

TRACK GROUPS (II)

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Introduction

IN a previous paper ((1), hereafter referred to as I) we defined the Track Groups $(P, Q)^m(X, x_0; x_0)$ of a pair $[P, Q]$, where Q is closed in P , and x_0 is a point of X , for $m \geq 1$. If K is an $(n+1)$ -dimensional CW-complex with but one vertex, and K^{n-1} is its $(n-1)$ th skeleton, we showed that the group $(K, K^{n-1})^m(X, x_0; x_0)$, written $(n+1, n-1)^m$, is a central extension of $H^{n+1}(\pi_{m+n+1})$ by $H^n(\pi_{m+n})$, where $H^i(G) = H^i(K, K^{n-1}; G)$, and

$$\pi_q = \pi_q(X, x_0),$$

for any space X . Here we mean by an extension of G by Q , a group E with subgroup G such that $E/G = Q$. In the first part of this paper (Chapter 5) we calculate this extension for a finite complex K , deducing the results (except for the commutators when $m = n = 1$) from the special cases $K = S^n$, $K = S^n \cup e^{n+1}$. It is found that the extension is non-trivial in general; for example, if K is the $(n-1)$ th, X the $(m+n-1)$ th, suspension of the real projective plane, then $(n+1, n-1)^m$ is cyclic (of order four) if $m+n > 2$. When $m = n = 1$, the groups may not be abelian: we show that the commutators are determined by the cup-products in K and the Whitehead Products in the homotopy groups of X . The method of finding the extensions is a geometric one, using Whitney's Tube Systems (13).

The second part of the paper (Chapter 6) deals with the Čech theory of Track Groups (following Spanier (7)), and extends the scope of the exact sequence of a triple $[P, Q, R]$ (see I, § 5) when the image space X is a compact ANR or a locally finite simplicial complex (with the weak topology).

In an appendix we obtain some results on group extensions which are required in Chapter 5. We study central extensions of an arbitrary abelian group G by a weak direct sum Q of abelian groups Q_α . We show that the group of extensions, $H^2(Q, G)$, is the strong sum of the groups $H^2(Q_\alpha, G)$ and a certain subgroup of $\text{Hom}(Q \otimes Q, G)$, generalizing a theorem of Lyndon (6). This enables us to give the structure of $(n+1, n-1)^m$ in a convenient way, without imposing restrictions on the space X .

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STRUCTURE OF TRACK GROUPS

10. The groups $(K^{n+1}, K^{n-1})^m(X, x_0; x_0)$

Let K be a finite CW-complex with only one vertex e^0 , let K^p be its p -skeleton, and let $(p, q)^m = (K^p, K^q)^m(X, x_0; x_0)$ for all $q \leq p$. Then we have seen (I, § 5) that there is an exact sequence

$$\begin{aligned} \rightarrow (n, n-1)^{m+1} \xrightarrow{\delta^*} (n+1, n)^m \\ \xrightarrow{i^*} (n+1, n-1)^m \xrightarrow{j^*} (n, n-1)^m \xrightarrow{\delta^*} (n+1, n)^{m-1} \rightarrow, \end{aligned}$$

and have shown (in (I, § 9)) that there is an isomorphism

$$\theta: (k, k-1)^m \rightarrow H^k(K^k, K^{k-1}; \pi_{m+k}) = C^k(\pi_{m+k})$$

such that $\delta^* = \pm \theta^{-1} \circ \delta \circ \theta$, where δ is the coboundary operator in the cochain sequence of the complex K , and so also (with a natural identification of $C^k(K)$ and $C^k(K^{n+1}, K^{n-1})$ for $k = n, n+1$) in the cochain sequence of the pair $[K^{n+1}, K^{n-1}]$. Thus, if $H^k(G)$ is the k th cohomology group of the pair $[K^{n+1}, K^{n-1}]$ with coefficients in G , $i^*(n+1, n)^m \approx H^{n+1}(\pi_{m+n+1})$, and $j^*(n+1, n-1)^m \approx H^n(\pi_{m+n})$ (identified with a subgroup of $C^n(\pi_{m+n})$). Let $\tilde{\theta}$ be the first isomorphism, and use θ for $\theta|j^*(n+1, n-1)$ for the second. Then in (I, § 9) it was shown that $(n+1, n-1)^m$ is a central extension of $\tilde{\theta}^{-1}H^{n+1}(\pi_{m+n+1})$ by $\theta^{-1}H^n(\pi_{m+n})$. We proceed to determine this extension.

10.1. Description of a central extension

Let E be a central extension† of an abelian group G by a weak direct sum Q of abelian groups Q_α . This corresponds to a unique element of the abstract cohomology group $H^2(Q, G)$ (see (4), or Appendix), in defining which the operations of Q on G are trivial. Now we show in the Appendix that $H^2(Q, G)$ has as a direct summand the strong sum $\Sigma^* H^2(Q_\alpha, G)$, embedded in a natural way, and that the other summand is mapped isomorphically into $\text{Hom}(Q \otimes Q, G)$ by a certain homomorphism defined from the commutators of E . To specify E , it suffices to determine the subgroups which are extensions of G by each Q_α in turn, and to determine the commutators of E .

If Q_α is cyclic, of order q , the extension of G by Q_α is necessarily abelian; if Q_α is generated by q_α of order p , we may select a *representative* \tilde{q}_α of q_α in E , that is, an element mapping onto q_α in the projection $E \rightarrow Q$, and calculate $p\tilde{q}_\alpha \in G$. Then the coset $p\tilde{q}_\alpha + pG$ depends only on the extension E and the choice of $q_\alpha \in Q_\alpha$. The determination of this coset is exactly equivalent to determining the extension.

† That is, G is a subgroup in the centre of E , and $E/G = Q$.

The extension of G by Q_α is called *trivial* if it determines the zero element of $H^2(Q_\alpha, G)$. In this case the extension contains a subgroup isomorphic to Q_α , and which projects isomorphically onto Q_α under the projection $E \rightarrow Q$. Thus the extension of G by Q_α is isomorphic to the direct sum $G + Q_\alpha$ in a natural way.

In our case, we have essentially a central extension of $H^{n+1}(\pi_{m+n+1})$ by $H^n(\pi_{m+n})$. The latter group has a decomposition

$$H^n(\pi_{m+n}) \approx H^n \otimes \pi_{m+n} + H^{n+1} * \pi_{m+n+1},$$

where H^k is the k th integral group of the pair $[K^{n+1}, K^{n-1}]$ and $+$ denotes direct summation.

More explicitly, we can take $\pi_k(K^k, K^{k-1})$ for the integral cochain group C^k , $k > 2$ (if $k = 2, 1$, we must 'kill' the operations of $\pi_1(K^1)$, as in (11)). Then $C^{n-1} = 0$, and C^n, C^{n+1} are finitely generated free abelian groups. As in (5), we can choose bases for C^n, C^{n+1} such that C^n has a basis consisting of cocycles u_i , and cochains b_j such that $\delta b_j = p_j z_j$, for some integers p_j , and some members z_j of a basis for C^{n+1} . Then $H^n(\pi_{m+n})$ can be identified with the kernel of $\delta: C^n \otimes \pi_{m+n} \rightarrow C^{n+1} \otimes \pi_{m+n}$, where $\delta(c \otimes g) = (\delta c) \otimes g$. Hence $H^n(\pi_{m+n})$ is the direct sum of groups $u_i \otimes \pi_{m+n} \approx \pi_{m+n}$, and of groups $b_j \otimes {}_p\pi_{m+n}$, where we denote by ${}_pG$ the subgroup of G of all elements of orders dividing p . By a theorem due to Prüfer, such groups ${}_pG$ are always the weak direct sum of cyclic groups; therefore we can express $H^n(\pi_{m+n})$ as a weak direct sum of cyclic groups, generated by classes of the type $b_j \otimes g_j$ (where $p_j g_j = 0$), and of isomorphs of π_{m+n} . We shall determine the extensions by each of these summands, and find the commutators. In (10.5) we shall give a system of generators and relations for the extension.

10.2. Commutators of $(n+1, n-1)^m$

We have seen that $(n+1, n-1)^m$ is abelian if $m > 1$; it is also abelian if $m = 1$ and $n > 1$.

LEMMA 10.21. $(n+1, n-1)^m$ is abelian if $m+n > 2$.

By the Factor Theorem (I, 4.2), $(K^{n+1}, K^{n-1})^1 \approx (K^{n+1}/K^{n-1}, e^0)^1$, where K^{n+1}/K^{n-1} is the complex obtained by shrinking K^{n-1} to a vertex e^0 . Let $K' = K^{n+1}/K^{n-1}$, and suppose that K' is of the same homotopy type as the suspension $\langle L \rangle$ of a complex L . Then we will show (12.1) that there is a vertex e of $\langle L \rangle$ such that $[K', e^0]$ is of the homotopy type of $[\langle L \rangle, e]$; therefore $(K', e^0)^1 \approx (\langle L \rangle, e)^1$. But there is a vertex e' of L such that $e \in \langle e' \rangle \subset \langle L \rangle$, and $\langle e' \rangle$ is contractible, and it follows from theorems in (I, 7.3) and (I, 8.2) that

$$(K', e^0)^1 \approx (\langle L \rangle, e)^1 \approx (\langle L \rangle, \langle e' \rangle)^1 \approx (L, e')^2,$$

and so these groups are abelian.

Now K' is $(n-1)$ -connected, and of dimension less than $n+2$, and it follows from Chang's results on A_n^2 -polyhedra (2) that if $n > 2$ K' is of the homotopy type of a union of *Elementary A_n^2 -polyhedra*, and, in particular, of spheres S^n , S^{n+1} , and spaces Y^{n+1} formed by attaching an $(n+1)$ -cell to a sphere. These spaces are clearly each of the homotopy type of the suspension of a similar space of one less dimension, and so therefore is their union. This is also true when $n = 2$.

In fact, J. H. C. Whitehead has proved that if K is an $(r-1)$ -connected complex of dimension not exceeding $2r-1$, then K is of the homotopy type of the suspension of a complex L of less dimension than K (an unpublished result). We can prove, assuming this theorem, that $(n+k, n-1)^1$ is abelian if $k < n$.

The only group which may not be abelian is thus $(2, 0)^1$, a central extension of $H^2(\pi_3)$ by $H^1(\pi_2)$. We pair π_2 with itself to π_3 by the Whitehead Product (a commutative pairing in this dimension) and so define the cup-product pairing

$$\cup: H^1(\pi_2) \otimes H^1(\pi_2) \rightarrow H^2(\pi_3),$$

written $\cup(u \otimes v) = u \cup v = -(v \cup u)$, for $u, v \in H^1(\pi_2)$. Recall that $\theta j^*: (2, 0)^1 \rightarrow H^1(\pi_2)$ has kernel $\tilde{\theta}^{-1}H^2(\pi_3)$; pick elements \tilde{u}, \tilde{v} such that $\theta j^*\tilde{u} = u$, $\theta j^*\tilde{v} = v$ (we say \tilde{u}, \tilde{v} are *representatives* of u, v), so that

$$\tilde{u} + \tilde{v} - \tilde{u} - \tilde{v}$$

is an element of $\tilde{\theta}^{-1}H^2(\pi_3)$ which depends only on u, v .

THEOREM 10.22. *In $(2, 0)^1$ let \tilde{u}, \tilde{v} represent $u, v \in H^1(\pi_2)$. Then*

$$\tilde{u} + \tilde{v} - \tilde{u} - \tilde{v} = \tilde{\theta}^{-1}(v \cup u) = -\tilde{\theta}^{-1}(u \cup v),$$

where the cup-product is defined using the Whitehead Product to pair π_2 with itself to π_3 .

The proof will be given in § 11.4.

10.3. The extension by $H^n \otimes \pi_{m+n}$

We shall show that, except for the non-commutative nature of $(2, 0)^1$, the extension of $H^{n+1}(\pi_{m+n+1})$ by $H^n \otimes \pi_{m+n}$ is trivial. Since H^n is free abelian and finitely generated, we can choose a basis u_i , so that any element u has a unique expression as a sum $\sum t_i u_i$ (the t 's being integers). Then any element of $H^n \otimes \pi_{m+n}$ has a unique expression as a sum

$$\sum u_i \otimes g_i, \quad g_i \in \pi_{m+n}.$$

Now a cohomology class $u \in H^n$ determines a homotopy class of maps $K^n \rightarrow S^n$ (in which K^{n-1} is mapped to a point w) which can be extended over K^{n+1} . Let $u'_i: [K^{n+1}, K^{n-1}] \rightarrow [S^n, w]$ be a map, such that if s^n is a

chosen generator of $H^n(S^n)$, $u_i^*(s^n) = u_i \in H^n$, and consider the commutative diagram:

$$\begin{array}{ccc} (S^n, w)^m & \xrightarrow{u_i^*} & (n+1, n-1)^m \xrightarrow{j^*} (n, n-1)^m \\ \downarrow \theta & & \downarrow \theta \\ C^n(S^n; \pi_{m+n}) & \xrightarrow{u_i^*} & C^n(K^n; \pi_{m+n}), \end{array}$$

in which the homomorphisms θ are isomorphisms onto. Then

$$\theta j^*(u_i^* \theta^{-1})(s^n \otimes g) = u_i^*(s^n) \otimes g = u_i \otimes g.$$

Therefore, if we define $(u_i, g) = u_i^* \theta^{-1}(s^n \otimes g)$, (u_i, g) is a representative for $u_i \otimes g$, is linear in $g \in \pi_{m+n}$ (since u_i^* and θ are homomorphisms) and defines an isomorphism

$$g \rightarrow (u_i, g)$$

of π_{m+n} into $(n+1, n-1)^m$.

THEOREM 10.31. *If $m+n > 2$, $(n+1, n-1)^m$ has a summand which is mapped isomorphically on $H^n \otimes \pi_{m+n}$ by $\theta \circ j^*$, and hence the extension of $H^{n+1}(\pi_{m+n+1})$ by this summand of $H^n(\pi_{m+n})$ is trivial. Also $(2, 0)^1$ has subgroups mapped isomorphically on each summand $u_i \otimes \pi_{m+n}$ of $H^n \otimes \pi_{m+n}$ by $\theta \circ j^*$.*

This gives $(n+1, n-1)^m$ if K^{n+1}/K^{n-1} is torsion-free, except for the commutators of $(2, 0)^1$.

10.4. The extension by $H^{n+1} * \pi_{m+n}$

Since K is a finite complex, H^{n+1} is finitely generated, and we can decompose the summand of elements of finite order into a direct sum of cyclic subgroups. Let $(n+1)$ -cocycles z_j be such that their cohomology classes are of order p_j and generate these cyclic summands, so that there are n -cochains b_j such that $\delta b_j = p_j z_j$. Let ${}_p \pi_{m+n}$ be the subgroup of π_{m+n} of elements whose orders divide p ; then (see Appendix; also (14)), ${}_p \pi_{m+n}$ is the weak direct sum of cyclic groups of orders q_v dividing p ; let β_v be chosen generating these cyclic subgroups. Then any element of $H^n(\pi_{m+n})$ can be expressed as a sum of $u_i \otimes g_i$ and of multiples of the cohomology classes of $b_j \otimes \beta_{j,v}$. Let curly brackets $\{ \}$ denote the cohomology classes of the contained cocycles; then, having chosen the $u_i, b_j, \beta_{j,v}$, any element u has a unique expression as a sum

$$u = \sum_i u_i \otimes g_i + \sum_{j,v} t_{j,v} \{b_j \otimes \beta_{j,v}\},$$

where $g_i \in \pi_{m+n}$ and $0 \leq t_{j,v} < q_{j,v}$, the order of $\beta_{j,v}$. Since we have calculated the commutators of $(n+1, n-1)^m$ and the extensions of $H^{n+1}(\pi_{m+n+1})$ by the groups $u_i \otimes \pi_{m+n}$, it remains only to calculate the extensions by the cyclic groups of orders $q_{j,v}$ generated by $\{b_j \otimes \beta_{j,v}\}$.

THEOREM 10.41. *Let $\{b_j \otimes \beta_{j,\nu}\}$ be as above, so that $\beta_{j,\nu}$ is of order $q_{j,\nu}$ dividing p_j , and $\delta b_j = p_j z_j$. Then we can choose $(b_j, \beta_{j,\nu}) \in (n+1, n-1)^m$ such that $\theta j^*(b_j, \beta_{j,\nu}) = \{b_j \otimes \beta_{j,\nu}\}$, and*

$$q_{j,\nu}(b_j, \beta_{j,\nu}) = \theta^{-1}(\lambda_{p_j, p_{j,\nu}}^{m+n} \{z_j \otimes \gamma^* \beta_{j,\nu}\})$$

where $\lambda_{p,q}^{m+n}$ is given in Table 1, and $\gamma^* \beta_{j,\nu} = \beta_{j,\nu} \circ \gamma^{m+n}$, if $\gamma^2 \in \pi_3(S^2)$ is the class of the Hopf map and γ^k is the non-zero element of $\pi_{k+1}(S^k)$.

TABLE 1

$\lambda_{p,q}^{m+n}$	$m+n = 2$	$m+n > 2$
$p \equiv 0 \pmod 4$	0	0
$p \equiv q \equiv 2 \pmod 4$	$q/2$	1
q odd	0	0

In this table, notice that q divides p ; therefore, if q is not odd, either $p \equiv q \equiv 2 \pmod 4$, or else $p \equiv 0 \pmod 4$, and hence all cases are covered.

When $m+n > 2$, we can express this result in terms of Steenrod Squares. In these cases, $\gamma^*(\beta)$ is of order two if it is not zero; following J. H. C. Whitehead (12), we define a pairing of π_{m+n} with itself to π_{m+n+1} : let $k: \pi_{m+n} \rightarrow \bar{\pi}_{m+n}$, the group reduced mod 2, be the natural map, and choose a basis \bar{g}_ρ for the vector space $\bar{\pi}_{m+n}$. Map

$$(\sum t_\rho \bar{g}_\rho) \otimes (\sum t'_\rho \bar{g}_\rho) \rightarrow \sum t_\rho t'_\rho \gamma^*(k^{-1} \bar{g}_\rho),$$

which is easily verified to be a single-valued homomorphism, and precede this with the map

$$k \otimes k: \pi_{m+n} \otimes \pi_{m+n} \rightarrow \bar{\pi}_{m+n} \otimes \bar{\pi}_{m+n}.$$

This is the required pairing: it is shown in (12) that it does not depend on the choice of the basis \bar{g}_ρ . In particular, it maps $\beta \otimes \beta$ to $\gamma^*(\beta)$. With this pairing, form Sq_{n-1} , which we write as $s^1: H^n(\pi_{m+n}) \rightarrow H^{n+1}(\pi_{m+n+1})$.

COROLLARY 10.42. *If $(b_j, \beta_{j,\nu})$ is as in Theorem 10.41, then*

$$q_{j,\nu}(b_j, \beta_{j,\nu}) = \tilde{\theta}^{-1} s^1 \{b_j \otimes \beta_{j,\nu}\}.$$

For the actual value of $s^1 \{b_j \otimes \beta_{j,\nu}\}$ is $\frac{1}{2} p_j (p_j - 1) \{z_j \otimes \gamma^* \beta_{j,\nu}\}$, which is zero if p_j is odd (since $\{z_j\}$ is of order p_j) or if p_j is divisible by 4 (since $\gamma^*(\beta_{j,\nu})$ is of order two) or if $q_{j,\nu}$ is odd (since $\gamma^*(\beta_{j,\nu})$ is zero); in the remaining cases, $\frac{1}{2} p_j (p_j - 1) \{z_j \otimes \gamma^*(\beta_{j,\nu})\} = \{z_j \otimes \gamma^*(\beta_{j,\nu})\}$.

To prove Theorem 10.41, it is sufficient to consider only the universal example, $Y_p^{n+1} = S^n \cup e^{n+1}$, a CW-complex formed by attaching† an $(n+1)$ -cell to an n -sphere by a map of degree $p > 0$; this plays the part of S^n in

† See (9), § 8, p. 235.

10.3. Let b, z be generators of $C^n(Y_p^{n+1})$, $C^{n+1}(Y_p^{n+1})$ respectively, such that $\delta b = pz$. Consider the integral cochain b_j in $C^n = C^n(K)$: this determines a class of maps $K^n \rightarrow S^n$ such that K^{n-1} is mapped to a point w , and the obstruction cocycle to the extension over K^{n+1} is δb_j . Now embed S^n in Y_p^{n+1} (where $p = p_j$): the obstruction to the extension vanishes, and we can take a cellular map

$$b': [K^{n+1}, K^{n-1}] \rightarrow [Y_p^{n+1}, w],$$

which induces chain mappings $b'^*: C^k(Y_p^{n+1}) \rightarrow C^k = C^k(K)$ ($k = n, n+1$) such that $b'^*(b) = b_j$, $b'^*(z) = z_j$. (This last follows from the fact that b'^* is a chain mapping, and so $b'^*(pz) = b'^*(\delta b) = \delta b'^*(b) = \delta b_j = pz_j$, so that in the free abelian group C^{n+1} , $p(z_j - b'^*(z)) = 0$, and hence $z_j - b'^*(z)$ is itself zero.)

It follows from an argument similar to that in (10.3), that Theorem 10.41 is true for any complex K if it is true for the complex Y_p^{n+1} . Now suppose $\beta \in \pi_{n+m}(X)$ is of order q dividing p . Then β is represented by a map $\beta': S^{n+m} \rightarrow X$ which has an extension over Y_q^{n+m+1} to X ; let ι^k generate $\pi_k(S^k)$ for all k , i_q^k generate $\pi_k(Y_q^{k+1})$, the image of ι^k under the injection $S^k \rightarrow Y_q^{k+1}$. Then we have a map $\beta'': Y_q^{n+m+1} \rightarrow X$ such that $\beta''_*(i_q^{n+m+1}) = \beta$; consider the homomorphism

$$\beta''_*: (Y_p^{n+1}, e^0)^m(Y_q^{n+m+1}, e^0; e^0) \rightarrow (Y_p^{n+1}, e^0)^m(X, x_0; x_0);$$

if Theorem 10.41 is true with i_q^{m+n+1} instead of $\beta_{j,\nu}$ ($q = q_{j,\nu}$), it is also true for $\beta_{j,\nu}$ itself. We have proved:

LEMMA 10.43. *Theorem 10.41 is true if and only if it is true in the special case $K = Y_p^{n+1}$, $X = Y_q^{n+m+1}$, where q divides p .*

Now let $G_{p,q}^{m+n} = (Y_p^{n+m}, e^0)^1(Y_q^{n+m+1}, e^0; e^0)$.

Since Y_p^{n+1} is of the homotopy type of the suspension $\langle Y_p^n \rangle$, it follows from the argument in the proof of Lemma 10.21 that

$$(Y_p^{n+1}, e^0)^m(Y_q^{n+m+1}, e^0; e^0) \approx (Y_p^n, e^0)^{m+1}(Y_q^{n+m+1}, e^0; e^0),$$

and hence by an inductive argument that for all $m > 1$, $n \geq 1$,

$$(Y_p^{n+1}, e^0)^m(Y_q^{n+m+1}, e^0; e^0) \approx G_{p,q}^{n+m}.$$

We showed in I (§ 8.1) that there was a homomorphism

$$S^*: (P, Q)^m(X, x_0; x_0) \rightarrow (P^*, Q^*)^m(X^*, x_0^*; x_0^*),$$

where the group on the right is obtained by replacing P, Q, X by their suspensions, and x_0 by a point x_0^* in the suspension of x_0 . Take $P = Y_p^{n+1}$, $Q = e^0$: then Q^* is contractible, and (by I, § 7.3) the group on the right

is isomorphic to $(Y_p^{n+2}, e^0)^m(X^*, x_0^*; x_0^*)$. Therefore we have a homomorphism, which we shall also write S^* ,

$$S^*: G_{p,q}^{n+m} \rightarrow G_{p,q}^{n+m+1}.$$

Fix the integers p, q (where q divides p), and set $\Pi_k^N = \pi_k(Y_{q_1}^N)$. Then the suspension S^* is a homomorphism

$$S^*: \Pi_k^N \rightarrow \Pi_{k+1}^{N+1},$$

regarding the homotopy group as the Track Group of a sphere. Let

$$\psi: H^{n-i}(K; G) \rightarrow H^{n-i+1}(\langle K \rangle; G)$$

be the suspension isomorphism, where $\langle K \rangle$ is the suspension of K , and define, for $i = 0, 1$,

$$E_i^* = S_*^* \circ \psi = \psi \circ S_*^*: H^{n-i}(Y_p^n; \Pi_{n+1-i}^{n+1}) \rightarrow H^{n-i+1}(Y_p^{n+1}; \Pi_{n+2-i}^{n+2}),$$

where S_*^* is the homomorphism induced by S^* on the coefficient group. Then we have a diagram which is easily verified to be commutative:

$$\begin{array}{ccccc} 0 \rightarrow H^n(Y_p^n; \Pi_{n+1}^{n+1}) & \xrightarrow{\tilde{\theta}^{-1}} & G_{p,q}^n & \xrightarrow{\theta \circ j_*} & H^{n-1}(Y_p^n; \Pi_n^{n+1}) \rightarrow 0 \\ \downarrow E_0^* & & \downarrow S^* & & \downarrow E_1^* \\ 0 \rightarrow H^{n+1}(Y_p^{n+1}; \Pi_{n+2}^{n+2}) & \xrightarrow{\tilde{\theta}^{-1}} & G_{p,q}^{n+1} & \xrightarrow{\theta \circ j_*} & H^n(Y_p^{n+1}; \Pi_{n+1}^{n+2}) \rightarrow 0. \end{array}$$

Notice that the horizontal sequences are exact at $G_{p,q}^N$ ($N = n+1, n$), that $\tilde{\theta}^{-1}$ is an isomorphism into, and $\theta \circ j_*$ is onto.

Now Π_{N-1}^N is cyclic of order q , generated by i_q^{N-1} ; therefore

$$S^*: \Pi_{N-1}^N \rightarrow \Pi_N^{N+1}$$

is always an isomorphism onto, and, since ψ is always an isomorphism onto, so is E_1^* . Next, Π_N^N is zero if q is odd, and cyclic of order two otherwise, if $N > 3$, generated by $\gamma^*(i_q^{N-1})$, so that S^* and hence E_0^* is an isomorphism onto if $n > 2$ in the diagram. By the 5-lemma $S^*: G_{p,q}^n \rightarrow G_{p,q}^{n+1}$ is an isomorphism onto if $n > 2$. Also Π_3^3 is generated by $i_q^2 \circ \gamma = \gamma^*(i_q^2)$ (where γ is now the class of the Hopf map), so that $S^*: \Pi_3^3 \rightarrow \Pi_4^4$ is onto. The kernel of this homomorphism is generated by $2\gamma^*(i_q^2)$. Therefore we have proved

LEMMA 10.44. $S^*: G_{p,q}^n \rightarrow G_{p,q}^{n+1}$ is an isomorphism onto if $n > 2$, and, if $n = 2$, is onto with kernel in $\tilde{\theta}^{-1}H^2(Y_p^2; \Pi_3^3)$ generated by the class of $z \otimes 2\gamma^*(i_q^2)$.

It follows from this lemma that if the first column of Table 1 is correct (i.e. $\lambda_{p,q}^2$ is as listed), then the second column is also correct. The effect of the last two lemmas is to reduce the verification of Theorem 10.41 to the finding of the structure of $G_{p,q}^2$. We now reduce this further, to the case $q = p$.

LEMMA 10.45. *We can deduce the structure of $G_{p,q}^{m+n}$ from the structure of $G_{r,r}^2$ (all r). That is, if $\lambda_{r,r}^2$ in Theorem 10.41 is zero or $\frac{1}{2}r$ according as r is not or is congruent to 2 mod 4, then $\lambda_{p,q}^{m+n}$ is given by Table 1.*

Let b, z generate the integral groups $C^k(Y_p^2)$ for $k = 1, 2$ respectively, and be such that $\delta b = pz$; let b', z' be the corresponding generators of $C^k(Y_q^2)$, $k = 1, 2$, so that $\delta b' = qz'$. Assume q divides p , and let $r = p/q$. Then there is a map

$$k: [Y_p^2, e^0] \rightarrow [Y_q^2, e^0]$$

such that k maps $S^1 \subset Y_p^2$ homeomorphically on $S^1 \subset Y_q^2$, and the 2-cell of Y_p^2 with degree r on that of Y_q^2 . Then k is a cellular map, inducing homomorphisms $k^*: C^k(Y_q^2) \rightarrow C^k(Y_p^2)$ such that $\delta \circ k^* = k^* \circ \delta$, and $k^*(b') = b$, and hence $k^*(z') = rz$.

k also induces a homomorphism of the exact sequence of $[Y_q^2, S^1, e^0]$ into that of $[Y_p^2, S^1, e^0]$ (see (I, § 5)), and clearly

$$\theta \circ k^* = k^* \circ \theta: (S^1, e^0)^1 \rightarrow C^1(Y_p^2; \pi_2),$$

$$\theta \circ k^* = k^* \circ \theta: (Y_q^2, S^1)^1 \rightarrow C^2(Y_p^2; \pi_3).$$

Suppose that $(b', i_q^2) \in G_{q,q}^3$ has been found such that

$$\Delta j_*(b', i_q^2) = \{b' \otimes i_q^2\}, \quad q(b', i_q^2) = \lambda_{q,q}^2 \tilde{\theta}^{-1}\{z' \otimes \gamma^*(i_q^2)\}.$$

Then

$$\theta j_* k^*(b', i_q^2) = k^*\{b' \otimes i_q^2\} = \{b \otimes i_q^2\},$$

$$\begin{aligned} qk^*(b', i_q^2) &= \lambda_{q,q}^2 \tilde{\theta}^{-1} k^*\{z' \otimes \gamma^*(i_q^2)\} \\ &= r\lambda_{q,q}^2 \tilde{\theta}^{-1}\{z \otimes \gamma^*(i_q^2)\}; \end{aligned}$$

therefore, if we take $k^*(b', i_q^2)$ as a representative for $\{b \otimes i_q^2\}$, we have, if q is odd or divisible by 4, $qk^*(b', i_q^2) = 0$, as asserted in Table 1 (which gives $\lambda_{p,q}^2 = 0$ if q is odd or p is divisible by 4). Suppose $q \equiv 2 \pmod{4}$; then there is an integer s such that $(p/2 + qs)$ is 0 or $q/2$ according as $p \equiv 0$ or $2 \pmod{4}$. Consider the element

$$(b, i_q^2) = k^*(b', i_q^2) + s\tilde{\theta}^{-1}\{z \otimes \gamma^*(i_q^2)\},$$

where s is to be zero if q is not congruent to 2 mod 4; this is another representative for $\{b \otimes i_q^2\}$ (i.e. is mapped on this by $\theta \circ j^*$), and

$$\begin{aligned} q(b, i_q^2) &= \tilde{\theta}^{-1}(r\lambda_{q,q}^2 + sq)\{z \otimes \gamma^*(i_q^2)\} \\ &= \tilde{\theta}^{-1}(p/2 + sq)\{z \otimes \gamma^*(i_q^2)\} \\ &= \tilde{\theta}^{-1}(\tfrac{1}{2}q)\{z \otimes \gamma^*(i_q^2)\}, \end{aligned}$$

which is the assertion of Theorem 10.41 (using Table 1) in this case. This proves the lemma.

It only remains to compute $G_{p,p}^2$ for all p : this will be done in the next section (§ 11).

10.5. $(n+1, n-1)^m$ given by generators and relations

Since K is a finite complex, the integral cohomology group

$$H^{n+1}(K^{n+1}, K^{n-1})$$

is finitely generated, and can be decomposed into the direct sum of a free abelian group and a finite number of cyclic groups. Let a set of generators w_k, z_j of cyclic summands be chosen, where the w_k 's are of infinite order, and z_j of order $p_j, j = 1, \dots$, the coefficients of torsion. Let the free abelian group $H^n(K^{n+1}, K^{n-1})$ have a basis u_i . For each j choose a decomposition of $p_j \pi_{m+n}$ into cyclic summands (the weak direct sum), where ${}_p G$ means the same as in (10.1), and let $\beta_{j,\nu}$ of order $q_{j,\nu}$ be a typical generator of one summand.

We denote by g, g', \dots elements of π_{m+n+1} , α, α', \dots arbitrary elements of π_{m+n} . Choose an n -cochain b_j such that $(1/p_j)\delta b_j$ is in the class z_j . Then any element of $H^{n+1}(K^{n+1}, K^{n-1}; \pi_{m+n+1}) = H^{n+1} \otimes \pi_{m+n+1}$ has an expression as a sum

$$w = \sum_k w_k \otimes g_k + \sum_j z_j \otimes g'_j,$$

where the g_k 's are unique, and the cosets of $p_j \pi_{m+n+1}$ containing g'_j are unique. Also any element of $H^n(K^{n+1}, K^{n-1}; \pi_{m+n})$ has an expression as a weak sum

$$u = \sum_i u_i \otimes \alpha_i + \sum_{j,\nu} t_{j,\nu} z_{j,\nu} * \beta_{j,\nu},$$

where $z_{j,\nu} * \beta_{j,\nu}$ is the class containing $b_{j,\nu} \otimes \beta_{j,\nu}, 0 \leq t_{j,\nu} < q_{j,\nu}$, and the α_i 's, $t_{j,\nu}$'s are unique.

We construct a group isomorphic to $(n+1, n-1)^m$. Let w be as above, and take new symbols $(u_i, \alpha_i), (z_j, \beta_{j,\nu})$. Let \bar{u} be given by

$$\bar{u} = \sum_i (u_i, \alpha_i) + \sum_{j,\nu} t_{j,\nu} (z_j, \beta_{j,\nu}),$$

whenever $u = \sum u_i \otimes \alpha_i + \sum_{j,\nu} t_{j,\nu} z_j * \beta_{j,\nu}$ is an element of

$$H^n(K^{n+1}, K^{n-1}; \pi_{m+n}).$$

When $m+n = 2$, we lay down a fixed order in which the sum \bar{u} is to be taken: for it follows from our definition that there are only a finite number of non-zero terms. Then our group is to be generated by these elements w, \bar{u} , subject to the following relations:

1. The subgroup generated by the w 's is abelian, and isomorphic to $H^{n+1}(K^{n+1}, K^{n-1}; \pi_{m+n+1})$ in the obvious way.
2. For all $\bar{u}, w, \bar{u} + w = w + \bar{u}$.
3. If \bar{u} is as above, $\bar{u} = 0$ if and only if $\alpha_i = 0, t_{j,\nu} = 0$, for each $i, (j, \nu)$.
- 4 (a). $(m+n > 2)$: If \bar{u} is as above and

$$\bar{u}' = \sum_i (u_i, \alpha'_i) + \sum_{j,\nu} t'_{j,\nu} (z_j, \beta_{j,\nu}),$$

then $\bar{u} + \bar{u}' = \bar{u}'' + w$, where

$$\bar{u}'' = \sum_i (u_i, \alpha_i'') + \sum_{j,\nu} t_{j,\nu}'' (z_j, \beta_{j,\nu}),$$

$$\alpha_i'' = \alpha_i + \alpha_i',$$

$$t_{j,\nu}'' = t_{j,\nu} + t_{j,\nu}' \text{ or } t_{j,\nu} + t_{j,\nu}' - q_{j,\nu}, \text{ according as } t_{j,\nu} + t_{j,\nu}' < q_{j,\nu} \text{ or not,}$$

$$w = \sum_{j,\nu} z_j \otimes \lambda_{p_j, q_{j,\nu}}^{m+n} ((t_{j,\nu} + t_{j,\nu}' - t_{j,\nu}'') / q_{j,\nu}) \gamma^*(\beta_{j,\nu}).$$

4 (b). ($m = n = 1$) If \bar{u} , \bar{u}' are as in 4 (a), then $\bar{u} + \bar{u}' = \bar{u}'' + w + w'$, where \bar{u}'' , w are as before, and

$$w' = \sum_{i < i'} (u_{i'} \cup u_i) \otimes [\alpha_i, \alpha_{i'}'] + \\ + \sum_{i,j,\nu} \{(c_{i'} \cup b_j) \otimes [\beta_{j,\nu}, \alpha_{i'}']\} + \sum_{(j',\nu') < (j,\nu)} \{(b_{j'} \cup b_j) \otimes [\beta_{j,\nu}, \beta_{j',\nu'}']\},$$

where c_i is a cochain in the class u_i , $[\xi, \eta]$ is the Whitehead Product, and brackets $\{ \}$ denote cohomology class.

This group is an extension of $H^{n+1}(\pi_{m+n+1})$ by $H^n(\pi_{m+n})$ if we define a projection onto the latter such that $\bar{u} \rightarrow u$, $w \rightarrow 0$. That the group is equivalent to the extension $(n+1, n-1)^m$ follows from Theorems 10.22, 10.31, 10.41, and the discussion in the Appendix.

10.6. Some special cases

We tabulate the Track Groups $(n+1, n-1)^m$ in certain simple cases, particularly the cases when $X = S^k$, a sphere. Let Z_p be a cyclic group of order p , and Z_∞ an infinite cyclic group. The generators of the summands are not described explicitly; in most cases they are easy to recognize (see the previous section).

$$10.61. G_{p,q}^{n+m} = (Y_p^{n+1}, e^0)^m (Y_q^{n+m+1}, e^0; e^0).$$

Let $p = 2^a p_0$, $q = 2^b q_0$, where p_0, q_0 are odd, and let $d = GCD(p, q)$. Then $d = 2^c d_0$, where d_0 is odd and $c = \text{Min}(a, b)$.

TABLE 2

$G_{p,q}^k$		$c = 0$	$c = 1$	$c \geq 2$
$k = 2$	$a > b = c$	$Z_d + Z_d$	$Z_{2d} + Z_d$	$Z_{2d} + Z_d$
	$a = b = c$	$Z_d + Z_d$	$Z_{2d} + Z_{2d}$	$Z_d + Z_d$
	$c = a < b$	$Z_d + Z_d$	$Z_d + Z_d$	$Z_d + Z_d$
$k > 2$	$a > b = c$	Z_d	$Z_2 + Z_d$	$Z_2 + Z_d$
	$a = b = c$	Z_d	Z_{2d}	$Z_2 + Z_d$
	$c = a < b$	Z_d	$Z_2 + Z_d$	$Z_2 + Z_d$

10.62. $S(k; n+m; p) = (Y_p^{n+1}, e^0)^m(S^k, e^0; e^0)$, $k \geq n+m-1$.
We suppose $k > 1$: all these groups are trivial for $k = 1$.

TABLE 3

$S(k; n+m; p)$	p odd	$p \equiv 2 \bmod 4$	$p \equiv 0 \bmod 4$
$k = n+m+1 \geq 2$	Z_p	Z_p	Z_p
$k = n+m \geq 3$	0	Z_2	Z_2
$k = n+m = 2$	Z_p	Z_p	Z_p
$k = n+m-1 \geq 3$	0	Z_4	$Z_2 + Z_2$
$k = n+m-1 = 2$	0	Z_2	Z_2

10.63. *Other A_n^2 -polyhedra*, $n \geq 3$

An A_n^2 -polyhedron is an $(n-1)$ -connected finite CW-complex of dimension not exceeding $n+2$. According to Chang (2), if $n \geq 3$, an A_n^2 -polyhedron is of the same homotopy type as a *cluster* (that is, a union of spaces with a single common point) of *elementary A_n^2 -polyhedra*. These are the spaces $S^n, S^{n+1}, S^{n+2}, Y_p^{n+1}, Y_p^{n+2}$ (p a power of a prime), and spaces of the four varieties:

$$\begin{aligned} B_4 &= S^n \cup e^{n+2}, \\ B_5(r) &= Y_r^{n+1} \cup e^{n+2} = B_4 \cup e^{n+1}, \quad r = 2^p, \\ B_6(s) &= S^n \cup S^{n+1} \cup e^{n+2}, \quad s = 2^q, \\ B_7(r, s) &= B_6(s) \cup e^{n+1}, \quad r = 2^p, \quad s = 2^q. \end{aligned}$$

In B_4 the $(n+2)$ -cell is attached essentially; in B_6 , by a map of the form $(a+b)$, where a is an essential map on S^n , and b is a map of degree 2^q on S^{n+1} ; in B_7 the $(n+1)$ -cell is attached to S^n by a map of degree 2^p .

We compute the groups $(B_t, e^0)^m(S^k, e^0; e^0)$ for $k \geq m+n$ (except for low values of $m+n$); if $k > m+n+2$, the group is trivial. Let $t = 4, 5, 6$, or 7 , and consider the sequence:

$$\xrightarrow{\delta^*} (B_t, S^n)^m(S^k, e^0; e^0) \xrightarrow{i^*} (B_t, e^0)^m(S^k, e^0; e^0) \xrightarrow{j^*} (S^n, e^0)^m(S^k, e^0; e^0) \xrightarrow{\delta^*}.$$

It is easy to see that i^* is onto when $k > m+n$, and also when $k = m+n$, and $t = 5, 7$; when $k = m+n$ and $t = 4, 6$, j^* maps onto $2\pi_{m+n}(S^{m+n}) = Z_\infty$. The groups are abelian, as B_t is of the homotopy type of the suspension of an analogous space one dimension less ($n \geq 3$); therefore when $k = m+n$ and $t = 4, 6$, $(B_t, e^0)^m(S^k, e^0; e^0) = i^*(B_t, S^n)^m(S^k, e^0; e^0) + Z_\infty$, since any abelian extension of a group by a cyclic infinite group is isomorphic to the direct sum. Therefore, in this range of k , it remains only to compute the image of i^* .

By the Factor Theorem (I, 4.2) $(B_t, S^n)^m \approx (B_t/S^n, e^0)^m$ which is given by the Cluster Theorem (I, § 7.4) in terms of groups already computed. It is not difficult to find the homomorphism

$$\delta^*: (S^n, e^0)^{m+1}(S^k, e^0; e^0) \rightarrow (B_t, S^n)^m(S^k, e^0; e^0)$$

directly from the definition, and it is given in (I, § 9) in terms of Steenrod Squares. The results are tabulated below, in which table 2^p , 2^q are as in the definitions of B_i .

TABLE 4

$B_i, e^0)^m(S^k, e^0; e^0)$	$t = 4$	$t = 5$	$t = 6$	$t = 7$
$k = m + n + 2 \geq 6$	Z_∞	Z_∞	Z_{2^q}	Z_{2^q}
$k = m + n + 1 \geq 5$	0	$Z_{2^{p+1}}$	0	$Z_{2^{p+1}}$
$k = m + n \geq 7$	Z_∞	Z_2	$Z_2 + Z_\infty$	$Z_2 + Z_2$

11.1. Tube systems in I^3

In § 11 we compute $G_{p,p}^2$ and the commutators of $(2,0)^1$ by purely geometric means. Instead of writing down explicit homotopies (which would, in any case, be extremely difficult) we use an illuminating but intuitive method of Whitney's.

In (13) Hassler Whitney defines a *tube* in I^3 as a map $T: \sigma^2 \times S^1 \rightarrow I^3$, where σ^2 is a circular disk (the points (y_1, y_2) in the Euclidean plane such that $y_1^2 + y_2^2 \leq 1$), and where T satisfies the conditions

- (i) T is orientation preserving and 1-1 into, a C^2 map with non-vanishing Jacobian,
- (ii) T is linear on each cross-section $\sigma^2 \times \theta$ ($\theta \in S^1$),
- (iii) $T(\sigma^2 \times S^1)$ does not meet \dot{I}^3 .

We shall vary this definition by using rectangular patches instead of circular disks. A *tube-system* in I^3 is a collection of disjoint tubes T_1, \dots, T_n (we use the same letter and same name 'tube' to stand for the map T and its image $T(\sigma^2 \times S^1)$). We wish to describe how two tubes are intertwined, and how one tube is twisted: Whitney calls the circles $\zeta_i = T_i(x_i \times S^1)$ (where $x_i \in \sigma^2$) *filaments*, and defines $LC(T_i, T_j)$ as $LC(\zeta_i, \zeta_j)$ for $i \neq j$ (this clearly does not depend on the choices of x_i, x_j), and $LC(T_i)$, the twist of the tube T_i , as $LC(\zeta_i, \zeta'_i)$, for filaments of the same tube with any choice of $x_i \neq x'_i$. He shows that these are invariant definitions, and also that there is an isotopic deformation of a tube T into a canonical tube with the same linking coefficient. Always, when we speak of deforming a tube, we mean deforming by an isotopy.

A *tube-system mapping* is defined by a system of tubes T_1, \dots, T_n , and n maps $\alpha_i: [\sigma^2, \dot{\sigma}^2] \rightarrow [X, x_0]$ ($i = 1, \dots, n$) representing elements α_i of

$$\pi_2(X, x_0) = \pi_2.$$

The mapping

$$\phi: [I^3, \dot{I}^3] \rightarrow [X, x_0],$$

representing an element of $\pi_3(X, x_0) = \pi_3$, is defined over each $T_i(\sigma^2 \times S^1)$ by

$$\phi T_i(y_1, y_2, \theta) = \alpha_i(y_1, y_2), \quad (y_1, y_2) \in \sigma^2, \quad \theta \in S^1,$$

and maps the rest of I^3 to x_0 . This is continuous since it is continuous over the closed sets $T_i(\sigma^2 \times S^1)$ and over the closure of the complement of all the tubes. We shall describe this map ϕ by saying that ϕ maps each cross-section of the tube T_i by a_i , the rest to x_0 ; a cross-section is the rectangle $T_i(\sigma^2 \times \theta)$, for any $\theta \in S^1$. *

We notice three results of Whitney's. First, if $\alpha \in \pi_2$, and $\gamma^*(\alpha) \in \pi_3$ is the composite $\alpha \circ \gamma$ (γ is the class of the Hopf map $S^3 \rightarrow S^2$), and ϕ is a tube system mapping defined by just one tube T with $LC(T) = +1$, and a map a representing α , then ϕ represents $\gamma^*(\alpha)$.

Secondly, if ϕ is the mapping defined by two tubes T_1, T_2 such that $LC(T_1, T_2) = +1$, but $LC(T_1) = 0 = LC(T_2)$ (that is, two untwisted tubes linking once positively), and two maps a_i representing α_i , then ϕ represents the Whitehead Product $[\alpha_1, \alpha_2]$.

These are particular cases of the third and serve to explain it. Let Π be any plane in I^3 , with a chosen direction along its normal (giving the positive normal) defining 'above' and 'vertical' (with respect to Π) in I^3 . We shall choose one or other of the faces of I^3 as Π . A tube system is called *standard* with respect to Π if each tube T_i in the system always has one filament on top. In view of the linearity condition (ii) on T_i , this may be replaced by the slightly stronger condition:

There is a line in σ^2 which is vertical under the composition of maps $\sigma^2 \rightarrow \sigma^2 \times \theta \rightarrow T_i(\sigma^2 \times \theta)$, for all θ .

Now select a filament ζ_i in each tube T_i , and let $p: I^3 \rightarrow \Pi$ (the normal projection on Π) carry ζ_i into the curve $p\zeta_i$ in Π . We may simultaneously deform all the tubes of a standard system into a standard system such that the curves $p\zeta_i$ meet only in isolated points, and there cut each other (or themselves) orthogonally, so that at each intersection there are only two curves, or two parts of a curve. The orientation of S^1 defines an orientation of ζ_i and so of $p\zeta_i$; if $z \in \Pi$ is a point of intersection of $p\zeta_i$ and $p\zeta_j$ (where i may be equal to j), these orientations define two vectors v_i, v_j , the positive tangents to $p\zeta_i$ and $p\zeta_j$ at z , respectively. If ζ_i is above ζ_j over z , we define the crossing to be positive or negative according as the ordered pair (v_i, v_j) defines the positive or negative orientation of Π .

Let N_{ij}^* ($i \neq j$) denote the algebraic sum of the crossings of $p\zeta_i$ with $p\zeta_j$ at which ζ_i crosses above ζ_j , and let $N_{ij} = N_{ij}^* + N_{ji}^*$ be the algebraic sum of all crossings of $p\zeta_i$ with $p\zeta_j$. Finally, let N_i be the algebraic sum of all crossings of $p\zeta_i$ with itself. Then it is clear that $N_{ij}^* = LC(T_i, T_j) = N_{ji}^*$ (consider a cylinder orthogonal to Π on the curve $p\zeta_i$).

Then Whitney proves that, if a tube-system mapping ϕ is defined by means of a standard tube system T_1, \dots, T_n and maps a_i representing α_i ,

THEOREM 11.11 (Whitney). ϕ represents the element

$$\sum_i N_i \gamma^*(\alpha_i) + \sum_{i < j} N_{ij}^* [\alpha_i, \alpha_j] \in \pi_3,$$

which clearly is independent of the ordering of the tubes. In particular, if $\alpha_i = \alpha$ for all i , and $N = \sum_i N_i + \sum_{i < j} N_{ij}$, the algebraic sum of all crossings, this reduces to $N\gamma^*(\alpha)$, since

$$N_{ij}^* [\alpha, \alpha] = 2N_{ij}^* \gamma^*(\alpha) = (N_{ij}^* + N_{ji}^*) \gamma^*(\alpha) = N_{ij}(\alpha).$$

11.2. Open tubes and junctions

By an *open tube* we mean a map $T: \sigma^2 \times I \rightarrow I^3$ satisfying the conditions of linearity and differentiability ((i), (ii) in 11.1). If $T(y_1, y_2, 1) = T(y_1, y_2, 0)$ for every (y_1, y_2) in σ^2 , and in addition the tube does not meet I^3 , then we may also consider the tube to be the (closed) tube given by

$$T(y_1, y_2, t) = T(y_1, y_2, e^{2\pi i t})$$

(where S^1 is the unit circle in the Argand Plane). In general, we shall impose conditions upon the *ends* $T(\sigma^2 \times 0)$, $T(\sigma^2 \times 1)$, and require them to be specified rectangles, or, as in the case above, to be the same. Also, if a *standard* system is defined in the same way, we shall only use standard systems. The direction of a tube is defined by a vector along I running from 0 to 1, and we call $T(\sigma^2 \times 0)$ the beginning, $T(\sigma^2 \times 1)$ the end, or say the tube *springs from* the patch $T(\sigma^2 \times 0)$, and *terminates* on the patch $T(\sigma^2 \times 1)$.

Whitney (loc. cit.) defines positive and negative *q-junctions* in I^3 : for convenience, we modify his definition slightly. Let $E: I^3 \rightarrow I^3$ be an orientation preserving linear map; we shall refer to the point (t_1, t_2, t_3) of E , meaning the point $E(t_1, t_2, t_3)$. In the face $t_2 = 0$ of E , take q rectangular patches $\sigma_j: \sigma^2 \rightarrow E$, similarly situated and arranged in a row parallel to the edge $t_2 = t_3 = 0$ so that the directions y_1, y_2 increasing in σ^2 are parallel to the directions t_1, t_3 increasing of E ; for example, let σ_j be the patch

$$\sigma_j(y_1, y_2) = E((4j-3+2y_1)/4q, 0, (1+2y_2)/4).$$

Now suppose that q disjoint tubes T_1, \dots, T_q spring from the patches $\sigma_1, \dots, \sigma_q$, so that $T_j(y_1, y_2, 0) = \sigma_j(y_1, y_2)$: then we shall call E a *positive q-junction*. A *negative q-junction* is defined similarly, with an orientation reversing linear map $E: I^3 \rightarrow I^3$, and with tubes terminating on the face $t_2 = 0$ of E ; for example, we may reflect the above construction in the plane $t_1 = \frac{1}{2}$, reparametrizing the tubes accordingly.

We now generalize our concept of tube system, to include collections of closed tubes, junctions, and open tubes, such that every open tube begins and ends on junctions or the face $y_2 = 0$ of I^3 , and no tube or junction meets any other face of I^3 .

A *mapping* of such a tube system is defined by assigning to each tube an element of π_2 and a map representing that element (satisfying certain conditions). We map each cross-section of a tube in the way defined for closed tubes, and map the exterior of all the tubes and junctions to the point x_0 . It may then be possible to extend this map over each of the junctions in such a way that the boundaries (except for the patches on one face) are mapped to x_0 . For example, if E is a q -junction, and tubes T_1, \dots, T_q all spring from or terminate on E , the map of the tubes can be extended over E if and only if $\alpha_1 + \dots + \alpha_q = 0$, where T_1 is mapped by means of α_i representing α_i . We shall not consider situations in which this extension is not possible.

Let $\psi: [I^3, I^3] \rightarrow [K^3, K^2]$ be the characteristic map of some 3-cell in the complex K^3 , such that all faces of I^3 except the face $y_2 = 0$ are mapped to a vertex e^0 . Then a tube system in I^3 , with the ends of open tubes on no face except $y_2 = 0$, defines a tube system in the 3-cell, with open tubes terminating or beginning in K^2 . Then it may happen that we can map $K^3 \rightarrow X$ so that we induce a tube system mapping of I^3 ; in this case, we say that we have a tube system mapping in the 3-cell.

Let E_+ , E_- be respectively a positive and negative q -junction in some system. We may deform the patches on their faces so that they are both given by the explicit formulae above (see the second paragraph of this section). Then it may happen that in a tube system mapping ϕ ,

$$\phi E_+(t_1, t_2, t_3) = \phi E_-(t_1, t_2, t_3);$$

in this case, we say that the junctions have been mapped *similarly*. We now show (following Whitney) how we can deform $\phi \text{ rel } I^3$ into another tube system mapping in which the junctions E_+ , E_- have been eliminated, at the expense, perhaps, of introducing closed tubes. Observe that a deformation of the tube system induces a deformation of ϕ in an obvious way; we may deform the system so that the two junctions are back to back, that is,

$$E_+(t_1, 1, t_2) = E_-(t_1, 1, t_2),$$

slightly relaxing our condition that junctions and tubes were to be disjoint. Consider the map ϕ' defined by an identical tube system, except that opposite patches on E_+ and E_- have been joined by open tubes, parallel to the edge $t_1 = t_3 = 0$ of both junctions, and the tubes suitably re-parametrized (so that the tube terminating on σ_j in E_- now runs through and is part of the tube springing from σ_j on E_+): ϕ' is to agree with ϕ outside the junctions E_+ , E_- and inside is to map each cross-section of the new parts of the tubes as their ends are mapped, and the rest to x_0 . Then we say that $\phi \simeq \phi'$, the homotopy being constant outside the junctions; it is

only necessary to define the homotopy inside the junctions, and such a homotopy is given by

$$\begin{aligned}\phi_t E_+(t_1, t_2, t_3) &= \begin{cases} \phi E_+(t_1, 0, t_3) & \text{if } 0 \leq t_2 \leq t, \\ \phi E_+(t_1, t_2 - t, t_3) & \text{if } t \leq t_2 \leq 1, \end{cases} \\ \phi_t E_-(t_1, t_2, t_3) &= \begin{cases} \phi E_-(t_1, 0, t_3) & \text{if } 0 \leq t_2 \leq t, \\ \phi E_-(t_1, t_2 - t, t_3) & \text{if } t_2 \leq t \leq 1. \end{cases}\end{aligned}$$

This process we call *eliminating the similar junctions* E_+ , E_- . In our cases, all junctions will be paired with a similarly mapped junction of opposite sign: we can therefore eliminate them all in turn, obtaining tube system mappings based on closed tubes, which can be evaluated by Whitney's formula.

11.3. On extending homotopies in a complex

We define a construction of general use; suppose $f_0: K^n \rightarrow X$, where K is a simplicial or a CW-complex, and f_0 has an extension f'_0 over K^{n+1} . Then it is known that a homotopy f_t also has an extension f'_t over K^{n+1} ; we describe a particular way of extending f_t .

First, if $f'_0: I^{n+1} \rightarrow X$, and $f_t: I^n \rightarrow X$, where $f_0 = f'_0|I^n \times 0$, then we have an extension of f_t to f'_t given by

$$f'_t(y_1, \dots, y_n) = \begin{cases} f_{(1-2y_{n+1})}(y_1, \dots, y_n) & \text{if } 0 \leq y_{n+1} \leq \frac{1}{2}t \\ f'_0(y_1, \dots, y_n, (2y_{n+1}-t)/(2-t)) & \text{if } \frac{1}{2}t \leq y_{n+1} \leq 1 \end{cases}$$

so that f'_0 is the original map, and f'_1 is given by

$$f'_1(y_1, \dots, y_n) = \begin{cases} f_{(1-2y_{n+1})}(y_1, \dots, y_n) & \text{if } 0 \leq y_{n+1} \leq \frac{1}{2} \\ f'_0(y_1, \dots, y_n, 2y_{n+1}-1) & \text{if } \frac{1}{2} \leq y_{n+1} \leq 1. \end{cases}$$

Now suppose that K is a simplicial or CW-complex, L is a closed subcomplex, and every $(n+1)$ -cell or simplex has a characteristic map

$$\phi: I^{n+1} \rightarrow K^{n+1}$$

which is a homeomorphism of $I^{n+1} - \dot{I}^{n+1}$ onto the interior, and maps $\dot{I}^{n+1} - I^n \times 0$ into L . For example, let $n = 1$, and let L be a subcomplex containing all the vertices: then every simplex has such a map, where $\phi(\dot{I}^2 - I \times 0)$ is a vertex, and $I \times 0 - \dot{I} \times 0$ is mapped homeomorphically on the rest of the boundary. Again, if K is a connected CW-complex, and L is a vertex, we can replace K by another complex of the same homotopy type, L by another vertex, in which the above condition is satisfied.

Now let $g'_0: [K^{n+1} \cup L, L] \rightarrow [X, x_0]$, and let $g_t: [K^n \cup L, L] \rightarrow [X, x_0]$ be a homotopy rel L of $g_0 = g'_0| [K^n \cup L, L]$. Let

$$f'_0 = g'_0 \circ \phi, f_t = g_t \circ (\phi|I^n \times 0),$$

and let f'_i be as above, for each cell of K^{n+1} ; then we can define g'_i , an extension of g_i , as the unique map such that $g'_i \circ \phi = f'_i$, for each cell of K^{n+1} . We shall describe this operation as *taking up the homotopy* g_i . Clearly g'_1 is given over $K^n \cup L$ by g_i , and over each cell of $K^{n+1} - L$ by the equations for f'_1 above, with f'_1, f'_0 replaced by $g'_1 \circ \phi, g'_0 \circ \phi, f_i$ by $g_i \circ (\phi | I^n \times 0)$.

11.4. The commutators of $(2, 0)^1$

If $K = K^2$ is a finite, connected CW-complex, we can suppose without loss of generality that K is a finite, connected simplicial complex with the weak topology, also of dimension 2 ((9) Theorem 13). In this simplicial complex, give the vertices a simple order: then a simplex is uniquely described by its vertices in this order. Also there is a tree T in the 1-skeleton which contains all the vertices of K (see (8), p. 322), and, since T is contractible to a vertex, we have seen (I, § 7.3) that

$$(K, T)^1(X, x_0; x_0) \approx (2, 0)^1.$$

Any cocycle $u \in C^1(K, T; \pi_2)$, which is unique in its cohomology class, determines a homotopy class of maps $[K^1 \times I, T \times I \cup K^1 \times \dot{I}] \rightarrow [X, x_0]$ which have extensions over $K^2 \times I$ carrying $K^2 \times \dot{I}$ to x_0 ; let $f: K^2 \times I \rightarrow X$ be such an extension, and let g be similarly defined from a cocycle v . We construct f, g , and the commutator $f+g-f-g$ in a certain way, and prove Theorem 10.22.

Let J^1 be the closure of $\dot{I}^2 - I \times 0$, and select a map

$$\psi: [I^2, I \times 0, J^1] \rightarrow [ABC, \partial ABC, A]$$

which maps the interior of I^2 homeomorphically on the interior of the 2-simplex ABC , and $I \times 0 - \dot{I} \times 0$ homeomorphically on

$$\partial ABC - A = (AB \cup BC \cup AC) - A,$$

where A is the first vertex of ABC . For convenience in the diagrams, we shall illustrate all constructions and homotopies on $ABC \times I$ with diagrams referring to $I^2 \times I$: these are to be mapped to $ABC \times I$ by the map $\psi \times 1$ (1 being the identity map $I \rightarrow I$). The face $I^2 \times 0$ will be known as the *bottom*, and the face $(I \times 0) \times I$ as the *front*; notwithstanding this convention, let the plane Π of (11.1) be $(0 \times I) \times I$, the left face of the cube: we construct tube systems in $I^2 \times I$ which are standard with respect to Π .

Take a positive 3-junction E in $I^2 \times I$, the face $t_2 = 0$ to the front, the face $t_1 = 0$ to the left, nearest Π ; this is projected by $\psi \times 1$ into a junction in $ABC \times I$. In each $AB \times I$, select a patch σ , with edges parallel to $AB \times 0$ and $A \times I$, and (if ψ is a suitable homeomorphism) select corresponding patches

$$\sigma_1 \subset (\psi \times 1)^{-1}(AB \times I), \quad \sigma_2 \subset (\psi \times 1)^{-1}(BC \times I), \quad \sigma_3 \subset (\psi \times 1)^{-1}(AC \times I)$$

which project homeomorphically onto the patches on the faces of the prism $ABC \times I$. In $I^2 \times I$, take three open tubes springing from the junction E and terminating on these patches: we may arrange the situation so that these run parallel to Π and to the bottom of the cube: these, projected into $ABC \times I$ define open tubes running from the junction to the rectangular faces of the prism (see Diagram 1).

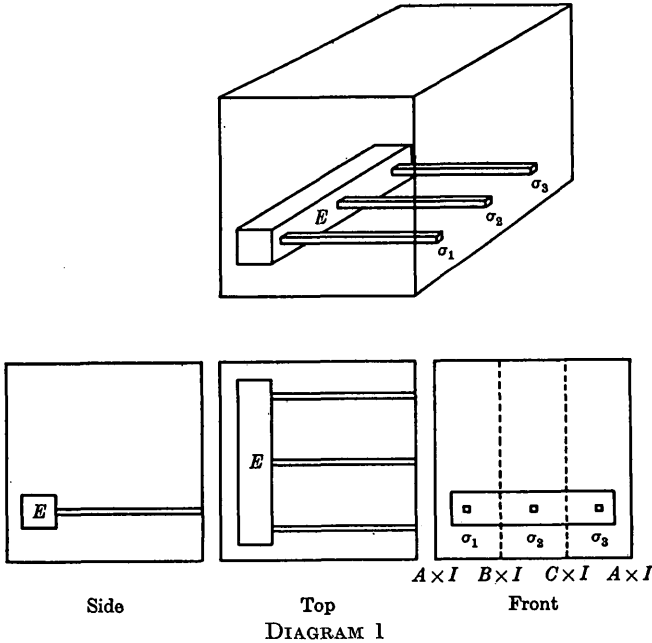


DIAGRAM 1

We choose a patch σ in each $AB \times I$ (all 1-cells of K), choose a ψ for each 2-simplex of K so that it is linear on each $\psi^{-1}(AB \times I)$, a junction in each $ABC \times I$, and carry out the above construction for each prism. Each simplex is to be oriented by the ordering of the vertices: each patch σ is to be oriented consistently with the rectangle $AB \times I$; then the open tubes above determine orientations of the patches $\sigma_1, \sigma_2, \sigma_3$ which are the same for σ_1, σ_2 , but the opposite orientation for σ_3 (for the orientation of $I^2 \times 0$ determines an orientation of $AB \times I, BC \times I, CA \times I$, and the latter is the opposite orientation to the one chosen).

Now the cochain $u \in C^1(K, T; \pi_2)$ determines an element $u(AB)$ of π_2 for each simplex AB , which is zero if $AB \subset T$; if u is a cocycle, then for every simplex ABC ,

$$u(AB) + u(BC) - u(AC) = 0.$$

Select a map $f_{AB}: [\sigma, \sigma] \rightarrow [X, x_0]$ which represents $u(AB)$ for each 1-simplex

AB , choosing the constant map x_0 if $u(AB) = 0$. Extend f_{AB} over $AB \times I$ for every simplex, by mapping the rest of $AB \times I$ to x_0 . Then this is a map $K^1 \times I \rightarrow X$ such that $(K^1 \times \dot{I}) \cup T \times I$ is mapped to x_0 . Extend this map over the tubes in $ABC \times I$, by mapping each cross-section of each tube by the same map as on its end patch; these are maps representing $u(AB)$, $u(BC)$, and $-u(AC)$, and since u is a cocycle this can be extended over the interiors of each junction. We map the rest of $K^2 \times I$ to x_0 , and obtain

$$f: [K \times I, K \times \dot{I} \cup T \times I] \rightarrow [X, x_0],$$

which is uniquely defined, in its homotopy class, over $K^1 \times I$ by u . Clearly, we have a large number of different extensions, one for each set of choices of the extensions over the junctions.

Now let g be similarly defined, using a cocycle v . Then let $h = f + g - f - g$ be given by

$$h(p, t) = \begin{cases} f(p, 4t) & \text{if } 0 \leq t \leq \frac{1}{4}, \\ g(p, 4t-1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2}, \\ f(p, 3-4t) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}, \\ g(p, 4-4t) & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Then h is also a tube system mapping, defined as follows (see Diagram 2a). Divide $K \times I$ into four homeomorphic copies, separated by $K \times \frac{1}{4}$, $K \times \frac{1}{2}$, $K \times \frac{3}{4}$, and take one of the above tube systems in each; map the first tube system by f , the second by g , and the third and fourth by f, g after reflecting in the surfaces $K \times \frac{5}{8}$, $K \times \frac{7}{8}$. This gives in each $AB \times I$ four patches σ_+ , σ'_+ , σ_- , σ'_- vertically above each other with σ_+ at the bottom and σ'_- at the top (nearest $AB \times 1$). In $ABC \times I$ we have, in order from $ABC \times 0$, E_+ , E'_+ , E_- , E'_- , the first two being positive 3-junctions, and the second two negative 3-junctions. Each of these are joined by open tubes to three patches, so that E_+ is joined to $\sigma_{1,+}$, $\sigma_{2,+}$, $\sigma_{3,+}$, and E'_- to $\sigma'_{1,-}$, $\sigma'_{2,-}$, $\sigma'_{3,-}$. Moreover, in the map h , E_+ and E_- , and also E'_+ , E'_- are mapped similarly.

This map h represents the commutator, an element of $i^*(K^2, K^1)^1$. We must therefore deform h into a map carrying $K^1 \times I$ to x_0 . In doing this, notice that any deformation of the tube systems defines a homotopy in the obvious way. First, on each $AB \times I$, slide the patches σ_+ , σ_- towards the first edge $A \times I$, and the other two, σ'_+ , σ'_- , towards the second edge $B \times I$. Now *take up* this homotopy (in the sense of (11.3)): the effect is to bend the various tubes in $ABC \times I$ slightly (Diagram 2b).

We now define a homotopy over $K^1 \times I$: identify $AB \times I$ with $AB \times I \times 0$ in $AB \times I^2$ in the obvious way, and let the patches σ_+ , σ_- and the patches σ'_+ , σ'_- be joined by open tubes T , T' respectively, so that each runs into

$AB \times I^2$ parallel to $AB \times 0 \times I$ and $A \times I^2$, turns up and runs parallel to $AB \times I \times 0$ and $A \times I^2$, then turns back towards $AB \times I \times 0$ and terminates on the upper patch (Diagram 3). Map each cross-section of each tube in

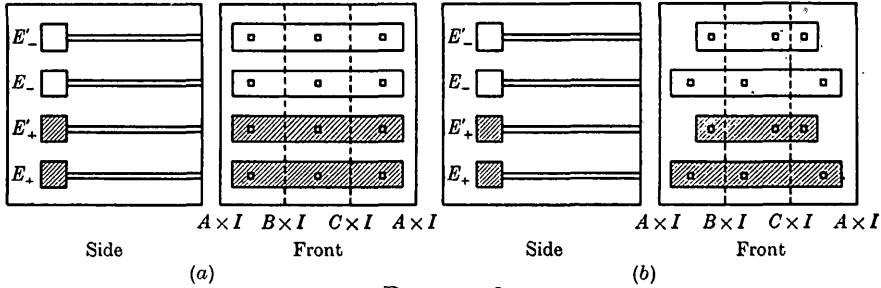


DIAGRAM 2

the same way as its ends, and map the rest of $AB \times I^2$ to x_0 ; do this for every simplex $AB \subset K^1$. The effect of taking up this homotopy in $ABC \times I$ is to connect the tubes running from E_+ to the tubes running to E_- , and the tubes running from E'_+ to the tubes terminating on E'_- , and not meeting $(\partial ABC) \times I$ (Diagram 3).

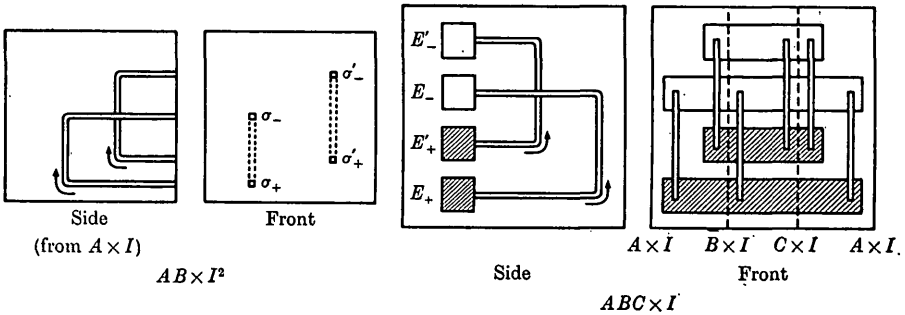


DIAGRAM 3

Before going further we must name the tubes; this we do in the obvious way so that T_1, T_2, T_3 spring from E_+ and run to E_- , and contain respectively the tubes that used to terminate on $\sigma_{1,+}, \sigma_{2,+}, \sigma_{3,+}$, and also those that used to run from $\sigma_{1,-}, \sigma_{2,-}, \sigma_{3,-}$; T'_1, T'_2, T'_3 are the similar tubes running from E'_+ to E'_- .

Now eliminate the junctions E_+, E_- , which are similarly mapped, in the following way. Pull both back (towards $I \times 1 \times I$ in $I^2 \times I$) until they are behind the other junctions, turn the lower up and the upper down, and so oppose them (back to back). Now eliminate the junctions as described in (11.2), replacing the tubes T_1 by closed tubes, each describing a simple loop parallel to the plane $\Pi = (0 \times I) \times I$ (Diagram 4 a).

In this situation the tube T_1 does not link anything; moving it slightly to the left to clear it from E'_+ we can contract the tube into a small loop (still parallel to Π) at the bottom of the cube. Similarly, T_3 may be moved

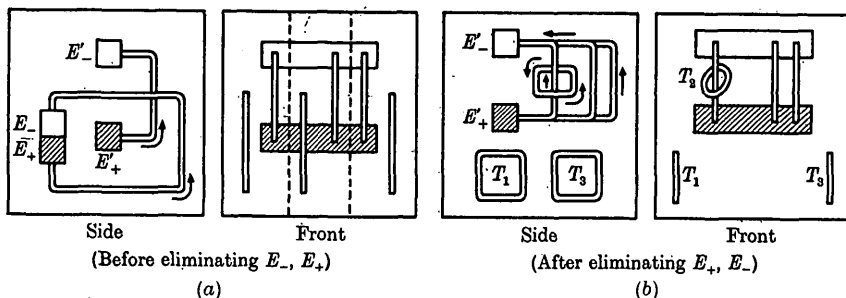


DIAGRAM 4

slightly to the right, contracted, and moved to the bottom. T_2 encircles E'_+ and cannot be removed from the remaining tubes; however, if we tilt its plane slightly, it can be brought over the end of E'_+ and contracted into a loop encircling T'_1 joining E'_+ to E'_- (Diagram 4b). Obviously, this may be done so that the tubes are still standard. There is now no impediment

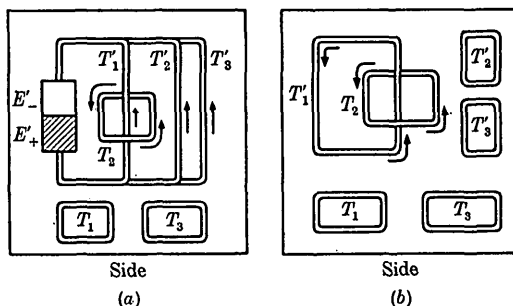


DIAGRAM 5

to eliminating the remaining junctions in the same way, turning E'_+ up and E'_- down, opposing them, and eliminating the junctions as in (11.2), obtaining closed tubes T'_i . Clearly T'_2, T'_3 link no other tubes, and may be contracted and pushed to the bottom as were T_1, T_3 (Diagram 5).

We now have a tube system mapping in every prism $ABC \times I$ in which the boundary is mapped to a point; this represents an element of π_3 which may be determined by Whitney's formula. The tubes T_1, T_3, T'_2, T'_3 have no crossings with any tube, seen from Π (the left), whereas T_2 and T'_1 cross twice, the lower crossing with T_2 nearer Π , the higher with T'_1 nearer Π . Both these are positive crossings, and T_2 carries a map representing $u(BC)$, T'_1 a map representing $v(AB)$. Therefore the whole represents the Whitehead Product $[u(BC), v(AB)] = [v(AB), u(BC)]$.

Now $AB \cup BC = ABC$ (considering these as integral cochains) and $v \cup u$ has value $[v(AB), u(BC)]$ on ABC . This cochain in $C^2(K, T; \pi_3)$ represents a class of maps in $(K, T)^2$ which contains the map constructed above. Therefore Theorem 10.22 is proved.

11.5. Structure of $G_{p,p}^2$

We now investigate the homotopy classes of maps

$$[Y_p^2 \times I, Y_p^2 \times \dot{I} \cup e^0 \times I] \rightarrow [Y_p^3, e^0].$$

Construct $Y_p^2 = S^1 \cup e^2$ as follows: in the face $I \times 0 \subset I^2$ select p 'windows' ξ_1, \dots, ξ_p , where $(y_1, 0)$ lies in ξ_j if and only if $4j-3 \leq 4py_1 \leq 4j-1$. Let ϕ map each window with degree $+1$ on S^1 , the rest of $I \times 0$ and of I^2 to $e^0 \in S^1$. Let Y_p^2 be the identification space $S^1 \cup I^2$ with the identifications defined by ϕ on \dot{I}^2 , and extend ϕ over the interior of I^2 to form a characteristic map of the 2-cell e^2 .

In $Y_p^2 \times I$ select a rectangular patch σ in $S^1 \times I - e^0 \times I$; this defines p rectangular patches $\sigma_1, \dots, \sigma_p$ in $I \times 0 \times I$, each of which is in some window ξ_j and is mapped homeomorphically on σ by $\phi \times 1$. For the moment, we take $\Pi = I^2 \times 0$ as the bottom of $I^2 \times I$, and the face $(I \times 0) \times I$ with the patches in as the front; in this way, the direction y_1 increasing (where (y_1, y_2, y_3) is a point in $I^2 \times I$) defines left to right.

Take a positive p -junction E in $I^2 \times I$ so that the face $t_2 = 0$ of E faces the front, and the face $t_3 = 0$ is underneath (and t_1 increases from left to right). Take p open tubes T_1, \dots, T_p running from the junction parallel to the bottom and the left face, terminating on the patches σ_i . This may be done so that the system is standard with respect to Π , the bottom. The map $\phi \times 1$ maps this into Y_p^2 , and defines a tube system in $e^2 \times (I - \dot{I})$, with p tubes running from a junction in the set, and terminating in its closure on the one patch σ in $S^1 \times I$. This tube system has the advantage that a tube system mapping in $I^2 \times I$ defines a map of $Y_p^2 \times I$ which carries $Y_p^2 \times \dot{I} \cup e^0 \times I$ to a point.

Now let $f: [\sigma, \dot{\sigma}] \rightarrow [Y_p^3, e^0]$ represent the generator i_p^2 of the cyclic group of order p , $\pi_2(Y_p^3)$. Extend f over the rest of $S^1 \times I$, mapping the rest to x_0 , and over the rest of $Y_p^2 \times I$ outside the junction, by mapping the cross-sections of the tubes by f (as the ends are already mapped) and the rest to x_0 . This map can be extended over the junction, since $\pi_2^p = 0$; we choose a definite extension. The resulting map f' represents the element (b, i_p^2) of Theorem 10.41; we must compute $p(b, i_p^2)$.

Let g be the map given by

$$g(q, t) = f(q, pt - j) \quad \text{if } q \in Y_p^2, j \leq pt \leq j+1, j = 0, \dots, p-1.$$

Then g represents $p(b, i_p^2)$, and is defined as a tube system mapping by taking p copies of the above tube system, inserting them in the various layers of $Y_p^2 \times I$ separated by $Y_p^2 \times (j/p)$ ($j = 1, \dots, p-1$), and mapping them all similarly. We need to name the junctions and tubes in this new

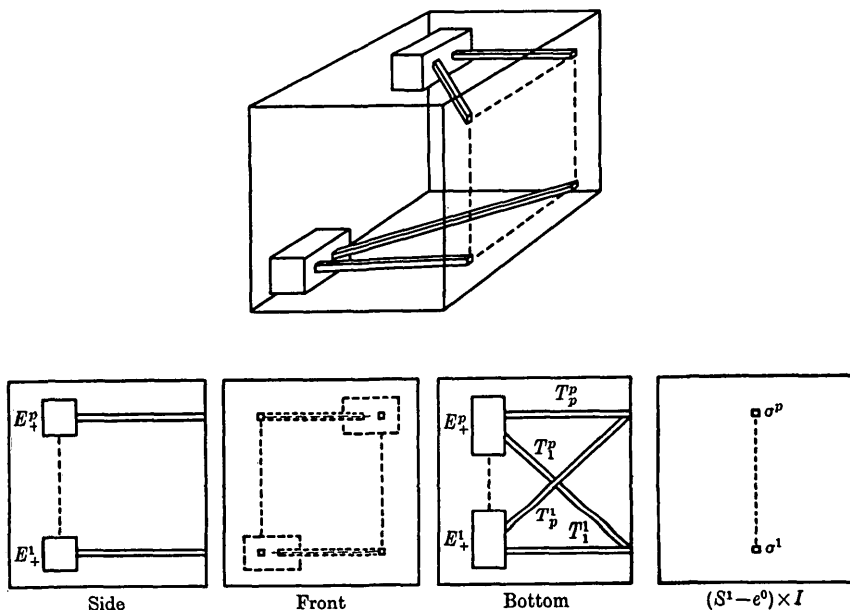


DIAGRAM 6

system. Let p rectangular patches $\sigma^1, \dots, \sigma^p$ be taken, vertically above each other, in $S^1 \times I$, so that σ^1 is nearest $S^1 \times 0$, σ^p nearest $S^1 \times 1$, and the others lie between in the natural order. Take similarly p positive p -junctions E_+^1, \dots, E_+^p in $I^2 \times I$, and map these to junctions in $Y_p^2 \times I$ by $\phi \times 1$. Each σ^j defines p patches $\sigma_1^j, \dots, \sigma_p^j$ on the face $I \times 0 \times I$ of I^2 , which we may suppose are level with the junction E_+^j . Join E_+^j by open tubes T_i^j to the patches σ_i^j , the tubes running parallel to $I^2 \times 0$ and $0 \times I \times I$, and so named that the subscript increases from left to right. Map this tube system into $Y_p^2 \times I$; this is the tube system on which g is a tube system mapping.

This tube system is inconvenient to draw in $I^2 \times I$, especially if we want views from the top. Therefore we make the junctions rather small, and displace them sideways to the right, the top farthest, so that their projections on $I^2 \times 0 = \Pi$ do not overlap, and occur in natural order from left to right (Diagram 6).

Seen from the top, there is still ambiguity in the ends of the tubes T_i^j , which, for fixed i , lie above each other. Therefore we stagger the patches $\sigma^1, \dots, \sigma^p$ in the same way, so that when projected on $I^2 \times 0$ the corresponding

patches σ_i^j lie in p groups of p , in each group i is the same and the index j increases from 1 to p from left to right; in this projection, the patches are arranged in lexicographic order of (i, j) (Diagram 7).

Now slide the patches $\sigma^1, \dots, \sigma^p$ parallel to $e^0 \times I$ until they lie in a line parallel to $S^1 \times 0$. When this homotopy is taken up in $Y_p^2 \times I$, the various

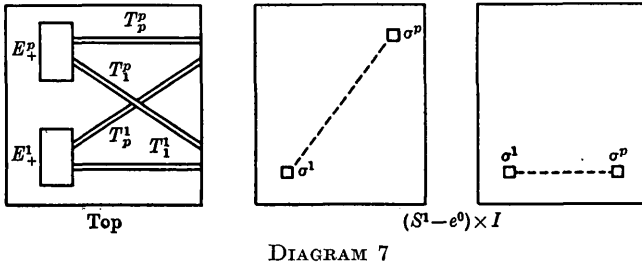


DIAGRAM 7

tubes are no longer parallel to the bottom, but incline in various directions. This can be arranged so that the tube system is standard with respect to Π ; it does not alter the first picture in Diagram 7. Notice that no twisting of the tubes has occurred, and that T_j^i only crosses above T_b^a if $a < i$ and $b > j$, and under T_b^a if these inequalities are reversed; in other cases, T_j^i and T_b^a do not cross in the projection on $\Pi = I^2 \times 0$.

The map on $S^1 \times I$ is inessential (rel $S^1 \times \dot{I} \cup e^0 \times I$); we define a particular homotopy of this to the constant map. In $S^1 \times I \times I$ take a positive p -junction E , with the face $t_2 = 0$ towards $S^1 \times I \times 0$, which we identify in the obvious way with $S^1 \times I$, and the face $t_3 = 0$ nearest $S^1 \times 0 \times I$. Join this by open tubes to the patches $\sigma^1, \dots, \sigma^p$; map each cross-section of each tube by f , and extend over the junction by the extension previously chosen, mapping the rest to x_0 . The tubes are supposed parallel to and standard with respect to $S^1 \times 0 \times I$; when this homotopy is taken up in $Y_p^2 \times I$ by the method of (11.3), we have an equivalent homotopy in $I^2 \times I$.

The effect of this construction is to introduce p negative p -junctions into $I^2 \times I$, in such a way that T_j^i all terminate on the negative junction $E_{i,-}$ for fixed i (Diagram 8a). Each negative junction is similarly mapped to each positive junction, and the tube T_j^i runs from E_{+}^j to $E_{i,-}$, and the scheme of crossings of their projections is the same as before, so that T_j^i crosses over T_a^b if and only if $b < j$ and $a > i$.

We now eliminate the junctions in pairs: it is clear that this can be done without introducing further crossings. For we can compress the system into the left half of the cube (Diagram 8b), and, keeping parallel to the bottom, bring in turn the pairs $E_{+}^j, E_{j,-}$, towards the right, and oppose them (back to back) (Diagram 8c). Then, when all have been eliminated, there remains a number of closed tubes, all mapped by the same map,

forming a standard system with respect to Π . This map represents an element of π_3 which is given by Whitney's formula; in this case, as the tubes are all mapped in the same way, all we have to do is to compute the total number of crossings (having regard to sign). The crossings are all of the same sign (it will appear that the actual sign is immaterial), and there are precisely $p^2(p-1)^2/4$. For there is a crossing for every two pairs of

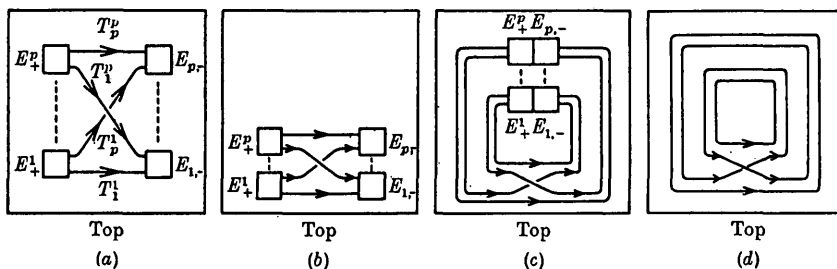


DIAGRAM 8

integers (i, j) , (a, b) for which $a > i$ and $b < j$; this is defined by two ordered pairs (i, a) , (b, j) , and there are obviously $\frac{1}{2}p(p-1)$ such ordered pairs. Therefore, by Whitney's formula the map represents $p^2(p-1)^2/4 \cdot \gamma^*(i_p^2)$; if p is odd or a multiple of 4, this is a multiple of $p\gamma^*(i_p^2)$; otherwise, it differs from $\frac{1}{2}p\gamma^*(i_p^2)$ by a multiple of $p\gamma^*(i_p^2)$. But in Theorem 10.41 the cocycle pz is a coboundary, and, passing to cohomology classes, we may neglect multiples of $p\gamma^*(i_p^2)$. This completes the proof of Theorem 10.41.

CHAPTER 6

NORMAL SPACES

12. Direct limit theorems

In proving the exactness of the sequence of $[P, Q, R]$ in (I, § 5), we were obliged to confine ourselves to the category of HE triples (I, 5.1), and exclude the cases where Dowker's Homotopy Extension Theorem ((3) Theorem 10.2) can be proved. We now wish to remedy this, and consider, in Theorem 12.21, triples $[P, Q; X]$, where P contains a closed subspace Q , is normal and paracompact (i.e. every covering of P has a locally finite refinement), and X is a compact ANR. We use a method analogous to Čech cohomology theory, following Spanier (7) who applied this to the cohomotopy groups of compact spaces P .

In § 12.1 we show that we may confine ourselves to spaces X that are simplicial complexes. Next, we consider the set $(P, Q)^0$ of homotopy classes of maps of $[P, Q]$ to $[X, x_0]$, and in § 12.3 replace $(P, Q)^0$ by the

direct limit of sets $(N, L)^0$, where N is the nerve of a covering of P and L is the subcomplex corresponding to Q . This is an application of a fundamental lemma of Spanier's (Lemma 12.31), and extends his results from compact to normal paracompact spaces (without using a group structure). Finally, in § 12.4, we replace $(P, Q)^0$ by $(P, Q)^m$, and show we can replace the previous direct limit by the direct limit of the groups $(N, L)^m$, where N is still a nerve of a covering of P .

12.1. Spaces of the homotopy type of a simplicial complex

By a simplicial complex, we mean (unless otherwise stated) a locally finite simplicial complex; among the spaces that are of the homotopy type of such complexes, we find the CW-complexes with a countable number of cells ((9) Theorem 13), and the spaces dominated by such complexes ((11) Theorem 24). This last class of spaces includes the path-components of a compact ANR. It is shown in ((9), p. 223) that a locally finite simplicial complex with the metric topology necessarily has the weak topology (in which a set F is closed if and only if its intersection with each closed simplex is closed); on the other hand, a simplicial complex which is not locally finite may have a metric topology other than the weak topology ((9), p. 224, Ex. 3).

Let X be a path-connected ANR, Y a locally finite simplicial complex, of the homotopy type of X ; we wish to show that there are points $x_0 \in X$, $y_0 \in Y$ such that $[X, x_0]$, $[Y, y_0]$ are of the same homotopy type.

THEOREM 12.11. *If X, Y are of the same homotopy type, and are each CW-complexes, polyhedra, or compact ANR's, then, given $x_0 \in X$, there is a $y_0 \in Y$ such that $[X, x_0]$ and $[Y, y_0]$ are of the same homotopy type.*

The conditions on X, Y are sufficient to justify the various homotopy extensions that are involved (see I (5.1) for references to these theorems, and the subsequent lemmas).

We are given a map $f_0: X \rightarrow Y$ and a two-sided homotopy inverse $g_0: Y \rightarrow X$. Let $1: X \rightarrow X$, $1': Y \rightarrow Y$, be the identity maps, and define† $y_0 = f_0(x_0)$, $x_1 = g_0(y_0)$. We first replace g_0 by a homotopic map g_1 such that $g_1(y_0) = x_0$. Since X is path-connected, there is a path

$$\sigma: [I, 0, 1] \rightarrow [X, x_0, x_1];$$

define $g_t(y_0) = \sigma(1-t)$, so that $g_0(y_0)$ has the extension g_0 over Y , and extend this homotopy to $g_t: Y \rightarrow X$. Then g_1 is also a homotopy inverse for f_0 .

Since $g_1 \circ f_0 \simeq 1$, there is a map $F: X \times I \rightarrow X$ such that $F(x, 0) = g_1 f_0(x)$, $F(x, 1) = x$, all $x \in X$. Let $\eta(t) = F(x_0, t)$, a loop $\eta: [I, \dot{I}] \rightarrow [X, x_0]$. We use this to define a g'_0 homotopic to $g'_1 = g_1$ such that $g'_0 \circ f_0 \simeq 1 \text{ rel } x_0$.

† Since X is path-connected, so is Y ; then we can deform f_0 so that $f_0(x_0)$ is an assigned point y_0 .

Let $g'_i(y_0) = \eta(1-t)$, so that g'_1 has the extension g'_1 over Y , and extend this homotopy to $g'_i: Y \rightarrow X$. Let $F': X \times I \rightarrow X$ be given by

$$F'(x, t) = \begin{cases} g'_{2t}f_0(x) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ F(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

so that F' is a homotopy between $g'_0 \circ f_0$ and 1. We deform this into a homotopy $\text{rel } x_0$. Let $\zeta(t) = F'(x_0, t)$, a loop given by

$$\zeta(t) = \begin{cases} \zeta(2-2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \zeta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

which is clearly inessential $\text{rel } \dot{I}$; let $\zeta_0 = \zeta$, and let $\zeta_s: [I, \dot{I}] \rightarrow [X, x_0]$ be a nul-homotopy such that $\zeta_1(t) = x_0$ for all t . Let $F' = F'_0$, and define F'_s over $X \times \dot{I} \cup x_0 \times I$ so that $F'_s(x_0, t) = \zeta_s(t)$, $F'_s|(X \times \dot{I}) = F'| (X \times \dot{I})$. Then this homotopy can be extended over $X \times I$, giving F'_1 , a homotopy $\text{rel } x_0$ between $g'_0 \circ f_0$ and 1.

Now g'_0 is a left homotopy inverse of $f_0: [X, x_0] \rightarrow [Y, y_0]$ and a right homotopy inverse of $f_0: X \rightarrow Y$. By the preceding argument, with X, Y interchanged, g'_0 instead of f_0 , and f_0 instead of g_0 , we construct a map $f'_0: [X, x_0] \rightarrow [Y, y_0]$ which is homotopic to f_0 and such that

$$f'_0 \circ g'_0 \simeq 1' \text{ rel } y_0.$$

Then we say that $g'_0: [Y, y_0] \rightarrow [X, x_0]$ is a homotopy equivalence of the pairs, for it has a left inverse f'_0 and a right inverse f_0 : therefore the map $f''_0 = f'_0 \circ g'_0 \circ f_0$ is a two-sided homotopy inverse for g'_0 , as is shown by the following homotopies $\text{rel } x_0, y_0$:

$$\begin{aligned} g'_0 \circ f''_0 &= g'_0 \circ f'_0 \circ g'_0 \circ f_0 \simeq g'_0 \circ 1' \circ f_0 = g'_0 \circ f_0 \simeq 1, \\ f''_0 \circ g'_0 &= f'_0 \circ g'_0 \circ f_0 \circ g'_0 \simeq f'_0 \circ 1 \circ g'_0 = f'_0 \circ g'_0 \simeq 1'. \end{aligned}$$

This proves the theorem.

12.2. Coverings and direct limits

A covering of P is called *locally finite* if every point of P has a neighbourhood which meets at most a finite number of the sets of the covering; we shall only be concerned with locally finite coverings of P by open sets, which form a system \mathfrak{Q} which can be directed (in the sense of (5), p. 4) by refinement. If Σ is a covering in \mathfrak{Q} consisting of sets U^α , the nerve of Σ is to be realized as a simplicial complex (which is not necessarily locally finite) with the weak topology,† whose vertices A^α are in 1-1 correspondence with the sets of Σ , and where vertices span a simplex if and only if the corresponding sets have non-empty intersection. The system \mathfrak{Q} has a

† We take the weak topology for the nerve, to ensure that a simplicial mapping is continuous.

subsystem $\mathfrak{L}\mathfrak{F}$ which consists of all coverings whose nerves have the property that the star of every vertex is of finite dimension. Dowker ((3), Lemma 3.3) proves that if P is normal, then $\mathfrak{L}\mathfrak{F}$ is cofinal in \mathfrak{L} . The system $\mathfrak{L}\mathfrak{F}$ itself contains the system \mathfrak{F} of all finite coverings: clearly if P is compact, \mathfrak{F} is cofinal in \mathfrak{L} and in $\mathfrak{L}\mathfrak{F}$. Hereafter all spaces will be normal, all coverings in $\mathfrak{L}\mathfrak{F}$.

Let Q be a closed subspace of P ; then a covering of P induces a covering of Q (by intersection), and the nerve of this may be embedded in a natural way in the nerve of the covering of P . If Σ_λ is a covering of P , let N_λ be its nerve, L_λ the subcomplex of N_λ defined by Q . Then it is well known† that if Σ_λ refines Σ_μ there are simplicial mappings $[N_\lambda, L_\lambda] \rightarrow [N_\mu, L_\mu]$ whose homotopy class is uniquely defined by $\Sigma_\lambda, \Sigma_\mu$. These induce the same homomorphisms

$$(N_\mu, L_\mu)^m(X, x_0; x_0) \rightarrow (N_\lambda, L_\lambda)^m(X, x_0; x_0),$$

and so define a direct limit $\lim_{\rightarrow} (N_\lambda, L_\lambda)^m$ over any system in $\mathfrak{L}\mathfrak{F}$ (including $\mathfrak{L}\mathfrak{F}$). Also, as P is normal, Dowker ((3), p. 202) shows that there is a unique homotopy class of natural maps‡ $[P, Q] \rightarrow [N_\lambda, L_\lambda]$, and these induce a homomorphism $\lim_{\rightarrow} (N_\lambda, L_\lambda)^m \rightarrow (P, Q)^m$ (where, for convenience, we omit reference to the space X).

A *paracompact* space is, according to Dowker, a space in which every covering has a locally finite refinement. We shall prove:

THEOREM 12.21. *If P is normal and paracompact, X is a compact ANR, locally finite simplicial complex or countable CW-complex, there is an isomorphism onto,*

$$\lim_{\rightarrow} (N_\lambda, L_\lambda)^m \rightarrow (P, Q)^m,$$

where the direct limit is over the system $\mathfrak{L}\mathfrak{F}$ (or \mathfrak{L}). If P is also compact, or if P is countably compact and X is a countable simplicial complex, we may replace $\mathfrak{L}\mathfrak{F}$ by \mathfrak{F} .

(A countably compact space is one in which each countable covering has a finite sub-covering.)

Let $P \supset Q \supset R$, where Q, R are closed in P . Then, as the direct limit of exact sequences is exact, we deduce from the previous theorem and the results of I (§ 5)

† The proof (see (5)) for coverings in \mathfrak{F} applies to coverings in $\mathfrak{L}\mathfrak{F}$, if the nerves have the weak topology.

‡ Theorem 1.1 in (3) demonstrates the existence of canonical maps if the nerve has the metric topology. This implies our assertion, as is shown in § 12.3 below.

THEOREM 12.22. *Under the above conditions, there is an exact sequence, terminating at $(Q, R)^1$,*

$$\dots \rightarrow (Q, R)^{m+1} \rightarrow (P, Q)^m \rightarrow (P, R)^m \rightarrow (Q, R)^m \rightarrow \dots$$

as in I (§ 5).

If we take the dimension of a space as that based on the order of its coverings (Lebesgue, (5)), it does not matter which of $\mathfrak{L}\mathfrak{F}$ or \mathfrak{F} we use, since P is normal (Dowker (3)). Let the cohomology groups be the Čech groups based on $\mathfrak{L}\mathfrak{F}$ or \mathfrak{F} according to circumstances; then Theorem 12.21 and the results of I (§ 9) imply

THEOREM 12.23. *Under the conditions of Theorem 12.21, if $\dim(P)$, $\dim(P-Q)$, or $\dim(P/Q) = k$, where Q is not empty, then*

($k = 1$), $(P, Q)^m \approx H^1(P, Q; \pi_{m+1})$,

($k = 2$), $(P, Q)^m$ is a central extension of $H^2(P, Q; \pi_{m+2})$ by $H^1(P, Q; \pi_{m+1})$,

($k = 3$), $(P, Q)^m$ is a central extension of $H^3(P, Q; \pi_{m+3})$ by a central extension of $H^2(P, Q; \pi_{m+2})$ by the subgroup of $H^1(P, Q; \pi_{m+1})$ consisting of all elements whose Postnikov Square is zero.

When Q is empty the group is a split extension of the above groups by π_m , which is the direct sum when $m > 1$.

Here the Postnikov Square is to be computed in each nerve, and the direct limit taken; the pairing of π_{m+1} with itself to π_{m+2} is defined in (12) or I (§ 9).

For under the conditions, we can find a cofinal system of coverings in the appropriate system, the nerves of whose coverings have the property that N_λ/L_λ is of dimension k ; the groups $(N_\lambda, L_\lambda)^m$ are given in I (§ 9) in terms of cohomology groups; passing to the limit and using Theorem 12.21, we obtain the above results. When P is compact the extension in the case $k = 2$ is given by the formulae in 10.22, 10.31, and 10.41. It can be shown that when Q is empty, the split extension of $H^1(P; \pi_2)$ by π_1 is determined by the operations of π_1 on the cohomology group acting through the usual operations on the coefficient group (cf. I, § 8). When Q is empty and $\dim P = 2$, the split extension is unknown.

12.3. Spanier's lemma

The rest of § 12 is devoted to a proof of Theorem 12.21. When P is compact, the result is an easy extension of the corresponding theorem in cohomotopy groups in (7); and we shall show that the proof that the homomorphism is onto is a direct application of a fundamental lemma due to Spanier (7).

Let Σ_λ be a covering of P by sets U_λ^α ; if p is a point of P there are only a finite number of sets containing p , and if these are $U_\lambda^0, \dots, U_\lambda^n$, let $\sigma(p)$ be the simplex of N_λ whose vertices are $A_\lambda^0, \dots, A_\lambda^n$. Then a *canonical map* $h_\lambda: [P, Q] \rightarrow [N_\lambda, L_\lambda]$ is a map such that $h_\lambda(p) \in \sigma(p)$ for all p ; Dowker proves in (3) that if P is normal, such a map exists† provided Σ_λ is in $\mathfrak{L}\mathfrak{F}$, and the nerve is given the metric topology. Let N'_λ be the nerve with the weak topology, $f: N_\lambda \rightarrow N'_\lambda$ the natural transformation. Then f is simplicial on each simplex, and so continuous on every finite sub-complex of N_λ ; hence, by Lemma 1.2 of (3), the composition $f \circ h_\lambda: P \rightarrow N'_\lambda$ is continuous. Therefore, every normal space P has a canonical map into the nerve of a covering in $\mathfrak{L}\mathfrak{F}$ with the weak topology. It is clear that any two canonical maps are homotopic, since if h_λ, h'_λ are two canonical maps, $h_\lambda(p)$ and $h'_\lambda(p)$ are in the same closed simplex, and so therefore is the segment joining them; if the points stand for their position vectors in the simplex we have a homotopy defined by $h_t(p) = (1-t)h_\lambda(p) + th'_\lambda(p)$. Therefore the homotopy class of canonical maps $h_\lambda: [P, Q] \rightarrow [N_\lambda, L_\lambda]$ is unique.

LEMMA 12.31 (Spanier). *If $h_\lambda: [P, Q] \rightarrow [N_\lambda, L_\lambda]$ is canonical, given any map $f: [P, Q] \rightarrow [X, x_0]$ there is a $\varphi_\lambda^f: [N_\lambda, L_\lambda] \rightarrow [X, x_0]$ such that $f \simeq \varphi_\lambda^f \circ h_\lambda$. Here X is a simplicial complex.*

According to Theorem 12.1, we may replace X by a compact ANR. φ_λ^f is constructed as follows: let $x_0 = B_0$ be a vertex of X , and let the other vertices be B_j . Cover P by $f^{-1}(\text{St } B_j)$, where $\text{St } B_j$ is the *open* star of the vertex; then we can find a Σ_μ in $\mathfrak{L}\mathfrak{F}$ refining both this covering and Σ_λ . According to (5) there is a map‡ $T_{\lambda\mu}: [N_\lambda, L_\lambda] \rightarrow [N_\mu, L_\mu]$ whose homotopy class is uniquely defined by $\Sigma_\lambda, \Sigma_\mu$, which maps each A_λ^α to an A_μ^β such that $U_\lambda^\alpha \subset U_\mu^\beta$, and is a simplicial map on each simplex. Construct

$$\varphi_\mu^f: [N_\mu, L_\mu] \rightarrow [X, x_0]$$

by the following method, and define $\varphi_\lambda^f = \varphi_\mu^f \circ T_{\lambda\mu}$ for some choice of $T_{\lambda\mu}$. We can choose Σ_μ so that if U_μ^β meets Q , $U_\mu^\beta \subset f^{-1}(\text{St}(x_0))$, for if Σ_ω refines Σ_λ and the covering of P by $f^{-1}(\text{St } B_j)$, we can cover P by the sets

$$U_\omega^\gamma \cap (P - Q), \quad U_\omega^\gamma \cap f^{-1}(\text{St } x_0).$$

For each $A_\mu^i \in N_\mu$, select a vertex B_{j_i} such that $U_\mu^i \subset f^{-1}(\text{St } B_{j_i})$, choosing x_0 if U_μ^i meets Q . If $A_\mu^0 \dots A_\mu^n$ is a simplex of N_μ , the sets U_μ^0, \dots, U_μ^n intersect, and therefore so do the sets $f^{-1}(\text{St } B_{j_0}), \dots, f^{-1}(\text{St } B_{j_n})$. Therefore B_{j_0}, \dots, B_{j_n} are among the vertices of a simplex of X , and the map $\varphi_\mu^f(A_\mu^i) = B_{j_i}$ can be extended to a simplicial map $\varphi_\mu^f: N_\mu \rightarrow X$, which clearly carries L_μ to x_0 . The proof that $f \simeq \varphi_\mu^f \circ h_\mu$ is in (7).

† See note ‡, p. 313.

‡ See note †, p. 313.

If $(P, Q)^0$ is the set of homotopy classes of maps of $[P, Q]$ to $[X, x_0]$, the maps $T_{\lambda\mu}$ induce transformations $T_{\lambda\mu}^*: (N_\mu, L_\mu)^0 \rightarrow (N_\lambda, L_\lambda)^0$ where the compositions $T_{\lambda\mu}^* T_{\mu\nu}^* = T_{\lambda\nu}^*$, and so define $\lim_{\rightarrow} (N_\lambda, L_\lambda)^0$; the canonical maps $h_\lambda: [P, Q] \rightarrow [N_\lambda, L_\lambda]$ induce $h_\lambda^*: (N_\lambda, L_\lambda)^0 \rightarrow (P, Q)^0$; since $T_{\lambda\mu} \circ h_\lambda$ is a canonical map, $h_\lambda^* \circ T_{\lambda\mu}^* = h_\mu^*$ and we obtain a transformation

$$h^*: \lim_{\rightarrow} (N_\lambda, L_\lambda)^0 \rightarrow (P, Q)^0.$$

Then Lemma 12.31 at once implies that h^* is onto if P is paracompact or X is locally finite; or, using the system \mathfrak{F} , if P is compact.

h^* is an isomorphism if $h^*x = h^*y$ implies $x = y$. Spanier (7) also proves that h^* is an isomorphism if P is compact, the limit set being a group. His proof can be modified to prove that h^* is an isomorphism if P is paracompact, without using the group property; however, this requires a discussion of coverings of the cylinder $P \times I$. We shall give the proof of the next theorem in (12.5).

THEOREM 12.32 (Spanier). *If P is normal, paracompact, and X is a compact ANR, or locally finite simplicial complex, then, over the system $\mathfrak{L}\mathfrak{F}$, $h^*: \lim_{\rightarrow} (N_\lambda, L_\lambda)^0 \rightarrow (P, Q)^0$ is an isomorphism onto. If P is compact, or countably compact, and X a countable simplicial complex, we can replace $\mathfrak{L}\mathfrak{F}$ by \mathfrak{F} .*

In order to apply this to Track Groups, we investigate the relations between coverings of P and coverings of $P \times I^m$.

12.4. Coverings of product spaces

We investigate the coverings of $P \times T$, where T will be specialized to I^m . We first deal with *product coverings* of $P \times T$, that is, a covering by product sets $U \times V$, for all U in Σ , a covering of P , all V in Σ' , a covering of T .

LEMMA 12.41 (Dowker). *If P is compact or countably compact, and T is compact and satisfies the first axiom of countability, then any countable covering of $P \times T$ has a refinement by a (finite) product covering.*

The proof is in (3).

Let Σ be a covering of P , Σ' a covering of T , $\Sigma \times \Sigma'$ the product covering; let $Q \subset P$, $S \subset T$, and let N , N' be the nerves of Σ , Σ' ; L , L' the subcomplexes corresponding to Q , S ; let N^* be the nerve of $\Sigma \times \Sigma'$ and L^* the subcomplex corresponding to $Q \times T \cup P \times S$.

THEOREM 12.42. *There is a homeomorphism into,*

$$\theta: [N \times N', N \times L' \cup L \times N'] \rightarrow [N^*, L^*],$$

whose image is a deformation retract of $[N^, L^*]$.*

That is, there is a homotopy $\text{rel } \theta(N \times N')$, $\sigma_i: [N^*, L^*] \rightarrow [N^*, L^*]$, such that σ_0 is the identity map, $\sigma_1(N^*) \subset \theta(N \times N')$, $\sigma_1(L^*) \subset \theta(N \times L' \cup L \times N')$.

Let U^i, V^j be sets of $\Sigma, \Sigma', A^i, B^j$ the corresponding vertices of N, N' , and $A^{i,j}$ the vertex of N^* corresponding to $U^i \times V^j$. If ξ is a simplex of N with vertices A^0, \dots, A^m, ξ' a simplex of N' with vertices B^0, \dots, B^n , the corresponding sets all intersect, so that $A^{0,0}, \dots, A^{m,n}$ are the vertices of a simplex ζ of N^* . Take barycentric coordinates in each simplex, and define $\theta(A^i \times B^j) = A^{i,j}$, and, if $a_{i,j}^*$ is the product $a_i b_j$,

$$\theta((a_0, \dots, a_m) \times (b_0, \dots, b_n)) = (a_{0,0}^*, \dots, a_{m,n}^*),$$

where $(a_0, \dots, a_m), (b_0, \dots, b_n), (a_{0,0}^*, \dots, a_{m,n}^*)$ are the barycentric coordinates of points in ξ, ξ', ζ respectively. This clearly defines a continuous map of $\xi \times \xi'$ into ζ which is in fact a homeomorphism into. If θ is similarly defined over every product of simplices in $N \times N'$, these maps fit together to form a homeomorphism of $N \times N'$ into N^* , which may be verified as mapping $L \times N' \cup N \times L'$ into L^* .

Let $\phi: N^* \rightarrow N, \psi: N^* \rightarrow N'$ be simplicial maps defined by $\phi(A^{i,j}) = A^i, \psi(A^{i,j}) = B^j$, and extending these barycentrically over each simplex of N^* . This is always possible, for every simplex of N^* is either of the type $A^{0,0}, \dots, A^{m,n} = \zeta$ (where $A^{i,j}$ and $A^{a,b}$ both occur if $A^{i,b}$ and $A^{a,j}$ occur), or else is a face of such a simplex. Over these simplices ζ, ϕ, ψ may be defined on the point with barycentric coordinates $(a_{0,0}^*, \dots, a_{m,n}^*)$ by setting $a_i = a_{i,0}^* + \dots + a_{i,n}^*$, and $b_j = a_{0,j}^* + \dots + a_{m,j}^*$, and

$$\phi(a_{0,0}^*, \dots, a_{m,n}^*) = (a_0, \dots, a_m),$$

$$\psi(a_{0,0}^*, \dots, a_{m,n}^*) = (b_0, \dots, b_n).$$

Then clearly $\phi \circ \theta, \psi \circ \theta$ are the projections $N \times N' \rightarrow N, N \times N' \rightarrow N'$.

Let ζ be a simplex of N^* of the above type; then $\theta(\phi(z), \psi(z))$ is in ζ for every z in ζ . Define $\sigma_t: N^* \rightarrow N^*$ by

$$\sigma_t(z) = (1-t)z + t\theta(\phi(z), \psi(z)),$$

where we use the symbol for a point to stand for the position vector of the point in the simplex. Then it may be verified that σ_t is a homotopy of the required type, retracting N^* on $\theta(N \times N') \text{ rel } \theta(N \times N')$, moving L^* over itself.

We have proved more than we need, for we have shown that θ is a homotopy equivalence (it has as inverse $\theta^{-1} \circ \sigma_1$, the map $z \rightarrow \phi(z) \times \psi(z)$). In fact, we only use the fact that θ has a *left* inverse in our application.

Now let P be paracompact. It is no longer possible to confine ourselves to product coverings of $P \times T$. By a *P-product covering* (see (3), p. 218) of $P \times T$ we mean a covering of P by sets $U_\lambda^\alpha \in \Sigma_\lambda$, and, for each α a covering of T by sets $V_\alpha^i \in \Sigma_\alpha$, and the covering of $P \times T$ by the product sets $U_\lambda^\alpha \times V_\alpha^i$. Thus, in general, the covering does not contain $U_\lambda^\alpha \times V_\beta^j$

($\alpha \neq \beta$). Moreover, Σ_λ is to be a locally finite covering of P , and Σ_α a finite covering of T .

LEMMA 12.43. *If P is paracompact, and T is compact, then every covering of $P \times T$ has a P -product refinement. In particular, $P \times T$ is paracompact.*

This is proved in (3) for the case $T = I$. Notice that we do not require that P is normal; if the first part is true, we have exhibited a locally finite covering of $P \times T$ refining an arbitrary covering, so the second part is true.

For all open U in P , all open V in T , the sets $U \times V$ form a sub-base for $P \times T$, so that every covering has a refinement by product sets. Let p be a point of T ; since T is compact, $p \times T$ is compact, and we can choose a finite number of these product sets which cover $P \times T$: let these be $U_{(p)}^1 \times V_{(p)}^1, \dots, U_{(p)}^n \times V_{(p)}^n$. Let $U_{(p)}$ be the intersection of the $U_{(p)}^i$, so that the collection of all $U_{(p)}$ for all p cover P and the collection of all $U_{(p)} \times V_{(p)}^i$ cover $P \times T$. Then we may choose a locally finite refinement Σ_λ of this covering of P ; in each U_λ^α in Σ_λ choose a point p , and define the covering Σ_α of T as the collection of sets $V_{(p)}^i$ constructed above. Then the totality of sets $U_\lambda^\alpha \times V_\alpha^i$ ($V_\alpha^i \in \Sigma_\alpha$) covers $P \times T$ and is a P -product covering refining the given covering. This proves the lemma.

If the above P -product covering of $P \times T$ is said to be based on Σ_λ , we can use this covering to construct a P -product covering based on any locally finite Σ_μ refining Σ_λ . For if $U_\mu^\beta \in \Sigma_\mu$, we can choose Σ_β to be the covering Σ_α of T for any α such that $U_\mu^\beta \subset U_\lambda^\alpha$. The new covering clearly is a refinement of the old. We may also refine each finite covering of T with similar effect. In particular, if T is a simplicial complex we may choose Σ_α so that its nerve is homeomorphic to T for each α (for every finite covering of a finite polyhedron has a refinement by the (open) stars of the vertices of some subdivision, whose nerve is an isomorph of the complex).

It follows that if $T = I^m$ we may use Dowker's Lemma 12.41 and refine each Σ_α (a covering of I^m) by a finite product covering $\Sigma_\alpha^1 \times \dots \times \Sigma_\alpha^m$ such that the nerve of Σ_α^t is a homeomorph of I (the sets of Σ_α^t are open segments).

COROLLARY 12.44. *If P is paracompact, any covering of $P \times I^m$ has a P -product refinement Σ^* whose sets are $U_\lambda^\alpha \times V_{\alpha^1, i_1}^1 \times \dots \times V_{\alpha^m, i_m}^m$, for all $U_\lambda^\alpha \in \Sigma_\lambda$, all $V_{\alpha^t, i}^t \in \Sigma_{\alpha^t}^{t, i}$ where $\Sigma_{\alpha^t}^{t, i}$ is a finite covering of I by open segments, whose nerve is a homeomorph of I .*

Let N^* be the nerve of such a covering Σ^* , and let L^* be the subcomplex corresponding to $Q \times I^m \cup P \times I^m$, where Q is closed in P . Let N_λ be the nerve of Σ_λ , L_λ the subcomplex corresponding to Q , and set

$$M_\lambda = N_\lambda \times I^m \cup L_\lambda \times I^m.$$

THEOREM 12.45. *There is a homeomorphism into,*

$$\theta: [N_\lambda \times I^m, M_\lambda] \rightarrow [N^*, L^*],$$

with a left inverse $\sigma: [N^, L^*] \rightarrow [N_\lambda \times I^m, M_\lambda]$. $N_\lambda \times I^m$ can be given a simplicial subdivision and θ can be a simplicial map.*

It is only necessary to prove this theorem in the case $m = 1$; for suppose it is true for m replaced by $(m-1)$, $m > 1$. Let $P \times I^{m-1}$ be covered by the sets in Corollary 12.44 with $V_\alpha^{m,im}$ omitted, let N^{**} be the nerve of this covering, L^{**} the corresponding subcomplex. This defines a $(P \times I^{m-1})$ -product covering of $P \times I^m$, and, applying the theorem in the case $m = 1$, we may find a simplicial subdivision of $N^{**} \times I$ and a simplicial map $\theta^*: [N^{**} \times I, N^{**} \times I \cup L^{**} \times I] \rightarrow [N^*, L^*]$ with a left inverse σ^* , and also a simplicial subdivision of $N_\lambda \times I^{m-1}$ and a simplicial map

$$\theta': N_\lambda \times I^{m-1} \rightarrow N^{**}$$

with left inverse σ' . Let θ', σ' define in the obvious way

$$\theta'': [N_\lambda \times I^{m-1} \times I, L_\lambda \times I^{m-1} \times I \cup N_\lambda \times I^m] \rightarrow [N^{**} \times I, L^{**} \times I \cup N^{**} \times I]$$

with left inverse σ'' . Then $\theta = \theta^* \circ \theta''$ is a map of the required sort, with left inverse $\sigma'' \circ \sigma^*$, and it follows from the argument in the case $m = 1$ that θ'' is simplicial with respect to some subdivision.

We now prove the theorem with $m = 1$. Let $L(V)$, $R(V)$ be, respectively, the lower and upper bounds of an open segment V of I . Then we can order the sets $V_\alpha^0, \dots, V_\alpha^n$ of Σ_α so that their lower bounds form a monotonic increasing sequence. Naturally, no three consecutive sets will intersect (the nerve of Σ_α is a homeomorph of I and contains no 2-simplices), and the upper bound of any set is greater than the lower bound of the next succeeding set.

We assume that we may give the vertices of N_λ a partial order such that the vertices of any simplex (which has finite dimension) are in simple order. For each vertex A_λ^α of N_λ we realize the nerve of Σ_α on $A_\lambda^\alpha \times I$, and then turn $N_\lambda \times I$ into a simplicial complex. If Σ_α has sets $V_\alpha^0, \dots, V_\alpha^n$ in order, take $B_\alpha^0 = A_\lambda^\alpha \times 0$, $B_\alpha^n = A_\lambda^\alpha \times 1$, and arrange B_α^i between, in order. The correspondence between the sets V_α^i and points B_α^i establishes an isomorphism between $A_\lambda^\alpha \times I$ (a simplicial complex in the obvious way with vertices B_α^i) and the nerve of Σ_α .

Let $\sigma = A_\lambda^0, \dots, A_\lambda^n$ be any simplex of N_λ , the vertices in the chosen order. Construct an ordered set of simplices $\pi(\sigma, m)$ on $\sigma \times I$ with vertices among the B_λ^i as follows.

$\pi(\sigma, 1) = B_\lambda^0 B_\lambda^1 \dots B_\lambda^n B_\lambda^1$ where the set $V_\lambda^1 \subset A_\lambda^i \times I$ is such that

- (i) $L(V_\lambda^1) \leq L(V_\lambda^t)$ for all $t = 0, \dots, n$;
- (ii) if $L(V_\lambda^i) = L(V_\lambda^t)$, then $i \leq t$.

Suppose we have constructed $\pi(\sigma, m-1)$ with vertices $B_p^{t_p-1}, B_0^{t_0}, \dots, B_n^{t_n}$ for some p , so that $A_\lambda^p \times I$ is the only edge of the prism $\sigma \times I$ with two vertices of $\pi(\sigma, m-1)$ on it (except possibly edges on $\sigma \times 0, \sigma \times 1$). We construct $\pi(\sigma, m)$ with a face $B_0^{t_0}, \dots, B_n^{t_n}$ in common with $\pi(\sigma, m-1)$.

$\pi(\sigma, m) = B_0^{t_0}, \dots, B_n^{t_n}, B_q^{t_q+1}$ where $V_q^{t_q+1}$ is such that

(i) $L(V_q^{t_q+1}) \leq L(V_i^{t_i+1})$ for all $i = 0, \dots, n$,

(ii) q is the least of $0, \dots, n$ such that $V_q^{t_q+1}$ has this property.

Then every $\pi(\sigma, m)$ has two faces which project onto σ in the natural projection of $\sigma \times I$ onto σ , each is of dimension $(n+1)$, and has an n -dimensional face in common with the immediately preceding and succeeding simplices (if any). Moreover, the construction is uniquely defined by the coverings Σ_α and the ordering of the vertices of σ (which is given by the partial ordering of the vertices of N_λ). Therefore, if σ' is a face of σ , the simplices on $\sigma' \times I$ are faces of the simplices on $\sigma \times I$, and hence $N_\lambda \times I$ is a simplicial complex with these simplices and their faces as its simplices. Notice that if we were to change the partial order of the vertices of N_λ , we should obtain a different simplicial decomposition.

In order to show that we can find a simplicial map of $N_\lambda \times I$ into N^* we must show that the sets of I corresponding to the vertices of any $\pi(\sigma, m)$ or its faces all intersect. Let $\sigma = A_\lambda^0, \dots, A_\lambda^n$, $\pi(\sigma, m) = B_0^{t_0}, \dots, B_n^{t_n}, B_q^{t_q+1}$. By construction, $L(V_q^{t_q+1}) \leq L(V_i^{t_i+1}) < R(V_i^{t_i})$ for all $i = 0, \dots, n$, so $L(V_q^{t_q+1})$ is less than $\min(R(V_i^{t_i}))$. If $m = 1$, all the sets V_i^0 intersect (they all contain 0) so that in this case their intersection meets V_q^1 . Suppose we have shown that all the sets of $\pi(\sigma, m-1) = B_p^{t_p-1}, B_0^{t_0}, \dots, B_n^{t_n}$ all intersect; in particular the sets $V_0^{t_0}, \dots, V_n^{t_n}$ all intersect, so that $\max(L(V_i^{t_i})) < \min(R(V_i^{t_i}))$, and so $\max(L(V_i^{t_i})) < R(V_q^{t_q+1})$ since the nerve of Σ_q is a homeomorph of I . Therefore the intersection of these sets meets $V_q^{t_q+1}$, and hence the sets of $\pi(\sigma, m)$ all meet. Since the sets U^0, \dots, U^n all intersect, the sets $U_0^0 \times V_0^{t_0}, \dots, U^n \times V_n^{t_n}, U_q \times V_q^{t_q+1}$ all intersect, and so the corresponding vertices $A^{0,t_0}, \dots, A^{n,t_n}, A^{q,t_q+1}$ of N^* are the vertices of a simplex. We call this simplex of N^* the simplex *corresponding to* $\pi(\sigma, m)$.

Now let θ be the map of $N_\lambda \times I$ into N^* which maps each $\pi(\sigma, m)$ simplicially onto its corresponding simplex in N^* , and maps each B_i^j onto $A^{i,j}$ in N^* . This is clearly a simplicial map with respect to the chosen simplicial decomposition of $N_\lambda \times I$, and is a homeomorphism into.

We now construct a left inverse σ to θ . Since Σ^* (the covering of $P \times I$) is locally finite, every simplex of N^* has a finite number of vertices; since no three sets of a Σ_α intersect, no more than two vertices of such a simplex

can have the same first index. Therefore, if we allow some vertices to be repeated, every simplex of N^* has the form

$$\zeta = A^{0,j_0}, \dots, A^{n,j_n}, A^{0,k_0}, \dots, A^{n,k_n},$$

where either $k_s = j_s$ or $k_s = j_s + 1$. If a point of this simplex has barycentric coordinates $(a_{0,j_0}^*, \dots, a_{n,j_n}^*, a_{0,k_0}^*, \dots, a_{n,k_n}^*)$ where $a_{s,k_s}^* = 0$ if $k_s = j_s$ (these spurious coordinates arise from our allowing duplication), we can consider the point as being given in terms of position vectors in the simplex†

$$\sum_s a_{s,j_s}^* A^{s,j_s} + \sum_s a_{s,k_s}^* A^{s,k_s}.$$

Let $\sigma(A^{s,i}) = B_s^i$: then the image of the vertices of ζ lie in $(A^0, \dots, A^n) \times I$, and, considering the B_s^i as position vectors in the prism, we make σ map the above point to†

$$\sum_s a_{s,j_s}^* B_s^{j_s} + \sum_s a_{s,k_s}^* B_s^{k_s},$$

defining a linear map of the simplex into the prism. If $k_s = j_s$ for all but at most one value of s , the vertices B_s^i span a simplex in the prism, though not necessarily one of the chosen ones of the simplicial decomposition; nevertheless, σ is a simplicial map in this case onto the new simplex in $N_\lambda \times I$.

If σ is constructed in this way over every simplex of N^* , it gives a single-valued and therefore continuous map of N^* into $N_\lambda \times I$, which can be seen to map L^* into $N_\lambda \times I \cup L_\lambda \times I$, and be such that $\sigma \circ \theta$ is the identity map. If we were to re-order the vertices of N_λ , we should alter θ but not alter σ , which would have the same properties with respect to the new θ . This proves the theorem in the case $m = 1$, and so completes its proof.

12.5. Proof of Theorem 12.32

We shall prove Theorem 12.32 in the case when P is paracompact; the other cases follow by similar but simpler arguments, where Dowker's Lemma (12.41) is used instead of our Lemma 12.43. The proof is modelled on the argument used by Spanier in (7), to prove the similar theorem in cohomotopy groups (P compact, $(N_\lambda, L_\lambda)^0$ a group).

Let $x_\lambda \in (N_\lambda, L_\lambda)^0$, and let $[x_\lambda]$ be the corresponding element of the direct limit, so that $[x_\lambda]$ is the set of all x_μ such that for some Σ_ν refining both Σ_λ and Σ_μ ,

$$T_{\nu\lambda}^* x_\lambda = T_{\nu\mu}^* x_\mu.$$

Then the canonical maps h_λ induce $h_\lambda^*: (N_\lambda, L_\lambda)^0 \rightarrow (P, Q)^0$, and the corresponding transformation on the direct limit is defined by

$$h^*[x_\lambda] = h_\lambda^* x_\lambda = h_\nu^* T_{\nu\lambda}^* x_\lambda.$$

† In these expressions, \sum_s means the sum over s , and not the covering defined by U^s .

If $f: [P, Q] \rightarrow [X, x_0]$, $\{f\}$ its homotopy class, define for some suitable λ , $x_\lambda = \{\varphi_\lambda\}$, the class of the map constructed in Spanier's Lemma 12.31. Then $h_\lambda^* x_\lambda = \{f\}$, according to that lemma, so that $h^*[x_\lambda] = \{f\}$, and hence h^* is onto.

Now suppose that $h^*[x_\lambda] = h^*[y_\mu]$. Since Σ_λ and Σ_μ have a common refinement, we can suppose without loss of generality that $\lambda = \mu$. Now we want a condition on Σ_λ akin to irreducibility: a *normal* covering is one that has a canonical map $h_\lambda: P \rightarrow N_\lambda$ such that $h_\lambda^{-1}(\text{St } A_\lambda^\alpha) = U_\lambda^\alpha$ for the open star of each vertex A_λ^α of N_λ , and also h_λ satisfies a condition that we do not require, that it is essential in each closed simplex of N_λ in the sense of (3), p. 209, bottom.

Then Dowker has shown ((3), Lemma 3.4) that the locally finite normal coverings of a normal P are cofinal in $\mathfrak{L}\mathfrak{F}$, and the finite normal coverings are cofinal in \mathfrak{F} . We therefore suppose that all coverings are normal.

We are given x_λ, y_λ such that $h_\lambda^* x_\lambda = h_\lambda^* y_\lambda$; by the simplicial approximation theorem we can suppose that there are representative simplicial maps $f \in x_\lambda, g \in y_\lambda$, and, choosing a normal canonical map

$$h_\lambda: [P, Q] \rightarrow [N_\lambda, L_\lambda],$$

suppose given a homotopy $F: [P \times I, Q \times I] \rightarrow [X, x_0]$ such that

$$F(p, 0) = f h_\lambda(p), \quad F(p, 1) = g h_\lambda(p).$$

Cover $P \times I$ by the sets $F^{-1}(\text{St } B^j)$ where B^j is a vertex of the simplicial complex X , and $\text{St } B^j$ is its open star. We can suppose that x_0 is one of these vertices. Refine this covering by a P -product covering based on a locally finite covering Σ_μ of P , which we may suppose refines Σ_λ (cf. the remarks leading to Corollary 12.44) and has the property of Σ_μ in the construction of φ_λ^f in Lemma 12.31, that if a set meets Q its product with I is in $F^{-1}(\text{St } x_0)$. We then refine each covering Σ_α of I , preserving the property that its nerve is a homeomorph of I ; for the moment we suppose only that the second set does not contain 0, nor the penultimate 1, but later we shall demand that the first and the last are sufficiently small, by a criterion given below.

We define $f', g': [N_\mu, L_\mu] \rightarrow [X, x_0]$ such that $f' \in T_{\mu\lambda}^* x_\lambda, g' \in T_{\mu\lambda}^* y_\lambda$, and $f' \simeq g'$. For each vertex A_μ^β in L_μ , choose the vertex x_0 in X ; for any other, choose a vertex $B^j = f(A_\lambda^\alpha)$, where $A_\lambda^\alpha \in N_\lambda$ is such that $U_\lambda^\alpha \supset U_\mu^\beta$. Then $f^{-1}(\text{St } B^j) \supset \text{St } A_\lambda^\alpha$, and, since Σ_λ is a normal covering, we can choose a normal canonical map h_λ such that $h_\lambda^{-1} f^{-1}(\text{St } B^j)$ contains U_λ^α and so U_μ^β . Therefore, if A_μ^0, \dots, A_μ^r are vertices of a simplex of N_μ , the corresponding vertices of X are vertices of a simplex; we can define a simplicial map $f': N_\mu \rightarrow X$ such that $f'(A_\mu^\beta) = B^j$. The choice of $U_\lambda^\alpha \supset U_\mu^\beta$ in this construction at the same time defines a simplicial map $T_{\mu\lambda}: N_\mu \rightarrow N_\lambda$ such

that $f' = f \circ T_{\mu\lambda}$. Similarly we may construct $g': N_\mu \rightarrow X$ and a map $T''_{\lambda\mu}: N_\mu \rightarrow N_\lambda$ such that $g' = g \circ T''_{\lambda\mu}$.

We now use $F: P \times I \rightarrow X$ to construct a homotopy between f' and g' . Let V_β^0 be the first set of Σ_β , the covering of I defined by the P -product covering of $P \times I$ and the set U_μ^β , and suppose that it and U_μ^β are sufficiently small that $U_\mu^\beta \times V_\beta^0 \subset F^{-1}(\text{St } B^i)$, where $f'(A_\mu^\beta) = B^i$; we can always choose the refinement Σ_μ of Σ_λ and then refine Σ_β so that this is true. Similarly we require that the last set V_β^n is such that $U_\mu^\beta \times V_\beta^n \subset F^{-1}(\text{St } B^k)$, where $B^k = g'(A_\mu^\beta)$. Let N^* be the nerve of the covering Σ^* of $P \times I$, and construct φ_*^F as in 12.31 so that $\varphi_*^F(L^*) = x_0$ (L^* being the subcomplex corresponding to $Q \times I$), $\varphi_*^F(A^{\beta,0}) = B^i = f'(A_\mu^\beta)$, $\varphi_*^F(A^{\beta,n}) = B^k = g'(A_\mu^\beta)$, and, for any other vertex $A^{\beta,s}$, $\varphi_*^F(A^{\beta,s}) = B^l$ where $F^{-1}(\text{St } B^l) \supset U_\mu^\beta \times V_\beta^s$ and φ_*^F is a simplicial mapping. Then if θ is the homeomorphism of Theorem 12.45, $\varphi_*^F \circ \theta$ is a homotopy, $\text{rel } L_\mu \times I$, between f' and g' , which proves that $[x_\lambda] = [y_\lambda]$, and so that h^* is an isomorphism, as required. This completes the proof of the theorem when P is normal and paracompact.

12.6. Application to Track Groups; Proof of Theorem 12.21

According to Lemmas 12.41, 12.43, the product or P -product coverings of $P \times I^m$ are cofinal in the system of \mathfrak{F} or $\mathfrak{L}\mathfrak{F}$ coverings of this space. We shall discuss the latter case only; the former are covered by similar arguments with a suitable change of wording.

Let N^* be the nerve of the covering Σ^* of Theorem 12.45, N_λ the nerve of the covering Σ_λ of P , L^* , L_λ the subcomplexes there defined. The homeomorphism θ and its left inverse σ induce transformations

$$(N^*, L^*)^0 \xrightleftharpoons[\sigma^*]{\theta^*} (N_\lambda \times I^m, L_\lambda \times I^m \cup N_\lambda \times \dot{I}^m)^0$$

such that θ^* is onto, σ^* an isomorphism into. We therefore have similar transformations of the direct limits

$$\lim_{\rightarrow} (N^*, L^*)^0 \xrightleftharpoons[\sigma^*]{\theta^*} \lim_{\rightarrow} (N_\lambda, L_\lambda)^m,$$

where the limit on the left is the direct limit over the $\mathfrak{L}\mathfrak{F}$ coverings of $P \times I^m$, and on the right over the $\mathfrak{L}\mathfrak{F}$ coverings of P . In this case also θ^* is onto, σ^* is an isomorphism into. The canonical maps $h_\lambda: [P, Q] \rightarrow [N_\lambda, L_\lambda]$, $h^*: [P \times I^m, Q \times I^m \cup P \times \dot{I}^m] \rightarrow [N^*, L^*]$ induce homomorphisms h^* , h^{**} respectively of the direct limits such that $h^* \circ \theta^* = h^{**}$, $h^{**} \circ \sigma^* = h^*$. According to 12.32, h^{**} is an isomorphism onto, so that h^* is onto by the first relation, and an isomorphism into by the second. Therefore h^* is also an isomorphism onto, which proves Theorem 12.21. Theorem 12.22 follows

at once, for if $P \supset Q \supset R$, and a covering Σ of P has nerve N with sub-complexes M, L corresponding respectively to Q, R , we have an exact sequence

$$\rightarrow (N, M)^m \rightarrow (N, L)^m \rightarrow (M, L)^m \rightarrow (N, M)^{m-1} \rightarrow \dots \rightarrow (N, M)^1,$$

whose maps all commute with the homomorphisms $T_{\lambda\mu}^*$; therefore, on passing to the direct limit and applying the isomorphisms h^* we have the exact sequence of Theorem 12.22.

APPENDIX: CENTRAL EXTENSIONS

If G, E, Q are groups, written additively, and

$$0 \rightarrow G \xrightarrow{i} E \xrightarrow{j} Q \rightarrow 0$$

is exact (that is, $j^{-1}(0) = i(G)$, j is onto, $i^{-1}(0) = 0$), we say that E is an *extension of G by Q* . If E' is another extension, and in the commutative diagram

$$\begin{array}{ccccc} & & E & & \\ & i \nearrow & \downarrow \phi & \nwarrow j & \\ 0 \rightarrow G & & & & Q \rightarrow 0 \\ & i' \searrow & \downarrow & \swarrow j' & \\ & & E' & & \end{array}$$

$\phi: E \approx E'$, we say that E, E' are *equivalent extensions*; this is an equivalence relation, and we write $\{E\}$ for the equivalence class of E . Notice that this depends on the homomorphisms i, j , as well as the group E . We shall always suppose that G is abelian, and eventually that Q is abelian.

Let A be the (contravariant) functor which assigns to every group X the group of automorphisms $A(X)$ of X , and let $\theta_0: X \rightarrow A(X)$ be the natural homomorphism which assigns to $x \in X$ the inner automorphism $x' \rightarrow x + x' - x$. Then, as G is abelian, the composition

$$E \xrightarrow{\theta_0} A(E) \xrightarrow{i^*} A(G)$$

(where i^* is induced by i) maps G to the identity automorphism, and so can be factored through Q . Let

$$\theta = i^* \circ \theta_0 \circ j^{-1}: Q \rightarrow A(G);$$

clearly θ is the same for equivalent extensions. Then E is a *central extension* if and only if $i(G)$ is in the centre of E , i.e. if and only if θ is trivial.

The classes of extensions which determine the same θ can be turned into an abelian group (see (4)); the sum $\{E'\} + \{E''\}$ is defined by taking the subgroup E_0 of the direct sum $E' + E''$ which consists of all pairs (e', e'') such that $j'e' = j''e''$, and factoring E_0 by the subgroup G_0 of all pairs $(i'g, -i''g)$, $g \in G$. Let $E = E_0/G_0$, and let $\{e', e''\}$ be the coset of (e', e'') ; then we may define

$$i(g) = \{i'g, 0\} = \{0, i''g\}, \quad \text{and} \quad j\{e', e''\} = j'e' = j''e'',$$

so that

$$0 \rightarrow G \xrightarrow{i} E \xrightarrow{j} Q \rightarrow 0$$

is exact. Notice that G_0 is invariant in E_0 because E', E'' (and hence also E) determine the same θ . $\{E\}$ is the required sum.

Let Q be abelian hereafter, let $H^2(Q, G)$ be the group of classes of central extensions (so that θ is trivial), and let $\text{Ext}(Q, G)$ be the subgroup of abelian extensions. We now describe an embedding of the factor group $H^2(Q, G)/\text{Ext}(Q, G)$ in $\text{Hom}(Q \otimes Q, G)$, the group of homomorphisms of the tensor product $Q \otimes Q$ to G .

Let $\chi: H^2(Q, G) \rightarrow \text{Hom}(Q \otimes Q, G)$ be defined by selecting E in a class $\{E\}$, and setting $\chi\{E\} = h$, where

$$h(je_1 \otimes je_2) = i^{-1}(e_1 + e_2 - e_1 - e_2),$$

and is extended by linearity to a homomorphism $h: Q \otimes Q \rightarrow G$. Clearly, h does not depend on the choice of E in $\{E\}$ and respects the bilinearity of the tensor product, since θ is trivial, and it may be verified that χ is a homomorphism. χ is not onto: let K be the subgroup of homomorphisms h such that

$$h(q \otimes q) = 0, \quad \text{all } q \in Q;$$

then $h(q_1 \otimes q_2) = -h(q_2 \otimes q_1)$ if $h \in K$.

LEMMA A.1. *The sequence*

$$0 \rightarrow \text{Ext}(Q, G) \xrightarrow{i} H^2(Q, G) \xrightarrow{\chi} \text{Hom}(Q \otimes Q, G)$$

is exact at the two centre groups, and $\chi H^2(Q, G) \subset K$. Here i is the identity map.

The proof is omitted. If we identify the pairings of Q with itself to G with the elements of $\text{Hom}(Q \otimes Q, G)$ in the obvious way, K is contained in the subgroup of anti-commutative pairings, but need not coincide with it. In general, we do not know if χ is onto K .

The *strong direct sum* $\Sigma^* Q_\alpha$ of additive groups Q_α is another name for the direct product; its elements are all collections $\Sigma^* q_\alpha$ ($q_\alpha \in Q_\alpha$, one for each α), and addition is defined by $\Sigma^* q_\alpha + \Sigma^* q'_\alpha = \Sigma^*(q_\alpha + q'_\alpha)$. q_α is called a *coordinate* of $\Sigma^* q_\alpha$. The *weak direct sum* Q of the Q_α is the subgroup of $\Sigma^* Q_\alpha$ consisting of all elements for which all but a finite number of coordinates are zero. We embed Q_α in Q in the usual way, and may write a non-zero element $\Sigma^* q_\alpha$ of Q as a sum $\Sigma_\alpha q_\alpha$, taken over all the non-zero coordinates.

Let Q be the weak direct sum of groups Q_α , and let $K_0 \subset K$ be the subgroup of $\text{Hom}(Q \otimes Q, G)$ of all homomorphisms h such that $h(q \otimes q') = 0$ if q, q' are contained in the same subgroup Q_α . Let $I_\alpha: Q_\alpha \rightarrow Q$ be the injection: then I_α induces a homomorphism

$$I_\alpha^*: H^2(Q, G) \rightarrow H^2(Q_\alpha, G)$$

such that $I_\alpha^*\{E\} = \{E_\alpha\}$, where $E_\alpha = j^{-1}(Q_\alpha) \subset E$. Therefore there is a homomorphism

$$I^*: H^2(Q, G) \rightarrow \Sigma^* H^2(Q_\alpha, G),$$

the strong sum, defined by $I^*\{E\} = \Sigma^* I_\alpha^*\{E\}$. Our principal result is

THEOREM A.2. $H^2(Q, G) = \phi^* \Sigma^* H^2(Q_\alpha, G) + \lambda K_0$,

where $I^* \circ \phi^*$ is the identical homomorphism and $\chi \circ \lambda$ maps K_0 identically on itself. Thus ϕ^*, λ are isomorphisms into. Also ϕ^* induces $\phi^*: \Sigma^* \text{Ext}(Q_\alpha, G) \approx \text{Ext}(Q, G)$.

We first construct ϕ^* . For each α , let E_α be a central extension of G by Q_α , so that there is an exact sequence

$$0 \rightarrow G \xrightarrow{i_\alpha} E_\alpha \xrightarrow{j_\alpha} Q_\alpha \rightarrow 0.$$

Let E' be the weak direct sum of the E_α , and let G' be the subgroup of E' generated by all elements $i_\alpha(g) - i_\beta(g)$, all $g \in G$, all α, β . Then G' is in the centre of E' , and

so is invariant; let $[\Sigma e_\alpha]$, an element of $E = E'/G'$, be the coset of G' containing Σe_α , $e_\alpha \in E_\alpha$, and define

$$0 \rightarrow G \xrightarrow{i} E \xrightarrow{j} Q \rightarrow 0$$

by $i(g) = [i_\alpha(g)]$ (any α), $j[\Sigma e_\alpha] = \Sigma j_\alpha e_\alpha \in Q$. Then we can verify that E is an extension of G by Q , whose equivalence class depends only on the equivalence classes of the extensions E_α . Moreover, $I^*\{E\} = \Sigma^*\{E_\alpha\}$. Define $\phi^*(\Sigma^*\{E_\alpha\}) = \{E\}$; then we may easily verify from the definition of addition that ϕ^* is a homomorphism. Since $I^* \circ \phi^*$ is the identical isomorphism (i.e. I^* has a right inverse) and $H^2(Q, G)$ is abelian,

$$H^2(Q, G) = \phi^* \Sigma^* H^2(Q_\alpha, G) + I^{*-1}(0),$$

the direct sum, where ϕ^* is an isomorphism into.

Next, we show that χ maps $I^{*-1}(0)$ isomorphically into K_0 . That $\chi(I^{*-1}(0)) \subset K_0$ is clear, for if $I^*\{E\} = 0$, $I_\alpha^*\{E\} = 0$ for every α , and contains the direct sum $G + Q_\alpha$, which is abelian: this implies $\chi\{E\} \in K_0$. Also, we see that for any $E_\alpha \in I_\alpha^*\{E\}$ the projection $j_\alpha: E_\alpha \rightarrow Q_\alpha$ has a right inverse, since E_α is equivalent to $G + Q_\alpha$.

Let $E_\alpha = j^{-1}(Q_\alpha)$, where $j: E \rightarrow Q$, and let τ_α be a right inverse for $j|E_\alpha$. Now, if $\{E\} \in I^{*-1}(0) \cap \chi^{-1}(0)$, E is abelian; hence the homomorphisms τ_α define $\tau: Q \rightarrow E$ by

$$\tau(\Sigma_\alpha q_\alpha) = \Sigma_\alpha \tau_\alpha(q_\alpha),$$

such that τ is a right inverse for j . Hence E is equivalent to the direct sum $G + Q$, and $\{E\} = 0$. This proves our assertion that χ maps $I^{*-1}(0)$ isomorphically.

We complete the proof of the theorem by constructing a homomorphism

$$\lambda: K_0 \rightarrow I^{*-1}(0)$$

such that $\chi \circ \lambda$ is the identical map $K_0 \rightarrow \text{Hom}(Q \otimes Q, G)$.

Let $h \in K_0$, and suppose the subscripts α are well ordered or, at any rate, a relation $<$ exists such that every finite subset is simply ordered, that is for every pair $\alpha \neq \beta$, either $\alpha < \beta$ or $\beta < \alpha$. Construct a free group F , written additively, with generators the symbols $(g; \Sigma q_\alpha)$ for all $g \in G$, $\Sigma q_\alpha \in Q$. Let R be the least invariant subgroup of F which contains all the expressions

$$(g; \Sigma q_\alpha) + (g'; \Sigma q'_\alpha) - (g''; \Sigma(q_\alpha + q'_\alpha)),$$

where

$$g'' = g + g' + \sum_{\beta < \alpha} h(q_\alpha \otimes q'_\beta),$$

for all g, g' and all $\Sigma q_\alpha, \Sigma q'_\alpha \in Q$. Notice that in the definition of g'' only a finite number of the terms $h(q_\alpha \otimes q'_\beta)$ are non-zero, and the expression is to be read with the other terms omitted.

Let $E = F/R$, and let $[g; \Sigma q_\alpha]$ be the coset of R containing $(g; \Sigma q_\alpha)$. The projection $j': F \rightarrow Q$ defined on the generators by

$$j'(g; \Sigma q_\alpha) = \Sigma q_\alpha$$

maps R to $0 \in Q$, and so induces a homomorphism $j: E \rightarrow Q$ such that $[g; \Sigma q_\alpha] \rightarrow \Sigma q_\alpha$. Then we may verify that there is an exact sequence

$$0 \rightarrow Q \xrightarrow{i} E \xrightarrow{j} Q \rightarrow 0,$$

where $i(g) = [g; 0]$. We now show that E is a central extension of G by Q .

Clearly
$$[g; \Sigma q_\alpha] + [-g + \sum_{\beta < \alpha} h(q_\alpha \otimes q_\beta); \Sigma(-q_\alpha)] = 0,$$

where the sum in the second bracket is interpreted as that with all $h(q_\alpha \otimes q_\beta)$ which are zero omitted. Hence

$$\begin{aligned} [g; \Sigma q_\alpha] + [g'; 0] - [g; \Sigma q_\alpha] \\ &= [g + g'; \Sigma q_\alpha] + \left[-g + \sum_{\beta < \alpha} h(q_\alpha \otimes q_\beta); \Sigma(-q_\alpha) \right] \\ &= \left[g + g' - g + \sum_{\beta < \alpha} h(q_\alpha \otimes q_\beta) + \sum_{\beta < \alpha} h(q_\alpha \otimes -q_\beta); 0 \right] \\ &= [g'; 0], \end{aligned}$$

since h is a homomorphism.

Define $\lambda(h) = \{E\}$, where E is the extension constructed above. Clearly

$$j^{-1}(Q_\alpha) = G + Q_\alpha,$$

so that $I_\alpha^*\{E\} = 0$, whence $I^*\{E\} = 0$, and λ is into $I^{*-1}(0)$. Also, $\chi\lambda(h) = h$, for

$$\begin{aligned} [g; \Sigma q_\alpha] + [g'; \Sigma q'_\alpha] - [g; \Sigma q_\alpha] - [g'; \Sigma q'_\alpha] \\ &= \left[g + g' + \sum_{\beta < \alpha} h(q_\alpha \otimes q'_\beta); \Sigma(q_\alpha + q'_\alpha) \right] - \left[g + g' + \sum_{\beta < \alpha} h(q'_\alpha \otimes q_\beta); \Sigma(q'_\alpha + q_\alpha) \right] \\ &= \left[g + g' + \sum_{\beta < \alpha} h(q_\alpha \otimes q'_\beta); \Sigma(q_\alpha + q'_\alpha) \right] + \left[-g - g' - \sum_{\beta < \alpha} h(q'_\alpha \otimes q_\beta) + \right. \\ &\quad \left. + \sum_{\beta < \alpha} h((q_\alpha + q'_\alpha) \otimes (q'_\beta + q_\beta)); -\Sigma(q_\alpha + q'_\alpha) \right] \\ &= \left[g + g' + \sum_{\beta < \alpha} h(q_\alpha \otimes q'_\beta) - g - g' - \sum_{\beta < \alpha} h(q'_\alpha \otimes q_\beta) + \right. \\ &\quad \left. + \sum_{\beta < \alpha} h((q_\alpha + q'_\alpha) \otimes (q'_\beta + q_\beta)) - \sum_{\beta < \alpha} h((q_\alpha + q'_\alpha) \otimes (q_\beta + q'_\beta)); 0 \right] \\ &= \left[\sum_{\beta < \alpha} h(q_\alpha \otimes q'_\beta) - \sum_{\beta < \alpha} h(q'_\alpha \otimes q_\beta); 0 \right] \\ &= \left[\sum_{\beta < \alpha} h(q_\alpha \otimes q'_\beta) + \sum_{\alpha < \beta} h(q_\alpha \otimes q'_\beta); 0 \right] \\ &= [h(\Sigma q_\alpha \otimes \Sigma q'_\alpha); 0]. \end{aligned}$$

Thus λ satisfies the required conditions, and

$$H^2(Q, G) = \phi^* \Sigma^* H^2(Q_\alpha, G) + \lambda K_0.$$

Possibly the easiest way to see that ϕ^* , I^* define isomorphisms onto between $\text{Ext}(Q, G)$ and $\Sigma^* \text{Ext}(Q_\alpha, G)$ is to recapitulate the argument, showing that

$$\text{Ext}(Q, G) = \phi^* \Sigma^* \text{Ext}(Q_\alpha, G) + I^{*-1}(0),$$

and that χ maps $I^{*-1}(0)$ isomorphically, so that I^* is also an isomorphism, ϕ^* onto (since by Lemma A.1 $\chi^{-1}(0) = \text{Ext}(Q, G)$).

A useful corollary to the theorems is

COROLLARY A.3. *If Q is the (weak) direct sum of cyclic groups, in particular, if Q is finitely generated, $H^2(Q, G) = i \text{Ext}(Q, G) + \lambda K_0$.*

For it follows from Lemma A.1 that if Q is cyclic, $i \text{Ext}(Q, G) = H^2(Q, G)$, since $K \subset \text{Hom}(Q \otimes Q, G)$ is zero. Thus, if Q is the direct sum of cyclic groups Q_α ,

$$\phi^* \Sigma^* H^2(Q_\alpha, G) = i \text{Ext}(Q, G).$$

Of course, i is an isomorphism into.

We may also deduce a theorem due to Lyndon (6).

THEOREM A.4. *If Q is the direct sum of a finite number of groups Q_i , then*

$$H^2(Q, G) \approx \sum_i H^2(Q_i, G) + \sum_{i < j} \text{Hom}(Q_i \otimes Q_j, G)$$

and

$$\text{Ext}(Q, G) \approx \sum_i \text{Ext}(Q_i, G).$$

Lyndon proved this when $Q = Q_1 + Q_2$, using normalized cochains; the generalization of his result to any finite sum is immediate. To obtain this form we examine the group $K_0 \subset \text{Hom}(Q \otimes Q, G)$. Since $Q = \sum_i Q_i$,

$$\text{Hom}(Q \otimes Q, G) = \sum_{i,j} \text{Hom}(Q_i \otimes Q_j, G).$$

Therefore there is a natural projection

$$\text{Hom}(Q \otimes Q, G) \rightarrow \sum_{i < j} \text{Hom}(Q_i \otimes Q_j, G),$$

and it is easy to see that this maps K_0 isomorphically onto.

In our application of these results in § 10 we consider an extension of $H^{n+1}(\pi_{m+n+1})$ by $H^n(\pi_{m+n})$, where the integral groups H^{n+1} , H^n are finitely generated. Thus $H(\pi_{m+n})$ is a finite sum of isomorphs of π_{m+n} and groups $p\pi_{m+n}$, where p ranges over the coefficients of torsion of the complex. Now, by a theorem due to Prüfer (see below) the latter groups are weak direct sums of cyclic groups. Hence $H^n(\pi_{m+n})$ is the direct sum of a finite number of isomorphs of π_{m+n} and a weak direct sum of cyclic groups. The extension by the first is given by Theorem 10.31 (it is trivial), and by the second in Theorem 10.41. It follows from Theorem A.2 and the definition of χ that the extension is determined by these results and the calculation of the commutators carried out in 10.22.

We conclude by observing that Theorem A.2 can be used to give an alternative proof of the theorem of Prüfer (15) used above: consider an abelian group G (not necessarily countable) such that for some integer k , $kG = 0$. Then the assertion is that G is the weak direct sum of cyclic groups. Let p be a prime divisor of k , and let $G(p)$ be the p -component of G , that is, the subgroup of all elements whose orders are powers of p . If $p \neq q$, then $G(p)$ and $G(q)$ intersect only in the unit element, and it is easy to show that $G = \sum G(p)$, the direct sum, summed over all prime divisors of k . Therefore it is sufficient to prove the assertion when $k = p^n$, p a prime. Our proof proceeds by induction on n ; for the result is clearly true when $n = 1$, as G is then a vector space over the field of integral residues mod p .

Suppose the result true when $n < m$, and that $p^m G = 0$. Then $p^{m-1}G$ is a vector space, with a basis (x_α) ; since $x_\alpha \in p^{m-1}G$, $x_\alpha = p^{m-1}y_\alpha$ for some $y_\alpha \in G$. Let H be the subgroup of G generated by the y_α : we say that H is the weak direct sum of cyclic groups, generated by the y_α . Let H' be the weak direct sum of cyclic groups of orders p^m generated by y'_α ; $\theta: H' \rightarrow H$ the homomorphism onto, such that $\theta(y'_\alpha) = y_\alpha$: then θ is an isomorphism onto. For if $y' = \sum_\alpha t_\alpha y'_\alpha$ (where all but a finite number of the t_α are zero) and $p^{m-1}(\theta y') = 0$, then $\sum_\alpha t_\alpha x_\alpha = 0$, and so each t_α is divisible by p . Therefore, if $\theta y' = 0$, y' is divisible by p , and, by applying this argument successively to y' , $(1/p)y'$, $(1/p^2)y'$, ..., we find that y' is divisible by p^m , and hence $y' = 0$.

Let $Z = G/H$, so that there is an exact sequence

$$0 \rightarrow H \xrightarrow{i} G \xrightarrow{\mu} Z \rightarrow 0,$$

where i is the injection, μ the natural map. By construction, $p^{m-1}G \subset H$, so that

$p^{m-1}Z = 0$; by hypothesis, Z is the weak direct sum of cyclic groups of orders p^t , $t < m$. We show that G is the direct sum of H and an isomorph of Z . By the last assertion of Theorem A.2, it is sufficient to consider a cyclic summand of Z , and we suppose Z is cyclic of order p^t , generated by z . Choose $g \in G$ such that $\mu(g) = z$; for some $h \in H$, $p^t g = h$. Since $p^m g = 0$, $p^{m-t} h = 0$, and since H is the weak direct sum of cyclic groups of order p^n , $h = p^t h'$ for some $h' \in H$. Therefore $\mu(g - h') = z$, and $p^t(g - h') = 0$. Therefore G is the direct sum, as required; hence, by the principle of finite induction, the assertion is true for all $n \geq 1$.

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