

# TRACK GROUPS (I)

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## Introduction

A *Track Group*  $(P, Q)^m(X, x_0; x_0)$  is, for a large class of spaces, the  $m$ th homotopy group of the function space  $F$  of maps of  $P$  into  $X$  which carry the closed (possibly empty) subset  $Q$  to  $x_0$ , a point of  $X$ ; the base point is, in general, the trivial map  $x_0$ , such that  $x_0(P) = x_0$ . These homotopy groups of function spaces have been studied by S. T. Hu (13) and by S. Wylie.

Following S. Wylie, to whom the concept of a track group is due, we define the group directly by means of maps of  $P \times I^m$  into  $X$ , which carry  $(Q \times I^m \cup P \times I^m)$  to  $x_0$ . Though  $(P, Q)^m$  depends only on the homotopy type of the pair  $[P, Q]$  (that is, not on the particular pair  $[P, Q]$  chosen) it is not true in general that it depends only on the homotopy type of  $[P^*, Q^*]$ , where  $P^*, Q^*$  are the  $m$ -fold suspensions of  $P, Q$ , respectively.

If  $P \supset Q \supset R$ , and  $P, Q, X$  satisfy certain conditions, there is an exact sequence

$$\dots \rightarrow (P, Q)^m \rightarrow (P, R)^m \rightarrow (Q, R)^m \rightarrow (P, Q)^{m-1} \rightarrow \dots \rightarrow (Q, R)^1,$$

where the space  $X$  is the same throughout; the last homomorphism maps into  $(Q, R)^1$ .

We use this exact sequence to determine the elements of various track groups; in particular, we obtain an alternative proof of Fox's theorem on the isomorphic embedding of homotopy groups in the Torus homotopy groups. We then apply these methods to the determination of the elements of track groups of CW complexes. These have already been shown to be solvable by S. T. Hu (13). If  $K^q$  is the  $q$ -skeleton of a complex  $K$ , we determine the elements of  $(K^{n+1}, K^{n-2})^m$  in terms of the cohomology groups of  $K$ , and the Steenrod squares, and the homotopy groups of  $X$ . We also show that  $(K^{n+1}, K^{n-1})^m(X, x_0; x_0)$  is a central extension of

$$H^{n+1}(K^{n+1}, K^{n-1}; \pi_{m+n+1})$$

by  $H^n(K^{n+1}, K^{n-1}; \pi_{m+n})$ , where  $\pi_q = \pi_q(X, x_0)$ , except when  $n = 0$ , when it is a split extension.

In a subsequent paper we shall determine these group extensions when the complex is finite, and extend the results to normal spaces that are compact or paracompact.

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## CHAPTER I

### GENERAL TRACK GROUPS

#### 1.1. Basic notations

Let  $[P_\lambda], [X_\lambda]$  be two sets of spaces indexed by the same set  $\Lambda$ ; no two spaces need be disjoint. A map  $f: [P_\lambda] \rightarrow [X_\lambda]$  is a set of continuous maps  $f_\lambda: P_\lambda \rightarrow X_\lambda$ , such that for all  $\lambda, \mu \in \Lambda$ ,  $f_\lambda$  and  $f_\mu$  agree on the intersection  $P_\lambda \cap P_\mu$ . Notice that, if  $P_\lambda \subset P_0$  and  $X_\lambda \subset X_0$  for all  $\lambda$ , then a map  $f_0: P_0 \rightarrow X_0$  such that  $f_0(P_\lambda) \subset X_\lambda$  for all  $\lambda$  induces a map  $f: [P_\lambda] \rightarrow [X_\lambda]$ .

If  $T, S$  are arbitrary spaces, and  $f: [P_\lambda] \rightarrow [X_\lambda]$ , we shall write

$$f^!: [P_\lambda \times T, P_\lambda \times S] \rightarrow [X_\lambda \times T, X_\lambda \times S]$$

for the map defined by  $f^!_\lambda(p, t) = (f_\lambda(p), t)$  for all  $p$  in  $P_\lambda$ , and for all  $t$  in  $T$  and all  $\lambda$  in  $\Lambda$ .

$I^m$  shall be the  $m$ -cube, consisting of all points  $(y_1, \dots, y_m)$  in Euclidean  $m$ -space such that  $0 \leq y_i \leq 1$  for each  $i$ . We embed Euclidean  $m$ -space in Euclidean  $n$ -space for every  $n > m$ , in the obvious way.  $\dot{I}^m$  is the boundary of  $I^m$ , consisting of all points  $(y_1, \dots, y_m)$  such that  $y_j = 0$ , or  $1$ , for some  $j$ . Since  $I^{m-1}$  has been embedded in  $I^m$  as the subset  $y_m = 0$ , we have a closed  $(m-1)$ -cell  $J^{m-1}$ , defined as the closure of  $\dot{I}^m - I^{m-1}$ ; this clearly meets  $I^{m-1}$  in  $\dot{I}^{m-1}$ . We shall often write  $I$  for the unit interval  $I^1$ .

#### 1.2. Homotopy type

Two maps  $f_0, f_1: [P_\lambda] \rightarrow [X_\lambda]$  are *homotopic* if there is a map

$$F: [P_\lambda \times I] \rightarrow [X_\lambda]$$

such that, for all  $\lambda$ , all  $p \in P_\lambda$ , and for  $t = 0, 1$ ,  $F_\lambda(p, t) = f_{t,\lambda}(p)$ . We use this last formula to define  $f_{t,\lambda}$ , and so  $f_t$ , for every  $t \in I$ . We write  $f_0 \simeq f_1$ , and call indifferently the map  $F$  or the family of maps  $f_t$  a *homotopy*. It is easily seen that  $f \simeq g$  defines an equivalence relation between maps, and we shall use curly brackets to denote the homotopy class  $\{f\}$  of a map  $f$ , and write  $(P_\lambda)^0(X_\lambda)$  for the set of homotopy classes of maps  $[P_\lambda] \rightarrow [X_\lambda]$ .

Let  $1$  stand for the identity maps  $[P_\lambda] \rightarrow [P_\lambda]$ ,  $[Q_\lambda] \rightarrow [Q_\lambda]$ . Then, the composite of maps  $f \circ g$  being defined in the obvious way, two maps  $f: [P_\lambda] \rightarrow [Q_\lambda]$ ,  $g: [Q_\lambda] \rightarrow [P_\lambda]$  such that  $f \circ g \simeq 1$ ,  $g \circ f \simeq 1$ , are called

*homotopy equivalences*, and each is said to be a (left and right) homotopy inverse to the other. We write  $f: [P_\lambda] \equiv [Q_\lambda]$ ,  $g: [Q_\lambda] \equiv [P_\lambda]$ , and  $[P_\lambda], [Q_\lambda]$  are said to be of the same *homotopy type*. Fox's theorem (8), that a map with a left and a right homotopy inverse is a homotopy equivalence, applies to these maps.

### 1.3. Factor spaces

Let  $Q$  be a closed subspace of  $P$ . The *factor space*  $P/Q$  ('Zerlegensraum' (2, 4)) is defined as follows. If  $Q$  is the empty set, written  $Q = 0$ , then  $P/Q = P$ , and  $\phi: P \rightarrow P/Q$  is the identity map. If  $Q$  is not empty, take a point  $q$  not in  $P - Q$ , and let  $P/Q$  be the point set  $(P - Q) \cup q$ ; let  $\phi: P \rightarrow P/Q$  be the identity map on  $P - Q$ , and map  $Q$  to the point  $q$ .  $P/Q$  is then given the identification topology under  $\phi$  (see (23, 4)), so that a set  $V \subset P/Q$  is open if and only if  $\phi^{-1}(V)$  is open. It is well known that, if  $f: P \rightarrow X$ , then  $f\phi^{-1}: P/Q \rightarrow X$  is continuous if and only if it is single-valued, i.e. if  $f(Q)$  is a single point.

### 2.1. Track groups

We now define what we call a *track group*, and write  $(P_\lambda)^m(X_\lambda; x_0)$ . For a large class of spaces,

$$(P_\lambda)^m(X_\lambda; x_0) \approx \pi_m(F, x_0),$$

where  $F$  is the function space of maps of  $[P_\lambda]$  to  $[X_\lambda]$ , and  $x_0$  is the trivial map given by  $x_0(P_\lambda) = x_0$  for all  $\lambda$ . Here  $F$  is given the compact-open topology, that is, a sub-base for the open sets consists of the sets of maps which carry a compact set  $Q_\lambda \subset P_\lambda$  into an open set  $U_\lambda \subset X_\lambda$ , for all choices of  $\lambda$ ,  $Q_\lambda$ , and  $U_\lambda$ .

Let  $x_0$  be a point in every  $X_\lambda$ , and define  $x_\lambda = x_0$  for every  $\lambda$ ; then an element of  $(P_\lambda)^m(X_\lambda; x_0)$  is a homotopy class of maps

$$f: [P_\lambda \times I^m, P_\lambda \times I^m] \rightarrow [X_\lambda, x_\lambda],$$

so that there is a natural 1-1 correspondence between the elements of  $(P_\lambda)^m(X_\lambda; x_0)$  and of  $(P_\lambda \times I^m, P_\lambda \times I^m)^0(X_\lambda, x_\lambda)$ .

We shall write the track group additively, although it is not necessarily abelian if  $m = 1$ , and define the addition of two classes  $\{f\}, \{g\}$  by

$$\{f\} + \{g\} = \{f + g\},$$

where the map  $f + g$  is given by

$$(f + g)_\lambda(p, y_1, \dots, y_m) = \begin{cases} f_\lambda(p, 2y_1, y_2, \dots, y_m) & \text{if } 0 \leq y_1 \leq \frac{1}{2}, \\ g_\lambda(p, 2y_1 - 1, y_2, \dots, y_m) & \text{if } \frac{1}{2} \leq y_1 \leq 1, \end{cases}$$

for all  $p \in P_\lambda$ , all  $\lambda$ . Formally, except for the presence of  $p$ , this is the same

definition as that of the operation of addition in homotopy groups, using cubes as antecedent spaces (see (10, 11)), and we find at first a close parallelism.

It may be verified by the usual means that  $f+g$  is a map in the sense defined, that its homotopy class is independent of the choices of  $f$  and  $g$  in their homotopy classes, and so defines an addition of homotopy classes. Similarly, we may verify that the addition is associative, commutative if  $m > 1$ , and that the class of the trivial map  $x_0$  is the identity element. Also, each class  $\{f\}$  has a unique inverse,  $-\{f\}$ , which is the class containing the map  $f^-$  given by

$$f^-(p, y_1, \dots, y_m) = f_\lambda(p, 1-y_1, y_2, \dots, y_m).$$

Thus this definition of addition turns  $(P_\lambda)^m(X_\lambda; x_0)$  into a group.

## 2.2. Induced homomorphisms

We define certain natural homomorphisms, and deduce that the track groups are invariants of the homotopy types of the sets of spaces involved in their definition.

Let  $\phi: [P_\lambda] \rightarrow [Q_\lambda]$ ; then  $\phi$  induces a map

$$\phi^b: [P_\lambda \times I^m, P_\lambda \times I^m] \rightarrow [Q_\lambda \times I^m, Q_\lambda \times I^m]$$

defined in (1.1), and it follows at once from the definition of addition that if we set  $\phi^*\{f\} = \{f \circ \phi^b\}$ , we have a homomorphism

$$\phi^*: (Q_\lambda)^m(X_\lambda; x_0) \rightarrow (P_\lambda)^m(X_\lambda; x_0),$$

which is independent of the choice of  $f$  or of  $\phi$  in their homotopy classes.

Likewise, if  $\psi: [X_\lambda, x_0] \rightarrow [Y_\lambda, y_0]$ , where  $y_0$  is a point in every  $Y_\lambda$ , we define a homomorphism

$$\psi_*: (P_\lambda)^m(X_\lambda; x_0) \rightarrow (P_\lambda)^m(Y_\lambda; y_0),$$

defined by  $\psi_*\{f\} = \{\psi \circ f\}$ ; this depends only on the homotopy class of  $\psi$ .

**THEOREM 2.21.**  *$\phi^*, \psi_*$  depend only on the homotopy classes of the maps  $\phi, \psi$ . The identity map induces the identity homomorphism, and, if  $\phi_1, \phi_2$  or  $\psi^1, \psi^2$  are successive maps, then*

$$(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*, \quad \phi^* \circ \psi_* = \psi_* \circ \phi^*, \quad \text{and} \quad (\psi^2 \circ \psi^1)_* = \psi_*^2 \circ \psi_*^1.$$

This follows from the fact that composition of maps is associative. We deduce at once that  $(P_\lambda)^m(X_\lambda; x_0)$  depends only on the homotopy type of  $[P_\lambda]$ . For suppose  $f: [P_\lambda] \equiv [Q_\lambda]$ ; this homotopy equivalence has a two-sided homotopy inverse  $g$ , and both  $g^* \circ f^*, f^* \circ g^*$  are isomorphisms onto, so that  $f^*$  is an isomorphism onto.

Let  $X'_\lambda$  be the path-component of  $x_0$  in  $X_\lambda$ . Then we show that  $(P_\lambda)^m(X_\lambda; x_0)$  depends only on the homotopy type of  $[X'_\lambda; x_0]$ . For let  $i: [X'_\lambda, x_0] \rightarrow [X_\lambda, x_0]$  be the identity map: we must show that  $i_*: (P_\lambda)^m(X'_\lambda; x_0) \rightarrow (P_\lambda)^m(X_\lambda; x_0)$  is an isomorphism onto. Now the factor space  $P_\lambda \times I^m / P_\lambda \times I^m$  is path-connected, so that the image of  $P_\lambda \times I^m$  in  $X_\lambda$  is path-connected, and hence lies in  $X'_\lambda$ . Therefore  $i_*$  is onto. Similarly, the image of  $P_\lambda \times I^m \times I$  lies in  $X'_\lambda$  for any homotopy, so that  $i_*$  is an isomorphism. By an argument analogous to that in the preceding paragraph, we see that  $(P_\lambda)^m(X'_\lambda; x_0)$  depends only on the homotopy type of  $[X'_\lambda, x_0]$ .

### 3. Base-point invariance

Let  $X_0$  be the intersection of all the spaces  $X_\lambda$ , and let  $X'_0$  be the path-component of  $x_0$  in  $X_0$ . We prove that the track groups  $(P_\lambda)^m(X_\lambda; x_i)$ , for all  $x_i$  in  $X'_0$ , form a system of local groups in  $X'_0$  (see (17)).

Let  $x_0, x_1$  be points of  $X'_0$ , so that there is a path  $\sigma: [I, 0, 1] \rightarrow [X'_0, x_0, x_1]$ ; we use  $\sigma$  to define a homomorphism

$$\sigma^*: (P_\lambda)^m(X_\lambda; x_1) \rightarrow (P_\lambda)^m(X_\lambda; x_0).$$

Take polar coordinates  $(s, \theta)$  in  $I^m$ , where  $\theta$  is a point of  $I^m$ , and  $(s, \theta)$  is a point on the segment joining  $\theta$  to the centre  $(\frac{1}{2}, \dots, \frac{1}{2})$  of  $I^m$ , and dividing it in the ratio  $(1-s):s$ , so that  $(1, \theta) = \theta$ ,  $(0, \theta) = (\frac{1}{2}, \dots, \frac{1}{2})$ . Define a map  $\sigma f$  by

$$(\sigma f)_\lambda(p, s, \theta) = \begin{cases} f_\lambda(p, 2s, \theta) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \sigma(2-2s) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

for all  $p \in P_\lambda$ , all  $\lambda$ , where  $\{f\} \in (P_\lambda)^m(X_\lambda; x_1)$ . This is a single-valued transformation, and so continuous since piece-wise continuous over the closed sets defined by  $0 \leq s \leq \frac{1}{2}$ ,  $\frac{1}{2} \leq s \leq 1$ . Now define  $\sigma^*\{f\} = \{\sigma f\}$ : this is obviously a transformation of the required sort, is independent of the choice of  $\sigma$  in its homotopy class, and, if  $x_0$  is the trivial path, then  $x_0^*$  is the identical transformation.

**LEMMA 3.1.**  $\sigma^*$  is a homomorphism.

The proof is akin to the proof of the corresponding assertion that a path in  $X'_0$  determines a homomorphism of the relative homotopy group  $\pi_m(X_0, X'_0, x_1)$  to  $\pi_m(X_0, X'_0, x_0)$ . A proof using an explicitly defined homomorphism is given in an appendix.

It follows quickly that the Fundamental Groupoid<sup>†</sup> of  $X'_0$  is a groupoid of operators on the track groups based at points of  $X'_0$ , and hence

**THEOREM 3.2.** *The groups  $(P_\lambda)^m(X_\lambda; x_0)$  form a system of local groups in  $X'_0$ ; in particular, they are all isomorphic, and admit the fundamental group of  $X'_0$  as a group of operators.*

<sup>†</sup> The fundamental groupoid is the multiplicative groupoid of homotopy classes  $\text{rel } I$  of maps of  $I$  into  $X$ , defined in a way similar to the fundamental group.

The correspondence between track groups and homotopy groups is developed in a different way in § 6, where it is proved that, under certain conditions,  $(P_\lambda)^m(X_\lambda; x_0)$  is isomorphic to  $\pi_m(F, x_0)$ , where  $F$  is as defined in (2.1). Assuming this, we have an alternative proof of Theorem 3.2, for it is known that the homotopy groups form a system of local groups (11). In particular, if  $Y \subset F$  is the set of all trivial maps  $x_i$ , for all points  $x_i$  of  $X'_0$ , the homotopy groups  $\pi_m(F, x_i)$  form a system of local groups in the path-components of  $Y$ . We show that  $Y$  is homeomorphic to  $X'_0$ .

Let  $\bar{p}: Y \rightarrow X'_0$  be such that  $\bar{p}(x_i) = x_i$ ; this is clearly a 1-1 correspondence. If  $p_\lambda$  is a point of  $P_\lambda$ , then  $p_\lambda$  is compact; if  $U$  is an open set of  $X'_0$ , then  $\bar{p}^{-1}(U)$  is open in  $Y$  by definition, being the set of all maps in  $Y$  which carry the compact set  $p_\lambda$  into  $U$ . Therefore  $\bar{p}$  is continuous. Also, the sets  $\bar{p}^{-1}(U)$  for all open  $U$  in  $X'_0$  form a sub-base for  $Y$ , and their images under  $\bar{p} = (\bar{p}^{-1})^{-1}$  are open. Therefore  $\bar{p}^{-1}$  is also continuous, and hence  $\bar{p}$  is a homeomorphism onto.

#### 4. The excision and factor theorems

First, suppose that  $X_\mu = x_0$  for some  $\mu$ , and that, for all  $\lambda$ ,

$$S \subset U \subset \bar{U} \subset P_\lambda,$$

where  $U$  is open in each  $P_\lambda$ .

**THEOREM 4.1.** *The inclusion mapping  $j: [P_\lambda - S] \rightarrow [P_\lambda]$  induces an isomorphism onto*

$$j^*: (P_\lambda)^m(X_\lambda; x_0) \rightarrow (P_\lambda - S)^m(X_\lambda; x_0).$$

We know that  $j^*$  is a homomorphism, and proceed to construct a two-sided inverse,  $k^*$ . Take any map  $f \in \{f\} \in (P_\lambda - S)^m(X_\lambda; x_0)$ , and extend  $f$  to  $f'$  over  $[P_\lambda]$  by mapping  $S \times I^m$  to  $x_0$  for every  $\lambda$ . We must show that each map  $f'_\lambda$  is continuous; since  $\bar{U} \subset P_\mu$ ,  $f'_\mu(\bar{U} \times I^m) = x_0$ , and so  $f'_\lambda(\bar{U} \times I^m) = x_0$  for every  $\lambda$ ; since  $f_\lambda$  is continuous, and agrees with  $f'_\lambda$  on  $(P_\lambda - S) \times I^m$ ,  $f'_\lambda|_{(P_\lambda - U) \times I^m}$  is continuous. Therefore  $f'_\lambda$  is continuous over each of the closed subspaces  $(P_\lambda - U) \times I^m$ ,  $\bar{U} \times I^m$  of  $P_\lambda \times I^m$ , and so, being single-valued, is continuous over the whole. Therefore, if we define  $k^*\{f\}$  to be the homotopy class of  $f'$ , we can easily verify that  $k^*$  is a two-sided inverse to  $j^*$ .

We now suppose that  $X_\mu = x_0$  as before, and that  $F$  is a closed subset of  $P_\lambda$  for all  $\lambda$ .

**THEOREM 4.2.** *The identification mapping  $\phi: [P_\lambda] \rightarrow [P_\lambda/F]$  induces an isomorphism onto*

$$\phi^*: (P_\lambda/F)^m(X_\lambda; x_0) \rightarrow (P_\lambda)^m(X_\lambda; x_0).$$

Since  $X_\mu = x_0$ , all maps carry  $F \times I^m$  to  $x_0$ . Therefore, from (1.3), we see that  $\phi^{-1}$  determines a homomorphism which is clearly a two-sided inverse to  $\phi^*$ .

## CHAPTER II

## THE EXACT SEQUENCE

## 5. The exact sequence

For simplicity, we shall confine ourselves for the majority of this paper to pairs of spaces  $[P, Q]$ , where  $Q$  is a sub-space of  $P$ , and write  $(P, Q)^m$  for the group  $(P, Q)^m(X, x_0; x_0)$ , where the space  $X$  and the point  $x_0$  will be the same throughout. Then, if  $R \subset Q \subset P$ , we shall prove, with certain restrictions on  $P, Q$ , and  $X$ , that there is an exact sequence†

$$\dots \rightarrow (P, Q)^m \rightarrow (P, R)^m \rightarrow (Q, R)^m \rightarrow (P, Q)^{m-1} \rightarrow \dots$$

## 5.1. Homotopy extension

We shall say that a set of spaces  $[T, S; X]$  satisfies the homotopy extension theorem if  $S \subset T$ , and if, for any maps  $f_0: T \rightarrow X$ ,  $g_0 = f_0|_S: S \rightarrow X$ , and any homotopy  $g_t: S \rightarrow X$ , there is also a homotopy  $f_t: T \rightarrow X$  such that  $g_t = f_t|_S$ . Various conditions on  $T, S, X$  ensure this property:

- (a)  $X$  is a compact ANR,  $S$  closed in  $T$ ,  $T$  metric separable (Borsuk (3)).
- (b)  $X$  is a compact ANR,  $S$  closed in  $T$ ,  $T$  normal and paracompact (Dowker (5)).
- (c)  $X$  arbitrary,  $S, T$  compact ANR's (Whitehead (23)).
- (d)  $X$  arbitrary,  $T$  a CW complex,  $S$  a closed subcomplex of  $T$  (Whitehead (24)).

Since we require a slightly stronger condition on  $[T, S; X]$  than that it satisfies the homotopy extension theorem, we shall not use (b) at first, and shall call any  $[T, S; X]$  satisfying (a), (c), or (d) an *HE triple*. In a later paper we shall be able to include triples satisfying (b). The importance of this restriction is explained by Lemma 5.12 below, but first we require

**LEMMA 5.11.** *If  $A = A_1 \cup A_2$ ,  $B = A_1 \cap A_2$ , and if  $A_1, A_2$ , and  $B$  are compact ANR's, then so is  $A$ .*

From any covering of  $A$  we may pick a finite number of sets whose intersections with  $A_i$  cover  $A_i$ , for  $i = 1, 2$ . Therefore  $A$  is compact. Let  $\phi: [A_1, B] \rightarrow [A, A_2]$  be the identity map, so that  $\phi$  is a homeomorphism of  $A_1 - B$  onto  $A - A_2$ , and  $A_1, A_2$  are compact ANR's. Then, by a theorem of Whitehead's (23), since  $A$  is compact and  $B$  is a compact ANR,  $A$  is a compact ANR. We can now prove

**LEMMA 5.12.** *If  $[P, Q; X]$  is an HE triple, so are the triples*

$$[P \times I^m, Q \times I^m; X], [P \times I^m, Q \times I^m \cup P \times J^{m-1}; X]$$

and

$$[P \times I^m, Q \times I^m \cup P \times \dot{I}^m; X].$$

*In particular, all satisfy the homotopy extension theorem.*

† I am indebted to R. L. Taylor, who first showed how the exact sequence can be extended to general spaces.

Under condition (a),  $P \times I^m$  is a separable metric space, and the various sub-spaces are closed in it. Under condition (c), the spaces  $P \times I^m$ ,  $Q \times I^m$ ,  $P \times \dot{I}^m$ ,  $Q \times \dot{I}^m$ ,  $P \times J^{m-1}$ ,  $Q \times J^{m-1}$  are all compact ANR's, so that the listed sub-spaces are compact ANR's by the previous lemma. Finally, we may turn  $I^m$  into a locally finite CW complex so that  $I^{m-1}$ ,  $J^{m-1}$ , and  $\dot{I}^m$  are closed subcomplexes; then, under condition (d), we may turn  $P \times I^m$  into a CW complex in a natural way so that the listed sub-spaces are closed sub-complexes (see Proposition H on p. 227 of (24)). The result then follows from the known theorems.

## 5.2. The injection homomorphisms

Let  $R \subset Q \subset P$ , where  $R$ ,  $Q$  are closed subspaces of  $P$ , and  $R$  may be empty. The inclusion maps

$$i: [P, R] \rightarrow [P, Q], \quad j: [Q, R] \rightarrow [P, R]$$

induce homomorphisms

$$(P, Q)^m \xrightarrow{i^*} (P, R)^m \xrightarrow{j^*} (Q, R)^m$$

with the obvious property that  $j^*i^*(P, Q)^m = 0$ . We shall write  $j^* \circ i^* = 0$ , the trivial homomorphism.

LEMMA 5.21. *If  $[P, Q; X]$  is an HE triple,  $i^* = j^{*-1}(0)$ , that is, the above sequence is exact at  $(P, R)^m$ .*

If  $f: [P \times I^m, R \times I^m \cup P \times \dot{I}^m] \rightarrow [X, x_0]$  is such that  $f|_{Q \times I^m}$  is inessential rel  $R \times I^m \cup Q \times \dot{I}^m$ , we extend a homotopy between

$$f|[Q \times I^m, R \times I^m \cup Q \times \dot{I}^m]$$

and the constant map  $x_0$  over the space  $Q \times I^m \cup P \times \dot{I}^m$  by mapping  $P \times \dot{I}^m$  constantly to  $x_0$ ; as in (4.1), this is a continuous map. By Lemma 5.12 this homotopy can be extended over  $P \times I^m$  to give a map

$$g: [P \times I^m, R \times I^m \cup P \times \dot{I}^m] \rightarrow [X, x_0]$$

which is homotopic to  $f$ , and has the property that  $g(Q \times I^m) = x_0$ . Clearly  $g$  also defines a map in a class in  $(P, Q)^m$ , and  $\{g\} \in i^*(P, Q)^m$ . Thus  $j^{*-1}(0) \subset i^*(P, Q)^m$ , and hence, since  $i^*(P, Q)^m \subset j^{*-1}(0)$  for all triples,  $j^{*-1}(0) = i^*(P, Q)^m$  in these circumstances.

## 5.3. The boundary homomorphism

We define a homomorphism  $\delta^*: (Q, R)^m \rightarrow (P, Q)^{m-1}$  whenever  $m > 1$  and  $[P, Q; X]$  is an HE triple, by extending a map

$$f: [Q \times I^m, R \times I^m \cup Q \times \dot{I}^m] \rightarrow [X, x_0]$$

to a map  $f': [P \times I^m, Q \times \dot{I}^m \cup R \times I^m \cup P \times J^{m-1}] \rightarrow [X, x_0]$ , and considering the homotopy class of  $f'|_{[P \times I^{m-1}, P \times \dot{I}^{m-1} \cup Q \times I^{m-1}]}$ .  $\delta^*\{f\}$



measures, in fact, the obstruction to the extension of  $f$  to a map  $f''$  representing an element of  $(P, R)^m$ . However, the discussion is simplified by turning the homotopy classes of these intermediate maps  $f'$  into a group. We define this group without restriction on  $P, Q, X$ .

Let  $S_m \subset P \times I^m$  be the sub-space  $R \times I^m \cup Q \times \dot{I}^m \cup P \times J^{m-1}$ , and consider the set of homotopy classes  $(P \times I^m, S_m)^0(X, x_0)$ ; when  $P = Q$ , this is the track group  $(Q, R)^m$ , but the addition defined in (2.1) turns this set into a group when  $m > 1$ , for all  $P$ . We call this group  $(P, R | Q)^m$  during this section: its elements are the homotopy classes of maps

$$f: [P \times I^m, R \times I^m, Q \times \dot{I}^m, P \times J^{m-1}] \rightarrow [X, x_0, x_0, x_0],$$

and the unit of the group is the class of the constant map  $x_0$ .

Now define three natural homomorphisms: the identity map

$$\theta: [P \times I^m, S_m] \rightarrow [P \times I^m, R \times I^m \cup P \times \dot{I}^m]$$

defines

$$\theta^*: (P, R)^m \rightarrow (P, R | Q)^m;$$

the identity map  $d: [P \times I^{m-1}, Q \times I^{m-1} \cup P \times \dot{I}^{m-1}] \rightarrow [P \times I^m, S_m]$  defines

$$d^*: (P, R | Q)^m \rightarrow (P, Q)^{m-1};$$

and the identity map  $h: [Q \times I^m, R \times I^m \cup Q \times \dot{I}^m] \rightarrow [P \times I^m, S_m]$  defines

$$h^*: (P, R | Q)^m \rightarrow (Q, R)^m.$$

Thus  $h^*, d^*$  are obtained by restricting the maps of  $(P, R | Q)^m$  to the subspaces  $Q \times I^m, P \times I^{m-1}$ , respectively, while  $\theta^*$  results from allowing  $P \times I^{m-1}$  to be mapped other than to  $x_0$ .

We have the following scheme of homomorphisms:

$$\begin{array}{ccccccc} \dots (P, Q)^m & \xrightarrow{i^*} & (P, R)^m & \xrightarrow{\theta^*} & (P, R | Q)^m & \xrightarrow{d^*} & (P, Q)^{m-1} \xrightarrow{i^*} (P, R)^{m-1} \dots \\ & & & & \downarrow h^* & & \\ & & & & (Q, R)^m & & \end{array}$$

$\searrow j^*$

LEMMA 5.31. *The above diagram is commutative, that is,  $j^* = h^* \circ \theta^*$ . It is exact at  $(P, Q)^{m-1}$ , so that  $d^* = i^{*-1}(0)$ , and at  $(P, R | Q)^{m-1}$  in the sense that  $\theta^* = d^{*-1}(0)$ .*

Notice that this lemma does not require restrictions on  $P, Q$ , or  $X$ . The naturality follows at once from the definitions, since  $j^*, h^*, \theta^*$  are induced by injections.

We now prove that  $\theta^* = d^{*-1}(0)$ . If  $z = \theta^*y$ , then  $z$  contains a map  $f: P \times I^m \rightarrow X$  such that  $f(P \times \dot{I}^m) = x_0$ ; therefore  $f|P \times I^{m-1}$  is the constant map, and  $d^*\theta^*y = 0$ . Conversely, if  $d^*z = 0$ , then, for any  $f$  in  $z$ ,

$f|P \times I^{m-1}$  is inessential rel  $Q \times I^{m-1} \cup P \times \dot{I}^{m-1}$ . Let  $F: P \times I^{m-1} \rightarrow X$  be such a homotopy, and define a deformation  $f_t$  of  $f$  by the equations

$$f_t(p, y_1, \dots, y_m) = \begin{cases} F(p, y_1, \dots, y_{m-1}, t - (1+t)y_m) & \text{if } 0 \leq y_m \leq t/(1+t), \\ f(p, y_1, \dots, y_{m-1}, y_m(1+t) - t) & \text{if } t/(1+t) \leq y_m \leq 1. \end{cases}$$

It may be seen that  $f_t$  is a homotopy rel  $S_m = R \times I^m \cup Q \times \dot{I}^m \cup P \times J^{m-1}$ . Then  $f_0 = f$ , and  $f_1$  maps  $P \times \dot{I}^m$  to  $x_0$ , so that  $\{f_1\} = z$  is in  $\theta^*(P, R)^m$ .

Finally, we prove that  $d^* = i^{*-1}(0)$ . If  $w = d^*z$ , choose a map  $f$  in  $z$  so that  $g = f|P \times I^{m-1}$  represents the class  $w$ . Then, using the natural identification  $P \times I^{m-1} \times I = P \times I^m$ , the map  $f$  actually defines a homotopy rel  $P \times \dot{I}^{m-1} \cup R \times I^m$  between  $g$  and the constant map, so that  $i^*w = 0$ . Conversely, if  $i^*w = 0$ , a homotopy between a representative map and the constant map defines an element of  $(P, R|Q)^m$ , whose image under  $d^*$  is  $w$ .

LEMMA 5.32. *When  $[P, Q; X]$  is an HE triple,  $h^*$  is an isomorphism onto.*

We must show that a map of  $Q \times I^m$  to  $X$  can be extended to  $P \times I^m$  in such a way that  $S_m = R \times I^m \cup Q \times \dot{I}^m \cup P \times J^{m-1}$  is mapped to  $x_0$ ; and also that any two such extensions are homotopic.

Firstly, then,  $h^*$  is onto: for  $J^{m-1}$  is a deformation retract of  $I^m$ , and any such deformation retraction induces a deformation retraction  $\sigma_t: P \times I^m \rightarrow P \times I^m$  of  $P \times I^m$  onto  $P \times J^{m-1}$ , where  $\sigma_0$  is the identity map,  $\sigma_1(P \times I^m) \subset P \times J^{m-1}$ , and  $\sigma_t|P \times J^{m-1}$  is the identity map for every  $t$ . Let  $\sigma'_t$  be  $\sigma_t$  restricted to  $Q \times I^m$ , so that  $\sigma'_t$  is a similar deformation retraction.

Now let  $f: [Q \times I^m, R \times I^m \cup Q \times \dot{I}^m] \rightarrow [X, x_0]$ , and consider the induced map  $f: Q \times I^m \rightarrow X$ ; then  $f = f \circ \sigma'_0 \simeq f \circ \sigma'_1$ , where  $f \circ \sigma'_1$  is the constant map  $x_0$ . We extend this last map  $f \circ \sigma'_1$  over  $P \times I^m$ , and extend the homotopy  $f \circ \sigma'_1 \simeq f \circ \sigma'_0$  over  $Q \times I^m \cup P \times J^{m-1}$ , by mapping  $(P - Q) \times J^{m-1}$  constantly to  $x_0$  throughout the homotopy. By Lemma 5.12,

$$[P \times I^m, Q \times I^m \cup P \times J^{m-1}, X]$$

is an HE triple, so that we can extend this homotopy farther over  $P \times I^m$ . Let  $g_t$  be this extension, so that, by construction,  $g_t$  is a homotopy rel  $P \times J^{m-1}$  between the constant map and an extension of  $f: Q \times I^m \rightarrow X$  over  $P \times I^m$ . Then  $g_1$  defines an element of  $(P, R|Q)^m$  which projects onto the class of  $f$  under  $h^*$ .

Secondly,  $h^*$  is an isomorphism; suppose  $h^*\{g\} = 0$ , so that  $g|Q \times I^m$  is homotopic rel  $R \times I^m \cup Q \times \dot{I}^m$  to the constant map. Extend this homotopy over  $Q \times I^m \cup P \times J^{m-1}$  by mapping  $P \times J^{m-1}$  to  $x_0$  throughout the homotopy; by Lemma 5.12 this homotopy has an extension over  $P \times I^m$ , rel  $S_m$ , to  $g_t$ , where  $g_0 = g$ , and  $g_1(Q \times I^m) = x_0$ . Therefore we can define a homotopy between  $g_1 = g_1 \circ \sigma_0$  and the constant map  $x_0 = g_1 \circ \sigma_1$ , since  $g_1 \circ \sigma_t$  maps  $S_m$  to  $x_0$ . Hence  $\{g\} = 0$ , and  $h^*$  is an isomorphism.

#### 5.4. The exact sequence of $[P, Q, R]$

Suppose  $P \supset Q \supset R$ , and that  $[P, Q; X]$  is an HE triple; identify  $(Q, R)^m$  and  $(P, R | Q)^m$  by means of  $h^*$ , and define  $\delta^* = d^* \circ (h^*)^{-1}$ . Then Lemmas 5.21, 5.31, 5.32 imply

**THEOREM 5.41.** *When  $[P, Q; X]$  is an HE triple, there is an exact sequence, terminating with a homomorphism into,*

$$\rightarrow (P, Q)^m \xrightarrow{i^*} (P, R)^m \xrightarrow{j^*} (Q, R)^m \xrightarrow{\delta^*} (P, Q)^{m-1} \rightarrow \dots \xrightarrow{j^*} (Q, R)^1.$$

*This sequence will be called the sequence of  $[P, Q, R]$ .*

The sequence is natural: for, if  $\phi: [P, Q, R] \rightarrow [P', Q', R']$ ,  $\phi$  induces maps  $\phi: [A, B] \rightarrow [A', B']$  for every ordered pair  $A, B$  from  $P, Q, R$ , and we may easily prove

**THEOREM 5.42.** *The map  $\phi: [P, Q, R] \rightarrow [P', Q', R']$  induces a homomorphism of the exact sequence of  $[P', Q', R']$  into that of  $[P, Q, R]$ , in the sense that the following diagram is commutative:*

$$\begin{array}{ccccccc} \rightarrow & (P', Q')^m & \xrightarrow{i^*} & (P', R')^m & \xrightarrow{j^*} & (Q', R')^m & \xrightarrow{\delta^*} & (P', Q')^{m-1} \rightarrow \\ & \downarrow \phi^* & & \downarrow \phi^* & & \downarrow \phi^* & & \downarrow \phi^* \\ \rightarrow & (P, Q)^m & \xrightarrow{i^*} & (P, R)^m & \xrightarrow{j^*} & (Q, R)^m & \xrightarrow{\delta^*} & (P, Q)^{m-1} \rightarrow. \end{array}$$

## 6. Function spaces

The set of maps of  $[P_\lambda]$  into  $[X_\lambda]$  can be turned into a topological space by the compact-open topology (9), as indicated in (2.1). Recall that a sub-base  $V(\lambda, A_\lambda, U_\lambda)$ , where  $A_\lambda$  is a compact set in  $P_\lambda$ ,  $U_\lambda$  an open set in  $X_\lambda$ , and the set  $V(\lambda, A_\lambda, U_\lambda)$  consists of all maps  $f: [P_\lambda] \rightarrow [X_\lambda]$  that map  $A_\lambda$  into  $U_\lambda$ , defines open sets as the union of finite intersections of the  $V$ 's. We write  $[X_\lambda]^{[P_\lambda]}$  for the function space. Then, for any  $T$ , there is an obvious continuous map

$$\theta: [X_\lambda]^{[P_\lambda \times T]} \rightarrow [[X_\lambda]^{[P_\lambda]}]^T;$$

a theorem of Fox's (9) may be stated in these circumstances as proving that  $\theta$  is a homeomorphism onto if either (i) the spaces  $P_\lambda$  are regular and locally compact, or (ii) if  $T = I$ , and the spaces  $P_\lambda$  satisfy the first axiom of countability. We may quickly deduce that, if  $x_\lambda = x_0$  for all  $\lambda$ ,

$$[[X_\lambda]^{[P_\lambda]}, x_0]^{[I^m, I^m]} \quad \text{and} \quad [X_\lambda, x_\lambda]^{[P_\lambda \times I^m, P_\lambda \times I^m]}$$

are homeomorphic. Let  $x_0$  be the constant map that maps each  $P_\lambda$  to  $x_0$ .

**THEOREM 6.1.** *If  $F = [X]^{[P_\lambda]}$ , and each  $P_\lambda$  is regular and locally compact or satisfies the first axiom of countability, there is a natural isomorphism*

$$\pi_m(F, x_0) \approx (P_\lambda)^m(X_\lambda; x_0).$$

Embed the two function spaces in the obvious way in  $[[X_\lambda]^{[P_\lambda]}]^{I^m}$  and in  $[X_\lambda]^{[P_\lambda \times I^m]}$  respectively, and notice that the map  $\theta$  above with  $T = I^m$  maps the one homeomorphically on the other, and so induces a natural 1-1 correspondence between their path-components, which are the elements of the two groups in the theorem. Inspection of the definitions of additions ((2.1) and (10)) shows that this 1-1 transformation is a homomorphism, and so an isomorphism onto.

We now confine ourselves to pairs of spaces  $[P, Q]$  such that  $Q \subset P$ . Let  $G$  be the subspace of the function space  $F = [X, x_0]^{[P, R]}$  which is  $[X, x_0]^{[P, Q]}$  after both these have been embedded in  $X^P$ , where  $R \subset Q \subset P$ . Then we can prove by the same means

**THEOREM 6.2.** *If  $P \supset Q \supset R$ , where  $P$  is regular and locally compact or satisfies the first axiom of countability, then*

$$\pi_m(F, G, x_0) \approx (P, R | Q)^m(X, x_0).$$

The exact homotopy sequence of the triple  $[F, G, x_0]$  (11) immediately gives an exact sequence involving the groups  $(P, Q)^m$ ,  $(P, R)^m$ , and  $(P, R | Q)^m$ , in the notation of § 5. It may be verified that the maps  $i^*$ ,  $\theta^*$ ,  $d^*$  in § 5 are the analogues of the homomorphisms of the homotopy sequence.

**COROLLARY 6.21.** *Under the above conditions, there is an exact sequence, terminating with a homomorphism into  $(P, R)^1$ ,*

$$\rightarrow (P, Q)^m \xrightarrow{i^*} (P, R)^m \xrightarrow{\theta^*} (P, R | Q)^m \xrightarrow{d^*} (P, Q)^{m-1} \rightarrow .$$

We can use Lemma 5.21 to extend the exact sequence down to  $(Q, R)^1$ .

### 6.3. Fibre spaces

The map  $j: [Q, R] \rightarrow [P, R]$  induces a map  $j': [X, x_0]^{[P, R]} \rightarrow [X, x_0]^{[Q, R]}$  such that  $j'(f) = f|_{[Q, R]}$ . It is clear that  $j'$  is continuous, since these spaces have the compact-open topology. If now  $j'$  is a fibre mapping of  $[X, x_0]^{[P, R]}$  into  $[X, x_0]^{[Q, R]}$  that carries the component of the trivial map onto the component of the trivial map, we at once obtain (see (14, 22)) that  $(P, R | Q)^m \approx (Q, R)^m$ ,  $m > 1$ , on using the previous theorem and corollary. Therefore, in these circumstances, we obtain the exact sequence of  $[P, Q, R]$  as a consequence of Theorem 6.1 (and Lemma 5.21 for the last homomorphism).

When  $[P, Q; X]$  is an HE triple, the path-component of the trivial map in  $X^P$  is mapped onto the path-component of the trivial map in  $X^Q$ , and a similar statement is true for  $[X, x_0]^{[P, R]}$  and  $[X, x_0]^{[Q, R]}$ . When  $X$  is a compact ANR, and  $P$  is a separable metric space,  $j'$  is a fibre mapping: Fox (7) observes that Borsuk's homotopy extension theorem (see 5.1 (a))

implies that  $X^P$  is a fibre space over  $j'X^P \subset X^Q$ , with fibre  $[X, x_0]^{[P, Q]}$ , and this leads us to a similar conclusion for  $[X, x_0]^{[P, R]}$  over  $[X, x_0]^{[Q, R]}$ .

However, R. L. Taylor pointed out that our isomorphism

$$(P, R \mid Q)^m \simeq (Q, R)^m$$

needs only the existence of the covering homotopy theorem for a fibre space, and that this can be demonstrated when  $[P, Q; X]$  is an HE triple. We now give his proof of this fact.

Let  $j': A \rightarrow B$ ,  $f: T \rightarrow B$  be given. Then a map  $f^*: T \rightarrow A$  such that  $j'f^* = f$  is called a *covering map for  $f$* ; likewise, if  $F: T \times I \rightarrow B$  is a homotopy, we call a covering map for  $F$  a *covering homotopy*. Further, we say that  $[T; A, B; j']$  *satisfy the covering homotopy theorem* if, given any map  $f_0: T \rightarrow B$  which possesses a covering map  $f_0^*$ , and any homotopy  $f_t$  of  $f_0$ , there is a covering homotopy  $f_t^*$ .

Let  $F, G$  be as in Theorem 6.2, and let  $F' = [X, x_0]^{[Q, R]}$ .

**LEMMA 6.31.** *If  $P$  is regular and locally compact, or satisfies the first axiom of countability, and if  $[P, Q; X]$  is an HE triple, then  $[I^m; F, F'; j']$  satisfy the covering homotopy theorem.*

Under conditions (a), (c) of (5.1),  $P$  satisfies the first axiom of countability, and so does any locally finite CW complex.

Any map  $f: I^m \rightarrow F'$  may be regarded as a map  $'f: Q \times I^m \rightarrow X$  such that  $'f(R \times I^m) = x_0$ . Similarly, a covering map  $f^*: I^m \rightarrow F$  for  $f$  may be regarded as a map  $'f^*: P \times I^m \rightarrow X$ , which is an extension of  $'f$  over  $P \times I^m$ . The lemma is then a consequence of the known theorems on homotopy extension, translated into theorems on covering maps.

**LEMMA 6.32.** *Under the hypotheses of the previous lemma, any map  $f: I^m \rightarrow F'$  with  $f(I^m) = x_0$ , the trivial map, has a covering map  $f^*: I^m \rightarrow F$  such that  $f^*(J^{m-1}) = x_0$ , and  $f^*(I^{m-1}) \subset G$ .*

This proof is a modification of Taylor's. There is a homeomorphism  $h: I^m \rightarrow I^m$  such that  $h(I^{m-1} \times 1) = J^{m-1}$ , while (of course)  $h(I^m) = I^m$ . Let  $f: I^m \rightarrow F'$  carry  $I^m$  to  $x_0$ , and define a homotopy  $g_t: I^{m-1} \rightarrow F'$  by

$$g_t(y_1, \dots, y_{m-1}) = fh(y_1, \dots, y_{m-1}, t).$$

Then  $g_1 = x_0$ , and has the covering map  $g_1^* = x_0: I^{m-1} \rightarrow F$ , and hence  $g_t$  has a covering homotopy  $g_t^*$ . We now reverse this construction to provide a covering map for  $f$ ; define  $f^*, f^{**}$  by  $f^* = f^{**}h^{-1}$ , where

$$f^{**}(y_1, \dots, y_m) = g_{y_m}^*(y_1, \dots, y_{m-1}).$$

Then  $f^*(J^{m-1}) = f^*h(I^{m-1} \times 1) = f^{**}(I^{m-1} \times 1) = g_1^*(I^{m-1}) = x_0$ , and  $f = j' \circ f^*$ . Then  $f^*$  is the required covering map.

We deduce

**THEOREM 6.33.** *When  $[P, Q; X]$  is an HE triple, and  $P$  is regular and locally compact, or satisfies the first axiom of countability, then*

$$(P, Q | R)^m \approx (Q, R)^m.$$

The isomorphism is that induced by the isomorphisms of Theorems 6.1, 6.2, and that induced by the map  $j'$ ,

$$j'_*: \pi_m(F, G, x_0) \approx \pi_m(F', x_0).$$

For the previous lemma shows that  $j'_*$  is onto; we show that Lemma 6.31 implies that  $j'_*$  is an isomorphism. Suppose that  $j'_*\{f^*\} = 0$ , so that, if  $f = j' \circ f^*$ , there is a homotopy between  $f$  and the constant map which possesses a covering homotopy between  $f^*$  and a map of  $I^m$  into  $G$ . Therefore  $\{f^*\} = 0$ .

#### 6.4. $\mu$ -based track groups

Let  $\mu: [P, Q] \rightarrow [X, x_0]$ . Then, by analogy with  $\pi_m(F, \mu)$ , where  $F$  is the function space  $[X, x_0]^{[P, Q]}$ , we can define a group  $(P, Q)^m(X, x_0; \mu)$ , whose elements are homotopy classes of maps

$$f: [P \times I^m, Q \times I^m] \rightarrow [X, x_0]$$

such that, for each map, and throughout each homotopy,

$$f(p, y_1, \dots, y_m) = \mu(p), \quad \text{for all } p, \text{ all } (y_1, \dots, y_m) \in I^m.$$

Thus, if  $\mu$  is the constant map  $x_0$ , the group is  $(P, Q)^m(X, x_0; x_0)$  as defined above.

Let  $R \subset Q \subset P$ , and suppose that  $\mu: [P, Q] \rightarrow [X, x_0]$ . Let  $\mu$  define also  $\mu: [P, R] \rightarrow [X, x_0]$ , the map with the same values. Then, under appropriate conditions, we obtain as before an exact sequence

$$\rightarrow (P, Q)_\mu^m \xrightarrow{i^*} (P, R)_\mu^m \xrightarrow{j_\mu^*} (Q, R)^m \xrightarrow{\delta_\mu^*} (P, Q)_\mu^{m-1} \rightarrow,$$

where  $(P, Q)_\mu^m$  stands for  $(P, Q)^m(X, x_0; \mu)$ , and similarly with  $Q$  replaced by  $R$ ;  $(Q, R)^m$  is  $(Q, R)^m(X, x_0; x_0)$  as before.

In general,  $(P, R)_\mu^m$  is not isomorphic to  $(P, R)^m$ . G. W. Whitehead has shown (19) that, if  $w$  is a point of  $S^n$ , then the path-components of  $[X, x_0]^{[S^n, w]}$  are of the same homotopy type; the proof uses addition of maps of  $S^n$  to introduce an operation into the function space. In the same way, we can show that the path components of  $[X, x_0]^{[P \times S^n, P \times w]}$  have all the same homotopy type, using  $S^n$  to introduce the operation into the function space instead of  $I^n$  as before (it is easy to see that it is unnecessary to use the metric topology for the function space).

Then we may deduce, by the methods of this section,

**THEOREM 6.41.** *If  $P$  is regular and locally compact, or satisfies the first axiom of countability, and  $[P \times S^n, P; X]$  is an HE triple, then, for all  $\mu: [P \times S^n, P] \rightarrow [X, x_0]$  and all  $m \geq 1$ , there is an isomorphism*

$$(P \times S^n, P)^m(X, x_0; \mu) \simeq (P \times S^n, P)^m(X, x_0; x_0).$$

However, if  $R$  is a proper subspace of  $P$  or the empty set,  $(P \times S^n, R)_\mu^m$  and  $(P \times S^n, R)^m$  are not necessarily isomorphic. G. W. Whitehead, in (19), gives an example which, translated into our notation, furnishes a counterexample when  $m = n = 2$ ,  $P$  is a point, and  $R$  is empty. In fact, it can be shown that, if  $m > 1$ ,  $(S^n, 0)^m$  is the direct sum of  $\pi_{n+m}(X)$  and  $\pi_m(X)$ , while  $(S^n, w)^m$  is isomorphic to  $\pi_{n+m}(X)$ ; the homomorphism  $\delta_\mu^*: (w)^{m+1} \rightarrow (S^n, w)^m$  is that in which  $\alpha \in (w)^{m+1} = \pi_{m+1}(X)$  goes to the Whitehead product  $\pm[\alpha, \{\mu\}] \in \pi_{m+n}(X)$ , where  $\{\mu\}$  is the homotopy class of  $\mu$ . Therefore  $(S^n)_\mu^m$  is an extension of a factor group of  $\pi_{n+m}(X)$  by a subgroup of  $\pi_m(X)$ , which groups are determined by  $\{\mu\}$ .

### CHAPTER 3

#### RETRACT THEOREMS

##### 7.1. The retract theorem

We make some applications of the exact sequence, and so suppose throughout this chapter that  $Q, R$  are closed subspaces of  $P$ , where  $P \supset Q \supset R$ , and, unless it is explicitly stated to be unnecessary, that  $[P, Q; X]$  is an HE triple.

Let  $j$  be the injection in

$$[Q, R] \xrightarrow{j} [P, R] \xrightarrow{\sigma} [Q, R];$$

we say that  $Q$  is a *retract* of  $P$  if  $\sigma \circ j = 1$ , the identity map, and, more generally, that  $[Q, R]$  is a *homotopy retract* of  $[P, R]$  if  $\sigma \circ j \simeq 1$ .

**THEOREM 7.1.** *If  $Q$  is a retract of  $P$ , or if  $[Q, R]$  is a homotopy retract of  $[P, R]$ , then, for  $m > 1$ ,*

$$(P, R)^m \simeq (P, Q)^m + (Q, R)^m,$$

and  $(P, R)^1$  is a split extension of  $(P, Q)^1$  by  $(Q, R)^1$ .

By an extension  $E$  of  $G$  by  $Q$ , we mean that there is an exact sequence

$$0 \rightarrow G \xrightarrow{i} E \xrightarrow{j} Q \rightarrow 0,$$

and by a split extension we mean an extension in which  $j$  has a right inverse, so that an abelian split extension is the direct sum, written  $G + Q$ . The

theorem follows from the fact that, in the exact sequence of  $[P, Q, R]$ ,  $j^*$  has a right inverse  $\sigma^*$ ; therefore  $j^*$  is onto,  $\delta^*$  is trivial, and  $i^*$  is an isomorphism into. Hence  $(P, R)^m$  is a split extension of  $(P, Q)^m \approx i^*(P, Q)^m$  by  $(Q, R)^m$ , which is the direct sum when abelian, and so when  $m > 1$ .

If there is a homotopy  $j_i: [Q, R] \rightarrow [P, R]$  such that  $j_0 = j$ , the identity map, while  $j_1(Q) \subset R$ , we say that  $[Q, R]$  is *contractible into  $R$  in  $[P, R]$* .

LEMMA 7.21. *Under these circumstances,  $(P, Q)^m$  is an extension of  $(Q, R)^{m+1}$  by  $(P, R)^m$ .*

For  $j^* = j_1^*$  is a trivial homomorphism in the exact sequence of  $[P, Q, R]$ .

For a large class of spaces, the extension of the lemma is a direct sum if  $m > 1$ : for example, under conditions (c), (d) of (5.1). Under condition (a), however, it seems that we must suppose that  $P$  itself is a compact ANR.

THEOREM 7.22. *If  $[Q, R]$  is contractible into  $R$  in  $[P, R]$ , and if  $[P, Q; P]$  satisfy the homotopy extension theorem, then, if  $m > 1$ ,*

$$(P, Q)^m \approx (Q, R)^{m+1} + (P, R)^m,$$

*while  $(P, Q)^1$  is a split extension of  $(Q, R)^2$  by  $(P, R)^1$ .*

For under these conditions we can extend the homotopy  $j_i$  to a homotopy  $h_i: [P, R] \rightarrow [P, R]$ , where  $h_0$  is the identity map, and  $h_1(Q) \subset R$ . Therefore  $h_1$  can be factored into  $h \circ i = h_1$ , where  $i$  is the identity map  $[P, R] \rightarrow [P, Q]$ . Therefore  $h^*$  is a right inverse for  $i^*$ , so that  $j^*$  is trivial,  $\delta^*$  an isomorphism into, and  $(P, Q)^m$  is a split extension, and so the direct sum when  $m > 1$ .

If now  $[P, Q]$  is contractible into  $Q$  in  $[P, Q]$ , we call  $Q$  a *weak deformation retract of  $P$* : then there is a homotopy  $\phi_i: [P, Q] \rightarrow [P, Q]$  such that  $\phi_0$  is the identity map and  $\phi_1(P) \subset Q$ . If  $\phi_i$  is constant over  $Q$ , we call  $Q$  a *deformation retract of  $P$* .

LEMMA 7.31. *If  $Q$  is a weak deformation retract of  $P$ , then*

$$(P, Q)^m(X, x_0; x_0) = 0 \quad \text{for all triples } [P, Q; X].$$

For, since  $\phi_0^* = \phi_1^*$ , the identity automorphism is also the trivial homomorphism; this does not require  $[P, Q; X]$  to be an HE triple.

THEOREM 7.32. *If  $[P, Q; X]$  is an HE triple, then, if  $Q$  is a weak deformation retract of  $P$ ,  $(Q, R)^m \approx (P, R)^m$ ; if  $R$  is a weak deformation retract of  $P$ ,  $(Q, R)^{m+1} \approx (P, Q)^m$ ; while if  $R$  is a weak deformation retract of  $Q$ ,*

$$(P, Q)^m \approx (P, R)^m.$$

The only difficulty is to show that when  $Q$  is a weak deformation retract of  $P$ , then  $j^*: (P, R)^1 \rightarrow (Q, R)^1$  is onto; the other results are trivial consequences of the exact sequence and the previous lemma. The most convenient way of getting round the difficulty is perhaps to extend the exact sequence by a set-map  $\delta^*: (Q, R)^1 \rightarrow (P, Q)^0(X, x_0)$ , the set of homotopy



classes of maps of  $[P, Q]$  to  $[X, x_0]$ . Then it can be shown that the elements of  $\delta^*(Q, R)^1$  are in natural correspondence with the cosets of

$$j^*(P, R)^1 \subset (Q, R)^1.$$

In the particular case considered, an extension of Lemma 7.31 to the case  $m = 0$  shows that  $j^*$  is onto, and hence an isomorphism onto.

We give one more application of the exact sequence. Suppose

$$P \supset S \supset Q \supset R,$$

where  $S$  is such that  $R$  is a weak deformation retract of  $S$ . Then clearly  $[Q, R]$  is contractible into  $R$  in  $[S, R]$ , and so in  $[P, R]$ ; under certain conditions, Theorem 7.22 asserts that  $(P, Q)^m$  is a split extension of  $(Q, R)^{m+1}$  by  $(P, R)^m$ . We now show that this is the direct sum.

**LEMMA 7.33.** *Let  $[P, S; X]$ ,  $[P, Q; X]$  be HE triples, and let  $[P, Q; P]$  satisfy the homotopy extension theorem. Then, if  $R$  is a weak deformation retract of  $S$  ( $P \supset S \supset Q \supset R$ ),  $(P, Q)^m \approx (P, R)^m + (Q, R)^{m+1}$ , all  $m \geq 1$ .*

From the exact sequences of  $[P, Q, R]$ ,  $[P, S, R]$ , and  $[P, S, Q]$  we have the commutative diagram:

$$\begin{array}{ccc}
 (P, S)^m & \xrightarrow{i_2^*} & (P, R)^m \\
 & \searrow i_1^* & \nearrow i^* \\
 & (P, Q)^m & \\
 \delta^* \nearrow & & \nwarrow j_1 \\
 (Q, R)^{m+1} & & (S, Q)^m
 \end{array}$$

in which (by Theorem 7.32)  $i_2^*$  is an isomorphism onto, so that  $k^*$  can be defined as  $i_1^* \circ (i_2^*)^{-1}$ . The image of  $k^*$  is therefore the kernel of  $j_1^*$ , which is invariant in  $(P, Q)^m$ .  $k^*$  is a right inverse for  $i^*$ , so that  $(P, Q)^m$  is a split extension of  $(Q, R)^{m+1} \approx \delta^*(Q, R)^{m+1}$  by  $k^*(P, R)^m$ , and so the direct sum.

As an example of the use of this lemma, take  $R = r$ , a point of  $P$ ,

$$Q = q \cup r,$$

two distinct points of  $P$  that are contained in a set  $S$  that is contractible in itself (e.g. two vertices of a connected CW complex). Then, using the Excision Theorem (4.1), we see

$$(P, q \cup r)^m \approx (P, r)^m + (q)^{m+1}.$$

Of course, in the lemma,  $(P, R)^m \approx (P, S)^m$  and  $(Q, R)^{m+1} \approx (S, Q)^m$ . The lemma can then be stated in the following useful form (on changing the notation slightly):

**COROLLARY 7.34.** *Let  $P \supset Q \supset R \supset S$ , where  $S$  is a weak deformation retract of  $Q$ , and  $[P, Q; X]$ ,  $[P, R; X]$  are HE triples and  $[P, R; P]$  satisfy the homotopy extension theorem. Then  $(P, R)^m \approx (P, Q)^m + (Q, R)^m$ ,  $m \geq 1$ .*

#### 7.4. The cluster theorem

We call  $P = \bigcup P_\lambda$  a *cluster* of spaces  $P_\lambda$  ( $\lambda \in \Lambda$ ) if, for every  $\lambda \neq \mu$ ,  $P_\lambda \cap P_\mu = p$ , a point common to all  $P_\lambda$ , and if  $P$  has the 'weak' topology, in the sense that a set  $F \subset P$  is closed if and only if  $F \cap P_\lambda$  is closed for every  $\lambda$ . We suppose also that  $p$  is a closed set in every  $P_\lambda$ . Then a map  $f: P \rightarrow X$  is continuous if  $f|P_\lambda$  is continuous for every  $\lambda$ , and similarly for maps of  $P \times I^m$ .

**THEOREM 7.41.**  $(P, p)^m \approx \sum (P_\lambda, p)^m$ , the strong sum, all  $m \geq 1$ .

The strong sum of a system of groups  $G_\lambda$  is analogous to the group of infinite cochains; an element  $[x_\lambda]$  of the sum assigns to every  $\lambda$  an element  $x_\lambda$  of  $G_\lambda$ , called a *coordinate*, and addition is defined by addition of coordinates.

The identity maps  $\phi_\lambda: P_\lambda \rightarrow P$  induce homomorphisms

$$\phi_\lambda^*: (P, p)^m \rightarrow (P_\lambda, p)^m,$$

and so define a homomorphism  $\phi^*: (P, p)^m \rightarrow \sum (P_\lambda, p)^m$  such that

$$\phi^* \xi = [\phi_\lambda^* \xi].$$

The natural retractions  $\sigma_\lambda: P \rightarrow P_\lambda$  define homomorphisms  $\sigma_\lambda^*$ , which are right inverses for the  $\phi_\lambda^*$ . We shall define  $\sigma^*: \sum (P_\lambda, p)^m \rightarrow (P, p)^m$  which will be a two-sided inverse for  $\phi^*$ . Let  $[\xi_\lambda]$  be an element of  $\sum (P_\lambda, p)^m$ , and in each  $\xi_\lambda$  choose a map  $f_\lambda: P_\lambda \times I^m \rightarrow X$  such that

$$f_\lambda(p \times I^m \cup P_\lambda \times \dot{I}^m) = x_0.$$

Let  $f: P \times I^m \rightarrow X$  be the map which is  $f_\lambda$  on each  $P_\lambda \times I^m$ ; by the weak topology of  $P$ ,  $f$  is continuous, and so determines an element of  $(P, p)^m$ . This element is  $\sigma^*[\xi_\lambda]$ : by construction,  $\phi_\lambda^* \sigma^*[\xi_\lambda] = \xi_\lambda$ , so that  $\phi^*$  is a left inverse for  $\sigma^*$ . It is clear that  $\sigma^*[\xi_\lambda]$  does not depend on the choices of maps  $f_\lambda$  in the  $\xi_\lambda$ , and that  $\sigma^*$  is a homomorphism. Then, if  $\xi$  is an element of  $(P, p)^m$ , choose a representative map  $f$ , and in constructing

$$\sigma^* \phi^* \xi = \sigma^*[\phi^* \xi]$$

choose  $f|P_\lambda \times I^m$  for the map  $f_\lambda$ . Then it is clear that  $\sigma^* \phi^* \xi = \xi$ , so that  $\phi^*$  is also a right inverse for  $\sigma^*$ , and so both are isomorphisms onto.

**COROLLARY 7.42.** *If  $P_\lambda \cap P_\mu = \circ Q$  for all  $\lambda \neq \mu$ , and if  $P = \bigcup P_\lambda$  is such that  $P/Q$  is a cluster of spaces  $P_\lambda/Q$ , then*

$$(P, Q)^m \approx \sum (P_\lambda, Q)^m \approx (P_\lambda, Q)^m(X_\lambda, x_0; x_0),$$

where  $X_\lambda = X$  for all  $\lambda$ , and the group on the right is the track group of the collection  $[P_\lambda, Q]$ .

Without any restrictions on the topology of  $P/Q$ , the other conditions imply that the track group of the collection  $[P_\lambda, Q]$ ,  $(P_\lambda, Q)^m(X_\lambda, x_0; x_0)$ , is isomorphic to the strong sum  $\sum (P_\lambda, Q)^m$ ; the proof is similar to the proof of Theorem 7.41. The corollary is otherwise an application of this theorem and the Factor Theorem 4.2.

The isomorphism  $(P, Q)^m \approx (P_\lambda, Q)^m(X_\lambda, x_0; x_0)$  no longer exists if  $P/Q$  has a stronger topology: it is not difficult to construct a counter-example.

Suppose now that  $[P, Q; X]$  and  $[P_\lambda, Q; X]$  are HE triples for all  $\lambda$ , where  $Q$  is as in (7.42). Then we have a commutative diagram

$$\begin{array}{ccccccc} \rightarrow & (P_\lambda, Q)^m & \xrightarrow{i^*} & (P_\lambda, R)^m & \xrightarrow{j_\lambda^*} & (Q, R)^m & \xrightarrow{\delta_\lambda^*} & (P_\lambda, Q)^{m-1} & \rightarrow \\ & \phi_\lambda^* \uparrow \downarrow \sigma_\lambda^* & & \phi_\lambda^* \uparrow & & \uparrow \approx & & \phi_\lambda^* \uparrow \downarrow \sigma_\lambda^* & \\ \rightarrow & (P, Q)^m & \xrightarrow{i^*} & (P, R)^m & \xrightarrow{j^*} & (Q, R)^m & \xrightarrow{\delta^*} & (P, Q)^{m-1} & \rightarrow \end{array}$$

in which the horizontal sequences are exact, and the  $\sigma_\lambda^*$  are induced by the retractions  $P/Q \rightarrow P_\lambda/Q$  and the isomorphisms induced by the factor maps  $P \rightarrow P/Q$ ,  $P_\lambda \rightarrow P_\lambda/Q$ . Obviously, if  $\sigma^*$  is defined by the  $\sigma_\lambda^*$  as in Theorem 7.41, we have

**LEMMA 7.43.** *In the above diagram,  $\delta^*z = \sigma^*[\delta_\lambda^*z]$ , for  $z \in (Q, R)^{m+1}$ , and the image of  $j^*$  is the intersection of the images of the  $j_\lambda^*$ .*

(In order to prove the last when  $m = 1$ , we have to examine the set transformation  $\delta^*: (Q, R)^1 \rightarrow (P, R)^0(X, x_0)$ , as in the proof of Theorem 7.32.) This gives  $(P, R)^m$  as some extension of a factor group of

$$(P, Q)^m = \sum (P_\lambda, Q)^m$$

by a subgroup of  $(Q, R)^m$ . We might hope that a knowledge of all the sequences of  $[P_\lambda, Q, R]$  would enable us to compute  $(P, R)^m$ : we obtain this group as an extension in a different way, and the next theorem shows that something more is required, in general.

Now assume that  $P/R$  has the weak topology, in the sense that  $F \subset P/R$  is closed if and only if  $F \cap P_\lambda/R$  is closed for every  $\lambda$ . Augment the set  $\Lambda$  by adding a new symbol  $o$ , and define  $P_o = Q$ ; partially order the new set  $\Lambda^*$  by the relations  $\lambda > o$  for all  $\lambda \neq o$ . Then the groups  $(P_\lambda, R)^m$  form an inverse system of groups indexed by  $\Lambda^*$  with homomorphisms  $j_\lambda^*$ ; let  $G = \varprojlim (P_\lambda, R)^m$  be the inverse limit of the system, which is a subgroup of the strong sum  $(Q, R)^m + \sum (P_\lambda, R)^m$  (the summation of  $\sum$  being over  $\Lambda$ ).

Since  $\delta_\lambda^*(Q, R)^{m+1} \subset (P_\lambda, Q)^m$ , we have a subgroup  $\sum \delta_\lambda^*(Q, R)^{m+1}$  of the strong sum  $\sum (P_\lambda, R)^m$ ; this subgroup is mapped by the isomorphism  $\sigma^*$  onto a subgroup  $H$  of  $(P, Q)^m$ :

$$H = \sigma^*(\sum \delta_\lambda^*(Q, R)^{m+1}) \subset (P, Q)^m.$$

Define a homomorphism

$$D^*: (Q, R)^{m+1} \rightarrow H$$

by the equation

$$D^*(z) = \sigma^*[\delta_\lambda^* z], \quad \text{all } z \in (Q, R)^{m+1}.$$

Finally, define  $i'^*: H \rightarrow (P, R)^m$  by  $i'^* = i^*|_H$ , and a homomorphism  $j'^*: (P, R)^m \rightarrow G$  by the equation

$$j'^*(y) = [\phi_\lambda^* y, j^* y] \quad \text{all } y \in (P, R)^m.$$

**THEOREM 7.44.** *The sequence*

$$(Q, R)^{m+1} \xrightarrow{D^*} H \xrightarrow{i'^*} (P, R)^m \xrightarrow{j'^*} G \rightarrow 0$$

*is exact at  $H$ ,  $(P, R)^m$ , and  $G$ . Thus  $(P, R)^m$  is an (undetermined) extension of  $H/D^*(Q, R)^{m+1}$  by  $G$ .*

Notice that the groups  $H$ ,  $G$  and the homomorphism  $D^*$  are determined by the collection of sequences  $[P_\lambda, Q, R]$ . We first prove that  $j'^*$  is onto: let  $[y_\lambda, y_0]$  be an element of  $G$ , so that  $y_\lambda \in (P_\lambda, R)^m$  and  $j_\lambda^* y_\lambda = y_0 \in (Q, R)^m$ . Select a map  $f_0 \in y_0$ ; *ex hypothesi*, this has an extension  $f_\lambda \in y_\lambda$  for each  $\lambda$ , and these maps fit together to form a map  $f$  which defines  $y \in (P, R)^m$ , as  $P/R$  has the weak topology and the identification topology as a factor space of  $P$ . Then  $j'^* y = [y_\lambda, y_0]$ , as required.

If  $j'^* y = 0$ , for a  $y \in (P, R)^m$ , we see from the exact sequences that  $y = i'^* z$  for some  $z \in H$ ; for  $j'^* y = [\phi_\lambda^* y, j^* y]$ , so that  $j'^* y = 0$  implies  $\phi_\lambda^* y = 0$  and  $j^* y = 0$ , i.e.  $y = i^* z$  for some  $z$  in  $(P, Q)^m$ . Then  $z$  is in the subgroup  $H$ , for  $i_\lambda^* \phi_\lambda^* z = \phi_\lambda^* i^* z = \phi_\lambda^* y = 0$ , so that  $\phi_\lambda^* z = \delta_\lambda^* w(\lambda)$  for some  $w(\lambda)$  in  $(Q, R)^{m+1}$ , and so  $z = \sigma^*[\delta_\lambda^* w(\lambda)]$ , an element of  $H$ . Therefore  $i'^* H$  contains  $j'^*-1(0)$ . On the other hand,  $j'^* i'^* H = 0$ , since

$$\begin{aligned} j'^* i'^* \sigma^*[\delta_\lambda^* w(\lambda)] &= [\phi_\lambda^* i'^* \sigma^*[\delta_\lambda^* w(\lambda)], j^* i'^* \sigma^*[\delta_\lambda^* w(\lambda)]] \\ &= [i_\lambda^* \phi_\lambda^* \sigma^*[\delta_\lambda^* w(\lambda)], 0] \\ &= [i_\lambda^* \delta_\lambda^* w(\lambda), 0] = [0, 0] = 0. \end{aligned}$$

Therefore the sequence is exact at  $(P, R)^m$ .

Finally, it is clear that  $i'^* D^* = 0$ ; conversely, if  $i^* z = 0$  for  $z \in (P, Q)^m$ , then  $z = \delta^*$  for some  $w \in (Q, R)^{m+1}$ , and  $z = \sigma^*[\delta_\lambda^* w] = D^* w$ , an element of  $H$ . We have in fact shown that  $D^*(Q, R)^{m+1} = i'^*-1(0) = i^*-1(0)$ . This completes the proof of Theorem 7.44.

Now  $\phi_\lambda^*: (P, R)^m \rightarrow (P_\lambda, R)^m$  can be factored through  $G$ , since it is the composite of  $j'^*$  and the projection of  $G$  in  $(P_\lambda, R)^m$ ; therefore  $H$  is in the kernel of every  $\phi_\lambda^*$ , so that the groups  $(P_\lambda, R)^m$  cannot give information about the extension of  $i'^* H$  by  $G$ .

### 8.1. Suspensions and products with a sphere

Let  $\langle P$  denote the join of  $P$  with a point  $A$  not in  $P$ ,  $P\rangle$  the join of  $P$  with a point  $B$  not in  $P$ , and  $\langle P\rangle$  the union of these spaces.  $\langle P\rangle$  is called the suspension of  $P$ , and contains  $\langle Q\rangle$ , the join of  $Q$  with  $A \cup B$ , for every non-empty  $Q$  in  $P$ . The suspension  $\langle 0\rangle$  of the empty set is defined to be  $A \cup B$ , and likewise  $\langle 0 = A, 0\rangle = B$ . Then it is easy to see that the elements of  $(\langle P\rangle, \langle Q\rangle)^0(X, x_0)$  and of  $(P, Q)^1$  are in 1-1 correspondence (even if  $Q$  is empty), and that the Excision Theorem (4.1) and Theorem 7.32 imply, under certain conditions, that

$$(\langle P\rangle, \langle Q\rangle)^m \approx (\langle P\rangle, \langle Q\rangle \cup P)^m \approx (\langle P, \langle Q \cup P\rangle)^m \approx (P, Q)^{m+1}.$$

Let  $p_0 \in P$ ,  $x_0 \in X$ , and let  $\phi: P \times I \rightarrow P^* = P \times I / (P \times \dot{I} \cup p_0 \times I)$ ,  $\phi: X \times I \rightarrow X^* = X \times I / (X \times \dot{I} \cup x_0 \times I)$  be the factor maps, and let  $\phi(x_0) = x_0^*$  be chosen as base-point in  $X_0^*$ . Then, if  $f$  represents  $\{f\}$  in  $(P, Q)^m(X, x_0; x_0)$  where  $p_0 \in Q \subset P$ , define  $Sf: P^* \times I^m \rightarrow X^*$  by

$$(Sf)(\phi(p, t), y_1, \dots, y_m) = \phi(f(p, y_1, \dots, y_m), t);$$

then  $\{Sf\}$  represents an element of  $(P^*, Q^*)^m(X^*, x_0^*; x_0^*)$ , and it is clear that  $S^*\{f\} = \{Sf\}$  defines a homomorphism

$$S^*: (P, Q)^m(X, x_0; x_0) \rightarrow (P^*, Q^*)^m(X^*, x_0^*; x_0^*),$$

where  $Q^* = \phi(Q \times I) \subset P^*$ . If  $h: [P, Q] \rightarrow [P', Q']$ , we define

$$Sh: [P^*, Q^*] \rightarrow [P'^*, Q'^*]$$

in a similar way, and state:

LEMMA 8.11.  *$S^*$  is a natural homomorphism, such that*

$$(Sh)^* \circ S^* = S^* \circ h^* \quad \text{and} \quad S^* \circ \delta^* = \delta^* \circ S^*.$$

Hence  $S^*$  maps the exact sequence of  $[P, Q, R]$  over  $[X, x_0]$  into that of  $[P^*, Q^*, R^*]$  over  $[X^*, x_0^*]$ , if  $R$  is not empty.

We now give some results on products with spheres:

LEMMA 8.12.

$$(P, Q)^{m+n} \approx (P \times I^n, Q \times I^n \cup P \times \dot{I}^n)^m \approx (P \times S^n, Q \times S^n \cup P \times w)^m,$$

where  $w$  is a point of  $S^n$ . Also, if  $Q$  is not empty,  $(\langle P\rangle, \langle Q\rangle)^m \approx (P, Q)^{m+1}$ ; while if  $Q$  is empty, so that  $\langle Q\rangle = A \cup B$ ,

$$(\langle P\rangle, \langle Q\rangle)^m \approx (P, Q)^{m+1} + (A)^m + (B)^{m+1},$$

when  $[P, p; X]$  and  $[P, p; P]$  are HE triples.

These results follow from the definitions and the Factor Theorem (4.2); the last uses also Lemma 7.33 (see the example given there) and the Retract Theorem applied to  $(\langle P\rangle, A)^m$ .

COROLLARY 8.13. *If  $[P, Q; X]$  is an HE triple, then*

$$(P \times S^n, Q \times S^n)^m \approx (P, Q)^{m+n} + (P, Q)^m, \quad \text{if } m > 1,$$

*while  $(P \times S^n, Q \times S^n)^1$  is a split extension of  $(P, Q)^{n+1}$  by  $(P, Q)^1$ .*

To prove this, we factor the spaces on the left by  $Q \times S^n$ , and those on the right by  $Q$ . Now  $P/Q$  is a retract of  $P \times S^n / Q \times S^n$ . The result then follows from the previous lemma, the Retract Theorem 7.1, and the Factor Theorem 4.2.

LEMMA 8.14. *If  $p$  is a point,  $(p)^m \approx \pi_m(X, x_0)$ .*

The lemma is obvious if we identify  $p \times I^m$  and  $I^m$  in the obvious way; the definitions of the two groups are then the same (cf. (10)). As a consequence of this and the previous lemma we have  $(S^n, w)^m \approx \pi_{n+m}(X, x_0)$ , while  $(S^n)^m$  is a split extension of this by  $\pi_m(X, x_0)$ , which is naturally the direct sum if  $m > 1$ . It is possible to show that the operations of  $\pi_1$  on  $\pi_{m+1}$  induced by the inner automorphisms of  $(S^n)^1$  by the elements of the subgroup  $\pi_1$  are the usual operations. In view of Theorem 6.1, these results may be found in (11, 13), and, as the next lemma shows, in (1, 12, 13).

LEMMA 8.15. *For all  $m \geq 1$ , and all  $n > r \geq 0$ , the abhomotopy group  $\kappa_{r+m}^{n+m}$  is  $(S^n, S^r)^m \approx (S^{n-r-1})^{m+r+1} \approx \pi_{m+n} + \pi_{m+r+1}$ .*

Let  $S^{-1}$  be the empty set. The Abhomotopy Group  $\kappa_r^n$  (1, 12) is defined as  $(S^n, S^r)^0(X, x_0)$ , for  $r \geq 0$ , and it is easily verified that the definitions of addition in the abhomotopy group and the track group are equivalent. The lemma may be proved by induction on  $r$ , using the previous lemma and the homotopy equivalence  $[S^n, S^r] \equiv [\langle S^{n-1} \rangle, \langle S^{r-1} \rangle]$ ; it is obviously true when  $r = -1$ .

## 8.2. Track groups of product spaces

If  $E$  is a split extension of  $A$  by  $B$ , write  $E = A \oplus B$ ; if  $E_1 = A_1 \oplus A_2$ , and  $E_n = E_{n-1} \oplus A_{n+1}$ , write  $E_n = \sum^* A_i$  (an ordered 'sum'); we call  $\sum^* A_i$  a repeated split extension. Then  $E = \sum^* A_i$  has subgroups  $A_i$ , and is generated by the elements of these subgroups; the structure of  $E$  is completely determined by these subgroups  $A_i$  and the commutators of  $E$ .

Let  $c_p^k$  ( $p = 0, 1, \dots, k$ ) denote a subset  $i_1, \dots, i_p$  of the integers  $1, \dots, k$  in their natural order; for fixed  $k$ , the  $c_p^k$  have an obvious partial ordering by inclusion. We write the product space  $P_{i_1} \times \dots \times P_{i_p} = P(c_p^k)$ , and embed it in  $P = P_1 \times \dots \times P_k$  in a natural way: choose points  $z_i \in P_i$  for each  $i$ , and let  $(p_{i_1}, \dots, p_{i_p})$  be identified with  $(q_1, \dots, q_k)$  where  $q_i = p_{i_j}$  if  $i$  is in the subset  $c_p^k$ , and  $z_i$  otherwise. Then it is clear that  $P(c_p^k)$  is a retract of  $P = P(c_k^k)$  for every  $c_p^k$ . For convenience, define  $c_0^k$  to be the empty set, and  $P(c_0^k)$  to be the point  $(z_1, \dots, z_k)$ .

Let  $Q(c_0^k)$  be the empty set, and let  $Q(c_p^k)$  be the union of spaces  $P(c_r^k)$  for which  $c_r^k$  is properly contained in  $c_p^k$ ; then  $Q(c_p^k)$  is a subset of  $P(c_p^k)$ . For example,  $P(c_2^2)$  is the product  $P_1 \times P_2$ , and  $Q(c_2^2)$  consists of the 'axes'  $P_1 \times z_2, z_1 \times P_2$ .

We wish to decompose the group  $(P)^m$  into the direct sum (or, if  $m = 1$ , into a repeated split extension) of the groups  $(P(c_p^k), Q(c_p^k))^m$ , by means of the retract theorem. For this we need various HE triples: we say that  $[P_i; X]$  has the HE property if all the spaces  $P_i$  are locally finite CW complexes, or are all compact ANR's, or are all separable metric spaces while  $X$  is a compact ANR.

**THEOREM 8.2.** *If  $P = P_1 \times \dots \times P_k$ , and  $[P_i; X]$  has the HE property, then  $(P)^m \approx \sum (P(c_p^k), Q(c_p^k))^m$ , a direct sum, if  $m > 1$ , while  $(P)^1$  is a repeated split extension  $\sum^* (P(c_p^k), Q(c_p^k))^1$ . Here  $c_p^k$  assumes all possible  $2^k$  values.*

We deduce this theorem from the next, more general, theorem; however, it can be proved directly from the Retract Theorem 7.1, by induction on  $k$ .

Suppose now that a space  $R$  contains a number of closed sub-spaces  $R_i$  containing a common point  $r_0$ . Let  $S_i = \bigcup_{t < i} R_t$ , the union of the first  $i$  sub-spaces, and let  $\phi_i: R \rightarrow R/S_i$  be the factor map which squeezes  $S_i$  to a point; for convenience we let  $\phi_0: R \rightarrow R$  be the identity map.

**THEOREM 8.3.** *If, for each  $s \geq 0$ ,  $[\phi_s R, \phi_s R_{s+1}; X]$  is an HE triple in which  $\phi_s R_{s+1}$  is a retract of  $\phi_s R$ , then when  $m > 1$ ,*

$$(R)^m \approx (R_1)^m + \sum_{1 \leq s \leq k} (\phi_s R_{s+1}, \phi_s R_s)^m + (\phi_k R, \phi_k R_{k+1})^m;$$

and when  $m = 1$ ,  $(R)^m$  is a repeated split extension of these groups, in reverse order.

If  $k = 0$ , this is exactly the retract theorem. Assume that the theorem has been proved true for  $k = n-1$ , so that  $(R)^m$  has a direct summand  $(\phi_{n-1} R, \phi_{n-1} R_n)^m$ , or, when  $m = 1$ ,  $(R)^m$  is a repeated split extension of this group by other groups. Then by the Retract Theorem 7.1 and the Factor Theorem 4.2,

$$\begin{aligned} (\phi_{n-1} R, \phi_{n-1} R_n)^m &\approx (\phi_n R, \phi_n R_n)^m \\ &\approx (\phi_n R, \phi_n R_{n+1})^m \oplus (\phi_n R_{n+1}, \phi_n R_n)^m, \end{aligned}$$

where the split extension is the direct sum if  $m > 1$ . Hence the theorem is true also for  $k = n$ , and so, by the principle of finite induction, for all finite  $k$ .

Notice, in the above notation, that if  $\phi_r R_n$  is a retract of  $\phi_r R$ , then also  $\phi_s R_n$  is a retract of  $\phi_s R$  for all  $s \geq r$ .

To apply Theorem 8.3 to the proof of Theorem 8.2, give the  $c_p^k$  any simple ordering consistent with their partial order, so that  $c_0^k$  is the first, and  $c_p^k$  the  $2^k$ th element. Take  $R = P$ ,  $R_i = P(c_p^k)$ , where  $c_p^k$  is the  $i$ th element in the simple order. Then, when  $R_i = P(c_p^k)$ ,  $Q(c_p^k)$  is contained in the union of the first  $(i-1)$  subspaces  $R_t$ , since it consists of the union of the  $P(c_r^k)$  for which  $c_r^k$  precede  $c_p^k$  in the natural partial order; conversely, the intersection of  $R_i$  and  $R_t$  for  $t < i$  is contained in  $Q(c_p^k)$ . Hence  $\phi_{i-1} R_i = P(c_p^k)/Q(c_p^k)$ , while  $R_1 = P(c_0^k) = P(c_0^k)/Q(c_0^k)$ . Therefore if we replace  $k$  in Theorem 8.3 by  $2^k$ , the groups of Theorem 8.3 are those of Theorem 8.2 after an application of the Factor Theorem 4.2, together with the trivial group.

Though we shall not determine the commutators of  $(P)^1$  in Theorem 8.2, we can prove

LEMMA 8.4. *If  $a \in (P(c_p^k), Q(c_p^k))^1$ ,  $b \in (P(c_q^k), Q(c_q^k))^1$ , the commutator  $a+b-a-b$  is in  $(P(c_r^k), Q(c_r^k))^1$ , where  $c_r^k = c_p^k \cup c_q^k$ , the set of integers in either  $c_p^k$  or  $c_q^k$  or both.*

If  $E = A \oplus B$ ,  $B$  can be identified with a subgroup of  $E$ , while  $A$  is already an invariant subgroup. Let  $c_p^k$  be the  $i$ th element in the chosen simple ordering of the  $c$ 's; an examination of the proof of Theorem 8.3 shows that  $(P)^1$  contains an invariant subgroup generated by all the groups  $(P(c_t^k), Q(c_t^k))^1$  for which  $c_t^k$  (including  $c_p^k$  itself) do not precede  $c_p^k$  in the chosen order. Therefore, the commutator  $a+b-a-b$  is generated by these elements; similarly, it is generated by elements from groups defined by  $c_t^k$ 's which do not precede  $c_q^k$  in this simple order, and so which do not precede either  $c_p^k$  or  $c_q^k$ .

On the other hand, the simple ordering may be chosen in any way that is consistent with the natural partial order. By considering all possible simple orderings of this type, we see that the  $c_t^k$ 's above can be made to satisfy the condition that they do not precede  $c_r^k$ , the least element greater than both  $c_p^k$  and  $c_q^k$ . But  $P(c_r^k)$  is a retract of  $P$ , so that

$$(P)^1 \approx (P, P(c_r^k))^1 \oplus (P(c_r^k))^1.$$

The second group is a subgroup of  $(P)^1$  containing both  $a$  and  $b$ , and so  $a+b-a-b$ ; however, the only elements of the above type are in

$$(P(c_r^k), Q(c_r^k))^1,$$

which completes the proof of the lemma.

In the special case when each  $P_i$  is a sphere  $S^{n_i}$ , the group

$$(P(c_p^k), Q(c_p^k))^m \approx \pi_N(X, x_0),$$

where  $N = m + n_{i_1} + \dots + n_{i_p}$  if  $c_p^k = i_1, \dots, i_p$ . This can be readily seen by expressing  $P$  as a product complex, when  $P(c_p^k)/Q(c_p^k)$  appears as a sphere  $S^{N-m}$ ; alternatively, this result will follow directly from the Retract



Theorem 7.1, Lemma 8.12, and Lemma 8.14. This other method shows that the commutator  $a+b-a-b$  of Lemma 8.4 is zero if  $c_p^k$  and  $c_q^k$  have a common integer. The commutators in this case can be determined by the methods of § 10 in the next paper, or may be deduced by Fox's theorem (10) on the structure of Torus homotopy groups. It is clear that the Torus homotopy group  $\tau_n$  is  $(S_1^1 \times \dots \times S_{n-1}^1)^1$ : Fox has shown that in this case  $a+b-a-b$  is  $\pm[a, b]$ , the Whitehead product, where the sign is determined by  $c_p^{n-1}, c_q^{n-1}$ . If  $n_1 + \dots + n_k = n-1$ , let

$$\phi: S_1^1 \times \dots \times S_{n-1}^1 \rightarrow S^{n_1} \times \dots \times S^{n_k}$$

be the identification map which maps the product of the first  $n_1$  circles on  $S^{n_1}$ , the next  $n_2$  on  $S^{n_2}$ , and so on, mapping the cells  $(S_1^1 - z_1) \times \dots \times (S_{n_1}^1 - z_{n_1})$ , etc., with degree  $+1$  on the cells  $(S^{n_1} - z_1)$ , etc. Then it is possible to show that the induced homomorphism  $\phi^*$  is an isomorphism into, and so that the commutators of  $(S^{n_1} \times \dots \times S^{n_k})^1$  are determined by Whitehead products; in this case, the determination of the sign is more complex, and the integers  $n_1, \dots, n_k$  enter into it as weighting factors.

#### CHAPTER 4

#### CW COMPLEXES

### 9. Track groups of CW complexes

Let  $K$  be a path-connected CW complex, as defined in (24); let  $K^n$  be the  $n$ -skeleton of  $K$  (the union of cells of dimension not exceeding  $n$ ) and let  $K^{-1}$  be the empty set. If  $L$  is any closed subcomplex, we may turn  $K/L$  into a complex in a natural way, such that  $K^n/(L \cap K^n)$  is the  $n$ -skeleton of  $K/L$ . Then there is a tree  $T$  in  $K/L$  which contains all the vertices, i.e.  $K^0/(L \cap K^0) \subset T \subset K^1/(L \cap K^1)$  (for proof, see (20), p. 322). We define  $\bar{K} = (K/L)/T$ , so that we first shrink  $L$  to a point, then  $T$ , obtaining a complex such that  $\bar{K}^0$  is a point.

Define  $(n+r, n)^m = (\bar{K}^{n+r}, \bar{K}^n)^m$  for all  $m, r > 0, n \geq -1$ , and call the exact sequence of  $[\bar{K}^{n+r}, \bar{K}^n, \bar{K}^{n-s}]$  the exact sequence of  $[n+r, n, n-s]$ .

**LEMMA 9.11.** *If  $n > 0$ , then  $(K^{n+r} \cup L, K^n \cup L)^m \approx (n+r, n)^m$ .*

This lemma follows immediately from the Factor Theorem 4.2.

Now select a vertex  $e^0$  of  $K$ , choosing  $e^0$  in  $L$  if  $L$  is not empty; let  $e_\lambda^0$  be the vertices of  $K - (L \cup e^0)$ .

**LEMMA 9.12.**  *$(r, 0)^m \approx (K^r \cup L, e^0 \cup L)^m$ , while, if  $\sum$  is the strong sum,*

$$(K^r \cup L, K^0 \cup L)^m \approx (r, 0)^m + \sum (e_\lambda^0)^{m+1}.$$

The first statement is a consequence of the Factor Theorem 4.2, and Theorem 7.32, since  $e^0$  is a deformation retract of  $T$ . Also, by the Factor Theorem,  $(K^r \cup L, K^0 \cup L)^m \approx ((K/L)^r, (K/L)^0)^m$ ; let  $e^0, e_\lambda^0$  be the vertices of  $K/L$  corresponding to the vertices  $e^0, e_\lambda^0$  of  $K$ , and apply Lemma 7.33 with  $S = T$ ,  $P = (K/L)^r$ ,  $Q = (K/L)^0$ ,  $R = e^0$ , obtaining

$$((K/L)^r, e^0)^m + ((K/L)^0, e^0)^{m+1}.$$

To this last group, apply the Cluster Theorem 7.41, treating  $(K/L)^0$  as a cluster of spaces  $(e^0 \cup e_\lambda^0)$  with  $e^0$  as a common point, and finally excise  $e^0$  by the Excision Theorem 4.1. This proves the lemma.

The group  $(r-1, -1)^m$  can be obtained from the retract theorem when  $m \geq 1$ ; when  $m = 1$ , it is a split extension of  $(r-1, 0)^1$  by  $\pi_1(X, x_0)$ . The particular case  $(1, -1)^1$  can be obtained from Theorem 7.44 and our knowledge (see Theorem 8.2, Lemma 8.4) of  $(S^1)^1$ ; however, the nature of  $(2, -1)^1$  is not known in general (for example, if  $K$  is the real projective plane).

In future, we shall deal only with the groups  $(n+r, n)$ ; thus we shall only have to consider complexes  $K$  such that  $K^0 = e^0$ , a single point.

The group of  $n-G$ -cochains over  $K$  we take as  $H^n(K^n, K^{n-1}; G)$ ; any rule associating an element of  $G$  with each  $n$ -cell of  $K$  uniquely determines such a cochain. The Factor and Cluster Theorems then immediately imply

LEMMA 9.13. *For all  $m+n > 1$ , there is a natural isomorphism*

$$\theta: (n, n-1)^m \approx C^n(K; \pi_{m+n}),$$

where  $\pi_{m+n} = \pi_{m+n}(X, x_0)$ .

For if  $n > 0$ ,  $K^n/K^{n-1}$  is a cluster of spheres  $S_\lambda^n$  in 1-1 correspondence with the cells  $e_\lambda^n$  of  $K$ . By Theorems 4.2 and 7.41,  $(n, n-1)^m \approx \sum (S_\lambda^n, e^0)^m$ , and, by (8.12, 8.14),  $(S_\lambda^n, e^0)^m \approx \pi_{m+n}$ . Let this isomorphism make  $z \in (n, n-1)^m$  correspond to  $[z_\lambda]$ , where  $z_\lambda \in \pi_{m+n}$ , and define  $\theta z(e_\lambda^n) = z_\lambda$ , so that  $\theta z$  is a cochain in  $C^n(K; \pi_{m+n})$ . When  $n = 0$ , a similar, easier, proof exists.

$\theta$  is natural in the sense that a cellular map  $g: K \rightarrow K^*$  induces

$$g^*: C^n(K^*; \pi_{m+n}) \rightarrow C^n(K; \pi_{m+n}),$$

as well as a homomorphism  $g^*$  of the track group, and

$$\theta \circ g^* = g^* \circ \theta.$$

Similarly, a map  $\psi: [X, x_0] \rightarrow [Y, y_0]$  induces homomorphisms  $\psi_*$  such that  $\theta \circ \psi_* = \psi_* \circ \theta$ .

## 9.2. The obstruction homomorphism

We now relate the homomorphism  $\delta^*: (n, n-s)^m \rightarrow (n+1, n)^m$  to the concept of obstruction cocycle (cf. (6)).

Let  $I = 0 \cup 1 \cup e^1$  be an expression of the unit interval as a CW complex, and let the product  $A \times I$  (where  $A$  is any CW complex) be the CW complex with cells  $e \times 0, e \times 1, e \times e^1$  (all cells  $e \subset A$ ), as described on p. 227 of (24). Treat  $I^m$  as the product of  $I$   $m$  times with itself, and define  $K^{n+1} \times I^m = P_m$  as a CW complex. For convenience, we name various subcomplexes of  $P_m$ :

$$\begin{aligned} P_m &= K^{n+1} \times I^m, & P_{m-1} &= K^{n+1} \times I^{m-1} \times 0 = K^{n+1} \times I^{m-1}; \\ Q_m &= K^n \times I^m, & Q_{m-1} &= K^n \times I^{m-1} \times 0 = K^n \times I^{m-1}; \\ Q_m^* &= (K^n \times I^m) \cup (K^{n+1} \times \dot{I}^m), \\ Q_{m-1}^* &= (K^n \times I^{m-1} \times 0) \cup (K^{n+1} \times \dot{I}^{m-1} \times 0); \\ R_m^* &= (K^{n-s} \times I^m) \cup (K^{n+1} \times \dot{I}^m), \\ R_m^{**} &= (K^{n-s} \times I^m) \cup (K^n \times \dot{I}^m), \\ S_m &= (K^{n-s} \times I^m) \cup (K^n \times \dot{I}^m) \cup (K^{n+1} \times J^{m-1}). \end{aligned}$$

Then  $Q_m^*$  is the  $(m+n)$ -skeleton of the  $(m+n+1)$ -dimensional complex  $P_m$ , whose  $m+n+1$ -cells are the products  $e_\lambda^{n+1} \times e^m$ , where  $e^m = I^m - \dot{I}^m$ , and  $e_\lambda^{n+1}$  are the cells of  $K^{n+1} - K^n$ .

Let  $\phi_\lambda: I^{n+1} \rightarrow K^{n+1}$  be a *characteristic map* for the cell  $e_\lambda^{n+1}$  (cf. (24), p. 221), a homeomorphism of  $I^{n+1} - \dot{I}^{n+1}$  onto  $e_\lambda^{n+1}$ . Since  $K^{n+1}$  has but one vertex, without loss of generality we may suppose that  $\phi_\lambda(I^0) = e^0$ ; this requirement can always be met by some choice of a complex in the homotopy type of  $[K^{n+1}, K^n, K^{n-s}, e^0]$  (see Lemma 3, p. 239, of (24)). Then

$$\phi_\lambda^y: I^{n+m+1} \rightarrow P_m,$$

as defined in (1.1), is a characteristic map for  $e_\lambda^{n+1} \times e^m$ , and

$$\phi_\lambda^y(I^0) = e^0 \times (0, \dots, 0);$$

define  $\phi_\lambda^y = \phi_\lambda^y| \dot{I}^{m+n+1}: \dot{I}^{m+n+1} \rightarrow Q_m^*$ , the *attaching map* of the cell  $e_\lambda^{n+1} \times e^m$ , and let the same letter stand for the same map of the pair

$$[\dot{I}^{m+n+1}, I^0] \rightarrow [Q_m^*, e^0 \times I^0].$$

Now let  $f: [Q_m^*, S_m] \rightarrow [X, x_0]$ ; the *obstruction cocycle* (6)  $c(f)$  to the extension of  $f$  over  $[P_m, S_m]$  to  $[X, x_0]$  is the cocycle in  $C^{m+n+1}(P_m; \pi_{m+n})$  with the value  $\{f \circ \phi_\lambda^y\}$  on the cell  $e_\lambda^{n+1} \times e^m$ . Clearly, such an extension exists if and only if  $c(f) = 0$ .

Now let  $f: [P_m, S_m] \rightarrow [X, x_0]$ , not necessarily mapping  $K^{n+1} \times I^m$  to  $x_0$ . We define three associated maps

$$f_0, f_1, f_2: [Q_m^*, S_m] \rightarrow [X, x_0];$$

$f_0$  will be  $f| [Q_m^*, S_m]$ ;  $f_1$  will be the extension of  $f| [Q_m, R_m^{**}]$  obtained by mapping  $K^{n+1} \times \dot{I}^m \subset S_m \cup P_{m-1} \subset Q_m^*$  to the point  $x_0$ ;  $f_2$  will map  $Q_m \cup S_m$  to  $x_0$ , and on  $P_{m-1}$  will agree with  $f$ . Thus  $f_0, f_1$  agree on  $Q_m$ ,  $f_0, f_2$  agree on  $P_{m-1}$ ; notice that  $Q_m^* = Q_m \cup S_m \cup P_{m-1}$ .

Consider  $\phi_\lambda^j: I^{m+n+1} \rightarrow Q_m^*$ ; it maps  $I^{n+1} \times J^{m-1}$  to  $S_m$ ,  $I^{n+1} \times I^{m-1}$  to  $P_{m-1}$ , and  $I^{n+1} \times I^m$  to  $Q_m$ . Then, if  $J^{m-1}$  is properly oriented, and as

$$I^{m+n+1} = I^{n+1} \times I^m + (-1)^{m+n+1} I^{n+1} \times I^{m-1} + (-1)^{n+1} I^{n+1} \times J^{m-1},$$

we have, for every  $\lambda$ ,

$$\{f_0 \circ \phi_\lambda^j\} = \{f_1 \circ \phi_\lambda^j\} + \{f_2 \circ \phi_\lambda^j\} \in \pi_{m+n}(X, x_0).$$

Therefore  $c(f_0) = c(f_1) + c(f_2)$  in  $C^{n+m+1}(P_m; \pi_{m+n})$ ; however,  $f_0$  has the extension  $f$ , so that  $c(f_0) = 0$  and we have proved

LEMMA 9.21. *If  $f: [P_m, S_m] \rightarrow [X, x_0]$  defines  $f_1, f_2: [Q_m^*, S_m] \rightarrow [X, x_0]$  as above, the obstruction cocycles to the extensions of  $f_1, f_2$  over  $P_m$  satisfy*

$$c(f_1) = -c(f_2).$$

Now  $C^{m+n+1}(P_m; \pi_{m+n})$  is defined as  $H^{m+n+1}(P_m, Q_m^*; \pi_{m+n})$ ; consider the relative chain groups  $C^{m+k}(P_m, R_m^*; G) = H^{m+k}(P_m^{m+k}, P_m^{m+k-1} \cup R_m^*; G)$ , and define

$$\psi: C^{m+k}(P_m, R_m^*; G) \simeq C^k(K^{n+1}, K^{n-s}; G)$$

as the map  $(\psi c)(e_\lambda^k) = c(e_\lambda^k \times e^m)$ . Then, as  $C^{m+n+1}(P_m; \pi_{m+n})$  and

$$C^{m+n+1}(P_m, R_m^*; \pi_{m+n})$$

are defined as the same groups, we have in particular

$$\psi: C^{m+n+1}(P; \pi_{m+n}) \simeq C^{n+1}(K^{n+1}, K^{n-s}; \pi_{m+n}).$$

Then  $\psi^{-1}$  is a type of suspension isomorphism, so that if  $\delta'$  is the coboundary operator in the cochain sequence of  $[P_m, R_m^*]$ , and  $\delta$  is the corresponding operator in the cochain sequence  $[K^{n+1}, K^{n-s}]$ ,  $\psi \circ \delta' = \delta \circ \psi$ . Also, if  $f_1, f_2$  are as in the previous lemma, so that  $f' = f|_{[P_{m-1}, Q_{m-1}^*]}$  represents an element of  $(n+1, n)^{m-1}$ , we have

$$\theta\{f'\} = \theta d^*\{f\} = (-1)^{m+n+1} \psi c(f_2),$$

where  $\theta$  is the isomorphism of Lemma 9.13, and  $d^*$  is the homomorphism defined in (5.3), with  $P = K^{n+1}$ ,  $Q = K^n$ ,  $R = K^{n-s}$ . (The correct value for the sign may be obtained from the formula for  $I^{m+n+1}$  given above.)

With these conventions, the previous lemma implies

LEMMA 9.22. *Let  $g: [Q_m, R_m^{**}] \rightarrow [X, x]_0$ , representing an element*

$$\{g\} \in (n, n-s)^m,$$

*define  $g_1: [Q_m^*, R_m^*] \rightarrow [X, x_0]$ , the extension of  $g$  which maps  $K^{n+1} \times I^m$  to  $x_0$ . Then, if  $c(g_1)$  is the obstruction to the extension of  $g_1$  over  $P_m$ , and*

$$\delta^*\{g\} \in (n+1, n)^m,$$

*we have*

$$\psi c(g_1) = (-1)^{n+m} \theta \delta^*\{g\}.$$

For we have seen (Lemma 5.32) that there is a map  $f: [P_m, S_m] \rightarrow [X, x_0]$  such that  $g = f| [Q_m, R_m^{**}]$ . Then  $g_1 = f_1$  as defined above, and

$$d^*\{f\} = \{f'\} = \delta^*\{g\}.$$

But  $\theta\{f'\} = (-1)^{m+n+1}\psi c(f_2)$ , so, by the previous lemma,

$$\theta\delta^*\{g\} = (-1)^{m+n}\psi c(f_1) = (-1)^{m+n}\psi c(g_1).$$

This lemma enables us to apply our knowledge of the first and second obstructions to the calculation of the elements of  $(n+1, n-1)^m$  and  $(n+1, n-2)^m$  respectively (though this does not determine the group structure, leaving an extension problem).

### 9.3. Central extensions

An extension  $E$  of  $G$  by  $Q$  (where  $Q = E/G$ ) is called *central* if  $G$  is in the centre of  $E$ ; clearly, an abelian extension is a central extension.

Consider  $(n+1, n)^m \xrightarrow{i^*} (n+1, n-s)^m \xrightarrow{j^*} (n, n-s)^m$  in the exact sequence of  $[n+1, n, n-s]$ : from the exactness,  $(n+1, n-s)^m$  is an extension of  $i^*(n+1, n)^m$  by  $j^*(n+1, n-s)^m$ .

**THEOREM 9.3.**  $(n+1, n-s)^m$  is a central extension of  $i^*(n+1, n)^m$  by  $j^*(n+1, n-s)^m \subset (n, n-s)^m$ , for all  $m \geq 1$ , and all  $n \geq s$ .

The theorem is trivial if  $m > 1$ , as the groups are all abelian. When  $m = 1$ , and  $P_1, Q_1^*$ , etc., are as in (9.2), we can represent any element  $a \in i^*(n+1, n)^1$  by a map  $f: [P_1, R_1^*]$  such that  $f(Q_1^*) = x_0$ . Let  $b$  be any element of  $(n+1, n-s)$ , represented by a map  $g: [P_1, R_1^*] \rightarrow [X, x_0]$ , and define a map  $h$  representing the commutator  $a+b-a-b$  by the equations:

$$\begin{aligned} h(p, y_1) &= f(p, 4y_1) & \text{if } 0 \leq y_1 \leq \tfrac{1}{4}, \\ &= g(p, 4y_1 - 1) & \text{if } \tfrac{1}{4} \leq y_1 \leq \tfrac{1}{2}, \\ &= f(p, 3 - 4y_1) & \text{if } \tfrac{1}{2} \leq y_1 \leq \tfrac{3}{4}, \\ &= g(p, 4 - 4y_1) & \text{if } \tfrac{3}{4} \leq y_1 \leq 1. \end{aligned}$$

Define  $h'$  by the same equations, with  $f$  replaced by the constant map  $x_0$ . Then  $h'$  represents  $0+b-0-b=0$ ; we show that  $h \simeq h' \text{ rel } Q_1^*$ , so that also  $a+b-a-b=0$ .

Turn  $P_1$  into a CW complex in two ways: let  $P_1'$  be the complex previously considered, with cells  $e \times 0, e \times 1, e \times e^1$ , for all cells  $e$  of  $K^{n+1}$ . Now let  $I$  be another complex, with five vertices  $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ , and four 1-cells  $I_1, \dots, I_4$ , the open intervals  $]0, \frac{1}{4}[, \dots, ]\frac{3}{4}, 1[$ . This gives a product complex  $P_1'' = K^{n+1} \times I$ , consisting of cells  $e \times (t/4), e \times I_s$ , for all integers  $t, s$ , such that  $0 \leq t \leq 4, 1 \leq s \leq 4$ .

Let  $\lambda, \mu$  be the identity maps in

$$P'_1 \xrightarrow{\mu} P''_1 \xrightarrow{\lambda} P_1 \xrightarrow{f, g, x_0, h, h'} X$$

so that  $\mu$  is a cellular map. As  $f(Q_1^*) = x_0$ , the maps  $f \circ \lambda \circ \mu, x_0 \circ \lambda \circ \mu$ , agree on the  $(n+1)$ -skeleton  $Q_1^*$  of  $P'_1$ , and so therefore  $h \circ \lambda \circ \mu, h' \circ \lambda \circ \mu$ , agree on this skeleton. Since  $f(K^{n+1} \times I) = g(K^{n+1} \times I) = x_0$ , a point, it follows also that  $h \circ \lambda, h' \circ \lambda$ , agree on the  $(n+1)$ -skeleton of  $P''_1$ , which is  $Q_1^* \cup \bigcup (K^{n+1} \times \frac{1}{2}I)$ . Therefore we have separation cochains

$$d = d(f \circ \lambda \circ \mu, x_0 \circ \lambda \circ \mu) \in C^{n+2}(P'_1; \pi_{n+2}),$$

$$d' = d(h \circ \lambda \circ \mu, h' \circ \lambda \circ \mu) \in C^{n+2}(P'_1; \pi_{n+2}),$$

$$d'' = d(h \circ \lambda, h' \circ \lambda) \in C^{n+2}(P''_1; \pi_{n+2});$$

these are related by the equations

$$d' = \mu_* d'',$$

$$d''(e^{n+1} \times I_{2l-1}) = (-1)^{l-1} d(e^{n+1} \times e^1), \quad i = 1, 2,$$

$$d''(e^{n+1} \times I_{2l}) = 0, \quad i = 1, 2.$$

The last relation follows from the fact that  $h, h'$  are the same on  $K^{n+1} \times y_1$ , if  $y_1 \in I_{2l}$ . The last two equations imply that

$$0 = \mu_* d'' = d',$$

and hence  $h \circ \lambda \circ \mu \simeq h' \circ \lambda \circ \mu$ , rel  $Q_1^*$ . Therefore  $\{h\} = \{h'\} = 0$ , as was to be proved.

#### 9.4. The elements of $(n+1, n-1)^m$

If  $k$  is an integer and  $c$  a cochain, define  $(k\delta)c = k(\delta c)$ . Then there is a diagram

$$\begin{array}{ccc} (n, n-1)^m & \xrightarrow{\delta^*} & (n+1, n)^{m-1} \\ \downarrow \theta & & \downarrow \theta \\ C^n(K; \pi_{m+n}) & \xrightarrow{(-1)^{m+n}\delta} & C^{n+1}(K; \pi_{m+n}). \end{array}$$

LEMMA 9.41. *The above diagram is commutative, i.e.  $\theta \circ \delta^* = (-1)^{m+n}\delta \circ \theta$  if  $n+m > 1$ .*

In the notation of (9.2), where we proved  $\theta\delta^*\{g\} = (-1)^{m+n}\psi c(g_1)$  (here  $s$  is specialized to  $s = 1$ ),  $g_1$  and the constant map  $x_0$  agree on  $R_m^*$ , and so possess a separation cochain  $d^{n+m}(g_1, x_0) \in C^{n+m}(P_m, R_m^*, \pi_{n+m})$  such that

$$\delta' d^{n+m}(g_1, x_0) = c(g_1) - c(x_0),$$

which is  $c(g_1)$ , as  $x_0$  possesses the extension  $x_0$ , the constant map of  $P_m$  to  $x_0$ . Now it is clear from the definitions that

$$\psi d^{n+m}(g_1, x_0) = \theta\{g\}$$

(see (25), Appendix B, where the separation cochain is defined as a homomorphism  $d^k: \pi_k(K^k, K^{k-1}) \rightarrow \pi_k(X, x_0)$ ). Therefore, for all  $m \geq 1$ ,

$$\delta\theta\{g\} = \delta\psi d^{n+m}(g_1, x_0) = \psi\delta' d^{n+m}(g_1, x_0) = \psi c(g_1) = (-1)^{n+m}\theta\delta^*\{g\},$$

which proves the lemma if  $m+n > 1$ .

**THEOREM 9.42.** *For all  $m+n > 1$ ,  $(n+1, n-1)^m$  is a central extension of  $H^{n+1}(K^{n+1}, K^{n-1}; \pi_{m+n+1})$  by  $H^n(K^{n+1}, K^{n-1}; \pi_{m+n})$ .*

This is a consequence of the exact sequence of  $[n+1, n, n-1]$ , Lemma 9.41, and Theorem 9.3.

Since  $i^* \circ \theta^{-1}: C^{n+1}(K; \pi_{m+n+1}) \rightarrow (n+1, n-1)^m$  has as kernel the subgroup of coboundaries, we may factor this through the natural homomorphism  $C^{n+1} \rightarrow H^{n+1}$ , which assigns to every cocycle its cohomology class. Therefore we can define  $\tilde{\theta}: i^*(n+1, n)^m \rightarrow H^{n+1}(K^{n+1}, K^{n-1}; \pi_{m+n+1})$  as the map which sends  $i^*u$  to the cohomology class of  $\theta u$ .

### 9.5. Elements of $(n+1, n-2)^m$

Consider the commutative diagram

$$\begin{array}{ccccc} j_2^* & (n, n-1)^m & \xrightarrow{\delta_2^*} & (n+1, n)^{m-1} & \xrightarrow{i_2^*} & (n+1, n-1)^m & \xrightarrow{j_2^*} \\ & \downarrow i_0^* & & \uparrow \approx & & \downarrow i_1^* & \\ j_0^* & (n, n-2)^m & \xrightarrow{\delta_0^*} & (n+1, n)^{m-1} & \xrightarrow{i_0^*} & (n+1, n-2)^{m-1} & \xrightarrow{j_0^*} \\ & \downarrow j_0^* & & \downarrow i_2^* & & \uparrow \approx & \\ j_1^* & (n-1, n-2)^m & \xrightarrow{\delta_1^*} & (n+1, n-1)^{m-1} & \xrightarrow{i_1^*} & (n+1, n-2)^{m-1} & \xrightarrow{j_1^*} \end{array}$$

in which the horizontal sequences are the exact sequences of  $[n+1, n, n-1]$ ,  $[n+1, n, n-2]$ , and  $[n+1, n-1, n-2]$ , while the vertical sequence on the left is part of the exact sequence of  $[n, n-1, n-2]$ .

Consider  $j_0^*(n-1, n-2)^m$ : Lemma 9.41 shows that the isomorphism  $\theta$  maps it onto the group of cocycles of  $[K^{n+1}, K^{n-2}]$ , i.e.

$$Z^{n-1}(K^n, K^{n-2}; \pi_{m+n-1}) \approx H^{n-1}(K^{n+1}, K^{n-2}; \pi_{m+n-1}),$$

which we shall write as  $H^{n-1}(\pi_{m+n-1})$ , the  $(n-1)$ th cohomology group of the pair  $[K^{n+1}, K^{n-2}]$ . Then

$$\delta_1^* \theta^{-1} H^{n-1}(\pi_{m+n-1}) = \delta_1^* j_0^*(n, n-2)^m = i_2^* \delta^*(n, n-2)^m;$$

but we have also seen that  $i_2^*(n+1, n)^m = \tilde{\theta}^{-1} H^{n+1}(K^{n+1}, K^{n-1}; \pi_{m+n})$ , where  $\tilde{\theta}$  is an isomorphism (see the end of (9.4)), and  $H^{n+1}(K^{n+1}, K^{n-1}; \pi_{m+n})$  is isomorphic (under the injection) to  $H^{n+1}(K^{n+1}, K^{n-2}; \pi_{m+n})$ , which we shall write as  $H^{n+1}(\pi_{m+n})$ , the  $(n+1)$ th cohomology group of the pair  $[K^{n+1}, K^{n-2}]$ . We have therefore a homomorphism

$$\tau^* = \tilde{\theta} \circ \delta_1^* \circ \theta^{-1} = \tilde{\theta} \circ i_2^* \circ \delta^* \circ (j_0^*)^{-1} \circ \theta^{-1}: H^{n-1}(\pi_{m+n-1}) \rightarrow H^{n+1}(\pi_{m+n}). \quad (9.51)$$

This homomorphism in fact determines the elements of  $(n+1, n-2)^m$ ; later we shall show that it can be computed in finite complexes by Steenrod squares.

**THEOREM 9.52.**  $(n+1, n-2)^m$  is a central extension of

$$H^{n+1}(\pi_{m+n+1})/\tau^*H^{n-1}(\pi_{m+n})$$

by a subgroup of  $(n, n-2)^m$  which is a central extension of  $H^n(\pi_{m+n})$  by  $\tau^{*-1}(0) \subset H^{n-1}(\pi_{m+n-1})$ . Here the cohomology groups are those of the pair  $[K^{n+1}, K^{n-2}]$ , with the coefficient group in the brackets.

The middle horizontal sequence in the diagram above shows that  $(n+1, n-2)^m$  is an extension of  $i^*(n+1, n)^m$  by  $j^*(n+1, n-2)^m$ , which is central by Theorem 9.3. Now  $i^* (= i_1^* \circ i_2^*)$  has kernel  $\delta^*(n, n-2)$  so that

$$\begin{aligned} i^*(n+1, n)^m &= i_2^*(n+1, n)^m / i_2^* \delta^*(n, n-2)^m \\ &= \tilde{\theta}^{-1} H^{n+1}(\pi_{m+n+1}) / \tilde{\theta}^{-1} \tau^* H^{n-1}(\pi_{m+n}) \end{aligned}$$

since  $i_2^* \circ \delta^* = \delta_1^* \circ j_0^*$ . Therefore  $i^*(n+1, n)^m$  is isomorphic with

$$H^{n+1}(\pi_{m+n+1})/H^{n-1}(\pi_{m+n}).$$

A similar argument shows that  $j_0^*$  maps the kernel of  $\delta^*$  onto  $\theta^{-1}\tau^{*-1}(0)$ , so that  $\delta^{*-1}(0)$  is a central extension of  $\delta^{*-1}(0) \cap i_0^*(n, n-1)^m$  by this group, which is  $i_0^*(\delta_2^{*-1}(0))$ , since  $\delta^* \circ i_0^* = \delta_2^*$ . But  $(n, n-1)^m = \theta^{-1}C^n(K; \pi_{n+m})$ , the kernel of  $\delta_2^*$  is the subgroup of cocycles in  $[K^{n+1}, K^{n-2}]$ , and the kernel of  $i_0^*$  the subgroup of coboundaries. Therefore this subgroup of  $(n, n-2)^m$  is isomorphic with  $H^n(\pi_{m+n})$ , the  $n$ th cohomology group of the pair  $[K^{n+1}, K^{n-2}]$ . This completes the proof.

### 9.6. Calculation of $\tau^*$

In the notation of 9.2, take  $s = 2$ , so that  $P_m = K^{n+1} \times I^m$ ,

$$Q_m^* = K^{n+1} \times \dot{I}^m \cup K^n \times I^m$$

as before, while

$$S_m = K^{n+1} \times J^{m-1} \cup K^n \times \dot{I}^m \cup K^{n-2} \times I^m,$$

$R_m^* = K^{n+1} \times \dot{I}^m \cup K^{n-2} \times I^m$ , etc., and let  $f: [P_m; S_m] \rightarrow [X, x_0]$ . Let  $g$  be  $f$  restricted to  $K^{n-1} \times I^m$ , representing the element

$$\{g\} = j_0^* h^* \{f\} \in (n-1, n-2)^m,$$

and let  $g_1$  be  $f$  restricted to  $K^{n+1} \times \dot{I}^m \cup K^{n-1} \times I^m$ , an extension of  $g$ . The first obstruction to the extension of  $g_1$  over  $Q_m^*$  vanishes (it has the extension  $f_1$  as defined in 9.21), and the second obstruction  $w_{g_1}$  (to the extension of  $g_1$  over  $P_m$ ) is the cohomology class of  $c(f_1)$ .



We have seen that  $\psi c(f_1) = (-1)^{m+n} \theta \delta^* h^* \{f\}$ , and the cohomology class of this is  $(-1)^{m+n} \theta i_2^* \delta^* h^* \{f\}$ , that is,  $(-1)^{m+n} \tau^* \theta j_0^* h^* \{f\}$ . Since  $\psi$  is a chain mapping, we deduce

$$\psi w_{g_1} = (-1)^{m+n} \tau^* \theta \{g\}.$$

Let  $\gamma \in \pi_{n+1}(S^n)$  be a non-zero element, containing maps of Hopf invariant  $+1$  if  $n = 2$ ; the composition  $\alpha \circ \gamma$  defines for  $n > 2$  a homomorphism  $\gamma^*: \pi_n(X) \rightarrow \pi_{n+1}(X)$  such that  $2\gamma^*(\alpha) = 0$  for all  $\alpha \in \pi_n(X)$ , and for  $n = 2$ , it defines a transformation  $\gamma^*: \pi_2(X) \rightarrow \pi_3(X)$  such that

$$\gamma^*(\alpha + \beta) - \gamma^*(\alpha) - \gamma^*(\beta) = [\alpha, \beta],$$

the Whitehead product (cf. (25)). Thus  $\gamma^*$  defines homomorphisms

$$i_*: \pi_n(X)/2\pi_n(X) \rightarrow \pi_{n+1}(X), \quad n > 2,$$

$$i_*: \Gamma(\pi_2(X)) \rightarrow \pi_3(X),$$

where  $\Gamma(G)$  is defined in (25) for any abelian group  $G$ .

Now let  $\phi': Q_m^* \rightarrow Q_m^*/R_m^*$  be the factor map which pinches  $R_m^*$  to a point, and let  $\phi$  be  $\phi'$  restricted to  $K^{n-1} \times I^m \cup K^{n+1} \times \dot{I}^m$ . Since  $\phi$  has the extension  $\phi'$  over  $Q_m^*$ , its first obstruction to the extension over  $P_m$  vanishes, and so  $\phi$  possesses a second obstruction  $w_\phi$  to its extension over  $P_m$  into  $Q_m^*/R_m^*$ . This image space is  $(m+n-2)$ -connected, so that  $w_\phi$  is given by Whitehead's formula (26)

$$(m+n-1 = 2) \quad w_\phi = w_\phi - w_{x_0} = i_* p_1 \{d^2(\phi, x_0)\},$$

$$(m+n-1 > 2) \quad w_\phi = w_\phi - w_{x_0} = i_* s^2 \{d^2(\phi, x_0)\},$$

where the separation cochain is an element of

$$C^{m+n-1}(P_m, R_m^*; \pi_{m+n-1}(Q_m^*/R_m^*)),$$

$s^2$  is the Steenrod square  $Sq_{m+n-3}$  (cf. (18, 26)),  $p_1$  is the Postnikov square (16, 26), the curly brackets indicate the cohomology class of the contained cocycle, and  $i_*$  is the homomorphism of the cohomology groups induced by the homomorphisms  $i_*$  defined above.

Now let  $g_1, f_1$  be as at the beginning of this section, so that  $f_1 \circ \phi'^{-1}$  is single-valued, and so continuous (cf. (1.3)). Hence

$$w_{g_1} = (f_1 \circ \phi'^{-1})_* w_\phi.$$

Also,  $(f_1 \circ \phi'^{-1})_* d^{m+n-1}(\phi, x_0) = d^{m+n-1}(g_1, x_0)$ , and (see the proof of Lemma 9.41)  $\psi d^{m+n-1}(g_1, x_0) = \theta \{g\}$ . Now  $\psi^{-1}$  may be considered as a suspension homomorphism, so that we have

$$\psi \circ s^2 = s^2 \circ \psi, \quad \psi \circ p_1 = p_0 \circ \psi,$$

where  $p_0$  is the Pontrjagin square (cf. (26), sections 5 and 8). Thus we may rewrite the formulae for  $w_{g_1}$  in terms of  $\psi w_{g_1}$

$$\psi w_{g_1} = \begin{cases} i_* \mathfrak{s}^2\{\theta\{g\}\}, & m+n > 3, \\ i_* p_0\{\theta\{g\}\}, & m+n = 3. \end{cases}$$

Combining our two expressions for  $\psi w_{g_1}$  in terms of  $\{g\}$ , we have

LEMMA 9.61. *If  $K^{n+1}$  is a finite complex, and  $u$  is an element of*

$$H^{n-1}(K^{n+1}, K^{n-2}; \pi_{m+n-1}),$$

*then, in  $H^{n+1}(K^{n+1}, K^{n-2}; \pi_{m+n})$ ,*

$$(-1)^{m+n} \tau^*(u) = \begin{cases} i_* \mathfrak{s}^2(u), & m+n > 3, \\ i_* p_0(u), & m+n = 3. \end{cases}$$

Now, if  $n = 2$ ,  $\mathfrak{s}^2 H^1(K^3, e^0; \pi_{m+1}) = 0$  (see Theorem 12.1 in (18)). Therefore we may restate Theorem 9.52 in the following form:

THEOREM A. *If  $K^3$  is a finite complex, and  $m > 1$ , then  $(3, 0)^m$  is an abelian extension of  $H^3(K^3; \pi_{m+3})$  by an abelian extension of  $H^2(K^3; \pi_{m+2})$  by  $H^1(K^3; \pi_{m+1})$ . Also  $(3, 0)^1$  is a central extension of  $H^3(K^3; \pi_4)$  by a central extension of  $H^2(K^3; \pi_3)$  by the subgroup of  $H^1(K^3; \pi_2)$  of elements  $u$  such that  $i_* p_0(u) = 0$ .*

THEOREM B. *If  $n > 2$ , and  $K^{n+1}$  is a finite complex,  $(n+1, n-2)^m$  is a central extension of  $H^{n+1}(\pi_{m+n+1})/i_* \mathfrak{s}^2 H^{n-1}(\pi_{m+n})$  by a central extension of  $H^n(\pi_{m+n})$  by the subgroup of  $H^{n-1}(\pi_{m+n-1})$  of elements  $u$  such that  $i_* \mathfrak{s}^2(u) = 0$ , the cohomology groups being those of the pair  $[K^{n+1}, K^{n-2}]$ .*

In a later paper we shall determine the extension mentioned in Theorem 9.42, which gives the structure of the groups  $(n+1, n-1)^m$  for all  $m$ . The structure of the groups  $(n+1, n-2)^m$  is not yet known in general, and an example will be given to show that we cannot expect to deduce the structure of this group from our knowledge of the groups  $(n+1, n-1)^m$ ,  $(n, n-2)^m$ , for all finite complexes  $K^{n+1}$ .

## APPENDIX

We now prove Lemma 3.1, that a path  $\sigma$  induces a homomorphism

$$\sigma^*: (P_\lambda)^m(X_\lambda; x_1) \rightarrow (P_\lambda)^m(X_\lambda, x_0),$$

where  $\sigma$  begins at  $x_0$  and ends at  $x_1$ . For convenience, we define an explicit homotopy between  $\sigma f + \sigma g$  and  $\sigma(f+g)$ , for any two maps  $f, g$ , representing elements of

$$(P_\lambda)^m(X_\lambda; x_1),$$

where  $+$  is as defined in (2.1). Let  $h_t$  be given by the equations (where  $p \in P_\lambda$ ),

$$h_{t,\lambda}(p, y_1, \dots, y_m) = \begin{cases} (\sigma f)_\lambda(p, y'_1, y_2, \dots, y_m) & \text{if } 0 \leq y_1 \leq \frac{1}{2}, \\ (\sigma g)_\lambda(p, y'_1, y_2, \dots, y_m) & \text{if } \frac{1}{2} \leq y_1 \leq 1, \end{cases}$$

where

$$y'_1 = \begin{cases} 2y_1/(1+t) & \text{if } 0 \leq y_1 \leq (1+t)/8, \\ 2y_1 - t/4 & \text{if } (1+t)/8 \leq y_1 \leq \frac{1}{2}; \end{cases}$$

$$y''_1 = \begin{cases} 2y_1 - 1 + t/4 & \text{if } \frac{1}{2} \leq y_1 \leq (7-t)/8, \\ (2y_1 - 1 + t)/(1+t) & \text{if } (7-t)/8 \leq y_1 \leq 1. \end{cases}$$

It is not difficult to see that  $h_t$  is single-valued and continuous, as where it is defined twice (for  $y_1 = \frac{1}{2}$ ) the maps  $\sigma f$ ,  $\sigma g$  depend only on  $\sigma$ , and agree.

Now  $h_0 = \sigma f + \sigma g$ , and  $h_1 = \sigma(f+g)$ ; thus these two maps are homotopic, so that  $\sigma*\{f\} + \sigma*\{g\} = \sigma*\{f+g\}$ , and the lemma is proved.

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