and fixed q; suppose A $(n-1)$-connected. The lemma applies to $\pi_q(EA_t)$ $(t > 2)$, and follows for $\pi_q(EA)$ from Lemma 6.4 and the inductive hypothesis; it is trivially true when $n > q$, and so true always.

Remark. Only a finite number of $H_\tau(\phi)$ can be non-zero; also, A_τ is a certain collapsed product of A's, and so a suspension.

Lemma 6.9. *If $\tau = (t_1,\ldots,t_r)$, and p is a prime greater than the product of the weights $w_{t_1}\ldots w_{t_r}$ of the symbolic basic products $\omega_{t_1},\ldots,\omega_{t_r}$, then p divides $\sigma_\tau(p)$.*

Proof. Since the type a_t of ω_t is less than w_t, it follows that it suffices to prove that p divides $\sigma(p;\, \alpha_1,\ldots,\alpha_r)$ when $\prod(\alpha_i+1) < p$. Now

$$\sigma(s;\alpha) = \sum_{0}^{s-1} t^{\alpha}$$

is a rational polynomial in s without a constant term, and the denominators of the coefficients are products of powers of primes not exceeding the degree $(\alpha+1)$. Such a polynomial will be called *stable*. Assume that $\sigma(s;\, \alpha_1,\ldots,\alpha_r)$ is a stable polynomial of degree $\prod(\alpha_i+1)$. Then the polynomial

$$\sigma(s;\, \alpha, \alpha_1,\ldots,\alpha_r) = \sum_{t=0}^{s-1} \sigma(t^{\alpha};\, \alpha_1,\ldots,\alpha_r)$$

is also stable, of degree $(\alpha+1)\prod(\alpha_i+1)$. This proves the lemma since $\sigma(s;\,\alpha)$ is of degree $(\alpha+1)$.

7. Proof of Theorem 2.3

By (5.1), $$(p1_A)\circ\psi = 0$$
if $\psi = p^{m-1}\phi$. By (6.8),
$$-p^m\phi = -p\psi = \sum \tilde{u}_\tau \circ \{\sigma_\tau(p)H_\tau(\psi)\}. \tag{7.1}$$
The terms in the braces are of two kinds: if $\tau = (t_1,\ldots,t_r)$ and $p > w_{t_1}\ldots w_{t_r}$, then p divides $\sigma_\tau(p)$ by Lemma 6.9, while, if $p \leqslant w_{t_1}\ldots w_{t_r}$,
$$H_\tau(\psi) \in p^{m-1}\pi_q(EA_\tau),$$
and A_τ is at least $(pn-1)$-connected. In the former case, the term in the braces is a multiple of $p^m H_\tau(\phi)$.

The theorem is trivially true if $q < n$. Assume that it is true for all q and all spaces of connectivity not less than n; suppose that A is $(n-1)$-connected and multiply (7.1) by p^k. The right-hand side becomes zero, and (7.1) becomes $p^{m+k}\phi = 0$. The theorem follows by the principle of induction.

Appendix

To ensure that the product $A \times B$ of complexes is a complex, it is convenient to restrict complexes and maps to the realization of CSS complexes satisfying the extension condition; this imposes no limitation on homotopy type in the category of CW complexes. If A, B are CW complexes with (vertex) base points, $A \divideontimes B$ is obtained by pinching $A \vee B$ to the base point; S' is the unit circle in the Argand plane, and the (reduced) suspension EA of A can be taken to be $A \divideontimes S'$. Define the *canonical pinching map*
$$\rho\colon EA \to EA \vee EA$$
as follows. Let $\rho_0\colon S' \to S' \vee S'$ be given by $\rho(e^{2\pi it}) = e^{4\pi it}$, with values in the first circle if $0 \leqslant t \leqslant \frac{1}{2}$, and in the second if $\frac{1}{2} \leqslant t \leqslant 1$ (1 being the base point of S'). Then $\rho = \iota_A \divideontimes \rho_0$, where $\iota_A\colon A \to A$ is the identity map.

The track-sum of maps α, $\beta\colon EA \to X$ is defined as the composition
$$EA \xrightarrow{\rho} EA \vee EA \xrightarrow{\alpha\vee\beta} X \vee X \xrightarrow{\mu} X,$$
where μ is the folding map, the identity on each copy of X; it is easily verified that this induces an associative addition of homotopy classes, and that addition is commutative if A is a suspension, or has the homotopy type of some suspension.

There is a natural one-to-one correspondence between the (base-point preserving) homotopy classes of maps $EA \to X$ and of $A \to \Omega X$, the loop-space on X. ΩX is an H-space with a homotopy associative multiplication, possessing a homotopy unit and homotopy inverses; it follows that the homotopy classes of maps $A \to \Omega X$ form a group $\pi_0^A(\Omega X)$. The track-addition is such that the correspondence is a homomorphism; thus the set $\pi_1^A(X)$ of homotopy classes of maps $EA \to X$ forms a group under track-addition.

Proof of Lemma 3.2. The naturality of the canonical pinching map ρ implies that

$\rho = \iota_A \divideontimes \rho$:

$$E(A \divideontimes B) = A \divideontimes EB \to E(A \divideontimes B) \vee E(A \divideontimes B) = A \divideontimes (EB \vee EB).$$

Similarly μ is natural:

$$\mu = \iota_A \divideontimes \mu \colon A \divideontimes X \vee A \divideontimes X = A \divideontimes (X \vee X) \to A \divideontimes X.$$

Therefore

$$\begin{aligned} \iota_A \divideontimes \alpha + \iota_A \divideontimes \beta &= \mu \circ (\iota_A \divideontimes \alpha \vee \iota_A \divideontimes \beta) \circ \rho \\ &= \iota_A \divideontimes (\mu \circ (\alpha \vee \beta) \circ \rho) = \iota_A \divideontimes (\alpha + \beta). \end{aligned}$$

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