A NOTE ON THE COHOMOLOGY OF SEMIGROUPS

M. G. Barratt

The semigroups considered are monoids, that is, they possess a unit element 1. There is a functor U from the category of monoids and homomorphisms to the subcategory of groups and homomorphisms, together with a natural transformation

$$u: M \to UM$$

defined as follows.

UM is the quotient group of the free group on the elements of M, by the invariant subgroup generated by all the elements xyz^{-1} for which $xy = z$ in $M (x, y \in M)$; u sends each element of M to its coset in UM. Then $u(M)$ generates UM, and u is an isomorphism if M is a group. Also, a homomorphism f of M into a semigroup N induces a homomorphism $Uf: UM \to UN$ in the obvious way.

It has been conjectured that u induces an isomorphism of cohomology groups

$$u^*: H^*(UM; G) \approx H^*(M; G);$$

this is known to be true if M is free, or if u is a monomorphism and every element of UM can be written as a right quotient $u(m_1)\,u(m_2)^{-1}$ [**1**, p. 191]. An example will be given where

$$H^2(UM; Z) = 0 \neq H^2(M; Z).$$

All topological monoids M possess a classifying space $\overline{W}M$; when M is discrete the topological cohomology groups of $\overline{W}M$ coincide with the abstract cohomology groups of M. It follows from the above example that there is a non-connected monoid M such that

$$(\overline{W}u)_*: \pi_2(\overline{W}M) \to \pi_2(\overline{W}UM)$$

is not a monomorphism. It is still not known if $\overline{W}u$ is a homotopy equivalence if M is connected, as has been conjectured by J. C. Moore.

Let M be an abstract monoid.

Lemma 1. *$u^*: H^2(UM; G) \to H^2(M; G)$ is a monomorphism.*

Proof. When the coefficients are simple, $H^2(M; G)$ can be identified with the equivalence classes of epimorphisms $f: L \to M$ such that

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$$f^{-1}(1) = G \subset \text{centre } (L);$$
$$f(x) = f(y) \Leftrightarrow x \in Gy;$$
$$g \neq g' \in G \Rightarrow gx \neq g'x$$

where $f_i: L_i \to M$ $(i = 1, 2)$ are equivalent if and only if there is a commutative diagram

$$\begin{tikzcd} & L_1 \arrow[dd, "\phi"] \arrow[dr, "f_1"] & \\ G \arrow[ur] \arrow[dr] & & M \\ & L_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

in which ϕ is an isomorphism. If E is a central extension of G by UM in a class α, the class $u^*(\alpha)$ contains an epimorphism $f: L \to M$ such that there is a commutative diagram

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"'] & UL \arrow[r, "\phi"] \arrow[d, "Uf"] & E \arrow[d, "g"] \\ M \arrow[r, "u"] & UM \arrow[r, "1"] & UM \end{tikzcd}$$

where ϕ is an isomorphism which is the identity on $(Uf)^{-1}(1) = G = g^{-1}(1)$. Thus L splits over M implies that E splits over UM. A similar argument proves the lemma when the coefficients are not simple.

Now let M be the monoid generated by a, b, x, y subject to the relations $ax = bx$, $ay = by$; thus UM is a free group on three generators and $H^*(UM; \mathrm{Z}) = 0$. Let L be the monoid generated by A, B, X, Y, P, P^{-1} subject to the relations

$$PP^{-1} = 1 = P^{-1}P,$$
$$PW = WP \quad (W = A, B, X, \text{ or } Y),$$
$$AX = BX, \; AY = PBY,$$

and let $f: L \to M$ be given by

$$f(P) = 1, \quad f(A) = a, \quad f(B) = b, \quad f(X) = x, \quad f(Y) = y.$$

As only P has an inverse, and P commutes with every element, it is easily seen that any element of L has a unique expression as a product of some power of P with a word in A, B, X, Y in which no B is followed by X or Y. Thus f is an epimorphism of the correct kind.

Then clearly $u(A) = u(B)$, $u(P) = 1$, so that $Uf: UL \approx UM$. Therefore L is not isomorphic to the direct sum $M \oplus Z$, and hence the class of the epimorphism f in $H^2(M; Z)$ is not zero, and so not in the image of u^*.

In the case cited, $\pi_2(\overline{W}UM) = 0$, since $\overline{W}UM$ has the homotopy type of three circles attached at a common point. If $\pi_2(\overline{W}M) = 0$, the map $\overline{W}u: \overline{W}M \to \overline{W}UM$ would induce an isomorphism of the groups π_k $(k = 1, 2)$ and so of the groups H_k $(k = 1, 2)$, and hence of the groups H^k $(k = 1, 2)$. Therefore $\pi_2(\overline{W}M) \neq 0$.

This example evolved in a conversation with D. Kan on Moore's conjecture.

Reference.

1. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, 1956).

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Manchester, 13.