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On the Homology of Non-Connected Monoids and Their Associated Groups

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§ 1. Introduction

It is well known that if X is a connected associative H-space of the homotopy type of a CW-complex then X has the structure of an H-group, i.e., $X \simeq \Omega BX$. For X not connected the result fails. However, one can still "adjoin" inverses to X and inquire about the homology algebra of the resulting space and its relation to the homology algebra of X.

Our purpose is to establish, under suitable hypotheses (see 3.5.1), the following isomorphism of Hopf algebras over a field k:

$$H_*(M; k) \,//\, k(\pi_0 M) \overset{\approx}{\to} H_*((UM)_0; k) \tag{1.1}$$

where M is a simplicial free monoid and $(UM)_0$ is the component of the identity of the simplicial group UM generated by M (see 2.2). The action of the monoid algebra $k(\pi_0 M)$ on $H_*(M; k)$ is by translation of components.

As an immediate application we obtain a new proof that the natural map $B\mathscr{S}_\infty \to (\Omega^\infty S^\infty)_0$ induces an isomorphism of Pontryagin algebras

$$H_* B\mathscr{S}_\infty \overset{\approx}{\to} H_*(\Omega^\infty S^\infty)_0 \tag{1.2}$$

where $\mathscr{S}_\infty$ is the infinite symmetric group. Our original proof using Dyer-Lashof operations is given in [2]. This paper arose from our attempt to understand the general phenomenon involved in (1.2).

The paper is organized as follows: In § 2 we give preliminary notions about the Pontryagin algebra and the relationship of M and UM. The main theorem 3.5.1 is formulated in § 3 and our application to $B\mathscr{S}_\infty$ and $(\Omega^\infty S^\infty)_0$ is given in § 4. The principal tool of the paper, the simplicial cobar spectral sequence of Bousfield and Curtis, is developed in § 5. Sections 6 and 7 contain proofs.

§ 2. Preliminaries

We shall work in the category of simplicial sets with basepoint which (unless otherwise noted) satisfy the extension condition of Kan. For the basic facts about this category the reader is referred to May's book [8].

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Let $|\cdot|$ denote the geometric realization functor and recall that if M is a countable simplicial monoid then $|M|$ is a topological monoid. Likewise, if X is a topological monoid and $\mathrm{Sin}(\cdot)$ denotes the total singular complex functor then $\mathrm{Sin}(X)$ is a simplicial monoid.

2.1. *Homology and the Pontryagin Algebra*

Let k be a commutative ring with unit and let $k(\cdot)$ denote the free k-module functor. If X is a simplicial set then its *homology groups with coefficients in k* are defined by $H_*(M; k) = \pi_* k(X)$. For brevity we shall simply write $H_*(X)$. If M is a simplicial monoid then $k(M)$ is also its monoid algebra. In this case $H_*(M)$ has a *(Pontryagin) algebra* structure defined by

$$\begin{aligned} H_i(M) \otimes H_j(M) &= \pi_i k(M) \otimes \pi_j k(M) \xrightarrow{E} \pi_{i+j}(k(M) \otimes k(M)) \\ &\xrightarrow{\pi_{i+j}(m)} \pi_{i+j} k(M) = H_{i+j}(M) \end{aligned}$$

where E is the Eilenberg-Zilber map and m is the multiplication map of $k(M)$. In references to the Hopf algebra structure of $H_*(M)$, we assume k is a field.

2.2. *The Universal Group UM*

If M is any monoid then there is a *universal group UM* generated by M. Let FM be the free group generated by the elements of M and let N be the normal subgroup generated by xyz^{-1} where $x, y, z \in M$ and $xy = z$ in M. Let $UM = FM/N$ and let $u: M \to UM$ be the composite $M \hookrightarrow FM \twoheadrightarrow UM$. Then u is a natural transformation and satisfies the following universal property: if $M \xrightarrow{\varphi} H$ is a homomorphism of monoids and H is a group then there is a unique group homomorphism $\bar{\varphi}: UM \to H$ such that $\varphi = \bar{\varphi} u$

$$\begin{array}{ccc} & & UM \\ & {}^{u}\nearrow & \downarrow \bar{\varphi} \\ M & \xrightarrow{\varphi} & H \end{array}$$

2.3. *The Relation Between M and UM*

Let $\bar{W}$ be the classifying functor of MacLane [8]. We observe that $\bar{W}$ is defined for simplicial monoids as well as simplicial groups. Note, however, that $\bar{W}M$ does not satisfy the extension condition unless M is a simplicial group.

2.3.1. PROPOSITION: *If M is a simplicial free monoid then*

$$(\bar{W}u)_*: H_* \bar{W}M \xrightarrow{\approx} H_* \bar{W}UM$$

is an isomorphism.

Proof: There are canonical first quadrant spectral sequences [10; p. 68]

$$E^2_{p,q} = \pi_p(\mathrm{Tor}_q^{k(M)}(K(k,0), K(k,0)) \Rightarrow \pi_{p+q}k(\bar{W}M))$$
$$E^2_{p,q} = \pi_p(\mathrm{Tor}_q^{k(UM)}(K(k,0), K(k,0)) \Rightarrow \pi_{p+q}k(\bar{W}UM))$$

Since M is free in each dimension so is UM and so $\mathrm{Tor}_q^{(kM_p)}.(k,k) \approx \mathrm{Tor}_q^{k(UM_p)}(k,k)$ and both are zero for $q > 1$ [4]. Hence the spectral sequences agree and collapse ($E^2 = E^\infty$). The result follows.

We shall now show that if M is connected and free then M, UM, and $\Omega B|M|$ have the "same" homotopy type.

2.3.2. THEOREM: *If M is a connected simplicial free monoid then*

$$u: M \to UM$$

is a homotopy equivalence.

The proof is given in § 6.

We now turn to the relation between $|UM|$ and $\Omega B|M|$ where B is the Dold-Lashof classifying space functor for associative H-spaces [6].

2.3.3. LEMMA: *If H is a countable simplicial group, then $|\bar{W}H|$ is naturally homotopy equivalent to $B|H|$.*

Proof: Since $|H| \to E|H| \to B|H|$ is a principal fibration it follows that $\mathrm{Sin}|H| \to$ $\to \mathrm{Sin}\, E|H| \to \mathrm{Sin}\, B|H|$ is a simplicial principal fibration. Now using the five lemma and the classification theorem for simplicial principal bundles [8] we have

$$\pi_* \,\mathrm{Sin}\, B|H| \overset{\approx}{\to} \pi_* \bar{W} \,\mathrm{Sin}\, |H|$$

and since the natural homomorphism $H \to \mathrm{Sin}|H|$ of simplicial groups is a homotopy equivalence, $\pi_* \,\mathrm{Sin} B|H| \overset{\approx}{\to} \pi_* \bar{W}H$. Hence $\mathrm{Sin} B|H| \simeq \bar{W}H$, and thus $|\bar{W}H| \simeq$ $\simeq |\mathrm{Sin} B|H|| \simeq B|H|$.

2.3.4. PROPOSITION: *If M is a countable connected simplicial free monoid then $\Omega B|M| \simeq |UM|$ by an H-map.*

Proof: By 2.3.2, $M \simeq UM$, hence $B|UM| \simeq B|M|$. Thus $\Omega B|M| \simeq \Omega B|UM| \simeq$ $\simeq \Omega|\bar{W}UM|$ by 2.3.3. Now $\Omega|\bar{W}UM| \simeq |G\bar{W}UM| \simeq |UM|$ by H-maps, hence the result.

§ 3. The Main Theorem

3.1. *Basic Assumptions*

Throughout this section we shall assume that M is a countable simplicial free monoid (free in each dimension) satisfying the extension condition. The components $\pi_0 M$ of M are also a monoid, which we assume to be *free.*

3.2. *Action of* π_0

Let $\varphi: M \to K(\pi_0 M, 0)$ be the natural projection. Since $\pi_0 M$ is free there is a monomorphism (cross section) $i: K(\pi_0 M, 0) \to M$ such that $\varphi \circ i = \text{id}$. In this way $\pi_0 M$ acts (uniquely up to homotopy) by multiplication on the right and left of M. We shall be concerned with the right action on homology. The homology ring $H_*(M) = H_*(M; k)$ (see 2.1) is thus a right module over the monoid algebra $k(\pi_0 M)$. We shall say that $H_*(M)$ *splits* if $H_*(M) = H_*(M) // k(\pi_0 M) \otimes k(\pi_0 M)$ as a Hopf algebra.

One easily verifies that $\pi_0 UM = U\pi_0 M$ and so $\pi_0 UM$ is a free group and $H_* UM$ is a right $k(\pi_0 UM)$-module. Let $(UM)_0$ be the component of the identity then

$$(UM)_0 \subsetneq UM \xrightarrow{\varphi} K(\pi_0 UM, 0)$$

is a fibration and $(UM)_0$ is sometimes called the universal cover. Since $UM = (UM)_0 \times K(\pi_0 UM, 0)$ as a simplicial set it follows that $H_* UM = H_*(UM)_0 \otimes k(\pi_0 UM)$ as a k-module.

3.3. *The Induced Map*

If the action of $\pi_0 UM$ on UM is homotopy commutative then $H_* UM = H_*(UM)_0 \otimes k(\pi_0 UM)$ as a Hopf algebra. If in addition $H_* M$ splits then the map

$$H_* M \xrightarrow{u_*} H_* UM = H_*(UM)_0 \otimes k(\pi_0 UM) \xrightarrow{1 \otimes \varepsilon} H_*(UM)_0$$

induces a map of Hopf algebras

$$v_*: H_*(M) // k(\pi_0 M) \to H_*(UM)_0 \tag{3.3.1}$$

3.4. *Strongly Homotopy Commutative Action*

We shall say that the action of $\pi_0 UM$ on UM is *strongly homotopy commutative* (shc) if the composite map

$$|UM| \times |K(\pi_0 UM, 0)| \xrightarrow{1 \times |i|} |UM| \times |UM| \xrightarrow{|m|} |UM|$$

is strongly homotopy multiplicative (shm) in the sense of Sugawara [14]. Recall that a shm map $f: X \to Y$ of associative H-spaces is a family of maps $f^n: X^{n+1} \times I^n \to Y$, $n = 0, 1, 2, \ldots$ such that $f^0 = f$ and

$$\begin{aligned} &f^n(x_0, \ldots, x_n, t_1, \ldots, t_n) \\ &\quad = f^{n-1}(x_0, \ldots, x_{i-1}x_i, \ldots, x_n, t_1, \ldots, \hat{t}_i, \ldots, t_n) \quad \text{if} \quad t_i = 0 \\ &\quad = f^{i-1}(x_0, \ldots, x_{i-1}, t_1, \ldots, t_{i-1})\, f^{n-i}(x_i, \ldots, x_n, t_{i+1}, \ldots, t_n) \quad \text{if} \quad t_i = 1\,. \end{aligned}$$

Such maps induce maps of the classifying spaces $BX \to BY$ ([14, Lemma 2.2]).

3.5. *Main Theorem*

We shall give sufficient conditions for the map $v_*: H_*M//k(\pi_0 M) \to H_*(UM)_0$ of (3.3.1) to be an isomorphism. We remind the reader of our basic assumptions (3.1).

3.5.1. THEOREM: *Let k be a field. If H_*M splits and the action of $\pi_0 UM$ on UM is strongly homoiopy commutative then*

$$v_*: H_*M \,//\, k(\pi_0 M) \xrightarrow{\approx} H_*(UM)_0$$

is an isomorphism of Hopf algebras.

The proof is given in § 7. The strong homotopy commutativity of the action of $\pi_0 UM$ is to insure the convergence of the cobar spectral sequence (see Remark 5.3.1) for UM.

3.6. *The Case $\pi_0 M = Z^+$*

In this case $\pi_0 UM = Z$. Denote the components of M by $M_0, M_1, M_2, \ldots$ and denote the elements of $\pi_0 M$ by $p^0 = 1, p^1, p^2, \ldots$. We are assuming that a cross section $i: \pi_0 M \to$ $\to M$ has been chosen (see 3.2) and so we shall also use $p^0 = 1, p^1, p^2, \ldots$ to denote their images in M. Let $M_\infty = \lim\limits_{\to} M_i$ be the direct limit of

$$M_0 \xrightarrow{xp} M_1 \xrightarrow{xp} \cdots \xrightarrow{xp} M_i \xrightarrow{xp} M_{i+1} \xrightarrow{xp} \cdots$$

where xp denotes right multiplication by p. There is a simplicial map $p^{-\infty}: M_\infty \to (UM)_0$ given by $p^{-\infty} = \lim\limits_{\to}(xp^{-i})$

$$\begin{array}{ccccccc} M_0 & \xrightarrow{xp} & M_1 & \xrightarrow{xp} \cdots \xrightarrow{xp} & M_i & \xrightarrow{xp} \cdots & M_\infty \\ & {\scriptstyle xp^0} \searrow & \downarrow {\scriptstyle xp^{-1}} & & \swarrow {\scriptstyle xp^{-i}} & \swarrow {\scriptstyle p^{-\infty}} & \\ & & & (UM)_0 & & & \end{array}$$

We shall also use p to denote the corresponding element in $k(\pi_0 M) \subset H_*M$.

3.6.1. LEMMA: *If $\pi_0 M = Z^+$ and $k(\pi_0 M)$ is normal in H_*M then $H_*M_\infty =$ $= H_*M//k(\pi_0 M)$ and so H_*M_∞ has the structure of a Hopf algebra, whose coalgebra structure agrees with its natural coalgebra structure.*

Proof: Since $\pi_0 M = Z^+$ the (unit) augmentation ideal $Ik(\pi_0 M)$ is a free $k(\pi_0 M)$-module with basis consisting of the single element $1-p$. Thus, since $k(\pi_0 M)$ is normal, $H_*M//k(\pi_0 M) = H_*M/H_*M \cdot (1-p)$ which is precisely the definition of $\lim\limits_{\to} H_*M_i = H_* \lim\limits_{\to} M_i = H_*M_\infty$.

3.6.2. THEOREM: *Suppose M satisfies the hypotheses of* 3.5.1. *If $\pi_0 M = Z^+$*

then H_*M_∞ has the structure of a Hopf algebra and

$$p_*^{-\infty}: H^*M_\infty \xrightarrow{\approx} H_*(UM)_0$$

is an isomorphism of Hopf algebras.

Proof: In view of 3.5.1 and 3.6.1, it suffices to show that $p_*^{-\infty}$ agrees with v_* of 3.3.1. Now $1\otimes\varepsilon: H_*(UM)_0\otimes k(\pi_0 UM)\to H_*(UM)_0$ may be decomposed as $\sum_{-\infty}^{\infty} p_*^{+i}: \sum_{+\infty}^{\infty} [H_*(UM)_0\otimes p^i]\to H_*(UM)_0$. Hence $(1\otimes\varepsilon)\circ u_* = (\sum_{-\infty}^{\infty} p_*^i)\circ u_*$ and so $v_* = p_*^{-\infty}$.

We say that a simplicial set H is an H-*space object* if its geometric realization $|H|$ is an H-space. If $\bar{W}UM$ is an H-space object then $\pi_0 UM = \pi_1 \bar{W}UM$ is abelian and therefore if $\pi_0 M$ is a free monoid then $\pi_0 UM$ is both free and abelian, i.e., $\pi_0 UM = Z$ and $\pi_0 M = Z^+$.

3.6.3. THEOREM: *Let k be a field. Suppose $\bar{W}UM$ is an* H-*space object and that multiplication by p in H_*M is monic and commutative then*

$$p_*^{-\infty}: H_*M_\infty \xrightarrow{\approx} H_*(UM)_0$$

is an isomorphism of Hopf algebras.

Proof: Since $|\bar{W}UM|$ is an H-space and $|\bar{W}UM| \simeq B|UM|$ (2.3.3) it follows that $B|UM|$ is also an H-space. Hence by a theorem of Sugawara [14, Th. 43], $|UM|\times|UM| \to |UM|$ is strongly homotopy multiplicative and thus by restriction $|UM|\times\pi_0 UM\to |UM|$ is shm and so the action of $\pi_0 UM$ is shc. The hypothesis on multiplication by p easily implies that H_*M splits. The result now follows from 3.6.2.

3.6.4. COROLLARY. *Let $k=Z$. Assume H_*M_∞ and $H_*(UM)_0$ are finitely generated and that the other hypotheses of* 3.6.3 *hold. Then $p_*^{-\infty}$ is an isomorphism of algebras.*

Proof: Let $M_{p^{-\infty}}$ be the mapping cone of $p^{-\infty}: M_\infty \to (UM)_0$ and use the universal coefficient theorem and 3.6.3 to show that $\tilde{H}_*M_{p^{-\infty}}=0$.

§ 4. Application: $\Omega^\infty S^\infty$ and the Infinite Symmetric Group

Let $\mathcal{S}_n$ be the symmetric group of order $n!$, i.e., all permutations of $\{1, 2, \ldots, n\}$. If we consider the elements of $\mathcal{S}_n$ as $n\times n$ permutation matrices then a group homomorphism

$$\mathcal{S}_n \times \mathcal{S}_m \xrightarrow{\mu} \mathcal{S}_{n+m}$$

is defined by juxtaposition of matrices:

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

for $A\in\mathscr{S}_n$, $B\in\mathscr{S}_m$. The composite

$$\mathscr{S}_n \approx \mathscr{S}_n \times \mathscr{S}_1 \xrightarrow{\mu} \mathscr{S}_{n+1}$$

gives a monomorphism $\mathscr{S}_n \to \mathscr{S}_{n+1}$. Using this map, the infinite symmetric $\mathscr{S}_\infty$ is defined by

$$\mathscr{S}_\infty = \varinjlim \mathscr{S}_n$$

Note that $\mathscr{S}_\infty$ is just the group of those infinite permutation matrices acting on $\{1,2,3,\ldots\}$ which permute finite many integers.

We now recall Barratt's Γ construction [1]. For any simplicial set X, Barratt has given a simplicial free group ΓX which is group homotopy equivalent to $G^\infty\Sigma^\infty X$. Thus $|\Gamma S^0|\simeq\Omega^\infty S^\infty$. Explicitly $\Gamma S^0 = U\Gamma^+S^0$ where

$$\Gamma^+S^0 = {}^*\cup\bigcup_{n\geqslant 1}\bar{W}\mathscr{S}_n$$

and where $\bar{W}\mathscr{S}_n$ is short for $\bar{W}K(\mathscr{S}_n, 0)$. For convenience we shall denote the basepoint of $\bar{W}\mathscr{S}_n$ by p^n and set $p^0 = *$. The map

$$\bar{W}\mathscr{S}_n \times \bar{W}\mathscr{S}_m = \bar{W}(\mathscr{S}_n \times \mathscr{S}_m) \xrightarrow{\bar{W}\mu} \bar{W}\mathscr{S}_{n+m}$$

defines an associative multiplication and if we let $*$ be the unit then Γ^+S^0 is a countable simplicial free monoid.

Clearly $\pi_0\Gamma^+S^0 = Z^+ = \{p^0, p^1, p^2, \ldots\}$ and so $M=\Gamma^+S^0$ falls under the province of § 3.6. Observe that

$$(\Gamma^+S^0)_\infty = \varinjlim \bar{W}\mathscr{S}_n = \bar{W}\mathscr{S}_\infty.$$

The following theorem gives our application. Topologically, it says that even though $B\mathscr{S}_\infty$ is not an H-space ($\pi_1 B\mathscr{S}_\infty = \mathscr{S}_\infty$ is not abelian), $H_* B\mathscr{S}_\infty$ nevertheless has a ring structure and that moreover it is isomorphic as a ring to $H_*(\Omega^\infty S^\infty)_0$ via an isomorphism induced by a map $B\mathscr{S}_\infty \to (\Omega^\infty S^\infty)$. Observe that $B\mathscr{S}_\infty$ is an Eilenberg MacLane space $K(\mathscr{S}_\infty, 1)$.

4.1. THEOREM: *Let the coefficient ring k be a field or the integers. Then*

$$p_*^{-\infty}: H_*\bar{W}\mathscr{S}_\infty \xrightarrow{\approx} H_*(\Gamma S^0)_0$$

is an isomorphism of Hopf algebras.

Proof: First consider the case of k a field. Recall the homomorphism of Steenrod [13; p. 53] for coefficients in any group:

$$\kappa: H_i\bar{W}\mathscr{S}_m \to H^{mn-i}(SP^m(S^n))$$

given by $a \to (-1)^{i(i-1)/2}(\iota_m)^m/a$, where n is even, ι_m generates $H^n(SP^m(S^n))$ and $SP^m(S^n)$ is the m-fold symmetric product of S^n. If $i<n$ then κ is an isomorphism [9; Th. 6.7]. It follows that

$$H_* \bar{W}\mathscr{S}_n \to H_* \bar{W}\mathscr{S}_{n+1} \tag{1}$$

is injective and for $* < (n+1)/2$ bijective for coefficients in any group.

Thus multiplication by p in $H_*\Gamma^+ S^0$ is monic. We also claim that such multiplication by p is commutative. The diagram

$$\begin{array}{ccc} \mathscr{S}_n \times \mathscr{S}_1 & \xrightarrow{\tau} & \mathscr{S}_1 \times \mathscr{S}_n \\ & \searrow^{\mu} \quad \swarrow^{\mu} & \\ & \mathscr{S}_{n+1} & \end{array}$$

does not commute. However, if $c : \mathscr{S}_{n+1} \to \mathscr{S}_{n+1}$ denotes conjugation by the cyclic permutation $(1, 2, \ldots, n+1)$ then $c \circ \mu \circ \tau = \mu$. Now conjugation by an element of a group induces a map homotopic to the identity on the classifying space [12]; hence $\bar{W}c \simeq \mathrm{id}: \bar{W}\mathscr{S}_{n+1} \to \bar{W}\mathscr{S}_{n+1}$ and so $\bar{W}(\mu \circ \tau) \simeq \bar{W}\mu$ and hence the claim.

To complete the proof of the theorem observe that $|\bar{W}\Gamma S^0| \simeq B|\Gamma S^0| \simeq B\Omega^\infty S^\infty \simeq \simeq \Omega^\infty \Sigma^\infty(S^1)$ and so $\bar{W}\Gamma S^0$ is an H-space object, thus the hypotheses of Theorem 3.6.3 are satisfied and the result follows.

Let $k=Z$. From (1) we have that $H_* \bar{W}\mathscr{S}_\infty$ is finitely generated. The result now follows from Corollary 3.6.4 since $H_*(\Gamma S^0)_0$ is also finitely generated.

§ 5. The Cobar Spectral Sequence

In this section we describe and expand upon the cobar spectral sequence of Bousfield and Curtis [3; § 10], which will be used to prove the main Theorem 3.5.1. Let k be a commutative ring and let $H_*(\cdot)$ denote homology with coefficients in k (see 2.1).

5.1. *Filtration by Powers of the Augmentation Ideal*

For any monoid M we suppose that the monoid algebra $k(M)$ is given the unit augmentation $\varepsilon: K(M) \to k$ defined by $\varepsilon(m)=1$ for $m \in M$. Then $k(M)$ is a Hopf algebra. Filter $k(M)$ by powers of the augmentation ideal $IM = \ker \varepsilon$

$$k(M) = IM^0 \supset IM^1 \supset IM^2 \supset \cdots \supset IM^p \supset IM^{p+1} \supset \cdots \tag{5.1.1}$$

The associated graded algebra $\sum_{p,q} E^0_{p,q} k(M) = \sum_{p,q} (IM^p/IM^{p+1})_{p+q}$ is also a Hopf algebra.

5.1.2. LEMMA: *Let $T(\cdot)$ denote the tensor algebra functor and let*

$$h: T(IM/IM^2) \to E^0 k(M)$$

be the natural homomorphism of Hopf algebras extending the identity on IM/IM^2. *If* M *is a free monoid on the set* $\{x_\alpha\}$ *then* IM/IM^2 *is a free k-module with basis* $\{x_\alpha - 1\}$ *and h is an isomorphism.*

5.1.3. Before proving the lemma we note that if M is free (on $\{x_\alpha\}$) then it can also be given the zero augmentation $\bar{\varepsilon}: k(M) \to k$ defined by $\bar{\varepsilon}(m)=0$ for $m \neq 1$ in M and $\bar{\varepsilon}(1)=1$. Let $\bar{I}M = \ker \bar{\varepsilon}$ and consider the isomorphism of augmented algebras

$$\varphi:(k(M), \varepsilon) \to (k(M), \bar{\varepsilon})$$

defined by setting $\varphi(m)=m-1$ for $m \neq 1$ in M and $\varphi(1)=1$. Since IM (resp. $\bar{I}M$) is a free $k(M)$-module on $\{x_\alpha - 1\}$ (resp. $\{x_\alpha\}$, (see [4, p. 192]), it follows that $\varphi: IM \to \bar{I}M$ is an isomorphism. Hence $IM^p \simeq \bar{I}M^p$ for $p \geqslant 0$, and if $\bar{E}^0 k(M) = \sum \bar{I}M^p/\bar{I}M^{p+1}$ then $E^0\varphi: E^0 k(M) \xrightarrow{\approx} \bar{E}^0 k(M)$. Finally as bigraded algebras $\bar{E}^0 k(M) \approx T\{x_\alpha\}$.

5.1.4. *Proof of* 5.1.2: Since $IM/IM^2 \overset{\varphi}{\approx} \bar{I}M/\bar{I}M^2 \approx \{x_\alpha\}$ it follows that IM/IM^2 is a free k-module on $\{x_\alpha - 1\}$. That h is an isomorphism is now obvious since

$$\begin{array}{ccc} T(IM/IM^2) & \xrightarrow{h} & E^0 k(M) \\ {\scriptstyle T(\varphi)}\downarrow\approx & & {\scriptstyle E^0\varphi}\downarrow\approx \\ T\{x_\alpha\} = T(\bar{I}M/\bar{I}M^2) & \xrightarrow{\approx} & \bar{E}^0 k(M) \end{array}$$

commutes.

If M is a simplical monoid then the filtration (5.11) of $k(M)$ induces a filtration of the Pontryagin algebra $H_*M = \pi_* k(M)$

$$H_*M = F^0 H_* \supset F^1 H_* \supset \cdots \supset F^p H_* \supset F^{p+1} H_* \supset \cdots \qquad (5.1.5)$$

given by $F^p H_* = \mathrm{Im}\{\pi_* IM^p \to \pi_* k(M)\}$. Let $E^0 H_* M = F^p H_* / F^{p+1} H_*$.

Since H_*M is an augmented k-algebra it is also filtered by powers of its augmentation ideal $IH_* = IH_*M = \pi_* IM$

$$H_*M = IH_*^0 \supset IH_*^1 \supset \cdots \supset IH_*^p \supset IH_*^{p+1} \supset \cdots \qquad (5.1.6)$$

5.1.7. LEMMA: *If k is a field then* $F^p H_* = IH_*^p$ *for* $p \geqslant 0$.

Proof: Clearly $F^1 H_* = IH_*^1$. Now the p-fold multiplication map $IH_* \otimes \cdots \otimes IH_* \to$ $\to IH_*$ factors (see (2.1))

$$IH_* \otimes \cdots \otimes IH_* = \pi_* IM \otimes \cdots \otimes \pi_* IM \xrightarrow{E} \pi_*(IM \otimes \cdots \otimes IM) \xrightarrow{\pi_* m} \pi_* IM = IH_*$$

where E is the p-fold Eilenberg-Zilber map. Since k is a field E is an isomorphism by Kunneth and the result follows.

5.2. *The Spectral Sequence for Simplicial Monoids*

Suppose M is a simplicial monoid. Let $\{E^r(M)\}$ denote the spectral sequence associated with the homotopy exact couple induced by 5.1.1. Then

$$E^1_{p,q}M = \pi_{p+q}(E^0_{p,*}k(M))$$
$$d^r: E^r_{p,q}M \to E^r_{p+r,q-r-1}M$$

and $E^\infty M = E^0 H_* M$, the graded algebra associated with 5.1.5.

5.2.1. LEMMA: *Suppose M is a free simplicial monoid and $\pi_0 M$ is free. Then $\{E^r M\}$ converges to $E^0 H_* M$.*

Proof: We shall show that $\bigcap_r \mathrm{Im}\{\pi_* IM^{p+r} \to \pi_* IM^p\} = 0$. Suppose $\pi_0 M$ is the free monoid on $\{x_\alpha\}$ and let $\bar{I}M$ denote the augmentation ideal of $k(M)$ with the zero augmentation (see 5.1.3.)

Let M_x denote the component of M corresponding to $x \in \pi_0 M$. If $z \in \bar{I}M^p$ then $z = z_{i_1} + z_{i_2} + \cdots + z_{i_n}$ where $z_{i_j} \varepsilon k(M'_{x_{ij}})$. Now let $q \geqslant p$ be an integer such that each $x_{ij}$is a product of less than q elements of $\{x_\alpha\}$. Then if z is a cycle $[z] \notin \mathrm{Im}\{\pi_* \bar{I}M^q \to \to \pi_* \bar{I}M^p\}$ since $\bar{I}M^q$ contains only elements of $k(M_x)$ where x is a product of at least q elements of $\{x_\alpha\}$. Hence $\bigcap_r \mathrm{Im}\{\pi_* IM^{p+r} \to \pi_* IM^p\} \approx \bigcap_r \mathrm{Im}\{\pi_* \bar{I}M^{p+r} \to \pi_* \bar{I}M^p\} = 0$.

For M connected we can apply the connectivity results of Curtis [5; Remark 4.10] and argue in the manner of Quillen [11; Theorem 3.7] to obtain

5.2.2. LEMMA: *If M is a connected simplicial free monoid then IM^p is $p-1$ connected and $\{E^r M\}$ converges strongly to $E^0 H_* M$.*

5.3. *The Spectral Sequence for Simplicial Groups*

If F is a simplicial group then it is also a simplicial monoid and so the spectral sequence of 5.2 is defined. The question of convergence, however, is apparently more delicate and we have been unable to prove a result for F similar to Lemma 5.2.1.

5.3.1. *Remark:* The hypothesis of the main Theorem 3.5.1 requiring the action of $\pi_0 UM$ on UM to be strongly homotopy commutative is necessary precisely because we don't know if $\{E^r(F)\}$ converges for a non-connected simplicial free group F. This hypothesis (in the presence of 3.1) forces the existence of a simplicial group homomorphism $UM \to (UM)_0 \times K(\pi_0 UM, 0)$ which is also a weak equivalence. We can then use the following result of Bousfield and Curtis [3; Th. 10.2]:

5.3.2. LEMMA: *If F is a connected simplicial free group then IF^p is $p-1$ connected and $\{E^r(F)\}$ converges strongly to $E^0 H_*(F)$.*

If F is free then the natural map $G\bar{W}F \to F$ is a group homotopy equivalence.

Furthermore, since $F/[F, F] \approx IF/IF^2$ we have

$$E^1_{p,q}F = (\otimes^p \tilde{H}_* \bar{W}F)_{2p+q} \tag{5.3.3}$$

for k a field.

Finally the spectral sequences for M and UM agree.

5.3.4. LEMMA: *If M is a simplicial free monoid then*

$$E^r u : E^r M \approx E^r UM \quad \text{for} \quad 0 \leqslant r$$

Proof: It suffices to prove the result for $r=0$. By 5.1.2 $E^0 M = T(IM/IM^2)$ and by [3; Lemma 10.1] $E^0 UM = T(IUM/IUM^2)$. However $IM/IM^2 = \mathrm{Tor}^{k(M)}(k, k) \approx \approx \mathrm{Tor}^{k(UM)}(k, k) = IUM/(IUM)^2$ [4; p. 192] and hence the result.

5.4. *Remark*

It should be noted (see [3, 10.3]) that the cobar spectral sequence (at least in the connected case) is closely related to Adams' cobar construction and the Eilenberg-Moore spectral sequence.

§ 6. Proof of Theorem 2.3.2

6.1. LEMMA: *If M is a connected simplicial free monoid then*

$$u_* : H_*(M) \overset{\approx}{\to} H_*(UM)$$

is an isomorphism of Hopf algebras.

Proof: Since M is connected the spectral sequences

$$\begin{array}{ccc} E^1(M) & \Rightarrow & E^0 H_*(M) \\ \downarrow E^1 u & & \downarrow E^0 u_* \\ E^1(UM) & \Rightarrow & E^0 H_*(UM) \end{array}$$

of § 5 are strongly convergent by 5.2.2 and 5.3.2. Now $E^1 u : E^1(M) \overset{\approx}{\to} E^1(UM)$ by 5.3.4 and the result follows.

6.2. *Proof of Theorem* 2.3.2

Since $u : M \to UM$ is a map of connected simplicial monoids the result follows from 6.1 by a simplicial "H-space" version of the Whitehead theorem.

§ 7. Proof of the Main Theorem 3.5.1

Throughout this section we shall assume the hypotheses of 3.5.1. Thus k is a field and $H_*(\cdot)$ denotes homology with coefficients in k.

7.1. LEMMA: *There is a homomorphism of simplicial groups*

$$\psi: UM \to (UM)_0 \times K(\pi_0 UM, 0)$$

which is a weak homotopy equivalence.

7.2. LEMMA: *The map* $u: M \to UM$ *induces an isomorphism*

$$E^0 u_*: E^0 H_* M \overset{\approx}{\to} E^0 H_* UM$$

of the graded algebras associated with filtration (5.1.6) *by powers of the augmentation ideal.*

7.3. *Proof of* 3.5.1: By the splitting of H_*M and the homotopy commutativity of the action of $\pi_0 UM$ we have

$$H_*M \approx H_*M \,//\, k(\pi_0 M) \otimes k(\pi_0 M)$$
$$H_*UM \approx H_*(UM)_0 \otimes k(\pi_0 UM)$$

as augmented algebras. Thus $u_*: H_*M \to H_*UM$ becomes

$$H_*M \,//\, k(\pi_0 M) \otimes k(\pi_0 M) \xrightarrow{\bar{u}_* \otimes k(\pi_0 u)} H_*(UM)_0 \otimes k(\pi_0 UM)$$

where $\bar{u}_*: H_*M//k(\pi_0 M) \to H_*(UM)_0 \otimes k(\pi_0 M)$ is the restriction of u_* to $H_*M// k(\pi_0 M) \otimes 1$ and $k(\pi_0 u): k(\pi_0 M) \to k(\pi_0 UM)$ is obtained by applying the monoid ring functor $k(\cdot)$ to the inclusion $\pi_0 u: \pi_0 M \to \pi_0 UM$.

Also

$$E^0 H_* M \approx E^0(H_*M \,//\, k(\pi_0 M)) \otimes E^0(k(\pi_0 M))$$
$$E^0 H_* UM \approx E^0(H_*(UM)_0) \otimes E^0(k(\pi_0 UM))$$

and so we claim $E^0 \bar{u}_* = E^0 v_*: E^0(H_*M//k(\pi_0 M)) \to E^0 H_*(UM)_0$. This is the key step: to prove it we observe that if $[m] \in H_i M_x$ where M_x is the component of M corresponding to $x \in \pi_0 M$ then $\bar{u}_*([m]) = [m \cdot s_0^i x^{-1}] \otimes [x] \in H_i(UM)_0 \otimes k(\pi_0 UM)$, however $[1] - [x] \varepsilon IK(\pi_0 UM)$ and hence the claim.

Thus

$$E^0 u_* = E^0 v_* \otimes E^0 k(\pi_0 u): E^0(H_*M \,//\, k(\pi_0 M)) \otimes E^0 k(\pi_0 M)$$
$$\overset{\approx}{\to} E^0 H_*(UM)_0 \otimes E^0 k(\pi_0 UM)$$

which is an isomorphism by 7.2. Now since $\pi_0 UM = U\pi_0 M = $ free group, we have

$$E^0 k(\pi_0 u): E^0 k(\pi_0 M) \overset{\approx}{\to} E^0 k(\pi_0 UM)$$

by 5.3.4 and thus $E^0 v_*$ is an isomorphism. To complete the proof observe that the filtrations of $H_*M//k(\pi_0 M)$ and $H_*(UM)_0$ are complete (actually finite in each dimension by connectedness) and so $v_*: H_*M//k(\pi_0 M) \xrightarrow{\approx} H_*(UM)_0$ is also an isomorphism.

Proof of 7.1: Let UM' denote the simplicial group $(UM)_0 \times K(\pi_0 UM, 0)$. Applying the geometric realization functor to the multiplication map

$$m: UM' \to UM$$

we obtain (by hypotheses of 3.5.1) a shm map of countable CW-groups, $|m|: |UM'| \to \to |UM|$. Hence by Sugawara [14; Lemma 2.2] there is a map of fibre spaces

$$\begin{array}{ccccc} |UM| & \to & E|UM| & \to & B|UM| \\ \uparrow{\scriptstyle |m|} & & \uparrow{\scriptstyle |\bar{m}|} & & \uparrow{\scriptstyle |\tilde{m}|} \\ |UM'| & \to & E|UM'| & \to & B|UM'| \end{array}$$

Since $|m|$ is a homotopy equivalence so is $|\tilde{m}|$. By 2.3.3, $|\bar{W}H|$ is naturally homotopy equivalent to $B|H|$ for any countable simplicial group H. Hence there is a homotopy equivalence $\varphi: \bar{W}UM' \to \bar{W}UM$ and a group homotopy equivalence $G\varphi: G\bar{W}UM' \to G\bar{W}UM$. Let $\varrho: G\bar{W}UM \to G\bar{W}UM'$ be a group homotopy inverse of $G\varphi$ and let $\psi: UM \to UM'$ be the composite

$$UM \xrightarrow{\eta} G\bar{W}UM \xrightarrow{\varrho} G\bar{W}UM' \xrightarrow{\operatorname{adj}(1\ \bar{W}_{UM'})} UM'$$

where η is a group homotopy inverse of the adjoint map $\operatorname{adj}(1_{\bar{W}UM}): G\bar{W}UM \to UM$.

Since each of η, ϱ, and $\operatorname{adj}(1_{\bar{W}UM'})$ is a group homomorphism and a weak equivalence so is ψ.

Proof of 7.2: The maps $M \xrightarrow{u} UM \xrightarrow{\psi} (UM)_0 \times K(\pi_0 UM, 0)$ induce maps of the cobar spectral sequences

$$\begin{array}{lll} E^1(M) & \Rightarrow E^0 H_* M & (1) \\ \downarrow{\scriptstyle E^1 u} & \quad\downarrow{\scriptstyle E^0 u_*} & \\ E^1(UM) & \Rightarrow E^0 H_* UM & (2) \\ \downarrow{\scriptstyle E^1 \psi} & \quad\downarrow{\scriptstyle E^0 \psi_*} & \\ E^1((UM)_0 \times K(\pi_0 UM, 0)) & \Rightarrow E^0 H_*((UM)_0 \times K(\pi_0 UM, 0)) & (3) \end{array}$$

where the convergence of (2) is unknown. The convergence of (1) follows from 5.2.1. Since $E^r((UM)_0 \times K(\pi_0 UM, 0)) \approx E^r(UM)_0 \otimes E^r(K(\pi_0 UM, 0))$ as differential algebras and since $E^0(K(\pi_0 UM, 0)) \approx E^\infty(K(\pi_0 UM, 0))$ the convergence of (3) follows from 5.3.2. Now $E^1(M) \xrightarrow{\approx} E^1(UM)$ by 5.3.4 and $E^1(UM) \xrightarrow{\approx} E^1((UM)_0 \times K(\pi_0 UM, 0))$ by 7.1 and 5.3.3. Hence by the convergence of (1) and (3) $E^0(\psi u)_*$ is an isomorphism. But $E^0\psi_*$ is an isomorphism by 7.1 and so $E^0 u_*$ is also an isomorphism.

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Added in proof: J. P. May points out that the assumption of countability can be omitted here and thus in the rest of the paper by restricting the range of $1 \cdot 1$ to the category of compactly generated spaces.