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Combinatorial Foundation of Homology and Homotopy

Applications to Spaces, Diagrams, Transformation Groups, Compactifications, Differential Algebras, Algebraic Theories, Simplicial Objects, and Resolutions

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Preface

The classical formulation of homology theory is based on the notion of ring and module or more generally on abelian categories. The homology that one considers, however, often comes from a group, or a Lie algebra, or a topological space, etc. which are non-abelian objects. Therefore a general treatment of homology should derive the abelian concept of homology from non-abelian data.

The notion of homology emerges in this book from a theory of cogroups or more generally from a theory of coactions. Such theories arise frequently in algebra and topology. For example, most algebraic objects like groups, algebras, Lie algebras, etc. are models of theories of cogroups. Moreover, each homotopy theory contains theories of coactions. A "theory of coactions" is a very general concept related to notions in the literature like near ring or Malcev variety. Nevertheless it has exactly those properties which are needed to obtain a homology theory suitable for obstruction theory.

Classical obstruction theory relies on the properties of CW-complexes. Here we will show that fundamental results on CW-complexes have generalizations in the realm of categorical algebra. For this we associate to a theory $\mathbf{T}$ of coactions the notion of a $\mathbf{T}$-complex in a cofibration category which is the categorical generalization of a CW-complex.

We present a homology and cohomology theory for $\mathbf{T}$-complexes which embodies numerous homology theories in various fields of algebra and topology. For example, by suitable specialization one obtains the homology of groups, the homology in a variety of groups, the Hochschild homology of an algebra, the homology of a Lie algebra, the homology of a topological space, the Bredon homology of a G-space where G is a group, the homology theory for diagrams of spaces, the homology theory for controlled spaces, or the homology theory for compactifications, and many more examples. All these examples are homology theories associated to theories $\mathbf{T}$ of coactions and $\mathbf{T}$-complexes.

The book consists of two parts. The first part (Chapters A, B, C, D) furnishes a long list of explicit examples and applications in various fields of topology and algebra. The second part (Chapters I, ... , VIII) develops the axiomatic theory of combinatorial homology and homotopy.

The unification in this book possesses all the usual advantages. One proof replaces many different proofs in all such fields. In addition, an interplay takes place among the various specializations, which thereby enrich one another. The unified theory also applies to various new situations. Moreover, all definitions, proofs and results in the second part use a categorical language, so that by a duality which reverses the direction of arrows one obtains the corresponding dual definitions, proofs and results, respectively.

May 1998

H.-J. Baues

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Part II. Combinatorial Homology and Homotopy

Leitfaden

The main concepts studied in the axiomatic theory of Part 2 are given by the following list. We start with a

$$\textit{theory of cogroups} \quad \mathbf{T}, \quad \text{or a} \tag{I.1.9}$$

$$\textit{theory of coactions} \quad \mathbf{T}. \tag{I.1.11}$$

All the results in Chapter I, II and in VII, § 3 deal with properties of **T**. This is pure categorical algebra. We derive from **T** the

$$\textit{enveloping functor} \quad U : \mathbf{Coef} \to \mathbf{Ringoids} \tag{I.5.11}$$

which is needed in all chapters. In order to introduce homotopy theory we recall from Baues [AH] some properties of a

$$\textit{cofibration category} \quad \mathbf{C}, \quad \text{or an} \tag{III.1.1}$$

$$I\textit{-category} \quad \mathbf{C}. \tag{III.7.1}$$

A **T**-complex can be defined in a

$$\textit{cofibration category under} \quad \mathbf{T} \tag{IV.2.1}$$

and homology of a **T**-complex can be obtained in a

$$\textit{homological cofibration category under} \quad \mathbf{T}. \tag{V.1.1}$$

Chapter IV deals with cofibration categories under **T**; in particular, we discuss the Whitehead theorem, cellular approximation, and the Blakers-Massey property in such categories. If the Blakers-Massey property holds then one obtains a homological cofibration category under **T** and all the results of Chapters V, VI, VII are available.

In particular, we prove the following results in a homological cofibration category:

– definition of homology and cohomology in terms of a chain functor
– obstruction theory for the extension of maps
– Whitehead's exact sequence for the Hurewicz homomorphism
– homotopy lifting property of the chain functor

- model lifting property of the twisted chain functor
- homological Whitehead theorem
- obstruction theory for the realizability of chain complexes and chain maps
- Hurewicz theorem
- Eilenberg-Mac Lane complex and Quillen (co-) homology
- finiteness obstruction theorem of Wall

Finally in Chapter VIII we deal with Whitehead torsion. For this we choose a

$$\textit{class of discrete objects} \quad \mathcal{D} \tag{VII.1.1}$$

in an I-category $\mathbf{C}$ and we describe the properties of $(\mathbf{C}, \mathcal{D})$ which define a

$$\textit{cellular } I\textit{-category} \quad (\mathbf{C}, \mathcal{D}). \tag{VIII.5.1}$$

The geometric Whitehead group can be defined in such an I-category. Moreover in a

$$\textit{homological cellular } I\textit{-category} \quad (\mathbf{C}, \mathcal{D}) \tag{VIII.12.3}$$

the geometric Whitehead group coincides with the algebraic Whitehead group. Here the algebraic Whitehead group is defined in terms of the enveloping functor U studied in Chapter I. The finiteness obstruction theorem also uses the enveloping functor U, for the definition of a reduced projective class group.

We point out that all the results above are proved in a new way since we do not use the universal covering of a CW-complex which was of crucial importance in the proofs of J.H.C. Whitehead.

Fields of Application

The results of the axiomatic theory in Part 2 can be applied in many different areas of algebra and topology. We here describe various fields of application, some of which already have been worked out in the literature. The theory was designed to cover all these specializations. It is worth while to formulate in each such field all the results which are implied by the axiomatic theory. We give various hints in this respect in the text. A complete discussion of such applications in the context of the abstract results in Part 2 was avoided in order not to obscure the axiomatic theory.

For the convenience of the reader we describe explicit examples and applications in the introductory chapters A, B, C and D of Part 1. These chapters can be read without knowing the results and notation of the general theory.

The first two chapters I and II of Part 2 can be applied for all theories of coactions and theories of cogroups. For example,

(1) varieties of groups, or
(2) algebras, commutative algebras, Lie algebras, and many other kinds of algebras defined by operads

give rise to theories of cogroups. Also

(3) groupoids

give rise to theories of coactions, see (I.2.11). Moreover, in each homotopy theory $\mathbf{C}$ the homotopy category of suspensions termed $\mathbf{susp}(*)$ is a theory of cogroups and the homotopy category of $*$-cones termed $\mathbf{cone}(*)$ is a theory of coactions; see (I.2.4) and (III, § 6).

The chapters III, ... , VIII of Part 2 deal with complexes in cofibration categories. There are many different homotopy theories which have the properties of a cofibration category, in particular each Quillen model category. We are mainly interested in the homotopy theories of

(4) topological spaces,
(5) simplicial objects in some category like (1) and (2),
(6) differential algebras of some kind like (2).

We also consider for a small category $\mathbb{A}$ the category of

(7) $\mathbb{A}$-diagrams in a category $\mathbf{C}$ like (4), (5), (6)

which are functors $\mathbb{A} \to \mathbf{C}$. Morphisms are natural transformations of such functors. If $\mathbb{A}$ is given by a discrete group G then (7) is the category of G-equivariant maps between G-objects in $\mathbf{C}$. Moreover, if G is a topological group we have the category of

(8) G-spaces

which leads to the homotopy theory of transformation groups. We can put restrictions on the maps in the categories above and again obtain new homotopy theories. For example, we may consider

(9) topological spaces and compact maps,
(10) topological spaces with some control (for example, bounded control or continuous control, etc.),
(11) shape theory.

Again we can consider $\mathbb{A}$-diagrams in (9) or (10) and the theory of transformation groups for (9) and (10), respectively.

One important feature is also the possibility of *relativization*. In fact, given a homotopy theory $\mathbf{C}$ and an object D in $\mathbf{C}$ then also the category

(12) $\mathbf{C}^D$ of objects under D in $\mathbf{C}$

is again a homotopy theory. We can apply this to all theories $\mathbf{C}$ in (4), ... , (11) above.

In the literature there are many further examples of homotopy theories. Most of them are candidates for the application of the abstract theory in this book. In particular, the recent

(13) "motivic homotopy theory"

of Morel-Voevodsky [HC] will lead to applications in algebraic geometry. Moreover, the homotopy theory of

(14) resolutions of spaces

due to Dwyer-Kan-Stover [E^2], [HG], Blanc [AI] and Goerss-Hopkins [RM] is a wonderful field of application for the methods and results of this book; see Chapter D.

This list, which is by no means complete, shows the wide range of different fields to which the theory of this book can be applied. It also shows the necessity of an axiomatic approach which separates a result from the specific environment where the result was proved for the first time. We consider classical and fundamental results of homotopy theory and we characterize axiomatically the assumptions under which such results hold. This leads to the concepts in the Leitfaden above. The non-axiomatic approach would try to prove the results in each case again and again.

For example, the *theorem on Whitehead torsion* was proved in the following categories:

a) for topological spaces by J.H.C. Whitehead [SH], Stöcker [W] and Cohen [C],

b) for G-spaces by Lück [TG],
c) for topological spaces and compact maps by Siebenmann [S] and Farell-Wagoner [S],
d) for bounded controlled spaces by Munkholm-Anderson [B].

All these cases a), b), c), d) are specializations of the general result in (VIII, § 12) below which holds in any homological cellular I-category. Here the axiomatic approach has a further advantage since it clarifies the definition of the algebraic Whitehead group. We give a definition which is valid simultaneously for all cases a), b), c) and d). The reader may compare the complicated definitions of algebraic Whitehead groups of Lück [TG], Siebenmann [S] and Munkholm-Anderson [B].

Similar remarks hold for the *finiteness obstruction theorem* in (VIII, § 2) or for the *homological Whitehead theorem* in (VI, § 7) which was recently proved for diagrams of spaces by Moerdijk-Svenson [D].

Concerning the homotopy theory of simplicial objects we point out that André [HS] and Swan [HA] use a kind of $\mathbf{T}$-complex to define André-Quillen homology, which is a special case of $(\mathbf{C}, \mathbf{T})$-homology in (VI, § 11).

The reader will find many further examples which connect the general theory in this book with the literature.

We point out that there are numerous results in this book which are new even if one specializes them, for example, to G-spaces in (8) or to other fields of application (4) ... (14). Already the specialization to spaces under D in (12) in the category of topological spaces leads to new and interesting facts in ordinary topology; compare § 1 in Chapter A.

Part I

Examples and Applications

The axiomatic theory of Part 2 is based on a theory of coactions which is embedded in a homotopy category $\mathbf{C}/\simeq$. Here $\mathbf{C}$ is an abstract category in which "homotopies" are defined satisfying suitable axioms. For example, $\mathbf{C}$ is a category of cofibrant objects in a Quillen model category. In the theory of Chapters I, ... , VIII in Part 2 we describe the notions and results concerning the combinatorial foundation of homology and homotopy.

In the following chapters A, B, C, D we consider the specialization of the axiomatic theory for various examples in topology and algebra. We discuss only the basic notation and results for these examples. This can be understood easily without knowing the axiomatic theory.

The reader will benefit from the presentation of the examples. It is useful to compare such examples in order to visualize the abstract theory and to clarify the motivation for the various abstract notions.

Chapter A: Examples and Applications in Topological Categories

In this chapter we describe the leading examples of combinatorial homology and homotopy theory which are well known fields of algebraic topology. We consider the homotopy theory of spaces, diagrams of spaces, spaces with a topological group of transformations, and spaces controlled at infinity. These examples are discussed in a highly parallel fashion so that the underlying abstract theory is a shining achievement.

1 Homotopy Theory of Spaces Under a Space

Ordinary homotopy theory is concerned with the category **Top** of topological spaces and (continuous) maps. Let

$$I = [0,1] \subset \mathbb{R} \tag{1.1}$$

be the unit <u>interval</u> of real numbers. Then a <u>homotopy</u> $H : f \simeq g$ of maps $f, g : X \to Y$ is a map $H : I \times X \to Y$ satisfying $H_0 = f$ and $H_1 = g$ with $H_t(x) = H(t,x)$ for $t \in I, x \in X$. Here $I \times X$ is the topological product of the spaces I and X. The relation of homotopy is an equivalence relation so that the set of homotopy classes

$$[X,Y]^{\emptyset} = \mathbf{Top}(X,Y)/\simeq \tag{1.2}$$

is defined. $[X,Y]^{\emptyset}$ is the set of morphisms $X \to Y$ in the <u>homotopy category</u> $\mathbf{Top}/\simeq$. Homotopies H as above are also called <u>free</u> homotopies or homotopies relative the empty space $\emptyset$. They have the disadvantage that they are not compatible with base points and therefore free homotopies are not suitable for the definition of the fundamental group and homotopy groups of a space. In order to obtain such groups one has to consider homotopies relative a point or more generally homotopies relative a non empty space D.

We choose a topological space D which may be any space in the category **Top**. As important special case $D = *$ is a point or D is a discrete space. The results achieved below for $D = *$ are well known and classical though for an arbitrary space D some of the results seem to be new.

Consider the homotopy theory in the category

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$$\mathbf{C} = \mathbf{Top}^D \tag{1.3}$$

of spaces under D. Objects in $\mathbf{C}$ are maps $D \to X$ in $\mathbf{Top}$ and morphisms f are <u>maps under</u> D, i.e. commutative triangles in $\mathbf{Top}$

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

A homotopy $H : f \simeq g \operatorname{rel} D$ of maps in $\mathbf{C}$ is a <u>homotopy relative</u> D; this is a homotopy for which H_t is a map under D for all $t \in [0,1]$. If $D \to X$ is a cofibration in $\mathbf{Top}$ we write

$$[X,Y]^D = \mathbf{C}(X,Y)/\simeq \operatorname{rel} D \tag{1.4}$$

for the set of homotopy classes relative D. Let

$$\mathbf{C}_c = \mathbf{Top}_c^D \subset \mathbf{Top}^D \tag{1}$$

be the full subcategory of $\mathbf{Top}^D$ for which the objects are cofibrations $D \rightarrowtail X$ in $\mathbf{Top}$. Here cofibrations in $\mathbf{Top}$ are defined by the homotopy extension property; see Baues [AH]. Then homotopy $\operatorname{rel} D$ is a natural equivalence relation on $\mathbf{Top}_c^D$ so that the homotopy category

$$\mathbf{C}_c/\simeq \; = \mathbf{Top}_c^D/\simeq \operatorname{rel} D \tag{2}$$

is defined. If $D = *$ is a point this is the homotopy category of "well pointed" spaces. The set of morphisms $X \to Y$ in $\mathbf{C}_c/\simeq$ coincides with $[X,Y]^D$ above. A <u>homotopy type under</u> D is a class of isomorphic objects in $\mathbf{C}_c/\simeq$. Homotopy relative D is also defined by the <u>cylinder object</u> $I(X,D)$ in $\mathbf{Top}_c^D$ which is given by the push out in $\mathbf{Top}$

$$\begin{tikzcd} {[0,1] \times X} \arrow[r] & I(X,D) \\ {[0,1] \times D} \arrow[u] \arrow[r, "pr"] & D \arrow[u] \end{tikzcd}$$

where pr is the projection. (Compare § 7 of chapter III below.)

Recall the following notation on spheres and balls. Let $\mathbb{R}^{n+1}$ be the Euclidean space with the norm $\| - \|$. Then the <u>Euclidean $(n+1)$-ball</u> is the subspace

$$\tilde{B}^{n+1} = \{x \in \mathbb{R}^{n+1},\ \|x\| \leq 1\} \quad \text{of } \mathbb{R}^{n+1}.$$

The <u>Euclidean n-sphere</u> is the subspace $\tilde{S}^n = \{x \in \mathbb{R}^{n+1}, \|x\| = 1\}$ which is the boundary of the Euclidean $(n+1)$-ball. A <u>sphere</u> S^n is a space homeomorphic to the $\tilde{S}^n$ and a <u>ball</u> B^{n+1} is a space homeomorphic to the Euclidean ball $\tilde{B}^{n+1}$. The boundary of B^{n+1} is an n-sphere S^n. For example the interval I is a 1-ball and also the cube I^{n+1} (given by the product of $n+1$ intervals) is an $(n+1)$-ball. We choose for each sphere S^n a basepoint $*$ so that we have

$$* \in S^n \subset B^{n+1} \quad \text{for } n \geq 0. \tag{1.5}$$

We say that a space Y is obtained from a space X by attaching $(n+1)$-cells if a discrete set Z together with a push out diagram

$$\begin{array}{ccc} Z \times B^{n+1} & \longrightarrow & Y \\ \cup & & \cup \\ Z \times S^n & \xrightarrow{f} & X \end{array} \tag{1.6}$$

in **Top** is given. Here f is called the attaching map. The disjoint union $A \amalg B$ is the coproduct of spaces A, B in the category **Top**. Clearly for a discrete set Z the product

$$Z \times A = \coprod_{z \in Z} A$$

is such a disjoint union of spaces.

We now recall the appropriate notion of CW-complex in the category of spaces under the space D. A (relative) CW-complex $(X_{\geq 0}, D)$ is given by a sequence of inclusions

$$D \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \ldots \tag{1.7}$$

Here X_0 is the disjoint union of D and a discrete set and X_{n+1} is obtained from the n-skeleton X_n by attaching $(n+1)$-cells, $n \geq 0$. We also write $X = \lim(X_{\geq 0})$ for the direct limit of the sequence and call (X, D) a relative CW-complex. The dimension of (X, D) is defined by $\dim(X, D) \leq n$ if $X = X_n$. We say that (X, D) is reduced if $X_0 = D$, that is, if the discrete set $X_0 - D$ of 0-cells in X is empty. Moreover (X, D) is normalized if all attaching maps carry base points $*$ of the sphere S^n to the 0-skeleton X_0. Clearly each 1-skeleton X_1 is normalized since X_1 is obtained by attaching 1-cells to X_0.

(1.8) Lemma. *Let (X, D) be a relative CW-complex. Then there exists a normalized relative CW-complex (Y, D) together with a homotopy equivalence $Y \to X$ under D. If $\pi_0 D \to \pi_0 X$ is surjective then (Y, D) can be chosen to be normalized and reduced.*

Proof. The proof uses standard arguments; compare the proof of (2.9) below for $\mathbb{A}$-spaces which yields (1.8) as a special case if $\mathbb{A} = *$ is the trivial category.

We now consider homotopy groups of a space A. Let $\pi_0(A)$ be the set of path components of A with $0 \in \pi_0(A)$ given by the basepoint $a_0 \in A$. For $n \geq 1$ the homotopy groups are given by the set of homotopy classes relative $*$

$$\pi_n(A, a_0) = [S^n, A]^*. \tag{1.9}$$

For a pair of spaces (A, B) with $a_0 \in B \subset A$ we also obtain the relative homotopy groups

$$\pi_{n+1}(A, B, a_0) = [(B^{n+1}, S^n), (A, B)]^*$$

defined by the set of homotopy classes relative $*$ of pair maps $(B^{n+1}, S^n) \to (A, B)$.

A weak equivalence in $\mathbf{Top}^D$ is a map $f : X \to Y$ under D which induces isomorphisms between homotopy groups

$$f_* : \pi_n(X, a_0) \cong \pi_n(Y, fa_0)$$

for all $a_0 \in X$ and $n \geq 0$. It is well known that for each space Y under D there exists a relative CW-complex (X, D) together with a map $X \to Y$ under D which is a weak equivalence. We call (X, D) a CW-approximation of (Y, D). This implies that the localized category $\mathrm{Ho}(\mathbf{Top}^D)$, in which weak equivalences are inverted, admits an equivalence of categories

$$\mathrm{Ho}(\mathbf{Top}^D) \xrightarrow{\sim} \mathbf{CW}^D/\simeq \mathrm{rel}\, D. \tag{1.10}$$

Here $\mathbf{CW}^D$ is the full subcategory of $\mathbf{Top}_c^D$ consisting of relative CW-complexes. Using the equivalence (1.10) each homotopy functor defined on relative CW-complexes (like homology and cohomology in (1.26), (1.27) below) yields a homotopy functor on $\mathbf{Top}^D$.

A groupoid G is a category in which all morphisms are isomorphisms. We write $a \in \mathrm{Ob}(G)$ or $a \in G$ if a is an object in G and for $a, b \in \mathrm{Ob}(G)$ let $G(a, b)$ be the set of morphisms from a to b. Then $G(a) = G(a, a)$ is a group, the vertex group of G at a.

For each space A we have the fundamental groupoid $\Pi(A) = \Pi A$. Objects in ΠA are the points of A and morphisms are homotopy classes of maps $f : [0, 1] \to A \;\mathrm{rel}\, S^0 = \{0, 1\}$ with $f(0) = a, f(1) = b$. Such a morphism is also called a track $t : b \to a \in \Pi A$. The vertex group $\Pi(A)(a_0)$ coincides with the fundamental group $\pi_1(A, a_0)$. If D is a subspace of X we write

$$\Pi(X, D) \subset \Pi X$$

for the full subgroupoid of ΠX consisting of objects which are points in D. We call $\Pi(X, D)$ the restricted fundamental groupoid. If $\pi_0 D \to \pi_0 X$ is surjective then the inclusion $\Pi(X, D) \subset \Pi X$ is an equivalence of categories. We shall use the assumption on the surjectivity of $\pi_0 D \to \pi_0 X$ frequently since this implies that each path component of X contains a point in D.

Let $\mathbf{Ab}$ be the category of abelian groups. For a category G let G^{op} be the opposite category. Then the homotopy groups (1.9) and (1.10) yield canonical functors $(n \geq 2)$

$$\begin{cases} \pi_n(A) : (\Pi A)^{\mathrm{op}} \to \mathbf{Ab} \\ \pi_{n+1}(A, B) : (\Pi B)^{\mathrm{op}} \to \mathbf{Ab} \end{cases} \tag{1.11}$$

Here $\pi_n(A)$ carries $a_0 \in A$ to the abelian group $\pi_n(A, a_0)$ and carries a track $t : b_0 \to a_0$ in ΠA to the induced map $t^\sharp : \pi_n(A, a_0) \to \pi_n(A, b_0)$ which is an isomorphism. The element $t^\sharp\{g\}$ with $\{g\} = \{g : S^n \to A\} \in \pi_n(A, a_0)$ is determined by the homotopy extension property of the cofibration $* \rightarrowtail S^n$. See Baues [AH] II.5.7. In a similar way the functor $\pi_{n+1}(A, B)$ in (1.11) is defined.

A functor

$$M : G^{\mathrm{op}} \to \mathbf{Ab} \tag{1.12}$$

is called a (right) G-module. Hence M is a contravariant functor from G to $\mathbf{Ab}$. If G is small (i.e. if the class of objects in G is a set) then such G-modules are the objects of the abelian category $\mathbf{Mod}(G)$. Morphisms are natural transformations. Hence by (1.11) we see that homotopy groups $\pi_n(A)$ and $\pi_n(A, B)$ are (ΠA)-modules and (ΠB)-modules respectively.

Next we consider the functorial property of the fundamental groupoid. For this let $\mathbf{Grd}$ be the category of small groupoids. Morphisms are functors. For a groupoid G let $\mathbf{Grd}(G)$ be the following category. Objects are functors $G \to H$ between groupoids which are the identity on objects (hence $\mathrm{Ob}\, G = \mathrm{Ob}\, H$). Morphisms are functors $H \to K$ under G that is, commutative triangles in $\mathbf{Grd}$:

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ H \arrow[rr] & & K \end{tikzcd}$$

For each cofibration $D \to X$ in $\mathbf{Top}^D_c$ we obtain the object

$$c(X) = (\Pi(D) \to \Pi(X, D))$$

in $\mathbf{Grd}(\Pi D)$ where $\Pi(X, D)$ is the restricted fundamental groupoid of X. This defines the coefficient functor

$$c : \mathbf{Top}^D_c/\simeq \mathrm{rel}\, D \to \mathbf{Grd}(\Pi D). \tag{1.13}$$

If $D = *$ is a point $*$ then $\mathbf{Grd}(\Pi *) = \mathbf{Gr}$ is the category of groups. Moreover the coefficient functor c for $D = *$ is just the functor which carries a pointed space X to its fundamental group $\pi_1 X$. In this sense the coefficient functor c is a canonical generalization of the fundamental group.

For each small groupoid H we have the abelian category $\mathbf{Mod}(H)$ of H-modules. We now define the full subcategory

$$\mathbf{mod}(H) \subset \mathbf{Mod}(H) \tag{1.14}$$

consisting of free H-modules. For this we use the category $\mathbf{Set}_{\mathrm{Ob}(H)}$ of sets over $\mathrm{Ob}(H)$; objects are functions $\alpha : Z \to \mathrm{Ob}(H)$ in the category $\mathbf{Set}$ of sets and morphisms f are functions over $\mathrm{Ob}(H)$, i.e. commutative triangles in $\mathbf{Set}$

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & Z' \arrow[dl, "\beta"] \\ & \mathrm{Ob}(H) & \end{tikzcd}$$

We have the forgetful functor

$$\varphi : \mathbf{Mod}(H) \to \mathbf{Set}_{\mathrm{Ob}(H)}$$

which carries $F : H \to \mathbf{Ab}$ to the function

$$f : Z = \coprod_{a \in \mathrm{Ob}(H)} F(a) \to \mathrm{Ob}(H)$$

with $f(x) = a$ for $x \in F(a)$. Let $L(H)$ be the left adjoint of φ which carries a function $\alpha : Z \to \mathrm{Ob}(H)$ to the H-module $L_\alpha(H) \in \mathbf{Mod}(H)$. We call $L_\alpha(H)$ the <u>free H-module</u> with basis α. Let $\mathbf{mod}(H)$ be the full subcategory of $\mathbf{Mod}(H)$ consisting of all free G-modules $L_\alpha(H)$ with $\alpha : Z \to \mathrm{Ob}(H)$ an object in $\mathbf{Set}_{\mathrm{Ob}\, H}$. A further description of $L_\alpha(H)$ is obtained as follows. Let $\mathbb{Z}[H(-,a)]$ be the H-module which carries $b \in \mathrm{Ob}(H)$ to the free abelian group generated by the set $H(b,a)$. Then $L_\alpha(H)$ is the direct sum

$$L_\alpha(H) = \bigoplus_{z \in Z} \mathbb{Z}[H(-, \alpha(z))]$$

in the abelian category $\mathbf{Mod}(H)$.

For a groupoid G and $H \in \mathbf{Grd}(G)$ we have $\mathrm{Ob}\, H = \mathrm{Ob}\, G$ and hence the class of objects of $\mathbf{mod}(H)$ admits the bijection

$$\mathrm{Ob}\, \mathbf{mod}(H) = \mathrm{Ob}\, \mathbf{Set}_{\mathrm{Ob}(G)}$$

which carries $L_\alpha(H)$ to α. Moreover each map $u : H \to K \in \mathbf{Grd}(G)$ yields a canonical additive functor

$$u_* : \mathbf{mod}(H) \to \mathbf{mod}(K) \tag{1.15}$$

which carries $L_\alpha(H)$ to $L_\alpha(K)$ and for which the following diagram in $\mathbf{Mod}(H)$ commutes with $\alpha, \beta \in \mathrm{Ob}\, \mathbf{Set}_{\mathrm{Ob}(G)}$,

$$\begin{tikzcd} L_\alpha(H) \arrow[r, "a"] \arrow[d, "u_\alpha"'] & L_\beta(H) \arrow[d, "u_\beta"] \\ L_\alpha(K) \arrow[r, "a_*"] & L_\beta(K) \end{tikzcd}$$

Here a K-module is an H-module via $u : H \to K$. Moreover u_α is the unique map which is the identity on the basis $\alpha : Z \to \mathrm{Ob}(G)$. The functor u_* in (1.15) carries the morphism a in $\mathbf{mod}(H)$ to the morphism a_* in $\mathbf{mod}(K)$ given by the diagram above.

A ringoid is a category in which all morphism sets are abelian groups and in which composition is bilinear. An additive category is a ringoid in which finite sums (coproducts) exist. A ring is a ringoid with exactly one object. By (1.14) we obtain a "functor" which carries $H \in \mathbf{Grd}(G)$ to the additive category $\mathbf{mod}(H)$ and which carries $u : H \to K \in \mathbf{Grd}(G)$ to u_* in (1.15). Here, however, $\mathbf{mod}(H)$ is not a small category. We therefore choose a subset

$$\mathcal{A} \subset \mathrm{Ob}\,\mathbf{Set}_{\mathrm{Ob}(G)} \tag{1.16}$$

that is, $\mathcal{A}$ is a set of elements α where $\alpha : Z_\alpha \to \mathrm{Ob}(G)$ is a function on a set Z_α. Let **Ringoids** be the category of small ringoids and additive functors. Then we obtain the enveloping functor

$$U_{\mathcal{A}} : \mathbf{Grd}(G) \to \mathbf{Ringoids} \tag{1.17}$$

which carries H to the full subcategory of $\mathbf{mod}(H)$ consisting of free H-modules $L_\alpha(H)$ with $\alpha \in \mathcal{A}$. Moreover $U_{\mathcal{A}}$ carries $u : H \to K \in \mathbf{Grd}(G)$ to the induced map $u_* : U_{\mathcal{A}}(H) \to U_{\mathcal{A}}(K)$ which is the restriction of u_* in (1.15).

(1.18) Example. If $\mathcal{A}$ in (1.16) is a set which has only one element $\alpha, \mathcal{A} = \{\alpha\}$, then $U_{\mathcal{A}}(H)$ is a ring. In particular we consider the case that $G = *$ is the trivial groupoid (consisting of one object $*$ and one morphism 1_*) and that $\mathcal{A}$ consists of the element α which is the identity of $\mathrm{Ob}(G) = *$. Then $U_{\mathcal{A}}$ in (1.17) yields as a special case the enveloping functor

$$U : \mathbf{Gr} \to \mathbf{Rings}$$

Here **Gr** is the category of groups which coincides with $\mathbf{Grd}(*)$ and **Rings** is the category of rings. Moreover U carries the group H to the group ring $\mathbb{Z}[H]$. Therefore the enveloping functor $U_{\mathcal{A}}$ in (1.17) is a canonical generalization of the well known group ring functor $H \mapsto \mathbb{Z}[H]$.

It is possible to describe free H-modules in $\mathbf{mod}\,(H)$ by use of homotopy groups. Let $G = \Pi D$ be the fundamental groupoid of the space D. Then any function $\alpha : Z \to D = \mathrm{Ob}(G)$ where Z is a discrete set yields the following push out diagram in **Top**

$$\begin{array}{ccc} Z \times S^n & \longrightarrow & S^n_\alpha \\ \cup & & \cup \\ Z \times * = Z & \xrightarrow{\alpha} & D \end{array} \tag{1.19}$$

We call S^n_α the n-dimensional spherical object in $\mathbf{Top}^D_c$ associated to α. The projection $Z \times S^n \to Z$ induces the retraction $0 : S^n_\alpha \to D$. Moreover S^n_α for $n \geq 1$ is a cogroup object in $\mathbf{Top}^D_c/\simeq$ which is abelian for $n \geq 2$. For objects $D \rightarrowtail X$ and

$D \rightarrowtail Y$ we define the <u>sum</u> $X \vee Y$ in $\mathbf{Top}_c^D$ by the push out of $X \leftarrow D \rightarrow Y$. In particular we may consider the sum $S_\alpha^n \vee X$ and the retraction map

$$(0,1) : S_\alpha^n \vee X \to X$$

which is a map in $\mathbf{Top}_c^D$. For a basepoint $a_0 \in D$ we define

$$\pi_n(S_\alpha^n \vee X, a_0)_2 = \operatorname{kernel}\{(0,1)_* : \pi_n(S_\alpha^n \vee X, a_0) \to \pi_n(X, a_0)\} \tag{1.20}$$

Using (1.11) each track $t : b_0 \to a_0 \in \Pi X$ between points $a_0, b_0 \in D$ yields an induced map $t^\sharp : \pi_n(S_\alpha^n \vee X, a_0)_2 \to \pi_n(S_\alpha^n \vee X, b_0)_2$. This shows that (1.20) defines a $\Pi(X, D)$-module $\pi_n(S_\alpha^n \vee X)$ which carries $a_0 \in D$ to the abelian group (1.20), $n \geq 2$.

(1.21) Proposition. *For $H = \Pi(X, D)$ and $n \geq 2$ the free H-module $L_\alpha(H)$ coincides with the $\Pi(X, D)$ module $\Pi_n(S_\alpha^n \vee X)_2$. Moreover given a map $f : X \to Y$ in $\mathbf{Top}_c^D$ which induces $u : H = \Pi(X, D) \to K = \Pi(Y, D)$ the following diagram commutes; see (1.15).*

$$\begin{tikzcd}
L_\alpha(H) \arrow[r, equal] \arrow[d, "u_\alpha"'] & \pi_n(S_\alpha^n \vee X)_2 \arrow[d, "(1 \vee f)_*"] \\
L_\alpha(K) \arrow[r, equal] & \pi_n(S_\alpha^n \vee Y)_2
\end{tikzcd}$$

A relative CW-complex (X, D) with $G = \Pi(D)$ which is reduced and normalized yields for $n \geq 1$ functions

$$\alpha_n : Z_n \to D = \mathrm{Ob}(G) \tag{1.22}$$

where Z_n is the set of n-cells in $X - D$. In fact, each n-cell $e \in Z_n$ has an attaching map which carries the basepoint $* \in S^{n-1}$ to a point $\alpha_n(e) \in D$. We point out that the restricted fundamental groupoid

$$H = \Pi(X, D) = \Pi(X_2, D) \tag{1.23}$$

depends only on the 2-skeleton of X. This follows from the cellular approximation theorem. The attaching map of 2-cells yields a map $\partial_X : S_{\alpha_2}^1 \to X_1$ which induces

$$\partial_X : \Pi(S_{\alpha_2}^1, D) \to \Pi(X_1, D) \in \mathbf{Grd}(G). \tag{1.24}$$

This is a <u>presentation</u> of H in the sense that

$$H = \Pi(X_1, D)/N(\operatorname{image}(\partial_X))$$

Here $N(\operatorname{image}(\partial_X))$ denotes the normal closure of image (∂_X). These facts are well known if $D = *$ is a point. Since (X_1, D) is reduced we see that X_1 is obtained from D by attaching 1-cells. This implies that $\Pi(X_1, D)$ is the <u>free groupoid under</u>

ΠD. See (I.2.10) below. This free groupoid admits a coaction induced by the coaction map

$$\mu : X_1 \to X_1 \vee S^1_{\alpha_1} \quad \text{in } \mathbf{Top}^D_c/\simeq \tag{1.25}$$

which is defined on each 1-cell by the map

$$\mu : [0,1] \to [0,1] \cup_{\{1\}} ([1,2]/\{1,2\})$$

obtained from the canonical homeomorphism $[0,1] \approx [0,2]$ of intervals in $\mathbb{R}$. Here 1 is the basepoint of $S^0 = \{0,1\} \subset B^1 = [0,1]$.

A chain complex C_* in an additive category $\mathbf{A}$ is a sequence of maps

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots, \qquad n \in \mathbb{Z},$$

in $\mathbf{A}$ with $dd = 0$. Chain maps and homotopies of chain maps in $\mathbf{A}$ are defined in the usual way.

(1.26) Definition. Let (X,D) be a relative CW-complex which is normalized and reduced so that $\alpha_n : Z_n \to D$ is defined for $n \geq 1$; see (1.22). Let $G = \Pi D$ and let $H = \Pi(X_2, D)$ be the restricted fundamental groupoid of X_2. Then there is a well defined chain complex (see (1.27))

$$\begin{cases} C_*(X,D) \quad \text{in } \mathbf{mod}(H) \quad \text{with} \\ C_n(X,D) = L_{\alpha_n}(H) \quad \text{for } n \geq 1 \end{cases} \tag{1}$$

and $C_n(X,D) = 0$ for $n \leq 0$. Moreover a cellular map $f : (X,D) \to (Y,D)$ under D induces a map

$$u : H = \Pi(X_2, D) \to K = \Pi(Y_2, D) \in \mathbf{Grd}(G)$$

and a chain map

$$f_* : u_*(C_*(X,D)) \to C_*(Y,D) \tag{2}$$

in $\mathbf{mod}(K)$. Here we use u_* in (1.15). If D is a discrete space we define

$$\alpha_0 : Z_0 = D$$

by the identity of D; in this case there is a well defined augmented chain complex

$$\begin{cases} C_*(X) = \operatorname{aug} C_*(X,D) \quad \text{in } \mathbf{mod}(H) \quad \text{with} \\ C_n(X) = L_{\alpha_n}(H) \quad \text{for } n \geq 0 \end{cases} \tag{3}$$

and $C_n(X) = 0$ for $n < 0$. A cellular map f as above induces f_* on $C_*(X)$ as in (2) such that f_* is the identity in degree 0. If $D = *$ is a point then $C_*(X)$ coincides with the cellular chain complex of the universal covering of the space X. We get $C_*(X,D)$ by the general procedure in (V, § 2). The augmentation functor aug is described in (II, § 6). In (1.27) we recall the classical method to obtain $C_*(X,D)$.

If (X, D) is any space under D for which $\pi_0 D \to \pi_0 X$ is surjective we choose a normalized reduced CW-approximation (Y, D) of (X, D). Hence in this case we can define the chains of (X, D) by the chains of (Y, D), that is:

$$\begin{cases} C_*(X, D) = C_*(Y, D) \\ C_*(X) = C_*(Y) \end{cases} \tag{4}$$

This yields below the notion of homology and cohomology of (X, D) which by standard arguments does not depend on the choice of (Y, D).

(1.27) Remark. Let $(X, D) = (X_{\geq 1}, D)$ be a relative CW-complex as in (1.26) and assume for all $v \in D$ the universal covering space $p_v : \hat{X}(v) \to X$ exists and let $\hat{X}_i(v) = (p_v)^{-1} X_i$ for $i \geq 0$. Then the chain complex $C_*(X)$ in (1.26) satisfies

$$C_n(X)(v) = H_n(\hat{X}_n(v), \hat{X}_{n-1}(v)) \tag{*}$$

where the right hand side is the singular relative homology. Equation (*) is an isomorphism of H-modules with $H = \Pi(X, D)$ and (*) is natural with respect to cellular maps $X \to Y$ under D. The isomorphism (*) follows from (5.2) in Brown-Higgins [CC]. In fact, we consider first the 'crossed complex' of $(X_{\geq 1}, D)$ given by the relative homotopy groups

$$\pi_n(X_n, X_{n-1}, v), \quad n \geq 2, \ v \in D \tag{**}$$

and the groupoid $\Pi(X_1, D)$. Then we apply the functor Δ of Brown-Higgins [CC] and we get a chain complex of H-modules which coincides with $C_*(X)$ in degree ≥ 1. If D is discrete then Δ applied to (**) yields the augmented chain complex $C_*(X)$. In this book we do not use (*) or (**) for the definition of the chain complex in (1.26) since $C_*(X_{\geq 1})$ is defined for any **T**-complex $X_{\geq 1}$ in (V, § 2).

Using the chain complexes $C_*(X, D)$ and $C_*(X)$ in $\mathbf{mod}(H)$ with $H = \Pi(X, D)$ in (1.26) we obtain for each object M of $\mathbf{Mod}(H)$ the chain complexes of abelian groups

$$\mathrm{Hom}(C_*(X, D), M) \quad \text{and} \quad \mathrm{Hom}(C_*(X), M).$$

Here Hom denotes the set of morphisms in the abelian category $\mathbf{Mod}(H)$. Hence the cohomology with coefficients in M

$$\begin{cases} H^n(X, D; M) = H^n \,\mathrm{Hom}(C_*(X, D), M) \\ \quad\; H^n(X; M) = H^n \,\mathrm{Hom}(C_*(X), M) \end{cases} \tag{1.28}$$

is defined.

Remark. Given a space X and any ΠX-module $\bar{M}$ then the singular cohomology $H^n(X, \bar{M})$ with local coefficients $\bar{M}$ is defined, see for example Spanier [AT]. Using the restriction M of $\bar{M}$ given by the inclusion $H = \Pi(X, D) \subset \Pi X$ we get the natural isomorphism

$$H^n(X, \bar{M}) = H^n(X, M)$$

where the right hand side is defined by (1.28).

On the other hand we can define the <u>homology</u>

$$H_n(X, D) = H_n(C_*(X, D)) \tag{1.29}$$

of the chain complex $C_*(X, D)$ in the abelian category $\mathbf{Mod}(H)$ with $H = \Pi(X, D)$. Hence $H_n(X, D)$ is an H-module, i.e. an object in $\mathbf{Mod}(H)$.

(1.30) Notation. Each H-module M yields a $\mathbf{mod}(H)$-module

$$\mathrm{Hom}(-, M) : \mathbf{mod}(H)^{\mathrm{op}} \to \mathbf{Ab}$$

which carries $L_\alpha(H)$ to $\mathrm{Hom}(L_\alpha(H), M)$. We denote $\mathrm{Hom}(-, M)$ as well by M. In particular $H_n(X, D)$ in (1.29) yields the $\mathbf{mod}(H)$-module

$$H_n(X, D) = \mathrm{Hom}(-, H_n(X, D)) : \mathbf{mod}(H)^{\mathrm{op}} \to \mathbf{Ab}$$

which is the homology of (X, D) considered in (V.3.3) since we have for $C_* = C_*(X, D)$ the canonical isomorphism

$$H_n \, \mathrm{Hom}(L_\alpha(H), C_*) = \mathrm{Hom}(L_\alpha(H), H_n C_*).$$

We now are ready to formulate the following <u>homological Whitehead theorem</u> for relative CW-complexes which is a special case of (VI, § 7).

(1.31) Theorem. *Let $f : (X, D) \to (Y, D)$ be a cellular map between normalized reduced relative CW-complexes in $\mathbf{Top}_c^D$. Then $f : X \to Y$ is a homotopy equivalence under D (i.e. an isomorphism in the homotopy category $\mathbf{Top}_c^D/\simeq rel\, D$) if and only if the coefficient functor c induces an isomorphism $u = c(f)$,*

$$u : H = \Pi(X, D) \xrightarrow{\cong} K = \Pi(Y, D) \in \mathbf{Grd}(G)$$

with $G = \Pi(D)$ and one of the following conditions (i), (ii), (iii) is satisfied:

(i) *$f_* : u_*(C_*(X, D)) \to C_*(Y, D)$ is a homotopy equivalence of chain complexes in $\mathbf{mod}(K)$, see (1.15).*
(ii) *$f_* : H_n(X, D) \to u^* H_n(Y, D)$ is an isomorphism of H-modules (or right $\mathbf{mod}(H)$-modules) for $n \geq 1$, see (1.30).*
(iii) *For all K-modules $N \in \mathbf{Mod}(K)$ the induced map*

$$f^* : H^n(Y, D; N) \to H^n(X, D; u^* N)$$

is an isomorphism for $n \geq 1$; see (1.28).

Part (iii) of this theorem is well known and for $D = *$ also part (i) and (ii) are well known.

We use homology (1.28) and homotopy groups (1.11) for the following certain exact sequence of J.H.C. Whitehead . Again let (X, D) be a normalized reduced relative CW-complex or more generally let (X, D) be a pair of spaces of which $\pi_0 D \to \pi_0 X$ is surjective. Let $H = \Pi(X, D)$ be the restricted fundamental groupoid. Then homotopy groups yield the H-modules (resp. $\mathbf{mod}(H)$-modules; see (1.30))

$$\begin{cases} \pi_n(X) : H^{\mathrm{op}} \to \mathbf{Ab}, & n \geq 2, \\ \Gamma_n(X, D) : H^{\mathrm{op}} \to \mathbf{Ab}, & n \geq 1, \end{cases}$$

with $\pi_n(X)(v) = \pi_n(X, v)$ for $v \in D = \mathrm{Ob}(H)$. Moreover Γ_n is defined for $n \geq 3$ by skeleta, that is

$$\Gamma_n(X, D)(v) = \mathrm{image}\{\pi_n(X_{n-1}, v) \to \pi_n(X_n, v)\}.$$

For $n = 1, 2$ the definition of Γ_n is more complicated, see (V.5.3) and (II, § 2). As a special case of (V.5.4) we get

(1.32) Theorem. *Let (X, D) be a pair of spaces for which $\pi_0 D \to \pi_0 X$ is surjective and let $H = \Pi(X, D)$. Then the following sequence is an exact sequence of H-modules (resp. right $\mathbf{mod}(H)$-modules)*

$$\longrightarrow \Gamma_n(X, D) \longrightarrow \pi_n(X) \xrightarrow{h} H_n(X, D) \longrightarrow \Gamma_{n-1}(X, D) \longrightarrow \dots$$
$$\longrightarrow \Gamma_2(X, D) \longrightarrow \pi_2(X) \xrightarrow{h} H_2(X, D) \longrightarrow \Gamma_1(X, D) \longrightarrow 0$$

Moreover this sequence is natural in $(X, D) \in \mathbf{Top}^D$. The homorphism h is the Hurewicz homomorphism.

If $D = *$ then Γ_1 and Γ_2 are trivial and in this case the theorem describes exactly J.H.C. Whitehead's result [CE].

The cohomology groups with local coefficients

$$\begin{cases} H^{n+1}(X, D; u^* \pi_n Y), \textit{ and} \\ H^{n+1}(X, D; u^* \Gamma_n(Y, D)) \end{cases}$$

are needed to define various properties of obstruction theory which we discuss in detail in (V, § 4) and chapter VI. For example we get by (V.4.4) the well known result:

(1.33) Theorem. *Let (X, D) be a normalized reduced relative CW-complex and let $f : D \to Y$ be a map in $\mathbf{Top}$ which admits an extension $g : X_n \to Y$, $n \geq 2$. Then the restriction $g \mid X_{n-1}$ admits an extension $\bar{g} : X_{n+1} \to U$ if and only if an obstruction element*

$$\mathcal{O}(g \mid X_{n-1}) \in H^{n+1}(X, D; u^* \pi_n Y)$$

vanishes. Here $u : \Pi(X, D) \to \Pi Y$ is induced by g.

We point out that this obstruction theorem requires the use of the restricted fundamental groupoid which satisfies $\Pi(X,D) = \Pi(X_2,D)$ so that the induced map u is well defined.

There is also an obstruction theory for the realizability of chain maps and chain complexes described by a tower of categories in (VI, § 5). Moreover there are the homotopy lifting property of the chain functor and the model lifting property of the twisted chain functor which have useful meaning for the chain functor in (1.26); see (VI, § 3) and (VI, § 8). We cannot describe all the results of this book in this section applied to the example $\mathbf{Top}^D$. We leave it to the reader to give the appropriate explicit interpretation in $\mathbf{Top}^D$ of the abstract results of the theory below. We here discuss only a few examples in order to illustrate the abstract theory.

As main applications of this book we now discuss special cases of results in chapter VII and VIII which relate problems of homotopy theory with algebraic K-theory.

(1.34) Definition. Let D be a space and let $\mathcal{A}$ be a set of functions α with $\alpha : Z_\alpha \to D$ where Z_α is a discrete set. We say that a function $\varphi : Z \to D$ is $\mathcal{A}$-finite if $\beta_1, \dots, \beta_k \in \mathcal{A}$ together with a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, equal, "\chi_\alpha"] \arrow[dr, "\varphi"'] & & Z_{\beta_1} \amalg \cdots \amalg Z_{\beta_k} \arrow[dl, "(\beta_1,\dots,\beta_k)"] \\ & D & \end{tikzcd}$$

of sets are given where χ_α is a bijection. Similarly we say that a normalized reduced relative CW-complex (X,D) is $\mathcal{A}$-finite if all functions $\alpha_n : Z_n \to D$, $n \geq 1$, in (1.22) are $\mathcal{A}$-finite and (X,D) is finite-dimensional.

(1.35) Examples. A) Let $D = *$ be a point and let $\mathcal{A} = \{1_*\}$ be given by the identity of $*$. Then $(X,*)$ is $\mathcal{A}$-finite if X is a finite CW-complex.

B) If D is discrete and $\mathcal{A} = \{1_D\}$ is given by the identity of D then (X,D) is $\mathcal{A}$-finite if all path components of X are finite CW-complexes with the same number of n-cells for $n \geq 1$.

Now let (X,D) and (Y,D) be normalized reduced relative CW-complexes. A domination (X,f,g,H) of Y in $\mathbf{Top}^D_c$ is given by maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y \quad \text{in } \mathbf{Top}^D_c$$

and a homotopy $H : gf \simeq 1 \ \mathrm{rel}\, D$. The domination has dimension $\leq n$ if $\dim(X,D) \leq n$ and the domination is $\mathcal{A}$-finite if (X,D) is $\mathcal{A}$-finite.

As a special case of theorem (VII.2.4) we get:

(1.36) Theorem. *Let (Y, D) be a normalized reduced relative CW-complex with restricted fundamental groupoid $K = \Pi(Y, D)$. If (Y, D) admits an $\mathcal{A}$-finite domination in $\mathbf{Top}_c^D$ then the finiteness obstruction*

$$[Y] = [C_*(Y, D)] \in K_0(U_{\mathcal{A}}(K))$$

is defined where $U_{\mathcal{A}}$ is the enveloping functor in (1.17) and K_0 is the reduced projective class group, see (VII, § 1). Moreover $[Y] = 0$ if and only if there exists an $\mathcal{A}$-finite normalized reduced relative CW-complex (X, D) and a homotopy equivalence $X \to Y$ under D.

If $D = *$ is a point this yields a classical result of Wall [FC], [FCII]; compare the first example in (1.35). If we consider the second example in (1.35) we get a new result.

The reader might wonder why we have chosen such a general form (using $U_{\mathcal{A}}$) for the description of the finiteness obstruction theorem of Wall. In fact, we describe the result here in the same way as the general result of the abstract theory which requires the enveloping functor $U_{\mathcal{A}}$. For $\mathbb{A}$-diagrams of spaces in § 2 below we shall see that the choice of the set $\mathcal{A}$ has a significant role. The same type of remark holds also for the choice of the set $\mathcal{D}$ in the next definition (1.37).

We now describe simple homotopy equivalences and Whitehead torsion under a space D:

(1.37) Definition. Let D be a space (which is allowed to be empty) and let

$$\mathbf{K} = \mathbf{Top}_c^D \tag{1}$$

be the category of cofibrations under D, see (1.4) (1). Moreover let $\mathcal{D}$ be a set of sets with the property that the empty set $\emptyset$ is in $\mathcal{D}$ and that the disjoint union $A \amalg B$ of A, $B \in \mathcal{D}$ is again in $\mathcal{D}$. Then each $A \in \mathcal{D}$ yields the disjoint union

$$A \amalg D \quad \text{in } \mathbf{Top}_c^D \tag{2}$$

which we call a <u>discrete object</u>. Here A has the discrete topology. The most important example of $\mathcal{D}$ is the set of finite sets $\{1, \dots, n\}$, $n \geq 0$. A $\mathcal{D}$-<u>complex</u> is a relative CW-complex (L, D) for which the set Z_n of n-cells in $L - D$ is an element in $\mathcal{D}$, $n \geq 0$. Let $\mathbb{R}_+^{n+1}, \mathbb{R}_-^{n+1} \subset \mathbb{R}^{n+1}$ be defined by elements $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ with $x_0 \geq 0$ and $x_0 \leq 0$ respectively. A <u>ball pair</u> is a tuple (B^{n+1}, S^n, P^n, Q^n) which is homeomorphic to the Euclidean ball pair (see (1.5))

$$(\tilde{B}^{n+1}, \tilde{S}^n, \tilde{S}^n \cap \mathbb{R}_+^{n+1}, \tilde{S}^n \cap \mathbb{R}_-^{n+1}) \tag{3}$$

Here $P^n \cap Q^n = S^{n-1}$ is a sphere and we assume that the basepoint of B^{n+1} is an element in $P^n \cap Q^n$. For $A \in \mathcal{D}$ we consider a push out diagram

$$\begin{array}{ccc} A \times B^{n+1} & \longrightarrow & K \\ \cup & & \cup \\ A \times P^n & \xrightarrow{f} & L \end{array} \tag{4}$$

where f is given by a pair map $f : (A \times P^n, A \times S^{n-1}) \to (L_n, L_{n-1})$. Then (K, D) is again a $\mathcal{D}$-complex which we call an elementary expansion of L. Clearly $L \subset K$ is a homotopy equivalence under D and we call a retraction $r : K \to L$ an elementary collapse. A simple homotopy equivalence $f : L \to L'$ under D is obtained by a finite sequence of elementary expansions and collapses respectively.

Let $\mathcal{D}$-**cell** be the full subcategory of $\mathbf{Top}_c^D$ consisting of finite dimensional $\mathcal{D}$-complexes (L, D). In (VIII, § 8) we define a functor

$$\mathrm{Wh} : \mathcal{D}\text{-}\mathbf{cell}/\simeq \mathrm{rel}\, D \to \mathbf{Ab}$$

which carries (L, D) to the Whitehead group $\mathrm{Wh}(L, D)$. As a special case of (VIII.8.3) we get the following result.

(1.38) Theorem. *There is a function τ assigning to any homotopy equivalence $f : Y \to L$ under D between finite dimensional $\mathcal{D}$-complexes Y, L an element $\tau(f) \in \mathrm{Wh}(L, D)$. Moreover $\tau(f) = 0$ if and only if f is homotopic* $\mathrm{rel}\, D$ *to a simple homotopy equivalence under D.*

The Whitehead group $\mathrm{Wh}(L, D)$ can be computed algebraically by the following result which is a special case of (VIII.12.7).

(1.39) Theorem. *Let (L, D) be a normalized finite dimensional $\mathcal{D}$-complex and let $H = \Pi(L, L_0)$ be the restricted fundamental groupoid. Then the algebraic Whitehead group*

$$\mathrm{Wh}(H) = K_1^{\mathrm{iso}}(U_{\mathcal{A}}(H))/\sim$$

is defined. Here $\mathcal{A}$ is the set of all functions $A \to L_0$ with $A \in \mathcal{D}$, the functor $U_{\mathcal{A}}$ is the enveloping functor in (1.17) and K_1^{iso} is the "isomorphism torsion group" in (VIII, § 10). Moreover there is an isomorphism

$$\tau : \mathrm{Wh}(L, D) \cong \mathrm{Wh}(H)$$

We now consider the special case that $D = \emptyset$ is empty and $L_0 = *$ is a point so that $H = \pi_1 L$ is the fundamental group. Moreover let $\mathcal{D}$ be the class of finite sets so that $U_{\mathcal{A}}(H)$ is the additive category of finite dimensional free $\mathbb{Z}[\pi_1 L]$-modules. In this case the theorems (1.38), (1.39) coincide with the classical results of J.H.C. Whitehead [SH] on simple homotopy equivalences; compare Cohen [C].

All the results in this section are examples and applications of the results of the general theory in the chpaters I, ... , VIII below. In order to translate the general theory to the special homotopy theory in $\mathbf{Top}^D$ one has to use the following glossary where on the left hand side we use the notation of the general theory.

T (I.1.11)	Category of coactions given by the full subcategory of $\mathbf{Top}_c^D/\simeq$ of reduced 1-dimensional relative CW-complexes (X_1, D). This is also the category of free groupoids under $\Pi D = G$. Cogroups in $\mathbf{T}$ are spherical objects S^1_α and the coaction map is given by (1.25).
Twist (I.3.5)	Category of presentation ∂_X, generalizes the category of free "pre crossed modules".
∂_X	Presentation of a groupoid H in $\mathbf{Grd}(G)$.
Coef **(I.4.1)**	This is a category equivalent to $\mathbf{Grd}(G)$. The equivalence carries the presentation ∂_X of H to H.
$\mathbf{mod}(\partial_X)$ (I.5.7)	This is the category $\mathbf{mod}(H)$ in (1.14) where ∂_X is a presentation of H. Here we use (1.21).
$U_{\mathcal{A}}$ (I.5.11)	This is the enveloping functor $U_{\mathcal{A}}$ in (1.17). Here we identify $\alpha \in \mathcal{A}$ with the spherical object $S^1_\alpha \in \mathbf{T}$.
$(\mathbf{C}, \mathbf{T})$ (V.11)	$(\mathbf{Top}_c^D, \mathbf{T})$ is a homological cofibration category if D is non empty. Here $\mathbf{T}$ is defined above.
Complex (IV.2.2)	This is the subcategory of $\mathbf{Top}_c^D$ consisting of normalized reduced relative CW-complexes $(X_{\geq 1}, D)$ and cellular maps.
C_* (V.2.3)	This is the chain functor in (1.26).
$(\mathbf{C}, \mathcal{D})$ (VIII.5.1) (VIII.12.3)	$\mathbf{Top}_c^D$ is a homological cellular I-category with the cylinder in (1.4) (3) and the class $\mathcal{D}$ of discrete objects in (1.37).
$(\square_x^{k+1}, \Sigma_X^k, P_X^k, Q_X^k)$ (VIII.4.5)	For X in $\mathbf{Top}_c^D$ this is the push out of $B \times X \supset B \times D \xrightarrow{pr} D$ where pr is the projection and $B = (B^{k+1}, S^k, P^k, Q^k)$ is the ball pair in (1.37) (3).

It is very useful to have these examples in mind in order to visualize the abstract and categorical theory in the second part of the book.

2 Homotopy Theory of Diagrams of Spaces

Let $\mathbb{A}$ be a fixed small category. For objects $a, b \in \mathbb{A}$ let $\mathbb{A}(a, b)$ be the set of morphisms (arrows) $a \to b$ in $\mathbb{A}$. If $\mathbf{C}$ is a category then an $\mathbb{A}$-diagram or an $\mathbb{A}$-object X in $\mathbf{C}$ is a functor

$$X : \mathbb{A}^{\text{op}} \to \mathbf{C}, \tag{2.1}$$

i.e. a contravariant functor from $\mathbb{A}$ to $\mathbf{C}$. Let $\mathbb{A}\mathbf{C}$ be the category of such $\mathbb{A}$-objects in $\mathbf{C}$; morphisms are natural tarnsformations. An object X in $\mathbf{C}$ yields the constant $\mathbb{A}$-object (also denoted by X) which carries each object in $\mathbb{A}$ to X and each morphism in $\mathbb{A}$ to the identity of X. This way we obtain the inclusion of categories

$$\mathbf{C} \subset \mathbb{A}\mathbf{C}$$

which carries the object X to the constant $\mathbb{A}$-object given by X.

In particular we need the category $\mathbb{A}\,\mathbf{Set}$ of $\mathbb{A}$-sets. We consider $\mathbf{Set}$ as a subcategory of the category $\mathbf{Top}$ of topological spaces by taking the discrete topology of a set. This yields also the inclusion of categories

$$\mathbb{A}\,\mathbf{Set} \subset \mathbb{A}\,\mathbf{Top} \tag{2.2}$$

where $\mathbb{A}\,\mathbf{Top}$ is the category of $\mathbb{A}$-spaces. We say that an object in $\mathbb{A}\,\mathbf{Set}$ is a discrete $\mathbb{A}$-space.

Notice that the notions of coproduct, product, pushout, pullback, colimit, and limit exist in the category $\mathbb{A}\,\mathbf{Set}$ and $\mathbb{A}\,\mathbf{Top}$ respectively. They are constructed by applying these notions objectwise in $\mathbf{Set}$, resp. $\mathbf{Top}$.

For each object a in $\mathbb{A}$ we have the $\mathbb{A}$-set

$$\mathbb{A}(-,a) : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set} \tag{2.3}$$

which carries $b \in \mathrm{Ob}(\mathbb{A})$ to the set $\mathbb{A}(b,a)$ of arrows in $\mathbb{A}$. We call the $\mathbb{A}$-set $\mathbb{A}(-,a)$ an $\mathbb{A}$-point. A coproduct of $\mathbb{A}$-points over an index set M,

$$Z = \coprod_{i \in M} \mathbb{A}(-,a_i) \tag{1}$$

in $\mathbb{A}\,\mathbf{Set}$, is termed a free $\mathbb{A}$-set. Let

$$\mathbb{A}\,\mathbf{set} \subset \mathbb{A}\,\mathbf{Set} \tag{2}$$

be the full subcategory consisting of free $\mathbb{A}$-sets.

(2.4) Remark. There is a covariant version of the theory which considers covariant diagrams $\mathbb{A} \to \mathbf{C}$ and for which $\mathbb{A}$-points in (2.3) are replaced by the covariant functors $\mathbb{A}(a,-)$. Accordingly all definitions and results below have a covariant analogue.

In this section we describe basic results of homotopy theory in $\mathbb{A}\,\mathbf{Top}$. A homotopy or more precisely an $\mathbb{A}$-homotopy between $\mathbb{A}$-spaces X, Y is a map

$$[0,1] \times X \to Y \quad \text{in } \mathbb{A}\,\mathbf{Top} \tag{2.5}$$

where $[0,1]$ is the constant $\mathbb{A}$-space given by the unit interval $[0,1] \subset \mathbb{R}$. Equivalently $[0,1] \times X$ is the composite of functors

$$\mathbb{A}^{\mathrm{op}} \xrightarrow{X} \mathbf{Top} \xrightarrow{I} \mathbf{Top}$$

where I with $I(Y) = [0,1] \times Y$ is the cylinder in $\mathbf{Top}$. Such homotopies are free homotopies or homotopies relative the empty $\mathbb{A}$-space $\emptyset$. As in (1.3) we have to consider homotopies relative a non-empty $\mathbb{A}$-space D in order to obtain algebraic objects like homotopy groups. In particular the case when D is a discrete $\mathbb{A}$-space is of interest. The example of $\mathrm{Or}(G)$-spaces for a topological group G in the next section § 3, however, shows that also the non-discrete case plays an important role. Let

$$\mathbf{C} = (\mathbb{A}\,\mathbf{Top})^D \tag{2.6}$$

be the category of $\mathbb{A}$-spaces under D. Using the cylinder (2.5) we define cofibrations in $\mathbb{A}\,\mathbf{Top}$ by the $\mathbb{A}$-homotopy extension property. They yield the full subcategory

$$\mathbf{C}_c = (\mathbb{A}\,\mathbf{Top})^D_c \tag{1}$$

consisting of cofibrations $D \rightarrowtail X$ in $\mathbb{A}\,\mathbf{Top}$. Accordingly we obtain the homotopy category

$$\mathbf{C}_c/\simeq\, = (\mathbb{A}\,\mathbf{Top})^D_c/\simeq \operatorname{rel} D \tag{2}$$

where homotopy relative D is defined by the cylinder object $I(X,D)$ defined by the push out in $\mathbb{A}\,\mathbf{Top}$

$$\begin{array}{ccc} [0,1]\times X & \longrightarrow & I(X,D) \\ \uparrow & & \uparrow \\ [0,1]\times D & \xrightarrow{pr} & D \end{array} \tag{3}$$

as in (1.4) (3).

We say that an $\mathbb{A}$-space Y is obtained from an $\mathbb{A}$-space X by <u>attaching</u> $(n+1)$-<u>cells</u> if a free $\mathbb{A}$-set Z together with a pushout diagram in $\mathbb{A}\,\mathbf{Top}$

$$\begin{array}{ccc} Z\times B^{n+1} & \longrightarrow & Y \\ \cup & & \cup \\ Z\times S^n & \xrightarrow{f} & X \end{array} \tag{2.7}$$

is given. Here S^n and B^{n+1} are the constant $\mathbb{A}$-spaces given by (1.5). A <u>relative</u> $\mathbb{A}$-<u>CW-complex</u> $(X_{\geq 0}, D) = (X, D)$ is given by a sequence of inclusions

$$D \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \ldots \tag{2.8}$$

in $\mathbb{A}\,\mathbf{Top}$. Here X_0 is the coproduct of D and a free $\mathbb{A}$-set and X_{n+1} is obtained from X_n by attaching $(n+1)$-cells for $n \geq 0$. Let $X = \lim(X_{\geq 0})$ be the direct limit of the sequence. We say that (X, D) is <u>reduced</u> if $X_0 = D$ and that (X, D) is <u>normalized</u> if the attaching maps

$$f_n : Z_n \times S^{n-1} \to X_{n-1} \tag{1}$$

carry $Z_n \times *$ to X_0, $n \geq 1$. Here the free $\mathbb{A}$-set Z_n is called the <u>set of n-cells</u> of the $\mathbb{A}$-CW-complex (X, D). We point out that for a space U in $\mathbf{Top}$ and an $\mathbb{A}$-space Y we have

$$\mathbb{A}\,\mathbf{Top}(\mathbb{A}(-,a)\times U, Y) = \mathbf{Top}(U, Y(a)). \tag{2}$$

Hence the attaching map f_n above is for each $\mathbb{A}$-point $\mathbb{A}(-,a) \in Z_n$ defined by a map $S^{n-1} \to X_{n-1}(a)$ in **Top** which is the attaching map of a generating cell in the relative CW-complex $(X_n(a), D(a))$. Using such generating cells we see that obstruction theory for $\mathbb{A}$-spaces X can be described by ordinary homotopy groups of the spaces $X(a), a \in \mathrm{Ob}(\mathbb{A})$. This in particular implies that the cellular approximation theorem holds for $\mathbb{A}$-CW-complexes. The next result is an analogue of (1.8).

(2.9) Lemma. *Let (X, D) be a relative $\mathbb{A}$-CW-complex. Then there exists a normalized relative $\mathbb{A}$-CW-complex (Y, D) together with a homotopy equivalence $Y \to X$ in $(\mathbb{A}\,\mathbf{Top})_c^D/\simeq \mathrm{rel}\, D$. Moreover if $\pi_0 D \to \pi_0 X$ is surjective then (Y, D) can be chosen to be normalized and reduced.*

Proof. Since the cellular approximation theorem holds we can find homotopies of attaching maps $f_n \simeq g_n$ where g_n carries $Z_n \times *$ to X_0. This yields inductively the $\mathbb{A}$-CW-complex (Y, D). If $\pi_0 D \to \pi_0 X$ is surjective we can choose a path for each generating 0-cell $\mathbb{A}(-,a)$ in X to a point in D. We glue a ball pair $\mathbb{A}(-,a) \times (B^2, S^1, P, Q)$ via Q to this path and we collapse P to a point. The resulting space (Y, D) is a reduced $\mathbb{A}$-CW-complex. q.e.d.

For each $\mathbb{A}$-space X one gets the $\mathbb{A}$-groupoid $H = \Pi X$ which is given by the composite of functors

$$H : \mathbb{A}^{\mathrm{op}} \xrightarrow{X} \mathbf{Top} \xrightarrow{\Pi} \mathbf{Grd} \tag{2.10}$$

Here the functor Π carries a space U to the fundamental groupoid of U. We use the $\mathbb{A}$-groupoid ΠX to define the following category $\int_{\mathbb{A}} \Pi X$ which we call the integrated fundamental groupoid of the $\mathbb{A}$-space X (compare § 2 in Moerdijk-Svenson [D]). The category

$$\int_{\mathbb{A}} H = \int_{\mathbb{A}} \Pi X \tag{1}$$

is the integration of $H = \Pi X$ along $\mathbb{A}$ which assembles the diagram of categories (2.10) into one large category. The objects are pairs (a, x) where $a \in \mathrm{Ob}(\mathbb{A})$ and $x \in X(a) = \mathrm{Ob}(\Pi X)(a)$. An arrow $(a, x) \to (a', x')$ between such objects is a pair (α, t) where $\alpha : a \to a'$ is an arrow in $\mathbb{A}$ and $t : x \to X(\alpha)(x') \in X(a)$ is an arrow in $\Pi X(a)$. Composition is defined in the evident way.

If X is an $\mathbb{A}$-space under D we also obtain the functor $H' = \Pi(X, D)$: $\mathbb{A}^{\mathrm{op}} \to \mathbf{Grd}$ which carries $a \in \mathrm{Ob}(\mathbb{A})$ to the restricted fundamental groupoid $\Pi(X(a), D(a))$. We clearly have the inclusion

$$\Pi(X, D) = H' \subset H = \Pi X \tag{2}$$

of $\mathbb{A}$-groupoids which yields the inclusion of integrations along $\mathbb{A}$

$$\int_{\mathbb{A}} \Pi(X,D) = \int_{\mathbb{A}} H' \subset \int_{\mathbb{A}} H = \int_{\mathbb{A}} \Pi X \tag{3}$$

If the morphism $\pi_0 D \to \pi_0 X$ of $\mathbb{A}$-sets is surjective (that is, if each path component of $X(a), a \in \operatorname{Ob}\mathbb{A}$, contains a point of $D(a)$) then the inclusion is an equivalence of categories. The assumption that $\pi_0 D \to \pi_0 X$ is surjective will be used frequently.

Homotopy groups (1.9) yield the canonical functor

$$\pi_n(X) : \Big(\int_{\mathbb{A}} \Pi X\Big)^{\mathrm{op}} \to \mathbb{A}b, \quad n \geq 2, \tag{2.11}$$

which carries (a, x) to $\pi_n(X(a), x)$ and which carries (α, t) to the induced map

$$\pi_n(X(a'), x') \xrightarrow{X(\alpha)_*} \pi_n(X(a), X(\alpha)(x')) \xrightarrow{t^\sharp} \pi_n(X(a), x)$$

Hence the homotopy group $\pi_n X$ of the $\mathbb{A}$-space X is a $\int_{\mathbb{A}} \Pi X$-module in the category $\mathbf{Mod}(\int_{\mathbb{A}} \Pi X)$; compare (1.12). In a similar way we see that for a pair (Y, X) in $\mathbb{A}\mathbf{Top}$ the relative homotopy group $\pi_{n+1}(Y, X)$ is a $\int_{\mathbb{A}} \Pi X$-module.

A weak equivalence in $(\mathbb{A}\mathbf{Top})^D$ is a map $f : X \to Y$ under D in $\mathbb{A}\mathbf{Top}$ which induces isomorphisms between homotopy groups

$$f_* : \pi_n(X(a), a_0) \cong \pi_n(Y(a), fa_0)$$

for all $a \in \operatorname{Ob}\mathbb{A}, a_0 \in X(a)$, $n \geq 0$. It is known (see for example Dror [HH]) that each $\mathbb{A}$-space Y under D admits a weak equivalence $f : X \to Y$ under D where (X, D) is a relative $\mathbb{A}$-CW-complex termed an $\mathbb{A}$-CW-approximation of (Y, D). It is easy to show that a weak equivalence $f : (X, D) \to (Y, D)$ under D between relative $\mathbb{A}$-CW-complexes is actually a homotopy equivalence in $(\mathbb{A}\,\mathbf{Top})^D_c/\!\simeq \operatorname{rel} D$; see (IV, § 3).

For example, if $D = \emptyset$ is empty and Y is a discrete diagram then an $\mathbb{A}$-CW-approximation EY of Y yields the classifying space $BY = EY/\!\sim$ where the equivalence relation on EY is generated by $x \sim \alpha^* x$ for $\alpha : a \to b$ in $\mathbb{A}$ and $x \in (EY)(b)$.

The $\mathbb{A}$-CW-approximation yields the equivalence of categories

$$\operatorname{Ho}(\mathbb{A}\mathbf{Top})^D \xrightarrow{\sim} \mathbb{A}\text{-}\mathbf{CW}^D/\!\simeq \operatorname{rel} D \tag{2.12}$$

Here the left hand side is the localization defined by inverting weak equivalences and the right hand side is the full subcategory of $(\mathbb{A}\mathbf{Top})^D_c/\!\simeq \operatorname{rel} D$ consisting of relative $\mathbb{A}$-CW-complexes (X, D). The equivalence shows that each homotopy functor defined on relative $\mathbb{A}$-CW-complexes (X, D) yields a homotopy functor on $(\mathbb{A}\mathbf{Top})^D$. Therefore it is sufficient to define homology and cohomology only for $\mathbb{A}$-CW-complexes. For an $\mathbb{A}$-groupoid G let $\mathbb{A}\mathbf{Grd}(G)$ be the following category which is a subcategory of $(\mathbb{A}\mathbf{Grd})^G$. Objects are maps $f : G \to H$ in $\mathbb{A}\mathbf{Grd}$ which induce the identity

$$\operatorname{Ob}(f) = 1 : \operatorname{Ob}(G) = \operatorname{Ob}(H)$$

where $\mathrm{Ob} : \mathbf{Grd} \to \mathbf{Set}$ carries a groupoid to its set of objects. Morphisms are maps in $\mathbb{A}\mathbf{Grd}$ under G. For each cofibration $D \rightarrowtail X$ in $\mathbb{A}\mathbf{Top}$ we obtain the object

$$c(X) = (\Pi D \to \Pi(X, D))$$

in $\mathbb{A}\mathbf{Grd}(\Pi D)$. This defines the coefficient functor

$$c : (\mathbb{A}\mathbf{Top})_c^D/\simeq \mathrm{rel}\, D \to \mathbb{A}\mathbf{Grd}(\Pi D) \tag{2.13}$$

For each $\mathbb{A}$-groupoid H we have integration $\int_{\mathbb{A}} H$ and the abelian category $\mathbf{Mod}(\int_{\mathbb{A}} H)$ of $(\int_{\mathbb{A}} H)$-modules. We now define the full subcategory

$$\mathbf{mod}\Big(\int_{\mathbb{A}} H\Big) \subset \mathbf{Mod}\Big(\int_{\mathbb{A}} H\Big) \tag{2.14}$$

consisting of free $(\int_{\mathbb{A}} H)$-modules. For this we use the category $(\mathbb{A}\mathbf{Set})_{\mathrm{Ob}\, H}$ of $\mathbb{A}$-sets over the $\mathbb{A}$-set $\mathrm{Ob}\, H$ given by (2.12). We have the forgetful functor

$$\varphi : \mathbf{Mod}\Big(\int_{\mathbb{A}} H\Big) \to (\mathbb{A}\mathbf{Set})_{\mathrm{Ob}\, H} \tag{1}$$

which carries $F : \int_{\mathbb{A}} H \to \mathbf{Ab}$ to the $\mathbb{A}$-set over $\mathrm{Ob}\, H$ given by

$$f(a) : \coprod_{x \in \mathrm{Ob}\, H(a)} F(a, x) \to \mathrm{Ob}\, H(a) \in \mathbf{Set} \tag{2}$$

for $a \in \mathrm{Ob}(\mathbb{A})$. Here $f(a)$ is the function which satisfies $f(a)(y) = x$ for $y \in F(a, x)$. Let $L(H)$ be the left adjoint of φ. Moreover consider a map

$$\alpha : Z \to \mathrm{Ob}\, H \quad \text{in } \mathbb{A}\,\mathbf{Set} \tag{3}$$

where Z is a free $\mathbb{A}$-set (2.3) (2). Then we call $L_\alpha(H) = L(H)(\alpha) \in \mathbf{Mod}(\int_{\mathbb{A}} H)$ the free $(\int_{\mathbb{A}} H)$-module generated by α. Let $\mathbf{mod}(\int_{\mathbb{A}} H)$ be the full subcategory in (2.14) consisting of such free modules.

Now let G be a fixed $\mathbb{A}$-groupoid. Each morphism $u : H \to K \in \mathbb{A}\mathbf{Grd}(G)$ yields a canonical functor

$$u_* : \mathbf{mod}\Big(\int_{\mathbb{A}} H\Big) \to \mathbf{mod}\Big(\int_{\mathbb{A}} K\Big) \tag{2.15}$$

which carries $L_\alpha(H)$ to $L_\alpha(K)$ where $\mathrm{Ob}(H) = \mathrm{Ob}(K) = \mathrm{Ob}(G)$. Moreover one has the commutative diagram in $\mathbf{Mod}(\int_{\mathbb{A}} H)$

$$\begin{tikzcd}
L_\alpha(H) \arrow[r, "\alpha"] \arrow[d, "u_\alpha"'] & L_\beta(H) \arrow[d, "u_\beta"] \\
L_\alpha(K) \arrow[r, "u_*(a)"] & L_\beta(K)
\end{tikzcd}$$

as in (1.15). Here $L_\alpha(K)$ is a $(\int_{\mathbb{A}} H)$-module via the induced map $\int_{\mathbb{A}} u : \int_{\mathbb{A}} H \to \int_{\mathbb{A}} K$ on integrations along $\mathbb{A}$ given by u. For H in $\mathbb{A}\mathbf{Grd}(G)$ we choose a subset

$$\mathcal{A} \subset \operatorname{Ob}\mathbf{mod}\Big(\int_{\mathbb{A}} H\Big) \tag{2.16}$$

that is, $\mathcal{A}$ is a set of elements α where $\alpha : Z_\alpha \to \operatorname{Ob}(H) = \operatorname{Ob}(G)$ is a function in $\mathbb{A}$-**Set** defined on a free $\mathbb{A}$-set Z_α. We define the <u>enveloping functor</u>

$$U_{\mathcal{A}} : \mathbb{A}\mathbf{Grd}(G) \to \mathbf{Ringoids} \tag{2.17}$$

which carries H to the full subcategory of $\mathbf{mod}(\int_{\mathbb{A}} H)$ consisting of free objects $L_\alpha(H)$ with $\alpha \in \mathcal{A}$. Moreover $U_{\mathcal{A}}$ carries $u : H \to K \in \mathbb{A}\mathbf{Grd}(G)$ to the induced map $u_* : U_{\mathcal{A}}(H) \to U_{\mathcal{A}}(K)$ which is the restriction of u_* in (2.15).

It is possible to describe the free $(\int_{\mathbb{A}} H)$-modules in $\mathbf{mod}(\int_{\mathbb{A}} H)$ by use of homotopy groups. For this we assume that an $\mathbb{A}$-space $D \rightarrowtail X$ under an $\mathbb{A}$-space D is given and that $H = \Pi(X, D)$. Then any function α as in (2.14) (3) yields the following push out in $\mathbb{A}\mathbf{Top}$

$$\begin{array}{ccc} Z \times S^n & \longrightarrow & S^n_\alpha \\ \cup & & \cup \\ Z \times * = Z & \xrightarrow{\alpha} & D \end{array} \tag{2.18}$$

We call S^n_α the n-dimensional <u>spherical object</u> in $(\mathbb{A}\mathbf{Top})^D_c$ associated to α. The projection $Z \times S^n \to Z$ induces the retraction $0 : S^n_\alpha \to D$. Moreover S^n_α for $n \geq 1$ is a cogroup object in $(\mathbb{A}\mathbf{Top})^D_c/\simeq$ which is abelian for $n \geq 2$. For the sum $S^n_\alpha \vee X$ in $(\mathbb{A}\mathbf{Top})^D_c$ we obtain the retraction map

$$(0, 1) : S^n_\alpha \vee X \to X$$

which is a map in $(\mathbb{A}\mathbf{Top})^D_c$. Now we define the $\int_{\mathbb{A}} H = \int_{\mathbb{A}} \Pi(X, D)$-module

$$\pi_n(S^n_\alpha \vee X)_2 = \operatorname{kernel}\{\pi_n(S^n_\alpha \vee X) \xrightarrow{(0,1)_*} \pi_n(X)\} \tag{2.19}$$

by use of (2.11), $n \geq 2$.

(2.20) Proposition. *For $H = \Pi(X, D)$ the free $(\int_{\mathbb{A}} H)$-module $L_\alpha(H)$ coincides with $\pi_n(S^n_\alpha \vee X)_2$ for $n \geq 2$. Moreover given $f : X \to Y$ in $(\mathbb{A}\mathbf{Top})^D_c$ we obtain the induced map*

$$u : H = \Pi(X, D) \to K = \Pi(Y, D)$$

in $\mathbb{A}\mathbf{Grd}(\Pi D)$ for which the following diagram commutes; see (2.15).

$$\begin{array}{ccc} L_2(H) & = & \pi_n(S^n_\alpha \vee X)_2 \\ {\scriptstyle u_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle (1\vee f)_*} \\ L_\alpha(K) & = & \pi_n(S^n_\alpha \vee Y)_2 \end{array}$$

Let (X, D) be a relative $\mathbb{A}$-CW-complex which is reduced and normalized where D is an $\mathbb{A}$-space. We obtain by the attaching maps f_n in (2.8) (1) the functions $(n \geq 1)$

$$\alpha_n : Z_n \to D \in \mathbb{A}\mathbf{Top} \tag{2.21}$$

where Z_n is the free $\mathbb{A}$-set of n-cells in (X, D). Here α_n is the restriction of f_n to $Z_n \times * \subset Z_n \times S^{n-1}$. The function α_n is well defined since we assume that (X, D) is reduced and normalized.

The cellular approximation theorem yields the following canonical isomorphism of $\mathbb{A}$-groupoids in $\mathbb{A}\mathbf{Grd}(\Pi D)$

$$H = \Pi(X, D) = \Pi(X_2, D). \tag{2.22}$$

Hence $\Pi(X, D)$ depends only on the 2-skeleton of X. The attaching map $\partial_X : S^1_{\alpha_2} \to X_1$ of 2-cells given by f_2 in (2.8) (1) yields a map (also denoted by ∂_X)

$$\partial_X : \Pi(S^1_\alpha, D) \to \Pi(X_1, D) \tag{2.23}$$

in $\mathbb{A}\mathbf{Grd}(\Pi D)$ which is a <u>presentation</u> of $H = \Pi(X_2, D)$ in the sense that

$$H = \Pi(X_1, D)/N\ \mathrm{image}(\partial_X)$$

where N denotes the normal closure; compare (1.24). Since (X_1, D) is reduced $\Pi(X_1, D)$ is a free object in $\mathbb{A}\mathbf{Grd}(\Pi D)$. Such <u>free $\mathbb{A}$-groupoids under</u> D admit a coaction induced by

$$\mu : X_1 \to X_1 \vee S^1_{\alpha_1} \quad \text{in } (\mathbb{A}\mathbf{Top})^D_c/\simeq \mathrm{rel}\, D \tag{2.24}$$

Here μ is defined in the same way as μ in (1.25).

(2.25) Definition. Let D be an $\mathbb{A}$-space and let (X, D) be a relative $\mathbb{A}$-CW-complex which is normalized and reduced. Hence the functions $\alpha_n : Z_n \to D$ are defined for $n \geq 1$; see (2.21). Let $H = \Pi(X_2, D) \in \mathbb{A}\mathbf{Grd}(\Pi D)$. Then there is a well defined <u>chain complex</u>

$$\begin{cases} C_*(X, D) & \text{in } \mathbf{mod}\left(\int_{\mathbb{A}} H\right) \quad \text{with} \\ C_n(X, D) = L_{\alpha_n}(H) & \text{for } n \geq 1 \end{cases} \tag{1}$$

and $C_n(X, D) = 0$ for $n \leq 0$. Moreover a cellular map $f : (X, D) \to (Y, D)$ induces a map

$$u : H = \Pi(X_2, X) \to K = \Pi(Y_2, D) \in \mathbb{A}\mathbf{Grd}(\Pi D)$$

and a chain map

$$f_* : u_*(C_*(X, D)) \to C_*(Y, D) \tag{2}$$

in $\mathbf{mod}(\int_{\mathbb{A}} K)$. Here we use u_* in (2.15). If D is a free $\mathbb{A}$-set (2.3) then we define

$$\alpha_0 : Z_0 = D$$

by the identity of D. In this case there is a well defined augmented chain complex

$$C_* = \operatorname{aug} C_*(X, D) \quad \text{in } \mathbf{mod}\left(\int_{\mathbb{A}} H\right) \quad \text{with}$$

$$\begin{cases} C_n(X) = L_{\alpha_n}(H) & \text{for } n \geq 0 \quad \text{and} \\ C_n(X) = 0 & \text{for } n < 0 \end{cases} \tag{3}$$

If (X, D) is an $\mathbb{A}$-space under D for which $\pi_0 D \to \pi_0 X$ is surjective we choose a normalized reduced $\mathbb{A}$-CW-approximation (Y, D) of (X, D) (see (2.9) and (2.12)). Hence in this case we can define the chains of X by the chains of Y, that is:

$$\begin{cases} C_*(X, D) = C_*(Y, D), & \text{see (1)}, \\ C_*(X) = C_*(Y), & \text{see (3)}. \end{cases} \tag{4}$$

This yields below the appropriate notion of homology and cohomology for any $\mathbb{A}$-space X under D for which $\pi_0 D \to \pi_0 X$ is surjective. It is easy to see that homology and cohomology of (X, D) does not depend on the choice of Y.

It is possible to obtain $C_*(X, D)$ along the lines in (1.27). We get $C_*(X, D)$ by the general procedure in (V, § 2). The augmentation functor aug used in (3) above is described in (II, § 6).

Using the chain complexes $C_*(X, D)$ and $C_*(X)$ in $\mathbf{mod}(\int_{\mathbb{A}} H)$ we obtain for each object M in $\mathbf{Mod}(\int_{\mathbb{A}} H)$ the chain complexes of abelian groups

$$\operatorname{Hom}(C_*(X, D), M) \quad \text{and} \quad \operatorname{Hom}(C_* X, M).$$

Here Hom denotes the set of morphisms in the abelian category $\mathbf{Mod}(\int_{\mathbb{A}} H)$. Hence the cohomology with coefficients in M

$$\begin{cases} H^n(X, D; M) = H^n \operatorname{Hom}(C_*(X, D), M) \\ H^n(X; M) = H^n \operatorname{Hom}(C_*(X), M) \end{cases} \tag{2.26}$$

is defined.

(2.27) Remark. Moerdijk-Svenson [D] have introduced for each $\mathbb{A}$-space X and $(\int_{\mathbb{A}} \Pi X)$-module $\bar{M}$ the cohomology $H^n(X, \bar{M})$. In fact (1.27) yields a further way to describe the Moerdijk-Svenson cohomology since for the restriction M of $\bar{M}$ given by the inclusion $\int_{\mathbb{A}} H = \int_{\mathbb{A}} \Pi(X, D) \subset \int_{\mathbb{A}} \Pi X$ we have the natural isomorphism $H^n(X, D; \bar{M}) = H^n(X, D; M)$ where the right hand side is defined by (1.27).

On the other hand we can define the homology

$$H_n(X, D) = H_n(C_*(X, D)) \tag{2.28}$$

of the chain complex $C_*(X, D)$ in the abelian category $\mathbf{Mod}(\int_{\mathbb{A}} H)$ with $H = \Pi(X, D)$. Hence $H_n(X, D)$ is an $(\int_{\mathbb{A}} H)$-module in $\mathbf{Mod}(\int_{\mathbb{A}} H)$.

(2.29) Remark. As in (1.30) we obtain by $H_n(X, D)$ the $\mathbf{mod}(\int_{\mathbb{A}} H)$-module

$$\mathcal{H}_n(X, D) = \mathrm{Hom}(-, H_n(X, D)) : \left(\mathbf{mod}\left(\int_{\mathbb{A}} H\right)\right)^{\mathrm{op}} \to \mathbf{Ab}$$

This is the homology of the complex (X, D) considered in (V.3.3).

We now are ready to formulate the following homological Whitehead theorem for diagrams of spaces which is exactly the analogue of (1.31).

(2.30) Theorem. *Let D be an $\mathbb{A}$-space and let $f : (X, D) \to (Y, D)$ be a cellular map between normalized reduced relative $\mathbb{A}$-CW-complexes in $(\mathbb{A}\,\mathbf{Top})_c^D$. Then f is a homotopy equivalence under D (i.e. an isomorphism in the homotopy category $(\mathbb{A}\mathbf{Top})_c^D/\simeq \mathrm{rel}\, D$) if and only if the coefficient functor c induces an isomorphism, $u = c(f)$,*

$$u : H = \Pi(X, D) \xrightarrow{\cong} K = \Pi(Y, D) \in \mathbb{A}\mathbf{Grd}(\Pi D)$$

and one of the following conditions (i), (ii), (iii) is satisfied:

(i) $f_* : u_*(C_*(X, D)) \to C_*(Y, D)$ is a homotopy equivalence of chain complexes in $\mathbf{mod}(\int_{\mathbb{A}} K)$; see (2.25).
(ii) $f_* : H_n(X, D) \to u^* H_n(Y, D)$ is an isomorphism of $\int_{\mathbb{A}} H$-modules for $n \geq 1$; see (2.28).
(iii) For all modules N in $\mathbf{Mod}(\int_{\mathbb{A}} K)$ the induced map

$$f^* : H^n(Y, D; N) \to H^n(X, D; u^* N)$$

is an isomorphism of abelian groups for $n \geq 1$, see (2.26).

Part (iii) of the theorem can also be derived from the Whitehead theorem 3.8 of Moerdijk-Svenson [D] which in turn can be derived from (1.30) (iii). For us theorem (1.30) is a special case of (VI, § 7) below.

We now use the homology (2.28) and homotopy groups (2.11) for the following certain exact sequence of J.H.C. Whitehead. Let D be an $\mathbb{A}$-space and let (X, D) be a normalized reduced relative $\mathbb{A}$-CW-complex or more generally let (X, D) be a pair of $\mathbb{A}$-spaces for which $\pi_0 D \to \pi_0 X$ is surjective. Let $H = \Pi(X, D)$ be the restricted fundamental $\mathbb{A}$-groupoid in (2.10) (2). Then homotopy groups yield the $\int_{\mathbb{A}} H$-modules (resp. $\mathbf{mod}(\int_{\mathbb{A}} H)$-modules; see (1.30))

$$\left\{\begin{array}{ll}\pi_n(X) : \left(\int_{\mathbb{A}} H\right)^{\mathrm{op}} \to \mathbf{Ab}, & n \geq 2, \\ \Gamma_n(X,D) : \left(\int_{\mathbb{A}} H\right)^{\mathrm{op}} \to \mathbf{Ab}, & n \geq 1.\end{array}\right. \tag{2.31}$$

Here Γ_n is defined for $n \geq 3$ by skeleta, that is

$$\Gamma_n(X,D) = \mathrm{image}\{\pi_n(X_{n-1}) \to \pi_n(X_n)\}.$$

For $n = 1, 2$ the definition of Γ_n is more complicated, see (V.5.3) and (II, § 2). As a special case of (V.5.4) we get

(2.32) Theorem. *Let (X, D) be a pair of $\mathbb{A}$-spaces for which $\pi_0 D \to \pi_0 X$ is surjective and let $H = \Pi(X, D)$. Then the following sequence is an exact sequence of $\int_{\mathbb{A}} H$-modules, $n \geq 2$,*

$$\begin{aligned}&\to \Gamma_n(X,D) \to \pi_n(X) \xrightarrow{h} H_n(X,D) \to \Gamma_{n-1}(X,D) \to \dots \\ &\to \Gamma_2(X,D) \to \pi_2(X) \xrightarrow{h} H_2(X,D) \to \Gamma_1(X,D) \to 0\end{aligned}$$

Moreover this sequence is natural for (X, D) in $\mathbb{A}\mathbf{Top}^D$. The homomorphism h is the Hurewicz homomorphism.

The cohomology groups (2.26) with local coefficients

$$\left\{\begin{array}{l}H^{n+1}(X,D;u^*\pi_nY), \quad \text{and} \\ H^{n+1}(X,D,u^*\Gamma_n(Y,D))\end{array}\right.$$

are needed to define various features of obstruction theory which we discuss in detail in (V, § 4) and chapter VI. For example we get by (V.4.4) the next result which is the analogue of (1.33).

(2.33) Theorem. *Let (X, D) be a normalized reduced relative $\mathbb{A}$-CW-complex and let $f : D \to Y$ be a map in $\mathbb{A}\mathbf{Top}$ which admits an $\mathbb{A}$-extension $g : X_n \to Y$, $n \geq 2$. Then the restriction $g \mid X_{n-1}$ admits an $\mathbb{A}$-extension $\bar{g} : X_{n+1} \to Z$ if and only if an obstruction element*

$$\mathcal{O}(g \mid X_{n-1}) \in H^{n+1}(X,D;u^*\pi_nY)$$

vanishes. Here $u : \int_{\mathbb{A}} \Pi(X,D) \to \int_{\mathbb{A}} \Pi Y$ is induced by g.

We point out that the result requires the use of the restricted fundamental $\mathbb{A}$-groupoid which satisfies $\Pi(X,D) = \Pi(X_2,D)$ so that the induced map u is well defined by $g : X_n \to Y$ for $n \geq 2$.

There is also an obstruction theory in $\mathbb{A}\mathbf{Top}^D$ for the realizability of chain maps and chain complexes described by a tower of categories in (VI, § 5). Moreover there are the homotopy lifting property of the chain functor and the model lifting property of the twisted chain functor which have useful meaning for the chain functor in (2.25); see (VI, § 3) and (VI, § 8). We leave it to the reader to give the appropriate explicit interpretation in $\mathbb{A}\mathbf{Top}^D$ of such results. We here discuss only a few examples in order to illustrate the theory in chapter I, ... , VIII.

As main application of this book we now describe special cases of results in chapter VII and VIII which relate problems of homotopy theory in $\mathbb{A}\mathbf{Top}$ with algebraic K-theory.

(2.34) Definition. Let D be an $\mathbb{A}$-space and let $\mathcal{A}$ (2.16) be a set of functions α in $\mathbb{A}\mathbf{Top}$ with $\alpha : Z_\alpha \to D$ where Z_α is a free $\mathbb{A}$-set. We say that a function $\varphi : Z \to D$ in $\mathbb{A}\mathbf{Top}$ is $\mathcal{A}$-finite if $\beta_1, \ldots, \beta_k \in \mathcal{A}$ together with a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, equal, "\chi_\alpha"] \arrow[dr, "\varphi"'] & & Z_{\beta_1} \amalg \cdots \amalg Z_{\beta_k} \arrow[dl, "(\beta_1,\ldots,\beta_k)"] \\ & D & \end{tikzcd}$$

in $\mathbb{A}\mathbf{Top}$ are given where χ_α is a bijection. Similarly we say that a normalized reduced relative $\mathbb{A}$-CW-complex (X, D) is $\mathcal{A}$-finite if all functions $\alpha_n : Z_n \to D$, $n \geq 1$, in (2.21) are $\mathcal{A}$-finite and (X, D) is finite dimensional. Using the various $\mathbb{A}$-points $\mathbb{A}(-, a)$ in (2.3) it is easy to obtain many different examples of sets $\mathcal{A}$ as above.

Now let (X, D) and (Y, D) be normalized reduced relative $\mathbb{A}$-CW-complexes. A domination (X, f, g, H) of Y in $(\mathbb{A}\mathbf{Top})^D_c$ is given by maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y \quad \text{under } D \tag{2.35}$$

and an $\mathbb{A}$-homotopy $H : gf \simeq 1 \operatorname{rel} D$. The domination has dimension $\leq n$ if $\dim(X, D) \leq n$ and the domination is $\mathcal{A}$-finite if (X, D) is $\mathcal{A}$-finite. As a special case of theorem (VII.2.4) we get:

(2.36) Theorem. *Let D be an $\mathbb{A}$-space and let (Y, D) be a normalized reduced relative $\mathbb{A}$-CW-complex with restricted fundamental $\mathbb{A}$-groupoid $K = \Pi(Y, D)$. If (Y, D) admits an $\mathcal{A}$-finite domination in $(\mathbb{A}\mathbf{Top})^D_c$ then the finiteness obstruction*

$$[Y] = [C_*(Y, D)] \in K_0(U_\mathcal{A}(K))$$

is defined. Here $U_\mathcal{A}$ is the enveloping functor in (2.17) and K_0 is the reduced projective class group, see (VII, § 1). Moreover $[Y] = 0$ if and only if there exists an $\mathcal{A}$-finite normalized reduced relative $\mathbb{A}$-CW-complex (X, D) and a homotopy equivalence $X \to Y$ in $(\mathbb{A}\mathbf{Top})^D_c$.

This is the analogue of the finiteness obstruction theorem (1.36) of Wall.

Remark. Theorem (2.36) only holds in the relative case when D is not empty. In order to obtain such a result in the non-relative case one has to apply the theorem to the pair (X, X_0) where $(X, \emptyset)$ is an $\mathbb{A}$-CW-complex relative the empty diagram $\emptyset$. The condition $(X, \emptyset)$ "dominated" by $(Y, \emptyset)$ has to imply that we may assume $X_0 = Y_0$ and that (X, X_0) is dominated by (Y, Y_0) relative $X_0 = Y_0 = D$.

Next we describe simple homotopy equivalences and Whitehead torsion for $\mathbb{A}$-spaces.

(2.37) Definition. Let D be an $\mathbb{A}$-space (which is allowed to be empty) and let

$$\mathbf{K} = (\mathbb{A}\mathbf{Top})_c^D \tag{1}$$

be the category in (2.6) (1). Moreover let $\mathcal{D}$ be a set of free $\mathbb{A}$-sets with the property that the empty set $\emptyset$ is in $\mathcal{D}$ and that the coproduct $A \amalg B$ of $A, B \in \mathcal{D}$ is again in $\mathcal{D}$. Then each $A \in \mathcal{D}$ yields the coproduct

$$A \coprod D \quad \text{in } (\mathbb{A}\mathbf{Top})_c^D \tag{2}$$

which we call a <u>discrete object</u> in $\mathbf{K}$. Here A has the discrete topology. A <u>$\mathcal{D}$-complex</u> is a relative $\mathbb{A}$-CW-complex (L, D) for which the free $\mathbb{A}$-set Z_n of n-cells in $L - D$ is an element in $\mathcal{D}$, $n \geq 0$. A ball pair is a tuple (B^{n+1}, S^n, P^n, Q^n) as defined in (1.37) (3) where $n \geq 0$. For $A \in \mathcal{D}$ we consider a push out diagram in $\mathbb{A}\mathbf{Top}(n \geq 0)$

$$\begin{array}{ccc} A \times B^{n+1} & \longrightarrow & K \\ \cup & & \cup \\ A \times P^n & \xrightarrow{f} & L \end{array} \tag{3}$$

where f is given by a pair map $f : (A \times P^n, A \times S^{n-1}) \to (L_n, L_{n-1})$. Then (K, D) is again a $\mathcal{D}$-complex which we call an <u>elementary expansion</u> of L. Clearly $L \subset K$ is a homotopy equivalence in $(\mathbb{A}\mathbf{Top})_c^D$ and we call a retraction $r : K \to L$ an <u>elementary collapse</u>. A <u>simple homotopy equivalence</u> $f : L \to L'$ under D is obtained by a finite sequence of elementary expansions and collapses respectively.

Let $\mathcal{D}$-**cell** be the full subcategory of $(\mathbb{A}\mathbf{Top})_c^D$ consisting of finite dimensional $\mathcal{D}$-complexes (L, D). In (VIII, § 8) we define a functor

$$\mathrm{Wh} : \mathcal{D}\text{-}\mathbf{cell}/\simeq \mathrm{rel}\, D \to \mathbf{Ab}$$

which carries (L, D) to the <u>Whitehead group</u> $\mathrm{Wh}(L, D)$. As a special case of (VIII.8.3) one has the following result.

(2.38) Theorem. *Let D be an $\mathbb{A}$-space which may be empty. There is a function τ assigning to any homotopy equivalence $f : Y \to L$ in $(\mathbb{A}\mathbf{Top})_c^D$ between finite dimensional $\mathcal{D}$-complexes Y, L an element $\tau(f) \in \mathrm{Wh}(L, D)$. Moreover $\tau(f) = 0$ if and only if f is $\mathbb{A}$-homotopic* rel D *to a simple homotopy equivalence under D.*

The Whitehead group $\mathrm{Wh}(L, D)$ can be computed algebraically by the following result which is a special case of (VIII.12.7).

(2.39) Theorem. *Let D be an $\mathbb{A}$-space which may be empty. Let (L, D) be a normalized finite dimensional $\mathcal{D}$-complex and let $H = \Pi(L, L_0)$ be the restricted fundamental $\mathbb{A}$-groupoid. Then the algebraic Whitehead group*

$$\mathrm{Wh}(H) = K_1^{\mathrm{iso}}(U_{\mathcal{A}}(H))/\sim$$

is defined. Here $\mathcal{A}$ is the set of all functions $A \to L_0$ in $\mathbb{A}\mathbf{Set}$ with $A \in \mathcal{D}$; the functor $U_{\mathcal{A}}$ is the enveloping functor in (2.17) and K_1^{iso} is the "isomorphism torsion group" in (VIII, § 10). Moreover there is an isomorphism of abelian groups

$$\tau : \mathrm{Wh}(L, D) \cong \mathrm{Wh}(H).$$

All the results in this section are examples and applications of the results of the general theory in the chapter I, ... , VIII below. In order to translate the general theory to the special homotopy theory in $(\mathbb{A}\mathbf{Top})^D$ one has to use the following glossary where on the left hand side we use the notation of the general theory.

$\mathbf{T}$ (I.1.11)	Category of coactions given by the full subcategory of $(\mathbb{A}\mathbf{Top})_c^D/\simeq$ of reduced 1-dimensional relative $\mathbb{A}$-CW-complexes (X_1, D). Cogroups in $\mathbf{T}$ are spherical objects S^1_α and the coaction map is defined by (2.24).
Twist (I.3.5)	Category of presentation ∂_X, generalizes the category of free "pre crossed modules".
∂_X	Presentation as in (2.23).
Coef	This is a category equivalent to $\mathbb{A}\mathbf{Grd}(\Pi D)$. The equivalence carries the presentation ∂_X of H to H.
$\mathbf{mod}(\partial_X)$ (I.5.7)	This is the category $\mathbf{mod}(\int_{\mathbb{A}} H)$ in (2.14) where ∂_X is a presentation of H. Here we use (2.20).
$U_{\mathcal{A}}$ (I.5.11)	This is the enveloping functor $U_{\mathcal{A}}$ in (2.17). Here we identify $\alpha \in \mathcal{A}$ with the spherical object $S^1_\alpha \in \mathbf{T}$.
$(\mathbf{C}, \mathbf{T})$ (V.1.1)	$((\mathbb{A}\mathbf{Top})_c^D, \mathbf{T})$ is a homological cofibration category if D is non empty. Here $\mathbf{T}$ is defined above.
Complex (IV.2.2)	This is the subcategory of $(\mathbb{A}\mathbf{Top})_c^D$ consisting of normalized reduced relative $\mathbb{A}$-CW-complexes (X, D) and cellular maps.
C_* (V.2.3)	This is the chain functor in (2.25).
$(\mathbf{C}, \mathcal{D})$ (VIII.5.1) (VIII.12.3)	$(\mathbb{A}\mathbf{Top})_c^D$ is a homological cellular I-category with the cylinder in (2.6) (3) and the class $\mathcal{D}$ of discrete objects in (2.37). Here D is allowed to be empty.
$(\Box_X^{k+1}, \Sigma_X^k, P_X^k, Q_X^k)$ (VIII.4.5)	For X in $(\mathbb{A}\mathbf{Top})_c^D$ this is the push out of $B \times X \supset B \times D \xrightarrow{pr} D$ in $\mathbb{A}\mathbf{Top}$ where pr is the projection and $B = (B^{k+1}, S^k, P^k, Q^k)$ is the ball pair in (1.37) (3).

It will be convenient to have these examples in mind in order to visualize the abstract and categorical theory in the second part of the book below.

3 Homotopy Theory of Transformation Groups

In this section let G be a fixed topological group which is locally compact Hausdorff (for example a Lie group). Let X be a topological space. A (left) <u>action</u> of G on X

is a continuous map $G \times X \to X, (g,x) \mapsto g \cdot x$ satisfying $e \cdot x = e$ for the neutral element $e \in G$ and $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for $g_1, g_2 \in G$ and $x \in X$. Here $G \times X$ is the product of spaces with the product topology. Given a G-action on X we call G a transformation group for the G-space X. A G-map or an equivariant map $f : X \to Y$ between G-spaces is a continuous map satisfying $f(g \cdot x) = g \cdot f(x)$. Let $G\mathbf{Top}$ be the category of G-spaces and equivariant maps. (There is an alternative approach using only compactly generated spaces, see Lück [TG].)

We are going to apply the theory of this book to the homotopy theory of G-spaces. This, in fact, leads to many new features and results on G-spaces; compare the books of tom Dieck [TG] and Lück [TG]. In particular, we obtain a new way in dealing with the twisted version of Bredon cohomology.

The trivial G-space X is a topological space X with the action $g \cdot x = x$ for $g \in G, x \in X$. The product of G-spaces X and Y is the G-space $X \times Y$ with the action $g \cdot (x,y) = (g \cdot x, g \cdot y)$ for $g \in G, x \in X, y \in Y$. The coproduct $X \coprod Y$ is the disjoint union of spaces with the obvious G-action. A G-homotopy is a G-map

$$H : [0,1] \times X \to Y$$

between G-spaces. Here $[0,1]$ is the interval considered as a trivial G-space. Here H is a "free" homotopy. For a G-space D let

$$\mathbf{C} = (G\mathbf{Top})^D \tag{3.1}$$

be the category of G-spaces under D and let

$$\mathbf{C}_c = (G\mathbf{Top})^D_c \tag{1}$$

be the full subcategory given by G-cofibrations $D \rightarrowtail X$ in $\mathbf{C}$. Such G-cofibrations are defined via the homotopy extension property in $G\mathbf{Top}$. The homotopy category

$$\mathbf{C}_c/{\simeq} = (G\mathbf{Top})^D_c/{\simeq} \operatorname{rel} D \tag{2}$$

is defined by homotopy relative D and the relative cylinder $I(X,D)$ as in (1.4) (3).

Given a closed subgroup H of G we obtain the homogeneous space G/H which is the quotient space of G consisting of cosets $g'H$ for $g' \in G$. Clearly G/H is a G-space with the action $g \cdot (g'H) = (g \cdot g')H$. We call any such homogeneous space G/H a G-orbit point. A G-orbit set Z is the coproduct of such G-orbit points, that is, Z is given by a set M and closed subgroups H_m of G for $m \in M$ such that

$$Z = \coprod_{m \in M} G/H_m \tag{3.2}$$

is a coproduct of G-orbit points in $G\mathbf{Top}$. G-orbit sets are the most elementary G-spaces.

We say that a G-space Y is obtained from a G-space X by attaching $(n+1)$-cells if a G-orbit set Z together with a push out diagram in $G\mathbf{Top}$

$$\begin{array}{ccc} Z \times B^{n+1} & \longrightarrow & Y \\ \cup & & \cup \\ Z \times S^n & \longrightarrow & X \end{array} \tag{3.3}$$

is given. Here S^n and B^{n+1} are trivial G-spaces A (relative) G-CW-complex $(X_{\geq 0}, D) = (X, D)$ is given by a sequence of inclusions in $G\mathbf{Top}$

$$D \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \ldots \tag{3.4}$$

Here X_0 is the coproduct of D and a G-orbit set Z_0 and X_{n+1} is obtained from X_n by attaching $(n+1)$-cells, $n \geq 0$. Let $X = \lim X_{\geq 0}$. We say that (X, D) is reduced if $X_0 = D$ and (X, D) is normalized if the attaching map

$$f_n : Z_n \times S^{n-1} \to X_{n-1}$$

of n-cells carries $Z_n \times *$ to X_0. Here Z_n is the G-orbit set of n-cells of $X - D$, $n \geq 0$. The cellular approximation theorem holds and also the Blakers-Massey property is satisfied; see tom Dieck [T].

(3.5) Definition. The orbit category $\mathrm{Or}(G)$ is the category consisting of G-orbit points and G-maps. This is the full subcategory of $G\mathbf{Top}$ consisting of homogeneous spaces G/H where H is a closed subgroup of G. Each G-space X yields an $\mathrm{Or}(G)$-space X°

$$\begin{aligned} X^\circ &: \mathrm{Or}(G)^{\mathrm{op}} \to \mathbf{Top} \\ X^\circ(G/H) &= \mathrm{Map}_G(G/H, X) = X^H \end{aligned}$$

where Map_G is the space of G-maps. Here X^H is the H-fixed point set of X. For the $\mathrm{Or}(G)$-space X° all the notation in section § 2 is available if we set $\mathbb{A} = \mathrm{Or}(G)$. We point out that for a discrete group G and a G-CW-complex X the $\mathrm{Or}(G)$-space X° is an $\mathrm{Or}(G)$-CW-complex in the sense of § 2. This does not hold if G is not discrete. In fact, if G is discrete the theory on G-spaces in this section is completely determined by the theory on $\mathrm{Or}(G)$-spaces in § 2; compare for example Moerdijk-Svenson [D].

(3.6) Lemma. *Let (X, D) be a relative G-CW-complex. Then there exists a normalized relative G-CW-complex (Y, D) together with a homotopy equivalence $Y \to X$ in $(G\mathbf{Top})^D_c/\simeq \operatorname{rel} D$. Moreover if $\pi_0 D^\circ \to \pi_0 X^\circ$ is surjective in* $\mathrm{Or}(G)\mathbf{Set}$ *then (Y, D) can be chosen to be reduced.*

The proof is similar to the proof of (2.9). A map $f : X \to Y$ in $G\mathbf{Top}$ is a weak equivalence if the induced map $f^\circ : X^\circ \to Y^\circ$ in $\mathrm{Or}(G)\mathbf{Top}$ is a weak equivalence; see (2.12). It is known (see for example Lück [TG] I.2.3) that each G-space Y under D admits a weak equivalence $f : X \to Y$ under D where (X, D) is a G-CW-complex termed a G-CW-approximation of Y. Moreover, it is easy to show that a weak equivalence $f : (X, D) \to (Y, D)$ under D between relative G-CW-complexes is actually a homotopy equivalence in $(G\mathbf{Top})_c^D/\simeq \operatorname{rel} D$; see (IV, § 3). This yields the equivalence of categories

$$\operatorname{Ho}(G\mathbf{Top})^D \xrightarrow{\sim} G\text{-}\mathbf{CW}^D/\simeq \operatorname{rel} D \tag{3.7}$$

Here the left hand side is the localization with respect to weak equivalences and the right hand side is the full subcategory of $(G\mathbf{Top})_c^D$ consisting of relative G-CW-complexes (X, D). The equivalence (3.7) shows that it is sufficient to define homology and cohomology for G-CW-complexes.

As in (2.13) we define for a G-space D the <u>coefficient functor</u>

$$c : (G\mathbf{Top})_c^D/\simeq \operatorname{rel} D \to \operatorname{Or}(G)\mathbf{Grd}(\Pi D^\circ) \tag{3.8}$$

which carries the object $D \rightarrowtail X$ to the restricted fundamental groupoid $c(X) = (\Pi D^\circ \to \Pi(X^\circ, D^\circ))$. Here we use (3.5). Let Z be a G-orbit set and D be a G-space. Then we observe that a G-map $\alpha : Z \to D$ can be identified with a collection of points $\alpha_m \in X^{H_m}$ with $Z = \coprod_{m \in M} G/H_m$. Such a collection of points as well can be identified with a map

$$\alpha : Z' \to \operatorname{Ob}(\Pi D^\circ) \in \operatorname{Or}(G)\mathbf{Set} \tag{3.9}$$

where Z' is the free $\operatorname{Or}(G)$-set given by $Z' = \coprod_{m \in M} \operatorname{Or}(G)(-, G/H_m)$. Hence by (2.14) the free modules

$$L_\alpha(H) \in \mathbf{mod}\Big(\int_{\operatorname{Or}(G)} H\Big) \subset \mathbf{Mod}\Big(\int_{\operatorname{Or}(G)} H\Big) \tag{3.10}$$

are defined for $H \in \operatorname{Or}(G)\mathbf{grd}(\Pi D^\circ)$. As in (2.17) we choose a set $\mathcal{A}$ consisting of elements α which are G-maps $\alpha : Z_\alpha \to D$ where Z_α is a G-orbit set. Then the enveloping functor

$$U_{\mathcal{A}} : \operatorname{Or}(G)\mathbf{Grd}(\Pi D^\circ) \to \mathbf{Ringoids} \tag{3.11}$$

is defined which carries H to the full subcategory $U_{\mathcal{A}}(H) \subset \mathbf{mod}(\int_{\operatorname{Or}(G)} H)$ consisting of free modules $L_\alpha(H)$ with $\alpha \in \mathcal{A}$. This is a special case of (2.17).

It is possible to describe the free modules (3.10) by use of homotopy groups. For this we introduce the <u>spherical object</u> S_α^n in $(G\mathbf{Top})_c^D$ which is the push out

$$\begin{array}{ccc} Z \times S^n & \longrightarrow & S_\alpha^n \\ \cup & & \cup \\ Z \times * & \xrightarrow{\alpha} & D \end{array} \tag{3.12}$$

in $G\mathbf{Top}$ with the retraction $0 : S_\alpha^n \to D$ given by the projection $Z \times S^n \to Z$. Now let X be an object in $(G\mathbf{Top})_c^D$ and let $S_\alpha^n \vee X$ be the sum of S_α^n and X under D with the retraction

$$(0,1) : S^n_\alpha \vee X \to X$$

in $(G\mathbf{Top})^D_c$. We now obtain for $n \geq 2$ and $H = \Pi(X^\circ, D^\circ)$ the $(\int_{\mathrm{Or}(G)} H)$-module

$$\pi_n(S^n_\alpha \vee X)^\circ_2 = \mathrm{kernel}\{\pi_n(S^n_\alpha \vee X)^\circ \xrightarrow{(0,1)_*} \pi_n(X^\circ)\} \tag{3.13}$$

which satisfies $L_\alpha(H) = \pi_n(S^n_\alpha \vee X)^\circ_2$; compare (2.20).

If (X, D) is a normalized reduced G-CW-complex then the attaching maps f_n of n-cells yield for $n \geq 1$ the G-maps

$$\alpha_n : Z_n \to D \tag{3.14}$$

where Z_n is the G-orbit set of n-cells in $X - D$. In fact, α_n is the restriction of f_n to $Z_n \times *$ which maps to D since (X, D) is normalized and reduced. Therefore f_n actually is given by a map

$$f_n : S^n_{\alpha_n} \to X_{n-1} \quad \text{in } (G\mathbf{Top})^D_c$$

where $S^n_{\alpha_n}$ is the spherical object in (3.12). We call

$$\partial_X = f_2 : S^1_{\alpha_2} \to X_1 \tag{3.15}$$

the <u>presentation</u> associated to (X, D). Here X_1 has a coaction

$$\mu : X_1 \to X_1 \vee S^1_{\alpha_1} \quad \text{in } (G\mathbf{Top})^D_c/{\simeq} \ \mathrm{rel}\, D \tag{3.16}$$

where $S^1_{\alpha_1}$ is a cogroup object. In fact μ is defined similarly as in (1.25).

(3.17) Definition. Let D be a G-space and let (X, D) be a relative G-CW-complex which is normalized and reduced. Hence the functions $\alpha_n : Z_n \to D$ in $G\mathbf{Top}$ are defined for $n \geq 1$ where Z_n is the G-orbit set of n-cells. Let

$$H = \Pi(X^\circ_2, D^\circ) \in \mathrm{Or}(G)\mathbf{Grd}(\Pi D^\circ)$$

Then there is a well defined <u>chain complex</u>

$$\begin{cases} C_*(X, D) \quad \text{in } \mathbf{mod}\left(\int_{\mathrm{Or}(G)} H\right) \quad \text{with} \\ C_n(X, D) = L_{\alpha_n}(H) \quad \text{for } n \geq 1 \end{cases} \tag{1}$$

and $C_n(X, D) = 0$ for $n \leq 0$. If D is a G-orbit set then we define

$$\alpha_0 : Z_0 = D$$

by the identity of D. In this case the <u>augmented chain complex</u>

$$\begin{cases} C_*(X) = \mathrm{aug}\, C_*(X, D) \quad \text{in } \mathbf{mod}\left(\int_{\mathrm{Or}(G)} H\right) \quad \text{with} \\ C_n(X) = L_{\alpha_n}(H) \quad \text{for } n \geq 0 \end{cases} \tag{2}$$

and $C_nX = 0$ for $n < 0$ is defined. These chain complexes have properties as in (2.25). In fact, if G is discrete we have $C_*(X, D) = C_*(X^\circ, D^\circ)$ and $C_*(X) = C_*(X^\circ)$ where the right hand side is defined in (2.25); compare the final remark in (3.5). We get $C_*(X, D)$ by the general procedure in (V, § 2). The augmentation functor aug used in (2) is described in (II, § 6).

We obtain for each $(\int_{\mathrm{Or}(G)} H)$-module M the cohomology with coefficients in M

$$\begin{cases} H^n(X, D; M) = H^n \operatorname{Hom}(C_*(X, D), M) \\ H^n(X; M) = H^n \operatorname{Hom}(C_*(X), M) \end{cases} \tag{3.18}$$

Here Hom is defined by the abelian category $\mathbf{Mod}(\int_{\mathrm{Or}(G)} H)$ with $H = \Pi(X^\circ, D^\circ) = \Pi(X_2^\circ, D^\circ)$. This is a twisted version of the cohomology of Bredon [EC]; see Moerdijk-Svenson [D] where this cohomology is studied if G is discrete. On the other hand we define the homology

$$H_n(X, D) = H_n(C_*(X, D)) \tag{3.19}$$

of the chain complex $C_*(X, D)$ in the abelian category $\mathbf{Mod}(\int_{\mathrm{Or}(G)} H)$ so that $H_n(X, D)$ is a $(\int_{\mathrm{Or}(G)} H)$-module (and hence a $\mathbf{mod}(\int_{\mathrm{Or}(G)} H)$-module; see (1.30)). We now are ready to formulate the homological Whitehead theorem for G-spaces.

(3.20) Theorem. *Let D be a G-space and let $f : (X, D) \to (Y, D)$ be a cellular map between normalized reduced relative G-CW-complexes in $(G\mathbf{Top})_c^D$. Then f is a homotopy equivalence under D (i.e. an isomorphism in the homotopy category $(G\mathbf{Top})_c^D/\simeq \operatorname{rel} D$) if and only if the coefficient functor c induces an isomorphism, $u = c(f)$,*

$$u : H = \Pi(X^\circ, D^\circ) \xrightarrow{\cong} K = \Pi(Y^\circ, D^\circ) \in \mathrm{Or}(G)\mathbf{Grd}(\Pi D^\circ)$$

and one of the following conditions (i), (ii), (iii) is satisfied:

(i)

$$f_* : u_* C_*(X, D) \to C_*(Y, D)$$

is a homotopy equivalence of chain complexes in $\mathbf{mod}(\int_{\mathrm{Or}(G)} K)$.

(ii)

$$f_* : H_n(X, D) \to u^* H_n(Y, D)$$

is an isomorphism of $(\int_{\mathrm{Or}(G)} H)$-modules (or of $\mathbf{mod}(\int_{\mathrm{Or}(G)} H)$-modules) for $n \geq 1$.

(iii) For all modules N in $\mathbf{Mod}(\int_{\mathrm{Or}(G)} K)$ the induced map

$$f^* : H^n(Y, D; N) \to H^n(X, D, u^* N)$$

is an isomorphism of abelian groups for $g \geq 1$.

Also the Hurewicz homomorphism and the exact sequence of J.H.C. Whitehead have an analogue for G-spaces. As a special case of (V.3.4) we get.

(3.21) Theorem. *Let (X, D) be a pair of G-spaces for which $\pi_0 D^\circ \to \pi_0 X^\circ$ is surjective. Then the following sequence is an exact sequence of $(\int_{\mathrm{Or}(G)} H)$-modules with $H = \Pi(X^\circ, D^\circ)$, $n \geq 2$.*

$$\begin{gathered}\cdots \to \Gamma_n(X,D) \to \pi_n(X^\circ) \xrightarrow{h} H_n(X,D) \to \Gamma_{n-1}(X,D) \to \dots \\ \Gamma_2(X,D) \to \pi_2(X^\circ) \to H_2(X,D) \to \Gamma_1(X,D) \to 0\end{gathered}$$

Moreover the sequence the sequence is natural for (X, D) in $(G\mathbf{Top})_c^D$.

The homomorphism h is the <u>Hurewicz homomorphism</u> and $\pi_n(X^\circ)$ is defined by (2.11) and $\Gamma_n(X, D)$ for $n \geq 3$ is defined by

$$\Gamma_n(X,D) = \mathrm{image}\{\pi_n(X^\circ_{n-1}) \to \pi_n(X^\circ_n)\}$$

The definition of Γ_1 and Γ_2 is more complicated. If G is a discrete group then (3.21) can be considered as being a special case of (2.32).

Concerning obstruction theory we get the following analogue of (2.33):

(3.22) Theorem. *Let (X, D) be a normalized reduced relative G-CW-complex and let $f : D \to Y$ be a G-map which admits a G-extension $g : X_n \to Y$, $n \geq 2$. Then the restriction $g \mid X_{n-1}$ admits a G-extension $\bar{g} : X_{n+1} \to Y$ if and only if an obstruction element*

$$\mathcal{O}(g \mid X_{n-1}) \in H^{n+1}(X, D, u^* \pi_n Y^\circ)$$

vanishes. Here $u : \int_{\mathrm{Or}(G)} \Pi(X^\circ, D^\circ) \to \int_{\mathrm{Or}(G)} \Pi Y^\circ$ is induced by g.

This result again shows that the restricted fundamental groupoid is needed which satisfies $\Pi(X^\circ, D^\circ) = \Pi(X^\circ_2, D^\circ)$ by the cellular approximation theorem. Hence $g : X_n \to Y$ with $n \geq 2$ yields a well defined map u in the theorem.

We leave it to the reader to translate further results from the obstruction theory in chapter VII, VIII to the category of G-spaces. We now consider main applications concerning connections with algebraic K-theory.

(3.23) Definition. Let D be a G-space and let $\mathcal{A}$ be a set of functions $\alpha : Z_\alpha \to D$ in $G\mathbf{Top}$ where Z_α is a G-orbit set. We say that a function $\alpha : Z \to D$ in $G\mathbf{Top}$ is $\mathcal{A}$-<u>finite</u> if $\beta_1, \dots, \beta_k \in \mathcal{A}$ together with a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, equal, "\chi_\alpha"] \arrow[dr, "\alpha"'] & & Z_{\beta_1} \amalg \dots \amalg Z_{\beta_k} \arrow[dl, "(\beta_1,\dots,\beta_k)"] \\ & D & \end{tikzcd}$$

in $G\mathbf{Top}$ are given where χ_α is an isomorphism. Similarly we say that a normalized reduced relative G-CW-complex (X, D) is $\mathcal{A}$-finite if all functions $\alpha_n : Z_n \to D$, $n \geq 1$, in (3.14) are $\mathcal{A}$-finite and (X, D) is finite dimensional.

Now let (X, D) and (Y, D) be normalized reduced relative G-CW-complexes. A domination (X, f, g, H) of Y in $(G\mathbf{Top})_c^D$ is given by G-maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y \quad \text{under} \quad D \tag{3.24}$$

and a G-homotopy $H : gf \simeq 1 \operatorname{rel} D$. The domination has dimension $\leq n$ if $\dim(X, D) \leq n$ and the domination is $\mathcal{A}$-finite if (X, D) is $\mathcal{A}$-finite. As a special case of theorem (VII.2.4) we get the following result which corresponds to (2.36) if G is discrete.

(3.25) Theorem. *Let D be a G-space and let (Y, D) be a normalized reduced relative G-CW-complex with $K = \Pi(Y^\circ, D^\circ)$. If (Y, D) admits an $\mathcal{A}$-finite domination in $(G\mathbf{Top})_c^D$ then the finiteness obstruction*

$$[Y] = [C_*(Y, D)] \in K_0(U_{\mathcal{A}}(K))$$

is defined. Here $U_{\mathcal{A}}$ is the enveloping functor in (3.11) and K_0 is the reduced projective class group; see (VII, § 1). Moreover $[Y] = 0$ if and only if there exists an $\mathcal{A}$-finite normalized reduced relative G-CW-complex (X, D) and a homotopy equivalence $X \to Y$ under D.

As in the remark following (2.36) we can obtain a non-relative version of this result for G-CW-complexes $(X, \emptyset)$ relative the empty G-space $\emptyset$.

Also the theory of Whitehead on simple homotopy equivalences has a generalization for G-spaces as follows.

(3.26) Definition. Let D be a G-space (which is allowed to be empty) and let

$$\mathbf{K} = (G\mathbf{Top})_c^D \tag{1}$$

be the category in which the objects are G-cofibrations $D \rightarrowtail X$ in $G\mathbf{Top}$, see (3.1) (1). Moreover let $\mathcal{D}$ be a set of orbit sets with the property that the empty G-orbit set $\emptyset$ is in $\mathcal{D}$ and that for $A, B \in \mathcal{D}$ also the coproduct $A \amalg B$ in $G\mathbf{Top}$ is in $\mathcal{D}$. Then each $A \in \mathcal{D}$ yields the object

$$A \amalg D \quad \text{in } (G\mathbf{Top})_c^D \tag{2}$$

which we call a discrete object in $\mathbf{K}$. A $\mathcal{D}$-complex is a relative G-CW-complex (L, D) for which the G-orbit set Z_n of n-cells in $L - D$ is an element in $\mathcal{D}$, $n \geq 0$. Let (B^{n+1}, S^n, P^n, Q^n) be a ball pair as defined in (1.37) (3) with $n \geq 0$. For $A \in \mathcal{D}$ we consider a push out diagram in $G\mathbf{Top}$

$$\begin{array}{ccc} A \times B^{n+1} & \longrightarrow & K \\ \cup & & \cup \\ A \times P^n & \xrightarrow{f} & L \end{array} \tag{3}$$

where f is given by a pair map $f : (A \times P^n, A \times S^{n-1}) \to (L_n, L_{n-1})$. Then (K, D) is again a $\mathcal{D}$-complex which we call an elementary expansion of L. Clearly $L \subset K$ is a homotopy equivalence in $(G\text{-}\mathbf{Top})_c^D$ and we call a retraction $K \to L$ an elementary collapse. A simple homotopy equivalence $f : L \to L'$ under D is obtained by a finite sequence of elementary expansions and collapses respectively. Let $\mathcal{D}$-**cell** be the full subcategory of $(G\mathbf{Top})_c^D$ consisting of finite dimensional $\mathcal{D}$-complexes (L, D). In (VIII, § 8) we define a functor

$$\mathrm{Wh} : \mathcal{D}\text{-}\mathbf{cell}/\simeq \mathrm{rel}\, D \to \mathbf{Ab}$$

which carries (L, D) to the Whitehead group $\mathrm{Wh}(L, D)$. As a special case of (VIII.8.3) we get:

(3.27) Theorem. *Let D be a G-space which is allowed to be empty. There is a function τ assigning to any homotopy equivalence $f : Y \to L$ in $(G\mathbf{Top})_c^D$ between finite dimensional $\mathcal{D}$-complexes Y, L an element $\tau(f) \in \mathrm{Wh}(L, D)$. Moreover, $\tau(f) = 0$ if and only if f is a G-homotopic* $\mathrm{rel}\, D$ *to a simple homotopy equivalence under D.*

The Whitehead group $\mathrm{Wh}(L, D)$ can be computed algebraically by the following result which is a special case of (VIII.12.7).

(3.28) Theorem. *Let D be a G-space which may be empty. Let (L, D) be a normalized finite dimensional $\mathcal{D}$-complex and let $H = \Pi(L^\circ, L_0^\circ)$ be the restricted fundamental groupoid given by the pair (L, L_0) where L_0 is the 0-skeleton of (L, D). Then the algebraic Whitehead group*

$$\mathrm{Wh}(H) = K_1^{\mathrm{iso}}(U_{\mathcal{A}}(H))/\sim$$

is defined. Here $\mathcal{A}$ is the set of all G-maps $A \to L_0$ with $A \in \mathcal{D}$. The functor $U_{\mathcal{A}}$ is the enveloping functor in (3.11) and K_1^{iso} "isomorphism torsion group" in (VIII, § 10). Moreover there is an isomorphism of abelian groups

$$\tau : \mathrm{Wh}(L, D) = \mathrm{Wh}(H).$$

All the results in this section are examples and applications of the results of the general theory in the chapters I, ... , VII below. For the translation of the general theory to the special case one has to use the following table.

T (I.1.11)	Category of coactions. This is the full subcategory of $(G\mathbf{Top})^D_c/\simeq\ \mathrm{rel}\, D$ of reduced 1-dimensional G-CW-complexes (X_1, D). Cogroups in $\mathbf{T}$ are spherical objects S^1_α and the coaction map on X_1 is given by (3.16).
Twist (I.3.5)	Category of presentations ∂_X in (3.15).
Coef **(I.4.1)**	This is a category equivalent to $\mathrm{Or}(G)\mathbf{Grd}(\Pi D^\circ)$. The equivalence carries the presentation ∂_X defining (X_2, D) to $H = \Pi(X_2^\circ, D^\circ)$.
$\mathbf{mod}(\partial_X)$	This is the category $\mathbf{mod}(\int_{\mathrm{Or}(G)} H)$ in (3.10) where ∂_X is a presentation of H.
$U_{\mathcal{A}}$ (I.5.11)	This is the enveloping functor $U_{\mathcal{A}}$ in (3.11). Here we identify $\alpha \in \mathcal{A}$ with the spherical object $S^1_\alpha \in \mathbf{T}$.
$(\mathbf{C}, \mathbf{T})$ (V.1.1)	$(G\mathbf{Top})^D_c$ is a homological cofibration category if D is not empty.
Complex (IV.2.2)	This is the subcategory of $(G\mathbf{Top})^D_c$ consisting of normalized reduced relative G-CW-complexes (X, D) and cellular maps.
C_* (V.2.3)	This is the chain functor in (3.17).
$(\mathbf{C}, \mathcal{D})$ (VIII.5.1) (VIII.12.3)	$(G\mathbf{Top})^D_c$ is a homological cellular I-category with the cylinder $I(X, D)$ and the class $\mathcal{D}$ of discrete objects in (3.26). Here D is allowed to be empty.
$(\square^{k+1}_X, \Sigma^k_X, P^k_X, Q^k_X)$ (VIII.4.5)	For $X \in (G\mathbf{Top})^D_c$ this is the push out of $B \times X \supset B \times D \to D$ where $B \times D \to D$ is the projection and where $B = (B^{k+1}, S^k, P^k, Q^k)$ is the ball pair in (1.37) (3).

4 Homotopy Theory Controlled at Infinity

We choose a fixed compact Hausdorff space which we denote by ∞. An ∞-space or infinity space is a tuple $X = (\hat{X}, X, \infty)$ where $\hat{X}$ is a compact space together with a closed embedding $\infty \subset \hat{X}$ such that X is the complement $X = \hat{X} - \infty$. The space X is termed the open part of the ∞-space. A point $e \in \infty$ is an end of X if there is a sequence of points $x_1, x_2, \ldots$ in the open part which converges in $\hat{X}$ to $e \in \infty$. Hence X is dense in $\hat{X}$ if all points of ∞ are ends; in this case $\hat{X}$ is called an ∞-compactification of the space X. An ∞-map $f : X \to Y$ between ∞-spaces is a contiuous map for which the following diagram commutes in $\mathbf{Top}$:

$$\begin{tikzcd} & \infty \arrow[dl] \arrow[dr] & \\ \hat{X} \arrow[rr, "\hat{f}"] & & \hat{Y} \\ X \arrow[u, phantom, "\cup" rotate=90] \arrow[rr, "f"] & & Y \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd} \tag{4.1}$$

Hence an ∞-map is a map under ∞ which carries the open part to the open part. The continuous map $\hat{f}$ is determined by f. Let $\infty\mathbf{End}$ be the category of

∞-spaces and ∞-maps. In this section we study the homotopy theory of ∞-spaces and ∞-maps. Details on this example are described in Baues-Quintero [HI].

(4.2) Example. Let $\infty = *$ be a point. Then any locally compact Hausdorff space X has a <u>one point compactification</u> $(\hat{X}, X, *)$ which is an object in $*$**End**. In this case a map $f : X \to Y$ between locally compact Hausdorff spaces is an ∞-map if and onyl if f is a compact map. <u>Compact maps</u> in **Top** are closed maps for which the inverse of each point is a compact space; such maps are also termed <u>proper maps</u>. This shows that proper homotopy theory is a special case of the homotopy theory of ∞-spaces.

(4.3) Example. Let T be a locally finite <u>tree</u> with Freudenthal compactification $\hat{T}$. Then $T = (T, \hat{T}, \infty_T)$ is an ∞-space where $\infty_T = \hat{T} - T$ is a <u>Cantor set</u>. As a special case we may consider the category ∞_T**End** containing the ∞_T-space $T = (T, \hat{T}, \infty_T)$.

Given a compact space K in **Top** and an ∞-space X we obtain the ∞-space $K \otimes X$ by the push out diagram in **Top**

$$\begin{array}{ccc} K \times \hat{X} & \longrightarrow & K \otimes X \\ \cup & & \cup \\ K \times \infty & \xrightarrow{pr} & \infty \end{array} \tag{4.4}$$

where $K \times \hat{X}$ is the product in **Top**. The open part of $K \otimes X$ is the product $K \times X$. An ∞-<u>homotopy</u> is an ∞-map

$$H : [0,1] \otimes X \to Y \tag{4.5}$$

The according homotopy relation yields the homotopy category $\infty\mathbf{End}/\simeq$. We now choose an ∞-space D and consider the category

$$\mathbf{C} = (\infty\mathbf{End})^D \tag{4.6}$$

of ∞-spaces under D. An object in $\mathbf{C}$ is an ∞-map $D \to X$ and an ∞-<u>homotopy relative</u> D is a homotopy as in (4.5) for which the composite $[0,1] \otimes D \to [0,1] \otimes X \to Y$ is the trivial homotopy. Using the homotopy extension property with respect to the cylinder $[0,1] \otimes X$ we define ∞-<u>cofibrations</u>. If $D \to X$ is such an ∞-cofibration we write

$$[X, Y]^D = \mathbf{C}(X, Y)/\simeq \operatorname{rel} D \tag{4.7}$$

for the set of ∞-homotopy classe relative D. Let

$$\mathbf{C}_c = (\infty\mathbf{End})_c^D \tag{1}$$

be the full subcategory of $\mathbf{C}$ for which the objects are ∞-cofibrations $D \rightarrowtail X$. Then ∞-homotopy relative D is a natural equivalence relation so that the homotopy category

$$\mathbf{C}_c/\simeq\; = (\infty\mathbf{End})_c^D/\simeq \operatorname{rel} D \tag{2}$$

is defined. The relative cylinder object $I(X, D)$ is the push out in $\infty\mathbf{End}$

$$\begin{array}{ccc} [0,1]\otimes X & \longrightarrow & I(X,D) \\ \uparrow & & \uparrow \\ [0,1]\otimes D & \xrightarrow{pr} & D \end{array} \tag{3}$$

We point out that a push out P of $(A \leftarrow B \rightarrow C)$ in $\infty\mathbf{End}$ is given by the push out $\hat{P}$ of $(\hat{A} \leftarrow \hat{B} \rightarrow \hat{C})$ in **Top**; the open part of P is the push out of the open parts in **Top**.

We say that an ∞-space Z is an ∞-<u>set</u> if the open part of Z is a discrete space in **Top**. The ∞-set Z is <u>empty</u> if the open part of Z is the empty space in **Top**. An ∞-space Y is obtained by <u>attaching $(n+1)$-cells</u> if an ∞-set Z together with a push out diagram

$$\begin{array}{ccc} B^{n+1}\otimes Z & \longrightarrow & Y \\ \cup & & \cup \\ S^n\otimes Z & \longrightarrow & X \end{array} \tag{4.8}$$

is given. Here (B^{n+1}, S^n) is a ball in **Top** with $n \geq 0$. We now define the appropriate notion of CW-complex in $\infty\mathbf{End}$. A <u>(relative) ∞-CW-complex</u> $(X_{\geq 0}, D)$ is given by a sequence of inclusions

$$D \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \ldots$$

in $\infty\mathbf{End}$ which is <u>finite dimensional</u>, that is, there is $N \geq 0$ such that $X_N \rightarrow X_{N+k}$ is the identity for $k \geq 0$. Here X_0 is the coproduct in $\infty\mathbf{End}$ of D and an ∞-set Z_0 and X_{n+1} is obtained from X_n by attaching $(n+1)$-cells as in (4.8). Since $(X_{\geq 0}, D)$ is finite dimensional the limit $X = \lim X_i = X_N$ is defined in $\infty\mathbf{End}$. We also write $(X_{\geq 0}, D) = (X, D)$. We say that (X, D) is <u>reduced</u> if the ∞-set Z_0 of 0-cells is empty, that is $X_0 = D$. Moreover (X, D) is <u>normalized</u> if all attaching maps, $n \geq 1$,

$$f_n : S^{n-1} \otimes Z_n \rightarrow X_{n-1}$$

carry $* \otimes Z_n$ to X_0. Here $*$ is the basepoint of S^{n-1}. The ∞-set Z_n is termed the ∞-set of n-cells in (X, D). It is shown in Baues-Quintero [HI] that the <u>cellular approximation theorem</u> and the analogue of the <u>Blakers-Massey theorem</u> hold for ∞-CW-complexes.

We denote by

$$\pi_0^Z(X) = [Z, X]^{\emptyset} \tag{4.9}$$

the set of ∞-homotopy classes $Z \rightarrow X$ in $\infty\mathbf{End}/\simeq$. We say that an object $D \rightarrow X$ in $(\infty\mathbf{End})^D$ is <u>connected</u> if the induced map $\pi_0^Z(D) \rightarrow \pi_0^Z(X)$ is a surjection for all ∞-sets Z. For example the pair (X, X_0) is connected if X is an ∞-CW-complex. The next result is an analogue of (1.8).

(4.10) Lemma. *Let (X,D) be a relative ∞-CW-complex. Then there exists a normalized ∞-CW-complex (Y,D) together with a homotopy equivalence $Y \to X$ under D in ∞***End***. If (X,D) is connected then (Y,D) can be chosen to be reduced.*

Given an ∞-set Z_α and an ∞-map $\alpha : Z_\alpha \to D$ we define the spherical object S^n_α by the push out diagram in ∞**End**

$$\begin{array}{ccc} S^n \otimes Z_\alpha & \longrightarrow & S^n_\alpha \\ \cup & & \cup \\ * \otimes Z_\alpha = Z_\alpha & \xrightarrow{\alpha} & D \end{array}$$

and we define for X in $(\infty\mathbf{End})^D$ the homotopy group $(n \geq 1)$

$$\pi^\alpha_n(X) = [S^n_\alpha, X]^D \tag{4.11}$$

where we use (4.7). Let $D \to X$ and $D \to Y$ be connected objects in $(\infty\mathbf{End})^D$. An ∞-map $f : X \to Y$ under D is a weak equivalence in ∞**End** if for all α and $n \geq 1$ the induced map $\pi^\alpha_n x \to \pi^\alpha_n Y$ is an isomorphism. As a consequence of the general Whitehead-theorem (IV.4.6) we get:

(4.12) Theorem. *Let (X,D) and (Y,D) be connected relative ∞-CW-complexes. Then a weak equivalence $X \to Y$ under D in ∞***End** *is a homotopy equivalence under D, that is, an isomorphism in* $(\infty\mathbf{End})^D_c/\simeq \mathrm{rel}\, D$.

The analogue of CW-approximation as in (1.10), however, does not hold. This shows that ∞-CW-complexes form typical examples for the definitions in chapter IV. Let $(\infty\mathbf{CW})^D$ be the full subcategory of $(\infty\mathbf{End})^D_c$ consisting of normalized reduced relative ∞-CW-complexes (X,D). We now define the coefficient functor

$$c : (\infty\mathbf{CW})^D/\simeq \mathrm{rel}\, D \to \infty\mathbf{Coef}(D) \tag{4.13}$$

Here the objects of $\infty\mathbf{Coef}(D)$ are presentations ∂_X which are elements

$$\partial_X \in [S^1_\alpha, X_1]^D$$

where S^1_α is a spherical object and (X_1, D) is a 1-dimensional reduced relative ∞-CW-complex. By choosing an attaching map f_2 representing ∂_X we obtain the 2-dimensional normalized reduced CW-complex X_2 associated to ∂_X. A map $u : \partial_X \to \partial_Y$ in $\infty\mathbf{Coef}(D)$ is an element $u \in [X_1, Y_2]^D$ which admits an extension $X_2 \to Y_2$. There is an obvious composition of such maps. The coefficient functor c in (4.13) carries the ∞-CW-complex (X,D) to ∂_X where ∂_X is represented by the attaching map of 2-cells. Moreover c carries a cellular map $f : X \to Y$ to the map u represented by the restriction $f_1 : X_1 \to Y_1$ of f. At this point it is quite complicated to give a more algebraic description of the coefficient category $\infty\mathbf{Coef}(D)$ above. In the examples (1.13), (2.13), (3.8) it was possible to describe the coefficient categories by use of groupoids. Such a description is a lot more complicated for the category $\infty\mathbf{Coef}(D)$.

We associate with each object ∂_X in $\infty\mathbf{Coef}(D)$ an additive category $\mathbf{mod}(\partial_X)$ as follows. Let $n \geq 2$. The objects in $\mathbf{mod}(\partial_X)$ are given by the coproduct under D $S^n_\alpha \vee X_2$ which is the push out in $\infty\mathbf{End}$ of $S^n_\alpha \leftarrow D \rightarrow X_2$. Here S^n_α is a spherical object (4.11) and X_2 is associated to ∂_X. Morphisms are commutative diagrams

$$\begin{tikzcd} X_2 \arrow[r, equal] \arrow[d] & X_2 \arrow[d] \\ S^n_\alpha \vee X_2 \arrow[r, "f"] \arrow[d, "(0,1)"] & S^n_\beta \vee X_2 \arrow[d, "(0,1)"] \\ X_2 \arrow[r, equal] & X_2 \end{tikzcd} \tag{4.14}$$

in the homotopy category $(\infty\mathbf{End})^D_c/\simeq \operatorname{rel} D$. Here $0 : S^n_\alpha \rightarrow D \rightarrow X_2$ is given by the retraction $S^n_\alpha \rightarrow D$ of the spherical object. Given α and β the sum in the additive category $\mathbf{mod}(\partial_X)$ is given by

$$(S^n_\alpha \vee X_2) \oplus (S^n_\beta \vee X_2) = S^n_{(\alpha,\beta)} \vee X_2$$

where $(\alpha, \beta) : Z_\alpha \amalg Z_\beta \rightarrow D$ is defined by α, β on the coproduct in $\infty\mathbf{End}$. The initial object in $\mathbf{mod}(\partial_X)$ is given by $0 = X_2 = S^n_\alpha \vee X_2$ where $\alpha : Z_\alpha \rightarrow D$ is defined on the empty ∞-set Z_α. The "partial suspension" shows that the category $\mathbf{mod}(\partial_X)$ does not depend on the choice of n with $n \geq 2$. Therefore we omit n in the notation and we write

$$S^n_\alpha \vee X_2 = S_\alpha \vee X_2 \in \mathbf{mod}(\partial_X)$$

for an object in the additive category $\mathbf{mod}(\partial_X)$. We point out that $\mathbf{mod}(\partial_X)$ here is not a subcategory of a canonical abelian category so that we do not have an obvious inifnity analogue of the embedding (1.14).

Each map $u : \partial_X \rightarrow \partial_Y$ induces a functor

$$u_* : \mathbf{mod}(\partial_X) \rightarrow \mathbf{mod}(\partial_Y) \tag{4.15}$$

which carries $S^n_\alpha \vee X_2$ to $S^n_\alpha \vee Y_2$ and which carries f in (4.14) to the map

$$((1 \vee u)f, 1) : S^n_\alpha \vee Y_2 \rightarrow S^n_\beta \vee Y_2$$

Now we choose a set $\mathcal{A}$ of elements α where $\alpha : Z_\alpha \rightarrow D$ is an ∞-map defined on an ∞-set Z_α. Then the enveloping functor

$$U_{\mathcal{A}} : \infty\mathbf{Coef}(D) \rightarrow \mathbf{Ringoids} \tag{4.16}$$

carries ∂_X to the full subcategory of $\mathbf{mod}(\partial_X)$ consisting of objects $S^n_\alpha \vee X_2$ with $\alpha \in \mathcal{A}$. Moreover $U_{\mathcal{A}}$ is defined on morphisms u by u_* in (4.15).

Each normalized reduced relative ∞-CW-complex (X, D) yields canonical ∞-maps

$$\alpha_n : Z_n \to D \tag{4.17}$$

where Z_n is the ∞-set of n-cells of (X, D). Here α_n is the restriction of the attaching map f_n. In fact

$$\begin{array}{ccc} S^{n-1} \otimes Z_n & \xrightarrow{f_n} & X_{n-1} \\ \uparrow & & \uparrow \\ * \otimes Z_n = Z_n & \xrightarrow{\alpha_n} & D \end{array}$$

commutes since (X, D) is normalized and reduced. For the 1-skeleton X_1 we also obtain the coaction map

$$\mu : X_1 \to X_1 \vee S^1_{\alpha_1} \quad \text{in } (\infty\mathbf{End})^D_c/\!\simeq \text{rel}\, D \tag{4.18}$$

which is defined as in (1.25).

(4.19) Definition. Let (X, D) be a relative ∞-CW-complex which is normalized and reduced so that the ∞-maps $\alpha_n : Z_n \to D$ are defined for $n \geq 1$; see (4.17). Let $\partial_X : S^1_{\alpha_1} \to X_1$ be given by the attaching map of 2-cells in X. Then there is a well defined <u>chain complex</u>

$$\begin{cases} C_*(X, D) \quad \text{in } \mathbf{mod}(\partial_X) \quad \text{with} \\ C_n(X, D) = S_{\alpha_n} \vee X_2 \quad \text{for } n \geq 1; \quad \text{see (4.14)} \end{cases} \tag{1}$$

and $C_n(X, D) = 0$ for $n \leq 0$. Moreover a cellular map $f : (X, D) \to (Y, D)$ under D induces $u : \partial_X \to \partial_Y$ in $\infty\mathbf{Coef}(D)$ and a chain map

$$f_* : u_* C_*(X, D) \to C_*(Y, D) \tag{2}$$

in $\mathbf{mod}(\partial_Y)$. Here we use u_* in (4.15). If D is an ∞-set we define

$$\alpha_0 : Z_0 = D$$

by the identity of D. In this case there is a well defined <u>augmented chain complex</u>

$$\begin{cases} C_*(X) = \operatorname{aug} C_*(X, D) \quad \text{in } \mathbf{mod}(\partial_X) \quad \text{with} \\ C_n(X) = S_{\alpha_n} \vee X_2 \quad \text{for } n \geq 0 \end{cases} \tag{3}$$

and $C_n(X) = 0$ for $n < 0$. These chain complexes are exactly the infinity-analogue of the chain complexes in (1.26). We define (1) by the general procedure in (V, § 2). The augmentation functor aug is described in (II, § 6).

Using the chain complexes in (4.19) we obtain for each (right) $\mathbf{mod}(\partial_X)$-module M the cochain complexes of abelian groups $M(C_*(X, D))$ and $M(C_*(X))$ so that the <u>cohomology with coefficients</u> in M

$$\begin{cases} H^n(X,D;M) = H^n M(C_*(X,D)) \\ H^n(X;M) = H^n M(C_*(X)) \end{cases} \tag{4.20}$$

is defined. As an example we observe that homotopy groups (4.11) yield canonically $\mathbf{mod}(\partial_X)$-modules as follows.

Let $D \to Y$ be an ∞-map and let (X,D) be a normalized reduced relative ∞-CW-complex. Moreover let

$$u : X_1 \to Y$$

be an ∞-map under D which admits an extension $X_2 \to Y$. Then the right $\mathbf{mod}(\partial_X)$-module

$$u^*\pi_n(Y) : \mathbf{mod}(\partial_X)^{\mathrm{op}} \to \mathbf{Ab} \tag{4.21}$$

is defined for $n \geq 2$. This module carries $S_\alpha \vee X_2$ to the homotopy group $\pi_n^\alpha(Y)$ in (4.11). Moreover a map $f : S_\beta \vee X_2 \to S_\alpha \vee X_2$ induces the homomorphism

$$f^* : \pi_n^\alpha(Y) \to \pi_n^\beta(Y)$$

which carries $(a : S_\alpha^n \to Y) \in \pi_n^\alpha(Y)$ to the composite

$$f^*(a) : S_\beta^n \xrightarrow{f'} S_\alpha^n \vee X_2 \xrightarrow{(a,u)} Y$$

Here $f' = f \mid S_\beta^n$ is the restriction of f. As a special case we get for the inclusion $u : X_1 \subset X$ the $\mathbf{mod}(\partial_X)$-module $\pi_n(X)$ defined by (4.21).

The coefficients $u^*\pi_n(Y)$ show that the cohomology $H^m(X,D;u^*\pi_n(Y))$ is defined for $m \in \mathbb{Z}$, $n \geq 2$. This is needed in the following theorem of obstruction theory.

(4.22) Theorem. *Let (X,D) be a normalized reduced relative ∞-CW-complex and let $f : D \to Y$ be an ∞-map which admits an extension $g : X_n \to Y$, $n \geq 2$. Then the restriction $g \mid X_{n-1}$ admits an extension $\bar{g} : X_{n+1} \to Y$ in ∞**End** if and only if an obstruction element*

$$\mathcal{O}(g \mid X_{n-1}) \in H^{n+1}(X,D,u^*\pi_n Y)$$

vanishes. Here $u : X_1 \to Y$ is the restriction of g.

This is the infinity-analogue of a classical theorem of obstruction theory. We can also define the homology $H_n(X,D)$ and $H_n(X)$ which are right $\mathbf{mod}(\partial_X)$-modules

$$H_n(X,D), H_n(X) : \mathbf{mod}(\partial_X)^{\mathrm{op}} \to \mathbf{Ab}. \tag{4.23}$$

They carry the object $S_\alpha \vee X_2$ to the abelian group

$$\begin{aligned} H_n(X,D)(S_\alpha \vee X_2) &= H_n \operatorname{Hom}(S_\alpha \vee X_2, C_*(X,D)) \\ H_n(X)(S_\alpha \vee X_2) &= H_n \operatorname{Hom}(S_\alpha \vee X_2, C_* X) \end{aligned}$$

Here Hom denotes the set of morphisms in $\mathbf{mod}(\partial_X)$. These $\mathbf{mod}(\partial_X)$-modules describe the ∞-analogue of (1.30). We use them for the following homological Whitehead theorem which is a special case of (VI, § 7).

(4.24) Theorem. *Let $f : (X, D) \to (Y, D)$ be a cellular map between normalized reduced relative ∞-CW-complexes in $(\infty\mathbf{End})_c^D$. Then $f : X \to Y$ is a homotopy equivalence under D (i.e. an isomorphism in the homotopy category $(\infty\mathbf{End})_c^D/\simeq \mathrm{rel}\, D$) if and only if the coefficient functor c in (4.13) induces an isomorphism $u : \partial_X \to \partial_Y$ in $\infty\mathbf{Coef}(D)$ and one of the following conditions (i), (ii) are satisfied.*

(i)

$$f_* : u_*(C_*(X, D)) \to C_*(Y, D)$$

is a homotopy equivalence of chain complexes in $\mathbf{mod}(\partial_Y)$.

(ii)

$$f_* : H_n(X, D) \to (u_*)^* H_n(Y, D)$$

is an isomorphism of $\mathbf{mod}(\partial_X)$-modules for $n \geq 1$.

Next we consider the Hurewicz homomorphism and the exact sequence of J.H.C. Whitehead for ∞-spaces. As a special case of (V.3.4) we get:

(4.25) Theorem. *Let (X, D) be a connected relative ∞-CW-complex. Then the following sequence is an exact sequence of $\mathbf{mod}(\partial_X)$-modules, $n \geq 2$.*

$$\begin{gathered} \cdots \to \Gamma_n(X, D) \to \pi_n X \xrightarrow{h} H_n(X, D) \to \Gamma_{n-1}(X, D) \to \dots \\ \Gamma_2(X, D) \to \pi_2 X \to H_2(X, D) \to \Gamma_1(X, D) \to 0 \end{gathered}$$

Moreover the sequence is natural in (X, D).

The homomorphism h is the Hurewicz homomorphism for the modules $\pi_n X$ and $H_n(X, D)$ defined in (4.21) and (4.23) respectively. The module $\Gamma_n(X, D)$ for $n \geq 3$ is defined by

$$\Gamma_n(X, D) = \mathrm{image}\{\pi_n(X_{n-1}) \to \pi_n(X_n)\}$$

The definition of Γ_1 and Γ_2 is more complicated; see (V.5.3) and (II, § 2). As in all the sections § 1, § 2, § 3 we also have the following results which describe a connection with algebraic K-theory.

(4.26) Definition. Let D be an ∞-space and let $\mathcal{A}$ be a set of ∞-maps $\alpha : Z_\alpha \to D$ where Z_α is an ∞-set. We say that an ∞-map $\alpha : Z \to D$ is $\mathcal{A}$-finite if $\beta_1, \dots, \beta_k \in \mathcal{A}$ together with a commutative diagram in $\infty\mathbf{End}$

$$\begin{tikzcd} Z \arrow[r, equal, "\chi_\alpha"] \arrow[dr, "\alpha"'] & Z_{\beta_1} \amalg \cdots \amalg Z_{\beta_k} \arrow[d, "(\beta_1,\dots,\beta_k)"] \\ & D \end{tikzcd}$$

are given where χ_α is an isomorphism and $\amalg$ is the coproduct in ∞**End**. A normalized reduced relative ∞-CW-complex (X, D) is $\mathcal{A}$-finite if all functions $\alpha_n : Z_n \to D$, $n \geq 1$, in (4.17) are $\mathcal{A}$-finite. Recall that in the definition of an ∞-CW-complex (X, D) we assume that (X, D) is finite dimensional.

Now let (X, D) and (Y, D) be normalized reduced relative ∞-CW-complexes. A domination (X, f, g, H) of Y in $(\infty\mathbf{End})_c^D$ is given by ∞-maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y \quad \text{under} \quad D \tag{4.27}$$

and an ∞-homotopy $H : gf \simeq 1 \operatorname{rel} D$. The domination has dimension $\leq n$ if $\dim(X, D) \leq n$ and the domination is $\mathcal{A}$-finite if (X, D) is $\mathcal{A}$-finite. As a special case of (VII.2.4) we get the following infinity version of a classical result of Wall; see (1.36).

(4.28) Theorem. *Let D be an ∞-space and let (Y, D) be a normalized reduced relative ∞-CW-complex with $\partial_Y = c(Y, D) \in \infty\mathbf{Coef}(D)$ defined by the coefficient functor (4.13). If (Y, D) admits an $\mathcal{A}$-finite domination in $(\infty\mathbf{End})_c^D$ then the finiteness obstruction*

$$[Y] = [C_*(Y, D)] \in K_0(U_{\mathcal{A}}(\partial_Y))$$

*is defined. Here $U_{\mathcal{A}}$ is the enveloping functor in (4.16) and K_0 is the reduced projective class group; see (VII, § 1). Moreover $[Y] = 0$ if and only if there exists an $\mathcal{A}$-finite normalized reduced ∞-CW-complex (X, D) and a homtoopy equivalence $X \to Y$ under D in ∞**End**.*

This result implies also a non relative version for a ∞-CW-complexes $(X, \emptyset)$ where $\emptyset$ is empty. Compare the remark following (2.36).

We now describe the infinity version of classical results of J.H.C. Whitehead for simple homotopy equivalences.

(4.29) Definition. Let D be an ∞-space which is allowed to be empty and let

$$\mathbf{K} = (\infty\mathbf{End})_c^D \tag{1}$$

be the category in which the objects are ∞-cofibrations $D \rightarrowtail X$; see (4.7) (1). Moreover let $\mathcal{D}$ be a set of ∞-sets with the property that the empty ∞-set is in $\mathcal{D}$ and that for $A, B \in \mathcal{D}$ also the coproduct $A \amalg B$ in ∞**End** is in $\mathcal{D}$. Then each $A \in \mathcal{D}$ yields the object

$$A \amalg D \quad \text{in } (\infty\mathbf{End})_c^D \tag{2}$$

which we call a discrete object in $\mathbf{K}$. A $\mathcal{D}$-complex is a relative ∞-CW-complex (L, D) for which the ∞-set Z_r of n-cells is an element in $\mathcal{D}$, $n \geq 0$. Let (B^{n+1}, S^n, P^n, Q^n) be a ball pair as defined in (1.37) (3) with $n \geq 0$. For $A \in \mathcal{D}$ we consider a pushout diagram in $\infty\mathbf{End}$ (compare (4.4))

$$\begin{array}{ccc} B^{n+1} \otimes A & \longrightarrow & K \\ \cup & & \cup \\ P^n \otimes A & \xrightarrow{f} & L \end{array}$$

where f is given by a pair map

$$f : (P^n \otimes A, S^{n-1} \otimes A) \to (L_n, L_{n-1}).$$

Then (K, D) is again a $\mathcal{D}$-complex which we call an elementary expansion of L. The inclusion $L \subset K$ is a homotopy equivalence in $\infty\mathbf{End}$. Any retraction $K \to L$ is termed an elementary collapse. A simple homotopy equivalence $f : L \to L'$ under D is a finite sequence of elementary collapses and expansions respectively.

Let $\mathcal{D}$-**cell** be the full subcategory of $(\infty\mathbf{End})_c^D$ consisting of $\mathcal{D}$-complexes (L, D). In (VIII, § 8) we define a functor

$$\mathrm{Wh} : \mathcal{D}\text{-}\mathbf{cell}/\simeq \operatorname{rel} D \to \mathbf{Ab}$$

which carries (L, D) to the Whitehead group $\mathrm{Wh}(L, D)$. As a special case of (VIII.8.3) we get:

(4.30) Theorem. *Let D be an ∞-space which is allowed to be empty. There is a function τ assigning to any homotopy equivalence $f : Y \to L$ in $(\infty\mathbf{End})_c^D$ between $\mathcal{D}$-complexes Y, L an element $\tau(f) \in \mathrm{Wh}(L, D)$. Moreover $\tau(f) = 0$ if and only if f is ∞-homotopic* $\operatorname{rel} D$ *to a simple homotopy equivalence under D.*

The Whitehead group $\mathrm{Wh}(L, D)$ can be computed algebarically by the following result which is a special case of (VIII.12.7).

(4.31) Theorem. *Let D be an ∞-space which may be empty. Let (L, D) be a normalized finite dimensional $\mathcal{D}$-complex and let $\partial_L = c(L, L_0) \in \infty\mathbf{Coef}(L_0)$ be given by the coefficient functor (4.13). Then the algebraic Whitehead group*

$$\mathrm{Wh}(\partial_L) = K_1^{\mathrm{iso}}(U_{\mathcal{A}}(\partial_L))/\sim$$

is defined. Here $\mathcal{A}$ is the set of all ∞-maps $A \to L_0$ with $A \in \mathcal{D}$. The functor $U_{\mathcal{A}}$ is the enveloping functor on $\infty\mathbf{Coef}(L_0)$ in (4.16) and K_1^{iso} is the "isomorphism torsion group" in (VIII, § 10). Moreover there is an isomorphism of abelian groups

$$\tau : \mathrm{Wh}(L, D) = \mathrm{Wh}(\partial_L)$$

In order to translate the general theory in the following chapters to the special case of ∞-spaces one needs the following list.

$\mathbf{T}$ (I.1.11)	This is the full subcategory of $(\infty\mathbf{End})_c^D/\simeq \operatorname{rel} D$ of reduced 1-dimensional ∞-CW-complexes (X_1, D). A cogroup is a spherical object S^1_α and the coaction is defined in (4.18).
Coef **(I.4.1)**	$\infty\mathbf{Coef}(D)$ in (4.13).
$\mathbf{mod}(\partial_X)$ (I.5.7)	$\mathbf{mod}(\partial_X)$ in (4.14).
$U_{\mathcal{A}}$ (I.5.11)	$U_{\mathcal{A}}$ in (4.16).
$(\mathbf{C}, \mathbf{T})$ (V.1.1)	$(\infty\mathbf{End})_c^D$ is a homological cofibration category with $\mathbf{T}$ defined above.
Complex **(IV.2.2)**	Contains the subcategory of $(\infty\mathbf{End})_c^D$ consisting of normalized reduced relative ∞-CW-complexes (X, D) and cellular maps.
C_* (V.2.3)	This is the chain functor in (4.19).
$(\mathbf{C}, \mathcal{D})$ (VIII.5.1) (VIII.13.3)	$(\infty\mathbf{End})_c^D$ is a homological cellular I-category with the cylinder $I(X, D)$ and the class $\mathcal{D}$ of discrete objects in (4.29). Here D is allowed to be empty.
$(\square_X^{k+1}, \Sigma_X^k, P_X^k, Q_X^k)$ (VIII.4.5)	For $X \in (\infty\mathbf{End})_c^D$ the ball pair is the push out of $B \otimes X \supset B \otimes D \to D$ where $B \otimes D \to D$ is the projection and where $B = (B^{k+1}, S^k, P^k, Q^k)$ is the ball pair in (1.37) (3).

Chapter B: Examples and Applications in Algebraic Homotopy Theories

In this chapter we describe algebraic categories in which the theory of this book can be applied. We consider the category of differential algebras and certain categories of simplicial objects. In such algebraic categories we can consider complexes which correspond to CW-complexes in topology. There are, however, no obvious ball pairs in the categories, so that we do not apply the results on simple homotopy equivalences in chapter VIII.

In chapter C and D we study also homotopy theories of simplicial objects from a different point of view. The theory in § 2 is based on the "normal" Blakers-Massey theorem for simplicial objects while the theory in chapter C is based on a "delicate" Blakers-Massey theorem for simplicial objects.

1 Homotopy Theory of Chain Algebras

Let R be a commutative ring with unit 1. Thus a left R-module M is also a right R-module and we have the tensor product $M \otimes N$ of R-modules M and N which is an R-module by $r \cdot (x \otimes y) = (rx) \otimes y = x \otimes (ry)$ for $r \in R, x \in M, y \in N$. An <u>algebra</u> A is an R-module A together with an R-linear map

$$A \otimes A \to A \quad \text{carrying} \quad x \otimes y \quad \text{to} \quad x \cdot y \tag{1.1}$$

and an element $1 \in A$ satisfying $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot 1 = 1 \cdot x = x$ for $x, y, z \in A$. Let **Alg** be the category of algebras; morphisms $f : A \to B$ in **Alg** are R-linear maps which satisfy $f(1) = 1$ and $f(x \cdot y) = (fx) \cdot (fy)$. The ring R is an algebra which is the initial object of **Alg** also denoted by $R = *$.

We have the forgetful functor $\varphi : \mathbf{Alg} \to \mathbf{Set}$ which carries the algebra A to the underlying set of the R-module A. The left adjoint F of φ yields for each set Z the <u>free algebra</u> $F(Z)$. This is the tensor algebra $F(Z) = T(V)$ where V is the free R-module generated by Z. For the empty set $\emptyset$ we set $F(\emptyset) = R = *$. Let **alg**ndex**alg** be the full subcategory of **Alg** consisting of free algebras $F(Z)$ where Z is a set. Each free algebra $F(Z)$ is an (abelian) cogroup in the category **Alg** with the comultiplication

$$\mu : F(Z) \to F(Z) \vee F(Z) \tag{1.2}$$

defined by $\mu(z) = i_1 z + i_2 z$ for $z \in Z$. Here $A \vee B$ denotes the coproduct of algebras satisfying $F(Z) \vee F(Z') = F(Z \amalg Z')$ where $Z \amalg Z'$ is the disjoint union of sets.

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Moreover i_1, i_2 are inclusions $i_1 : A \to A \vee B$ and $i_2 : B \to A \vee B$. This shows that the subcategory

$$\mathbf{T} = \mathbf{alg} \subset \mathbf{Alg} \tag{1.3}$$

is a theory of cogroups in the sense of (I.1.9) below.

(1.4) Remark. Let $\mathbf{alg}^\sharp$ be the full subcategory of $\mathbf{Alg}$ consisting of free algebras $F(Z)$ where $Z = \{1, \dots, n\}$ is a finite set with $n \geq 0$. Then $\mathbf{S} = \mathbf{alg}^\sharp$ is a single sorted theory and the category of models of $\mathbf{alg}^\sharp$ satisfies $\mathbf{model}(\mathbf{S}) = \mathbf{Alg}$; compare (I.1.5). The category of free objects in $\mathbf{model}(\mathbf{S})$ is the category $\mathbf{alg}$; compare (I.2.6).

For each algebra A we can choose a surjective map $q : X \to A$ in $\mathbf{Alg}$ where X is a free algebra. Moreover we can choose a free algebra X'' and a map $\partial_X : X'' \to X$ in $\mathbf{alg}$ for which the ideal $I(\mathrm{image}(\partial_X))$ in X generated by $\mathrm{image}(\partial_X)$ coincides with $\mathrm{kernel}(q)$. We call ∂_X a presentation of the algebra

$$A = X/I(\mathrm{image}(\partial_X)). \tag{1.5}$$

This shows that the category of coefficients $\mathbf{Coef}$ defined for $T = \mathbf{alg}$ in (I.4.1) below admits an equivalence of categories

$$\mathbf{Coef} \xrightarrow{\sim} \mathbf{Alg} \tag{1.6}$$

which carries the presentation ∂_X to the quotient A in (1.5). Hence the category $\mathbf{Alg}$ can be obtained in two ways from free algebras: On the one hand side $\mathbf{Alg}$ in the category of models of the single sorted theory $\mathbf{S} = \mathbf{alg}^\sharp$ in (1.4), on the other hand $\mathbf{Alg}$ is the category of coefficients associated to the theory of cogroups $\mathbf{T} = \mathbf{alg}$ in (1.6).

We say that M is an A-bimodule if M is a R-module together with actions of A on M from the right and from the left, that is, we have an R-linear map

$$A \otimes M \otimes A \to M \quad \text{carrying} \quad x \otimes m \otimes y \quad \text{to} \quad x \cdot m \cdot y \tag{1.7}$$

which satisfies $1 \cdot m \cdot y = m \cdot y, x \cdot m \cdot 1 = x \cdot m$ and $(x \cdot m) \cdot y = x \cdot (m \cdot y)$. If V is a free R-module then $M = A \otimes V \otimes A$ is the free A-bimodule generated by V (with the obvious action of A from the left and from the right). Let A^{op} be the opposite algebra of A. As modules we have $A^{\mathrm{op}} = A$ and we denote by $x^* \in A^{\mathrm{op}}$ the element corresponding to $x \in A$. The multiplication on A^{op} is defined by $x^* \cdot y^* = (y \cdot x)^*$ for $x, y \in A$. The algebra $A \otimes A^{\mathrm{op}}$ with the multiplication $(x \otimes y^*) \cdot (x_1 \cdot y_1^*) = (xx_1) \otimes (y_1 y)^*$ is called the enveloping algebra of A, see for example Cartan-Eilenberg [HA]. This yields the enveloping functor

$$U : \mathbf{Alg} \to \mathbf{Rings} \tag{1.8}$$

which carries A to $U(A) = A \otimes A^{\mathrm{op}}$. Here $\mathbf{Rings}$ denotes the category of rings. An A-bimodule M may be regarded as a right $U(A)$-module by setting $m \cdot (x \otimes y^*) = y \cdot m \cdot x$. Let ∂_X be a presentation of A then

$$\mathbf{mod}(\partial_X) = \mathbf{mod}(A) \tag{1.9}$$

is the category of free A-bimodules or equivalently of free right $U(A)$-modules. One can check that (1.9) coincides with $\mathbf{mod}(\partial_X)$ in (I.5.7) below by using (I.5.10). Therefore the enveloping functor (1.8) is a special case of (I.5.11). Each map $u : A \to B$ in **Alg** induces the functor

$$u_* : \mathbf{mod}(A) \to \mathbf{mod}(B)$$

which carries the $U(A)$-module M to $M \otimes_{U(A)} u^*U(B)$.

Next we consider graded R-modules and graded algebras. A graded R-module (positively graded) is a sequence $V = \{V_n,\ n \in \mathbb{Z}\}$ of R-modules with $V_i = 0$ for $i < 0$. An element $v \in V_n$ has degree $|v| = n$ and we write $v \in V$. A map $f : V \to W$ of degree r between graded modules is a sequence of R-linear maps $f_n : V_n \to W_{n+r}$ for $n \in \mathbb{Z}$. The suspension sV of V is defined by $(sV)_n = V_{n-1}$; let $s : V \to sV$ be the corresponding map of degree $+1$, that is $|sv| = |v| + 1$ for $v \in V$. A chain complex V is a graded module together with a map $d : V \to V$ of degree -1 satisfying $dd = 0$ and the homology of (V, d) is the graded R-module $H(V, d) = \operatorname{kernel} d / \operatorname{image} d$. The tensor product $V \otimes W$ of graded modules is defined by

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes V_j$$

If V and W are chain complexes than $V \otimes W$ is a chain complex with $d(v \otimes w) = (dv) \otimes w + (-1)^{|v|} v \otimes (dw)$. A chain map is a map $f : V \to W$ of degree 0 between chain complexes satisfying $df = fd$.

(1.10) Definition. A graded algebra A is a graded R-module A together with a map of degree 0

$$\mu : A \otimes A \to A \quad \text{with } \mu(x \otimes y) = x \cdot y$$

and an element $1 \in A_0$ such that the multiplication μ is associative and has 1 as a unit. This is a chain algebra if A is a chain complex and μ is a chain map, that is

$$d(x \cdot y) = (dx) \cdot y + (-1)^{|x|} x \cdot dy.$$

Let **DA** be the category of chain algebras; morphisms $f : A \to B$ in **DA** are maps of degree 0 satisfying $f(x \cdot y) = (fx) \cdot (fg)$, $f(1) = 1$ and $df = fd$.

The homology HA of a chain algebra A is a graded algebra with the multiplication

$$HA \otimes HA \xrightarrow{j} H(A \otimes A) \xrightarrow{\mu_*} HA \tag{1.11}$$

where $j(\{x\} \otimes \{y\}) = \{x \otimes y\}$. We point out that we have the natural map

$$\lambda : A \to H_0 A \tag{1.12}$$

with $\lambda x = 0$ for $|x| > 0$ and $\lambda x = \{x\}$ for $|x| = 0$. Here we use the fact that in a chain algebra each element of degree 0 is a cycle. We say that a graded module V is concentrated in degree k if $V_i = 0$ for $i \neq k$. We have the functors

$$\mathbf{Alg} \xrightarrow{i} \mathbf{DA} \xrightarrow{c} \mathbf{Alg} \tag{1.13}$$

Here i carries the algebra A to the corresponding chain algebra which is concentrated in degree 0 and c is the coefficient functor which carries the chain algebra A to $c(A) = H_0(A)$, compare (1.25) below.

(1.14) Definition. For a graded module V we have the tensor algebra $T(V)$ which is the graded algebra given by

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

where $V^{\otimes n} = V \otimes \cdots \otimes V$ is the n-fold tensor product. The multiplication in $T(V)$ is given by $V^{\otimes n} \otimes V^{\otimes m} = V^{\otimes (n+m)}$. The tensor algebra $T(V)$ is a free graded algebra if V_n is a free R-module for all $n \in \mathbb{Z}$. A chain algebra A is termed a free chain algebra if the underlying graded algebra admits an isomorphism $T(V) \cong A$ where $T(V)$ is a free graded algebra. Let

$$\mathbf{DFA} \subset \mathbf{DA}$$

be the full subcategory of free chain algebras. For a free chain algebra $A = (T(V), d)$ we define the cylinder

$$IA = (T(V' \oplus V'' \oplus sV), d) \tag{1.15}$$

where V' and V'' are two copies of V, that is $V' = V'' = V$. Let $i_0 : A \to IA$ and $i_1 : A \to IA$ be given by $i_0(x) = x'$ and $i_1(x) = x''$. Here $x' \in V'$ and $x'' \in V''$ are the elements which correspond to $x \in V$. We define the differential d on IA on generators by $dx' = i_0 dx, dx'' = i_1 dx$ and

$$dsx = x'' - x' - Sdx$$

where $S : A \to IA$ is the unique map of degree $+1$ satisfying $Sx = sx$ for $x \in V$ and

$$S(x \cdot y) = (Sx)(i_1 y) + (-1)^{|x|}(i_0 x)Sy$$

for $x, y \in A$. Two maps $f, g : A \to B$ in $\mathbf{DA}$ are homotopic, $f \simeq g$, if there exists a map $H : IA \to B$ in $\mathbf{DA}$ with $Hi_0 = f$ and $Hi_1 = g$.

(1.16) Definition. An algebra A in $\mathbf{Alg}$ is supplemented if an algebra map $\varepsilon : A \to * = R$ is given. A map $f : A \to B$ in $\mathbf{Alg}$ is supplemented if $\varepsilon f = \varepsilon$. Hence supplemented algebras and maps form the category $\mathbf{Alg}_*$ of objects over $*$ in $\mathbf{Alg}$. A chain algebra A is supplemented if $H_0 A$ is supplemented so that one has the composite

$$A \xrightarrow{\lambda} H_0A \xrightarrow{\varepsilon} R = *$$

in **DA**. Hence the category of supplemented chain algebras $\mathbf{DA}_*$ is the same as the category of objects over $*$ in **DA**. Accordingly let $\mathbf{DFA}_*$ be the category of objects over $*$ in **DFA**.

Using the notions of I-category in (II. Appendix A) we prove in Baues [AH] I, § 7 the following result.

(1.17) Proposition. *The categories* **DFA** *and* $\mathbf{DFA}_*$ *with the cylinder (1.15) are I-categories. Cofibrations are maps in* **DFA** *which carry generators to generators.*

According to III.7.4 the proposition implies that **DFA** and $\mathbf{DFA}_*$ are also cofibration categories. Moreover using (1.13) we have full inclusions

$$\mathbf{alg} \subset \mathbf{DFA}/\simeq \quad \text{and} \quad \mathbf{alg}_* \subset \mathbf{DFA}_*/\simeq \tag{1.18}$$

where $\mathbf{T} = \mathbf{alg}$, resp. $\mathbf{T} = \mathbf{alg}_*$ are theories of cogroups. A map $f : A \to B$ in **DA** is a <u>weak equivalence</u> if f induces an isomorphism $f_* : HA \cong HB$. The next result corresponds to the classical Whitehead theorem for CW-complexes.

(1.19) Theorem. *Let $f : A \to B$ be a map in* **DFA**, *resp.* $\mathbf{DFA}_*$. *Then f is a weak equivalence if and only if f is a homotopy equivalence (i.e. an isomorphism in the quotient category* $\mathbf{DFA}/\simeq$, *resp.* $\mathbf{DFA}_*/\simeq$*).*

We obtain (1.19) as a special case of (IV.3.11) below. For this we observe that a **T**-complex X in **DA** is the same as a free chain algebra $X = (T(V), d)$ with the filtration of <u>skeleta</u> $X_{(i)}$ given by the subalgebras

$$X_{(i)} = (T(V^{i-1}), d)\,, \quad i \geq 1, \tag{1.20}$$

of X. Here V^i is the submodule of the graded module V with $(V^i)_j = V_j$ for $j \leq i$ and $(V^i)_j = 0$ otherwise. Using the filtration (1.20) it is easy to see that one has an equivalence of categories

$$\mathbf{DFA} = \mathbf{Complex}$$

where the right hand side is the category of **T**-complexes in (IV.2.2). We point out that the inclusion $\mathbf{alg} \subset \mathbf{DFA}$ carries a free algebra concentrated in degree 0 to a **T**-complex of dimension 1 since there is a shift in degree in (1.20).

Let $\Sigma = F(\{\sigma\})$ be the free algebra generated by one element σ in degree 0. Then Σ is supplemented by the map $\Sigma \to *$ which carries σ to $0 \in R$. One readily checks that for a pair (X, Y) of objects in **DA** (resp. $\mathbf{DA}_*$) one has natural isomorphisms ($n \geq 1$)

$$\begin{cases} \pi_n^{\Sigma}(X) = H_n(X) \\ \pi_{n+1}^{\Sigma}(X,Y) = H_{n+1}(X,Y) \end{cases} \tag{1.21}$$

where the left hand side is a homotopy group in **DA** (resp. $\mathbf{DA}_*$) and the right hand side is a homology group of the underlying chain complex. Since each free algebra $F(Z)$ is a coproduct $F(Z) = \bigvee_Z \Sigma$ we thus obtain for $A = F(Z) \in \mathbf{alg}$ the homotopy groups

$$\begin{cases} \pi_n^A(X) = \underset{Z}{\times} H_n(X) \\ \pi_{n+1}^A(X,Y) = \underset{Z}{\times} H_{n+1}(X,Y) \end{cases}$$

where the right hand side is a product over the set Z of generators in A. Now it is an easy exercise to show that (IV.3.11) below implies (1.19).

(1.22) Remark. Let $\mathbf{DA}(\mathit{flat})$ be the full subcategory of **DA** consisting of all chain algebras A for which all A_n are flat R-modules, $n \in \mathbb{Z}$. We show in Baues [AH] I, § 7 that $\mathbf{DA}(\mathit{flat})$ is a cofibration category provided R is a principal ideal domain. Compare also Munkholm [DGA] and Gugenheim-Munkholm [Tor]. In this case **DFA** is the category of cofibrant objects in $\mathbf{DA}(\mathit{flat})$ and hence one has the equivalence of categories

$$\mathrm{Ho}(\mathbf{DA}(\mathit{flat})) \xrightarrow{\sim} \mathbf{DFA}/\simeq$$

where the left hand side denotes the localization with respect to weak equivalences. This result implies (1.19) in case R is a principal ideal domain. The equivalence carries a chain algebra Y in $\mathbf{DA}(\mathit{flat})$ to a free chain algebra X for which one has chosen a weak equivalence $X \xrightarrow{\sim} Y$. Here X is termed a free approximation of Y.

Next we consider the pushout diagram in **DA**

$$\begin{tikzcd} K \arrow[r] & K \cup_L Y \\ L \arrow[u, "i"] \arrow[r, "j"'] & Y \arrow[u] \end{tikzcd}$$

for which we get the following Blakers-Massey theorem where R is an arbitrary commutative ring.

(1.23) Theorem. *Let L, K, Y be free chain algebras and assume i and j carry generators to generators. For $n, m \geq 1$ let $H_s(K,L) = 0$ for $s \leq n-1$ and $H_t(Y,L) = 0$ for $t \leq m-1$. Then the induced map*

$$H_r(K,L) \to H_r(K \cup_L Y, Y)$$

is surjective for $r \leq n+m-1$ and bijective for $r \leq n+m-2$.

(1.24) Remark. Theorem (1.23) implies that the category **DFA** with $\mathbf{T} = \mathbf{alg}$ has the Blakers-Massey property in (IV.5.3) below. Here we use (1.20) and (1.22). Hence n and m in (1.23) correspond exactly to n and m in (IV.5.3) since there

is a shift of degree by +1. By (V.1.2) the Blakers-Massey property implies that $(\mathbf{DFA}, \mathbf{T} = \mathbf{alg})$ and $(\mathbf{DFA}_*, \mathbf{T} = \mathbf{alg}_*)$ are homological cofibration categories. It is easy to check the axioms of a homological cofibration category in (V.1.1) directly without the use of (1.23). We leave this to the reader; compare Baues [DA] C2.0.17 and C2.0.19.

Proof of (1.23). Using Baues [AH] I.7.21 and the glueing lemma Baues [AH] II.1.2 we may assume that the generators v in $K - L$ satisfy $|v| \geq n$ and that the generators w in $Y - L$ satisfy $|w| \geq m$. Now the spectral sequence of a cofibration Baues [AH] I.7.23 yields (1.23); compare Baues [AH] I.7.5 where we can omit the summand given by "$n = 0$" since we consider relative homology groups. q.e.d.

We have the coefficient functor

$$c : \mathbf{DFA}/\simeq \;\to \mathbf{Alg} \tag{1.25}$$

which carries A to the algebra $c(A) = H_0A$ which is the degree 0 part of the graded algebra HA in (1.11). This is a special case of the coeffcient functor (V.1.3).

(1.26) Definition. Let $X = (T(V), d)$ be a free chain algebra in **DFA** and let $A = H_0X$ be the associated coefficient algebra in **Alg**; see (1.25). Then there is a well defined chain complex

$$\begin{cases} C_*(X,*) \quad \text{in } \mathbf{mod}(A), \quad \text{see (1.9), with} \\ C_n(X,*) = A \otimes V_{n-1} \otimes A \quad \text{for } n \geq 1 \end{cases}$$

and $C_n(X,*) = 0$ otherwise. Moreover there is a well defined augmented chain complex

$$C_*(X) = \operatorname{aug} C_*(X,*) \quad \text{in } \mathbf{mod}(A) \quad \text{with}$$

$$C_n(X) = \begin{cases} A \otimes V_{n-1} \otimes A \quad \text{for } n \geq 1, \\ A \otimes A \quad \text{for } n = 0, \\ 0 \quad \text{for } n < 0. \end{cases}$$

Moreover a map $f : X \to Y$ in **DFA** induces a chain map

$$f_* : u_*C_*X \to C_*Y \quad \text{in } \mathbf{mod}(B)$$

where $B = H_0Y$ and $u = H_0(f) : A \to B$, see (1.9).

We get the chain complex $C_*(X,*)$ by the general procedure in (V, § 2). The augmented chain complex $C_*(X)$ is defined by (II, § 6) since $\mathbf{T} = \mathbf{alg}$ is weakly augmented by the maps

$$\varepsilon : F(Z) \to F(Z) \vee \Sigma \quad \text{in } \mathbf{alg} \tag{1.27}$$

with $\Sigma = F(\{\sigma\})$ generated by one element σ and $\varepsilon(z) = (1+\sigma)\cdot z\cdot(1-\sigma)$ for $z \in Z$; see (1.2). We leave it to the reader to check the properties in (I.7.11).

We get the differential ∂ of C_*X explicitly by the differential d of $X = (T(v), d)$ as follows. For $v \in V_n \subset C_{n+1}X$ with $n \geq 0$ let $\partial(v) \in C_nX$ be given by the formula

$$\partial(v) = \begin{cases} (\lambda v)\otimes 1 - 1\otimes(\lambda v) & \text{for } n = 0 \\ -\Lambda dv & \text{for } n = 1 \\ -(\lambda\otimes 1\otimes\lambda)qdv & \text{for } n \geq 2 \end{cases} \tag{1.28}$$

Here $\lambda : X \to H_0X = A$ is the map in (1.12) and

$$\Lambda : T(V_0) \to A\otimes V_0\otimes A$$

is the unique function with $\Lambda(1) = 0, \Lambda(v) = 1\otimes v\otimes 1$ for $v \in V_0$ and

$$\Lambda(x\cdot y) = (\lambda x)\cdot(\Lambda y) + (\Lambda x)\cdot(\lambda y)$$

for $x, y \in T(V_0)$. For $|v| = 1$ we have $dv \in X_0 = T(V_0)$ so that Λdv is well defined. Finally for $n \geq 2$ let q be the projection of

$$X_{n-1} = T(V_0)\otimes V_{n-1}\otimes T(V_0) \oplus (X_{(n-1)})_{n-1}$$

onto the direct summand $T(V_0)\otimes V_{n-1}\otimes T(V_0)$. Then for $v \in V_n$ with $|v| = n \geq 2$ the element $qdv \in T(V_0)\otimes V_{n-1}\otimes T(V_0)$ is well defined. This yields the formula for $\partial(v)$ in (1.28) for $n \geq 2$. One can check that the chain complex $C_*(X)$ given by (1.28) coincides with the chain complex defined by (V, § 2) and (II, § 6).

(1.29) Remark. There is a classical functor

$$\Omega B : \mathbf{DA}_* \to \mathbf{DFA}_* \tag{1}$$

which carries a supplemented chain algebra Y to the cobar construction Ω of the bar construction B of Y. Compare Husemoller-Moore-Stasheff [DN]. Then $X = \Omega B(Y)$ is actually a free chain algebra and one has a natural weak equivalence $\Omega B(Y) \to Y$ which is a homotopy equivalence in $\mathbf{DFA}_*$ provided Y is a free chain algebra by (1.19). Hence $\Omega B(Y)$ is a functorial free approximation of Y. If $A \in \mathbf{Alg} \subset \mathbf{DA}$ then $B(A)$ is the <u>reduced bar resolution</u> of A and

$$B(A, A) = C_*(\Omega BA) \tag{2}$$

is the <u>normalized bar resolution</u> of A; compare Mac Lane [H] chapter X. Here C_* is the chain functor in (1.26) and using (1.28) one can check that (2) holds. The <u>Hochschild homology</u> and the <u>Hochschild cohomology</u> of A with coefficients in an A-bimodule M is defined by

$$\begin{cases} HH_n(A, M) = H_n(B(A, A)\otimes_{U(A)} M) \\ HH^n(A, M) = H^n(\mathrm{Hom}_{U(A)}(B(A, A), M)) \end{cases} \tag{3}$$

See Mac Lane [H] X § 3, § 4.

According to this remark we define for any free chain algebra X in **DFA** with $A = H_0X$ the following homology and cohomology with coefficients in an A-bimodule M:

$$\begin{cases} HH_n(X,M) = H_n(C_*(X) \otimes_{U(A)} M) \\ HH^n(X,M) = H^n(\mathrm{Hom}_{U(A)}(C_*(X),M)) \end{cases} \tag{1.30}$$

For example if $A = H_0X$ is supplemented then R is an A-bimodule so that $HH_n(X,R)$ and $HH^n(X,R)$ are defined. Moreover we obtain the A-bimodule

$$\mathrm{Tor}_n^X(A,A) = HH_n(X,U(A)) = H_n(C_*(X))$$

which is the homology of the chain complex $C_*(X)$ in the abelian category of $U(A)$-modules.

If Y is a supplemented chain algebra which is not free we have the free approximation $X \xrightarrow{\sim} Y$ of Y given by $X = \Omega BY$. Then we define the (co-) homology of Y by

$$\begin{cases} HH_n(Y,M) = HH_n(X,M) \\ HH^n(Y,M) = HH^n(X,M) \end{cases}$$

It is often the case that this does not depend on the choice of the free approximation X of Y (in particular, if R is a principal ideal domain and Y is flat, see (1.22)). We now are ready to formulate the following homological Whitehead theorem which is a special case of (VI, § 7).

(1.31) Theorem. *Let $f : X \to Y$ be a map in* **DFA** *(resp.* **DFA**$_*$*). Then f is a homotopy equivalence if and only if the induced map*

$$u = f_* : A = H_0X \to B = H_0Y$$

is an isomorphism and one of the following conditions (i), (ii), (iii) is satisfied.

(i) $f_* : u_* C_*(X,*) \to C_*(Y,*)$ *is a homotopy equivalence of chain complexes in* $\mathbf{mod}(B)$.

(ii) $f_* : \mathrm{Tor}_n^X(A,A) \to u^*\mathrm{Tor}_n^Y(B,B)$ *is an isomorphism of A-bimodules for* $n \geq 1$.

(iii) For all B-bimodules N the induced map

$$f^* : HH^n(Y,N) \to HH^n(X;u^*N)$$

is an isomorphism for $n \geq 1$.

Let X be a free chain algebra with skeleta $X_{(i)}$ in (1.20). Then we obtain for $n \geq 2$ the $H_0(X)$-bimodule

$$\Gamma_n^{\Sigma}(X) = \mathrm{image}\{H_nX_{(n)} \to H_nX_{(n+1)}\}$$

This corresponds to the $\mathbf{mod}(H_0X)$-module $\Gamma_n^{\Sigma}(X) = \Gamma_{n+1}(X)$ in (V.5.3) where this module is also defined for $n = 0, 1$.

(1.32) Theorem. *For each free chain algebra X with $A = H_0X$ the following sequence is an excat sequence of A-bimodules* $(n \geq 1)$

$$\begin{aligned}&\to \Gamma_n^\Sigma(X) \to H_n(X) \xrightarrow{h} \mathrm{Tor}_{n+1}^X(A,A) \to \Gamma_{n-1}^\Sigma(X) \to \dots\\ &\to \Gamma_1^\Sigma(X) \to H_1(X) \to \mathrm{Tor}_2^X(A,A) \to \Gamma_0^\Sigma(X) \to 0\end{aligned}$$

The sequence is natural in X.

This sequence is a special case of the exact sequence of J.H.C. Whitehead in (V.5.4) and h is the analogue of the Hurewicz homomorphism.

(1.33) Remark. In Baues-Felix-Thomas [PA] we study the sequence (1.32). If R is a principal ideal domain and if H_0X is free as an R-module we show that $\Gamma_0^\Sigma X = \Gamma_1^\Sigma X = 0$ and that

$$\Gamma_2^\Sigma(X) = H_1(X) \otimes_{H_0X} H_1(X)$$

where the right hand side is the tensor product of the right $H_0(X)$-module $H_1(X)$ with the left $H_0(X)$-module $H_1(X)$ so that the tensor product has the obvious structure of an H_0X-bimodule. Moreover we show that in case $H_iX = 0$ for $0 < i < n$ then $\Gamma_i^\Sigma(X) = 0$ for $i < 2n$ and

$$\Gamma_{2n}^\Sigma X = H_n(X) \otimes_{H_0X} H_n(X).$$

Using (V.4.4) we get the next result concerning obstruction theory of chain algebras. Let $Y \subset X$ be an inclusion of free chain algebras which carries generators to generators. Let $(X,Y)_{(n)}$ be the relative n-skeleton given by the subalgebra of X generated by Y and $X_{(n)}$ in (1.20).

(1.34) Theorem. *Let $f : Y \to U$ be a map in* **DA** *which admits an extension $g : (X,Y)_{(n)} \to U$ with $n \geq 2$. Then the restriction $g_{n-1} = g \mid (X,Y)_{(n-1)}$ admits an extension $\bar{g} : (X,Y)_{(n+1)} \to U$ in* **DA** *if and only if an obstruction element*

$$\mathcal{O}(g_{n-1}) \in HH^{n+1}(X,Y;u^*H_{n-1}U)$$

vanishes. Here $u : H_0X \to H_0U$ is induced by g since $n \geq 2$.

This is the analogue of the classical result of obstruction theory for CW-complexes. By (VI.3.1) we have the following homotopy lifting property of the chain functor C_* in (1.26), (1.28).

(1.35) Theorem. *Let $\bar{f} : X \to Y$ be a map in* **DFA** *with $f = C_*(\bar{f}) : C_*(X,*) \to C_*(Y,*)$ given by (1.26). Let $\alpha : f \simeq g$ be a homotopy of the chain map f. Then there exists a homotopy $H : \bar{f} \simeq \bar{g}$ in* **DFA** *satisfying $C_*(\bar{g}) = g$ and $C_*(H) = \alpha$.*

There are many further results on the homotopy theory of **DFA** which can be deduced from the abstract theory; for example the tower of categories in (VI, § 5) and the model lifting property of the twisted chain functor; see (VI, § 3, § 8).

We leave it to the reader to formulate the appropriate explicit interpretation of such results in the homotopy theory of chain algebras. As a final application of the abstract theory to chain algebras we describe the following result on finiteness obstructions.

Let X, Y be free chain algebras. A domination (X, f, g, H) of Y in **DFA** is given by maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y$$

and a homotopy $H : gf \simeq 1$ in **DFA**. The domination is finite if X is finitely generated. As a special case of (VII.2.4) we get

(1.36) Theorem. *Let Y be a free chain algebra which admits a finite domination in* **DFA**. *Then the finiteness obstruction*

$$[Y] = [C_*(Y, *)] \in K_0(U(H_0Y))$$

is defined. Here U is the enveloping functor (1.8) and K_0 is the reduced projective class group; see (VII, § 1). Moreover $[Y] = 0$ if and only if there exists a finitely generated free chain algebra X and a homotopy equivalence $X \to Y$ in **DFA**.

This is the chain algebra analogue of the finiteness obstruction theorem of Wall.

2 Homotopy Theory of Connected Simplicial Objects in Algebraic Theories

In this section we discuss simplicial objects in the category of models of a single sorted theory with zero object **S**. The corresponding homotopy theory was recently considered by Schwede [SH].

The homotopy theory for **S**, however, behaves very different to the homotopy theory for a theory **T** of coactions in chapter C. The basic example for **S** is the category of pointed simplicial sets while the basic example for **T** is the category of simplicial groups.

Pointed simplicial sets have all the homotopy theoretic properties as pointed spaces in $\mathbf{Top}^*$ which are considered in (A, § 1) above. Accordingly one obtains for the homotopy theory of simplicial **S**-models similar reults as for pointed spaces in (A, § 1).

(2.1) Definition. A single sorted theory **S** is a category with a distinguished "generating" object $S \in \mathrm{Ob}(\mathbf{S})$ such that the objects of **S** are exactly the finite n-fold coproducts

$$nS = \underbrace{S \vee \cdots \vee S}_{n\text{-times}} \tag{1}$$

with $n \geq 0$. Here $0S = *$ is the initial object of **S** and $1S = S$. We say that **S** has a zero object if $*$ is also the final object of **S**. Let $\mathbf{S}^{\mathrm{op}}$ be the opposite category of

$\mathbf{S}$. A functor $\mathbf{S}^{\mathrm{op}} \to \mathbf{C}$ is the same as a contravariant functor $\mathbf{S} \to \mathbf{C}$. A model M of $\mathbf{S}$ is a functor

$$M : \mathbf{S}^{\mathrm{op}} \to \mathbf{Set} \tag{2}$$

which carries coproducts in $\mathbf{S}$ to products of sets. We call $M(S)$ the underlying set of M. Let

$$\mathbf{M} = \mathbf{model}(\mathbf{S}) \tag{3}$$

be the category of models; morphisms are natural transformations. We have the distinguished model

$$S = \mathbf{S}(-, S) : \mathbf{S}^{\mathrm{op}} \to \mathbf{Set} \tag{4}$$

which as well is denoted by S. The category $\mathbf{M}$ has arbitrary coproducts $\coprod$ so that for a set Z the free object

$$S \otimes Z = \coprod_{z \in Z} S \tag{5}$$

is defined. Let $\mathbf{M}_{\mathrm{free}} \subset \mathbf{M}$ be the full subcategory of free objects. If $\mathbf{S}$ has a zero object then also $\mathbf{M}$ has a zero object.

(2.2) Example. (a) Let $S = \{1\}$ be the set consisting of the number 1. Then the n-fold coproducts in $\mathbf{Set}$ $(n \geq 0)$

$$n\{1\} = \{1, \ldots, n\}$$

form the full subcategory $\mathbf{set}^\sharp$ of $\mathbf{Set}$ for which $\mathbf{model}(\mathbf{set}^\sharp) = \mathbf{Set}$.

(b) Let $S = S^0 = \{0, 1\}$ be the pointed set consisting of 2-elements with basepoint $* = 0$. Then the n-fold coproducts in $\mathbf{Set}^*(n \geq 0)$

$$nS^0 = \{0, 1, \ldots, n\}$$

form the full subcategory $(\mathbf{set}^*)^\sharp$ of $\mathbf{Set}^*$ for which $\mathbf{model}(\mathbf{set}^*)^\sharp = \mathbf{Set}^*$ is the category of pointed sets. Here $(\mathbf{set}^*)^\sharp$ is a single sorted theory with zero object $* = \{0\}$.

(c) Let $S = \mathbb{Z}$ be the group of integers. Then the n-fold coproducts in the category $\mathbf{Gr}$ of groups are the free groups

$$n\mathbb{Z} = \langle \{1, \ldots, n\} \rangle$$

which form the full subcategory $\mathbf{gr}^\sharp$ of $\mathbf{Gr}$ for which $\mathbf{model}(\mathbf{gr}^\sharp) = \mathbf{Gr}$. Here $\mathbf{gr}^\sharp$ is again a single sorted theory with zero object $*$ which is the trivial group consisting of one element.

(2.3) Remark. Schwede [SH] considers the "simplicial enriched" version of a single sorted theory with zero object which he calls (in the dual language) a "simplicial theory". The single sorted theories with zero objects considered in this section correspond exactly to "discrete simplical theories" in the sense of Schwede. All results in this section have a canonical generalization for the more general simplicial theories.

In the following let $\mathbf{S}$ be a single sorted theory with zero object. Then the category $\mathbf{M}$ of models of $\mathbf{S}$ has also a zero object $*$ and hence $* = \mathrm{const}(*)$ is the zero object of the category $\Delta\mathbf{M}$ of simplicial objects in $\mathbf{M}$. See (C.1.1) below. We have the inclusion

$$\mathrm{const} : \mathbf{M} \subset \Delta\mathbf{M} \tag{2.4}$$

which carries the model M to the constant simplicial object $\mathrm{const}(M)$ also denoted by M. The functor $\mathbf{M} \to \mathbf{Set}^*$ which carries M to the underlying set $M(S)$ yields the functor

$$\Delta\mathbf{M} \to \Delta\mathbf{Set}^* \tag{2.5}$$

which carries the simplicial model X to the underlying pointed simplicial set $X(S)$. The simplicial set $X(S)$ has the additional algebraic structure given by the theory $\mathbf{S}$. For example for $\mathbf{S} = \mathbf{gr}^\sharp$ in (2.2) (c) a simplicial model is the simplicial analogue of a topological group. We say that X is connected if $X(S)$ is connected; i.e. $\pi_0 X = *$. A map $f : X \to Y$ in $\Delta\mathbf{M}$ is a weak equivalence if the underlying map $f : X(S) \to Y(S)$ between simplicial sets is a weak equivalence. According to Dwyer-Hirschhorn-Kan [MC] or Schwede [SH] we have the following result.

(2.6) Theorem. *Let $\mathbf{S}$ be a single sorted theory with zero object and let $\mathbf{M}$ be the category of models of $\mathbf{S}$. Then the category $\Delta\mathbf{M}$ of simplicial models has the structure of a closed simplicial model category with weak equivalences defined above.*

The fibrations in $\Delta\mathbf{M}$ are the maps $f : X \to Y$ for which $f : X(S) \to Y(S)$ is a fibration of simplicial sets. Hence the fibrant objects X in $\Delta\mathbf{M}$ are exactly the simplicial models X for which the underlying simplicial set $X(S)$ satisfies the Kan extension condition (May [SO]). Let $\mathrm{Ho}(\Delta\mathbf{M})$ be the homotopy category of $\Delta\mathbf{M}$ obtained by localizing with respect to weak equivalences in $\Delta\mathbf{M}$. Moreover for X, Y in $\Delta\mathbf{M}$ let

$$[X, Y] = \mathrm{Ho}(\Delta\mathbf{M})(X, Y) \tag{2.7}$$

be the set of morphisms $X \to Y$ in $\mathrm{Ho}(\Delta\mathbf{M})$. For a cofibrant object X in $\Delta\mathbf{M}$ we can define the suspension ΣX by the push out diagram in $\Delta\mathbf{M}$

$$\begin{tikzcd} X \otimes S[1] \arrow[r] & \Sigma X \\ X \otimes * \arrow[u] \arrow[r] & * \arrow[u] \end{tikzcd}$$

Here we use (C.1.8) and (C.1.4) below. As usual the set $[\Sigma X, Y]$ is a group. In particular we have the equation of homotopy groups

$$\pi_n^S(X) = [\Sigma^n S, X] = \pi_n(X(S)), \quad n \geq 1, \tag{1}$$

where $\Sigma^n S$ is the n-fold suspension of $S = \mathrm{const}(S) \in \Delta\mathbf{M}$. Similarly relative homotopy groups satisfy

$$\pi_{n+1}^S(X, Y) = \pi_{n+1}(X(S), Y(S)) \tag{2}$$

The following crucial result is due to Schwede [SH]; this is the (non-delicate) Blakers-Massey theorem for the category $\Delta\mathbf{M}$. Clearly since pointed simplicial sets $\Delta\mathbf{Set}^*$ form an example of $\Delta\mathbf{M}$ by (2.2) (b) we cannot expect that a delicate version of the Blakers-Massey theorem holds in $\Delta\mathbf{M}$; though this is the case for $\Delta\mathbf{Gr}$ given by the example (4.2) (c); see chapter C below.

(2.8) Theorem. *Consider a push out diagram in $\Delta\mathbf{M}$ of cofibrant objects*

$$\begin{array}{ccc} K & \xrightarrow{\bar{j}} & K \cup_L Y \\ {\scriptstyle i}\uparrow & & \uparrow \\ L & \xrightarrow{j} & Y \end{array}$$

where i is a cofibration. Assume further that all objects are connected and that $\pi_i^S(K, L) = 0$ for $i \leq m$ and $\pi_i^S(Y, L) = 0$ for $i \leq n$. Then the induced map

$$\bar{j}_* : \pi_i(K, L) \to \pi_i(K \cup_L Y, Y)$$

is surjective for $i \leq n + m$ and bijective for $i \leq n + m - 1$.

For the example (2.2) (b) this is exactly the classical Blakers Massey theorem for connected spaces which are equivalent to connected pointed simplicial sets.

(2.9) Definition. Let $\mathbf{S}$ be a single sorted theory with zero object and generating object S and let $\mathbf{M}$ be the category of models of $\mathbf{S}$. Then the full subcategory

$$\mathbf{T}(\mathbf{S}) \subset \mathrm{Ho}(\Delta\mathbf{M})$$

consisting of the suspensions $\Sigma(S \otimes Z)$ of free objects $S \otimes Z = \mathrm{const}(S \otimes Z)$ is the theory of cogroups associated to $\mathbf{S}$; see (2.1) (5) and (2.4). Using (III.6.8) we set that $\mathbf{T}(\mathbf{S})$ is augmented by ΣS with augmentation maps induced by $Z \to \{*\}$.

For example for $\mathbf{S}$ in (2.2) (b) we get $\mathbf{T}(\mathbf{S}) = \mathbf{gr}$ the category of free groups; while for $\mathbf{S}$ in (2.2) (c) we get $\mathbf{T}(\mathbf{S}) = \mathbf{ab}$ the category of free abelian groups. As a consequence of (2.6) and (2.8) we have the following result where we use the notation in (IV.2.1), (IV.5.3), (V.1.1).

(2.10) Theorem. *Let $\mathbf{S}$ be a single sorted theory with zero object and let $\mathbf{M}$ be the category of models of $\mathbf{S}$. Then $\Delta\mathbf{M}$ is a cofibration category under $\mathbf{T}(\mathbf{S})$ which satisfies the Balkers-Massey property (IV.5.3). Hence $\Delta\mathbf{M}$ is a homological cofibration category under $\mathbf{T}(\mathbf{S})$.*

This theorem shows that we can apply all the notation and results of the general theory in chapter I, ... , VII for the category $\Delta\mathbf{M}$. We now describe a selection of such applications.

Let $\mathbf{T}^\sharp(\mathbf{S})$ be the full subcategory of $\mathrm{Ho}(\Delta\mathbf{M})$ consisting of the finite coproducts

$$n\Sigma S = \underbrace{\Sigma S \vee \cdots \vee \Sigma S}_{n\text{-times}} = \Sigma(S \otimes \{1, \ldots, n\})$$

Then $\mathbf{T}^\sharp(\mathbf{S})$ is a single sorted theory of cogroups with zero object and we obtain the category of models

$$\mathbf{Coef} = \mathbf{model}(\mathbf{T}^\sharp(S)) \tag{2.11}$$

which coincides with the category of coefficients in (I.4.1). The category $\mathbf{T}(\mathbf{S})$ is the full subcategory of $\mathbf{Coef}$ consisting of all free models of $\mathbf{T}^\sharp(S)$. We have the coefficient functor

$$c : \mathrm{Ho}(\Delta\mathbf{M}) \longrightarrow \mathbf{Coef} \tag{2.12}$$

which carries the simplicial model X to $c(X) : \mathbf{T}^\sharp(\mathbf{S})^{\mathrm{op}} \to \mathbf{Set}$ such that $c(X)(n\Sigma S) = [n\Sigma S, X]$. For the example (4.2) (b) we see that $c(X) = \pi_1(X)$ is the fundamental group of X.

Next we define the analogue of a CW-complex in the category $\Delta\mathbf{M}$; this is a sequence of cofibrations

$$X^1 \rightarrowtail X^2 \rightarrowtail \cdots \rightarrowtail X^n \rightarrowtail X^{n+1} \rightarrowtail \tag{2.13}$$

where $X^1 = \Sigma(S \otimes Z_1)$ is an object in $\mathbf{T}(\mathbf{S})$ and where $X^n \rightarrowtail X^{n+1}$ is a principal cofibration with attaching map $f_n : \Sigma^n(S \otimes Z_n) \to X^n$; i.e. X^{n+1} is up to weak equivalence the mapping cone of f_n. Here the set Z_n is called the set of n-cells of X. We call $X = \lim X^i$ a $\mathbf{T}(\mathbf{S})$-complex in $\Delta\mathbf{M}$. We can choose all X^i and X to be fibrant and cofibrant in $\Delta\mathbf{M}$ for $i \geq 1$, see (IV, § 1, § 2). Let **Complex** be the full subcategory of $\Delta\mathbf{M}$ consisting of $\mathbf{T}(\mathbf{S})$-complexes. Then we have the equivalence of categories

$$\mathrm{Ho}(\Delta\mathbf{M})_0 \xrightarrow{\sim} \mathbf{Complex}/\simeq \tag{2.14}$$

where $(\Delta\mathbf{M})_0$ is the full subcategory of connected objects in $\Delta\mathbf{M}$ and $\simeq$ is the relation of homotopy of maps between cofibrant and fibrant objects. We point out that the cellular approximation theorem holds in $\Delta\mathbf{M}$ by (IV.5.8).

For each object G in **Coef** we can choose a $\mathbf{T}(\mathbf{S})$-complex X^2 with $c(X^2) \simeq G$. We call the attaching map $\partial_X = f_2$ a presentation of G. Moreover we define for G the following additive category $\mathbf{mod}(G)$. Let $n \geq 2$. Objects are the coproducts $\Sigma^n(S \otimes Z) \vee X^2$ in $\Delta\mathbf{M}$ where Z is a set and morphisms are commutative diagrams in $\mathrm{Ho}(\Delta\mathbf{M})$

$$\begin{tikzcd} X^2 \arrow[r, equal] \arrow[d, "i_2"] & X^2 \arrow[d, "i_2"] \\ \Sigma^n(S \otimes Z) \vee X^2 \arrow[r, "f"] \arrow[d, "(0,1)"] & \Sigma^n(S \otimes Z') \vee X^2 \arrow[d, "(0,1)"] \\ X^2 \arrow[r, equal] & X^2 \end{tikzcd}$$

The initial object is given by the empty set $Z = \emptyset$. The partial suspension shows that the category does not depend on the choice of $n \geq 2$; moreover the category does not depend on the choice of the presentation of G. We define the enveloping functor

$$U : \mathbf{Coef} \to \mathbf{Rings} \tag{2.15}$$

which carries G to the ring of endomorphisms of the object $\Sigma^n(S) \vee X^2$ in $\mathbf{mod}(G)$. Moreover U carries a map $u : G \to H$ in $\mathbf{Coef}$ to the ring homomorphism $u_* : U(G) \to U(H)$ which carries $f : \Sigma^n S \vee X^2 \to \Sigma^n S \vee X^2 \in U(G)$ to

$$u_* f = ((1 \vee \bar{u}) f \mid \Sigma^n S, i_2) : \Sigma^n S \vee Y^2 \to \Sigma^n \vee Y^2$$

where $\bar{u} : X^2 \to Y^2$ is a map with $c(\bar{u}) = u$. Let $\mathbf{Mod}(G)$ be the abelian category of all right $U(G)$-modules. Then $\mathbf{mod}(G) \subset \mathbf{Mod}(G)$ is the full subcategory of all free $U(G)$-modules.

(2.16) Definition. Let X be a $\mathbf{T}(\mathbf{S})$-complex in $\Delta\mathbf{M}$ and let Z_n be the set of n-cells of X for $n \geq 1$. Moreover let $G = c(X)$ be given by the coefficient functor (2.12). Then there is a well defined chain complex $C_*(X)$ in the category $\mathbf{mod}(G)$ of free right $U(G)$-modules satisfying

$$C_n(X) = \begin{cases} \bigoplus_{Z_n} U(G) & n \geq 1 \\ U(G) & n = 0 \end{cases}$$

and $C_n(X) = 0$ for $n < 0$. Moreover a filtration preserving map $f : X \to Y$ between $\mathbf{T}(\mathbf{S})$-complexes induces $u : c(X) \to c(Y)$ and a chain map

$$f_* : u_* C_*(X) \to C_*(Y)$$

in $\mathbf{mod}(H)$ with $H = c(Y)$. Since $\mathbf{T}(\mathbf{S})$ is augmented we obtain $C_*(X)$ by (V, § 2) and (II, § 6).

(2.17) Example. If $\mathbf{S}$ is defined as in (2.2) (b) then X is a pointed simplicial set corresponding to a CW-complex Y with $|X| \simeq Y$ and cells as in the $\mathbf{T}(\mathbf{S})$-complex X, in particular $Y^0 = *$. In this case $G = c(X) = \pi_1(X)$ is the fundamental group of X and $U(G) = \mathbb{Z}[\pi_1 X]$ is the group ring. Moreover $C_* X$ with

$$C_n(X) = H_n(\tilde{Y}^n, \tilde{Y}^{n-1})$$

is the cellular chain complex of the universal cover of Y.

If $\mathbf{S}$ is defined as in (2.2) (c) then X is a connected simplicial group corresponding to a CW-complex Y with $Y^1 = *$ such that $X \simeq G_Y$ where G_Y is the Kan loop group of Y. In this case we have $G = c(X) = \pi_1 X = \pi_2 Y$ and $U(G) = \mathbb{Z}$; i.e. U is the constant functor $\mathbb{Z}$. Moreover $C_* X$ satisfies

$$C_n X = \begin{cases} H_{n+1}(Y^{n+1}; Y^n) & \text{for } n \geq 1 \\ \mathbb{Z} & \text{for } n = 0 \end{cases}$$

for $n \in \mathbb{Z}$; i.e. up to the shift of degree $C_* X / C_0 X$ is the reduced cellular chain complex of Y.

Using the chain complex $C_* X$ in (2.16) we define for each right $U(G)$-module M with $G = c(X)$ the cochain complex $\mathrm{Hom}(C_* X, M)$ of abelian groups where Hom is defined in the category $\mathbf{Mod}(G)$ of right $U(G)$-modules. The <u>cohomology with coefficients</u> in M is

$$H^n(X; M) = H^n(\mathrm{Hom}(C_* X, M)) \tag{2.18}$$

Moreover the <u>homology</u> of X is the homology of $C_* X$ in the abelian category $\mathbf{Mod}(G)$, that is

$$H_n(X) = \frac{\text{kernel}\, d : C_n X \to C_{n-1} X}{\text{image}\, d : C_{n+1} X \to C_n X} \tag{2.19}$$

Hence $H_n X$ is a right $U(G)$-module.

Let X be a $\mathbf{T}(\mathbf{S})$-complex and let Y be an object in $\Delta\mathbf{M}$. Moreover let $u : X_1 \to Y$ be a map which admits an extension $X_2 \to Y$. Then u induces a map $u_* : c(X) = G \to c(Y)$. Moreover we obtain the right $\mathbf{mod}(G)$-module $(n \geq 2)$

$$u^* \pi_n Y : \mathbf{mod}(G)^{\mathrm{op}} \to \mathbf{Ab} \tag{2.20}$$

which is given by a right $U(G)$-module $u^* \pi_n(Y)$ as follows. The module carries the object $\Sigma^n(S \otimes Z) \vee X_2$ to the abelian group

$$[\Sigma^n(S \otimes Z), Y] = \bigoplus_Z \pi_n(Y)$$

Moreover a map f in $\mathbf{mod}(G)$ induces the homomorphism

$$f^* : [\Sigma^n(S \otimes Z'), Y] \to [\Sigma^n(S \otimes Z), Y]$$

which carries $a : \Sigma^n(S \otimes Z') \to Y$ to the composite

$$f^*(a) : \Sigma^n(S \otimes Z) \xrightarrow{f'} \Sigma^n(S \otimes Z') \vee X_2 \xrightarrow{(a,u)} Y.$$

Here $f' = f \mid \Sigma^n(S \otimes Z)$ is the restriction of f. If we take $Z = Z' =$ point then we obtain this way the $U(G)$-module $u^* \pi_n(Y)$.

(2.21) Theorem. *Let* $\mathbf{S}$ *be a single sorted theory with zero object and let* $\mathbf{M}$ *be the category of models of* $\mathbf{S}$. *Then obstructions for the extension of maps in* $\Delta\mathbf{M}$ *are defined as follows. Let* A *be a subcomplex of the* $\mathbf{T}(\mathbf{S})$*-complex* X *and let* $f : A \to Y$ *be a map in* $\Delta\mathbf{M}$ *such that* $f \mid A_n$ *admits an extension* $g : X_n \to Y$, $n \geq 2$. *Then the restriction* $g \mid X_{n-1}$ *admits an extension* $\bar{g} : X_{n+1} \to Y$ *with* $\bar{g} \mid A_{n+1} = f \mid A_{n+1}$ *if and only if an obstruction element*

$$\mathcal{O}(g \mid X_{n-1}) \in H^{n+1}(X, A; u^*\pi_n Y)$$

vanishes. Here $u : X_1 \to Y$ *is the restriction of* g.

For the example (2.2) (b) of pointed simplicial sets this is a classical result of obstruction theory; compare (2.17). We obtain (2.21) as a special case of (V, § 4). Next we obtain by (VI, § 7) the following homological Whitehead theorem:

(2.22) Theorem. *Let* $\mathbf{S}$ *be a single sorted theory with zero object and let* $\mathbf{M}$ *be the category of models of* $\mathbf{S}$. *Let* $f : X \to Y$ *be a filtration preserving map between* $\mathbf{T}(\mathbf{S})$*-complexes in* $\Delta\mathbf{M}$. *Then* f *is a homotopy equivalence, i.e. an isomorphism in* $\mathrm{Ho}(\Delta\mathbf{M})$, *if and only if the coefficient functor* c *induces an isomorphism*

$$u : G = c(X) \xrightarrow{\cong} H = c(Y)$$

and one of the following conditions (i), (ii), (iii) is satisfied.

(i) $f_* : u_*C_*X \to C_*Y$ *is a homotopy equivalence of chain complexes in* $\mathbf{mod}(H)$.
(ii) $f_* : H_n(X) \to (u_*)^*H_n(Y)$ *is an isomorphism of* $U(G)$*-modules for* $n \geq 1$.
(iii) For all right $U(H)$*-modules* M *and* $n \geq 1$ *the induced map*

$$f^* : H^n(Y, M) \longrightarrow H^n(X, u^*M)$$

is an isomorphism.

In Schwede [SH] 2.2.2 the result is proved for the special case that X and Y are simply connected, that is $G = H = 0$. Next we consider the Hurewicz homomorphism and the exact sequence of J.H.C. Whitehead in $\Delta\mathbf{M}$.

(2.23) Theorem. *Let* $\mathbf{S}$ *be a single sorted theory with zero object and let* $\mathbf{M}$ *be the category of models of* $\mathbf{S}$. *For each connected object* X *in* $\Delta\mathbf{M}$ *one has the following exact sequence of right* $U(G)$*-modules,* $G = c(X)$, $n \geq 2$.

$$\begin{aligned} \cdots \to \Gamma_n(X) \to \pi_n(X) \to H_n(X) \to \Gamma_{n-1}(X) \to \ldots \\ \cdots \to \Gamma_2(X) \to \pi_2(X) \to H_2(X) \to \Gamma_1(X) \to 0 \end{aligned}$$

Here we use (2.14) in order to replace X by a $\mathbf{T}(\mathbf{S})$-complex. Hence it is sufficient to prove (2.23) only for $\mathbf{T}(\mathbf{S})$-complexes X for which we define

$$\Gamma_n(X) = \mathrm{image}\{\pi_n(X_{n-1}) \to \pi_n(X_n)\}$$

if $n \geq 3$. The definition of Γ_1 and Γ_2 is more complicated; see (V, § 5). If the $(n-1)$-skeleton X_{n-1} is trivial ($n \geq 2$) then it is easy to see that $\Gamma_i X = 0$ for $i \leq n$ and hence we get the following Hurewicz theorem; compare Schwede [SH] 2.4.3.

(2.24) Corollary. *Let $n \geq 2$ and let X be $(n-1)$-connected. Then $\pi_n(X) \to H_n(X)$ is an isomorphism and $\pi_{n+1}(X) \to H_{n+1}(X)$ is surjective.*

Finally we consider the theorem of Wall in $\Delta\mathbf{M}$. We say that a $\mathbf{T}(\mathbf{S})$-complex X is finite if X has only finitely many cells; that is the set $Z_1 \cup Z_2 \cup \dots$ is finite. Now let X and Y be $\mathbf{T}(\mathbf{S})$-complexes. Then a domination (X, f, g) of Y is given by maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y \quad \text{in } \mathrm{Ho}(\Delta\mathbf{M})$$

with $gf = 1$. The domination if finite if X is finite. As a special case of (VII.2.4) we get:

(2.25) Theorem. *Let $\mathbf{S}$ be a single sorted theory with zero object and let $\mathbf{M}$ be the category of models of $\mathbf{S}$. Let Y be a $\mathbf{T}(\mathbf{S})$-complex and let $H = c(Y)$ be given by the coefficient functor. If Y admits a finite domination then the finiteness obstruction*

$$[Y] = [C_* Y] \in \tilde{K}_0(U(H))$$

is defined. Here U is the enveloping functor in (2.15) and $\tilde{K}_0$ is the reduced projective class group; see (VII, § 1). Moreover $[Y] = 0$ if and only if there exists a finite $\mathbf{T}(\mathbf{S})$-complex X and an isomorphism $X \to Y$ in $\mathrm{Ho}(\Delta\mathbf{M})$.

For the example (2.2) (b) of pointed simplicial sets this corresponds exactly to Wall's result on finiteness obstructions.

Chapter C: Applications and Examples in Delicate Homotopy Theories of Simplicial Objects

In this chapter we consider homotopy theories of simplicial objects which resemble the homotopy theory of simplicial groups. It is well known that in the Quillen model category of simplicial groups $\Delta\mathbf{Gr}$ all objects are fibrant; i.e. all simplicial groups satisfy the Kan extension condition. Moreover the free simplicial groups form a sufficiently large class of cofibrant objects in the sense that the homotopy category of free simplicial groups is equivalent to the homotopy category $\mathrm{Ho}(\Delta\mathbf{Gr})$ defined by localization with respect to weak equivalences. Since free groups are cogroups we see that free simplicial groups are simplicial objects in a special theory $\mathbf{T}$ of cogroups. In this chapter we study the homotopy theory of "free" simplicial objects in any theory of cogroups, or more generally in any theory of coactions. Such homotopy theories are canonical generalizations of the homotopy theory of simplicial groups.

Simplicial groups have the additional property that a delicate Blakers-Massey theorem holds which corresponds to the ordinary Blakers-Massey theorem for connected spaces via the Kan-equivalence of homotopy categories:

$$\mathrm{Ho}(\mathbf{Top}_0^*) \xrightarrow{\sim} \mathrm{Ho}(\Delta\mathbf{Gr})$$

Here $\mathbf{Top}_0^*$ is the category of pointed connected spaces. In section § 2 and § 3 we describe many further examples of theories of cogroups (resp. coactions) in which such a delicate Blakers-Massey theorem holds. We therefore call the homotopy theory considered in this chapter the "delicate homotopy theory of simplicial objects". The non-delicate or normal theory of simplicial objects resembles the homotopy theory of simplicial sets and is discussed in (B, § 2) above. The delicate homotopy theory has many features which are different to the normal theory. In particular a homology and cohomology theory with twisted coefficients is defined in the delicate theory which is not defined in the normal theory. In the case of a simplicial group G the homology of G is the homology of the classifying space of G with twisted coefficients; see (A, § 1).

1 Homotopy Theory of Free Simplicial Objects in Theories of Coactions

We study the simplicial objects in a theory $\mathbf{T}$ of coactions and we describe basic results for the homotopy theory of such simplicial objects. In the next section we consider various examples of such theories $\mathbf{T}$. In particular if $\mathbf{T} = \mathbf{gr}$ is the category of free groups then the results of this section correspond to well known facts of classical homotopy theory since a free simplicial object in $\mathbf{gr}$ is a model of a connected space. We discuss only some of the main applications of the general theory of chapter I, ... , VIII to simplicial objects. We leave it to the reader to work out results like the "model lifting property" or the obstruction theory for the "realizability of chain complexes" and many further applications.

Let Δ be the category of ordered sets $[n] = \{0 < \cdots < n\}$ and order preserving functions $\alpha : [n] \to [m]$ in **Set** where $n, m \geq 0$. For $0 \leq i \leq n$ we have the functions

$$\begin{aligned} d_i' &: [n-1] \rightarrowtail [n] - \{i\} \subset [n] \\ s_i' &: [n+1] \twoheadrightarrow [n] \quad \text{with } s_i'(i) = s_i'(i+1) \end{aligned}$$

in Δ where d_i' is injective and s_i' is surjective. A simplicial object in a category $\mathbf{C}$ is a functor

$$X : \Delta^{\mathrm{op}} \to \mathbf{C} \tag{1.1}$$

We write $X_n = X[n]$ and the face maps are $d_i = (d_i')^* : X_n \to X_{n-1}$ and the degeneracy maps are $s_i = (s_i')^* : X_n \to X_{n+1}$. The collection (X_n, d_i, s_i) satisfies the usual simplicial identities; see May [SO]. Let $\Delta\mathbf{C}$ be the category of simplicial objects in $\mathbf{C}$; morphisms are natural transformations. Hence $\Delta\mathbf{C}$ is an example of a category of Δ-diagrams as considered in (A.2.1). We have the standard full embedding of categories

$$\mathrm{const} : \mathbf{C} \subset \Delta\mathbf{C} \tag{1.2}$$

which carries the object $X \in \mathrm{Ob}(\mathbf{C})$ to the constant simplicial object $\mathrm{const}(X)$ with $\mathrm{const}(X)_n = X$ and $d_i = s_i = 1_X$. We often identify the constant simplicial object $\mathrm{const}(X)$ with the object X in $\mathbf{C}$ and we write $X = \mathrm{const}\,(X) \in \Delta\mathbf{C}$ in this case.

Let $\Delta\mathbf{Set}$ be the category of simplicial sets. The standard n-simplex $\Delta[n] = \Delta(-, [n])$ in $\Delta\mathbf{Set}$ is generated by $\sigma_n \in \Delta[n]_n$ where σ_n is the identity of $[n]$. The n-simplex $\Delta[n]$ has the universal property that, for every $X \in \Delta\mathbf{Set}$ and $x \in X_n$ there is a unique map $i_x : \Delta[n] \to X$ in $\Delta\mathbf{Set}$ with $i_x(\sigma_n) = x$. For every $n \geq 0$ let

$$\partial\Delta[n] \subset \Delta[n] \tag{1.3}$$

be the largest subobject of $\Delta[n]$ not containing σ_n. The base point $*$ of $\Delta[n]$ is given by $* = d_0 \ldots d_0(\sigma_n) \in (\Delta[n])_0$ which defines

$$* = \mathrm{const}(*) \to \Delta[n].$$

The inclusion (1.3) is a simplicial model of the n-ball since the geometric realization of $\Delta(n)$ is the standard n-simplex Δ^n in Euclidean space with boundary $\partial\Delta^n$ the realization of $\partial\Delta[n]$. We need a further simplicial model of the n-ball given by the simplicial n-ball

$$S[n-1] \subset D[n]. \tag{1.4}$$

Here $S[n-1]$ is the push out of $\Delta[n-1] \supset \partial\Delta[n-1] \to *$ where $*$ is the constant point. Moreover $D[n]$ is the push out of $\Delta[n] \supset \Lambda^0[n] \to *$ where $\Lambda^0[n]$ is the subobject of $\Delta[n]$ generated by $d_i\sigma_n$ with $i > 0$. We have the canonical push out diagram in $\Delta\mathbf{Set}$

$$\begin{array}{ccc} \Delta[n] & \longrightarrow & D[n] \\ \cup & & \cup \\ \partial\Delta[n] & \xrightarrow{q} & S[n-1] \end{array}$$

where q carries $\Lambda^0[n]$ to $*$.

An element $x \in X_n$ is <u>degenerate</u> if it is of the form $x = s_i y$ for some $y \in X_{n-1}$ and $0 \leq i \leq n$ and <u>non-degenerate</u> otherwise. The simplicial n-ball has the non-degenerate elements σ_n and $d_0\sigma_n$. The geometric realization of (1.4) is the usual n-ball (D^n, S^{n-1}) which is a CW-complex with two nontrivial cells; see (A, § 1). Recall from May [SO] or Fritsch-Piccini [CS] that the geometric realization $|X|$ of a simplicial set X is a CW-complex in which the cells of $|X|$ are in 1-1 correspondence to the non-degenerate elements of X. A simplicial set X is <u>finite</u> if X has only finitely many non-degenerate elements; this implies that X_n is a finite set for all n.

Let $\mathbb{D} \subset \Delta$ be the subcategory consisting of all surjective maps in Δ. The morphisms s'_i in (1.1) generate $\mathbb{D}$. Each simplicial set X yields a $\mathbb{D}$-set $X_{\mathbb{D}}$ by the composite

$$X_{\mathbb{D}} : \mathbb{D}^{\mathrm{op}} \subset \Delta^{\mathrm{op}} \xrightarrow{X} \mathbf{Set} \tag{1.5}$$

It is known that $X_{\mathbb{D}}$ has the following "pull back property".

(1.6) Definition. We say that a $\mathbb{D}$-set $X : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ has the <u>pull back property</u> if for $n \geq 0$ and $0 \leq j \leq n$ all degeneracy maps $s_j = (s'_j)^* : X_n \to X_{n+1}$ are injective and if for $i < j$ the diagram

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{s_i} & X_n \\ {\scriptstyle s_{j-1}}\downarrow & & \downarrow{\scriptstyle s_j} \\ X_n & \xrightarrow{s_i} & X_{n+1} \end{array} \tag{1}$$

is a pull back diagram in the category of sets, i.e. $s_iX_n \cap s_jX_n = s_js_iX_{n-1}$. In particular the $\mathbb{D}$-set $(i \geq 0)$

$$\mathbb{D}(-,[i]) : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set} \tag{2}$$

which carries $[n] \in \mathbb{D}$ to the set $\mathbb{D}([n],[i])$ has the pull back property.

We now consider a <u>theory of coactions</u> $\mathbf{T}$ as defined in (I.1.11) below. Hence coproducts $X \vee Y$ and an initial object $*$ exist in $\mathbf{T}$ and each object X in $\mathbf{T}$ is endowed with the structure of a coaction $\mu_X : X \to X \vee X'$ where X' is a cogroup in $\mathbf{T}$ such that one has a canonical isomorphism

$$(i_X, \mu_X) : X \vee X = X \vee X' \tag{1.7}$$

in $\mathbf{T}$, see (I.1.12). For a finite set Z and an object X in $\mathbf{T}$ we define the tensor product $X \otimes Z$ which is the coproduct

$$X \otimes Z = \bigvee_{z \in Z} X \tag{1.8}$$

of as many copies of X as there are elements in the set Z. Clearly the tensor product defines a bifunctor

$$\otimes : \mathbf{T} \times \mathbf{Set}(\mathrm{fin}) \to \Delta\mathbf{T} \tag{1}$$

where $\mathbf{Set}(\mathrm{fin})$ is the category of finite sets. If $\mathbf{T}$ has arbitrary coproducts then we can omit the finiteness of sets. If Z is a (finite) simplicial set then $X \otimes Z$ is a simplicial object in $\Delta\mathbf{T}$ with face and degeneracy maps induced by those of Z. Moreover we obtain the functor

$$\otimes : (\Delta\mathbf{T}) \times \Delta\mathbf{Set}(\mathrm{fin}) \to \Delta\mathbf{T} \tag{2}$$

which carries the pair (X, Z) to the simplicial object $X \otimes Z$ with $(X \otimes Z)_n = X_n \otimes Z_n$. Here face and degeneracy maps are defined by $\sigma_i \otimes \sigma_i$ and $d_i \otimes d_i$ respectively; i.e. $X \otimes Z$ is the "diagonal" of the corresponding bisimplicial object. If Z is a pointed simplicial set and if X is a based object in $\Delta\mathbf{T}$ (i.e. a map $0 : X \to \mathrm{const}(*)$ is given) then we define the half smash product $X \wedge Z$ by the push out diagram in $\Delta\mathbf{T}$

$$\begin{array}{ccc} X \otimes Z & \longrightarrow & X \wedge Z \\ \uparrow & & \uparrow \\ X \otimes * = X & \xrightarrow{0} & * \end{array} \tag{3}$$

Clearly any cogroup A in $\mathbf{T}$ is based by the trivial map $0 : A \to *$ so that $A \wedge Z$ is defined.

We shall need the following lemma which we derive from (1.7).

(1.9) Lemma. *Let $Z : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}(\mathrm{fin})$ be a $\mathbb{D}$-set with the pull back property in (1.6) and let $* \in Z_0$ be a basepoint. Then the inclusion $\mathrm{const}(*) \subset Z$ has the complement $Z - \mathrm{const}(*)$ which is again a $\mathbb{D}$-set with pull back property and for a coaction $X \to X \vee X'$ in $\mathbf{T}$ one has via (1.7) the canonical isomorphism in $\mathbb{D}\mathbf{T}$*

$$X \otimes Z \cong \mathrm{const}(X) \vee X' \otimes (Z - \mathrm{const}(*))$$

We can omit the finiteness of Z_n in (1.9) if we assume that $\mathbf{T}$ has arbitrary coproducts.

For $X \in \mathbf{T}$ the object $X \otimes \Delta[n]$ in $\Delta\mathbf{T}$ has the following universal property: For each object Y in $\Delta\mathbf{T}$ and each map $f : X \to Y_n$ in $\mathbf{T}$ there exists a unique map

$$\bar{f} : X \otimes \Delta[n] \to Y \quad \text{in } \Delta\mathbf{T} \tag{1.10}$$

which extends $f : X = X \otimes \{\sigma_n\} \to Y_n$ in degree n. According to this universal property we define "free" objects in $\Delta\mathbf{T}$ as in the following definition.

(1.11) Definition. Let $\mathbf{T}$ be a theory of coactions and let $\Delta\mathbf{T}$ be the category of simplicial objects in $\mathbf{T}$. We consider a diagram $X^{\geq 0}$:

$$X^0 \to X^1 \to \cdots \to X^n \to X^{n+1} \to \ldots$$

in $\Delta\mathbf{T}$ with the following properties:

(i) $X^0 = \mathrm{const}(X^0)$ is the constant simplicial object given by $X^0 \in \mathbf{T}$ with coaction $\mu : X^0 \to X^0 \vee A^0$.
(ii) For $n \geq 1$ a push out diagram in $\Delta\mathbf{T}$ of the form

$$\begin{tikzcd} A^n \otimes \Delta[n] \arrow[r] & X^n \\ A^n \otimes \partial\Delta[n] \arrow[u] \arrow[r, "\partial_n"] & X^{n-1} \arrow[u] \end{tikzcd}$$

is given where A^n is a cogroup in $\mathbf{T}$ termed the n-cell of $X^{\geq 0}$.

Then the direct limit $X = \lim X^{\geq 0}$ in $\Delta\mathbf{T}$ is defined and X is called a free object in $\Delta\mathbf{T}$. We say that X is a pointed free object if for $n \geq 1$ the attaching map ∂_n admits a factorization

$$\partial_n : A^n \otimes \partial\Delta[n] \to A^n \wedge \partial\Delta[n] \to X^{n-1}$$

where we use the base point of $\Delta[n]$ in (1.3). Each free object is actually isomorphic to a pointed free object; see (1.21) below.

Let Y be a free object in $\Delta\mathbf{T}$ with cells B^n, $n \geq 0$. We say that a map $f : X \to Y$ between free objects is a free inclusion if isomorphisms in $\mathbf{T}$

$$\begin{aligned} Y^0 &\cong X^0 \vee \bar{Y}^0 \\ B^n &\cong A^n \vee \bar{B}^n \end{aligned}$$

are given such that the map $f : X^n \to Y^n$, $n \geq 0$, is inductively induced by the inclusions

$$\begin{aligned} X^0 &\to X^0 \vee \bar{Y}^0 \cong Y^0, \\ A^n &\to A^n \vee \bar{B}^n \cong B^n. \end{aligned}$$

Here $\bar{B}^n$ is a cogroup and $\bar{Y}^0$ is a coaction and the isomorphisms are compatible with the cogroup (resp. coaction) structure on both sides. We call $\bar{B}^n$ the relative n-cell of (Y, X). Now assume that for all $n \geq 1$ the attaching map ∂_n in (ii) has a factorization

$$\partial_n : A^n \otimes \partial\Delta[n] \xrightarrow{1\otimes q} A^n \wedge S[n-1] \xrightarrow{\partial'_n} X^{n-1}$$

Then we call X a CW-object in $\Delta\mathbf{T}$. For a CW-object the push out (ii) induces the push out diagram (see (1.4))

$$\begin{tikzcd} A^n \wedge D[n] \arrow[r] & X^n \\ A^n \wedge S[n-1] \arrow[u] \arrow[r, "\partial'_n"] & X^{n-1} \arrow[u] \end{tikzcd} \tag{iii}$$

Hence we obtain the full subcategories

$$(\Delta\mathbf{T})_{CW} \subset (\Delta\mathbf{T})_{\text{free}} \subset \Delta\mathbf{T} \tag{iv}$$

Here $(\Delta\mathbf{T})_{\text{free}}$ consists of all objects X in $\Delta\mathbf{T}$ which are isomorphic in $\Delta\mathbf{T}$ to a free object. Moreover $(\Delta\mathbf{T})_{CW}$ consists of all CW-objects in $\Delta\mathbf{T}$.

(1.12) Lemma. *Let X be a free object in $\Delta\mathbf{T}$ with $X^0, A^n \in \mathrm{Ob}(\mathbf{T})$ for $n \geq 0$ defined as in (1.11). Then there is an isomorphism of $\mathbb{D}$-objects*

$$X_{\mathbb{D}} \cong \mathrm{const}(X^0) \vee \bigvee_{i\geq 1} A^i \otimes \mathbb{D}(-, [i])$$

Proof. We have a disjoint union of sets

$$\Delta[n]_k = (\partial\Delta[n])_k \amalg \mathbb{D}([k], [n])$$

Hence the isomorphism in (1.12) is a consequence of the push out (1.11) (ii). We do not claim that all push outs exist in $\Delta\mathbf{T}$, but push outs as in (1.11) exist since there are coproducts in $\mathbf{T}$. q.e.d.

The next result is a kind of converse of Lemma (1.12).

(1.13) Proposition. *Let $\mathcal{J}$ be a (finite) index set and for $j \in \mathcal{J}$ let $Z(j)$ be a pointed $\mathbb{D}$-set in* $\mathbf{Set}(\text{fin})$ *which has the pull back property. Moreover let $X(j), j \in \mathcal{J}$, be objects in the theory of coactions $\mathbf{T}$. If X is an object in $\Delta\mathbf{T}$ together with an isomorphism of $\mathbb{D}$-objects*

$$X_{\mathbb{D}} \cong \bigvee_{j\in\mathcal{J}} X(j) \otimes Z(j)$$

then X is isomorphic to a free object in $\Delta\mathbf{T}$.

If arbitrary coproducts exist in $\mathbf{T}$ we can omit in (1.13) the finiteness assumptions.

(1.14) Example. Let $\mathbf{T} = \mathbf{gr}$ be the category of free groups. Hence $\mathbf{gr}$ is a theory of cogroups in which arbitrary coproducts exist. Kan [HG] and Curtis [SH] say that a <u>simplicial group</u> G <u>is free</u> if there exists a $\mathbb{D}$-set Z together with an isomorphism of $\mathbb{D}$-groups

$$G_{\mathbb{D}} \cong \mathbb{Z} \otimes Z \tag{*}$$

Here $\mathbb{Z}$ is the group of integers and each free group $F = \langle Z_0 \rangle$ with basis Z_0 can be written $F = \mathbb{Z} \otimes Z_0$. Jardine-Goerss [SH] V.1.8 show that $(*)$ implies that Z has the pull back property. Hence by (1.13) we see that $(*)$ implies that the simplicial group X is also free in the sense of (1.11). In fact, this was already proved by Jardine-Goerss [SH] V.1.9 and our proof of (1.13) below generalizes the argument of Jardine-Goerss.

Proof of (1.13). Since $Z(j)$ is pointed we have the $\mathbb{D}$-set $U(j) = Z(j) - \mathrm{const}(*)$. Moreover for the object $X(j)$ in $\mathbf{T}$ we have the coaction

$$\mu : X(j) \to X(j) \vee A(j)$$

where $A(j)$ is a cogroup. Let

$$X^0 = \bigvee_{j \in \mathcal{J}} X(j) \tag{1}$$

then (1.9) and the assumption in (1.13) show that we have an isomorphism of $\mathbb{D}$-objects

$$X_{\mathbb{D}} \cong \mathrm{const}(X^0) \vee \bigvee_{j \in \mathcal{J}} A(j) \otimes U(j) \tag{2}$$

Let $U'(j)_n$ be the subset of $U(j)_n$ consisting of non-degenerate elements and let $A^n (n \geq 1)$ be the coproduct

$$A^n = \bigvee \{A(j);\ j \in \mathcal{J},\ U'(j)_n \neq \emptyset\} \tag{3}$$

Then (2) shows

$$X_{\mathbb{D}} \cong \mathrm{const}(X^0) \vee \bigvee_{n \geq 1} A^n \otimes \mathbb{D}(-, [n]) \tag{4}$$

and this is the form of a free object in (1.12). Following the argument of Jardine-Goerss [SH] V.1.9 we see that one has a push out in $\Delta\mathbf{T}$

$$\begin{tikzcd}
A^n \otimes \Delta[n] \arrow[r] & \mathrm{sk}_n(X) \\
A^n \otimes \partial\Delta[n] \arrow[u] \arrow[r, "\partial_n"] & \mathrm{sk}_{n-1}(X) \arrow[u]
\end{tikzcd}$$

where $sk_n(X)$ is the "n-skeleton" of X, compare Jardine-Goerss [SH] V, § 1. Hence the filtration in (1.11) coincides with

$$X^n = \mathrm{sk}_n(X) \tag{1.15}$$

if X is a free object in $\Delta\mathbf{T}$. q.e.d.

(1.16) Corollary. *Let X be a free object in $\Delta\mathbf{T}$ and let S be a (finite) simplicial set. Then $X \otimes S$ is isomorphic to a free object in $\Delta\mathbf{T}$.*

Proof. Since X is free we have (1.12) and hence we get

$$(X \otimes S)_{\mathbb{D}} = X^0 \otimes S_{\mathbb{D}} \vee \bigvee_{i \geq 1} A^i \otimes \mathbb{D}(-,[i]) \otimes S_{\mathbb{D}} \quad \text{with}$$

$$A^i \otimes \mathbb{D}(-,[i]) \otimes S = A^i \otimes (\mathbb{D}(-,[i]) \times S_{\mathbb{D}})$$

Here we use the product of $\mathbb{D}$-sets; compare Quillen [HA] II. page 1.9. Since $S_{\mathbb{D}}$ and $\mathbb{D}(-,[i])$ satisfy the pull back property we see that $Z^i = \mathbb{D}(-,[i]) \times S_{\mathbb{D}}$ satisfies the pull back property. Hence we can apply (1.13). q.e.d.

Following Quillen [HA] II. page 1.6 we define the <u>cylinder functor</u>

$$I : (\Delta\mathbf{T})_{\text{free}} \to (\Delta\mathbf{T})_{\text{free}} \tag{1.17}$$

which carries X to $I(X) = X \otimes \Delta[1]$. Here we use (1.16) to see that $I(X)$ is actually again an object in $(\Delta\mathbf{T})_{\text{free}}$. We have the obvious maps between simplicial sets $\Delta[0] \amalg \Delta[0] \to \Delta[1] \to \Delta[0]$ which induce the structure maps of the cylinder

$$X \vee X \xrightarrow{(i_0,i_1)} I(X) \xrightarrow{\;p\;} X$$

Compare (III. 7.1). Let $*$ be the initial object of $\mathbf{T}$ then const$(*)$ is the initial object of $\Delta\mathbf{T}$. Moreover we say that a map $f : X \to Y$ in $(\Delta\mathbf{T})_{\text{free}}$ is a <u>cofibration</u> if f is isomorphic in $\Delta\mathbf{T}$ to a free inclusion; see (1.11).

(1.18) Theorem. *Let $\mathbf{T}$ be a theory of coactions. Then the category $(\Delta\mathbf{T})_{\text{free}}$ with the cylinder (1.17) and cofibrations given by free inclusions (1.11) satisfies the axioms of a $\mathbf{I}$-category in (III, § 7).*

This implies by (III.7.4) that $(\Delta\mathbf{T})_{\text{free}}$ is also a cofibration category.

Proof of (1.18). The axioms (I1), (I2), (I4) and (I5) are obviously satisfied. We only have to check the homotopy extension property in (I3). For this it suffices to show that

$$A \otimes \partial\Delta[n] \to A \otimes \Delta[n]$$

has the homotopy extension property for all cogroups A in $\mathbf{T}$. But this is a consequence of the fact that for any U in $\Delta\mathbf{T}$ the simplicial set $\mathbf{T}(A,U)$ is actually a simplicial group which satisfies the Kan-extension condition; see May [SO] 17.1. q.e.d.

Theorem (1.18) shows that we have for any theory of coactions a well defined homotopy category

$$\mathbf{C}_c/\simeq\ = (\Delta\mathbf{T})_{\text{free}}/\simeq \tag{1.19}$$

We obtain this result readily from the simple proofs above.

(1.20) Remark. Dwyer-Hirschhorn-Kan [MC] 9.7 obtain a Quillen model category for simplicial objects in certain categories of universal algebras. If the universal algebras are defined on underlying groups then the associated homotopy theory is up to equivalence of categories of the form (1.19) where $\mathbf{T}$ is the category of cogroups given by free universal algebras. Compare also theorem II.5.4 in Jardine-Goerss [SH].

(1.21) Proposition. *Each free object in $\Delta\mathbf{T}$ is isomorphic to a pointed free object and is homotopy equivalent to a CW-object in $\Delta\mathbf{T}$. Hence there is an equivalence of categories*

$$(\Delta\mathbf{T})_{\text{free}}/\simeq\ \to (\Delta\mathbf{T})_{CW}/\simeq$$

Proof. Let X be a free object defined as in (1.11). The attaching map ∂_n yields a map

$$\alpha : A \otimes * \to A \otimes \partial[n] \xrightarrow{\partial_n} X^{n-1} \tag{1}$$

with $A = A^n$. Using the comultiplication μ of A we get for a simplicial set Z

$$\bar{\mu} : A \otimes Z \to (A \vee A) \otimes Z = A \otimes Z \vee A \otimes Z \to A \otimes Z \vee A \otimes * \tag{2}$$

Here the first map is $\mu \otimes Z$ and the second map is $1 \vee (A \otimes 0)$ where $0 : Z \to *$ is the trivial map. Using $\bar{\mu}$ we get the following commutative diagram

$$\begin{tikzcd}
A \otimes \Delta[n] \arrow[d, "\bar{\mu}"] & A \otimes \partial\Delta[n] \arrow[l] \arrow[r, "\partial_n \cdot \beta"] \arrow[d, "\bar{\mu}"] & X^{n-1} \arrow[d, equal] \\
A \otimes \Delta[n] \vee A \otimes * & A \otimes \partial\Delta[n] \vee A \otimes * \arrow[l] \arrow[r, "(\partial_n, \beta)"] & X^{n-1} \\
A \otimes \Delta[n] \arrow[u, "i_1"] & A \otimes \partial\Delta[n] \arrow[l] \arrow[r, "\partial_n"] \arrow[u, "i_1"] & X^{n-1} \arrow[u, equal]
\end{tikzcd}$$

The push outs of the rows are denoted by X^n_β, P, and X^n respectively. Now the lower left hand square is a push out and hence $i_1 : X^n \to P$ is an isomorphism. Therefore we get the canonical map

$$\mu_\beta : X^n_\beta \xrightarrow{\bar{\mu}} P \cong X^n \tag{3}$$

satisfying $\mu_\beta\mu_\alpha = \mu_{\beta+\alpha} : X^n_{\beta+\alpha} \to X^n_\beta \to X^n$. Hence for $\beta = -\alpha$ we see that μ_β is the inverse of μ_α. Moreover $\partial_n \cdot \beta$ is pointed in this case by (1). Hence the free object X is isomorphic to a pointed free object.

Next we consider the second statement in (1.21). Using the push outs in (1.11) (ii), (iii) it now suffices to prove that

$$1 \otimes q : A \wedge \partial\Delta[n] \simeq A \wedge S[n-1]$$

is a homotopy equivalence in $(\Delta\mathbf{T})_{\text{free}}/\simeq$ for each cogroup A in $\mathbf{T}$. To obtain a homotopy inverse we see that for $U = A \otimes \partial\Delta[n]$ there exists a map λ with $\lambda(*) = 0$ for which the following diagram homotopy commutes in $\Delta\mathbf{Set}$

$$\begin{tikzcd} & \mathbf{T}(A,U) & \\ \partial\Delta[n] \arrow[ur, "\bar{1}"] \arrow[rr, "q"] & & S[n-1] \arrow[ul, "\lambda"'] \end{tikzcd}$$

Here $\bar{1}$ is given by the identity of U. Since $\mathbf{T}(A,U)$ is a Kan complex and hence fibrant and since q is a weak equivalence under $*$ we obtain λ. q.e.d.

Next we define for each object X in $\mathbf{T}$ and for a simplicial object $U \in \Delta\mathbf{T}$ the set of homotopy classes

$$[X,U] = \pi_0\mathbf{T}(X,U) \tag{1.22}$$

where the right hand side is the set of path components of the simplicial set $\mathbf{T}(X,U)$ obtained by U. One can check that (1.22) is the same as the set of homotopy classes $\text{const}(X) \to U$ defined by the cylinder (1.17). Moreover we define for each cogroup A in $\mathbf{T}$ the homotopy groups ($n \geq 0$)

$$\begin{cases} \pi_n^A(U) = \pi_n\mathbf{T}(A,U) \\ \pi_{n+1}^A(U,V) = \pi_{n+1}(\mathbf{T}(A,U),\mathbf{T}(A,V)) \end{cases} \tag{1.23}$$

where $V \to U$ is a morphism in $\Delta\mathbf{T}$. The right hand side denotes the usual homotopy groups in $\Delta\mathbf{Set}$. Since A is a cogroup $\mathbf{T}(A,U)$ is a simplicial group and hence a Kan complex. We now are ready to state the following Whitehead theorem which is a direct consequence of (IV.3.11) below.

(1.24) Theorem. *Let $\mathbf{T}$ be a theory of coactions and let X and Y be free objects in $\Delta\mathbf{T}$. Then a map $f : X \to Y$ is a homotopy equivalence (i.e. an isomorphism in the quotient category $(\Delta\mathbf{T})_{\text{free}}/\simeq$) if and only if f induces bijections*

$$f_* : [Z,X] \to [Z,Y]$$
$$f_* : \pi_n^A(X) \to \pi_n^A(Y)$$

for all objects Z in $\mathbf{T}$ and all cogroups A in $\mathbf{T}$ and $n \geq 1$.

Proof. By (1.21) we may assume that X and Y are CW-objects in $\Delta\mathbf{T}$. Such a CW-object X yields a $\mathbf{T}$-complex

$$X_{(1)} \subset X_{(2)} \subset \cdots \subset X_{(n)} \subset X_{(n+1)} \subset \dots$$

with the shift in the dimension of skeleta defined by $X_{(n)} = X^{n-1} = \mathrm{sk}_{n-1}(X)$; see (1.11) and (1.15). It is clear that these $\mathbf{T}$-complexes are $\mathbf{T}$-good; hence the result follows from (IV.3.11). q.e.d.

(1.25) Definition. We say that $f : X \to Y$ in $\Delta\mathbf{T}$ is $(m, \mathbf{T})$-connected with $m \geq 0$ if for all objects Z in $\mathbf{T}$ the induced map

$$f_* : [Z, X] \to [Z, Y]$$

is surjective and if for all cogroups A in $\mathbf{T}$ the relative homotopy groups $\pi_r^A(Y, X) = 0$ are trivial for $r \leq m$.

For example let $i : X \to Y$ be a free inclusion in $(\Delta\mathbf{T})_{\mathrm{free}}$ with $X^m = Y^m$ then i is easily seen to be $(m, \mathbf{T})$-connected since the "cellular approximation theorem" obviously holds. The following result is a kind of converse of this fact.

(1.26) Proposition. *Let $i : X \to Y$ be a free inclusion which is $(m, \mathbf{T})$-connected. Then there exists a free inclusion $X \to \bar{Y}$ with $X^m = \bar{Y}^m$ and a map $\bar{Y} \to Y$ under X which is a homotopy equivalence in $(\Delta\mathbf{T})_{\mathrm{free}}/\simeq$.*

Proof. We define for Z in $\mathbf{T}$ a ball pair by the following push out diagram (compare VIII, § 4)

$$\begin{array}{ccc}
Z \otimes D[n] \otimes I & \longrightarrow & \square_Z^{n+1} \\
\uparrow & & \uparrow \\
Z \otimes S[n-1] \otimes I & \xrightarrow{1 \otimes pr} & Z \otimes S[n-1]
\end{array}$$

Here $pr : S[n-1] \times I \to S[n-1]$ is the projection. We have two inclusions

$$P_Z^n = Z \otimes D[n] \xrightarrow{i_0} \square_Z^{n+1}$$
$$Q_Z^n = Z \otimes D[n] \xrightarrow{i_1} \square_Z^{n+1}$$

Now assume $Y^0 = X^0 \vee Z$. Then there exists a homotopy

$$P_Z^1 = Z \otimes D[1] = Z \otimes \Delta[1] \to Y$$

from $Z \subset Y^0 \subset Y$ to a map $Z \to X$ since $X \to Y$ is $(m, \mathbf{T})$-connected. We now define V by the double push out diagram

$$\begin{tikzcd} & Q^1_Z \arrow[r, "pr"] \arrow[d, "qi_1"] & Z \arrow[d] \\ \square^2_Z \arrow[r, "q"] & W \arrow[r] & V \\ P^1_Z \arrow[u, "i_0"] \arrow[r] & Y \arrow[u] & \end{tikzcd}$$

Then $Y \to V$ is a homotopy equivalence and $X^0 = V^0$ holds. Now assume for $0 \leq r < m$ we have $X^r = Y^r$ and let $Y^{r+1} = X^{r+1} \vee A$ where A is a cogroup. Then a similar argument as above yields a homotopy equivalence $Y \to V$ with $X^{r+1} = V^{r+1}$. These are classical ball pair arguments going back to J.H.C. Whitehead. q.e.d.

(1.27) Definition. We say that the theory $\mathbf{T}$ of coactions has the delicate Blakers Massey property if the following holds. Consider a push out diagram in $(\Delta\mathbf{T})_{\text{free}}$

$$\begin{tikzcd} K \arrow[r, "\bar{\jmath}"] & K \cup_L Y \\ L \arrow[u, "i"] \arrow[r, "j"] & Y \arrow[u] \end{tikzcd}$$

where i and j are free inclusions of CW-objects with $K^{m-1} = L^{m-1}$ and $Y^{n-1} = L^{n-1}$ and $m, n \geq 1$. Then for all cogroups A in $\mathbf{T}$ the induced map

$$\bar{\jmath}_* : \pi_r^A(K, L) \to \pi_r^A(K \cup_L Y, Y)$$

is surjective for $1 \leq r \leq n + m - 1$ and bijective for $1 \leq r \leq n + m - 2$. (The usual (non-delicate) Blakers-Massey property requires only surjectivity of $\bar{\jmath}_*$ for $r \leq n + m - 2$ and bijectivity of $\bar{\jmath}_*$ for $r \leq n + m - 3$; compare (B.2.8).)

(1.28) Remark. We define $\mathbf{T}$-complexes $X_{(\geq 1)}$ in $\mathbf{C}_c = (\Delta\mathbf{T})_{\text{free}}$ by CW-objects X as in (1.11) however with the shift $+1$ in dimension

$$X_{(n)} = X^{n-1} \quad \text{for } n \geq 1.$$

Then all objects of $\mathbf{T}$ are $\mathbf{T}$-complexes of dimension 1 and we can proof that the delicate Blakers-Massey property in (1.27) implies the Blakers-Massey property in (IV.5.3) for $\mathbf{T}$-complexes. For this we use (1.26) and the glueing lemma Baues [AH] II.1.2. The shift $+1$ above implies that $(m-1, \mathbf{T})$-connected corresponds to m-connected in (IV.5.2). Hence the delicate Blakers-Massey property of $\mathbf{T}$ implies by (V.1.2) that the inclusion

$$\text{const} : \mathbf{T} \subset (\Delta\mathbf{T})_{\text{free}}/\simeq$$

yields a homological cofibration category in which all results of chapter I, ... , VII hold.

Below we describe many examples of theories of coactions which satisfy the delicate Blakers-Massey property. For all these examples one can use the following results which are special cases of the general theory in this book.

For the theory $\mathbf{T}$ of coactions we obtain the category **Coef** of coeffcients as in (I, § 4). Then the coefficient functor

$$c : (\Delta\mathbf{T})_{CW}/\simeq \,\to \mathbf{Coef} \tag{1.29}$$

is defined which carries X to the attaching map $\partial_X : A_1 \to X_0$ and which carries a map $f : X \to Y$ to the ∂-equivalence class of $f_0 : X_0 \to Y_0$. One can check that c is a well defined functor. In fact c is obtained by the commutative diagram

$$\begin{tikzcd} (\Delta\mathbf{T})_{CW} \arrow[r, "\bar{c}"] \arrow[d, "q"] & \mathbf{Twist} \arrow[d, "q"] \\ (\Delta\mathbf{T})_{CW}/\simeq \arrow[r, "c"] & \mathbf{Coef} \end{tikzcd}$$

where q denotes the quotient functor and where $\bar{c}$ carries X to the simplicial 1-diagram given by X; see (I.3.7). The functor $\bar{c}$ has a left adjoint

$$\overline{\mathrm{sk}}_1 : \mathbf{Twist} \to (\Delta\mathbf{T})_{CW} \tag{1.30}$$

for which $\overline{\mathrm{sk}}_1(\bar{c}(X)) = \mathrm{sk}_1(X)$ is the 1-skeleton of X. We also write

$$\overline{\mathrm{sk}}_1(\partial_X) = X^1$$

where $\partial_X : A^1 \to X^0$ is the attaching map of the CW-object X^1.

For an object ∂_X in **Coef** we define the additive category $\mathbf{mod}(\partial_X)$ as follows. Let $n \geq 1$. The objects of $\mathbf{mod}(\partial_X)$ are given by the coproducts $A \wedge S[n] \vee X^1$ in $\Delta\mathbf{T}$ where A is a cogroup in $\mathbf{T}$ and X^1 is given by the attaching map ∂_X. Morphisms are commutative diagrams

$$\begin{tikzcd} X^1 \arrow[r, equal] \arrow[d] & X^1 \arrow[d] \\ A \wedge S[n] \vee X^1 \arrow[r, "f"] \arrow[d, "(0,1)"'] & B \wedge S[n] \vee X^1 \arrow[d, "(0,1)"'] \\ X^1 \arrow[r, equal] & X^1 \end{tikzcd} \tag{1.31}$$

in the homotopy category $(\Delta\mathbf{T})_{\mathrm{free}}/\simeq$. The initial object in $\mathbf{mod}(\partial_X)$ is given by the trivial cogroup $*$ in $\mathbf{T}$ for which $* \wedge S[n] \vee X^1 = X^1$. If $\mathbf{T}$ satisfies the delicate Blakers-Massey property then the partial suspension shows that the category $\mathbf{mod}(\partial_X)$ does not depend on the choice of n with $n \geq 1$. Therefore we omit $[n]$ in the notation and we write

$$A \wedge S[n] \vee X^1 = A \wedge S \vee X^1$$

for an object in the additive category $\mathbf{mod}(\partial_X)$. The sum in $\mathbf{mod}(\partial_X)$ is given by

$$(A \wedge S \vee X^1) \oplus (B \wedge S \vee X^1) = (A \vee B) \wedge S \vee X^1$$

We point out that $\mathbf{mod}(\partial_X)$ is not in an obvious way the category of free objects in an abelian category.

Each map $u : \partial_X \to \partial_Y$ in **Coef** induces an additive functor

$$u_* : \mathbf{mod}(\partial_X) \to \mathbf{mod}(\partial_Y) \tag{1.32}$$

which carries $A \wedge S \vee X^1$ to $A \wedge S \vee Y^1$ and which carries f in (2.31) to the map

$$((1 \vee \bar{u})f, 1) : A \wedge S \vee Y^1 \to B \wedge S \vee Y^1$$

Here $\bar{u} : X^1 \to Y^1$ is a map for which $c(\bar{u}) = u$. The delicate Blakers-Massey property implies that (1.32) is well defined. Hence we get for each set $\mathcal{A}$ consisting of cogroups in $\mathbf{T}$ the enveloping functor

$$U_{\mathcal{A}} : \mathbf{Coef} \to \mathbf{Ringoids} \tag{1.33}$$

which carries ∂_X to the full subcategory of $\mathbf{mod}(\partial_X)$ consisting of objects $A \otimes S \vee X^1$ with $A \in \mathcal{A}$. Moreover $U_{\mathcal{A}}$ is defined on morphisms by u_* in (1.32).

(1.34) Remark. Let $\mathbf{T}$ be a theory of coactions satisfying the delicate Blakers-Massey property. Then we claim that $\mathbf{mod}(\partial_X)$ in (1.32) actually coincides with the additivization of $\mathbf{premod}(\partial_X)$ in (I.5.10).

(1.35) Definition. Let $\mathbf{T}$ be a theory of coactions satisfying the delicate Blakers-Massey property. Let X be a CW-object in $\Delta\mathbf{T}$ with attaching maps $A^n \wedge S[n-1] \to X^{n-1}$. Then there is a well defined chain complex (see (V, § 2))

$$\begin{cases} C_*(X) & \text{in } \mathbf{mod}(\partial_X) \quad \text{with} \\ C_n(X) = A^{n-1} \wedge S \vee X^1 & \text{for } n \geq 1 \end{cases} \tag{1}$$

and $C_n(X) = 0$ for $n \leq 0$. We point out that there is a shift $+1$ in degree in this definition of C_*X. Moreover a map $f : X \to Y$ between CW-objects induces a chain map

$$f_* : u_*C_*(X) \to C_*(Y) \quad \text{in } \mathbf{mod}(\partial_Y) \tag{2}$$

where $u = c(f)$ is defined by the coefficient functor c in (1.29).

If $\mathbf{T}$ is an augmented theory (or more generally a weakly augmented theory) then also the augmented chain complex $\operatorname{aug} C_*(X)$ in $\mathbf{mod}(\partial_X)$ is defined; see (II, § 6). In this case it is suitable to denote (1) by $C_*(X, *)$ and to write $C_*(X)$ for the augmented chain complex; compare (B.1.26).

We say that the CW-object X is of type $\mathcal{A}$ if for all $n \geq 0$ the objects A^n in $\mathbf{T}$ given by X in (1) are coproducts of objects in $\mathcal{A}$. In this case $C_*(X)$ in (1) is considered as a chain complex of $U_{\mathcal{A}}(\partial_X)$-modules; see (1.33).

Using the chain complexes in (1.35) we define for each (right) $\mathbf{mod}(\partial_X)$-module M the cochain complexes of abelian groups $M(C_*X)$ so that the cohomology with coefficients in M

$$H^n(X;M) = H^n M(C_*X) \tag{1.36}$$

is defined. As an example we observe that homotopy groups (1.23) yield canonically $\mathbf{mod}(\partial_X)$-modules as follows. Let U be an object in $\Delta\mathbf{T}$ and let

$$u : X^0 \to U$$

be a map which admits an extension $X^1 \to U$. If the delicate Blakers-Massey property holds one obtains the right $\mathbf{mod}(\partial_X)$-module

$$u^*\pi_n(U) : \mathbf{mod}(\partial_X)^{\mathrm{op}} \to \mathbf{Ab} \tag{1.37}$$

for $n \geq 2$. This module carries $A \wedge S \vee X^1$ to the homotopy group $\pi_{n-1}^A(U)$ in (1.23). Here we use again a shift in degree since A is considered to be of dimension 1; see (2.28). A map $f : B \wedge S \vee X^1 \to A \wedge S \vee X^1$ in $\mathbf{mod}(\partial_X)$ induces the homomorphism

$$f^* : \pi_{n-1}^B(U) \to \pi_{n-1}^A(U)$$

which carries $(a : B \wedge S[n-1] \to U) \in \pi_{n-1}^B(U)$ to the composite

$$f^*(a) : A \wedge S[n-1] \xrightarrow{f'} B \wedge S[n-1] \vee X^1 \xrightarrow{(a,u)} U$$

where f' is determined by f. As a special case we obtain for the inclusion $i : X^0 \subset X$ of a CW-object X the $\mathbf{mod}(\partial_X)$-module $i^*\pi_n(X)$ by the homotopy groups $\pi_{n-1}^A(X)$. The $\mathbf{mod}(\partial_X)$-module (1.37) is needed in the next result where we again use the shifted dimension of skeleta in (1.28).

(1.38) Theorem. *Let $\mathbf{T}$ be a theory of coactions in which the delicate Blakers-Massey holds. Let $Y \subset X$ be a free inclusion where X is a CW-object in $\Delta\mathbf{T}$ and let $f : Y \to U$ be a map in $\Delta\mathbf{T}$ such that $f^{n-1} = f \mid Y^{n-1}$ admits an extension $g : X^{n-1} \to U$, $n \geq 2$. Then the restriction $g \mid X^{n-2}$ admits an extension $\bar{g} : X^n \to U$ with $\bar{g} \mid Y^n = f^n$ if and only if an obstruction element*

$$\mathcal{O}(g \mid X^{n-2}) \in H^{n+1}(X, Y; u^*\pi_n U)$$

vanishes. Here $u : X^0 \to U$ is the restriction of g.

This typical result of obstruction theory is a special case of (V, § 4). We define the homology $H_n(X)$, resp. $\hat{H}_n(X)$, which is a right $\mathbf{mod}(\partial_X)$-module

$$H_n(X) : \mathbf{mod}(\partial_X)^{\mathrm{op}} \to \mathbf{Ab}. \tag{1.39}$$

Here $H_n(X)$ carries the object $A \wedge S \vee X^1$ to the abelian group

$$H_n(X)(A \wedge S \vee X^1) = H_n \operatorname{Hom}(A \wedge S \vee X^1, C_*(X))$$

where Hom denotes the abelian group of morphisms in $\mathbf{mod}(\partial_X)$. If we replace $C_*(X)$ by $\hat{C}_*(X)$ we obtain $\hat{H}_n(X)$.

Now we are ready to state the following homological Whitehead theorem for simplicial objects in theories of coactions. This is a special case of (VI, § 7).

(1.40) Theorem. *Let* $\mathbf{T}$ *be a theory of coactions satisfying the delicate Blakers-Massey property and let* $f : X \to Y$ *be a map between CW-objects in* $\Delta\mathbf{T}$. *Then* f *is a homotopy equivalence (or equivalently* f *is an isomorphism in the homotopy category* $(\Delta\mathbf{T})_{\text{free}}/\simeq$) *if and only if the coefficient functor carries* f *to an isomorphism* $u : \partial_X \cong \partial_Y$ *in* **Coef** *and one of the following conditions (i), (ii), (iii) is satisfied.*

$$f_* : u_* C_*(X) \to C_* Y \tag{i}$$

is a homotopy equivalence of chain complexes in $\mathbf{mod}(\partial_Y)$.

$$f_* : H_n(X) \to u^* H_n(Y) \tag{ii}$$

is an isomorphism of $\mathbf{mod}(\partial_X)$*-modules for* $n \geq 1$.

$$f^* : H^n(Y; N) \to H^n(X, u^* N) \tag{iii}$$

is an isomorphism of abelian groups for all right $U(\mathcal{A})$*-modules* N *where* X *and* Y *are of type* $\mathcal{A}$, $n \geq 1$.

Next we consider the Hurewicz homomorphism h and the exact sequence of J.H.C. Whitehead. As a special case of (V.3.4) we get:

(1.41) Theorem. *Let* $\mathbf{T}$ *be a theory of coactions satisfying the Blakers-Massey property and let* X *be a CW-object in* $\Delta\mathbf{T}$. *Then the following sequence is an exact sequence of* $\mathbf{mod}(\partial_X)$*-modules,* $n \geq 2$.

$$\begin{aligned} \cdots \longrightarrow \Gamma_n(X) \longrightarrow \pi_n(X) \xrightarrow{h} H_n(X) \longrightarrow \Gamma_{n-1}(X) \longrightarrow \dots \\ \longrightarrow \Gamma_2(X) \longrightarrow \pi_2(X) \longrightarrow H_2(X) \longrightarrow \Gamma_1(X) \longrightarrow 0 \end{aligned}$$

Moreover the sequence is natural in X.

The modules $\pi_n(X)$ and $H_n(X)$ are defined in (1.37) and (1.39) respectively. For $n \geq 3$ the module $\Gamma_n(X)$ is defined by

$$\Gamma_n(X) = \operatorname{image}\{\pi_n X^{n-2} \to \pi_n X^{n-1}\}$$

Here we again use the shift of degree $X^{n-1} = X_{(n)}$ in (1.28). The definition of Γ_1 and Γ_2 is more complicated; see (V.5.3) and (II, § 2).

(1.42) Definition. Let $\mathcal{A}$ be a set of cogroups in $\mathbf{T}$. We say that a CW-object X in $\Delta\mathbf{T}$ is $\mathcal{A}$-finite if the n-cells A_n of X are finite coproducts of objects in $\mathcal{A}$ for $n \geq 0$ and if X is finite dimensional, that is $X = X^n$ for some $n \geq 0$. This implies that X is of type $\mathcal{A}$; see (1.35).

Now let X and Y be CW-objects in $\Delta\mathbf{T}$. A domination (X, f, g, H) of Y in $(\Delta\mathbf{T})_{\text{free}}$ is given by maps

$$Y \xrightarrow{f} X \xrightarrow{g} Y \quad \text{in } (\Delta\mathbf{T})_{\text{free}} \tag{1.43}$$

and a homotopy $H : gf \simeq 1$. The domination is finite dimensional if X is finite dimensional and the domination is $\mathcal{A}$-finite if X is $\mathcal{A}$-finite. As a special case of (VII.2.4) we get the following simplicial version of a classical result of Wall; see (A.1.36).

(1.44) Theorem. *Let* $\mathbf{T}$ *be a theory of coactions satisfying the delicate Blakers-Massey property. If the CW-object* Y *in* $\Delta\mathbf{T}$ *admits an* $\mathcal{A}$*-finite domination in* $(\Delta\mathbf{T})_{\text{free}}$ *then the finiteness obstruction*

$$[Y] = [C_*(Y)] \in \tilde{K}_0(U_{\mathcal{A}}(\partial_Y))$$

is defined. Here ∂_Y *is the attaching map of* Y^1 *which is an object in* **Coef** *and* $U_{\mathcal{A}}$ *is the enveloping functor (1.33). Moreover* $\tilde{K}_0$ *is the reduced projective class group; see (VII, § 1). One has* $[Y] = 0$ *if and only if there exists an* $\mathcal{A}$*-finite CW-object* X *in* $\Delta\mathbf{T}$ *and a homotopy equivalence* $X \to Y$ *in* $(\Delta\mathbf{T})_{\text{free}}$.

2 Examples of Theories of Coactions Satisfying the Delicate Blakers-Massey Property

We have seen in section § 1 that the homotopy theory of simplicial objects in a theory $\mathbf{T}$ of coactions satisfies basic results of homotopy theory if the delicate Blakers-Massey property (1.27) holds in $\mathbf{T}$. In the following three sections we describe many examples of theories with this property. The first example considers the category $\mathbf{T} = \mathbf{gr}$ of free groups studied by Kan. It is, in fact, worthwhile to consider for each example below more carefully all results of section § 1 and of the general theory in chapter I, ... , VII.

(2.1) Example. Let $\mathbf{T} = \mathbf{gr}$ be the category of free groups which is a full subcategory of the category $\mathbf{Gr}$ of groups. Then $\mathbf{gr}$ is a theory of cogroups which satisfies the delicate Blakers-Massey property. To see this we use the result of Kan [HG] that there are equivalences of homotopy theories

$$\mathrm{Ho}(\mathbf{Top}_0^*) \xrightarrow{\sim} \mathrm{Ho}(\Delta\mathbf{Gr}) \xrightarrow{\sim} (\Delta\mathbf{gr})_{\text{free}}/\simeq$$

Here $\mathbf{Top}_0^*$ is the category of path connected pointed spaces. A reduced CW-complex X with $X^0 = *$ corresponds by the equivalence to a CW-object G_X in $\Delta\mathbf{gr}$. Moreover one has the shift of dimension since X^n corresponds to $(G_X)^{n-1}$ for $n \geq 1$. Hence the Blakers-Massey theorem in $\mathbf{Top}_0^*$ (see for example Gray [HT]) yields the delicate Blakers-Massey property of $\mathbf{gr}$.

We now describe two generalizations of the classical Kan example on simplicial groups in (2.1).

(2.2) Example. Let D be a discrete set and let $\mathbf{Top}_0^D$ be the category of spaces X under D for which $D \to X$ is path connected (i.e. $D = \pi_0 D \to \pi_0 X$ is surjective). We have the theory

$$\mathbf{T} = \mathbf{grd}(D) \subset \mathbf{Top}_0^D/\simeq \operatorname{rel} D$$

which is the full subcategory consisting of 1-dimensional CW-complexes X with 0-skeleton $X^0 = D$. Hence $\mathbf{grd}(D)$ is the category of free groupoids G with $\mathrm{Ob}(G) = D$ and functors which are the identity on objects. It is the result of Dwyer-Kan [SG] that one has an equivalence of categories

$$\mathrm{Ho}(\mathbf{Top}_0^D) \xrightarrow{\sim} (\Delta\mathbf{grd}(D))_{\mathrm{free}}/\simeq$$

For $D = *$ this is the classical result of Kan in (2.1). The equivalence again implies that $\mathbf{grd}(D)$ satisfies the delicate Blakers-Massey property. The homotopy theory of $\mathbf{Top}_0^D$ is studied in (A, § 1) above. Hence all results of (A, § 1) with D a discrete space have a transformation to the category $(\Delta\mathbf{grd}(D))_{\mathrm{free}}$. For example the Whitehead theorem (1.24) for $\mathbf{T} = \mathbf{grd}(D)_{\mathrm{free}}$ corresponds to the classical Whitehead theorem for CW-complexes X with $X^0 = D$.

The next example also generalizes the result of Kan (2.1) but seems to be new.

(2.3) Example. Let G be a group and let $\mathbf{Gr}^G$ be the category of groups under G. Let $\mathbf{gr}(G) \subset \mathbf{Gr}^G$ be the full subcategory consisting of objects $G \to G \vee F$ where $G \vee F$ is the coproduct of the group G and a free group F. See (I.2.3) below. For $G = *$ the trivial group we clearly have $\mathbf{gr}(*) = \mathbf{gr}$ as in (2.1). We claim that $\mathbf{gr}(G)$ has the delicate Blakers-Massey property and that one has an equivalence of categories

$$\mathrm{Ho}(\mathbf{Top}_0^{K(G,1)}) \xrightarrow{\sim} (\Delta\mathbf{gr}(G))_{\mathrm{free}}/\simeq$$

Here $K(G,1)$ is the Eilenberg-Mac Lane space of G and $\mathbf{Top}_0^D$ is the category of spaces X under D for which $D \to X$ is connected. The homotopy category $\mathbf{Top}^D$ is studied in (A, § 1). All the results there concerning $\mathbf{Top}_0^{K(G,1)}$ have a transformation to $(\Delta\mathbf{gr}(G))_{\mathrm{free}}$. If $G = *$ is the trivial group then this is again the classical correspondence of Kan (2.1) between connected spaces and simplicial groups.

(2.4) Conjecture. The theorem of Kan in (2.1) and the result of Dwyer-Kan (2.2) should be generalized for the category of $\mathbb{A}$-diagrams in § 2 of chapter A. More precisely let $\mathbb{A}$ be a small category and let D be a discrete $\mathbb{A}$-diagram. Let $\mathbf{T}_{\mathbb{A}}^D$ be the full subcategory of $(\mathbb{A}\mathbf{Top})_c^D/\simeq \operatorname{rel} D$ given by the 1-dimensional reduced relative $\mathbb{A}$-CW-complexes (X_1, D). Then $\mathbf{T}_{\mathbb{A}}^D$ is a theory of coactions. We conjecture that $\mathbf{T}_{\mathbb{A}}^D$ satisfies the delicate Blakers-Massey property and that one has an equivalence of categories

$$\mathrm{Ho}(\mathbb{A}\mathbf{Top})_0^D \xrightarrow{\sim} (\Delta\mathbf{T}_{\mathbb{A}}^D)_{\mathrm{free}}/\simeq.$$

Here the category $(\mathbb{A}\mathbf{Top})_0^D$ is the full subcategory of $\mathbb{A}$-diagrams X under D for which $D = \pi_0 D \to \pi_0 X$ is surjective. If $\mathbb{A}$ is the trivial category then the equivalence coincides with (2.1).

(2.5) Conjecture. The theorem of Kan (2.1) and the result of Dwyer-Kan (2.2) should also be true for the category of G-spaces in § 3 of chapter A. More precisely let G be a discrete group and let D be a G-space for which the associated $\mathrm{Or}(G)$-space D° in (A.3.5) is homotopy equivalent to a discrete $\mathrm{Or}(G)$-space. Let $\mathbf{T}_G^D$ be the full subcategory of $(G\mathbf{Top})_c^D/\simeq \operatorname{rel} D$ consisting of 1-dimensional reduced relative G-CW-complexes (X_1, D). Then (A.3.16) shows that $\mathbf{T}_G^D$ is a theory of coactions. We conjecture that $\mathbf{T}_G^D$ satisfies the delicate Blakers-Massey property and that one has an equivalence of categories

$$\mathrm{Ho}(G\mathbf{Top})_0^D \xrightarrow{\sim} (\Delta\mathbf{T}_G^D)_{\mathrm{free}}/\simeq$$

Here the category $(G\mathbf{Top})_0^D$ is the full subcategory of $(G\mathbf{Top})^D$ consisting of G-spaces X under D for which $\pi_0 D^\circ \to \pi_0 X^\circ$ is a surjective map between $\mathrm{Or}(G)$-sets. In fact this conjecture is a special case of (2.4) if we set $\mathbb{A} = \mathrm{Or}(G)$.

(2.6) Conjecture. Next we consider the theorem of Kan (3.1) in the context of spaces controlled at infinity in § 4 of chapter A. Let D be a locally finite tree and let ∞ be the Cantor set of ends of D. Hence D is an object in $\infty\mathbf{End}$. Let $\mathbf{T}^D$ be the full subcategory of $(\infty\mathbf{End})_c^D/\simeq \operatorname{rel} D$ consisting of 1-dimensional reduced relative ∞-CW-complexes (X_1, D). Then $\mathbf{T}^D$ is a theory of coactions. We conjecture that $\mathbf{T}^D$ satisfies the delicate Blakers-Massey property and that one has a full and faithful functor

$$lf\mathbf{CW}_0^T/\simeq \operatorname{rel} T \to (\Delta\mathbf{T}^D)_{\mathrm{free}}/\simeq.$$

Here $lf\mathbf{CW}_0^T$ is the category consisting of finite dimensional locally finite CW-complexes X (for which T is a maximal tree in X^1) and proper maps under T. Moreover homotopies $\operatorname{rel} T$ are also proper.

3 Polynomial Theories of Cogroups

We now describe a method which shows that polynomial theories of cogroups have the delicate Blakers-Massey property. For this we need the following notions of "polynomial functor" and "linear extension" of categories.

(3.1) Definition. (Compare Eilenberg-Mac Lane [H].) Let $\mathbf{C}$ be a category with sums and zero object. Let $\Gamma : \mathbf{C} \to \mathbf{Ab}$ be a functor with $\Gamma(*) = 0$. Then for $X_1, \dots, X_q \in \mathbf{C}$ the q-th cross effect $\Gamma(X_1 \mid \cdots \mid X_q)$, $q \geq 1$, is the kernel of the map

$$\Gamma(X_1 \vee \cdots \vee X_q) \to \bigoplus_{i=1}^{q} \Gamma(\partial_i)$$

for which the i-th coordinate is induced by

$$\begin{cases} 1 \vee 0 \vee 1 : X_1 \vee \cdots \vee X_q \to \partial_i \quad \text{with} \\ \partial_i = X_1 \vee \cdots \vee X_{i-1} \vee * \vee X_{i+1} \vee \cdots \vee X_q \end{cases}$$

We say that Γ has degree q if $\Gamma(X_1 \mid \cdots \mid X_{q+1}) = 0$ for all $X_1, \ldots, X_{q+1} \in \mathbf{C}$. The functor Γ is linear, resp. quadratic if Γ has degree 1, resp. 2. Moreover Γ is polynomial if there is $1 \leq q < \infty$ such that Γ has degree q. We also call q the polynomial degree of Γ.

(3.2) Definition. (Compare Baues [AH]). Let $\mathbf{C}$ be a category and let $D : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$ be a bifunctor (also termed $\mathbf{C}$-bimodule). We say that

$$D \overset{+}{\rightarrowtail} \mathbf{E} \overset{p}{\twoheadrightarrow} \mathbf{C}$$

is a linear extension of the category $\mathbf{C}$ by D if (a)–(c) hold.

(a) $\mathbf{E}$ and $\mathbf{C}$ have the same objects and p is a full functor which is the identity on objects.
(b) For each $f : A \to B$ in $\mathbf{C}$ the abelian group $D(A, B)$ acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\mathbf{E}$. We write $f_0 + \alpha$ for the action of $\alpha \in D(A, B)$ on $f_0 \in p^{-1}(f)$. Any $f_0 \in p^{-1}(f)$ is called a lift of f.
(c) The action satisfies the linear distributivity law:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.$$

We now consider a linear extension of categories

$$D \overset{+}{\rightarrowtail} \mathbf{E} \xrightarrow{p} \mathbf{C}$$

where $\mathbf{E}$ and $\mathbf{C}$ are theories of cogroups in which the initial object $*$ is also a final object (i.e. a zero object). Moreover p carries sums to sums and carries the cogroup structure of X in $\mathbf{E}$ to the cogroup structure of X in $\mathbf{C}$ and for all X, Y in $\mathbf{E}$ we have a central extension of groups

$$D(X, Y) \overset{i}{\rightarrowtail} \mathbf{E}(X, Y) \twoheadrightarrow \mathbf{C}(X, Y) \tag{3.3}$$

where $i(\alpha) = 0 + \alpha$ is given by the action of D on $\mathbf{E}$ in (3.2). More generally we have

$$(f + \alpha) +_\mu (g + \beta) = (f +_\mu g) + \alpha + \beta$$

for $f, g \in \mathbf{E}(X, Y)$ and $\alpha, \beta \in D(X, Y)$. Here $+$ is the action in (3.2) and $+_\mu$ is the group structure on $\mathbf{E}(X, Y)$ defined by the cogroup $\mu : X \to X \vee X$. Compare (6.3) in Baues-Hartl-Pirashvili [QC]. If in addition D is a bifunctor which is left linear and right polynomial then we say that $\mathbf{E}$ is polynomial related to $\mathbf{C}$.

(3.4) Theorem. *Let* $\mathbf{E}$ *be polynomial related to* $\mathbf{C}$. *If* $\mathbf{C}$ *satisfies the delicate Blakers-Massey property then also* $\mathbf{E}$ *satisfies the delicate Blakers-Massey property.*

Proof. Consider the diagram

$$\mathbf{X} = \left(\begin{array}{ccc} K & \longrightarrow & K \cup_Y L \\ \uparrow & & \uparrow \\ L & \longrightarrow & Y \end{array} \right)$$

in (1.27) defined in $\Delta\mathbf{E}$. Then the induced diagram $p\mathbf{X}$ in $\Delta\mathbf{C}$ is again a diagram as in (1.27) since the functor $p : \Delta\mathbf{E} \to \Delta\mathbf{C}$ given by $p : \mathbf{E} \to \mathbf{C}$ carries CW-objects to CW-objects. Now let A be a cogroup in $\mathbf{E}$ and apply the functors

$$\mathbf{E}(A, -) : \mathbf{E} \to \mathbf{Gr}$$
$$\mathbf{C}(A, -) : \mathbf{C} \to \mathbf{Gr}$$

to the diagram $\mathbf{X}$ and $p\mathbf{X}$ respectively. Then we get the cubical diagram of simplicial groups

$$\mathbf{E}(A, \mathbf{X}) \xrightarrow{p} \mathbf{C}(A, p\mathbf{X}) \tag{1}$$

which by (3.3) is surjective and which has the kernel diagram

$$D(A, p\mathbf{X}) = \mathrm{kernel}(p). \tag{2}$$

This is also the fiber diagram of (1). Since we assume that $\mathbf{C}$ satisfies the delicate Blakers-Massey property we know that $\mathbf{C}(A, p\mathbf{X})$ is $(n+m-2)$-homotopy cartesian. For this recall that a diagram of spaces or simplicial sets

$$\begin{array}{ccc} X_1 & \longrightarrow & X_{12} \\ \uparrow & & \uparrow \\ X & \longrightarrow & X_2 \end{array}$$

is k-<u>homotopy cartesian</u> if the induced map

$$\pi_r(X_1, X) \to \pi_r(X_{12}, X_2)$$

is surjective for $r \leq k+1$ and bijective for $r \leq k$. Using lemma (3.5) below we know that $D(A, p\mathbf{X})$ is $(n+m-2)$-homotopy cartesian. This implies by (1) and (2) that also $\mathbf{E}(A, \mathbf{X})$ is $(n+m-2)$-homotopy cartesian, compare Goodwillie calculus. Hence we see that the delicate Blakers-Massey property also holds for $\mathbf{E}$. q.e.d.

(3.5) Lemma. *Let* $\mathbf{T}$ *be a theory of cogroups with zero object and let* $\Gamma : \mathbf{T} \to \mathbf{Ab}$ *be a polynomial functor. Then diagram (1.27) yields a diagram*

$$\begin{array}{ccc} \Gamma(K) & \longrightarrow & \Gamma(K \cup_L Y) \\ \uparrow & & \uparrow \\ \Gamma(L) & \longrightarrow & \Gamma(Y) \end{array}$$

which is $(n+m-2)$*-homotopy cartesian.*

Proof. Since $*$ is the zero object in $\mathbf{T}$ we have the quotient K/L in $\Delta\mathbf{T}$ which is a push out of $K \leftarrow L \rightarrow \mathrm{const}(*)$. Moreover since (1.27) with $X = K \cup_L Y$ is a push out diagram we know that

$$K/L \cong X/Y \tag{1}$$

are isomorphic. Moreover the sequence

$$L \xrightarrow{i} K \xrightarrow{q} K/L \tag{2}$$

is split in each degree, that is for $n \geq 0$ we have the equation

$$K_n = L_n \vee (K/L)_n \tag{3}$$

such that i is the inclusion of L_n and q is the projection $(0,1)$ onto $(K/L)_n$. The equation (3), however, is not compatible with the simplicial operators while the maps i and q are defined in $\Delta\mathbf{T}$. Similar properties as in (2) and (3) hold for the pair (X, Y).

We now consider the functor $\Gamma : \mathbf{T} \rightarrow \mathbf{Ab}$. We have the natural map $(U \in \mathbf{T})$

$$P : \Gamma(U \mid U) \subset \Gamma(U \vee U) \xrightarrow{(1,1)_*} \Gamma(U) \tag{4}$$

induced by the folding map $(1,1) : U \vee U \rightarrow U$. This map yields for the simplicial objects K, L the following commutative diagram

$$\begin{tikzcd}
\Gamma(L) \arrow[r, "\Gamma i"] & \Gamma(K) \arrow[r, "j"] & \Gamma(K)/\Gamma(L) \arrow[r, "q_*"] & \Gamma(K/L) \\
\Gamma(L \mid L) \arrow[u, "P"] \arrow[r, "\Gamma(1|i)"] & \Gamma(L \mid K) \arrow[u, "\bar{P}"] \arrow[r] & \Gamma(L \mid K)/\Gamma(L \mid L) \arrow[u, "P_*"] &
\end{tikzcd} \tag{5}$$

Here $\bar{P}$ is the composite

$$\bar{P} : \Gamma(L \mid K) \xrightarrow{\Gamma(i|1)} \Gamma(K \mid K) \xrightarrow{P} \Gamma(K).$$

Moreover $\Gamma(K)/\Gamma(L)$ is the quotient of simplicial abelian groups and j is the quotient map. Since we assume $\Gamma(*) = 0$ we see that the map $\Gamma(q)$ has the factorization $\Gamma(q) = q_* j$. Moreover P_* in (5) is induced by $\bar{P}$. Now (3) implies that

$$\Gamma(L \mid K)/\Gamma(L \mid L) \xrightarrow{P_*} \Gamma(K)/\Gamma(L) \xrightarrow{q_*} \Gamma(K/L) \rightarrow 0 \tag{6}$$

is an exact sequence of simplicial abelian groups. This follows from the definition of the cross effect (3.1). We now apply (5) inductively, that is, we replace Γ by the functor Γ_U with $\Gamma_U(V) = \Gamma(U \mid V)$ for U, $V \in \mathbf{T}$. Then we get inductively the following system

$$
\begin{array}{ccccccc}
\dots & \xrightarrow{p_*} & \Gamma_2(K)/\Gamma_2(L) & \xrightarrow{p_*} & \Gamma_1(K)/\Gamma_1(L) & \xrightarrow{p_*} & \Gamma(K)/\Gamma(L) \\
 & & \dots \Big\downarrow q_* & & \Big\downarrow q_* & & \Big\downarrow q_* \\
 & & \Gamma_2(K/L) & & \Gamma_1(K/L) & & \Gamma(K/L)
\end{array}
\tag{7}
$$

Here $\Gamma_i(K)$ is the $(i+1)$-th cross effect

$$\Gamma_i(K) = \Gamma_i^L(K) = \Gamma(L \mid \cdots \mid L \mid K) \tag{8}$$

where L appears i times and $\Gamma_0(K) = \Gamma(K)$. Moreover as in (6) the sequence

$$\Gamma_{i+1}(K)/\Gamma_{i+1}(L) \xrightarrow{p_*} \Gamma_i(K)/\Gamma_i(L) \xrightarrow{q_*} \Gamma_i(K/L) \to 0 \tag{9}$$

is exact for $i \geq 0$. Since Γ is polynomial we see that $\Gamma_n = 0$ for all n sufficiently large. We now consider the map

$$\bar{\jmath}_* : \Gamma_i^L(K)/\Gamma_i^L(L) \to \Gamma_i^Y(X)/\Gamma_i^Y(Y) \tag{10}$$

induced by $\bar{\jmath} : (K, L) \to (X, Y)$. For $i \geq 1$ the simplicial group $\Gamma_i^L(K)$ is the diagonal of a multi simplicial abelian group (8). Hence by the Eilenberg-Zilber theorem (see Dold-Puppe [H] 2.9) we can compute $\pi_n \Gamma_i^L(K)$ by the homology of the total complex of $\Gamma_i^L(K)$. Now the grading in the total complex and the assumptions $L^{m-1} = K^{m-1}$ and $Y^{n-1} = L^{n-1}$ show that (10) induces a surjection $\pi_i(\bar{\jmath}_*)$ for $i \leq m+n-1$ and an isomorphism $\pi_i(\bar{\jmath}_*)$ for $i \leq m+n-2$. This implies inductively by (7) and (9) and (1) that also

$$\bar{\jmath}_* : \Gamma(K)/\Gamma(L) \to \Gamma(X)/\Gamma(Y) \tag{11}$$

induces a surjection $\pi_i(\bar{\jmath}_*)$ for $i \leq m + n - 1$ and an isomorphism $\pi_i(\bar{\jmath}_*)$ for $i \leq m + n - 2$. Hence the diagram in (3.5) is $(n + m - 2)$ homotopy cartesian.
q.e.d.

(3.6) Definition. We say that a theory $\mathbf{T}$ of cogroups is polynomial if $\mathbf{T}$ has a zero object and if there is $n \geq 0$ and a sequence

$$\mathbf{T} = \mathbf{E}_n \to \mathbf{E}_{n-1} \to \cdots \to \mathbf{E}_0$$

of theories of cogroups where $\mathbf{E}_{k+1}$ is polynomial related to $\mathbf{E}_k$ for $0 \leq k < n$ and where $\mathbf{E}_0$ is an additive category (i.e. $\mathbf{E}_0$ is a theory of cogroups in which all morphisms are linear; see (I.1.10)). For example the "quadratic categories" in Baues-Hartl-Pirashvili [QC] 5.4 are such polynomial theories of cogroups.

(3.7) Theorem. *A polynomial theory of cogroups satisfies the delicate Blakers-Massey property.*

Proof. It is clear that an additive category satisfies the delicate Blakers-Massey property. Therefore (3.7) is an inductive application of (3.4). q.e.d.

(3.8) Example. Let $\langle Z\rangle$ be the free group generated by the set Z and let

$$\langle Z\rangle_n = \langle Z\rangle/\Gamma_{n+1}\langle Z\rangle \tag{1}$$

be the free group of nilpotency degree n. Here $\Gamma_{n+1}G$ is the subgroup of the group G given by all $(n+1)$-fold commutators in G. Let

$$\mathbf{nil}_n \subset \mathbf{Gr} \tag{2}$$

be the full subcategory consisting of all $\langle Z\rangle_n$ where Z is a set. Then $\mathbf{nil}_n$ is a polynomial theory of cogroups. In fact we have for $n \geq 2$ the well known central extension of groups

$$L_n(Z) \rightarrowtail \langle Z\rangle_n \to \langle Z\rangle_{n-1} \tag{3}$$

where $L_n(Z)$ is the degree n part of the free Lie algebra generated by Z. By (3) we see that

$$\mathrm{Hom}(-, L_n) \overset{+}{\rightarrowtail} \mathbf{nil}_n \to \mathbf{nil}_{n-1} \tag{4}$$

is a linear extension; compare Baues [AH]. Moreover (4) shows that $\mathbf{nil}_n$ is polynomial related to $\mathbf{nil}_{n-1}$. Since $\mathbf{nil}_1 = \mathbf{ab}$ is the category of free abelian groups we see that $\mathbf{nil}_n$ is polynomial. Hence for all n the category $\mathbf{nil}_n$ satisfies the delicate Blakers-Massey property. This example can be generalized for many other varieties of groups.

The next examples can be applied to simplicial "$(r-1)$-connected π-algebras" and simplicial "modules over the Steenrod algebra" respectively.

(3.9) Example. Let $r \geq 2$ and let

$$\mathcal{S}_r = \{S^r, S^{r+1}, \dots\}$$

be the set of all spheres S^n with $n \geq r$. We define the subcategory

$$\mathbf{T}(\mathcal{S}_r) \subset \mathbf{Top}^*/\simeq$$

consisting of all (not necessarily finite) one point unions of spheres in $\mathcal{S}_r$. Then $\mathbf{T}(\mathcal{S}_r)$ is a theory of cogroups which satisfies the delicate Blakers-Massey property.

(3.10) Example. Let $r \geq 1$ and let $\mathcal{A}$ be a set of abelian groups. Moreover let

$$\mathcal{K}_r(\mathcal{A}) = \{K(A, n+r);\ n \geq 0, A \in \mathcal{A}\}$$

be the corresponding set of Eilenberg-Mac Lane spaces $K(A, n+r)$. We define the subcategory

$$\mathbf{K}_r(\mathcal{A}) \subset \mathbf{Top}^*/\simeq$$

consisting of all (not necessarily finite) products of spaces in $\mathcal{K}_r(\mathcal{A})$. Then the dual of $\mathbf{K}_r(\mathcal{A})$ is a theory of cogroups which satisfies the delicate Blakers-Massey property.

In fact (3.9) and (3.10) are special cases of the following result.

(3.11) Theorem. *Let* $\mathbf{T}$ *be a polynomial graded theory in the sense of Baues [DF] 8.1, 9.4 and let* $\mathbf{free}(\mathbf{T})$ *be the full subcategory of* $\mathbf{model}(\mathbf{T})$ *consisting of free objects. Then* $\mathbf{free}(\mathbf{T})$ *is a theory of cogroups which satisfies the delicate Blakers-Massey property.*

Proof. By Baues [DF] 8.4 we see that $\mathbf{T}_{n+1}$ is polynomial related to $Add(\mathbf{R}_{n+1}) \times \mathbf{T}_n$. Moreover it suffices to check the condition on $\bar{\jmath}_*$ in (1.27) for all generating cogroups A in $\mathbf{T}$. Since by the assumption on $\mathbf{T}$ we have

$$\mathbf{T}(A, X) = \mathbf{T}_n(A, X)$$

where $n = |A|$ we see that we can apply (3.7). q.e.d.

4 Algebras over an Operad

We consider commutative algebras and more generally algebras over an operad. The category of free algebras is a theory of cogroups which satisfies the delicate Blakers-Massey property.

Let R be a commutative ring and let **Calg** be the category of all commutative R-algebras with unit. Let

$$\mathbf{calg} \subset \mathbf{Calg} \tag{4.1}$$

be the full subcategory consisting of free commutative R-algebras $S(V)$ where V is a free R-module. Then **calg** is a theory of cogroups. The cogroup structure of $S(V)$ is given by the diagonal $(1,1) : V \to V \oplus V$, that is

$$\mu = S(1,1) : S(V) \to S(V \oplus V) = S(V) \vee S(V)$$

(4.2) Proposition. *The theory* **calg** *of cogroups given by free commutative R-algebras satisfies the delicate Blakers-Massey property (1.27).*

We have an equivalence of homotopy theories

$$(\Delta\mathbf{calg})_{\mathrm{free}}/\simeq \xrightarrow{\sim} \mathrm{Ho}(\Delta\mathbf{Calg}) \tag{4.3}$$

Given an object A in **Calg** we can choose a weak equivalence $K(A,1) \xrightarrow{\sim} A$ where $K(A,1)$ is in $(\Delta\mathbf{calg})_{\mathrm{free}}$. Then the (co-) homology (1.36) of $K(A,1)$ coincides with the André-Quillen (co-) homology of A with coefficients in the A-module M.

Proof of (4.2). The following proof was pointed out to me by Paul Goerss. Consider the diagram (1.27) in $(\Delta\mathbf{calg})_{\mathrm{free}}$. Let K/L be the quotient of the R-module K by the R-module L. Then the sequence

$$L \to K \to K/L \tag{1}$$

is an exact sequence of simplicial L-modules. Moreover in each degree n

$$0 \to L_n \to K_n \to (K/L)_n \to \tag{3}$$

is a split short exact sequence of L_n-modules. In fact we know that $K_n = L_n \otimes S(V)$ with an appropriate V and we obtain for the short exact sequence

$$S^+(V) \rightarrowtail S(V) \xrightarrow{\varepsilon} R$$

with augmentation ε a splitting of ε so that $(K/L)_n = L_n \otimes S^+(V)$ is contained in K_n. For the push out

$$X = K \cup_L Y = K \otimes_L Y \tag{3}$$

in (1.27) we obtain by (1) and (2) the short exact sequence

$$0 \to L \otimes_L Y \to K \otimes_L Y \to (K/L) \otimes_L Y \to 0 \tag{4}$$

with $L \otimes_L Y = Y$. By (4) and (1) we get

$$\begin{cases} \pi_n(X,Y) = \pi_n((K/L) \otimes_L Y) \\ \pi_n(K,L) = \pi_n(K/L) \end{cases} \tag{5}$$

Now using Quillen [HA] II, § 6 we obtain a spectral sequence

$$\mathrm{Tor}_p^{\pi_* L}(\pi_*(K/L), \pi_* Y) \Rightarrow \pi_{p+q}((K/L) \otimes_L Y)$$

and hence since $K^{m-1} = L^{m-1}$ and $Y^{m-1} = L^{n-1}$ we get by (5) the conclusion on $\bar{\jmath}_*$ in (1.27). q.e.d.

Proposition (4.1) has a generalization for algebras over any operad; in particular for associative algebras and Lie algebras. For an exposition on operads we refer to Getzler-Jones [O], Ginzberg-Kapranov [K] and Fresse [C], and Loday [R]. Let R be a commutative ring and let $\mathcal{P}$ be an operad over R. Then the category $\mathcal{P}$-**Alg** of <u>algebras over</u> $\mathcal{P}$ (or $\mathcal{P}$-algebras) is defined. Moreover let

$$\mathcal{P}\text{-}\mathbf{alg} \subset \mathcal{P}\text{-}\mathbf{Alg} \tag{4.4}$$

be the full subcategory of free $\mathcal{P}$-algebras $T(V)$ where V is a free R-module. Then $T(V)$ is a cogroup with the cogroup structure

$$\mu : T(V) \xrightarrow{T(1,1)} T(V \oplus V) = T(V) \vee T(V)$$

Hence $\mathcal{P}$-**alg** is a theory of cogroups.

(4.5) Remark. The categories in (4.3) can also be described by the use of a single sorted theory **S**. In fact, let **S** be the full subcategory of $\mathcal{P}$-**alg** consisting of finitely generated free $\mathcal{P}$-algebras. Then we have $\mathcal{P}$-**Alg** = **model**(**S**) and $\mathcal{P}$-**alg** = **free**(**S**) as in (I.2.5).

(4.6) Proposition. *The theory $\mathcal{P}$-**alg** of cogroups given by free algebra over the operad $\mathcal{P}$ satisfies the delicate Blakers-Massey property (1.27).*

A proof of this result was obtained by P. Goerss (private communication). Using the result of Quillen [HA] chapter II, § 4 we see that $\Delta(\mathcal{P}\text{-}\mathbf{Alg})$ is a closed model category and that

$$(\Delta\mathcal{P}\text{-}\mathbf{alg})_{\mathrm{free}}/\simeq \;\to \mathrm{Ho}(\Delta\mathcal{P}\text{-}\mathbf{Alg}) \tag{4.7}$$

is an equivalence of categories which generalizes (4.3). Given an object A in $\mathcal{P}$-**Alg** we can choose a weak equivalence $K(A,1) \to A$ in $\Delta\mathcal{P}$-**Alg** where $K(A,1) \in (\Delta\mathcal{P}\text{-}\mathbf{alg})_{\mathrm{free}}$. Then the (co-) homology (1.36) of $K(A,1)$ is the <u>Quillen homology</u> of A.

It is of interest to study in detail all the implications of the theory in chapters I, ..., VII for the homotopy theory of $\Delta\mathcal{P}$-**Alg**.

Chapter D: Resolutions in Model Categories

The purpose of this chapter is to revisit, expand and simplify the "E^2-homotopy theory" of Dwyer-Kan-Stover [E^2], [HG] which built simplicial resolutions of pointed path connected topological spaces out of spheres. Using the notion of a spiral model category $\mathbf{Q}$ in (2.4) we prove the existence of the spiral homotopy category $\mathrm{Ho}(\Delta\mathbf{Q})_s$ of simplicial objects in $\mathbf{Q}$ in (3.5). This is the analogue of an E^2-homotopy theory. Simplicial resolutions of objects in $\mathbf{Q}$ live in the spiral homotopy category.

We introduce two essential assumptions on the closed model category $\mathbf{Q}$. First a full subcategory $\mathbf{C}$ of $\mathbf{Q}$ is given and the notion of homotopy is defined in $\mathbf{C}$ by a natural cylinder I. Second a small subcategory $\mathbf{T}$ of $\mathbf{C}$ is given such that each object A of $\mathbf{T}$ is a cogroup; hence $A \in \mathbf{T}$ is not only a cogroup in the homotopy category $\mathrm{Ho}(\mathbf{Q})$ but also in the category $\mathbf{Q}$. These assumptions are basic properties of a spiral model category

$$\mathbf{Q} = (\mathbf{T} \subset \mathbf{C} \subset \mathbf{Q})$$

The spiral homotopy category $\mathrm{Ho}(\Delta\mathbf{Q})_s$ depends on this structure of $\mathbf{Q}$. The model category is termed spiral since the spiral exact sequences form the crucial ingredient of the theory.

In topology there are, however, no cogroups in $\mathbf{Top}^*$ but we have cogroups in the homotopy category $\mathbf{Top}^*/\simeq \mathrm{rel}\,*$. Cogroups defined up to homotopy do not suffice for the development of the "spiral homotopy theory" in this chapter. We therefore replace the category $\mathbf{Top}_0^*$ of connected pointed spaces by the category $\Delta\mathbf{Gr}$ of simplicial groups. There are many cogroups in $\Delta\mathbf{Gr}$ defined by constant free groups. The category of simplicial groups is the typical example of a spiral model category. The spiral homotopy category $\mathrm{Ho}(\Delta\mathbf{Q})_s$ for $\mathbf{Q} = \Delta\mathbf{Gr}$ is equivalent to the E^2-homotopy category of simplicial pointed spaces of Dwyer-Kan-Stover [E^2]. The theory in this chapter was also motivated by recent papers of Blanc [AI] and Goerss-Hopkins [RM].

1 Quillen Model Categories

We recall the notion of a closed model category (Quillen [HA] and Dwyer-Kan-Stover [E^2]) and we describe the Reedy model category of simplicial objects.

(1.1) Definition. A closed model category structure on a category $\mathbf{C}$ consists of three classes of maps in $\mathbf{C}$, called fibrations, cofibrations and weak equivalences, satisfying axioms CM1-CM5 below. Note that axiom CM1 implies that $\mathbf{C}$ has an initial object $*$ as well as a terminal object $\circledast$. An object $U \in \mathbf{C}$ is called fibrant if the map $U \to \circledast \in \mathbf{C}$ is a fibration and cofibrant if the map $* \to U \in \mathbf{C}$ is a cofibration. A map is called a trivial (co-) fibration if it is a weak equivalence as well as a (co-) fibration. A map $i : A \to B \in \mathbf{C}$ is said to have the left lifting property with respect to a map $p : X \to Y \in \mathbf{C}$ (and the map p is said to have the right lifting property with respect to the map i) if in every commutative square in $\mathbf{C}$ of the shape

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

there exists a diagonal arrow $B \to X$ such that the two resulting triangles are also commutative.

CM1 The category $\mathbf{C}$ has finite limits and colimits.
CM2 If f and g are maps such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.
CM3 If f is a retract of g and g is a fibration, a cofibration or a weak equivalence, then so is f.
CM4 (i) Every cofibration has the left lifting property with respect to every trivial fibration.
(ii) Every fibration has the right lifting property with respect to every trivial cofibration.
CM5 Every map f can be factored
(i) $f = qj$, where j is a cofibration and q is a trivial fibration, and
(ii) $f = qj$, where q is a fibration and j is a trivial cofibration.

Of course if $\mathbf{C}$ is a closed model category, then so its opposite $\mathbf{C}^{\mathrm{op}}$ with as weak equivalences, cofibrations and fibrations the opposites of the weak equivalences, the fibrations and the cofibrations (respectively) of $\mathbf{C}$ itself.

If $\mathbf{C}$ is a category with finite limits and colimits one can construct the following "latching objects" and "matching objects".

(1.2) Definition. Let $\mathbf{L}_n (n \geq 0)$ be the category which has as objects the maps $[j] \to [n] \in \Delta^{\mathrm{op}}$ with $j < n$ and which has as maps the obvious commutative triangles. Given an object $X \in \Delta\mathbf{C}$, let, by a slight abuse of notation, $X \mid \mathbf{L}_n : \mathbf{L}_n \to \mathbf{C}$ denote the composition of the forgetful functor $\mathbf{L}_n \to \Delta^{\mathrm{op}}$ with the functor $X : \Delta^{\mathrm{op}} \to \mathbf{C}$. The n-th <u>latching object</u> $L_n X$ of X then is defined by $L_n X = \varinjlim (X \mid \mathbf{L}_n)$. In particular $L_0 X$ is the initial object of $\mathbf{C}$. Note that there is an obvious natural map $L_n X \to X_n$ $(n \geq 0)$.

(1.3) Definition. In a similar way, let $\mathbf{M}_n$ ($n \geq 0$) be the category which has as objects the maps $[n] \to [j] \in \Delta^{\mathrm{op}}$ with $j < n$ and which has as maps the obvious commutative triangles. Given an object $X \in \Delta\mathbf{C}$, let, again by a slight abuse of notation, $X \mid \mathbf{M}_n : \mathbf{M}_n \to \mathbf{C}$ be the composition of the forgetful functor $\mathbf{M}_n \to \Delta^{\mathrm{op}}$ with the functor $X : \Delta^{\mathrm{op}} \to \mathbf{C}$. The n-th matching object M_nX of X then is defined by $M_nX = \varprojlim(X \mid \mathbf{M}_n)$. In particular M_0X is the terminal object of $\mathbf{C}$ and M_1X is the product $M_1X = X_0 \times X_0$. There is an obvious natural map $X_n \to M_nX \in \mathbf{C}$.

(1.4) Theorem of Reedy [M]. *Let* $\mathbf{C}$ *be a closed model category. Then* $\Delta\mathbf{C}$ *admits a closed model category structure in which*

(i) *a map* $X \to Y \in \Delta\mathbf{C}$ *is a weak equivalence (called "Reedy weak equivalence") whenever, for every* $n \geq 0$*, the restriction* $X_n \to Y_n \in \mathbf{C}$ *is a weak equivalence,*
(ii) *a map* $X \to Y \in \Delta\mathbf{C}$ *is a (trivial) cofibration (called "(trivial) Reedy cofibration") whenever, for every* $n \geq 0$*, the induced map* $(X_n \amalg_{L_nX} L_nY) \to Y_n \in \mathbf{C}$ *is a (trivial) cofibration, and*
(iii) *a map* $X \to Y \in \Delta\mathbf{C}$ *is a (trivial) fibration (called "(trivial) Reedy fibration") whenever, for every* $n \geq 0$*, the induced map* $X_n \to (Y_n \Pi_{M_nY} M_nX) \in \mathbf{C}$ *is a (trivial) fibration.*

We shall use the Reedy model category structure for $\Delta\mathbf{C}$ in § 3 below.

2 Spiral Model Categories

Let $\mathbf{C}$ be a category with initial object $*$ and coproducts $X \vee Y$ for $X, Y \in \mathbf{C}$. A natural cylinder I on $\mathbf{C}$ is a functor $I : \mathbf{C} \to \mathbf{C}$ together with a diagram

$$X \vee X \xrightarrow{(i_0,i_1)} IX \xrightarrow{p} X \tag{2.1}$$

which is natural in X and satisfies $p(i_0, i_1) = (1,1)$. We assume that $I(*) = *$ and that I commutes with coproducts; i.e. $I(X \vee Y) = I(X) \vee I(Y)$.

A based object in $\mathbf{C}$ is an object A in $\mathbf{C}$ together with a map $0 : A \to *$ termed the zero map. (If $\mathbf{C}$ has a zero object $*$, i.e. if the initial object is also a final object, then each object in $\mathbf{C}$ is based. In general, however, we do not assume that a zero object exists in $\mathbf{C}$.) Given a based object X we define the suspension ΣX and the cone CX by the push out diagrams in $\mathbf{C}$

$$\begin{tikzcd} IX \arrow[r] & CX \arrow[r, "q"] & \Sigma X \\ X \vee X \arrow[u] \arrow[r, "{(1,0)}"] & X \arrow[u, "i"] \arrow[r, "0"] & * \arrow[u] \end{tikzcd} \tag{2.2}$$

A map $f : X \to Y$ between based objects is a based map if $0f = 0$. Clearly a based map f induces maps $Cf : CX \to CY$ and $\Sigma f : \Sigma X \to \Sigma Y$ in $\mathbf{C}$ since I

is a natural cylinder. These maps Cf and Σf are again based since the zero map $IX \to X \to *$ of the cylinder given by (2.1) induces zero maps for CX and ΣX respectively.

For example, let $(A, 0, \mu, \nu)$ be a cogroup in $\mathbf{C}$ given by maps

$$0 : A \to *, \quad \mu : A \to A \vee A, \quad \nu : A \to A$$

for which the diagrams in (I.1.3) commute in $\mathbf{C}$. Then A is based and μ and ν are based maps and the assumptions on the natural cylinder I in (2.1) show that also the suspension ΣA and the cylinder CA,

$$(\Sigma A, 0, \Sigma\mu, \Sigma\nu) \quad \text{and} \quad (CA, 0, C\mu, C\nu), \tag{2.3}$$

are cogroups in $\mathbf{C}$.

(2.4) Definition. A spiral model category is a category $\mathbf{Q}$ together with full subcategories

$$\mathbf{T} \subset \mathbf{C} \subset \mathbf{Q}$$

having the following properties:

(i) $\mathbf{Q}$ is a closed model category in which all objects are fibrant; see (1.1).
(ii) $\mathbf{C}$ is an I-category with $* \in \mathbf{C}$ and cofibrations (resp. homotopy equivalences) in $\mathbf{C}$ are cofibrations (resp. weak equivalences) in $\mathbf{Q}$. Moreover arbitrary (not just finite) coproducts exist in $\mathbf{C}$ and the cylinder $I : \mathbf{C} \to \mathbf{C}$ and the inlcusion $\mathbf{C} \subset \mathbf{Q}$ commute with arbitrary coproducts; in particular $I* = *$. The interchange map $IIX \to IIX$ is an isomorphism. See (III.7.1).
(iii) $\mathbf{T}$ is a small theory of cogroups which is closed under suspension and finite coproducts in $\mathbf{C}$; that is $X, Y \in \mathbf{T}$ implies $\Sigma X, X \vee Y \in \mathbf{T}$. Compare (2.3) and (I.1.9). In addition for $A \in \mathbf{T}$ the functor $\mathbf{C}(A, -) : \mathbf{C} \to \mathbf{Set}$ commutes with filtered inductive limits; see (3.1) (2) below.

(2.5) Example. Let $\mathbf{Q} = \Delta\mathbf{Gr}$ be the category of simplicial groups and let $\mathbf{C} = (\Delta\mathbf{gr})_{\text{free}}$ be the category of free simplicial groups with the cylinder $IX = X \otimes \Delta[1]$ in (C.1.18). Let $\mathbf{T}$ be the full subcategory of $\mathbf{C}$ consisting of finite coproducts of spherical objects $\mathbb{Z} \wedge S[n]$, $n \geq 0$. Then this is a spiral model category in the sense of (2.4). More general examples are discussed in § 5 below.

We now assume that a spiral model category as in (2.4) is given. Let $X \in \mathbf{C}$ and $Y \in \mathbf{Q}$. An I-homotopy denoted by $\simeq_I$ is a map

$$H : IX \to Y \quad \text{in } \mathbf{Q} \quad \text{with } Hi_0 = f \text{ and } Hi_1 = g. \tag{2.6}$$

Then (2.4) implies that $\simeq_I$ is an equivalence relation so that the set

$$\tilde{\mathbf{Q}}(X, Y) = \mathbf{Q}(X, Y)/\simeq_I \tag{1}$$

of I-homotopy classes is defined. Here we use the fact that by (2.4) the object X is cofibrant and Y is fibrant. This set is also the set of morphisms $X \to Y$ in the homotopy category

$$\tilde{\mathbf{Q}} = \mathrm{Ho}(\mathbf{Q}) \tag{2}$$

obtained by localizing weak equivalences in $\mathbf{Q}$. Let $\tilde{\mathbf{C}} = \mathbf{C}/\simeq_I$ and $\tilde{\mathbf{T}} = \mathbf{T}/\simeq_I$ be the homotopy categories. Then we have the commutative diagram

$$\begin{array}{ccccc} \mathbf{T} & \subset & \mathbf{C} & \subset & \mathbf{Q} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \tilde{\mathbf{T}} & \subset & \tilde{\mathbf{C}} & \subset & \tilde{\mathbf{Q}} \end{array} \tag{3}$$

The inclusions are full inclusions. The functor p is the identity on objects and p carries the morphism $f : X \to Y$ to the homotopy class $\tilde{f}$ represented by f. The functor p induces the functor

$$p : \Delta\mathbf{Q} \to \Delta\tilde{\mathbf{Q}} \tag{4}$$

between categories of simplicial objects. This functor carries the simplicial object X in $\mathbf{Q}$ to the simplicial object $\tilde{X} = pX$ in $\tilde{\mathbf{Q}}$.

The functor $p : \mathbf{Q} \to \tilde{\mathbf{Q}}$ carries coproducts in $\mathbf{C}$ to coproducts and therefore p carries a cogroup in $\mathbf{C}$ to a cogroup in $\tilde{\mathbf{C}}$. Hence $\tilde{\mathbf{T}}$ is also a theory of cogroups. Moreover for a cogroup B in $\mathbf{C}$ and $X \in \mathbf{Q}$ the morphism sets $\mathbf{Q}(B, X)$ and $\tilde{\mathbf{Q}}(B, X)$ are groups and the quotient map

$$p : \mathbf{Q}(B, X) \to \tilde{\mathbf{Q}}(B, X) \quad \text{given by (1)} \tag{5}$$

is a surjective group homomorphism which is natural in X. If X is a simplicial object in $\Delta\mathbf{Q}$ with $pX = \tilde{X} \in \Delta\tilde{\mathbf{Q}}$ then p yields the surjection between simplicial groups

$$p : \mathbf{Q}(B, X) \to \tilde{\mathbf{Q}}(B, \tilde{X}) \tag{6}$$

We shall use the following notation.

(2.7) Definition. A simplicial group G is contractible if all homotopy groups of G vanish (i.e. $\pi_* G = 0$) or equivalently if the realization $|G|$ of G is a contractible space. We say that a simplicial object X in $\Delta\mathbf{Q}$ is spiral if for all $A \in \mathbf{T}$ the simplicial group $\mathbf{Q}(CA, X)$ is contractible. In § 3 we show that each Reedy fibrant object in $\Delta\mathbf{Q}$ is spiral.

(2.8) Theorem. *(Spiral exact sequence) Let $X \in \Delta\mathbf{Q}$ be spiral and $A \in \mathbf{T}$. Then one has the long exact sequence of homotopy groups with $n \in \mathbb{Z}$:*

$$\cdots \to \pi_{n-1}\mathbf{Q}(\Sigma A, X) \to \pi_n \mathbf{Q}(A, X) \xrightarrow{p} \pi_n \tilde{\mathbf{Q}}(A, \tilde{X}) \to \pi_{n-2}\mathbf{Q}(\Sigma A, X) \to \dots$$

Here the homotopy groups π_n are trivial if $n < 0$ so that for $n = 0$ we get the isomorphism

$$\pi_0\mathbf{Q}(A,X) \cong \pi_0\tilde{\mathbf{Q}}(A,\tilde{X}).$$

The sequence is natural in X.

Proof. One has for each cogroup A in $\mathbf{T}$ the exact sequence of groups

$$0 \longrightarrow \mathbf{Q}(\Sigma A,X) \xrightarrow{q^*} \mathbf{Q}(CA,X) \xrightarrow{i^*} \mathbf{Q}(A,X) \xrightarrow{p} \tilde{\mathbf{Q}}(A,X) \longrightarrow 0$$

which is natural in $X \in \mathbf{Q}$; here i and q are defined in (2.2) and p is the quotient map. This easy fact is the reason for the spiral exact sequence. Let

$$\mathbf{Q}(A,X)_0 = \{f \in \mathbf{Q}(A,X);\ f \simeq_I 0\}$$

Then we have $\mathbf{Q}(A,X)_0 = \operatorname{kernel} p = \operatorname{image} i^*$ so that we get short exact sequences of groups

$$\begin{gathered}0 \longrightarrow \mathbf{Q}(\Sigma A,X) \xrightarrow{q^*} \mathbf{Q}(CA,X) \longrightarrow \mathbf{Q}(A,X)_0 \longrightarrow 0\\ 0 \longrightarrow \mathbf{Q}(A,X)_0 \longrightarrow \mathbf{Q}(A,X) \xrightarrow{p} \tilde{\mathbf{Q}}(A,X) \longrightarrow 0\end{gathered}$$

which are natural in X. Hence if X is in $\Delta\mathbf{Q}$ we obtain accordingly short exact sequences of simplicial groups. This implies (2.8). q.e.d.

The proof shows that the existence of a spiral exact sequence for X implies that X is spiral.

(2.9) Definition. A map $f : X \to Y$ in $\Delta\mathbf{Q}$ is a vertical equivalence if f induces an isomorphism

$$f_* : \pi_n\mathbf{Q}(A,X) \xrightarrow{\cong} \pi_n\mathbf{Q}(A,Y)$$

for $n \geq 0$ and all A in $\mathbf{T}$. The map f is a spiral equivalence if f induces an isomorphism

$$\tilde{f}_* : \pi_n\tilde{\mathbf{Q}}(A,\tilde{X}) \xrightarrow{\cong} \pi_n\tilde{\mathbf{Q}}(A,\tilde{Y})$$

for $n \geq 0$ and all A in $\mathbf{T}$. Finally f is a horizontal equvalence if

$$f_n : X_n \longrightarrow Y_n$$

is a homotopy equivalence in $\mathbf{Q}$ (i.e. an isomorphism in $\tilde{\mathbf{Q}}$) for all $n \geq 0$.

It is clear that each Reedy weak equivalence (1.4) is a horizontal equivalence and that each horizontal equivalence is a spiral equivalence. Moreover we get from (2.8):

(2.10) Corollary. *Let $f : X \to Y$ be a map in $\Delta\mathbf{Q}$ between spiral objects X, Y. Then f is a vertical equivalence if and only if f is a spiral equivalence.*

Proof. We start the induction for $n = 0$ by the isomorphism for π_0 in (2.8). Since $\Sigma A \in \mathbf{T}$ we get the case $n = 1$ by the five lemma, and so on. q.e.d.

Since $\mathbf{T}$ and $\tilde{\mathbf{T}}$ are both theories of cogroups we know by (C.1.18) that

$$(\Delta\mathbf{T})_{\text{free}} \quad \text{and} \quad (\Delta\tilde{\mathbf{T}})_{\text{free}} \tag{2.11}$$

are both I-categories with the cylinder $- \otimes \Delta[1]$ and cofibrations given by free inclusion. For $X \in \Delta\mathbf{Q}$ we call $X \otimes \Delta[1]$ the vertical cylinder of X and accordingly a simplicial map $X \otimes \Delta[1] \to Y$ is termed a vertical homotopy in $\Delta\mathbf{Q}$. Similarly a map $\tilde{X} \otimes \Delta[1] \to \tilde{Y}$ is a vertical homotopy in $\Delta\tilde{\mathbf{Q}}$. We denote vertical homotopies by $\simeq_v$. If $X \in (\Delta\mathbf{T})_{\text{free}} \subset \Delta\mathbf{Q}$ then (C.1.18) shows that $\simeq_v$ is an equivalence relation so that the sets of vertical homotopy classes

$$\begin{cases} [X, Y]_v = (\Delta\mathbf{Q})(X, Y)/\simeq_v \\ [\tilde{X}, \tilde{Y}]_v = (\Delta\tilde{\mathbf{Q}})(\tilde{X}, \tilde{Y})/\simeq_v \end{cases} \tag{2.12}$$

are defined for $X \in (\Delta\mathbf{T})_{\text{free}}$ and $Y \in \Delta\mathbf{Q}$. Here we use the fact that $\tilde{X}$ is free if X is free. Moreover one readily checks the following lemma.

(2.13) Lemma. *The functor p in (2.6) induces a well defined functor*

$$p : (\Delta\mathbf{T})_{\text{free}} \longrightarrow (\Delta\tilde{\mathbf{T}})_{\text{free}}$$

which carries X to $\tilde{X}$ and which commutes with $\otimes\Delta[1]$; that is

$$(X \otimes \Delta[1])^{\sim} = \tilde{X} \otimes \Delta[1].$$

Moreover p carries free inclusions in $(\Delta\mathbf{T})_{\text{free}}$ to free inclusions.

The lemma shows that $p : \mathbf{Q} \to \tilde{\mathbf{Q}}$ induces for $X \in (\Delta\mathbf{T})_{\text{free}}$ and $Y \in \Delta\mathbf{Q}$ a well defined map

$$p : [X, Y]_v \longrightarrow [\tilde{X}, \tilde{Y}]_v \tag{2.14}$$

between sets of vertical homotopy classes. Moreover if A is a cogroup in $\mathbf{T}$ then A is based and one gets $A \wedge S[n] \in (\Delta\mathbf{T})_{\text{free}}$. Now one readily checks the natural isomorphisms:

$$\begin{cases} [A \wedge S[n], Y]_v = \pi_n \mathbf{Q}(A, Y) \\ [\tilde{A} \wedge S[n], \tilde{Y}]_v = \pi_n \tilde{\mathbf{Q}}(A, \tilde{Y}) \end{cases} \tag{2.15}$$

The right hand side denotes the homotopy groups which appear in the spiral exact sequence (2.8). This leads to the following result.

(2.15) Lemma. *For X in $(\Delta\mathbf{T})_{\text{free}}$ the projection $q : X \otimes \Delta[1] \to X$ is a spiral equivalence.*

Proof. By (2.15) we have to show that for all $A \in \mathbf{T}$, $n \geq 0$, and $Y = X \otimes \Delta[1]$ the map

$$\tilde{q}_* : [\tilde{A} \wedge S[n], \tilde{Y}]_v \longrightarrow [\tilde{A} \wedge S[n], \tilde{X}]_v$$

is an isomorphism. But $\tilde{Y} \to \tilde{X}$ coincides by (2.13) with the projection $\tilde{X} \otimes \Delta[1] \to \tilde{X}$ which is a vertical homotopy equivalence by (C.1.18). q.e.d.

Next we consider the category $\mathbf{Pair}(\mathbf{Q}) = \mathbf{Q}_2$ of pairs in $\mathbf{Q}$. Objects are morphisms $f : Y \to X$ also denoted by (X, Y) and morphisms are pair maps; see (III.1.4). The cylinder of a pair (X, Y) in $\mathbf{Pair}(\mathbf{C}) = \mathbf{C}_2$ is defined by $I(f) : I(Y) \to I(X)$, that is

$$I(X, Y) = (IX, IY) \tag{2.17}$$

If A is a cogroup in $\mathbf{C}$ then (CA, A) is a cogroup in $\mathbf{C}_2$ and the cone of (CA, A) in $\mathbf{C}_2$ is

$$C(CA, A) = (CCA, CA) \tag{1}$$

Moreover since the interchange map (2.4) (ii) is an isomorphism we get the suspension of (CA, A) by

$$\Sigma(CA, A) = (\Sigma CA, \Sigma A) = (C\Sigma A, \Sigma A) \tag{2}$$

This leads to the following result.

(2.18) Proposition. *Consider a pair (X, Y) in $\Delta\mathbf{Q}_2$ for which X and Y are spiral. Then the simplicial group*

$$\mathbf{Q}_2(C(CA, A), (X, Y))$$

is contractible for all $A \in \mathbf{T}$.

Proof. Since $j : CA \rightarrowtail CCA$ is a cofibration and an I-homotopy equivalence in the I-category $\mathbf{C}$ we know that there exists a retraction $r : CCA \to CA$ of j. For a morphism $i : Y \to X$ in $\Delta\mathbf{Q}$ we obtain the following pull back diagram of simplicial groups

$$\begin{array}{ccc}
\mathbf{Q}_2((CCA, CA), (X, Y)) & \longrightarrow & \mathbf{Q}(CA, Y) \\
\downarrow & & \downarrow{\scriptstyle i_*} \\
\mathbf{Q}(CCA, X) & \xrightarrow{j^*} & \mathbf{Q}(CA, X)
\end{array}$$

Here j^* is surjective since the retraction r exists. Hence the diagram is actually a homotopy pull back. The fiber of j^* is $\mathbf{Q}(\Sigma CA, X)$. This yields the result by (2.17) (2) since $\mathbf{Q}(CA, Y), \mathbf{Q}(CA, X)$ and $\mathbf{Q}(C\Sigma A, X)$ are contractible. q.e.d.

(2.19) Definition. We say that a map $f : G \to H$ between simplicial groups is a 0-fibration if the sequence

$$G \xrightarrow{f} H \xrightarrow{q} \pi_0 H \longrightarrow 0 \quad \text{in } \Delta\mathbf{Gr}$$

is exact. Here q is the canonical quotient map to the constant simplicial group given by $\pi_0 H$. Moreover we say that a map $f : Y \to X$ in $\Delta\mathbf{Q}$ is spiral if Y and X are spiral and if f induces 0-fibrations

$$\begin{aligned} f_* &: \mathbf{Q}(A, Y) \to \mathbf{Q}(A, X) \quad \text{and} \\ \tilde{f}_* &: \tilde{\mathbf{Q}}(A, \tilde{Y}) \to \tilde{\mathbf{Q}}(A, \tilde{X}) \end{aligned}$$

for all A in $\mathbf{T}$.

(2.20) Theorem. *Let $f : Y \to X$ be a spiral map in $\Delta\mathbf{Q}$. Then one gets for $A \in \mathbf{T}$ and $n \geq 1$ the following exact sequence of abelian relative homotopy groups with $\tilde{X} = pX, \tilde{Y} = pY \in \Delta\tilde{\mathbf{Q}}$.*

$$\begin{aligned} \ldots \longrightarrow \pi_n(\mathbf{Q}(\Sigma A, X), \mathbf{Q}(\Sigma A, Y)) \longrightarrow \pi_{n+1}(\mathbf{Q}(A, X), \mathbf{Q}(A, Y)) \longrightarrow \\ \longrightarrow \pi_{n+1}(\tilde{\mathbf{Q}}(A, \tilde{X}), \tilde{\mathbf{Q}}(A, \tilde{Y})) \longrightarrow \pi_{n-1}(\mathbf{Q}(\Sigma A, X), \mathbf{Q}(\Sigma A, Y)) \end{aligned}$$

Here the relative homotopy group $\pi_k(-,-)$ is trivial for $k \leq 0$. Moreover for $n = 0$ one has the following commutative diagram of groups in which the row and the columns are exact.

$$\begin{array}{ccccccccc}
\pi_2\tilde{\mathbf{Q}}(A,\tilde{Y}) & \longrightarrow & \pi_0\tilde{\mathbf{Q}}(\Sigma A,\tilde{Y}) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\pi_2\tilde{\mathbf{Q}}(A,\tilde{X}) & \longrightarrow & \pi_0\tilde{\mathbf{Q}}(\Sigma A,\tilde{X}) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\Lambda' & \longrightarrow & \Lambda & \longrightarrow & \pi_1(\mathbf{Q}(A,X),\mathbf{Q}(A,Y)) & \longrightarrow & \pi_1(\tilde{\mathbf{Q}}(A,\tilde{X}),\tilde{\mathbf{Q}}(A,\tilde{Y})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & \\
0 & & 0 & & & & & &
\end{array}$$

If X is trivial, i.e. $X = \text{const}(*)$, then the exact sequence coincides with the spiral exact sequence in (2.8). Accordingly we call (2.20) the spiral exact sequence for relative homotopy groups.

Proof of (2.20). We first consider the case $n = 0$. For $\pi_1 = \pi_1(\mathbf{Q}(A, X), \mathbf{Q}(A, Y))$ and $\tilde{\pi}_1 = \pi_1(\tilde{\mathbf{Q}}(A, \tilde{X}), \tilde{\mathbf{Q}}(A, \tilde{Y}))$ one has the following commutative diagram in which rows and columns are exact.

$$
\begin{array}{ccccccccc}
\pi_2(\tilde{\mathbf{Q}}(A,\tilde{Y}) & \longrightarrow & \pi_2\tilde{\mathbf{Q}}(A,\tilde{X}) & \longrightarrow & A' & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\pi_0\mathbf{Q}(\Sigma A,Y) & \longrightarrow & \pi_0\mathbf{Q}(\Sigma A,X) & \longrightarrow & A & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\pi_1\mathbf{Q}(A,Y) & \longrightarrow & \pi_1\mathbf{Q}(A,X) & \longrightarrow & \pi_1 & \longrightarrow & \pi_0\mathbf{Q}(A,Y) & \longrightarrow & \pi_0\mathbf{Q}(A,X) \\
\downarrow & & \downarrow & & \downarrow & & \| & & \| \\
\pi_1\tilde{\mathbf{Q}}(A,\tilde{Y}) & \longrightarrow & \pi_1\tilde{\mathbf{Q}}(A,\tilde{Y}) & \longrightarrow & \tilde{\pi}_1 & \longrightarrow & \pi_0\tilde{\mathbf{Q}}(A,\tilde{Y}) & \longrightarrow & \pi_0\tilde{\mathbf{Q}}(A,\tilde{X}) \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & 0 & & & &
\end{array}
$$

The columns are given by (2.8) and the rows are the exact sequences for relative homotopy groups. Since by (2.8) also

$$\pi_0\mathbf{Q}(\Sigma A, Z) = \pi_0\tilde{\mathbf{Q}}(\Sigma A, \tilde{Z})$$

with $Z = X, Y$ we get the diagram in (2.20). Next we consider the case $n \geq 1$. Since $\mathbf{C}$ is an I-category and $\mathbf{Q}$ is a cofibration category we can apply (III.7.4) and (III.1.4) to $\mathbf{C}_2$ and $\mathbf{Q}_2$ respectively with $\tilde{\mathbf{Q}}_2 = \mathrm{Ho}\,\mathbf{Q}_2$. Now the arguments in (2.8) applied to $\mathbf{Q}_2$ and (2.18) show that we obtain for $A \in \mathbf{T}$, $n \in \mathbb{Z}$, the exact sequence:

$$
\begin{aligned}
\ldots \to \pi_{n-1}\mathbf{Q}_2(\Sigma(CA,A),(X,Y)) &\to \pi_n\mathbf{Q}_2((CA,A),(X,Y)) \\
\to \pi_n\tilde{\mathbf{Q}}_2((CA,A),(X,Y)) &\to \pi_{n-2}\mathbf{Q}_2(\Sigma(CA,A),(X,Y)) \to \ldots
\end{aligned} \tag{1}
$$

For each pair (X_n, Y_n) in $\mathbf{Q}_2$ we have the long exact sequence of homotopy groups of the pair (see (III.2.4)) given by

$$\tilde{\mathbf{Q}}(\Sigma A, Y_n) \to \tilde{\mathbf{Q}}(\Sigma A, X_n) \to \tilde{\mathbf{Q}}_2((CA,A),(X_n,Y_n)) \to \tilde{\mathbf{Q}}(A, Y_n) \to \tilde{\mathbf{Q}}(A, X_n)$$

Since A is a cogroup this is an exact sequence of groups. Since the sequence is natural in X_n and Y_n this is also an exact sequence of simplicial groups. Since $Y \to X$ is spiral this exact sequence yields the following exact sequence of simplicial groups

$$0 \to \pi_0\tilde{\mathbf{Q}}(\Sigma A, X_n) \to \tilde{\mathbf{Q}}_2((CA,A),(X,Y)) \to \tilde{\mathbf{Q}}(A,\tilde{Y}) \xrightarrow{\tilde{f}_*} \tilde{\mathbf{Q}}(A,\tilde{X}) \to \pi_0\tilde{\mathbf{Q}}(A,\tilde{X}) \to 0$$

Here we use the assumption that $\tilde{f}_*$ in (2.19) is a 0-fibration. This implies for $n \geq 1$ that

$$\pi_n\tilde{\mathbf{Q}}_2((CA,A),(X,Y)) = \pi_n(\mathrm{kernel}\,\tilde{f}_*) = \pi_{n+1}(\tilde{\mathbf{Q}}(A,\tilde{X}), \tilde{\mathbf{Q}}(A,\tilde{Y})) \tag{2}$$

Moreover we have for the simplicial group $\mathbf{Q}_2((CA,A),(X,Y))$ the following pull back diagram of simplicial groups

$$\begin{array}{ccccccc}
\mathbf{Q}_2((CA,A),(X,Y)) & \longrightarrow & \mathbf{Q}(CA,X) & & & & \\
\downarrow & & \downarrow & & & & \\
\mathbf{Q}(A,Y) & \xrightarrow{f_*} & \mathbf{Q}(A,X) & \longrightarrow & \pi_0\mathbf{Q}(A,X) & \longrightarrow & 0
\end{array}$$

Here the row is an exact sequence of simplicial groups since f_* is a 0-fibration by (2.19). Since $\mathbf{Q}(CA,X)$ is contractible this yields for $n \geq 1$ the isomorphism

$$\pi_n\mathbf{Q}_2((CA,A),(X,Y)) = \pi_{n+1}(\mathbf{Q}(A,X),\mathbf{Q}(A,Y)) \tag{3}$$

Now (1), (2) and (3) yield the exact sequence in (2.20) for $n \geq 1$. q.e.d.

(2.21) Definition. We define an analogue of the pathlike construction of Dwyer-Kan-Stover [E[2]] 4.3. Here we need the assumption that there are arbitrary coproducts in $\mathbf{C}$. For A in $\mathbf{T}$ and $n \geq 1$ we define the object $P(A,n)$ in $\Delta\mathbf{C}$ by the push out diagram ($* = \mathrm{const}(*)$)

$$\begin{array}{ccc}
(CA)\otimes * \vee A\otimes\Delta[n] & \longrightarrow & P(A,n) \\
\uparrow & & \uparrow \\
A\otimes\partial\Delta[1] = A\otimes * \vee A\otimes * & \xrightarrow{(i_0,i_1)} & A\otimes\Delta[1]
\end{array}$$

where we use the basepoint of $\Delta[n]$. For X in $\Delta\mathbf{Q}$ let $P(X)$ be the coproduct in $\Delta\mathbf{C}$ of all objects $P(A,n)$ indexed by tuples (A,n,a,b,c) with

$$\begin{cases} A \in \mathbf{T} \\ n \geq 1 \\ a : CA \to X_0 \in \mathbf{Q} \\ b : A \to X_n \in \mathbf{Q} \\ c : A \to X_1 \in \mathbf{Q} \end{cases}$$

such that

$$\begin{cases} a \mid A = d_0c \quad \text{and} \\ d_0 \dots d_0 b = d_1 c. \end{cases}$$

Then one has a canonical simplicial map

$$\varepsilon : PX \to X \quad \text{in } \Delta\mathbf{Q}$$

given by a,b,c above.

(2.22) Proposition. *For Y and X in $\Delta\mathbf{Q}$ the map $Y \to Y \vee PX$ is a spiral equivalence and for any map $f : Y \to X$ in $\Delta\mathbf{Q}$ the map*

$$g = (f,\varepsilon) : Z = Y \vee PX \to X$$

induces 0-fibrations $\mathbf{Q}(A,g)$ and $\tilde{\mathbf{Q}}(A,\tilde{g})$; see (2.19).

Proof. Let $\bar{P}(A,n) = P(A,n)/(CA)\otimes *$ and let $\bar{P}X$ be the coproduct of all $\bar{P}(A,n)$ indexed by (A,n,a,b,c) in (2.21). Since the cylinder I commutes with arbitrary coproducts in $\mathbf{C}$ we see that $PX \to \bar{P}X$ is a horizontal equivalence. This implies that also $Y \vee PX \to Y \vee \bar{P}X$ is a horizontal equivalence since $Y_n \to Y_n \vee (PX)_n$ is a cofibration in $\mathbf{Q}$. Moreover $\bar{P}X$ is an object in $(\Delta\tilde{\mathbf{L}})_{\text{free}}$ where $\mathbf{L}$ is the full subcategory of $\mathbf{C}$ consisting of arbitrary coproducts of objects in $\mathbf{T}$ and $\tilde{\mathbf{L}} = \mathbf{L}/{\simeq_I}$ is a theory of cogroups. We can apply (C.1.18) for $(\Delta\tilde{\mathbf{L}})_{\text{free}}$. Now

$$\begin{array}{ccc} A\otimes\Delta[1]\cup_{i_1} A\otimes\Delta[n] & \longrightarrow & \bar{P}(A,n) \\ \simeq\uparrow i_0 & & \uparrow\simeq \\ A\otimes * & \longrightarrow & * \end{array}$$

is a push out diagram in $(\Delta\tilde{\mathbf{L}})_{\text{free}}$ and i_0 is a vertical homotopy equivalence in $((\Delta\tilde{\mathbf{L}})_{\text{free}}, \otimes\Delta[i])$; see (2.16). Hence also $* \to \bar{P}(A,n)$ is a vertical homotopy equivalence in $(\Delta\tilde{\mathbf{L}})_{\text{free}}$. This shows by (2.15) that $* \to \bar{P}X$ is a vertical homotopy equivalence and therefore also $Y \to Y \vee \bar{P}X$ is a vertical homotopy equivalence in $\Delta\tilde{\mathbf{Q}}$. This implies by the second equation in (2.15) that $Y \to Y \vee \bar{P}X$ is a spiral equivalence.

Finally it is easy to see that the maps $\mathbf{Q}(A,g)$ and $\tilde{\mathbf{Q}}(A,\tilde{g})$ are 0-fibrations since PX is defined appropriately. To see that $\mathbf{Q}(A,g)$ is a 0-fibration we use the fact that $\pi_0\mathbf{Q}(A,Z) = \pi_0\tilde{\mathbf{Q}}(A,\tilde{Z})$ in (2.8). q.e.d.

3 Spiral Homotopy Theory

Given a spiral model category

$$\mathbf{T} \subset \mathbf{C} \subset \mathbf{Q}$$

as in (2.4) we have the notion of spiral equivalence in $\Delta\mathbf{Q}$ by (2.9). In this section we study the homotopy theory obtained from spiral equivalences.

Let $\mathbf{L} \subset \mathbf{C}$ be the full subcategory of $\mathbf{C}$ consisting of arbitrary coproducts of objects in $\mathbf{T}$. Then $\mathbf{L}$ and also the quotient category $\tilde{\mathbf{L}} = \mathbf{L}/{\simeq_I}$ are theories of cogroups. We have the commutative diagram of functors

$$\begin{array}{ccccccc} \mathbf{T} & \subset & \mathbf{L} & \subset & \mathbf{C} & \subset & \mathbf{Q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow p \\ \tilde{\mathbf{T}} & \subset & \tilde{\mathbf{L}} & \subset & \tilde{\mathbf{C}} & \subset & \tilde{\mathbf{Q}} \end{array} \tag{3.1}$$

Compare (2.6) (3). The functor p carries a simplicial object X in $\Delta\mathbf{Q}$ to the simplicial object $\tilde{X}$ in $\Delta\tilde{\mathbf{Q}}$. For the theory $\mathbf{T}$ the category of models $\mathbf{model}(\mathbf{T})$ is defined. Objects are functors $\mathbf{T}^{\text{op}} \to \mathbf{Set}$ which carry finite coproducts in $\mathbf{T}$ to products in $\mathbf{Set}$. In particular $A \in \mathbf{T}$ yields the model $\mathbf{T}(-,A) \in \mathbf{model}(\mathbf{T})$. Arbitrary coproducts exist in $\mathbf{model}(\mathbf{T})$. A free model is a coproduct

$$F = \bigvee_{i \in J} \mathbf{T}(-, A_i)$$

where J is a set and $A_i \in \mathbf{T}$ for $i \in J$; see also (I.2.5). Let

$$\mathbf{free}(\mathbf{T}) \subset \mathbf{model}(\mathbf{T}) \tag{1}$$

be the full subcategory of free models. The condition on filtered inductive limits in (2.4) (iii) implies that one has canonical isomorphism of categories

$$\begin{cases} \mathbf{L} = \mathbf{free}(\mathbf{T}) \quad \text{and} \\ \tilde{\mathbf{L}} = \mathbf{free}(\tilde{\mathbf{T}}). \end{cases} \tag{2}$$

Here $\mathbf{L}$ and $\tilde{\mathbf{L}}$ are the categories in (3.1).

(3.2) Definition. Consider a sequence of maps

$$Y = X^{-1} \to X^0 \to \cdots \to X^{n-1} \to X^n \to \ldots$$

in $\Delta\mathbf{Q}$ where for $n \geq 0$ one has a push out diagram in $\Delta\mathbf{Q}$

$$\begin{array}{ccc} \bar{A}_n \otimes \Delta[n] & \longrightarrow & X^n \\ \uparrow & & \uparrow \\ \bar{A}_n \otimes \partial\Delta[n] & \longrightarrow & X^{n-1} \end{array} \tag{3.3}$$

Here $\bar{A}_n$ is an object in $\mathbf{C}$ which is homotopy equivalent in $\mathbf{C}$ to an object A_n in L; see (3.1). Then we call the induced map $Y \to \lim\{X^n\}$ a <u>spiral inclusion</u>. We say that a map f in $\Delta\mathbf{Q}$ is a <u>spiral cofibration</u> if f is a finite composite of spiral inclusions and trivial Reedy cofibrations.

Let $(\Delta\mathbf{Q})_s$ be the full subcategory of all objects X in $\Delta\mathbf{Q}$ for which $* \to X$ is a spiral cofibration. Then we obtain for $X \in (\Delta\mathbf{Q})_s$ an object $\tilde{X} \in (\Delta\tilde{\mathbf{L}})_{\text{free}}$ with $pX \cong \tilde{X}$. This yields the canonical functor

$$p_L : (\Delta\mathbf{Q})_s \to (\Delta\tilde{\mathbf{L}})_{\text{free}} \tag{3.4}$$

with $p_L(X) = \tilde{X}$. This functor carries spiral cofibrations to free inclusions; see (C.1.11). A map $f : X \to Y$ in $(\Delta\mathbf{Q})_s$ is a spiral equivalence if and only if $p_L(f)$ is an isomorphism in the homotopy category $(\Delta\tilde{\mathbf{L}})_{\text{free}}/\simeq_v$.

We call $(\Delta\mathbf{Q})_s$ together with spiral equivalences and spiral cofibrations a <u>spiral homotopy theory</u> since one has the following fundamental result.

(3.5) Theorem. *Let $\mathbf{T} \subset \mathbf{C} \subset \mathbf{Q}$ be a spiral model category as in (2.4). Then the category $(\Delta\mathbf{Q})_s$ with spiral equivalences and spiral cofibrations is a cofibration category satisfying the axioms in (III.1.1). Moreover all fibrant models in this cofibration category (see (III.1.1)) are spiral objects in $\Delta\mathbf{Q}$; see (2.7).*

Proof. The composition axiom (C1) and the push out axiom (C2) are satisfied in $(\Delta\mathbf{Q})_s$ since these axioms hold in $(\Delta\tilde{\mathbf{L}})_{\text{free}}$ by (C.1.18) and (III.7.4). Here we use (3.4). Next we obtain the factorization axiom (C3) by (3.6) below. We check the axiom on fibrant models (C4) as follows. We have by (1.4) for each object X in $(\Delta\mathbf{Q})_s$ a factorization

$$X \to RX \to \circledast$$

where $X \to RX$ is a trivial Reedy cofibration (and hence a spiral cofibration) and where $RX \to \circledast$ is a Reedy fibration. Here $\circledast$ is the final object in $\Delta\mathbf{Q}$. Hence RX is Reedy fibrant and we show in (3.7) that all Reedy fibrant objects are fibrant models in the sense of (III.1.1). This proves (C4). Moreover we show in (3.9) that Reedy fibrant objects in $\Delta\mathbf{Q}$ are spiral. q.e.d.

(3.6) Theorem of Stover. *Let* $\mathbf{T} \subset \mathbf{C} \subset \mathbf{Q}$ *be a spiral model category. Then any map* $f : Y \to X$ *in* $\Delta\mathbf{Q}$ *admits a factorization*

$$f : Y \to M_X \to X \quad \textit{in } \Delta\mathbf{Q}$$

where $Y \to M_X$ *is a spiral inclusion and* $M_X \to X$ *is a spiral equivalence.*

Proof. This is the analogue of the key construction in Stover [VK]; compare Dwyer-Kan-Stover [E2] 4.5. Given a map $R \to S$ in $\mathbf{Q}$ let $W(S) = R \vee V(S) \in \Delta\mathbf{Q}$ where $V(S)$ is obtained by taking a wedge of objects $A = A_\alpha$ for every $A \in \mathbf{T}$ and map $\alpha : A \to S$ in $\mathbf{Q}$, and attaching a cone $CA = (CA)_\beta$ for every map $\beta : CA \to S$ in $\mathbf{Q}$. Here $(CA)_\beta$ is attached to A_α with $\alpha = \beta \mid A$. As $W(S)$ comes with an obvious map $R \to W(S)$ in $\mathbf{Q}$, one can repeat this construction and obtain an object $W_\bullet(S) \in \Delta\mathbf{Q}$ with $W_n(S) = W^{n+1}(S)$ for all $n \geq 0$ and a factorization of the map $R \to S$ into a spiral cofibration $R \to W_\bullet(S)$ followed by a spiral equivalence $W_\bullet(S) \to S$. Here we use the arguments in Stover [VK]. Now given a map $Y \to X$ in $\Delta\mathbf{Q}$ we obtain the factorization

$$Y \to \operatorname{diag} W_\bullet(X) \to X \quad \text{in } \Delta\mathbf{Q}$$

with $M_X = \operatorname{diag} W_\bullet(X)$ being the diagonal of the bisimplicial object $W_\bullet(X)$. This factorization has the property in (3.6); see Dwyer-Kan-Stover [E2] 4.6. q.e.d.

(3.7) Theorem. *Let* X *be a Reedy fibrant object in* $(\Delta\mathbf{Q})_s$. *Then any spiral cofibration* $f : Y \to X$ *which is a spiral equivalence admits a retraction.*

Proof. We have the following factorization of f:

$$f : Y \xrightarrow{i} Y \vee PX \xrightarrow{j} R \xrightarrow{q} X$$

Here i is the inclusion in (2.22) and $qj = (f, \varepsilon)$ is obtained by (1.4). That is j is a trivial Reedy cofibration and q is a Reedy fibration. By (2.22) we see that

$$q_* : \mathbf{Q}(A, R) \to \mathbf{Q}(A, X)$$

is a fibration of simplicial sets for all $A \in \mathbf{T}$. Since i and j and f are spiral equivalences also q is a spiral equivalence. By lemma (3.8) we get the lift α in the square

$$\begin{tikzcd} Y \arrow[r, "ij"] \arrow[d, "f"'] & R \arrow[d, "q"] \\ X \arrow[ur] \arrow[r, equal] & X \end{tikzcd}$$

Moreover by (1.4) we get the lift β in the square

$$\begin{tikzcd} Y \vee PX \arrow[r, "{(1,0)}"] \arrow[d, "j"'] & Y \arrow[d] \\ R \arrow[ur] \arrow[r] & \circledast \end{tikzcd}$$

Here $0 : PX \to * \to Y$ is the zero map; see (2.21). Hence $\beta\alpha$ is a retraction of f. q.e.d.

(3.8) Lemma. *Let L be Reedy fibrant and let $q : R \to L$ in $\Delta\mathbf{Q}$ be a Reedy fibration and a spiral equivalence such that*

$$q_* : \mathbf{Q}(A, R) \to \mathbf{Q}(A, L)$$

is a fibration of simplicial sets for all $A \in \mathbf{T}$. Then each commutative square diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & R \arrow[d, "q"] \\ X \arrow[ur] \arrow[r] & L \end{tikzcd}$$

where f is a spiral cofibration admits a lift.

Proof. By (1.4) it suffices to consider the case where f is a spiral inclusion. Hence it suffices to consider the case when

$$f : \bar{A} \otimes \partial\Delta[n] \to \bar{A} \otimes \Delta[n]$$

is given by the inlcusion $\partial\Delta[n] \to \Delta[n]$ where $\bar{A}$ is homotopy equivalent in $\mathbf{C}$ to $A \in \mathbf{L}$. We consider a diagram

$$\begin{tikzcd} \bar{A} \otimes \partial\Delta[n] \arrow[r, "a"] \arrow[d] & R \arrow[d, "q"] \\ \bar{A} \otimes \Delta[n] \arrow[r, "b"] & L \end{tikzcd} \tag{1}$$

A lift α for (1) exists if and only if there is a lift in the following diagram in $\mathbf{Q}$

$$\begin{array}{ccccc} & & * & \longrightarrow & R_n \\ & & \big\downarrow & & \big\downarrow{\scriptstyle q_n} \\ A & \xrightarrow{j} & \bar{A} & \xrightarrow{(b,a)_*} & L_n \times_{M_nL} M_nR \end{array} \tag{2}$$

Here q_n is a fibration by (1.4) and j is a homotopy equivalence in $\mathbf{C}$. Since q_n has the homotopy lifting property with respect to the cylinder I by (2.4) we see that (2) has a lift if and only if

$$\begin{array}{ccc} * & \longrightarrow & R_n \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow{(b,a)_*j} & L_n \times_{M_nL} M_nR \end{array} \tag{3}$$

has a lift. This is equivalent to the existence of a lift in

$$\begin{array}{ccc} A \otimes \partial\Delta[n] & \xrightarrow{a'} & R \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ A \otimes \Delta[n] & \xrightarrow{b'} & L \end{array} \tag{4}$$

where $a' = a(j \otimes 1), b' = b(j \otimes 1)$. Now the existence of a lift in (4) is equivalent to the existence of a lift in

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{a''} & \mathbf{Q}(A, R) \\ \big\downarrow & & \big\downarrow{\scriptstyle q_*} \\ \Delta[n] & \xrightarrow{b''} & \mathbf{Q}(A, L) \end{array} \tag{5}$$

with a'' and b'' induced by a and b respectively. Since R and L are Reedy fibrant and hence spiral by (3.9) below we can apply the spiral exact sequence (2.8) which shows that q_* in (5) induces an isomorphism on homotopy groups. Here we use the assumption that q is a spiral equivalence. Hence q_* in (5) is actually a trivial fibration of simplicial groups and hence a lift in (5) exists since simplicial sets form a closed model category. q.e.d.

(3.9) Theorem. *Let X be a Reedy fibrant object in $\Delta\mathbf{Q}$. Then X is spiral.*

Proof. We have to show that $\mathbf{Q}(CA, X)$ is contractible for all $A \in \mathbf{T}$. In fact, $\pi_0\mathbf{Q}(CA, X) = 0$ is equivalent to the existence of a lift in

$$\begin{array}{ccc} * & \longrightarrow & X_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle (d_0,d_1)} \\ CA & \xrightarrow[(f,0)]{} & X_0 \times X_0 \end{array}$$

with $f \in \mathbf{Q}(CA, X_0)$. Here $* \to CA$ is a weak equivalence in $\mathbf{Q}$ by (2.4) and (d_0, d_1) is a fibration in $\mathbf{Q}$ since X is Reedy fibrant. Hence the lift exists since $\mathbf{Q}$ is a closed model category. Similarly we see that $\pi_n \mathbf{Q}(CA, X) = 0$. This is equivalent to the existence of a lift in

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X_n \arrow[d] \\ CA \arrow[r, "{(f,0,\ldots,0)}"'] & M_n X \end{tikzcd}$$

with $f : CA \to X_n$ representing an element in $\pi_n \mathbf{Q}(CA, X)$. q.e.d.

4 Spiral Homotopy Groups

Let $\mathbf{T} \subset \mathbf{C} \subset \mathbf{Q}$ be a spiral model category. Then theorem (3.5) shows that the localization

$$\mathrm{Ho}(\Delta\mathbf{Q})_s = (se)^{-1}(\Delta\mathbf{Q})_s \tag{4.1}$$

with respect to the class se of spiral equivalences exists. This is the spiral homotopy category. Recall that the completion of $\mathbf{T}$ with respect to arbitrary coproducts in $\mathbf{C}$ yields the theory $\mathbf{L}$ of cogroups with $\mathbf{T} \subset \mathbf{L} \subset \mathbf{C}$. By (3.4) we obtain the functor

$$p_L : \mathrm{Ho}(\Delta\mathbf{Q})_s \to (\Delta\tilde{\mathbf{L}})_{\mathrm{free}}/\simeq_v$$

which carries X to $\tilde{X}$. The definition of spiral equivalences shows that this functor reflects isomorphisms.

(4.2) Lemma. *Let X be an object in $(\Delta\mathbf{L})_{\mathrm{free}}$. Then $X \vee X \to X \otimes \Delta[1]$ is a spiral cofibration and the projection $X \otimes \Delta[1] \to X$ is a spiral equivalence. Hence $X \otimes \Delta[1]$ is a cylinder object in the cofibration category $(\Delta\mathbf{Q})_s$ in (3.5); see (III.1.7).*

For the proof we use the same argument as in (2.16). The lemma implies for X in $(\Delta\mathbf{L})_{\mathrm{free}}$ and for a fibrant object Y in $(\Delta\mathbf{Q})_s$ that

$$[X, Y]_s = [X, Y]_v \tag{4.3}$$

is the same as the set of vertical homotopy classes $X \to Y$ in $\Delta\mathbf{Q}$; see (2.12) and Baues [AH] II.3.13.

An object X in $\Delta\mathbf{Q}$ is based if $* \to X$ is a spiral cofibration and if a map $0 : X \to *$ is given. For example for each cogroup A in $\mathbf{T}$ or $\mathbf{L}$ the object $A = \mathrm{const}\,(A)$ is a based object in $\Delta\mathbf{Q}$. Given a based object X the spiral homotopy group ($n \geq 0$)

$$\pi_n^X(Y)_s, \quad \pi_{n+1}^X(Z, Y)_s \tag{4.4}$$

are defined in the spiral cofibration category $(\varDelta\mathbf{Q})_s$; see (III,§ 2). Here Y is an object in $(\varDelta\mathbf{Q})_s$ and $Y \to Z$ is an object in $\mathbf{Pair}\,(\varDelta\mathbf{Q})_s$. If X is in $(\varDelta\mathbf{L})_{\mathrm{free}}$ and if Y and Z are fibrant (compare Baues [AH] II.1.5) then we obtain by (4.2) the canonical bijections

$$\begin{cases} \pi_n^X(Y)_s = [X \wedge S[n], Y]_v \\ \pi_{n+1}^X(Z,Y)_s = [(X \wedge D[n+1], X \wedge S[n]), (Z,Y)]_v \end{cases} \tag{4.5}$$

Hence such homotopy groups are given by vertical homotopy classes of maps or pair maps. As a special case of (4.5) we get for a cogroup A in $\mathbf{T}$ the canonical isomorphisms of groups ($n \geq 0$)

$$\begin{cases} \pi_n^A(Y)_s = \pi_n\mathbf{Q}(A,Y) \\ \pi_{n+1}^A(Z,Y)_s = \pi_{n+1}(\mathbf{Q}(A,Z), \mathbf{Q}(A,Y)) \end{cases} \tag{4.6}$$

where the right hand side denotes homotopy groups of simplicial groups. The equation (4.6), (4.5) only hold if Y and Z are fibrant. Given objects $\bar{Y}$ and $(\bar{Z}, \bar{Y})$ in $(\varDelta\mathbf{Q})_s$ we can choose <u>fibrant replacements</u> which are spiral equivalences

$$\bar{Y} \to Y, \qquad (\bar{Z}, \bar{Y}) \to (Z,Y) \tag{4.7}$$

where Y and Z are fibrant. Then we have

$$\begin{cases} \pi_n^A(\bar{Y})_s \cong \pi_n^A(Y)_s = \pi_n\mathbf{Q}(A,Y) \\ \pi_n^A(\bar{Z},\bar{Y})_s \cong \pi_n^A(Z,Y)_s = \pi_n(\mathbf{Q}(A,Z), \mathbf{Q}(A,Y)) \end{cases}$$

Here the left hand isomorphism is induced by (4.7).

(4.7) Lemma. *Each pair $(\bar{Z}, \bar{Y})$ in $(\varDelta\mathbf{Q})_s$ admits a fibrant replacement $(\bar{Z}, \bar{Y}) \to (Z,Y)$ for which $Y \to Z$ is a spiral map; see (2.19).*

Proof. First we choose a fibrant replacement $\alpha : \bar{Z} \to Z$. Then we get the commutative diagram

$$\begin{array}{ccccc} \bar{Z} & \xrightarrow{\alpha} & Z & = & Z \\ {\scriptstyle \bar{f}}\uparrow & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle f} \\ \bar{Y} & \xrightarrow{i} & \bar{Y} \vee PZ & \xrightarrow{j} & Y \end{array}$$

Here $g = (\bar{\alpha}\bar{f}, \varepsilon)$ is defined as in (2.22) and i is a spiral equivalence by (2.22). Now we choose a fibrant object Y together with a spiral cofibration j which is a spiral equivalence, i.e. j is a fibrant model in the cofibration category $(\varDelta\mathbf{Q})_s$. Then we obtain f by Baues [AH] II.1.6. We claim that f is a spiral map. In fact Z and Y are fibrant and hence spiral by (3.5). Moreover $\mathbf{Q}(A,g)$ and $\tilde{\mathbf{Q}}(A,\tilde{g})$ are 0-fibrations by (2.22). This implies that also $\mathbf{Q}(A,f)$ and $\tilde{\mathbf{Q}}(A,\tilde{f})$ are 0-fibrations. q.e.d.

(4.9) Theorem (spiral exact sequence). *For any object Y in $(\Delta\mathbf{Q})_s$ and $n \in \mathbb{Z}$ one has the following exact sequence with $\tilde{Y} = p_L(Y) \in \Delta\tilde{\mathbf{L}}$ and $A \in \mathbf{T}$.*

$$\ldots \longrightarrow \pi_{n-1}^{\Sigma A}(Y)_s \longrightarrow \pi_n^A(Y)_s \longrightarrow \pi_n\tilde{\mathbf{Q}}(A, \tilde{Y}) \longrightarrow \pi_{n-2}^{\Sigma A}(Y)_s \longrightarrow \ldots$$

Here homotopy groups π_n are trivial for $n < 0$ so that for $n = 0$ we get the isomorphism

$$\pi_0^A(Y)_s = [A, Y]_s = \pi_0\tilde{\mathbf{Q}}(A, \tilde{Y})$$

The sequence is natural in Y.

Proof. We choose for Y a fibrant replacement $Y \to Y'$. Then we know by the definition of spiral equivalences that $Y \to Y'$ induces the isomorphism

$$\pi_n\tilde{\mathbf{Q}}(A, \tilde{Y}) \cong \pi_n\tilde{\mathbf{Q}}(A, \tilde{Y}')$$

Moreover we can use (4.7) and (1.8) since fibrant objects are spiral by (3.5). q.e.d.

Using similar arguments we get by (4.8) and (2.20) the next result.

(4.10) Theorem (spiral exact sequence for relative homotopy groups). *For any pair (X, Y) in $(\Delta\mathbf{Q})_s$ one has the following exact sequence of abelian groups with $n \geq 1$ and $A \in \mathbf{T}$.*

$$\cdots \to \pi_n^{\Sigma A}(X, Y)_s \to \pi_{n+1}^A(X, Y)_s \to \pi_{n+1}(\tilde{\mathbf{Q}}(A, \tilde{X}), \tilde{\mathbf{Q}}(A, \tilde{Y})) \to \pi_{n-1}^{\Sigma A}(X, Y)_s$$

Here relative homotopy groups π_k are trivial for $k \leq 0$. Moreover for $n = 0$ one has the following commutative diagram of groups in which the row and the columns are exact, see (2.20).

$$\begin{array}{ccccccccc}
\pi_2\tilde{\mathbf{Q}}(A, \tilde{Y}) & \longrightarrow & \pi_0\tilde{\mathbf{Q}}(\Sigma A, \tilde{Y}) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\pi_2\tilde{\mathbf{Q}}(A, \tilde{X}) & \longrightarrow & \pi_0\tilde{\mathbf{Q}}(\Sigma A, \tilde{X}) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\Lambda' & \longrightarrow & \Lambda & \longrightarrow & \pi_1^A(X, Y)_s & \longrightarrow & \pi_1(\tilde{\mathbf{Q}}(A, \tilde{X}), \tilde{\mathbf{Q}}(A, \tilde{Y})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & \\
0 & & 0 & & & & & &
\end{array}$$

Similarly as in (C.1.25) we define:

(4.11) Definition. We say that a map $f : Y \to X$ in $(\Delta\mathbf{Q})_s$ is $(m, \mathbf{T})$-connected with $m \geq 0$ if for all objects A in $\mathbf{T}$ the induced map

$$f_* : \pi_0^A(Y)_s \to \pi_0^A(X)_s$$

is surjective and the relative homotopy groups $\pi_r^A(X, Y)_s = 0$ are trivial for $r \leq m$.

(4.12) Lemma. *The map $f : Y \to X$ in $(\Delta\mathbf{Q})_s$ is $(m, \mathbf{T})$-connected if and only if the induced map $\tilde{f} : \tilde{Y} \to \tilde{X}$ in $(\Delta\tilde{\mathbf{L}})_{\text{free}}$ is $(m, \tilde{\mathbf{T}})$-connected in the sense of (C.1.25).*

Proof. For $m = 0$ this is clear by (4.9). Moreover for $m > 0$ we can use inductively theorem (4.10) where the relative homotopy groups $\pi_{n+1}(\tilde{\mathbf{Q}}(A, \tilde{X}), \tilde{\mathbf{Q}}(A, \tilde{Y}))$ coincide with the homotopy groups in (C.1.25) defined in $\Delta\tilde{\mathbf{L}}$. q.e.d.

The next result is crucial for the application of the general theory of this book to spiral homotopy theory. It is the spiral analogue of the Blakers-Massey theorem.

(4.13) Theorem. *Assume the theory $\tilde{\mathbf{L}}$ of cogroups in (3.1) satisfies the delicate Blakers-Massey property in (C.1.27) and consider a push out diagram in $(\Delta\mathbf{Q})_s$*

$$\begin{array}{ccc} K & \xrightarrow{\bar{\jmath}} & K \cup_L Y \\ \uparrow{\scriptstyle i} & & \uparrow \\ L & \xrightarrow{j} & Y \end{array}$$

where i and j are spiral cofibrations and i is $(m, \mathbf{T})$-connected and j is $(n, \mathbf{T})$-connected with $m, n \geq 0$. Then $(K \cup_L Y, Y)$ is $(m, \mathbf{T})$-connected and the induced map

$$\bar{\jmath}_* : \pi_r^A(K, L)_s \to \pi_r^A(K \cup_L Y, Y)_s$$

is surjective for $1 \leq r \leq n + m - 1$ and bijective for $1 \leq r \leq n + m - 2$ and $A \in \mathbf{L}$.

Proof. We use (C.1.26) for objects in $(\Delta\tilde{\mathbf{L}})_{\text{free}}$. Then we apply inductively (4.10). q.e.d.

(4.14) Corollary. *Assume $\tilde{\mathbf{L}}$ satisfies the delicate Blakers-Massey property. Then the spiral cofibration category $(\Delta\mathbf{Q})_s$ is a cofibration category under $\tilde{\mathbf{L}}$ in the sense of (IV.2.1) which has the Blakers-Massey property (IV.5.3). This implies that $(\Delta\mathbf{Q})_s$ is a homological cofibration category under $\mathbf{L}$ in the sense of (V.1.1). See (V.1.2).*

The corollary shows that we can apply all the theory of chapters I, ... , VII to spiral homotopy theories $(\Delta\mathbf{Q})_s$ provided $\tilde{\mathbf{L}}$ satisfies the delicate Blakers-Massey property.

5 Examples of Spiral Model Categories

Let $\mathbf{S}$ be a small theory of cogroups and let $\mathbf{M} = \mathbf{model}(\mathbf{S})$ be the category of models of $\mathbf{S}$; see (I. § 1). Moreover let $\mathbf{free}(\mathbf{S})$ be the category of free models as in (3.1) (1). Then we have full inclusions of categories

$$\mathbf{S} \subset \mathbf{free}(\mathbf{S}) \subset \mathbf{M} = \mathbf{model}(\mathbf{S}) \tag{5.1}$$

Here **free**(**S**) is again a theory of cogroups so that by (C.1.18) the I-category

$$\mathbf{C} = (\Delta\mathbf{free}(\mathbf{S}))_{\text{free}} \quad \text{with } I(X) = X \otimes \Delta[1] \tag{1}$$

is defined. Moreover the category of simplicial objects in **M**

$$\mathbf{Q} = \Delta\mathbf{M} \tag{2}$$

is a closed model category in which all objects are fibrant; see Quillen [HA] chapter II. § 4. A map $f : X \to Y$ in $\mathbf{Q}$ is a fibration (resp. weak equivalence) if for all $A \in \mathbf{S}$ the map $\mathbf{Q}(A, f)$ is a fibration (resp. weak equivalence) in the category of simplicial sets or, since A is a cogroup, in the category of simplicial groups. According to Quillen [HA] (chapter II page 4.11) a map f is a cofibration in $\mathbf{Q}$ if and only if f is a retract of a "free map". This shows that free inclusions in $\mathbf{C}$ are also cofibrations in $\mathbf{Q}$. Now we get for $k \geq 0$ <u>the spiral model category associated to</u> **S**

$$\mathbf{T}_k \subset \mathbf{C} \subset \mathbf{Q}. \tag{5.2}$$

Here $\mathbf{C}$ and $\mathbf{Q}$ are defined as above and $\mathbf{T}_k$ is the full subcategory of $\mathbf{C}$ consisting of finite coproducts of spherical objects $A \wedge S[n]$, $n \geq k$, $A \in \mathbf{T}$; see (C.1.8) (3). Now it is readily clear from the definition in (2.4) that $\mathbf{Q}_k = (\mathbf{T}_k, \mathbf{C}, \mathbf{Q})$ with $k \geq 0$ is spiral model category. Therefore the associated spiral cofibration category

$$(\Delta\mathbf{Q}_k)_s \subset \Delta(\Delta\mathbf{M}) \tag{5.3}$$

is well defined. This is a category of bisimplicial objects depending on $k \geq 0$. Let $\mathbf{L}_k$ be the completion of $\mathbf{T}_k$ with respect to arbitrary coproducts in $\mathbf{C}$. Then $\tilde{\mathbf{L}}_k$ satisfies the delicate Blakers-Massey property if and only if $\tilde{\mathbf{T}}_k$ does. Hence by (4.14) we get the following result.

(5.4) Theorem. *Let* **S** *be a theory of cogroups for which* $\tilde{\mathbf{T}}_k = \mathbf{T}_k/\simeq_I$ *defined in (5.2) satisfies the delicate Blakers-Massey property. Then* $(\Delta\mathbf{Q}_k)_s$ *in (5.3) is a homological cofibration under* $\tilde{\mathbf{L}}_k$ *as in (V.1.1) which has the Blakers-Massey property (IV.5.3).*

(5.5) Example. Let $\mathbf{S} = \mathbf{gr}^\sharp$ be the category of finitely generated free groups. Then $\tilde{\mathbf{T}}_k$ is the homotopy category of finite wedges of spheres S^n with $n \geq k+1$. For $k \geq 1$ this is a polynomial graded theory of cogroups and therefore (C.2.17) shows that $\tilde{\mathbf{T}}_k$ with $k \geq 1$ satisfies the delicate Blakers-Massey property. Hence we have the spiral model category

$$\mathbf{Q}_k = (\mathbf{T}_k \subset (\Delta\mathbf{gr})_{\text{free}} \subset \Delta\mathbf{Gr})$$

where **Gr** is the category of free groups and **Gr** is the category of groups. We get the associated spiral cofibration category $(\Delta\mathbf{Q}_k)_s$. For $k \geq 1$ this is a homological cofibration category under $\tilde{\mathbf{L}}_k$ satisfying the Blakers-Massey property (IV.5.3).

(5.6) Example. Let $\mathbf{Top}^*$ be the category of pointed spaces. The E^2-homotopy theory of Dwyer-Kan-Stover $[E^2]$ yields a closed model category structure for $\Delta\mathbf{Top}^*$ with E^2-weak equivalences, E^2-cofibrations and E^2-fibrations. The associated E^2-homotopy category $\mathrm{Ho}(\Delta\mathbf{Top}^*)$ is actually equivalent to $\mathrm{Ho}(\Delta\mathbf{Q}_0)_s$ with $\mathbf{Q} = \Delta\mathbf{Gr}$ as defined in (5.5). It is convenient to replace $\mathbf{Top}^*$ by $\Delta\mathbf{Gr}$ via the Kan equivalence since there are cogroups in $\Delta\mathbf{Gr}$ which correspond to spheres in $\mathbf{Top}^*$. The spheres, however, are cogroups in $\mathbf{Top}^*/\simeq \mathrm{rel}\,*$ but they are not cogroups in $\mathbf{Top}^*$. Due to the existence of such cogroups in $\Delta\mathbf{Gr}$ we can apply the theory of a spiral model category described above. This is a lot easier than the E^2-model category of simplicial pointed spaces of Dwyer-Kan-Stover $[E^2]$. A similar remark also holds for the E^2-homotopy theory of Goerss-Hopkins [RM]. It seems that in many cases when one has an E^2-homotopy theory $\mathbf{Q}'$ there is a "replacement" $\mathbf{Q}$ of $\mathbf{Q}'$ where $\mathbf{Q}$ is a spiral model category and

$$\mathrm{Ho}_{E^2}(\Delta\mathbf{Q}') = \mathrm{Ho}(\Delta\mathbf{Q})_s.$$

In particular $\mathbf{Q} = \Delta\mathbf{Gr}$ is such a replacement for $\mathbf{Q}' = \mathbf{Top}^*$.

6 Homology and Cohomology in Spiral Homotopy Theory

In this section let

$$\mathbf{Q} = (\mathbf{T} \subset \mathbf{C} \subset \mathbf{Q}) \tag{6.1}$$

be a spiral model category for which $\tilde{\mathbf{L}} = \mathbf{L}/\simeq_I$ satisfies the delicate Blakers-Massey property. Here $\mathbf{L}$ is the full subcategory of $\mathbf{C}$ consisting of arbitrary coproducts of objects in $\mathbf{T}$. For example for $k \geq 1$ the spiral model category

$$\mathbf{Q}_k = (\mathbf{T}_k \subset (\Delta\mathbf{gr})_{\mathrm{free}} \subset \Delta\mathbf{Gr})$$

in (5.5) satisfies the assumption on $\mathbf{Q}$ above. The assumptions in (6.1) imply that $(\Delta\mathbf{Q})_s$ is a homological cofibration category under $\tilde{L}$; see (4.14). Hence we can apply all results of chapter I, ... , VII. The cofibration category $(\Delta\mathbf{Q})_s$ is an example in which all objects are cofibrant but not fibrant. Hence the notion of principal cofibration in (III.3.1) (1) uses fibrant models.

(6.2) Lemma. *Let $X = \lim\{X^n\}$ be given by a spiral inclusion $* \to X$ as in (3.2). Then $X^{n-1} \to X^n$ is a principal cofibration with attaching map*

$$\partial_n \in [A_n \wedge S[n-1], X^{n-1}]_s = \pi_{n-1}^{A_n}(X^{n-1})_s$$

where $A_n \in \mathbf{L}$. Hence $X_{(\geq 1)}$ is a complex in the sense of (IV.2.2) with $X_{(n)} = X^{n-1}$; compare (C.1.28).

Proof. Consider the push out in (3.2) with A_n homotopy equivalent in $\mathbf{C}/\simeq_I$ to $\bar{A}_n$. Then the composite

$$f_n : A_n \otimes \partial\Delta[n] \to \bar{A}_n \otimes \partial\Delta[n] \to X^{n-1}$$

defines the element $f_n \cdot \beta$ with $\beta = -\alpha$ and $\alpha = f_n \mid A_n \otimes *$ as in the proof of (C.1.21). Moreover using q in this proof we obtain ∂_n from $f_n \cdot \beta$. q.e.d.

Lemma (6.2) shows by use of (3.6) that each object in $(\Delta\mathbf{Q})_s$ is spiral equivalent to an $\tilde{\mathbf{L}}$-complex in $(\Delta\mathbf{Q})_s$. Moreover (IV.5.10) implies that one has an equivalence of categories at the left hand side of the following commutative diagram

$$\begin{tikzcd} \mathrm{Ho}(\Delta\mathbf{Q})_s \arrow[r, "\tilde{c}"] & \mathbf{model}(\tilde{\mathbf{T}}) \\ \mathbf{Complex}/\overset{1}{\simeq} \arrow[u, "\wr"'] \arrow[r, "c"] & \mathbf{Coef} \arrow[u, "\wr"'] \end{tikzcd} \tag{6.3}$$

Here **Complex** is the category of $\tilde{\mathbf{L}}$-complexes in $(\Delta\mathbf{Q})_s$ as defined in (IV. § 2) and the coefficient functor c is defined in (V.1.3). The <u>coefficient functor</u> $\tilde{c}$ carries $X \in (\Delta\mathbf{Q})_s$ to $\tilde{c}(X) : \tilde{\mathbf{T}}^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$\tilde{c}(X)(A) = \pi_0 \tilde{\mathbf{L}}(A, \tilde{X})$$

with $\tilde{X} = p_L X$ given by (3.4) and $A \in \tilde{\mathbf{T}} \subset \tilde{\mathbf{L}}$. The equivalence on the right hand side of (6.3) is given as in (I.4.6) below. The spiral exact sequence shows that (6.3) is well defined and commutative. Moreover $\tilde{c}$ is compatible with the coefficient functor c in (C.1.29) since $\tilde{c}$ is defined by $\tilde{X} \in (\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$. Using the spiral exact sequence we see that ∂_1 in (6.2) yields an element

$$\partial_1 = \partial_X \in [A^1 \wedge S[0], X^0]_s = \tilde{\mathbf{L}}(A^1, A^0) \tag{6.4}$$

Here ∂_X represents an object in **Coef** which is a presentation of $\tilde{c}(X) \in \mathbf{model}(\tilde{\mathbf{T}})$. We shall identify ∂_X and $M = \tilde{c}(X)$. For each $\partial_X \in \tilde{\mathbf{L}}(A^1, A^0)$ we can choose X^1 in $(\Delta\mathbf{Q})_s$ such that $X^0 \to X^1$ is a principal cofibration with attaching map $\partial_X \in [A^1 \wedge S[0], X^0]_s$ where $X^0 = \mathrm{const}(\bar{A}^0)$ with $\bar{A}^0 \simeq A^0$ in $\mathbf{C}$.

(6.5) Lemma. *Let $A, B \in \tilde{\mathbf{L}}$ and $n \geq 1$. Then the maps $A \wedge S[n] \vee X^1 \to B \wedge S[n] \vee X^1$ in $\mathrm{Ho}(\Delta\mathbf{Q})_s$ under and over X^1 can be identified with the maps $A \wedge S[n] \vee \tilde{X}^1 \to B \wedge S[n] \vee \tilde{X}^1$ in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}/\simeq$ under and over $\tilde{X}^1$.*

Proof. It suffices to show that

$$\begin{array}{c} \pi_n^A(B \wedge D[n+1] \vee X^1, B \wedge S[n] \vee X^1)_s \\ \downarrow \\ \pi_n(\tilde{\mathbf{Q}}(A, B \wedge D[n+1] \vee \tilde{X}^1), \tilde{\mathbf{Q}}(A, B \wedge S[n] \vee \tilde{X}^1)) \end{array}$$

is an isomorphism. But this is a consequence of the spiral exact sequence for relative homotopy groups in (4.10). q.e.d.

(6.6) Definition. Given a small theory of cogroups $\tilde{\mathbf{T}}$ we define the <u>enveloping functor</u>

$$U : \mathbf{model}(\tilde{\mathbf{T}}) \to \mathbf{Ringoids}$$

as follows. The category $\mathbf{M} = \mathbf{model}(\tilde{\mathbf{T}})$ has coproducts so that for $A \in \tilde{\mathbf{T}}$ and $M \in \mathbf{M}$ the coproduct $A \vee M = \tilde{\mathbf{T}}(-, A) \vee M \in \mathbf{M}$ is defined. Hence

$$M \longrightarrow A \vee M \xrightarrow{(0,1)} M$$

defines a cogroup in the category $\mathbf{M}_M^M$ of objects under and over M in $\mathbf{M}$. Let $\mathbf{premod}(M) \subset \mathbf{M}_M^M$ be the full subcategory of all objects $A \vee M$ with $A \in \tilde{\mathbf{T}}$. Then $\mathbf{premod}(M)$ is a theory of cogroups for which the additivization $U(M) = \mathbf{premod}(M)^{\mathrm{ad}}$ is defined by (I.1.16). Compare the definition of the enveloping functor in (I.5.10).

For a ringoid $\mathbf{R}$ let $\mathbf{Mod}(\mathbf{R})$ be the category of (right) $\mathbf{R}$-modules, i.e. additive functors $\mathbf{R}^{\mathrm{op}} \to \mathbf{Ab}$. Morphisms are natural transformations. In particular, one gets for $A \in \mathbf{R}$ the R-module $\mathbf{R}(-, A)$. An arbitrary sum of such $\mathbf{R}$-modules $\mathbf{R}(-, \Delta_i)$, $i \in J$, is called a <u>free</u> $\mathbf{R}$-module. Let

$$\mathbf{mod}(\mathbf{R}) \subset \mathbf{Mod}(\mathbf{R}) \tag{6.7}$$

be the full subcategory of free $\mathbf{R}$-modules. Lemma (6.4) shows that the category $\mathbf{mod}(\partial_X)$ defined in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$ as in (C.1.31) coincides with the corresponding category $\mathbf{mod}(\partial_X)$ defined in $(\Delta\mathbf{Q})_s$; see (V.1.6). Moreover by (C.1.34) we get:

(6.8) Proposition. *Let ∂_X be a presentation of $M \in \mathbf{model}(\tilde{\mathbf{T}})$ as in (6.4). Then there is a canonical isomorphism of categories*

$$\mathbf{mod}(\partial_X) = \mathbf{mod}(U(M))$$

where $U(M)$ is the enveloping ringoid in (6.6).

Using (6.4) we see that the chain complex of X in $(\Delta\mathbf{Q})_s$ depends only on the chain complex of $\tilde{X}$ in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$. This leads to the following observation.

(6.9) Definition. Let X be an object in $(\Delta\mathbf{Q})_s$ such that $\tilde{X}$ is a CW-object in $\Delta\tilde{\mathbf{L}}$. Let $M = \tilde{c}(X) \in \mathbf{model}(\tilde{\mathbf{T}})$ be given by the coefficient functor $\tilde{c}$ in (6.3). Then the chain complex of X

$$C_*X = C_*\tilde{X} \quad \text{in } \mathbf{mod}(U(M))$$

coincides with the chain complex of $\tilde{X}$ in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$ defined in (C.1.35). Accordingly homology and cohomology of X coincide with homology and cohomology of $\tilde{X}$ in (C.1.36) and (C.1.39).

Let U be a fibrant object in $(\Delta\mathbf{Q})_s$ and let ∂_X and $X^0 \to X^1$ be given as in (6.4) and let $M = \tilde{c}(X^1)$ so that ∂_X is a presentation of M. Let $u : X^0 \to U$ be a map which admits an extension $X^1 \to U$. We define the right $U(M)$-module

$$u_*\pi_n(U) : U(M)^{\mathrm{op}} \longrightarrow \mathbf{Ab} \tag{6.10}$$

for $n \geq 2$ as follows. This module carries $A \vee M$ to the spiral homotopy group $\pi_{n-1}^A(U)_s$ in (4.6). Here we use a shift in degree since A is considered to be of dimension 1; see (C.1.28). A map $f : B \vee M \to A \vee M$ in $U(M)$ corresponds to a map $\bar{f} : B \wedge S[n-1] \vee X^1 \to A \wedge S[n-1] \vee X^1$ in $\mathrm{Ho}(\Delta\mathbf{Q})_s$ under and over X^1. See (6.8). Hence f induces the homomorphism

$$f^* : \pi_{n-1}^B(U)_s \longrightarrow \pi_{n-1}^A(U)_s$$

which carries $(a : B \wedge S[n-1] \to U) \in \pi_{n-1}^B(U)_s$ to the composite

$$f^*(a) : A \wedge S[n-1] \xrightarrow{f'} B \wedge S[n-1] \vee X^1 \xrightarrow{(a,u)} U$$

where f' is determined by $\bar{f}$. As a special case we obtain for a spiral inclusion $* \to X$ with fibrant model $X \overset{\sim}{\rightarrowtail} \bar{X}$ the map $u : X_0 \to \bar{X}$ which defines the $U(M)$-module $\pi_n(X) = u^*\pi_n(\bar{X})$ with $M = \tilde{c}(X)$.

(6.11) Theorem. *Let $Y \to X$ be a spiral inclusion with $Y = X^{-1}$ as in (3.2) and let U be a fibrant object in $(\Delta\mathbf{Q})_s$. Let $f : Y \to U$ be a map in $(\Delta\mathbf{Q})_s$ which admits an extension $g : X^{n-1} \to U$, $n \geq 2$. Then the restriction $g \mid X^{n-2}$ admits an extension $\bar{g} : X^n \to U$ if and only if an obstruction element*

$$\mathcal{O}(g \mid X^{n-2}) \in H^{n+1}(\tilde{X}, \tilde{Y}; u^*\pi_n(U)_s)$$

vanishes. Here $u : X^0 \to U$ is the restriction of g.

This is a special case of (V.§ 4); compare (C.1.38) where the corresponding result for maps in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$ is formulated. Next we consider the Hurewicz homomorphism h and the exact sequence of J.H.C. Whitehead in spiral homotopy theory $(\Delta\mathbf{Q})_s$ and we compare this exact sequence with the corresponding exact sequence in (C.1.41).

(6.12) Theorem. *Let X be an object in $(\Delta\mathbf{Q})_s$ with $\tilde{X} = p_L(X) \in (\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$ and $M = \tilde{c}(X) \in$ **model**$(\tilde{\mathbf{T}})$. Then one has the following commutative diagram ($n \geq 2$) in which the rows are exact sequences of $U(M)$-modules.*

$$\begin{array}{ccccccccccccc}
\dots \longrightarrow & H_{n+1}(\tilde{X}) & \longrightarrow & \Gamma_n(X)_s & \longrightarrow & \pi_n(X)_s & \longrightarrow & H_n(\tilde{X}) & \longrightarrow & \dots & \longrightarrow & \Gamma_1(X)_s & \longrightarrow 0 \\
& \| & & \downarrow & & \downarrow & & \| & & & & \downarrow & \\
\dots \longrightarrow & H_{n+1}(\tilde{X}) & \longrightarrow & \Gamma_n(\tilde{X}) & \longrightarrow & \pi_n(\tilde{X}) & \longrightarrow & H_n(\tilde{X}) & \longrightarrow & \dots & \longrightarrow & \Gamma_1(X) & \longrightarrow 0
\end{array}$$

The bottom row coincides with the sequence in (C.1.41). The diagram is natural in X.

The module $\pi_n(X)_s$ is defined in (6.10). Moreover we obtain for $n \geq 3$ the module $\Gamma_n(X)_s$ by

$$\Gamma_n(X)_s = \mathrm{image}\{\pi_n(X^{n-2})_s \to \pi_n(X^{n-1})_s\}$$

Here we use the shift of degree $X^{n-1} = X_{(n)}$ as in (6.2). For the definition of $\Gamma_1(X)_s$ and $\Gamma_2(X)_s$ see (V.5.3) and (II.§ 2).

Since $p_L : \mathrm{Ho}(\Delta\mathbf{Q})_s \to (\Delta\tilde{\mathbf{L}}_{\mathrm{free}})/\simeq$ reflects isomorphisms we see that the homological Whitehead theorem for $(\Delta\mathbf{Q})_s$ coincides with the corresponding theorem for $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$ in (C.1.40). Moreover we leave it to the reader to formulate a finiteness obstruction theorem for $(\Delta\mathbf{Q})_s$ which is the analogue of (C.1.44).

7 Spiral Resolutions and Spiral Realizations

Let $\mathbf{Q}$ be a spiral model category for which $\tilde{\mathbf{L}}$ satisfies the delicate Blakers-Massey property as in (6.1).

(7.1) Definition. Let Q be an object in $\mathbf{Q}$. A spiral resolution of Q is an object X in $(\Delta\mathbf{Q})_s$ together with a spiral equivalence $X \to \mathrm{const}(Q)$ in $\Delta\mathbf{Q}$.

The theorem of Stover (3.6) shows that spiral resolutions of Q exist. In fact, the construction in the proof of (3.6) yields a spiral resolution which is functorial in Q. (One can check that spiral resolutions are well defined up to isomorphism in $\mathrm{Ho}(\Delta\mathbf{Q})_s$.)

(7.2) Definition. Let $M \in \mathbf{model}(\tilde{\mathbf{T}})$. We say that $X \in (\Delta\mathbf{Q})_s$ is a spiral realization of M if one has an isomorphism

$$\pi_n\tilde{\mathbf{Q}}(A,\tilde{X}) = \begin{cases} M(A) & \text{for } n = 0 \\ 0 & \text{for } n > 0 \end{cases}$$

which is natural in $A \in \tilde{\mathbf{T}}$. Here $\tilde{X} = p_L(X)$ is a "resolution of M" in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$. In fact, $\tilde{X}$ is well defined up to homotopy equivalence in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$; but two spiral realizations X, X' of M in $(\Delta\mathbf{Q})_s$ need not to be isomorphic in $\mathrm{Ho}(\Delta\mathbf{Q})_s$.

Each object Q in $\mathbf{Q}$ yields the model

$$M = \tilde{\mathbf{Q}}(-,Q) \in \mathbf{model}(\tilde{\mathbf{T}})$$

represented by Q. That is, M carries $A \in \tilde{\mathbf{T}}$ to the set of morphisms $\tilde{\mathbf{Q}}(A,Q)$ in $\tilde{\mathbf{Q}}$.

(7.3) Lemma. *If X is a spiral resolution of $Q \in \mathbf{Q}$ then X is a spiral realization of $M = \tilde{\mathbf{Q}}(-,Q)$.*

The lemma is an immediate consequence of the definition of spiral equivalence.

(7.4) Remark. We say that $M \in \mathbf{model}(\tilde{\mathbf{T}})$ is realizable in $\mathbf{Q}$ if $M \cong \tilde{\mathbf{Q}}(-,Q)$ for some Q in $\mathbf{Q}$. Hence if M is realizable in $\mathbf{Q}$ then M is also spiral realizable in $(\Delta\mathbf{Q})_s$. In certain cases such as $\mathbf{Q} = \Delta\mathbf{Gr}$ also the converse holds; that is, the spiral realizability of M implies the realizability of M in $\mathbf{Q}$. For this one needs the "realization" $|X| \in \mathbf{Q}$ of $X \in \Delta\mathbf{Q}$. Compare Dwyer-Kan-Stover [HG], Blanc [AI], Goerss-Hopkins [RM].

(7.5) Definition. Let $M \in \mathbf{model}(\tilde{\mathbf{T}})$ and let ∂ be a presentation of M as in (6.4). By (2.4) (iii) we have the suspension functor $\Sigma : \tilde{\mathbf{T}} \to \tilde{\mathbf{T}}$ which yields the model $\Omega^n M \in \mathbf{model}(\tilde{\mathbf{T}})$ by

$$(\Omega^n M)(A) = M(\Sigma^n A)$$

for $A \in \tilde{\mathbf{T}}$. Here Σ has a factorization $\Sigma : \tilde{\mathbf{T}} \to \tilde{\mathbf{T}}^{\mathrm{add}} \to \tilde{\mathbf{T}}$ where $\tilde{\mathbf{T}}^{\mathrm{add}}$ is the additivization of $\tilde{\mathbf{T}}$; see (I.1.16). Hence $\Omega^n M$ for $n \geq 1$ is actually a (right) $\tilde{\mathbf{T}}^{\mathrm{add}}$-module. We have the canonical inclusion

$$\tilde{\mathbf{T}}^{\mathrm{add}} \subset U(M)$$

which carries $f : A \to B \in \tilde{\mathbf{T}}$ to $f \vee 1 : A \vee M \to B \vee M \in U(M)$. See (6.6). The next result shows that $\Omega^n M$ is in addition a (right) $U(M)$-module if M is spiral realizable.

(7.6) Proposition. *Let X be a spiral realization of M. Then one has a canonical isomorphism of abelian groups $(n \geq 1)$*

$$\pi_n^A(X)_s = (\Omega^n M)(A) \quad \textit{for } A \in \tilde{\mathbf{T}}.$$

Here the left hand side is a $U(M)$-module by (6.10). The isomorphism thus yields a $U(M)$-module structure of $\Omega^n M$ denoted by $\Omega^n(M)_X$. At this point it is not clear whether this module structure $\Omega^n(M)_X$ actually depends on X.

Proof of (7.6). Since X is a spiral realization of M we can use the spiral exact sequence and get the isomorphisms of abelian groups $(n \geq 1)$

$$\pi_n^A(X)_s \cong \pi_0^{\Sigma^n A}(X)_s \cong \pi_0 \tilde{\mathbf{Q}}(\Sigma^n A, \tilde{X}) = M(\Sigma^n A) = (\Omega^n M)(A).$$

q.e.d.

The proof shows that the $U(M)$-module $\Omega^n(M)_X$ restricted to $\tilde{\mathbf{T}}^{\mathrm{add}} \subset U(M)$ yields the $\tilde{\mathbf{T}}^{\mathrm{add}}$-module $\Omega^n M$ described in (7.4) which does not depend on X.

(7.7) Definition. Let $M \in \mathbf{model}(\tilde{\mathbf{T}})$. Then a resolution $K(M,1) = \tilde{X}$ of M is an object in $(\Delta\tilde{\mathbf{L}})_{\mathrm{free}}$ with $\tilde{\mathbf{L}} = \mathbf{free}(\tilde{\mathbf{T}})$ such that one has an isomorphism

$$\pi_n(A, \tilde{X}) \cong \begin{cases} M(A) & \text{for } n = 0 \\ 0 & \text{for } n > 0 \end{cases}$$

which is natural in $A \in \tilde{\mathbf{T}}$. Compare (VI. Appendix § 11). Let $C_*\tilde{X}$ be the chain complex of $\tilde{X}$ as defined in (C.1.35). Then $C_*\tilde{X}$ is a chain complex of free $U(M)$-modules with $M = c(\tilde{X}) \in \mathbf{Coef} = \mathbf{model}(\tilde{\mathbf{T}})$; see (6.3), (6.8) and (C.1.29). Hence the homology of $C_*\tilde{X}$ in the abelian category $\mathbf{Mod}(U(M))$ is defined. This is the Quillen homology

$$H_n(M) = H_n(C_*\tilde{X})$$

of the model M. The Quillen homology $H_n(M)$ is a (right) $U(M)$-module which only depends on M. For every right $U(M)$-module K also the Quillen cohomology

$$H^n(M,K) = H^n \operatorname{Hom}_{U(M)}(C_*\tilde{X}, K)$$

is defined.

As a consequence of (6.10) and (7.6) one readily gets the following result.

(7.8) Theorem. *Let X be a spiral realization of $M \in \mathbf{model}(\tilde{\mathbf{T}})$. Then one has isomorphisms of $U(M)$-modules*

$$\Gamma_n(X)_s \cong \begin{cases} H_2(M) & \text{for } n = 1 \\ H_{n+1}(M) \oplus \Omega^{n-1}(M)_X & \text{for } n \geq 2 \end{cases}$$

The result has many implications concerning the homological tower of categories in (VI. § 6) and the obstructions described in (VI. § 9). In particular the obstructions for the realizability of chain complexes in (VI. § 4) lead to the following result.

(7.9) Theorem. *Let $M \in \mathbf{model}(\tilde{\mathbf{T}})$. Then M is spiral realizable if and only if one can define inductively $X(n)$ such that an obstruction*

$$\mathcal{O}(X(n)) \in H^{n+2}(M, \Omega^{n-1}(M)_{X(n)})$$

vanishes for $n \geq 2$. Here $X(2)$ is determined by a presentation ∂_X of M and $X(n+1)$ can be defined if $\mathcal{O}(X(n)) = 0$. Moreover the group $H^{n+1}(M, \Omega^{n-1}(M)_{X(n)})$ acts transitively on the set of all possible choices of $X(n+1)$ which extend $X(n)$.

A result of this type with different assumptions was recently obtained by Blanc [AI] using completely different methods. Theorem (7.9) describes the obstructions which were anticipated by Dwyer-Kan-Stover $[E^2]$ 1.3.

Proof of (7.9). We use (VI.9.1) and (VI.9.4) and observe that the cohomology splits in two parts by (7.8). The first part is the corresponding obstruction in $(\Delta\tilde{\mathbf{L}})_{\text{free}}$ which is trivial since we choose for M an object $\tilde{X} = K(M,1)$ as in (7.7). The second part describes the obstruction for the existence of X in $(\Delta\mathbf{Q})_s$ with $p_L(X) = K(M,1)$. We build X inductively by constructing the skeleta $X(n) = X^{n-1}$ with $p_L(X(n)) = \tilde{X}^{n-1}$. q.e.d.

The homological tower of categories in (VI. § 6) yields further interesting results on the homotopy category of spiral realizations of M considered as a full subcategory of $\operatorname{Ho}(\Delta\mathbf{Q})_s$.

Part II

Combinatorial Homology and Homotopy

The long list of examples in Part 1 shows the necessity of an axiomatic theory of combinatorial homology and homotopy. In the following chapters I, ... , VIII we discuss the notions and results of such a foundational theory. The results are considerably more sophisticated than previous results achieved in axiomatic homotopy theory. For example, Wall's finiteness obstruction theorem and Whitehead's results on simple homotopy equivalences and Whitehead torsion are deep results of classical homotopy theory which we prove in the axiomatic context.

Chapter I: Theories of Coactions and Homology

In order to obtain homology for a theory **T** of coactions we introduce in this chapter the categories

Twist, **Coef**, **premod**, **mod** and **chain**

which we derive from the theory **T**. These categories are fundamental for the general treatment of homology theory.

The category **Coef** of coefficients is used to describe the "coefficients" of the homology and cohomology theory associated to **T**. The objects of **Twist** are presentations of the objects in **Coef** where a presentation ∂_X is a map in **T**. We derive from the category **Coef** the category **mod** of modules. The link between **Coef** and **mod** is the category **premod** of pre-modules. There is an enveloping functor U which carries an object in **Coef** to a ringoid such that **mod** is the Grothendieck construction of U. The category **chain** is the category of chain complexes in **mod**. Such chain complexes are used to define homology and cohomology for **T**.

For example let $\mathbf{T} = \mathbf{gr}$ be the category of free groups which is a theory of cogroups. Then $\mathbf{Coef} = \mathbf{Gr}$ in the category of groups and the enveloping functor carries a group G to the group ring $\mathbb{Z}[G]$. In this case a presentation of G is a map $\partial_X : X'' \to X$ between free groups in **T** and **Twist** is the category of free pre-crossed modules given by such presentations. We can consider ∂_X also as the attaching map of 2-dimensional cells in a CW-complex X^2 such that $\pi_1 X^2 = G$.

1 Theories of Cogroups and Theories of Coactions

We introduce basic notation concerning cogroups and coactions in categories. Also we consider theories and models of such theories. All the notation and results of the following sections are available in theories of cogroups and more generally in theories of coactions.

Let $\mathbf{C}$ be a category and let X, Y be objects in $\mathbf{C}$. Then $\mathbf{C}(X,Y)$ denotes the set of *morphisms* or *maps* $X \to Y$ in $\mathbf{C}$ and $\mathrm{Ob}(\mathbf{C})$ is the class of objects in $\mathbf{C}$. A *sum* or *coproduct* $X \vee Y$ in $\mathbf{C}$ is an object $X \vee Y$ together with morphisms $i_X : X \to X \vee Y, i_Y : Y \to X \vee Y$ such that for all objects Z in $\mathbf{C}$ one has the bijection

H.-J. Baues, *Combinatorial Foundation of Homology and Homotopy*

$$\mathbf{C}(X \vee Y, Z) = \mathbf{C}(X, Z) \times \mathbf{C}(Y, Z) \tag{1.1}$$

where the right hand side denotes the product of sets. The bijection carries $f : X \vee Y \to Z$ to $(i_X f, i_Y f)$. Hence any pair of maps $a : X \to Z, b : Y \to Z$ yields a unique map $(a, b) : X \vee Y \to Z$.

(1.2) Definition. A *theory* $\mathbf{T}$ is a category with an initial object $*$ and with finite sums denoted by $X \vee Y$. We consider $*$ as the empty sum. A *map* between theories is a functor $F : \mathbf{T} \to \mathbf{T}'$ which preserves sums. This is an *equivalence of theories* if there is a map $G : \mathbf{T}' \to \mathbf{T}$ with FG and GF natural isomorphic to the corresponding identical functors.

(1.3) Definition. Let $\mathbf{T}$ be a theory. A *based object* in $\mathbf{T}$ is an object X endowed with a map $0_X = 0 : X \to *$. This map defines for all objects Y in $\mathbf{T}$ the zero-map $0 : X \to * \to Y$. A *cogroup* $X = (X, 0, \mu, \nu)$ in $\mathbf{T}$ is a based object $(X, 0)$ together with a comultiplication $\mu_X = \mu : X \to X \vee X$ and a map $\nu_X = \nu : X \to X$ such that the following diagrams commute.

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, "\mu"'] & X \vee X \arrow[d, "1 \vee \mu"] \\
X \vee X \arrow[r, "\mu \vee 1"] & X \vee X \vee X
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \arrow[dl, "1"'] \arrow[d, "\mu"] \arrow[dr, "1"] & \\
X & X \vee X \arrow[l, "(0,1)"] \arrow[r, "(1,0)"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \arrow[dl, "0"'] \arrow[d, "\mu"] \arrow[dr, "0"] & \\
X & X \vee X \arrow[l, "(1,\nu)"] \arrow[r, "(\nu,1)"'] & X
\end{tikzcd}$$

We say that the cogroup X is *abelian* if the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\mu"'] \arrow[dr, "\mu"] & \\
X \vee X \arrow[rr, "T"] & & X \vee X
\end{tikzcd}$$

commutes where T is the interchange map with $Ti_1 = i_2$ and $Ti_2 = i_1$. Moreover a map $f : X \to Y$ between cogroups is *linear* if the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ X \vee X & \xrightarrow{f \vee f} & Y \vee Y \end{array}$$

commutes.

(1.4) Definition. Let $\mathbf{T}$ be a theory. A *coaction* $X = (X, X', \mu)$ in $\mathbf{T}$ is an object X together with a map $\mu_X = \mu : X \to X \vee X'$ where X' is a co-group such that the following diagrams commute.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & X \vee X' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle 1\vee\mu} \\ X \vee X' & \xrightarrow{\mu \vee 1} & X \vee X' \vee X' \end{array}$$

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr, "1"'] & X \vee X' \arrow[d, "{(1,0)}"] \\ & X \end{tikzcd}$$

Clearly each cogroup X yields a coaction with $X' = X$.

Let $\mathbf{Set}$ be the category of sets and let $\mathbf{T}^{\mathrm{op}}$ be the opposite category of $\mathbf{T}$. A *model* of a theory $\mathbf{T}$ is a functor

$$M : \mathbf{T}^{\mathrm{op}} \to \mathbf{Set} \tag{1.5}$$

(i.e. a contravariant functor $\mathbf{T} \to \mathbf{Set}$) such that M carries sums in $\mathbf{T}$ to products. This means M carries the inclusions $i_X : X \to X \vee Y, i_Y : Y \to X \vee Y$ to the projections $p_X = i_X^* : M(X \vee Y) \to M(X)$ and $p_Y = i_Y^* : M(X \vee Y) \to M(Y)$ of the product

$$M(X \vee Y) = M(X) \times M(Y)$$

In particular M carries the empty sum in $\mathbf{T}$ which is the initial object $*$ to the empty product in $\mathbf{Set}$ which is the final object $*$ in $\mathbf{Set}$ consisting of a single point.

If the theory $\mathbf{T}$ is a small category we define the category **model**. Objects are the models of $\mathbf{T}$ and morphisms are the natural transformations between models. Smallness of $\mathbf{T}$ is only needed in order to make sure that all morphisms $M \to M'$ between models M and M' form a set.

If X is a co-group in $\mathbf{T}$ then the set $M(X)$ has the structure of a group which we write additively though the group $M(X)$ needs not to be abelian. The neutral element $0 \in M(X)$ is given by $0^*(*) = 0$ and the group structure of $M(X)$ is the map

$$+ : M(X) \times M(X) = M(X \vee X) \xrightarrow{\mu^*} M(X) \tag{1.6}$$

with the inverse $- = \nu^* : M(X) \to M(X)$.

If $X = (X, X', \mu)$ is a coaction in $\mathbf{T}$ then the group $M(X')$ acts on the set $M(X)$ by the map

$$+ : M(X) \times M(X') = M(X \vee X') \xrightarrow{\mu^*} M(X) \tag{1.7}$$

with $x + (a + b) = (x + a) + b$ and $x + 0 = x$ for $x \in M(X)$ and $a, b \in M(X')$. As an example of a model on $\mathbf{T}$ we have for each object Y in $\mathbf{T}$ the functor

$$\mathrm{Mor}_Y : \mathbf{T}^{\mathrm{op}} \to \mathbf{Set} \tag{1.8}$$

which carries X to the set of morphisms $\mathrm{Mor}_Y(X) = \mathbf{T}(X, Y)$ in $\mathbf{T}$. We also write $\mathrm{Mor}_Y = \mathbf{T}(-, Y)$. Hence if X is a cogroup then $\mathbf{T}(X, Y)$ has the structure of a group and a morphism $f : Y \to Z$ induces a group homomorphism $f_* : \mathbf{T}(X, Y) \to \mathbf{T}(X, Z)$, that is for $a, b \in \mathbf{T}(X, Y)$ we have

$$f(a + b) = fa + fb.$$

Here the composition is written multiplicatively. Therefore it is suitable to write the group structure in $\mathbf{T}(X, Y)$ and hence in $M(X)$ additively.

(1.9) Definition. A *theory of cogroups* is a theory $\mathbf{G}$ for which each object X in $\mathbf{G}$ is endowed with the structure of a cogroup which is compatible with sums; that is, the cogroup structure of a sum $X \vee Y$ is given by the cogroup structures of X and Y respectively by $0_{X \vee Y} = (0_X, 0_Y), \nu_{X \vee Y} = \nu_X \vee \nu_Y$, and

$$\mu_{X \vee Y} : X \vee Y \xrightarrow{\mu_X \vee \mu_Y} (X \vee X) \vee (Y \vee Y) = (X \vee Y) \vee (X \vee Y).$$

(1.10) Example. A *ringoid* (or a pre-additive category) is a category $\mathbf{R}$ with the property that all morphism sets $\mathbf{R}(A, B)$ with $A, B \in \mathbf{R}$ are abelian groups and composition in $\mathbf{R}$ is bilinear. A ringoid with only one object is the same as a ring. An *additive category* $\mathbf{A}$ is a ringoid in which sums $A \vee B = A \oplus B$ exist. Such sums are also products in $\mathbf{A}$; see Mac Lane [C]. Each object A in $\mathbf{A}$ is an abelian cogroup with the comultiplication

$$\mu_A = i_1 + i_2 : A \to A \vee A$$

In fact, an additive category is the same as a theory of cogroups in which each cogroup is abelian and all morphisms are linear. A functor $F : \mathbf{R} \to \mathbf{S}$ between ringoids is *additive* if $F : \mathbf{R}(A, B) \to \mathbf{S}(FA, FB)$ is a homomorphism between abelian groups for all $A, B \in \mathbf{R}$. A functor $F : \mathbf{A} \to \mathbf{B}$ between additive categories which is a map between theories (i.e. F preserves sums) is the same as an additive functor since $F\mu_A = \mu_{FA}$.

(1.11) Definition. A *theory of coactions* is a theory $\mathbf{T}$ for which each object X is endowed with a cogroup object X' and a coaction

$$\mu_X : X \to X \vee X'$$

in $\mathbf{T}$. This structure of coaction on X is compatible with sums; that is for $X \vee Y$ the coaction $\mu_{X \vee Y}$ is the composite

$$\mu_{X\vee Y} : X \vee Y \xrightarrow{\mu_X \vee \mu_Y} (X \vee X') \vee (Y \vee Y') = (X \vee Y) \vee (X' \vee Y')$$

where $(X \vee Y)' = X' \vee Y'$. Moreover each coaction μ_X has the following *affine property*. For all objects Y and all maps $f, g : X \to Y$ in $\mathbf{T}$ there exists a unique $\alpha : X' \to Y$ with $g = f + \alpha$. Then X is also termed a *cotorsor*.

Clearly a theory of cogroups is an example of a theory of coactions. A model M on a theory of coactions $\mathbf{T}$ is *affine* if for all objects X in M and $x, y \in M(X)$ there is a unique $\alpha \in M(X')$ with $y = x + \alpha$; then M is also called a *torsor*. For example Mor_Y for $Y \in \mathbf{T}$ is an affine model by the affine property of all objects in $\mathbf{T}$.

(1.12) Lemma. *Let $\mu_X : X \to X \vee X'$ be a coaction in $\mathbf{T}$. Then μ_X has the affine property if and only if*

$$(i_X, \mu_X) : X \vee X \to X \vee X'$$

is an isomorphism in $\mathbf{T}$. Here i_X is the inclusion of X.

Proof. Consider the pair of maps $i_1 = i_X : X \to X \vee X'$ and $i_2 = \mu_X : X \to X \vee X'$. The affine property shows that for each pair $f, g : X \to Y$ of maps in $\mathbf{T}$ there is a unique map $\alpha : X' \to Y$ with $g = f + \alpha$. Hence there is a unique map $(f, \alpha) : X \vee X' \to Y$ with $(f, \alpha)i_1 = f$ and $(f, \alpha)i_2 = g$. Therefore i_1, i_2 satisfies the universal properties of the inclusions of a sum. q.e.d.

(1.13) Corollary. *Each model M of a theory of coactions is affine.*

Proof. We have $M(X) \times M(X) = M(X \vee X) = M(X \vee X') = M(X) \times M(X')$ by (1.12). q.e.d.

(1.14) Remark. The collection of models M of a theory of coactions yields a *Malcev variety* in the sense of Smith [MV]. In fact for $x, y, z \in M(X)$ with $X \in \mathbf{T}$ we have a unique $\beta \in M(X')$ with $y + \beta = z$. Now we define P by $P(x, y, z) = x + \beta$. Then $P(x, y, y) = x$ and $P(x, x, z) = z$. As a special case one obtains varieties of groups with operators as in (2.12) below.

We shall need the "additivization" of a theory of cogroups which is an additive category. For this we define for a set T the full inclusion of categories

$$\mathbf{add}(T) \subset \mathbf{cogr}(T) \tag{1.15}$$

as follows. An object in $\mathbf{cogr}(T)$ is a theory $\mathbf{T}$ of cogroups with the property that the objects of $\mathbf{T}$ are the elements of the set T, that is $\mathrm{Ob}(\mathbf{T}) = T$. Morphisms in $\mathbf{cogr}(T)$ are maps between theories which are the identity on objects. An object in $\mathbf{add}(T)$ is an additive category $\mathbf{A}$ with $\mathrm{Ob}(\mathbf{A}) = T$. Morphisms in $\mathbf{add}(T)$ are additive functors which are the identity on objects. Using the final remark in (1.10) we see that (1.15) is a full inclusion.

(1.16) Lemma. *The inclusion functor (1.15) has a left adjoint*

$$(\quad)^{\mathrm{ad}} : \mathbf{cogr}(T) \to \mathbf{add}(T)$$

This functor which carries $\mathbf{T}$ to $\mathbf{T}^{\mathrm{ad}}$ is termed the *additivization functor.*

Proof. For two maps $a, b : A \to B$ in $\mathbf{T}$ we write $a \sim b$ if there exists $\alpha \in \mathbf{T}(A, B_1 \vee B_2)$ with $B_1 = B_2 = B$ such that $(0,1)_*\alpha = 0$ and $(1,0)_*\alpha = 0$ and $a = b + (1,1)_*(\alpha)$. Then $\sim$ is a natural equivalence relation on $\mathbf{T}$ and one gets the quotient category $\mathbf{T}^{\mathrm{ad}} = \mathbf{T}/\!\sim$ which is the additivization of $\mathbf{T}$. q.e.d.

2 Examples

We consider various examples of theories of cogroups and theories of coactions. The basic example is the category $\mathbf{gr}$ of free groups which is a theory cogroups. This theory can be topologically described as the homotopy category of one point unions of 1-dimensional spheres. Therefore all results for the theory $\mathbf{gr}$ have a topological interpretation.

(2.1) Example. Let $\mathbf{Gr}$ be the *category of groups* and let $\mathbf{gr}$ be the full *subcategory of free groups* $F = \langle Z_F \rangle$ with a given set of generators Z_F. The trivial group $* = \{0\}$ is the initial and the final object of $\mathbf{Gr}$. Sums in $\mathbf{Gr}$ exists and for $F = \langle Z_F \rangle$ and $E = \langle Z_E \rangle$ in $\mathbf{gr}$ the sum is simply

$$F \vee E = \langle Z_F \,\dot{\cup}\, Z_E \rangle$$

where $Z_F \,\dot{\cup}\, Z_E$ is the disjoint union of sets. Each object in $\mathbf{gr}$ is a cogroup by the homomorphism

$$\mu_F : F \to F \vee F$$

defined on generators by $\mu(x) = i_1(x) \circ i_2(x)$ for $x \in Z_F$. Here i_1 and i_2 are the inclusions $F \to F \vee F$ and $\circ$ is the group law in $F \vee F$. This shows that $\mathbf{gr}$ is a theory of cogroups.

(2.2) Definition. Let D be an object in a category $\mathbf{C}$. Then we define the *category* $\mathbf{C}^D$ *of objects under* D as follows. Objects are morphisms $D \to X$ in $\mathbf{C}$ and morphisms in $\mathbf{C}^D$ are commutative triangles in $\mathbf{C}$

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Composition in $\mathbf{C}^D$ is defined as in $\mathbf{C}$. If push outs exist in $\mathbf{C}$ then sums exist in $\mathbf{C}^D$. In fact the sum $D \to X \cup_D Y$ of $D \to X$ and $D \to Y$ in $\mathbf{C}^D$ is given by the push out in $\mathbf{C}$

$$\begin{tikzcd} D \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \cup_D Y \end{tikzcd}$$

(2.3) Example. Let G be a group and let $\mathbf{Gr}^G$ be the *category of groups under* G as defined in (2.2). A *free group under* G is an inclusion $G \to F \vee G$ where $F = \langle Z_F \rangle$ is a free group. Let $\mathbf{gr}(G)$ be the full subcategory of $\mathbf{Gr}^G$ consisting of such free groups under G. Push outs exist in $\mathbf{Gr}$ so that sums exist in $\mathbf{Gr}^G$ and $\mathbf{gr}(G)$. One readily checks that a sum in $\mathbf{gr}(G)$ is given by

$$(F \vee G) \cup_G (E \vee G) = (E \vee F) \vee G$$

Each object in $\mathbf{gr}(G)$ is a cogroup in $\mathbf{gr}(G)$ by the homomorphism

$$\mu = \mu_F \vee 1_G : F \vee G \to F \vee F \vee G$$

where μ_F is the cogroup structure of F in $\mathbf{gr}$. This shows that $\mathbf{gr}(G)$ is a theory of cogroups. Clearly for the trivial group $G = *$ the theory of cogroups $\mathbf{gr}(*)$ coincides with $\mathbf{gr}$ in the example (2.1).

(2.4) Example. Let $\mathbf{Top}^*$ be the category of pointed topological spaces and let $\mathbf{Top}^*/\simeq$ be its homotopy category. Let

$$\mathbf{susp}(*) \subset \mathbf{Top}^*/\simeq$$

be the full subcategory consisting of all suspensions ΣX in $\mathbf{Top}^*$. It is well known that a suspension has a cogroup structure $\mu : \Sigma X \to \Sigma X \vee \Sigma X$ in $\mathbf{Top}^*/\simeq$ so that $\mathbf{susp}(*)$ is a theory of cogroups. Let $\mathcal{D}$ be the class of discrete sets in $\mathbf{Top}^*$ and let

$$\mathbf{susp}(*, \mathcal{D}) \subset \mathbf{susp}(*)$$

be the subcategory of suspension ΣX of discrete sets X in $\mathbf{Top}^*$. Then ΣX is just a one point union of one dimensional spheres and it is well known that the fundamental group π_1 yields an isomorphism

$$\pi_1 : \mathbf{susp}(*, \mathcal{D}) \cong \mathbf{gr}$$

of theories of cogroups. This example is generalized in (2.8) below.

Next we describe further algebraic examples of theories of cogroups.

(2.5) Definition. A theory $\mathbf{S}$ is *single sorted* if there is an object A in $\mathbf{S}$ such that all objects of $\mathbf{S}$ are given by the n-fold sums $A \vee \cdots \vee A$ with $n \geq 0$. If A is a cogroup in $\mathbf{S}$ then $\mathbf{S}$ is a theory of cogroups. In this case the models of $\mathbf{S}$ form a *variety of groups with operators.* Let $\mathbf{free}(\mathbf{S})$ be the full subcategory of $\mathbf{model}(\mathbf{S})$ consisting of sums $\bigvee_E \mathrm{Mor}_A$ where E is an index set and Mor_A is the model in (1.8). Such arbitrary sums exist in $\mathbf{model}(\mathbf{S})$; see 3.4.2 in Borceux [CA]. The objects of $\mathbf{free}(\mathbf{S})$ are termed the *free models* of $\mathbf{S}$ and $\mathbf{S}$ is the full subcategory of $\mathbf{free}(\mathbf{S})$ consisting of *finitely generated* free models. The inclusion $\mathbf{S} \subset \mathbf{free}(\mathbf{S})$ carries A to Mor_A. If $\mathbf{S}$ is a theory of cogroups then also $\mathbf{free}(\mathbf{S})$ is a theory of cogroups.

(2.6) Examples. Recall that $\mathbf{Gr}$ is the category of groups and $\mathbf{Ab}$ is the category of abelian groups. The category $\mathbf{Nil}_n$ is the full subcategory of $\mathbf{Gr}$ consisting of groups of nilpotency degree n. The free objects in $\mathbf{Nil}_n$ are the groups $\langle Z\rangle/\Gamma_{n+1}\langle Z\rangle$ where $\langle Z\rangle$ is a free group and $\Gamma_{n+1}\langle Z\rangle$ is the subgroup of $(n+1)$-fold commutators in $\langle Z\rangle$. We have $\mathbf{Nil}_1 = \mathbf{Ab}$. The category $\mathbf{Nil}_n$ is an example of a variety of groups.

A *variety of groups* $\mathbf{Var}$ is a full subcategory of $\mathbf{Gr}$ which is closed under taking subobjects, quotient objects and arbitrary categorical products. Given a subset $\mathcal{L}$ in the free group $F_\infty = \langle x_1, x_2, \dots\rangle$ generated by a sequence of elements $x_1, x_2, \dots$ we say that a group G *satisfies the laws* in $\mathcal{L}$ if for all homomorphisms $\alpha : F_\infty \to G$ we have $\alpha(\mathcal{L}) = \{0\}$ where 0 is the neutral element in G. The subcategory $\mathbf{Var}(\mathcal{L}) \subset \mathbf{Gr}$ consisting of all groups which satisfy the laws in $\mathcal{L}$ is a variety and each variety can be described this way; see Stammbach [HG]. A variety of groups $\mathbf{Var}$ yields the theory $\mathbf{S} = \mathbf{var}^\sharp$ of finitely generated free objects in $\mathbf{Var}$ with $\mathbf{Var} = \mathbf{model}(\mathbf{S})$; then $\mathbf{free}(\mathbf{S}) = \mathbf{var}$ is the category of all free objects in $\mathbf{Var}$.

Moreover for a fixed commutative ring R let $\mathbf{Alg}, \mathbf{Calg}$ and $\mathbf{Lie}$ be the categories of algebras, commutative algebras and Lie-algebras over R respectively. We obtain theories of cogroups as in the following table:

$\mathbf{S}$	$\mathbf{free}(\mathbf{S})$	$\mathbf{model}(\mathbf{S})$
$\mathbf{gr}^\sharp$	$\mathbf{gr}$	$\mathbf{Gr}$
$\mathbf{nil}_n^\sharp$	$\mathbf{nil}_n$	$\mathbf{Nil}_n,\ n \geq 1$
$\mathbf{ab}^\sharp$	$\mathbf{ab}$	$\mathbf{Ab}$
$\mathbf{var}^\sharp$	$\mathbf{var}$	$\mathbf{Var}$
$\mathbf{alg}^\sharp$	$\mathbf{alg}$	$\mathbf{Alg}$
$\mathbf{calg}^\sharp$	$\mathbf{calg}$	$\mathbf{Calg}$
$\mathbf{lie}^\sharp$	$\mathbf{lie}$	$\mathbf{Lie}$

The column in the middle of the table denotes the full subcategories of free objects and the column on the left hand side denotes the full subcategories of finitely generated free objects. Hence $\mathbf{gr}, \mathbf{nil}_n, \mathbf{ab}, \mathbf{var}, \mathbf{alg}, \mathbf{calg}$ and $\mathbf{lie}$ are examples of theories of cogroups. Theories of cogroups are considerably more general than varieties of groups.

(2.7) Example. The following modification of the examples in (2.6) was pointed out to me by M. Jibladze.

Let us fix a Grothendieck topos $\mathbf{E}$, with an explicit cite of definition (C, J). This means that C is a small category, J is a Grothendieck topology on it, and $\mathbf{E}$ is $\mathrm{Sh}(C, J)$, the category of set-valued sheaves on the site (C, J). Each object $c \in C$ determines a sheaf in $\mathbf{E}$ which we also denote by c, namely, the associated sheaf of the representable presheaf $\hom_C(-, c) : C^{\mathrm{op}} \to \mathbf{Sets}$. The particular important case is determined by a topological space X, when C is the poset of open sets of X considered as a category in the usual way, and J is the canonical topology (a family of opens covers its union); in this case $\mathbf{E}$ is $\mathrm{Sh}(X)$, the category of sheaves on X.

Now for any single sorted theory $\mathbf{T}$, one can consider the notion of a *sheaf of models of* $\mathbf{T}$ in $\mathbf{E}$. This is nothing else than a model of $\mathbf{T}$ in $\mathbf{E}$, i.e. a product preserving functor $\mathbf{T}^{\mathrm{op}} \to \mathbf{E}$. We denote the category of such models by $\mathbf{model}(\mathbf{T}, \mathbf{E})$. Take in particular a single sorted theory $\mathbf{S}$ of cogroups with zero object. Then one knows (see for example Johnstone [TT]) that the category $\mathbf{model}(\mathbf{S}, \mathbf{E})$ shares many good properties with the category of ordinary models: it is an exact category, is complete and cocomplete, and the forgetful functor $\mathbf{model}(\mathbf{S}, \mathbf{E}) \to \mathbf{E}$ is monadic; in particular it has a left adjoint $\mathrm{Free} : \mathbf{E} \to \mathbf{model}(\mathbf{S}, \mathbf{E})$.

Now for any sheaf $I \in \mathbf{E}, \mathrm{Free}(I)$ has the corresponding universal property: the functor $\hom_{\mathbf{mod}(\mathbf{S},\mathbf{E})}(\mathrm{Free}(I), -)$ is isomorphic to the functor $\hom_{\mathbf{E}}(I, -)$. But since $\mathbf{S}$ is a theory of cogroups, this latter functor factors through the category of groups. This means that $\mathrm{Free}(I)$ has a cogroup structure. Explicitly, the comultiplication can be dsecribed as follows: take the morphism

$$(j \circ \iota_1, j \circ \iota_2) : I \to \mathrm{Free}(I \amalg I) \times \mathrm{Free}(I \amalg I),$$

where $j : I \amalg I \to \mathrm{Free}(I \amalg I)$ is the adjunction unit, and $\iota_1, \iota_2 : I \to I \amalg I$ are coproduct inclusions. Compose the map above with the group multiplication $\mathrm{Free}(I \amalg I) \times \mathrm{Free}(I \amalg I) \to \mathrm{Free}(I \amalg I)$. Finally observe that since Free is a left adjoint, it preserves coproducts, so $\mathrm{Free}(I \amalg I)$ is isomorphic to $\mathrm{Free}(I) \vee \mathrm{Free}(I)$. So we obtain a map $I \to \mathrm{Free}(I) \vee \mathrm{Free}(I)$ in $\mathbf{E}$, which then by adjointness extends uniquely to a morphism $\mathrm{Free}(I) \to \mathrm{Free}(I) \vee \mathrm{Free}(I)$ in $\mathbf{model}(\mathbf{S}, \mathbf{E})$. This is the desired cogroup comultiplication.

It follows that the full subcategory of $\mathbf{model}(\mathbf{S}, \mathbf{E})$ on the image of the functor Free is a theory of cogroups. We can take a smaller category, namely its smallest subcategory containing all the $\mathrm{Free}(c)$, for $c \in C$, and closed under coproducts. Equivalently, it may be described as the full subcategory of $\mathbf{model}(\mathbf{S}, \mathbf{E})$ with objects of the form $\mathrm{Free}(\coprod_i c_i)$, for all families (c_i) of objects of C. This is again a theory of cogroups.

(2.8) Example. Let D be a topological space and let $\mathbf{Top}^D$ be the category of spaces under D; see (2.2). Two maps in $\mathbf{Top}^D$ are homotopic if they are homotopic relative D. We define full subcategories

$$\mathbf{susp}(D, \mathcal{D}) \subset \mathbf{cone}(D, \mathcal{D}) \subset \mathbf{Top}^D/\simeq \mathrm{rel}\, D$$

of the homotopy category where $\mathcal{D}$ is the class of discrete sets in **Top**. Here $\mathbf{susp}(D,\mathcal{D})$ is a theory of cogroups and $\mathbf{cone}(D,\mathcal{D})$ is a theory of coactions. These examples are used in chapter A, § 1.

The objects S^1_α of $\mathbf{susp}(D,\mathcal{D})$ are obtained by push out diagrams in **Top**

$$\begin{tikzcd} S^1 \times E \arrow[r] & S^1_\alpha \\ E \arrow[u, "i"] \arrow[r, "\alpha"] & D \arrow[u] \end{tikzcd}$$

where E is a discrete set. The inclusion i carries $e \in E$ to the pair $(*, e)$ in the product $S^1 \times E$ where $*$ is the basepoint in the 1-sphere S^1. We call S^1_α the 1-dimensional spherical object obtained by the function α. The cogroup structure $\mu : S^1 \to S^1 \vee S^1$ in $\mathbf{Top}^*/\simeq \mathrm{rel} *$ induces a cogroup structure $\mu : S^1_\alpha \to S^1_\alpha \cup_D S^1_\alpha$ of S^1_α in the homotopy category $\mathbf{Top}^D/\simeq \mathrm{rel}\, D$. Hence the homotopy category $\mathbf{susp}(D,\mathcal{D})$ consisting of all 1-dimensional spherical objects S^1_α as above is a theory of cogroups.

The objects $C_{\alpha,\beta}$ of $\mathbf{cone}(D,\mathcal{D})$ are obtained by push out diagrams in **Top**

$$\begin{tikzcd} I \times E \arrow[r] & C_{\alpha,\beta} \\ E \dot\cup E \arrow[u, "(i_0,i_1)"] \arrow[r, "(\alpha,\beta)"] & D \arrow[u] \end{tikzcd}$$

Here E is again a discrete set and $I = [0,1]$ is the unit interval. Moreover $E \dot\cup E$ is the disjoint union with $i_\varepsilon(e) = (\varepsilon, e)$ for $e \in E$, $\varepsilon \in \{0,1\}$. Hence $C_{\alpha,\beta}$ is obtained by attaching one cells to D. The pair $(C_{\alpha,\beta}, D)$ is the same as a 1-dimensional relative CW-complex with trivial 0-skeleton. We obtain a canonical coaction map

$$\mu : C_{\alpha,\beta} \to C_{\alpha,\beta} \cup_D S^1_\beta$$

which is induced by the map $\mu : I \to I \cup_{\{1\}} S^1$ which is the addition of the path from $\{0\}$ to $\{1\}$ in I and the path $I \to I/\{0,1\} = S^1$. Hence the full homotopy category $\mathbf{cone}(D,\mathcal{D})$ consisting of all objects $C_{\alpha,\beta}$ as above is a theory of coactions.

(2.9) Proposition. *Let D be a path connected space with fundamental group $\pi_1(D) = G$. Then there are equivalences of theories*

$$\mathbf{gr}(G) \xrightarrow{\sim} \mathbf{susp}(D,\mathcal{D}) \xrightarrow{\sim} \mathbf{cone}(D,\mathcal{D})$$

where $\mathbf{gr}(G)$ *is the theory in (2.3).*

It is, however, more intricate to describe for any space D the theory $\mathbf{cone}(D,\mathcal{D})$ purely algebraically. For this we generalize example (2.3) from groups to groupoids.

(2.10) Example. Let **Grd** be the category of groupoids and let $G \in \mathbf{Grd}$. A *free groupoid under G* is an inclusion $G \to X$ of groupoids which is the identity on

objects such that there exists a set $E \subset \mathrm{Mor}(X)$ of non-identity maps (called a *basis*) which has the following property. Every non-identity map of X can uniquely be written as a reduced composition of maps in E, their inverses and non-identity maps in G. Here *reduced* means that no map of E appears next to its inverse and that no two non-identity maps of G appear next to each other. If G consists only of identity maps then we call X a *free* groupoid. Let $\mathbf{grd}(G)$ be the full subcategory of $\mathbf{Grd}^G$ consisting of free groupoids X under G. Free groupoids are also considered in Dwyer-Kan [SG].

For any pair of spaces (X, D) we obtain the *fundamental groupoid* $\Pi(X, D)$. Objects are the points of D and morphisms are the homotopy classes of paths in X between two points in D. Hence $\Pi(D) = \Pi(D, D)$ is the usual fundamental groupoid of the space D. If $X \in \mathbf{cone}(D, \mathcal{D})$ then $\Pi(X, D)$ is a free groupoid under $\Pi(D)$. A basis of $\Pi(X, D)$ is given by the set of oriented 1-cells in $X - D$. In particular if D is discrete then $\Pi(X, D)$ is a free groupoid.

(2.11) Proposition. *Let G be a groupoid. Then the category* $\mathbf{grd}(G)$ *is a theory of coactions in such a way that for a space D and $G = \Pi(D)$ there is a canonical isomorphism of theories*

$$\mathbf{cone}(D, \mathcal{D}) = \mathbf{grd}(G).$$

3 The Category of Twisted Maps

We introduce for each theory of coactions $\mathbf{T}$ the category $\mathbf{Twist}$ of twisted maps which will be used in the next section for the definition of the category $\mathbf{Coef}$ of coefficients. We also introduce the difference operator ∇ which we apply to morphisms in $\mathbf{Twist}$. This operator is of crucial importance in the whole book.

Let A, B be based objects in the theory $\mathbf{T}$. We say that a map $f : A \to B \vee Y$ is *trivial on* Y if the composite $(0,1)f : A \to B \vee Y \to Y$ is the zero map for A; i.e. the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \vee Y \arrow[d, "(0,1)"] \\ & Y \end{tikzcd} \tag{3.1}$$

Let $\mathbf{T}(A, B \vee Y)_2$ be the set of all such maps f which are trivial on Y. Let A be a cogroup in a theory $\mathbf{T}$ and let X be a coaction in $\mathbf{T}$. Then we associate with each map $f : A \to Y$ in $\mathbf{T}$ the *difference map*

$$\nabla f : A \to Y' \vee Y \tag{3.2}$$

in $\mathbf{T}$ defined by

$$\nabla f = -i_Y f + (i_Y + i_{Y'}) f$$

One readily checks that for $g : Y \to Z$ and $\beta : Y' \to Z$ in $\mathbf{T}$ one has

$$(g+\beta)f = gf + (\beta, g)(\nabla f)$$

Moreover ∇f is a map trivial on Y. If $\mathbf{T}$ is a theory of coactions we also define for each map $f : X \to Y$ in $\mathbf{T}$ the *difference map*

$$\nabla f : X' \to Y' \vee Y \tag{3.3}$$

by the equation

$$i_Y f + \nabla f = (i_Y + i_{Y'})f$$

in the set $\mathrm{Mor}(X, Y' \vee Y)$. Here we use the affine property of $\mathbf{T}$. The difference map ∇f is the unique map for which the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X + i_{X'}"'] & Y \arrow[d, "i_Y + i_{Y'}"] \\ X' \vee X \arrow[r, "(\nabla f, i_Y f)"] & Y' \vee Y \end{tikzcd}$$

Here $i_X + i_{X'}$ is up to an interchange of summands the same as the coaction map μ on X. Clearly we get for $g : Y \to Z$ and $\beta : Y' \to Z$ again the equation

$$(g+\beta)f = gf + (\beta, g) \nabla f \tag{1}$$

and ∇f is trivial on Y. This implies for $\alpha, \beta : Y' \to Z$ the equation

$$(\alpha+\beta, g) \nabla f = (\alpha, g) \nabla f + (\beta, g+\alpha) \nabla f \tag{2}$$

since we have

$$\begin{aligned} (g+\alpha+\beta)f &= gf + (\alpha+\beta, g) \nabla f \\ &= (g+\alpha)f + (\beta, g+\alpha) \nabla f \\ &= gf + (\alpha, g) \nabla f + (\beta, g+\alpha) \nabla f \end{aligned}$$

Then the affine property of $\mathbf{T}$ yields (2).

(3.4) Lemma. *Let $\mathbf{T}$ be a theory of coactions. Then a composite $gf : X \to Y \to Z$ in $\mathbf{T}$ satisfies*

$$\nabla(gf) = (\nabla g, i_Z g) \nabla f : X' \to Y' \vee Y \to Z' \vee Z$$

Moreover for $\alpha : X' \to Y$ we have

$$\nabla(f+\alpha) = -i_Y\alpha + \nabla f + i_Y\alpha + \nabla\alpha.$$

The first equation in (3.4) shows that the isomorphism in (1.12) is natural is the sense that the following diagram commutes

$$\begin{array}{ccc} X \vee X & \xrightarrow{f \vee f} & Y \vee Y \\ \| & & \| \\ X' \vee X & \xrightarrow{(\nabla f, i_Y f)} & Y' \vee Y \end{array}$$

Proof. We have

$$\begin{aligned} i_Z gf + \nabla(gf) &= (i_Z + i_{Z'})gf \\ &= (i_Z g + \nabla g)f \\ &= i_Z gf + (\nabla g, i_Z g) \nabla f \end{aligned}$$

The second equation in (3.4) is proved as follows. For $h = f + \alpha : X \to Y$ we get

$$i_Y h + \nabla h = (i_Y + i_{Y'})h$$

and hence

$$i_Y f + i_Y \alpha + \nabla h = (i_Y + i_{Y'})f + (i_Y + i_{Y'})\alpha$$

where $i_Y f = (i_Y + i_{Y'})f - \nabla f$. Now affineness implies

$$- \nabla f + i_Y \alpha + \nabla h = (i_Y + i_{Y'})\alpha = i_Y \alpha + \nabla \alpha$$

or equivalently the second equation in (3.4). q.e.d.

(3.5) Definition. Let $\mathbf{T}$ be a theory of coactions. Then we define *the category* **Twist** *of twisted maps* in $\mathbf{T}$ as follows. Objects are maps $\partial_X : X'' \to X$ in $\mathbf{T}$ where X'' is a cogroup in $\mathbf{T}$. Morphisms $(f'', f) : \partial_X \to \partial_Y$ are given by commutative diagrams

$$\begin{array}{ccc} X'' & \xrightarrow{f''} & Y'' \vee Y \\ {\scriptstyle \partial_X}\downarrow & & \downarrow{\scriptstyle (\partial_Y, 1)} \\ X & \xrightarrow{f} & Y \end{array}$$

where f'' is trivial on Y. Composition is defined by

$$(f'', f)(g'', g) = (\bar{f} g'', fg)$$

where $\bar{f} : X'' \vee X \to Y'' \vee Y$ is given by $(f'', i_Y f)$. One readily checks that this is a well defined category. In fact an alternative description of the morphisms $\partial_X \to \partial_Y$ of **Twist** is given by pairs of commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X'' \vee X & \xrightarrow{\bar{f}} & Y'' \vee Y \\ {\scriptstyle (0,1)}\downarrow & & \downarrow{\scriptstyle (0,1)} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} X'' \vee X & \xrightarrow{\bar{f}} & Y'' \vee Y \\ {\scriptstyle (\partial_X,1)}\downarrow & & \downarrow{\scriptstyle (\partial_Y,1)} \\ X & \xrightarrow{f} & Y \end{array}$$

with the composition given by horizontal composition of these diagrams. We say that $f : X \to Y$ is *∂-compatible* if there exists f'' as above such that $(f'', f) : \partial_X \to \partial_Y$ is a morphism in **Twist**.

The category **Twist** has sums defined by the sum of maps

$$\partial_X \vee \partial_Y : X'' \vee Y'' \to X \vee Y$$

in **T**. The initial object in **Twist** is the identity $1 : * \to *$ where $*$ is the initial object in **T** which is a cogroup in **T**. We obtain a full embedding of categories

$$\mathbf{T} \subset \mathbf{Twist} \tag{3.6}$$

which carries X to the object $* \to X$ in **Twist** which is also denoted by X.

(3.7) Definition. A *simplicial 1-diagram* or a *graph* in a category **T** is a diagram

$$X_0 \xrightarrow{s_0} X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0$$

in **T** with $d_0 s_0 = d_1 s_0 = \mathrm{id}$. This is the 1-dimensional part of a simplicial object in **T**. Each object ∂_X in **Twist** yields a simplicial 1-diagram

$$X \xrightarrow{i_X} X'' \vee X \underset{(\partial_X, 1)}{\overset{(0,1)}{\rightrightarrows}} X$$

in **T** so that **Twist** is a full subcategory of the category of simplicial 1-diagrams in **T**. Compare chapter B, § 2 where we consider free simplicial objects in **T** which generalize the presentations considered in **Twist**; see (B.2.30).

An *action of a group* G on a group M is given by a homomorphism from the opposite group G^{op} to the group of automorphisms of M in the category **Gr**. For $g \in G$ and $m \in M$ we denote the action by m^g. Let $\varphi : G' \to G$ be a homomorphism between groups. Then a function $h : G' \to M$ is a *φ-crossed* homomorphism if for $x, y \in G'$

$$h(x \cdot y) = h(x)^{\varphi(y)} \cdot h(y)$$

holds. A *pre-crossed module*

$$\partial : M \to G \tag{3.8}$$

is a homomorphism of groups together with an action of G on M such that

$$\partial(m^g) = g^{-1} \partial(m) g \tag{1}$$

A morphism between pre-crossed modules $(\xi, \eta) : \partial \to \partial'$ is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\xi} & M' \\ {\scriptstyle\partial}\big\downarrow & & \big\downarrow{\scriptstyle\partial'} \\ G & \xrightarrow{\eta} & G' \end{array} \tag{2}$$

in the category of groups such that ξ is *η-equivariant*, that is $\xi(m^g) = \xi(m)^{\eta g}$. The image of ∂ is a normal subgroup of G so that the quotient group $G/\partial M$ and the exact sequence

$$M \xrightarrow{\partial} G \xrightarrow{q} G/\partial M \longrightarrow 0 \tag{3}$$

in **Gr** are defined. A pre-crossed module is termed a *crossed module* if in addition to (1) the following equation is satisfied with $m, n \in M$

$$m^{\partial n} = n^{-1} m\, n \tag{4}$$

Each pre-crossed module ∂ defines an *associated crossed module*

$$\partial^{\mathrm{cr}} : M^{\mathrm{cr}} \to G \tag{5}$$

Here M^{cr} is the quotient group M/P where P is the subgroup of M generated by all *Peiffer commutators*

$$\langle m, n\rangle = m^{-1} n^{-1} m (n^{\partial m}) \in M \tag{6}$$

One can check that P is a normal subgroup of M. If $\partial : M \to G$ is a crossed module then kernel(∂) is abelian and the action of G on M induces an action of $G/\partial M$ on kernel(∂).

(3.9) Remark. Each simplicial 1-diagram in the category of groups yields a pre-crossed module

$$d_1 : \mathrm{kernel}(d_0) \to X_0$$

Here the action of $g \in X_0$ on $m \in \mathrm{kernel}(d_0)$ is defined by

$$m^g = s_0(g)^{-1} m\, s_0(g)$$

where the right hand side is defined in the group X_1. It is well known that this construction yields an equivalence of categories between the category of simplicial 1-diagrams in **Gr** and the category of pre-crossed modules.

If A and B are cogroups in $\mathbf{T}$ then the group $\mathbf{T}(A, X)$ acts on the group $\mathbf{T}(A, B \vee X)_2$ by setting

$$f^a = -i_X a + f + i_X a \tag{3.10}$$

where the right hand side is defined in the group $\mathbf{T}(A, B \vee X)$. Moreover if $\partial_X : X'' \to X$ is an object in **Twist** we obtain the pre-crossed module

$$(\partial_X, 1)_* : \mathbf{T}(A, X'' \vee X)_2 \to \mathbf{T}(A, X) \tag{3.11}$$

by applying the functor $\mathbf{T}(A, -)$ to the simplicial 1-diagram in (3.7); compare (3.9). Hence the image of $(\partial_X, 1)_*$ in (3.11) is a normal subgroup.

We now apply the difference operator $\bigtriangledown$ to morphisms in **Twist**

(3.12) Lemma. *Let $(f'', f) : \partial_X \to \partial_Y$ be a morphism in* **Twist** *and let $Y_1'' = Y_2'' = Y''$. Consider the composite*

$$X'' \xrightarrow{\xi} Y_2'' \vee Y' \vee Y_1'' \vee Y \xrightarrow{q} Y' \vee Y$$

with $q = (\bigtriangledown\partial_Y, i_{Y'}, \partial_Y, i_Y)$. There exists a map ξ which is trivial on $Y_2'' \vee Y, Y' \vee Y$ and $Y_1'' \vee Y$ such that

$$(\bigtriangledown f, i_Y f) \bigtriangledown \partial_X - (\bigtriangledown\partial_Y, i_Y) f'' = q\xi.$$

Here the left hand side is given by the composites in the diagram

$$\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "\bigtriangledown\partial_X"'] & Y'' \vee Y \arrow[d, "(\bigtriangledown\partial_Y, i_Y)"] \\
X' \vee X \arrow[r, "(\bigtriangledown f, i_Y f)"] & Y' \vee Y
\end{tikzcd}$$

Proof. We have $(\partial_Y, 1)f'' = f\partial_X$. This implies that $\bigtriangledown(f\partial_X) = \bigtriangledown((\partial_Y, 1)f'')$ where

$$\begin{aligned} \bigtriangledown(f\partial_X) &= (\bigtriangledown f, i_Y f) \bigtriangledown \partial_X \\ \bigtriangledown((\partial_Y, 1)f'') &= (\bigtriangledown(\partial_Y, 1), i_Y(\partial_Y, 1)) \bigtriangledown f'' = q(\bigtriangledown f'') \end{aligned} \tag{1}$$

since $\bigtriangledown(\partial_Y, 1) = (\bigtriangledown\partial_Y, \bigtriangledown 1_Y) = (\bigtriangledown\partial_Y, i_{Y'})$. Here $\bigtriangledown f''$ is a map

$$X'' \to (Y_2'' \vee Y') \vee (Y_1'' \vee Y).$$

We define ξ by

$$\xi = \bigtriangledown f'' - (i_{Y_2''}, i_Y) f''. \tag{2}$$

Hence we get

$$\begin{aligned} q\xi &= q \bigtriangledown f'' - q(i_{Y_2''}, i_Y) f'' \\ &= q \bigtriangledown f'' - (\bigtriangledown\partial_Y, i_Y) f'' \\ &= (\bigtriangledown f(i_Y f) \bigtriangledown \partial_X - (\bigtriangledown\partial_Y, i_Y) f'' \end{aligned}$$

We now check that ξ is trivial one $Y_2'' \vee Y, Y' \vee Y$ and $Y_1'' \vee Y$. In fact, f'' is trivial on Y, that is $(0,1)f'' = 0$. We now replace in (1) the map ∂_Y by $0 : Y'' \to Y$ so that we get

$$0 = \bigtriangledown 0 = \bigtriangledown((0,1)f'') = q'(\bigtriangledown f'')$$

where $q' = (0, i_{Y'}, 0, i_Y)$. Hence $\bigtriangledown f''$ is trivial on $Y' \vee Y$. This implies that ξ is trivial on $Y' \vee Y$. Moreover by definition of $\bigtriangledown f''$ we know that $\bigtriangledown f''$ is trivial on $Y_1'' \vee Y$ so that also ξ is trivial on $Y_1'' \vee Y$. Finally we get for $q'' = (i_{Y_2''}, 0, 0, i_Y)$

$$\begin{aligned} q'' \bigtriangledown f'' &= q''(-(i_{Y_1''}, i_Y)f'' + (i_{Y_1''}, i_Y)f'' + (i_{Y_1''} + i_{Y_2''}, i_Y + i_{Y'})f'' \\ &= (0,1)f'' + (i_{Y_2''}, i_Y)f'' \\ &= (i_{Y_2''}, i_Y)f'' \end{aligned}$$

so that $q''\xi = 0$ and hence ξ is trivial on $Y_2'' \vee Y$. q.e.d.

4 The Category of Coefficients

Given a theory of coactions **T** we define the category of coefficients **Coef** by use of twisted maps. In fact, if **T** is the theory of free groups then **Coef** is equivalent to the category of groups.

(4.1) Definition. Let **T** be a theory of coactions. We define the *category* **Coef** *of coefficients* as follows. Objects are the same as in **Twist**, namely maps $\partial_X : X'' \to X$ in **T** where X'' is a cogroup. We say that two maps $f, f_1 : X \to Y$ are ∂ *-equivalent*, $f \sim f_1$, if there exists $\alpha : X' \to Y'' \vee Y$ trivial on Y with

$$f_1 = f + (\partial_Y, 1)\alpha$$

A morphism $\{f\} : \partial_X \to \partial_Y$ in **Coef** is the ∂-equivalence class of a ∂-compatible map $f : X \to Y$; see (3.5). Composition is defined by $\{f\}\{g\} = \{fg\}$. An object ∂_X in **Coef** is also termed a *presentation.* We also write **Coef** = **Coef**(**T**).

The next lemma shows that **Coef** is a well defined category. Moreover the category **Coef** has sums $\partial_X \vee \partial_Y$ and one has the full inclusion of categories

$$\mathbf{T} \subset \mathbf{Coef} \tag{1}$$

which carries X to the object $* \to X$ also denoted by X. In particular we have for a cogroup A in **T** the set of morphisms $\mathbf{Coef}(A, \partial_X)$ in the category **Coef**. This set is actually a group and it is clear by the definition of ∂-equivalences that one obtains an exact sequence of groups

$$\mathbf{T}(A, X'' \vee X)_2 \xrightarrow{(\partial_X,1)_*} \mathbf{T}(A, X) \xrightarrow{j} \mathbf{Coef}(A, \partial_X) \longrightarrow 0 \tag{4.2}$$

where $(\partial_X, 1)_*$ is the pre-crossed module in (3.11).

Example. If $\mathbf{T} = \mathbf{gr}$ is the theory of free groups then $\mathbf{Coef}(\mathbf{gr})$ is easily seen to be equivalent to the category **Gr** of groups. The equivalence

$$\pi_1 : \mathbf{Coef}(\mathbf{gr}) \xrightarrow{\sim} \mathbf{Gr} \tag{1}$$

carries ∂_X to the group $\mathbf{Coef}(A, \partial_X) = G$ where $A = \mathbb{Z} \in \mathbf{gr}$. The exact sequence (4.2) with $A = \mathbb{Z}$ describes a *presentation* of the group G. Hence $\mathbf{Coef}(\mathbf{gr})$ is precisely the category of groups G for which G has a fixed presentation ∂_X. The inclusion (4.1) (1) corresponds to $\mathbf{gr} \subset \mathbf{Gr}$. The equivalence π_1 above is generalized for theories in (4.6) below.

(4.3) Lemma. *Let* $\mathbf{T}$ *be a theory of coactions. If f is ∂-equivalent to f_1 and if f is ∂-compatible then f_1 is ∂-compatible. Moreover ∂-equivalence is a natural equivalence relation on the category consisting of ∂-compatible maps.*

Proof. For $f_1 = f + (\partial_Y, 1)\alpha$ let

$$f_1'' = f'' + (\alpha, i_Y\eta) \bigtriangledown \partial_X.$$

Then we get for $\beta = (\partial_Y, 1)\alpha$ the equations

$$\begin{aligned} f_1\partial_X &= (f+\beta)\partial_X = f\partial_X + (\beta, f) \bigtriangledown \partial_X \\ &= (\partial_Y, 1)f'' + (\partial_Y, 1)(\alpha, i_Y f) \bigtriangledown \partial_X \\ &= (\partial_Y, 1)(f'' + (\alpha, i_Y f) \bigtriangledown \partial_X) = (\partial_Y, 1)(f_1'') \end{aligned}$$

so that the first proposition of the lemma is proved. Now let $h : Y \to Z$ and $g : V \to X$ be ∂-compatible maps and $f \sim f_1$. Then we show $hf \sim hf_1$ and $fg \sim f_1 g$. In fact, we have

$$\begin{aligned} hf_1 &= h(f+\beta) = hf + h\beta \\ &= hf + h(\partial_Y, 1)\alpha \\ &= hf + (\partial_Z, 1)\bar{h}\,\alpha \end{aligned}$$

where $\bar{h} : Y'' \vee Y \to Z'' \vee Z$ is the map given by $(h'', h) : \partial_Y \to \partial_Z$ in $\mathbf{Twist}(\mathbf{T})$, that is $\bar{h} = (h'', i_Z h)$. On the other hand we have

$$\begin{aligned} f_1 g &= (f+\beta)g = fg + (\beta, f) \bigtriangledown g \\ &= fg + (\partial_Y, 1)(\alpha, 1_Y f) \bigtriangledown g \end{aligned}$$

Here $\bigtriangledown g$ is defined since we assume that $\mathbf{T}$ satisfies the affine property. q.e.d.

We now compare the category **Coef** associated to $\mathbf{T}$ with the category **model** of models of $\mathbf{T}$. We include the following lemmas (4.4) and (4.5) and the remark (4.6) since in many cases the category **Coef** actually coincides with the category of models.

(4.4) Lemma. *Let* $\mathbf{T}$ *be a theory of coactions and let* $M : \mathbf{T}^{\mathrm{op}} \to \mathbf{Set}$ *be a model of* $\mathbf{T}$. *Then M yields a well defined functor*

$$M_\sharp : \mathbf{Coef}^{\mathrm{op}} \to \mathbf{Set}$$

defined by $M_\sharp(\partial_X) = \mathrm{kernel}(\partial_X^* : M(X) \to M(X''))$. *Here $M(X'')$ is a group so that* $\mathrm{kernel}(\partial_X^*)$ *is defined by the set* $(\partial_X^*)^{-1}(0)$.

Proof. Let $f : X \to Y$ be ∂-compatible then we have $f\partial_X = (\partial_Y, 1)f''$ where f'' is trivial on Y. Hence for $y \in M_\sharp(\partial_Y)$ we have $\partial_Y^*(y) = 0$. We claim that $f^* : M(Y) \to M(X)$ induces a map

$$f^* : M_\sharp(\partial_Y) \to M_\sharp(\partial_X)$$

In fact, for f^*y we get

$$\begin{aligned}\partial_X^* f^*y &= (f\partial_X)^*y \\ &= ((\partial_Y, 1)f'')^*y \\ &= (f'')^*(\partial_Y, 1)^*y \\ &= (f'')^*(0, y) = (f'')^*(0, 1)^*y = ((0, 1)f'')^*y = 0^*y = 0\end{aligned}$$

Now assume $f \sim f_1$, that is $f_1 = f + (\partial_Y, 1)\alpha$ where α is trivial on Y. Then we get

$$\begin{aligned}f_1^*(y) &= (f + (\partial_Y, 1)\alpha)^*y \\ &= f^*y + ((\partial_Y, 1)\alpha)^*y \\ &= f^*y + \alpha^*(\partial_Y, 1)^*y \\ &= f^*y + \alpha^*(0, y) = f^*y\end{aligned}$$

since $\alpha^*(0, y) = \alpha^*(0, 1)^*y = ((0, 1)\alpha)^*y = 0^*y = 0$. q.e.d.

(4.5) Lemma. *Let* $\mathbf{T}$ *be a small theory of cogroups and let* **model** *be the category of models of* $\mathbf{T}$*. Then we obtain a well defined functor*

$$\pi_1 : \mathbf{Coef} \to \mathbf{model}$$

by the cokernel in the category of groups

$$\pi_1(\partial_X)(A) = \mathrm{cokernel}(\mathbf{T}(A, X'' \vee X)_2 \xrightarrow{(\partial_X, 1)_*} \mathbf{T}(A, X))$$

where A *is an object of* $\mathbf{T}$*. Moreover for* $M \in$ **model** *the functor* $M_\sharp$ *in (4.4) can be described by a canonical isomorphism*

$$M_\sharp(\partial_X) = \mathbf{model}\,(\pi_1(\partial_X), M)$$

Proof. It is clear that a ∂-compatible map $f : X \to Y$ induces a well defined map f_* for which the following diagram commutes

$$\begin{array}{ccc}\mathbf{T}(A, X) & \xrightarrow{f_*} & \mathbf{T}(A, Y) \\ \downarrow & & \downarrow \\ \pi_1(\partial_X)(A) & \xrightarrow{f_*} & \pi_1(\partial_Y)(A)\end{array}$$

The vertical arrows are the quotient maps. In fact $f_*(\partial_X, 1)_* = (\partial_Y, 1)_* \bar{f}_*$ where $\bar{f}_*$ carries an element in $\mathbf{T}(A, X'' \vee X)_2$ to an element in $\mathbf{T}(A, Y'' \vee Y)_2$ since f''

is trivial on Y. Clearly f_* is natural in A. Now let $f \sim f_1$ with $f_1 = f + (\partial_Y, 1)\alpha$. Then we have for $g \in \mathbf{T}(A, X)$ the equation

$$f_1 g = fg + (\partial_Y, 1)_* \delta$$

with $\delta = (\alpha, i_Y f) \bigtriangledown g$ trivial on Y. Hence $f_1 g$ and fg represent the same element in $\pi_1(\partial_Y)(A)$. This finishes the proof that π_1 is a well defined functor.

Next let $t \in \mathbf{model}(\pi_1(\partial_X), M)$ then we obtain for $\{1_X\} \in \pi_n(\partial_X)(X)$ the element $t\{1_X\} \in M(X)$ with $\partial_X^* t\{1_X\} = 0$ so that $t\{1_X\} \in M_\sharp(\partial_X)$. In fact $\partial_X^* t\{1_X\} = t\,\partial_X^*\{1_X\} = t\{\partial_X\} = 0$ since $\{\partial_X\} = 0$ in $\pi_1(\partial_X)(X'')$. Therefore $t \mapsto t\{1_X\}$ is a well defined function. The inverse of this function carries $x \in M_\sharp(\partial_X) \subset M(X)$ to the natural transformation $t_X : \pi_1(\partial_X) \to M$ defined by $t_X\{g\} = g^*\{X\}$ with $g \in \mathbf{T}(A, X)$. One readily checks that t_X is well defined.

q.e.d.

(4.6) Remark. Let $\mathbf{S}$ be a *single sorted* theory and let $\mathbf{free}(\mathbf{S})$ be the full subcategory of $\mathbf{model}(\mathbf{S})$ consisting of free models. If $\mathbf{S}$ is a theory of cogroups then so is $\mathbf{T} = \mathbf{free}(\mathbf{S})$ and one obtains an equivalence of categories

$$\pi_1 : \mathbf{Coef}(\mathbf{T}) \xrightarrow{\sim} \mathbf{model}(\mathbf{S})$$

The functor π_1 is defined as in (4.5) by restricting to objects A in $\mathbf{S}$. The equivalence of categories π_1 yields for the examples in (2.12) the following list:

$\mathbf{T}$	$\mathbf{Coef}$
$\mathbf{gr}$	$\mathbf{Gr}$
$\mathbf{nil}_n$	$\mathbf{Nil}_n$
$\mathbf{ab}$	$\mathbf{Ab}$
$\mathbf{var}$	$\mathbf{Var}$
$\mathbf{alg}$	$\mathbf{Alg}$
$\mathbf{calg}$	$\mathbf{Calg}$
$\mathbf{lie}$	$\mathbf{Lie}$

In the next proposition we use the inclusion $\mathbf{T} \subset \mathbf{Coef}$ in (4.1) (1). Moreover $\mathbf{Coef}(Z, \partial_X)$ denotes the set of morphisms $Z \to \partial_X$ in $\mathbf{Coef}$.

(4.7) Proposition. *A map* $\{f\} : \partial_X \to \partial_Y$ *in* $\mathbf{Coef}$ *is an equivalence in* $\mathbf{Coef}$ *if and only if for all objects* Z *in* $\mathbf{T}$ *the induced map*

$$f_* : \mathbf{Coef}(Z, \partial_X) \to \mathbf{Coef}(Z, \partial_Y)$$

is a bijection.

Proof. Since $f_* : \mathbf{Coef}(Y, \partial_X) \to \mathbf{Coef}(Y, \partial_Y)$ is surjective there exists $g : Y \to X$ in $\mathbf{T}$ such that $f_*\{g\} = \{1_Y\}$. The following diagram commutes

$$\begin{array}{ccc}
\mathbf{Coef}(Y, \partial_X) & \xrightarrow{f_*} & \mathbf{Coef}(Y, \partial_Y) \\
{\scriptstyle \partial_Y^*}\downarrow & & \downarrow{\scriptstyle \partial_Y^*} \\
\mathbf{Coef}(Y'', \partial_X) & \xrightarrow{f_*} & \mathbf{Coef}(Y'', \partial_Y)
\end{array}$$

Here we have $\partial_Y^*\{1_Y\} = \{0\}$. Hence by the injectivity of f_* in the bottom row we get $\partial_Y^*\{g\} = \{0\}$. This shows that g is a ∂-compatible $\partial_Y \to \partial_X$ map so that $\{g\} \in \mathbf{Coef}(\partial_Y, \partial_X)$ is well defined with $\{f\}\{g\} = \{fg\} = 1_{\partial_Y}$. We now use the injectivity of

$$f_* : \mathbf{Coef}(X, \partial_X) \to \mathbf{Coef}(X, \partial_Y)$$

which shows by

$$f_*\{gf\} = \{fgf\} = \{f\} = f_*\{1_X\}$$

that $\{gf\} = \{1_X\}$ in $\mathbf{Coef}(X, \partial_X)$. This shows that also $\{g\}\{f\} = 1_{\partial_X}$.

q.e.d.

5 Enveloping Functors and the Categories of Premodules and Modules

We deduce from the category **Coef** of coefficients the category of premodules and the category of modules.

(5.1) Definition. Let $\mathbf{T}$ be a theory of coactions. We define *the category* **premod** *of premodules* as follows. Objects are sums $A \vee \partial_X$ in **Coef** where A is a cogroup in $\mathbf{T}$ and where ∂_X is an object in **Coef**; that is $A \vee \partial_X$ is the composite

$$A \vee \partial_X = i_X\, \partial_X : X'' \to X \to A \vee X$$

The inclusion i_X and projection $(0, 1)$

$$X \xrightarrow{i_X} A \vee X \xrightarrow{(0,1)} X$$

are ∂-compatible and therefore represent maps in **Coef**. Now a morphism $(\bar{v}, u) :$ $A \vee \partial_X \to B \vee \partial_Y$ in **premod** is a commutative diagram in **Coef**:

$$\begin{array}{ccc}
\partial_X & \xrightarrow{u} & \partial_Y \\
{\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\
A \vee \partial_X & \xrightarrow{\bar{v}} & B \vee \partial_Y \\
{\scriptstyle (0,1)}\downarrow & & \downarrow{\scriptstyle (0,1)} \\
\partial_X & \xrightarrow{u} & \partial_Y
\end{array}$$

Composition is defined by horizontal composition of such diagrams. We have the canonical *coefficient functor*

$$c : \mathbf{premod} \to \mathbf{Coef} \tag{5.2}$$

which carries $A \vee \partial_X$ to ∂_X and $(\bar{v}, u)$ to u. A morphism $(\bar{v}, u)$ is equivalently given by the pair (v, u) with $v = \bar{v}\, i_A : A \to B \vee \partial_Y$ trivial on ∂_Y. We also write

$$(\bar{v}, u) = v \odot u = f \odot u \tag{5.3}$$

and we say that $v \odot u$ is an *u-equivariant* map in **premod**. Here v is represented by f with $v = j(f)$ and $f \in \mathbf{T}(A, B \vee Y)_2$; compare lemma (5.5) below. Hence we get the identification

$$\mathbf{premod}\,(A \vee \partial_X, B \vee \partial_Y)_u = \mathbf{Coef}\,(A, B \vee \partial_Y)_2 : v \odot u \mapsto v \tag{5.4}$$

where the left hand side denotes the set of all morphisms $A \vee \partial_X \to B \vee \partial_Y$ in **premod** which are u-equivariant. This set is a subgroup of the group $\mathbf{Coef}\,(A, B \vee \partial_Y)$. Here the group structure is obtained by the cogroup A in $\mathbf{T}$ which is also a cogroup in **Coef**; see (4.2).

The group $\mathbf{Coef}(A, \partial_Y)$ acts on the group (5.4) by setting for $a \in \mathbf{Coef}(A, \partial_Y)$ and $f \in \mathbf{Coef}(A, B \vee \partial_Y)_2$

$$f^a = -ia + f + ia \tag{1}$$

where $i : \partial_Y \to B \vee \partial_Y$ is the inclusion in **Coef**. The right hand side is defined in the group $\mathbf{Coef}(A, B \vee \partial_Y)$. This implies that the image of the pre-crossed module $(\partial_X, 1)_*$ in (3.11) and (4.2) acts trivially on the group (5.4).

(5.5) Lemma. *One has the short exact sequence of groups*

$$\begin{array}{ccccc}
N & \xrightarrow{\ \delta\ } & \mathbf{T}(A, B \vee Y)_2 & \xrightarrow{\ j\ } & \mathbf{Coef}\,(A, B \vee \partial_Y)_2 \longrightarrow 0 \\
\cap & & \cap & & \\
\mathbf{T}(A, Y'' \vee B \vee Y) & \xrightarrow{\ \delta\ } & \mathbf{T}(A, B \vee Y) & &
\end{array}$$

Here δ is induced by $(i_Y \partial_Y, i_B, i_Y)$ and N is the subgroup of all elements in $\mathbf{T}(A, Y'' \vee B \vee Y)$ which are trivial on both $Y'' \vee Y$ and $B \vee Y$. Moreover j is equivariant with respect to the action in (5.4) (1) and (2.10); that is $j(f^a) = (jf)^{ja}$.

Proof. The map j carries $f : A \to B \vee Y$ trivial on Y to the class $\{f\}$ in **Coef**. We now show that j is surjective. Let $f' : A \to B \vee Y$ be a map representing $v \in \mathbf{Coef}\,(A, B \vee \partial_Y)_2$. Hence we have $(0,1)f' \sim 0$ and therefore there is $\alpha : A \to Y''$ with $(0,1)f' = (\partial_Y, 1)\alpha$. Then we obtain $f = f' - (B \vee \partial_Y, 1)\alpha$ satisfying $(0,1)f = 0$ and $f \sim f'$. This proves that j is surjective. Next let $f \in \ker(j)$ that is f is trivial on Y and $f = (B \vee \partial_Y, 1)\alpha$ where $\alpha : A \to Y'' \vee (B \vee Y)$ is trivial on $B \vee Y$. Let $i : Y'' \vee Y \subset Y'' \vee B \vee Y$ be the inclusion and let r be the obvious retraction of i; that is $r = (i_{Y''}, 0, i_Y)$. Then we have for $\alpha' = r\alpha$ the equation $(\partial_Y, 1)\alpha' = 0$ since f is trivial on Y. This shows that

$$f = (B \vee \partial_Y, 1)\alpha = (B \vee \partial_Y, 1)(\alpha - i\,\alpha')$$

where $\alpha - i\,\alpha' \in N$. This shows that kernel$(j) = \delta(N)$. q.e.d.

In all examples of chapter A and B we defined the categories $\mathbf{mod}(\partial_X)$ of free modules with $\partial_X \in \mathbf{Coef}$. Moreover we defined for $u : \partial_X \to \partial_Y$ the induced additive functors $u_* : \mathbf{mod}(\partial_X) \to \mathbf{mod}(\partial_Y)$. The associated "Grothendieck construction" of $\partial_X \mapsto \mathbf{mod}(\partial_X)$ is a category $\mathbf{mod}$ with the following properties; see (5.8).

(5.6) Definition. A *category of modules* is a category $\mathbf{mod}$ together with a diagram of functors

$$\begin{tikzcd} \mathbf{premod} \arrow[rr, "E"] \arrow[dr, "c"'] & & \mathbf{mod} \arrow[dl, "c"] \\ & \mathbf{Coef} & \end{tikzcd}$$

with the following properties:

(i) The objects of $\mathbf{mod}$ are the same as in the category $\mathbf{premod}$ and E is a full functor which is the identity on objects.

(ii) The diagram commutes, that is $cE = c$. We say that $f : A \vee \partial_X \to B \vee \partial_Y$ in $\mathbf{mod}$ is *u-equivariant* if $c(f) = u : \partial_X \to \partial_Y \in \mathbf{Coef}$; moreover f is *∂_X-equivariant* if $\partial_X = \partial_Y$ and $u = 1$ is the identity of ∂_X. Let

$$\mathbf{mod}(A \vee \partial_X, B \vee \partial_Y)_u$$

be the set of all morphisms $f : A \vee \partial_X \to B \vee \partial_Y$ in $\mathbf{mod}$ which are u-equivariant. Then this set is an abelian group together with an action of the group $\mathbf{Coef}(A, \partial_Y)$ and E induces a surjective homomorphism of groups

$$E : \mathbf{premod}\,(A \vee \partial_X, B \vee \partial_Y)_u \longrightarrow \mathbf{mod}\,(A \vee \partial_X, B \vee \partial_Y)_u$$

which is equivariant with respect to the action of $\mathbf{Coef}(A, \partial_Y)$ in (5.4) (1). For each morphism $w : \partial_Z \to \partial_X$ the following diagram commutes

$$\begin{tikzcd} \mathbf{premod}(A \vee \partial_X, B \vee \partial_Y)_u \arrow[r, "E"] \arrow[d, "(1_A \vee w)^*"', "\cong"] & \mathbf{mod}(A \vee \partial_X, B \vee \partial_Y)_u \arrow[d, "E(1_A \vee w)^*", "\cong"'] \\ \mathbf{premod}(A \vee \partial_Z, B \vee \partial_Y)_{uw} \arrow[r, "E"] & \mathbf{mod}(A \vee \partial_Z, B \vee \partial_Y)_{uw} \end{tikzcd}$$

Here $(1_A \vee w)^*$ is an isomorphism by (5.4) and we assume that also $E(1_A \vee w)^*$ on the right hand side of the diagram is an isomorphism. We denote the morphism $E(1_A \vee w)$ in $\mathbf{mod}$ also by $1_A \vee w$.

(iii) For cogroups A, B, A_1, A_2, B_1, B_2 one has isomorphisms of groups

$$\begin{aligned}\mathbf{mod}(A_1 \vee A_2 \vee \partial_X, B \vee \partial_Y)_u &= \mathbf{mod}(A_1 \vee \partial_X, B \vee \partial_Y)_u \oplus \mathbf{mod}(A_2 \vee \partial_X, B \vee \partial_Y)_u \\ \mathbf{mod}(A \vee \partial_X, B_1 \vee B_2 \vee \partial_Y)_u &= \mathbf{mod}(A \vee \partial_X, B_1 \vee \partial_Y)_u \oplus \mathbf{mod}(A \vee \partial_X, B_2 \vee \partial_Y)_u\end{aligned}$$

These isomorphisms are induced by the inclusions $i_1 : A_1 \to A_1 \vee A_2$, $i_2 : A_2 \to A_1 \vee A_2$ and by the retractions $r_1 = (1,0) : B_1 \vee B_2 \to B_1$, $r_2 = (0,1) : B_1 \vee B_2 \to B_2$.

The conditions (i), (ii), (iii) imply that for $f \in \mathbf{mod}\,(B \vee \partial_Y, B_1 \vee \partial_W)_w$ and $g \in \mathbf{mod}\,(A_1 \vee \partial_V, A \vee \partial_X)_v$ the induced functions

$$\begin{cases} f_* : \mathbf{mod}\,(A \vee \partial_X, B \vee \partial_Y)_u \longrightarrow \mathbf{mod}\,(A \vee \partial_X, B_1 \vee \partial_W)_{wu} \\ g^* : \mathbf{mod}\,(A \vee \partial_X, B \vee \partial_Y)_u \longrightarrow \mathbf{mod}\,(A_1 \vee \partial_V, B \vee \partial_Y)_{uv} \end{cases} \tag{iv}$$

are homomorphisms satisfying $(f + f^1)_* = f_* + f^1_*$ and $(g + g_1)^* = g^* + g_1^*$. In this sense **mod** is an "additive category over **Coef**".

In (5.10) we show that such a category of modules always exists. The category **mod**, however, is not uniquely determined by the theory **T** of coactions. Later we shall use a category **mod** = **mod**(**C**) which is given by a cofibration category **C** under **T**.

Remark. The composite

$$\begin{array}{ccc} \mathbf{T}(A, Y'' \vee Y)_2 \xrightarrow{\ j\ } & \mathbf{Coef}(A, Y'' \vee \partial_Y)_2 & \\ & \Vert & \\ & \mathbf{premod}(A \vee \partial_X, Y'' \vee \partial_Y)_u & \xrightarrow{\ E\ } \mathbf{mod}(A \vee \partial_X, Y'' \vee \partial_Y)_u \end{array}$$

carries Peiffer commutators of the pre-crossed module (3.11) to the trivial element. In fact for $m, n \in \mathbf{T}(A, Y'' \vee Y)_2$ the Peiffer commutator is

$$\langle m, n \rangle = -m - n + m + n^{(\partial_Y, 1)_* m}$$

where we use the action of $(\partial_Y, 1)_* m \in \mathbf{T}(A, Y)$. But this action is killed by j since j is j-equivariant; see (5.5). Moreover E carries commutators to 0.

For an object ∂_X in **Coef** let

$$\begin{aligned} \mathbf{premod}(\partial_X) &\subset \mathbf{premod} \\ \mathbf{mod}(\partial_X) &\subset \mathbf{mod} \end{aligned} \tag{5.7}$$

be the subcategories consisting of objects $A \vee \partial_X$ where A is a cogroup in **T** and of maps $A \vee \partial_X \to B \vee \partial_X$ which are ∂_X-equivariant. Here **premod**(∂_X) is a theory of cogroups since the cogroup structure of A in **T** induces a cogroup structure $\mu_A \vee 1 : A \vee \partial_X \to A \vee A \vee \partial_X$ in **premod**(∂_X). Moreover **mod**(∂_X) is an additive category. A morphism $u : \partial_X \to \partial_Y$ induces functors u_* such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{premod}(\partial_X) & \xrightarrow{u_*} & \mathbf{premod}(\partial_Y) \\ \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} \\ \mathbf{mod}(\partial_X) & \xrightarrow{u_*} & \mathbf{mod}(\partial_Y) \end{array} \tag{1}$$

Here u_* is a map between theories so that u_* in the bottom row of the diagram is an additive functor. The functor u_* carries the object $A \vee \partial_X$ to the object $A \vee \partial_Y$. Moreover on morphisms u_* is defined by the following commutative diagrams. The initial object ∂_* in **Coef** is given by the identity $\partial_* = 1 : * \to *$ of the initial object $*$ in **T**. Hence we have the unique map $0 : \partial_* \to \partial_X$ in **Coef**.

$$\begin{array}{ccc} \mathbf{premod}(A \vee \partial_X, B \vee \partial_X)_1 & \xrightarrow{u_*} & \mathbf{premod}(A \vee \partial_Y, B \vee \partial_Y)_1 \\ {\scriptstyle (1\vee 0)^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle (1\vee 0)^*} \\ \mathbf{premod}(A \vee \partial_*, B \vee \partial_X)_0 & \xrightarrow{(1\vee u)_*} & \mathbf{premod}(A \vee \partial_*, B \vee \partial_Y)_0 \end{array} \tag{2}$$

$$\begin{array}{ccc} \mathbf{mod}(A \vee \partial_X, B \vee \partial_X)_1 & \xrightarrow{u_*} & \mathbf{mod}(A \vee \partial_Y, B \vee \partial_Y)_1 \\ {\scriptstyle E(1\vee 0)^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle E(1\vee 0)^*} \\ \mathbf{mod}(A \vee \partial_*, B \vee \partial_X)_0 & \xrightarrow{E(1\vee u)_*} & \mathbf{mod}(A \vee \partial_*, B \vee \partial_Y)_0 \end{array} \tag{3}$$

The vertical arrows in these diagrams are isomorphisms by (5.6) (ii).

(4) Lemma. *Both functors u_* above are well defined and carry sums to sums.*

Proof. Clearly u_* carries identities to identities. Moreover $u_*(fg) = (u_*f)(u_*g)$ is obtained by the commutativity of the following diagram in **premod** and **mod** respectively.

$$\begin{array}{ccccc} & & B \vee \partial_X & \xrightarrow{1\vee u} & B \vee \partial_Y \\ & & {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle u_*f} \\ A \vee \partial_* & \xrightarrow{1\vee 0} & A \vee \partial_X & \xrightarrow{1\vee u} & A \vee \partial_Y \\ & & {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle u_*g} \\ & & A \vee \partial_X & \xleftarrow{1\vee 0} A \vee \partial_* \xrightarrow{1\vee 0} & A \vee \partial_Y \end{array}$$

Here the top square of the diagram commutes since $(1 \vee u)(1 \vee 0) = 1 \vee 0$. In a similar way one shows that u_* carries sums to sums. Here a sum of $A \vee \partial_X$ and $B \vee \partial_X$ in $\mathbf{premod}(\partial_X)$ or $\mathbf{mod}(\partial_X)$ is $A \vee B \vee \partial_X$ where $A \vee B$ is the sum in **T**, compare also (5.6) (iii). q.e.d.

Let T_0 be the set of all cogroups in $\mathbf{T}$. Then we have the category of theories of cogroups $\mathbf{cogr}(T_0)$ and the category of additive categories $\mathbf{add}(T_0)$ defined in (1.15). By identifying the object $A \vee \partial_X$ with $A \in T_0$ we see that

$$\begin{aligned} \mathbf{premod}(\partial_X) &\in \mathbf{cogr}(T_0) \\ \mathbf{mod}(\partial_X) &\in \mathbf{add}(T_0) \end{aligned} \tag{5}$$

Moreover using u_* in (1) above we obtain the *enveloping functors*

$$\begin{aligned} \mathrm{premod} : \mathbf{Coef} &\to \mathbf{cogr}(T_0) \\ \mathrm{mod} : \mathbf{Coef} &\to \mathbf{add}(T_0) \end{aligned} \tag{6}$$

which carry ∂_X to $\mathbf{premod}(\partial_X)$ and $\mathbf{mod}(\partial_X)$ respectively. These functors actually determine the categories **premod** and **mod** by the so-called Grothendieck construction; see Gray [FC] and Thomason [HC].

(5.8) Definition. Let $\mathbf{C}$ be a small category and $F : \mathbf{C} \to \mathbf{Cat}$ be a functor where $\mathbf{Cat}$ is the category of small categories. Then the *Grothendieck construction* of F is the category $\mathbf{Gro}(F)$ defined as follows. The objects are pairs (A, ∂) with $\partial \in \mathrm{Ob}(\mathbf{C}), A \in \mathrm{Ob}(F\partial)$; and a morphism $(A, \partial) \to (A', \partial')$ is a pair (w, u) where $u : \partial \to \partial' \in \mathbf{C}$ and $w : F(u)(A) \to A' \in F(\partial')$.

One readily checks that one has isomorphisms of categories

$$\begin{aligned} \mathbf{premod} &= \mathbf{Gro}(\mathrm{premod}) \\ \mathbf{mod} &= \mathbf{Gro}(\mathrm{mod}) \end{aligned} \tag{5.9}$$

which carry $A \vee \partial_X$ to (A, ∂_X) and which carry the morphism $f = w \odot u$ to the pair $(w \odot 1, u)$. Here $w \odot 1$ and $w \odot u$ are defined by the commutative diagram

$$\begin{tikzcd} A \vee \partial_* \arrow[r, "1\vee 0"] \arrow[dr, "w"'] & A \vee \partial_Y \arrow[d, "w\odot 1"] & A \vee \partial_X \arrow[l, "1\vee u"'] \arrow[dl, "w\odot u"] \\ & A \vee \partial_Y & \end{tikzcd} \tag{1}$$

For **premod** this coincides with the notation in (5.3). We also write $w \odot u = (w, i_2 u)$ so that the composition formula is given by

$$(w \odot u)(w' \odot u') = (w, i_2 u)w' \odot uu' \tag{2}$$

We now are ready to prove the existence of a category **mod** of modules with the properties in (5.6). For this we use the additivization functor $(\ \ )^{\mathrm{ad}}$ in (1.16).

(5.10) Proposition. *The Grothendieck construction of the composite*

$$U : \mathbf{Coef} \xrightarrow{\mathrm{premod}} \mathbf{cogr}(T_0) \xrightarrow{(\ \ )^{\mathrm{ad}}} \mathbf{add}(T_0)$$

denoted by

$$\mathbf{premod}^{\mathrm{ad}} = \mathbf{Gro}(U)$$

is a category of modules in the sense of (5.6).

Proof. All properties in (5.6) are readily verified except the action of $\mathbf{Coef}(A, \partial_Y)$ in (5.6) (ii). But $(0,1)_*, (1,0)_*$ and $(1,1)_*$ in the proof of (1.16) applied to $\mathbf{T} = \mathbf{premod}(\partial_X)$ are easily seen to be equivariant with respect to this action. This implies that this action is well defined for $\mathbf{premod}^{\mathrm{ad}}$. q.e.d.

For a set $\mathcal{A}$ let $\mathbf{Ringoids}(\mathcal{A})$ be the category consisting of ringoids $\mathbf{R}$ with $\mathrm{Ob}(\mathbf{R}) = \mathcal{A}$ and of additive functors which are the identity on objects. If $\mathcal{A}$ consists of only one object this is the category $\mathbf{Rings}$ of rings with unit. If $\mathcal{A}$ is a subset of T_0 we have the canonical forgetful functor

$$\varphi_{\mathcal{A}} : \mathbf{add}(T_0) \longrightarrow \mathbf{Ringoids}(\mathcal{A})$$

which carries the additive category $\mathbf{K}$ with $\mathrm{Ob}(\mathbf{K}) = T_0$ to the full subcategory $\varphi_{\mathcal{A}}(\mathbf{K}) = \mathbf{R}$ of $\mathbf{K}$ with $\mathrm{Ob}(\mathbf{R}) = \mathcal{A}$. Given a category of modules as in (5.6) we call the composite

$$U_{\mathcal{A}} : \mathbf{Coef} \xrightarrow{\mathrm{mod}} \mathbf{add}(T_0) \xrightarrow{\varphi_{\mathcal{A}}} \mathbf{Ringoids}(\mathcal{A}) \tag{5.11}$$

the $\mathcal{A}$-*enveloping functor* associated to $\mathbf{mod}$. In particular if $\mathcal{A} = A$ consists of only one object we get the functor

$$U_A : \mathbf{Coef} \longrightarrow \mathbf{Rings}$$

which carries ∂_X to the A-*enveloping ring* of ∂_X.

(5.12) Definition. Let $\mathbf{C}$ be a category and let $U : \mathbf{C} \to \mathbf{Rings}$ be a functor. Then we define the category $\mathbf{Mod}(U)$ as follows. Objects are pairs (M_X, X) where X is an object in $\mathbf{C}$ and M_X is a right $U(X)$-module. Morphisms are pairs $(\alpha, v) : (M_X, X) \to (M_Y, Y)$ where $v : X \to Y$ is a morphism in $\mathbf{C}$ and where $\alpha : M_X \to M_Y$ is a $U(v)$-equivariant homomorphism of modules, that is $\alpha(m \cdot t) = \alpha(m) \cdot U(v)(t)$ for $m \in M_X, t \in U(X)$. Let $\mathbf{mod}(U)$ be the full subcategory of $\mathbf{Mod}(U)$ consisting of objects (M_X, X) where M_X is a free $U(X)$-module.

The category $\mathbf{Mod}(U)$ is again a Grothendieck construction of U; see (5.8). As a special case of (5.9) we obtain the following result.

(5.13) Proposition. *Let* $\mathbf{T}$ *be a theory of coactions such that the cogroups of* $\mathbf{T}$ *are sums* $\bigvee_E A$ *of an object* A *where* E *is a set. Then there is an isomorphism of categories*

$$\mathbf{mod} = \mathbf{mod}(U_A)$$

where U_A *is the enveloping functor in (5.11).*

(5.14) Examples. Let $\mathbf{S}$ be a single sorted theory of cogroups so that $\mathbf{model}(\mathbf{S})$ is a variety of groups with operators as considered in (2.12). Then $\mathbf{T} = \mathbf{free}(\mathbf{S})$ is a theory of cogroups satisfying the assumption in (5.13) so that

$$\mathbf{mod} = \mathbf{premod}^{\mathrm{ad}} = \mathbf{mod}(U)$$

can be described by the enveloping functor

$$U = U_A : \mathbf{Coef} = \mathbf{model(S)} \to \mathbf{Rings}$$

Here A is the free model of $\mathbf{S}$ generated by one element. For the examples in the table of (2.12) and (4.6) one gets the following description of U.

$\mathbf{T}$	$X \in \mathbf{Coef}$	$U(X) \in \mathbf{Rings}$
gr	$G \in \mathbf{Gr}$	$U(G) = \mathbb{Z}[G]$ group ring
$\mathbf{nil}_n$	$G \in \mathbf{Nil}_n$	$U(G) = \mathbb{Z}[G]/I(G)^n$
ab	$G \in \mathbf{Ab}$	$U(G) = \mathbb{Z}$, U is constant functor
var	$G \in \mathbf{Var}$	$U(G) = \mathbb{Z}[G]/\mathcal{V}(G)$ factor ring of $\mathbb{Z}[G]$
alg	$A \in \mathbf{Alg}$	$U(A) = A \otimes A^{\mathrm{op}}$
calg	$A \in \mathbf{Calg}$	$U(A) = A$
lie	$L \in \mathbf{Lie}$	$U(L) =$ universal enveloping algebra of L

Here $I(G)^n$ denotes the n-th power of the augmentation ideal $I(G)$ in the group ring $\mathbb{Z}[G]$. Moreover $\mathcal{V}(G)$ is the ideal of $\mathbb{Z}[G]$ defined by the variety **Var** as follows. If $\mathbf{Var} = \mathbf{Var}(\mathcal{L})$ is given by a set $\mathcal{L}$ of laws, see (2.6), then $\mathcal{V}(G)$ is the ideal generated by all elements $\alpha_* c(x) \in \mathbb{Z}[G]$ with $x \in N(\mathcal{L})$ and $\alpha \in \mathbf{Gr}(F_\infty, G)$. Here $N(\mathcal{L}) \subset F_\infty$ is the normal subgroup of F_∞ generated by $\mathcal{L}$ and $c : F_\infty \to \mathbb{Z}[F_\infty]$ is the unique function (crossed homomorphism) satisfying $c(X_2) = [x_i]$ for generators x_i, $i \geq 0$, of F_∞ and $c(a \cdot b) = c(a)^b + c(b)$. Compare for example Leedham-Green I.§ 1 [HV].

(5.15) Remark. The functor U in (5.14) can also be obtained as follows. Let $\mathbf{C}$ be a category and for an object $G \in \mathbf{C}$ let $\mathbf{C}_G$ be the category of objects over G. Then a *module M over G* is an abelian group object in the category $\mathbf{C}_G$. Let $\mathbf{Mod}(G)$ be the category of all abelian group objects in $\mathbf{C}_G$. If $\mathbf{C} = \mathbf{model(S)}$ as above then the forgetful functor $\varphi : \mathbf{Mod}(G) \to \mathbf{Set}$ with $\varphi(M) = M(A)$ has a left adjoint

$$\mathrm{free}_G : \mathbf{Set} \to \mathbf{Mod}(G)$$

which carries a set to a *free module over* G. Let $\bar{U}(G)$ be the endomorphism ring of the free module $\mathrm{free}_G(*)$ generated by one element $*$. One can check that $\bar{U}(G)$ coincides with $U(G)$ defined in (5.14) above. Compare Quillen [CA].

(5.16) Remark. In universal algebra there are also means to define an enveloping functor U; see Rowan [ER] and Day-Kiss [FR]. The list of examples of Rowan essentially agrees with the list in (5.14).

6 Chain Complexes and Homology

Given a category of modules **mod** as in § 5 we introduce the category of chain complexes in **mod** and the notion of homology for such chain complexes.

We first consider graded objects and chain complexes in an additive category $\mathbf{A}$. A *graded object* V in $\mathbf{A}$ is a sequence $V = \{V_i,\ i \in \mathbb{Z}\}$ of objects V_i in $\mathbf{A}$. A *map* $f : V \to W$ *of degree* k between graded objects is a sequence of maps $f = \{f_i : V_i \to W_{i+k},\ i \in \mathbb{Z}\}$ of maps f_i in $\mathbf{A}$. The category $\mathrm{gr}(\mathbf{A})$ of graded objects and degree 0-maps is again an additive category; the sum is given by

$$(V \oplus W)_i = V_i \oplus W_i$$

where the right hand side denotes the sum in $\mathbf{A}$ for $i \in \mathbb{Z}$. Let $M \subset \mathbb{Z}$ be a subset and let $*$ be the initial object of $\mathbf{A}$. We say that V is *concentrated in degree* M if $V_i = *$ for $i \in \mathbb{Z} - M$. We write $V_{\geq n}$ if V is concentrated in degree $\geq n$. Moreover V is *bounded below* if there exists $n \in \mathbb{Z}$ with $V = V_{\geq n}$. The *dimension* of V is given by $\dim(V) \leq n$ if $V_i = *$ for $i > n$.

A *chain complex* V in $\mathbf{A}$ is a graded object V wih a map $d : V \to V$ of degree -1 satisfying $dd = 0$. A *chain map* $f : V \to V'$ is a map of degree 0 with $df = fd$. A *homotopy* $f \simeq g$ between chain maps is a map $\alpha : V \to V'$ of degree 1 satisfying

$$-f + g = d\alpha + \alpha d$$

A *subcomplex* W of a chain complex V is a chain map $i : W \to V$ with the property that V as a graded object is a sum $W \oplus W'$ and i is the inclusion of the first summand.

(6.1) Definition. Let $\mathbf{A}$ be an additive category (or more generally a ringoid). A *left* $\mathbf{A}$*-module* M is an additive functor $M : \mathbf{A} \to \mathbf{Ab}$ where $\mathbf{Ab}$ is the category of abelian groups. A *right* $\mathbf{A}$*-module* is an additive contravariant functor from $\mathbf{A}$ to $\mathbf{Ab}$ or equivalently an $\mathbf{A}^{\mathrm{op}}$-module. For example if $\mathbf{A}$ is an additive subcategory of an additive category $\mathbf{M}$ then we obtain for each object M in $\mathbf{M}$ the right $\mathbf{A}$-module

$$\mathrm{Hom}(-, M) : \mathbf{A}^{\mathrm{op}} \to \mathbf{Ab}$$

which carries $A \in \mathbf{A}$ to the abelian group $\mathrm{Hom}(A, M)$ of morphisms $A \to M$ in $\mathbf{M}$. Given an additive functor $u : \mathbf{B} \to \mathbf{A}$ between ringoids we obtain for a left (resp. right) $\mathbf{A}$-module N the $\mathbf{B}$-module u^*N by the composition of functors $u^*N = Nu : \mathbf{B} \to \mathbf{A} \to \mathbf{Ab}$.

For a chain complex V in $\mathbf{A}$ and a left $\mathbf{A}$-module M we obtain the chain complex of abelian groups MV given by

$$\longrightarrow MV_{n+1} \xrightarrow{d_*} MV_n \xrightarrow{d_*} MV_{n-1} \longrightarrow$$

Hence the *homology* of V *with coefficients* in M is

$$H_n(V; M) = H_n(MV) = \mathrm{kernel}\, d_* / \mathrm{image}\, d_* \tag{6.2}$$

Similarly we obtain for a right $\mathbf{A}$-module N the cochain complex of abelian groups NV given by

$$\longleftarrow NV_{n+1} \xleftarrow{d^*} NV_n \xleftarrow{d^*} NV_{n-1} \longleftarrow$$

and the *cohomology* of V *with coefficients* in N is

$$H^n(V;N) = H^n(NV) = \text{kernel}\, d^* / \text{image}\, d^*.$$

If W is a subcomplex of the chain complex V then one has the short exact sequences of chain complexes in **Ab**

$$0 \longrightarrow MW \xrightarrow{i_*} MV \longrightarrow \text{cokernel}\, i_* \longrightarrow 0$$

$$0 \longrightarrow \text{kernel}\, i^* \longrightarrow NV \xrightarrow{i^*} NW \longrightarrow 0$$

We define the *relative (co-) homology groups* by

$$H_n(V,W;M) = H_n(\text{cokernel}\, i_*)$$
$$H^n(V,W;N) = H^n(\text{kernel}\, i^*)$$

Clearly we have the associated long exact sequences

$$\longrightarrow H_n(W;M) \longrightarrow H_n(V;M) \longrightarrow H_n(V,W;M) \xrightarrow{\partial} H_{n-1}(W;M) \longrightarrow$$

$$\longleftarrow H^n(W;N) \longleftarrow H^n(V;N) \longleftarrow H^n(V,W;N) \xleftarrow{\partial} H^{n-1}(W;N) \longleftarrow$$

Now let **T** be a theory of coactions and let **mod** be a category of modules for **T** as defined in (5.6). Many definitions and results below depend only on **T** and **mod**. We first introduce chain complexes in **mod** and (co-) homology for such chain complexes as follows. Recall that for ∂_X we have the additive category $\mathbf{mod}(\partial_X)$ in which the sum is given by

$$(A \vee \partial_X) \oplus (B \vee \partial_X) = A \vee B \vee \partial_X$$

where A, B are cogroups in **T**; see (5.7).

(6.3) Definition. Let **mod** be a category of modules for **T**. Then the following *category* **chain** *of chain complexes* is defined. Objects are pairs (A, ∂_X) where ∂_X is an object in **Coef** and A is a chain complex in $\mathbf{mod}\,(\partial_X)$; that is A is given by a sequence of cogroups A_i in **T** with $i \in \mathbb{Z}$ and ∂_X-equivariant maps

$$d_i : A_i \vee \partial_X \to A_{i-1} \vee \partial_X$$

in **mod** such that $d_{i-1} \circ d_i = 0$ in the abelian group $\mathbf{mod}(A_i \vee \partial_X, A_{i-2} \vee \partial_X)_1$ where 1 is the identity of ∂_X. Morphisms $(A, \partial_X) \to (B, \partial_Y)$ in **chain** are pairs (f, u) where $u : \partial_X \to \partial_Y$ is a morphism in **Coef** and where f is a sequence of u-equivariant maps f_i, $i \in \mathbb{Z}$, for which the diagram

$$\begin{tikzcd}
A_i \vee \partial_X \arrow[r, "d_i"] \arrow[d, "f_i"'] & A_{i-1} \vee \partial_X \arrow[d, "f_{i-1}"] \\
B_i \vee \partial_Y \arrow[r, "d_i"] & B_{i-1} \vee \partial_Y
\end{tikzcd}$$

commutes in **mod**. Such a morphism is called a u-equivariant *chain map*. Two such chain maps $(f,u),(g,v)$ from (A,∂_X) to (B,∂_Y) are *homotopic* if $u=v$ and if there exists a sequence α of morphisms of u-equivariant maps in **mod**

$$\alpha_i : A_i \vee \partial_X \to B_{i+1} \vee \partial_Y$$

with $i \in \mathbb{Z}$ such that in $\mathbf{mod}\,(A_i \vee \partial_X, B_i \vee \partial_Y)_u$ we have the equation

$$-f_i + g_i = d_{i+1}\alpha_i + \alpha_{i+1}d_i$$

We also write $\alpha : (f,u) \simeq (g,u)$ and one readily checks that homotopy is a natural equivalence relation so that the homotopy category **chain**/$\simeq$ is defined. One has the canonical functors

$$\mathbf{chain} \xrightarrow{q} \mathbf{chain}/\simeq \xrightarrow{c} \mathbf{Coef}$$

where q is the quotient functor and c is the coefficient functor which carries (A,∂_X) to ∂_X.

Example. Let $\mathbf{S}$ be a single sorted theory of cogroups and let $\mathbf{T} = \mathbf{free}(\mathbf{S})$. Then we have the equivalence

$$\mathbf{mod} = \mathbf{mod}(U) \tag{1}$$

where the right hand side is the category defined by the enveloping functor U, see (5.12). We define accordingly the category $\mathbf{chain}(U)$ of chain complexes in $\mathbf{mod}(U)$ such that (1) induces

$$\mathbf{chain} = \mathbf{chain}(U) \tag{2}$$

Objects in $\mathbf{chain}(U)$ are pairs (A,G) with $G \in \mathbf{model}(\mathbf{S})$ where A is a chain complex of free right $U(G)$-modules.

For $\mathbf{T} = \mathbf{gr}$ the functor U carries a group $G \in \mathbf{Gr}$ to the group ring $U(G) = \mathbb{Z}[G]$. In this case $\mathbf{chain}(U)$ is the category of free chain complexes over group rings. In particular the cellular chain complex of the universal covering of a reduced CW-complex is an object in $\mathbf{chain}(U)$. The coefficient functor carries (A,G) to G.

(6.4) Definition. Let (A,∂_X) be a chain complex in **chain** so that A is a chain complex in the additive category $\mathbf{mod}(\partial_X)$. Then the *homology* $H_n(A;M)$ with coefficients in a left $\mathbf{mod}(\partial_X)$-module M and the *cohomology* $H^n(A;N)$ with coefficients in a right $\mathbf{mod}(\partial_X)$-module N are defined as in (6.2). A u-equivariant chain map $f:(B,\partial_Y) \to (A,\partial_X)$ in **chain** induces the maps in (co-) homology

$$f_* : H_n(B;u^*M) \to H_n(A;M)$$
$$f^* : H^n(A;N) \to H_n(B;u^*N)$$

Here $u^*M = (u_*)^*M$ is given by the additive functor $u_* : \mathbf{mod}(\partial_Y) \to \mathbf{mod}(\partial_X)$ in (5.7) (1). We define f_* and f^* by the canonical factorization in **chain**

$$f = w \odot u : (B, \partial_Y) \xrightarrow{1 \vee u} (u_* B, \partial_X) \xrightarrow{w \odot 1} (A, \partial_X)$$

obtained from (5.9) (1). Here u_* carries the chain complex B in $\mathbf{mod}(\partial_Y)$ to the chain complex $u_* B$ in $\mathbf{mod}(\partial_X)$. The definition in (6.2) shows that

$$H_n(u_* B; M) = H_n(B; u^* M)$$
$$H^n(u_* B; N) = H^n(B; u^* N)$$

so that $f_* = (w \odot 1)_*$ and $f^* = (w \odot 1)^*$ are well defined. Clearly f_* and f^* depend only on the homotopy class of f in $\mathbf{chain}/\simeq$. We say that $i = w \odot u : (B, \partial_Y) \to (A, \partial_X)$ is the inclusion of a *subcomplex* if the associated map $w \odot 1 : u_* B \to A$ of chain complexes in $\mathbf{mod}(\partial_X)$ is the inclusion of a subcomplex. In this case we get the *relative homology* $H_n(A, B; M)$ and the *relative cohomology* $H^n(A, B; N)$ and the following long exact sequences as in (6.2)

$$\cdots \to H_n(B; u^* M) \xrightarrow{i_*} H_n(A; M) \to H_n(A, B; M) \to H_{n-1}(B; u^* M) \to \ldots$$
$$\cdots \leftarrow H^n(B; u^* N) \xleftarrow{i^*} H_n(A; N) \leftarrow H_n(A, B; N) \leftarrow H_{n-1}(B; u^* N) \leftarrow \ldots$$

(6.5) Example. For objects $D \vee \partial_X$ and $D' \vee \partial_X$ in $\mathbf{mod}(\partial_X)$ let

$$\mathrm{Hom}_{\partial_X}(D, D') = \mathbf{mod}(D \vee \partial_X, D' \vee \partial_X)_1$$

be the abelian group of ∂_X-equivariant maps $D \vee \partial_X \to D' \vee \partial_X$. Then $\mathrm{Hom}_{\partial_X}(-, D')$ is a right $\mathbf{mod}(\partial_X)$-module and $\mathrm{Hom}_{\partial_X}(D, -)$ is a left $\mathbf{mod}(\partial_X)$-module. Therefore we obtain the (co-) homology

$$H_n(A, \partial_X)(D) = H_n(A; \mathrm{Hom}_{\partial_X}(D, -))$$
$$H^n(A, \partial_X)(D') = H^n(A; \mathrm{Hom}_{\partial_X}(-, D'))$$

Here $H_n(A, \partial_X)$ is a right $\mathbf{mod}(\partial_X)$-module and $H^n(A, \partial_X)$ is a left $\mathbf{mod}(\partial_X)$-module.

Let $(D, \partial_X)_n$ be the chain complex in $\mathbf{mod}(\partial_X)$ which is concentrated in degree n and which is $D \vee \partial_X$ in degree n. Then one readily checks that

$$H_n(A, \partial_X)(D) = [(D, \partial_X)_n, A]$$
$$H^n(A, \partial_X)(D') = [A, (D', \partial_X)_n]$$

Here the right hand side denotes the corresponding sets of homotopy classes of ∂_X-equivariant chain maps.

A map $f : (B, \partial_Y) \to (A, \partial_X)$ in $\mathbf{chain}$ is a *homotopy equivalence* if there exist a map $g : (A; \partial_X) \to (B, \partial_Y)$ and homotopies $fg \simeq 1$ and $gf \simeq 1$ in $\mathbf{chain}$.

(6.6) Theorem. *Let (A, ∂_X) and (B, ∂_Y) be chain complexes in* **chain** *which are bounded below and let $f : (B, \partial_Y) \to (A, \partial_X)$ be a u-equivariant chain map. Then (a) and (b) are equivalent.*
(a) f is a homotopy equivalence in **chain**.
(b) u is an isomorphism in **Coef** *and*

$$f_* : H_n(B, \partial_Y) \to u^* H_n(A, \partial_X)$$

is an isomorphism of right $\mathbf{mod}(\partial_Y)$*-modules,* $n \in \mathbb{Z}$.

Moreover, if $\mathbf{mod}(\partial_X)$ *is the additive subcategory of an abelian category* $\mathbf{M}$ *such that all objects of* $\mathbf{mod}(\partial_X)$ *are projective in* $\mathbf{M}$ *then (a) is equivalent to (c).*
(c) u *is an isomorphism in* **Coef** *and*

$$f^* : H^n(A, \partial_X; N) \to H^n(B, \partial_Y; u^*N)$$

is an isomorphism for all right $\mathbf{mod}(\partial_X)$*-modules* N *of the form* $N = \mathrm{Hom}(-, M)$ *where* M *is an object in* $\mathbf{M}$ *and* $n \in \mathbb{Z}$*; see (6.1).*

Proof of (6.6). Since $f = w \odot u = (w \odot 1)(u \vee 1)$ it suffices to prove the theorem for $\partial_X = \partial_Y$ and $u = 1$. In this case (6.6) is a special case of the corresponding result for additive categories in (III.9.6) below. q.e.d.

Theorem (6.6) is needed for the various homological Whitehead theorems discussed in chapter A and B.

7 Augmented Theories of Coactions

In topology we have the action of the fundamental group $\pi_1(X)$ on the set of homotopy classes $[X, U]$ in $\mathbf{Top}^*/\simeq$. This action is well defined if $* \to X$ is a cofibration and then the action is induced by a canonical map

$$\varepsilon_X : X \to X \vee S^1. \tag{7.1}$$

For $\alpha \in \pi_1(X)$ on $\xi \in [X, U]$ we write $\xi^\alpha = (\xi, \alpha)\varepsilon_X$. Let $[X, U]^{\mathrm{free}}$ be the set of homotopy classes of non-pointed, or *free* maps from X to U in $\mathbf{Top}/\simeq$ and let

$$\varphi : [X, U] \to [X, U]^{\mathrm{free}}$$

be the forgetful map. Then we have for $\xi, \xi' \in [X, U]$ the equation $\varphi(\xi) = \varphi(\xi')$ if and only if there exists α with $\xi' = \xi^\alpha$. Hence the action of $\pi_1(X)$ on $[X, U]$ determines the difference between pointed homotopy classes and free homotopy classes. Compare also (III.§ 6) below.

Maps as ε_X in (7.1) are used to define the general notion of an "augmented theory of coactions". The augmentation ε_X will be needed to define homology groups of complexes in degree 0. If no augmentation is given such homology groups are only defined in degree ≥ 1. There are many examples of theories of coactions which are augmented; see (III.§ 6). In particular if D is a discrete space the theory $\mathbf{cone}(D, \mathcal{D}) = \mathbf{grd}(G)$ in (I.2.11) is augmented by $\Sigma = S^1 \times D$.

(7.2) Definition. Let $\mathbf{T}$ be a theory (i.e. a category in which sums $X \vee Y$ exist) and let Σ be a cogroup in $\mathbf{T}$. We say that the theory $\mathbf{T}$ is *Σ-augmented* or *augmented* by Σ if for each object X in $\mathbf{T}$ a coaction

$$\varepsilon_X : X \to X \vee \Sigma \tag{1}$$

in $\mathbf{T}$ is given such that for all $f : X \to Y$ in $\mathbf{T}$ the diagram

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & X \vee \Sigma \arrow[d, "f \vee 1"] \\ Y \arrow[r, "\varepsilon_Y"] & Y \vee \Sigma \end{tikzcd} \tag{2}$$

commutes in $\mathbf{T}$.

We call ε_X the *augmentation map* and for $\sigma : \Sigma \to U$ and $\xi : X \to U$ in $\mathbf{T}$ we write

$$\xi^\sigma = (\xi, \sigma)\varepsilon_X : X \to U. \tag{7.3}$$

Then the following formulas hold.

$$\xi^{\sigma_1+\sigma_2} = (\xi^{\sigma_1})^{\sigma_2} \quad \text{and} \quad \xi^0 = \xi \tag{a}$$

where $\sigma_1, \sigma_2, \sigma_1 + \sigma_2 \in \mathbf{T}(\Sigma, U)$ with the group structure $+$ of $\mathbf{T}(\Sigma, U)$ defined by the cogroup structure of Σ. Next we have

$$(\xi, \eta)^\sigma = (\xi^\sigma, \eta^\sigma) \tag{b}$$

where $(\xi, \eta) : X \vee Y \to U$. Moreover for a coaction $\mu_X : X \to X \vee X'$ in $\mathbf{T}$ we obtain

$$\xi + \alpha = (\xi, \alpha)\mu_X : X \to U$$

with $\xi : X \to U$ and $\alpha : X' \to U$ such that

$$(\xi + \alpha)^\sigma = \xi^\sigma + \alpha^\sigma \tag{c}$$

We also get for $f : Y \to X$ and $g : U \to V$ in $\mathbf{T}$ the formulas:

$$f^*(\xi^\sigma) = (f^*\xi)^\sigma \tag{d}$$
$$g_*(\xi^\sigma) = (g_*\xi)^{g_*\sigma} \tag{e}$$

Here (e) is a consequence of (3) and (d) follows from (2). Now (d) implies (b) and then (c). If $\mathbf{T}$ is a theory of coactions (or cogroups) then a coaction μ_X is given for all objects X in $\mathbf{T}$ and we can use (c), (d) and (e) if $\mathbf{T}$ is augmented by Σ as in (7.2). In this case we say that $\mathbf{T}$ or $(\mathbf{T}, \Sigma)$ is an *augmented theory of coactions.*

(7.4) Definition. Let $\mathbf{T}$ be a theory of coactions as in (1.11) so that for each object X in $\mathbf{T}$ a coaction

$$\mu_X : X \to X \vee X' \tag{1}$$

is given. Here X is termed a cogroup in **T** if $X = X'$. Let (Σ, μ_Σ) be a cogroup in **T** and assume that **T** is Σ-augmented as in (7.2) and (7.3) so that for each object X in **T** also a coaction

$$\varepsilon_X : X \to X \vee \Sigma \tag{2}$$

is given satisfying (7.2) (2). Then we say that **T** is *strongly augmented by* Σ if for each cogroup X' in **T** there exists a linear map (see (1.3))

$$\bar{\varepsilon}_{X'} : X' \to \Sigma \tag{3}$$

such that ε_X in (2) is given by the formula

$$\varepsilon_X = -i_\Sigma \bar{\varepsilon}_{X'} + i_X + i_\Sigma \bar{\varepsilon}_{X'} : X \to X \vee \Sigma \tag{4}$$

Here $+$ is defined by the cogroup structure μ_X of X and $i_\Sigma : \Sigma \to X \vee \Sigma, i_X : X \to X \vee \Sigma$ are the inclusions. Moreover for $X' = \Sigma$ we assume $\bar{\varepsilon}_\Sigma = 1$. For $\xi : X \to U, \alpha : X' \to U$ and $\sigma : \Sigma \to U$ in **T** we write

$$\begin{aligned} \xi + \alpha &= (\xi, \alpha)\mu_X \\ \xi^\sigma &= (\xi, \sigma)\varepsilon_X \end{aligned}$$

Then clearly all formulas (7.2) (a) ... (e) hold in an augmented theory of coactions. Moreover by (4) we get

$$\alpha^\sigma = -\sigma\bar{\varepsilon}_{X'} + \alpha + \sigma\bar{\varepsilon}_{X'} \tag{5}$$

in the group $\mathbf{T}(X', U)$.

(7.5) Remark. Each theory of coactions **T** is trivially augmented by the initial object $*$. In this case $\varepsilon_X : X \to X \vee * = X$ is the identity and $\bar{\varepsilon}_{X'} : X' \to *$ is the zero map of X'. Hence **T** is also trivially strongly augmented.

(7.6) Example. Let $\mathcal{D}$ be the class of discrete sets in **Top** and let $D \in \mathcal{D}$. Consider the theory of coactions

$$\mathbf{cone}(D, \mathcal{D}) = \mathbf{grd}(G)$$

in (I.2.11) which is the category of free groupoids G with $\mathrm{Ob}(G) = D$. Then $\mathbf{cone}(D, \mathcal{D})$ is augmented by the object

$$\Sigma = S^1 \times D \tag{1}$$

which coincides with S^1_α where $\alpha : D \to D$ is the identity; see (I.2.8). We have for each point $x \in D$ the inclusion $S^1_x = S^1 \times \{x\} \subset \Sigma$. For an object $X = C_{\alpha,\beta}$ in $\mathbf{cone}(D, \mathcal{D})$ we define the map

$$\varepsilon_X : C_{\alpha,\beta} \to C_{\alpha,\beta} \vee \Sigma \tag{2}$$

under D as follows. For $\alpha, \beta : E \to D$ and $e \in E$ let $I_e = I \times \{e\} \to C_{\alpha,\beta}$ be the map given by the definition of $C_{\alpha,\beta}$. Then the restriction of ε_X to the arc I_e is

$$\varepsilon_X \mid I_e = -S^1_{\alpha(e)} + I_e + S^1_{\beta(e)} \tag{3}$$

Since $C_{\alpha,\beta}$ is the union of D and such arcs $I_e, e \in E$, the map ε_X is well defined by (3). Now one can check that $\mathbf{cone}(D, \mathcal{D})$ with the augmentation maps (2) is an strongly augmented theory of coactions satisfying the properties in (7.4). If $D = *$ is a point then (2) above is a special case of (7.1). Moreover for a cogroup $A = C_{\alpha,\alpha} = S^1_\alpha$ in $\mathbf{cone}(D, \mathcal{D})$ we have the canonical map

$$\bar{\varepsilon}_A : S^1_\alpha \to \Sigma \tag{4}$$

which carries $S^1_e \subset S^1_\alpha$, $e \in E$, identically to $S^1_{\alpha(e)} \subset \Sigma$. Clearly by (3) we see that (7.4) (4) is satisfied. The strongly augmented theory of coactions $\mathbf{cone}(D, \mathcal{D})$ is a special case of (III.6.8) below; see (III.6.9).

Let $(\mathbf{T}, \Sigma)$ be an augmented theory of coactions. Then we obtain for each object X in $\mathbf{T}$ the morphism

$$\delta_X : X' \to \Sigma \vee X \tag{7.7}$$

which via the affine property is uniquely determined by the equation

$$i_X + \delta_X = (i_X, i_\Sigma)\varepsilon_X = i_X^{i_\Sigma} \tag{1}$$

Then we have for $(0,1) : \Sigma \vee X \to X$ the equation

$$(0,1)\delta_X = 0 : X' \to X \tag{2}$$

so that δ_X is trivial on X. In fact

$$\begin{aligned} (0,1)_*(i_X^{i_\Sigma}) &= 1_X^0 = 1_X \\ (0,1)_*(i_X + \delta_X) &= (0,1)_* i_X + (0,1)_* \delta_X \\ &= 1_X + (0,1)_* \delta_X \end{aligned}$$

Hence the affine property shows (2) by use of (1). In case X is a cogroup with trivial map $0 : X \to \Sigma$ we get also

$$(1,0)\delta_X = 0 : X' \to \Sigma \tag{3}$$

so that in this case δ_X is also trivial on Σ. We get (3) by the following equations

$$\begin{aligned} (1,0)(i_X^{i_\Sigma}) &= (0_* i_X)^{i_\Sigma} = 0^{i_\Sigma} = 0 \quad \text{see (a)} \\ (1,0)(i_X + \delta_X) &= 0_* i_X + (1,0)_* \delta_X = 0 + (1,0)_* \delta_X \end{aligned}$$

This yields by (1) equation (3).

For a morphism $f : X \to Y$ in $\mathbf{T}$ we consider the following diagram in $\mathbf{T}$.

$$\begin{array}{ccc} X' \vee X & \xrightarrow{(\delta_X, i_X)} & \Sigma \vee X \\ {\scriptstyle (\nabla f, f)}\downarrow & & \downarrow{\scriptstyle 1 \vee f} \\ Y' \vee Y & \xrightarrow{(\delta_Y, i_Y)} & \Sigma \vee Y \end{array}$$

(7.8) Lemma. *This diagram commutes in each augmented theory of coactions.*

Proof. We have the following equations

$$\begin{aligned} i_Y f + (1 \vee f)\delta_X &= (1 \vee f)(i_X + \delta_X) \\ &= (1 \vee f)(i_X^{i_\Sigma}) \\ &= (i_Y f)^{i_\Sigma} \quad \text{see (7.3) (e),} \end{aligned}$$

$$\begin{aligned} i_Y f + (\delta_Y, i_Y) \nabla f &= (i_Y + \delta_Y) f \quad \text{see (I.3.3) (1)} \\ &= (i_Y^{i_\Sigma}) f \\ &= (i_Y f)^{i_\Sigma} \quad \text{see (7.3) (d).} \end{aligned}$$

Hence the affine property shows

$$(1 \vee f)\delta_X = (\delta_Y, i_Y) \nabla f.$$

q.e.d.

We derive from (7.8) the following two results on morphisms in **premod**.

(7.9) Proposition. *Let* $(\mathbf{T}, \Sigma)$ *be an augmented theory of coactions and let* $\partial_X : X'' \to X$ *be an object in* **Coef**. *Then the composite*

$$X'' \vee \partial_X \xrightarrow{\nabla \partial_X \odot 1} X' \vee \partial_X \xrightarrow{\delta_X \odot 1} \Sigma \vee \partial_X$$

is the trivial ∂_X*-equivariant morphism in* **premod**(∂_X) *and hence via the functor* E *also in* **mod**(∂_X).

Proof. The morphism $\delta_X \odot 1$ in **premod** is well defined by (7.7) (2); see (I.5.5). Moreover we get

$$(\delta_X \odot 1)(\nabla \partial_X \odot 1) = v \odot 1$$

Here v is given via the quotient map j in (I.5.5) by the equation

$$\begin{aligned} v &= j((\delta_X, i_X) \nabla \partial_X) \\ &= j((1 \vee \partial_X)\delta_{X''}) \quad \text{see (7.8)} \\ &= 0 \end{aligned}$$

since $(1 \vee \partial_X)\delta_{X''} \in N$ in (I.5.5). This follows from (7.7) (2), (3). q.e.d.

(7.10) Proposition. *Let $(\mathbf{T}, \Sigma)$ be an augmented theory of coactions and let $u : \partial_X \to \partial_Y$ be a morphism in* **Coef** *represented by $f : X \to Y$. Then the diagram*

$$\begin{array}{ccc} X' \vee \partial_X & \xrightarrow{\delta_X \odot 1} & \Sigma \vee \partial_X \\ {\scriptstyle \nabla f \odot u}\downarrow & & \downarrow{\scriptstyle 1 \vee u} \\ Y' \vee \partial_Y & \xrightarrow{\delta_Y \odot 1} & \Sigma \vee \partial_Y \end{array}$$

commutes in **premod** *and hence via the functor E the diagram commutes also in* **mod**.

Proof. The diagram is the j-image of the diagram in (7.8) where j is the quotient map in (I.5.5). q.e.d.

We use the properties in (7.9) and (7.10) for the following definition of a weakly augmented theory of coactions which suffices to obtain the augmentation of the chain functor in (II.§ 6) below.

(7.11) Definition. We say that a theory $\mathbf{T}$ of coactions is *weakly augmented* by a cogroup Σ in $\mathbf{T}$ if for all X in $\mathbf{T}$ maps

$$\varepsilon_X : X \to X \vee \Sigma$$

are given such that $(1, 0)\varepsilon_X = 1_X$ and the composite in (7.9) is via the functor E trivial in $\mathbf{mod}(\partial_X)$ and the diagram in (7.10) is via the functor E commutative in **mod**. Here we do not assume that these properties hold in **premod**.

For example we see in (B.1.27) that the category **alg** of free algebras is weakly augmented.

(7.12) Lemma. *Let $(\mathbf{T}, \Sigma)$ be an augmented theory of coactions. Then the group $\mathbf{Coef}(\Sigma, \partial_Y)$ acts on the set $\mathbf{Coef}(\partial_X, \partial_Y)$. If $\sigma \in \mathbf{Coef}(\Sigma, \partial_Y)$ is represented by $\sigma_0 : \Sigma \to Y$ and if $u \in \mathbf{Coef}(\partial_X, \partial_Y)$ is represented by $f : X \to Y$ then the action is defined by $u^\sigma = \{f^{\sigma_0}\}$. One has the rules $u^\sigma \circ v = (u \circ v)^\sigma$ and $u \circ v^\sigma = (u \circ v)^{u_* \sigma}$.*

Proof. By (7.2) (2) the left hand side of the following diagram commutes.

$$\begin{array}{ccccc} X'' & \longrightarrow & X'' \vee \Sigma & \xrightarrow{\bar{f}, i_Y \sigma_0} & Y'' \vee Y \\ {\scriptstyle \partial_X}\downarrow & & \downarrow{\scriptstyle \partial_X \vee 1} & & \downarrow{\scriptstyle (\partial_Y, 1)} \\ X & \longrightarrow & X \vee \Sigma & \xrightarrow{(f, \sigma_0)} & Y \end{array}$$

Since f is ∂-compatible we obtain $\bar{f}$ such that also the right hand side commutes. Hence also $f^{\sigma_0} = (f, \sigma_0)\varepsilon_X$ is ∂-compatible. Now it is easy to see by the definition of ∂-equivalenc in (4.1) that $\xi^\sigma = \{f^{\sigma_0}\}$ does not depend on the choice of f and σ_0. q.e.d.

(7.13) Definition. Let $(\mathbf{T}, \Sigma)$ be an augmented theory of coactions and let $u : \partial_X \to \partial_Y$ be a morphism in **Coef** and $\sigma \in \mathbf{Coef}(\Sigma, \partial_Y)$. Then a function

$$\mathbf{mod}(A \vee \partial_X, B \vee \partial_Y)_u \to \mathbf{mod}(A \vee \partial_X, B \vee \partial_Y)_{u^\sigma}$$

is defined which carries $\xi \odot u$ to

$$(\xi \odot u)^\sigma = E(\{\xi_0^{i_Y \sigma_0}\} \odot u^\sigma) \tag{1}$$

Here u^σ is defined by (7.9) and $\xi_0 : A \to B \vee Y$ represents ξ and $\sigma_0 : \Sigma \to Y$ represents σ so that

$$\xi_0^{i_Y \sigma_0} = -i_Y \sigma_0 \bar{\varepsilon}_A + \xi_0 + i_Y \sigma_0 \bar{\varepsilon}_A \tag{2}$$

is defined by (7.4) (5). This is a special case of the action (5.4) (1) and therefore (1) is well defined since E in (5.6) is equivariant with respect to the action (5.4) (1). One readily checks the following rules by use of (7.3).

$$(\xi \odot u)^{\sigma_1 + \sigma_2} = ((\xi \odot u)^{\sigma_1})^{\sigma_2} \quad \text{and} \quad (\xi \odot u)^0 = \xi \odot u \tag{3}$$

$$((\xi_1 + \xi_2) \odot u)^\sigma = (\xi_1 \odot u)^\sigma + (\xi_2 \odot u)^\sigma \tag{4}$$

$$(\xi \odot u)^\sigma \circ (\eta \odot \nu) = ((\xi \odot u) \circ (\eta \odot \nu))^\sigma \tag{5}$$

$$(\xi \odot u) \circ (\eta \odot \nu)^\tau = ((\xi \odot u) \circ (\eta \odot \nu))^{u_* \tau} \tag{6}$$

Here (5) and (6) describe the compatibility of the action with the composition of morphisms in the category **mod**.

(7.14) Definition. Let $(\mathbf{T}, \Sigma)$ be an augmented theory of coactions and let $f : (B, \partial_Y) \to (A, \partial_X)$ be a chain map in **chain**. Then for $\sigma \in \mathbf{Coef}(\Sigma, \partial_X)$ the chain map

$$f^\sigma : (B, \partial_Y) \to (A, \partial_X) \tag{1}$$

is defined as follows. If f in degree n is given by $f_n \odot u$ then f^σ is given by the commutative diagram in $\mathbf{mod}(n \in \mathbb{Z})$

$$\begin{tikzcd}
B_{n+1} \vee \partial_Y \arrow[r, "d_{n+1} \odot 1"] \arrow[d, "(f_{n+1} \odot u)^\sigma"'] & B_n \vee \partial_Y \arrow[d, "(f_n \odot u)^\sigma"] \\
A_{n+1} \vee \partial_Y \arrow[r, "d_n \odot 1"] & A_n \vee \partial_Y
\end{tikzcd} \tag{2}$$

This diagram commutes if we omit the action of σ since f is a chain map. Hence by (7.10) (5), (6) diagram (2) commutes and hence f^σ is well defined. Clearly f^σ is u^σ-equivariant and not u-equivariant. Let $g : (B, \partial_Y) \to (A, \partial_X)$ be a v-equivariant chain map. Then we write

$$f \simeq_{\text{free}} g \iff \exists \sigma \text{ with } v^\sigma = u \text{ and } g^\sigma \simeq f \tag{3}$$

This is the notion of *free homotopy* in **chain**. One readily checks that free homotopy $\simeq_{\text{free}}$ is a natural equivalence relation on **chain**.

(7.15) Remark. For $\mathbf{T} = \mathbf{gr}$ a free homotopy corresponds exactly to the notion of homotopy of chain maps used by J.H.C. Whitehead [CHII] (10.1). This notion is relevant in the context of "Whitehead torsion". Clearly for $\mathbf{T} = \mathbf{gr}$ free chain homotopies correspond to free homotopies in (7.1); compare theorem 13 in J.H.C. Whitehead [CHII].

(7.16) Proposition. *Let* $\mathbf{T}$ *be an augmented theory of coactions. Then the quotient functor*

$$\varphi : \mathbf{chain}/\simeq \longrightarrow \mathbf{chain}/\simeq_{\text{free}}$$

reflects equivalences.

Proof. Assume f and g are chain maps with $(gf)^\sigma \simeq 1$ and $(fg)^\tau \simeq 1$. Let f be u-equivariant and let g be v-equivariant. Then $(vu)^\sigma = 1$ and $(uv)^\tau = 1$. This implies that $u^\tau v = 1$. Hence for $u : \partial_X \to \partial_Y \in \mathbf{Coef}$

$$u_*^\tau : \mathbf{Coef}(\Sigma, \partial_X) \to \mathbf{Coef}(\Sigma, \partial_Y)$$

is surjective. Hence for $\gamma, \gamma^\tau \in \mathbf{Coef}(\Sigma, \partial_Y)$ we obtain $\alpha \in \mathbf{Coef}(\Sigma, \partial_X)$ with $u^\tau \alpha = \gamma^\tau$. Since $u^\tau \alpha = (u\alpha)^\tau$ we get $u\alpha = \gamma$ and hence

$$u_* : \mathbf{Coef}(\Sigma, \partial_X) \to \mathbf{Coef}(\Sigma, \partial_Y)$$

is surjective. Let $\lambda \in \mathbf{Coef}(\Sigma, \partial_X)$ with $u_*(\lambda) = \tau$. Then we have

$$\begin{aligned} 1 &= (uv)^\tau = (uv)^{u_*\lambda} = uv^\lambda \\ 1 &= v^\sigma u \end{aligned}$$

Hence $v^\lambda = v^\sigma$ is the inverse of u so that u is an ismorphism in $\mathbf{Coef}$. Moreover we get

$$\begin{aligned} (gf)^\sigma &= g^\sigma f \simeq 1 \\ (fg)^\tau &= (fg)^{u_*\lambda} = f(g^\lambda) \simeq 1 \end{aligned}$$

Hence f is an isomorphism in $\mathbf{chain}/\simeq$. q.e.d.

(7.17) Remark. In chapter A we consider four examples of topological homotopy theories under an object D, see (A.§ 1), (A.§ 2), (A.§ 3) and (A.§ 4). In each case we obtain the theory $\mathbf{T}$ of coactions which is obtained by the 1-dimensional reduced CW-complexes (X^1, D). If D is discrete then $\mathbf{T}$ is always augmented and, in fact, strongly augmented by the spherical object $\Sigma = S^1_\alpha$ where α is the identity of D. Hence we can apply (7.16) in all these cases; compare (7.6).

Chapter II: Twisted Chain Complexes and Twisted Homology

Let $\mathbf{T}$ be a theory of coactions and let **chain** be the category of chain complexes as defined in (I.6.3). We introduce in this chapter the functor

$$K : \mathbf{Twist} \to \mathbf{chain}$$

which carries a presentation ∂_X to a chain complex d_X concentrated in degree 1 and 2.

In topology (i.e. for the theory $\mathbf{T} = \mathbf{gr}$ of free groups) one obtains d_X by the Fox derivative of the presentation ∂_X or equivalently, if X^2 is the 2-dimensional CW-complex given by the presentation ∂_X of the group $G = \pi_1 X^2$, then d_X is the differential $C_2(\tilde{X}^2) \to C_1(\tilde{X}^2)$ of the cellular chain complex of the universal covering $\tilde{X}^2$ of X^2.

The functor K leads to the definition of a "twisted chain complex" which is a pair $A \mid \partial_X$ consisting of a presentation ∂_X in **Twist** and a chain complex A which in degree ≤ 2 coincides with d_X. In topology such twisted chain complexes are the "admissible chain complexes" used by Wall [FC II]. The functor K induces the canonical functor

$$K : \mathbf{TWIST}_2^c \to \mathbf{TWIST}_1^c$$

where $\mathbf{TWIST}_2^c$ is the category of twisted chain complexes and $\mathbf{TWIST}_1^c$ is a subcategory of the category **chain** of chain complexes. Most results in this chapter are concerned with this functor K. As one of the main results we show in (5.4):

Theorem. *A map in $\mathbf{TWIST}_2^c$ is a twisted homotopy equivalence if and only if the induced chain map is a homotopy equivalence in* **chain**.

This result is needed in the proof of the homological Whitehead theorem in chapter VI. If $\mathbf{T} = \mathbf{gr}$ is the category of free groups then $\mathbf{TWIST}_2^c$ is isomorphic to the category of homotopy systems of J.H.C. Whitehead [CH], which are now termed crossed (chain) complexes in Baues [CH] and Brown-Higgins [CC]. The theorem above yields as a specialization theorem 12 of J.H.C. Whitehead [CH]. In fact, part of this chapter may be considered as an extension of Whitehead's classical Combinatorial Homotopy II paper [CH] to categorical algebra.

The general situation, however, is more complicated than the case $\mathbf{T} = \mathbf{gr}$ since the module Γ_1 in § 2 vanishes for $\mathbf{T} = \mathbf{gr}$. This module is used to describe the obstruction for the realizability of a chain map by a twisted chain map; see § 3.

We use the category $\mathbf{TWIST}_1^c$ to define the homology and cohomology of objects in the category **Coef** of coefficients; see Appendix A. In a similar way we use $\mathbf{TWIST}_2^c$ for the definition of the twisted homology of such objects in Appendix B. In fact Γ_1 is a special twisted homology group. Moreover in low degrees twisted homology specializes to Leedham-Green [HV] and André-Quillen [CR] homology.

We point out that all constructions and results in this chapter are available whenever a theory **T** of coactions is given. In section § 6 we consider the case of an augmented theory of coactions.

1 Twisted Chain Complexes

We here combine the category **Twist** and the category **chain** of chain complexes in **mod** to obtain the category of twisted chain complexes.

(1.1) Definition. We define a functor

$$K : \mathbf{Twist} \to \mathbf{chain}$$

as follows. For an object $\partial_X : X'' \to X$ in **Twist** let $K(\partial_X)$ be the chain complex (concentrated in degree 1 and 2) given by the ∂_X-equivariant map in **mod**

$$d_X = E(\nabla \partial_X \odot 1) : X'' \vee \partial_X \to X' \vee \partial_X$$

A map $(f'', f) : \partial_X \to \partial_Y$ in **Twist** is carried via K to the chain map (f_1, f_2)

$$\begin{array}{ccc} X'' \vee \partial_X & \xrightarrow{d_X} & X' \vee \partial_X \\ {\scriptstyle f_2 = E(f'' \odot u)} \downarrow & & \downarrow {\scriptstyle E(\nabla f \odot u) = f_1} \\ Y'' \vee \partial_Y & \xrightarrow{d_Y} & Y' \vee \partial_Y \end{array}$$

where $u : \partial_X \to \partial_Y \in \mathbf{Coef}$ is represented by f.

(1.2) Lemma. *The functor K is well defined.*

Proof. We have to show $f_1 d_X = d_Y f_2$. For this we use lemma (I.3.8) where ξ yields a u-equivariant map

$$\xi \odot u : X'' \vee \partial_X \to Y_2'' \vee Y' \vee Y_1'' \vee \partial_Y$$

in **premod** which satisfies $E(\xi \odot u) = 0$ by the second isomorphism in (I.5.6) (iii). Therefore the diagram in (I.3.8) induces via E a commutative diagram in **mod** which coincides with the diagram (1.1). Moreover K is a functor since for $(g'', g) : \partial_Y \to \partial_Z$ in **Twist** with $v = \{g\}$ we get

$$E(\nabla g \odot v) \circ E(\nabla f \odot u) = E(\nabla(gf) \odot vu)$$

This follows from (I.3.4). q.e.d.

(1.3) Definition. We introduce a natural equivalence relation $\sim_E$ on the category **Twist**. Let $(f'', f), (g'', g) : \partial_X \to \partial_Y$ be maps in **Twist**. Then we set $(f'', f) \sim_E (g'', g)$ if $f = g$ and $E(f'' \odot \{f\}) = E(g'' \odot \{g\})$ where E is the quotient functor for **mod** in (I.5.6). Let

$$E : \mathbf{Twist} \to \mathbf{Twist}/{\sim_E}$$

be the quotient functor. We denote the equivalence class $E(f'', f)$ by (Ef'', f). Clearly the functor K in (1.1) induces a well defined functor

$$K : \mathbf{Twist}/{\sim_E} \to \mathbf{chain}.$$

Remark. Let $1_A : A \to A$ be the identity of the cogroup A in $\mathbf{T}$. Then 1_A is an object in **Twist** and one gets the pre-crossed module

$$\begin{tikzcd} \mathbf{Twist}(1_A, \partial_Y) \arrow[r, "\delta"] \arrow[d, equal] & \mathbf{T}(A, Y) \\ \mathbf{T}(A, Y'' \vee Y)_2 \arrow[ur, "(\partial_Y, 1)_*"'] & \end{tikzcd}$$

Here δ carries (f'', f) to f. Moreover δ induces the homomorphism of groups

$$\delta : \mathbf{Twist}(1_A, \partial_Y)/{\sim_E} \to \mathbf{T}(A, Y)$$

which is a crossed module.

(1.4) Definition. We introduce a natural equivalence relation $\simeq_E$ on **Twist** as follows. Let $(f'', f), (g'', g) : \partial_X \to \partial_Y$ be maps in **Twist**. We say that these maps are *E-homotopic*, $(f'', f) \simeq_E (g'', g)$, if there exists

$$\alpha : X' \to Y'' \vee Y \in \mathbf{T}$$

trivial on Y such that

$$g = f + (\partial_Y, 1)\alpha \in \mathbf{T}(X, Y) \tag{1}$$

and in $\mathbf{mod}\,(X'' \vee \partial_X, Y'' \vee \partial_Y)_u$

$$-E(f'' \odot u) + E(g'' \odot u) = E(\alpha \odot u)d_X \tag{2}$$

Here $u = \{f\} = \{g\}$ is the map in **Coef** represented by f and g since (1) holds. Clearly $\simeq_E$ is actually a natural equivalence relation on $\mathbf{Twist}/{\sim_E}$ so that one gets the sequence of quotient functors

$$\mathbf{Twist} \xrightarrow{E} \mathbf{Twist}/{\sim_E} \xrightarrow{q} \mathbf{Twist}/{\simeq_E} \xrightarrow{c} \mathbf{Coef}$$

All functors $E, q,$ and c are the identity on objects and c carries the homotopy class of (f'', f) to the ∂-equivalence class of f. For this we point out that (1) above implies ∂-equivalence $g \sim f$.

(1.5) Lemma. *The functor K in (1.1) induces a well defined functor*

$$K : \mathbf{Twist}/{\simeq_E} \to \mathbf{chain}/{\simeq}$$

between homotopy categories. Moreover K is compatible with coefficient functors, that is $cK = c$.

Proof. We have to show that $g = f + (\partial_Y, 1)\alpha$ implies

$$E(\nabla g \odot u) = E(\nabla f \odot u) + E(\nabla\partial_Y \odot 1)E(\alpha \odot u) \tag{1}$$

For this we compute ∇g by the second equation in (I.3.4). Hence for $\beta = (\partial_Y, 1)\alpha$ we get

$$\begin{aligned} \nabla(g) &= -i_Y\beta + \nabla f + i_Y\beta + \nabla\beta \\ &= (\nabla f)^\beta + \nabla\beta \end{aligned} \tag{2}$$

We know that $(\nabla f)^\beta \odot u = \nabla f \odot u$ in **premod** since β represents the trivial element in $\mathbf{Coef}(X', \partial_Y)$; see (I.5.5). Hence we get

$$E(\nabla g \odot u) = E(\nabla f \odot u) + E(\nabla\beta \odot u) \tag{3}$$

Here $\nabla\beta$ is computed as in (I.3.12) (1) where we replace f'' by α. Hence for

$$\xi = \nabla\alpha - (i_{Y_2''}, i_Y)\alpha \tag{4}$$

we have $E(\xi \odot u) = 0$ and

$$q\xi = \nabla\beta - (\nabla\partial_Y, i_Y)\alpha \tag{4}$$

Hence we get

$$\begin{aligned} E(\nabla\beta \odot u) &= E((\nabla\partial_Y, i_Y)\alpha \odot u) \\ &= E(\nabla\partial_Y \odot 1)E(\alpha \odot u) \end{aligned} \tag{5}$$

and the proof of (1) is complete. q.e.d.

We now use the functor K for the definition of a new category of twisted chain complexes. Let $\mathbf{chain}_{1,2}$ and $\mathbf{chain}_{\geq 1}$ be the full subcategories of **chain** consisting of chain complexes concentrated in degree $\{1, 2\}$ and in degree ≥ 1 respectively. Then we have the following pull back diagram of categories

$$\begin{tikzcd} \mathbf{TWIST}_2 \arrow[r] \arrow[d] & \mathbf{chain}_{\geq 1} \arrow[d, "r"] \\ \mathbf{Twist}/{\sim_E} \arrow[r, "K"'] & \mathbf{chain}_{1,2} \end{tikzcd} \tag{1.6}$$

Here r is the forgetful functor and K is the functor in (1.1). We now describe the category $\mathbf{TWIST}_2$ and a full subcategory $\mathbf{TWIST}_2^c$ explicitly as follows.

(1.7) Definiton. We define the category $\mathbf{TWIST}_2$. Objects are chain complexes (A, ∂_X) in $\mathbf{chain}_{\geq 1}$ with the properties

$$\begin{cases} A_1 = X',\ A_2 = X'' \\ d_2 = d_X = E(\nabla\partial_X \odot 1) : X'' \vee \partial_X \to X' \vee \partial_X \end{cases} \tag{1}$$

A morphism, termed a *twisted chain map*,

$$\bar{f} = (f_{\geq 1}, Ef'', f) : (A, \partial_X) \to (B, \partial_Y) \tag{2}$$

is given by a morphism $(Ef'', f) : \partial_X \to \partial_Y$ in $\mathbf{Twist}/{\sim_E}$ and a u-equivariant chain map $f_{\geq 1} : (A, \partial_X) \to (B, \partial_Y)$ in $\mathbf{chain}$. Here $u = \{f\}$ is represented by f and

$$\begin{cases} f_1 = E(\nabla f \odot u), \\ f_2 = E(f'' \odot u). \end{cases} \tag{3}$$

(1.8) Definition. Let

$$\mathbf{TWIST}_2^c \subset \mathbf{TWIST}_2$$

be the full subcategory of all objects (A, ∂_X) satisfying the following *cocycle condition*: For d_3 in (A, ∂_X) there exists $\partial_3 \in \mathbf{T}(A_3, X'' \vee X)_2$ such that

$$\begin{cases} d_3 = E(\partial_3 \odot 1) \\ (\partial_X, 1)\partial_3 = 0 \quad \text{in } \mathbf{T}(A_3, X) \end{cases}$$

In this case we denote the object (A, ∂_X) by $A|\partial_X \in \mathbf{TWIST}_2^c$.

(1.9) Definition. We define the following subcategory

$$\mathbf{TWIST}_1^c \subset \mathbf{chain}_{\geq 1}$$

Objects are chain complexes (A, ∂_X) with the property

$$\begin{cases} A_1 = X',\ A_2 = X'' \\ d_2 = d_X = E(\nabla\partial_X \odot 1) : X'' \vee \partial_X \to X' \vee \partial_X \end{cases} \tag{1}$$

A morphism, termed a *∂-compatible chain map*,

$$\tilde{f} = (f_{\geq 1}, u) : (A, \partial_X) \to (B, \partial_Y) \tag{2}$$

is a u-equivariant chain map in $\mathbf{chain}$ for which there exists a ∂-compatible map $f : X \to Y$ representing $u = \{f\} : \partial_X \to \partial_Y$ in $\mathbf{Coef}$ such that $f_1 = E(\nabla f \odot u)$.

We have the canonical functor

$$K : \mathbf{TWIST}_2^c \to \mathbf{TWIST}_1^c \tag{1.10}$$

which is the identity on objects and which carries $\bar{f} = (f_{\geq 1}, Ef'', f)$ to $\tilde{f} = (f_{\geq 1}, u)$ with $u = \{f\}$. In the next two sections we study properties of this functor.

2 The Module Γ_1

Let $\mathbf{T}$ be a theory of coactions and let $\mathbf{mod}$ be an associated theory of modules as defined in I.§ 5.

(2.1) Definition. Let $\partial_Y : Y'' \to Y$ be an object in $\mathbf{Twist}$. We define for each object $A \vee \partial_X$ in $\mathbf{mod}$ and $u \in \mathbf{Coef}\,(\partial_X, \partial_Y)$ the abelian group $\Gamma_1(\partial_Y)(A \vee \partial_X)_u = \text{cokernel}(E_3)$ by the following diagram in which rows and columns are exact sequences of groups.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\Gamma'' \arrow[r, "E_3"] \arrow[d] & \Gamma' \arrow[r, "\gamma"] \arrow[d] & \Gamma_1(\partial_Y)(A \vee \partial_X)_u \\
\mathbf{T}(A, Y'' \vee Y)_2 \arrow[r, "E_2"] \arrow[d, "(\partial_Y, 1)_*"] & \mathbf{mod}(A \vee \partial_X, Y'' \vee \partial_Y)_u \arrow[r] \arrow[d, "(d_Y)_*"] & 0 \\
\mathbf{T}(A, Y) \arrow[r, "E_1"] & \mathbf{mod}(A \vee \partial_X, Y' \vee \partial_Y)_u &
\end{tikzcd}$$

Here E_1 and E_2 are defined by

$$\begin{cases} E_1(a) = E(\nabla a \odot u) \\ E_2(b) = E(b \odot u) \end{cases}$$

Here E is the quotient functor in (I.5.6) and $\nabla a \odot u$ and $b \odot u$ with $a \in \mathbf{T}(A, Y)$ and $b \in \mathbf{T}(A, Y'' \vee Y)$, denote morphisms in $\mathbf{premod}$ by (I.5.3). The operator E_1 in general is not a homomorphism but satisfies $E_1(0) = 0$. Using (I.5.6) (ii) we see that the map $E(1_A \vee u) : A \vee \partial_X \to A \vee \partial_Y$ in $\mathbf{mod}$ induces the isomorphism

$$(1_A \vee u)^* : \Gamma_1(\partial_Y)(A \vee \partial_Y)_1 \cong \Gamma_1(\partial_Y)(A \vee \partial_X)_u$$

which we use as an identification. This abelian group yields the *right* $\mathbf{mod}(\partial_Y)$ *-module*

$$\Gamma_1(\partial_Y) : \mathbf{mod}(\partial_Y)^{\mathrm{op}} \to \mathbf{Ab}$$

which carries the object $A \vee \partial_Y$ to $\Gamma_1(\partial_Y)(A) = \Gamma_1(\partial_Y)(A \vee \partial_Y)_1$ and which carries the ∂_Y-equivariant map g to g^* with $g^*\gamma(\xi) = \gamma(\xi g)$; see (2.3) below.

(2.2) Lemma. *The diagram in (2.1) commutes and E_1 is a j-crossed homomorphism where $j : \mathbf{T}(A, Y) \twoheadrightarrow \mathbf{Coef}(A, \partial_Y)$ is the quotient map.*

Proof. We have

$$E_1(\partial_Y, 1)_* b = E(\nabla((\partial_Y, 1)b) \odot u)$$

Here we see as in the proof of (I.3.8) that

$$\nabla((\partial_Y, 1)b)) = q(\nabla b)$$

and that $\xi = \nabla b - (i_{Y_2''}, i_Y)b$ is trivial on $Y_1'' \vee Y, Y' \vee Y$ and $Y_2'' \vee Y$. Hence by the second equation in (I.5.6) (iii) we see that $E(\xi \odot u) = 0$ so that we get

$$\begin{aligned} E_1(\partial_Y, 1)_* b &= E(q(\nabla b) \odot u) \\ &= E(q(i_{Y_2''}, i_Y)b \odot u) \\ &= E((\nabla\partial_Y, i_Y)b \odot u) \\ &= E(\nabla\partial_Y \odot 1)E(b \odot u) \\ &= d_Y\, E_2(b) \end{aligned}$$

and therefore the diagram in (2.1) commutes. We know by (I.5.6) that E is equivariant with respect to the action of $\mathbf{Coef}(A, \partial_Y)$. Therefore E_1 is a j-crossed homomorphism since we can apply the second equation in (I.3.4). q.e.d.

(2.3) Lemma. *$\Gamma_1(\partial_Y)$ is a well defined right* $\mathbf{mod}(\partial_Y)$*-module.*

Proof. Let $g : B \vee \partial_Z \to A \vee \partial_X$ be a v-equivariant map in $\mathbf{mod}$. We have to show that

$$g^* : \Gamma_1(\partial_Y)(A \vee \partial_X)_u \to \Gamma_1(\partial_Y)(B \vee \partial_Z)_{uv}$$

defined by $g^*\,\gamma(x) = \gamma\, g^*(x)$ is a well defined homomorphism. Here we have $x \in \Gamma' = \mathrm{kernel}(d_Y)_*$. Since clearly $g^*(d_Y)_* = (d_Y)_* g^*$ we have also $g^* x \in \mathrm{kernel}(d_Y)_*$ so that $\gamma(g^* x)$ is defined. Morover we have to check that for $x = E_3\, y$ there is $\tilde{y}$ with $g^* x = E_3\, \tilde{y}$. Here we have $y \in \mathbf{T}(A, Y'' \vee Y)_2$ with $(\partial_Y, 1)_* y = 0$. On the other hand E in (I.5.6) is full so that $g = E(\tilde{g} \odot v)$ with $\tilde{g} \in \mathbf{T}(B, A \vee X)_2$. Assume $u = \{f\}$ is represented by $f : X \to Y$. Then we have the following commutative diagram in $\mathbf{T}$

$$\begin{tikzcd} B \arrow[r, "\tilde{g}"] \arrow[dr, "0"'] & A \vee X \arrow[r, "{(y, i_Y f)}"] \arrow[d, "{(0,1)}"] & Y'' \vee Y \arrow[d, "{(\partial_Y, 1)}"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

Here we see that $\tilde{y} = (y, i_Y f)\tilde{g}$ satisfies $(\partial_Y, 1)\tilde{y} = 0$ and

$$\begin{aligned} E_3\, \tilde{y} &= E((y, i_Y f)\tilde{g} \odot uv) \\ &= E(y \odot u)E(\tilde{g} \odot v) \\ &= g^*(E_3 y) = g^*(x) \end{aligned}$$

q.e.d.

(2.4) Proposition. *Each map $(g'', g) : \partial_Y \to \partial_W$ in* **Twist** *representing $w = \{g\}$ in* **Coef** *induces a map*

$$(g'', g)_* : \Gamma_1(\partial_Y) \to w^* \Gamma_1(\partial_W)$$

of right $\mathbf{mod}(\partial_Y)$*-modules.*

Proof. By (1.1) the map (g'', g) induces a chain map $K(g'', g) = (g_1, g_2) : d_Y \to d_W$. Hence we see that $(g_2)_*$ carries kernel$(d_Y)_*$ to kernel$(d_W)_*$ so that

$$(g'', g)_* : \Gamma_1(\partial_Y)(A \vee \partial_X)_u \to \Gamma_1(\partial_W)(A \vee \partial_X)_{wu} \tag{1}$$

is defined by

$$(g'', g)_* \gamma x = \gamma (g_2)_* x \tag{2}$$

with $g_2 = E(g'' \odot w)$. We have to check that for $x = E_3\, y$ there is $\tilde{y}$ with $(g_2)_* x = E_3\, \tilde{y}$. Now the diagram

$$\begin{tikzcd}
A \arrow[r, "y"] \arrow[dr, "0"'] & Y'' \vee Y \arrow[r, "{(g'', i_W g)}"] \arrow[d, "{(\partial_Y, 1)}"] & W'' \vee W \arrow[d, "{(\partial_W, 1)}"] \\
 & Y \arrow[r, "g"] & W
\end{tikzcd}$$

commutes so that we have for $\tilde{y} = (g'', i_W g) y$ the equation $(\partial_W, 1)_* \tilde{y} = 0$ and

$$\begin{aligned}
E_3\, \tilde{y} &= E(\tilde{y} \odot uw) \\
&= E((g'', i_w g) y \odot wu) \\
&= E(g'' \odot w) E(y \odot u) \\
&= (g_2)_* E_3(y) = (g_2)_*(x).
\end{aligned}$$

q.e.d.

(2.5) Lemma. *Let $(h'', h), (g'', g) : \partial_Y \to \partial_W$ be maps in* **Twist** *with $h = g$. Then the maps*

$$(g'', g)_* = (h'', h)_* : \Gamma_1(\partial_Y) \to w^* \Gamma_1(\partial_W)$$

coincide.

This lemma shows that each ∂-compatible map $g : Y \to W$ induces a well defined natural transformation $g_* : \Gamma_1(\partial_Y) \to w^* \Gamma_1(\partial_W)$. In fact, if $\mathbf{T}$ has "enough objects and modules" then $g_* = w_* : \Gamma_1(\partial_Y) \to w^* \Gamma_1(\partial_W)$ depends only on the induced map $w = \{g\}$ in **Coef**, see (7.10) below.

Proof. The condition $h = g$ implies

$$(\partial_W, 1)h'' = (\partial_W, 1)g''$$

and hence we get an element

$$\xi = -h'' + g'' \in \mathbf{T}(Y'', W'' \vee W)_2$$

with $(\partial_W, 1)\xi = 0$. Hence we get

$$\begin{aligned}(g'', g)_*\gamma x &= \gamma\, E(g'' \odot w)_* x \\ &= \gamma\, E((h'' + \xi) \odot w)_* x \\ &= \gamma\, E(h'' \odot w)_* x + \gamma\, E(\xi \odot w)_* x \\ &= \gamma\, E(h'' \odot w)_* x \\ &= (h'', h)_* \gamma\, x\end{aligned}$$

Here we use the following facts. Choose for x an element y with $E_2 y = x$. Then we get:

$$\begin{aligned}E(\xi \odot w)_* x &= E(\xi \odot w) E_2 y \\ &= E(\xi \odot w) E(y \odot u) \\ &= E((\xi, g) y \odot wu) \\ &= E_3((\xi, g) y)\end{aligned}$$

where the last equation holds since

$$\begin{aligned}(\partial_W, 1)((\xi, g) y) &= ((\partial_W, 1)\xi, g) y \\ &= (0, g) y = 0\end{aligned}$$

q.e.d.

3 The Obstruction for the Twisted Realization of a Chain Map

We consider the “realizability” of morphisms with respect to the functor

$$K : \mathbf{TWIST}_2^c \to \mathbf{TWIST}_1^c \tag{3.1}$$

in (2.8). Let $A|\partial_X$ and $B|\partial_Y$ be objects in $\mathbf{TWIST}_2^c$ and let $(A, \partial_X) = K(A|\partial_X)$ and $(B, \partial_Y) = K(B|\partial_Y)$ be the corresponding chain complexes in $\mathbf{TWIST}_1^c$. Let

$$\tilde{f} : (A, \partial_X) \to (B, \partial_Y)$$

be a u-equivariant map in $\mathbf{TWIST}_1^c$. A *K-realization* of $\tilde{f}$ is a map

$$\bar{f} : A|\partial_X \to B|\partial_Y$$

in $\mathbf{TWIST}_2^c$ with $K(\bar{f}) = \tilde{f}$.

(3.2) Theorem. *A K-realization of $\tilde{f}$ exists if and only if an obstruction element*

$$\mathcal{O}\{\tilde{f}\} \in H^2(A, \partial_X; u^*\Gamma_1(\partial_Y))$$

vanishes.

Here we use the cohomology of the chain complex (A, ∂_X) with coefficients in the right $\mathbf{mod}(\partial_Y)$-module $\Gamma_1(\partial_Y)$ in (2.1).

(3.3) Definition of the obstruction. Let $\tilde{f}$ be the chain map $\tilde{f} = (f_{\geq 1}, u)$. If $\tilde{f}$ is in $\mathbf{TWIST}_1^c$ there exists a map $(f'', f) : \partial_X \to \partial_Y$ in **Twist** with $u = \{f\}$ and $f_1 = E(\nabla f \odot u)$. Hence we have

$$\begin{aligned} d_2 f_2 &= f_1 d_2 = f_1 d_X \\ &= E(\nabla f \odot u) d_X \\ &= d_Y\, E(f'' \odot u) = d_2\, E(f'' \odot u) \end{aligned}$$

Therefore the element

$$-f_2 + E(f'' \odot u) \in \mathbf{mod}(X'' \vee \partial_X, Y'' \vee \partial_Y)_u$$

satisfies $(d_Y)_*(-f_2 + E(f'' \odot u)) = 0$ so that

$$\beta_f = \gamma(-f_2 + E(f'' \odot u)\} \in \Gamma_1(\partial_Y)(X'' \vee \partial_X)_u$$

with $X'' = A_2$ is defined. In lemma (3.4) we show that β_f is a cocycle so that β_f represents a cohomology class

$$\mathcal{O}(\tilde{f}) = \{\beta_f\} \in H^2(A, \partial_X, u^*\Gamma_1(\partial_Y))$$

(3.4) Lemma. $d_3^*\beta_f = 0$.

Proof. We consider the elements

$$d_3^* f_2, d_3^* E(f'' \odot u) \in \mathbf{mod}(A_3 \vee \partial_X, Y'' \vee \partial_Y)_u. \tag{1}$$

We have

$$(d_Y)_* d_3^* f_2 = d_Y f_2 d_3 = d_Y d_3 f_3 = 0 \tag{2}$$

since $d_2 = d_Y$. Since $(d_Y)_*(-f_2 + E(f'' \odot u)) = 0$ we also get

$$(d_Y)_* d_3^* E(f'' \odot u) = 0 \tag{3}$$

Hence the elements

$$\gamma(d_3^* f_2), \gamma(d_3^* E(f'' \odot u)) \in \Gamma_1(\partial_Y)(A_3 \vee \partial_X)_u \tag{4}$$

are defined. We claim that both elements in (4) are trivial; this implies $d_3^*\beta_f = 0$. Let $\tilde{f}_3 \in \mathbf{T}(A_3, B_3 \vee Y)_2$ be an element with $f_3 = E(\tilde{f}_3 \odot u)$; such an element exists since E is full. On the other hand we have by the cocycle condition in (1.8) an element ∂_3 with $E(\partial_3 \odot 1) = d_3$. Hence we get

$$\begin{aligned} d_3^* f_2 &= d_3 f_3 \\ &= E(\partial_3 \odot 1) E(\tilde{f}_3 \odot u) \\ &= E((\partial_3, i_Y)\tilde{f}_3 \odot u) \end{aligned} \tag{5}$$

Here $(\partial_Y, 1)\partial_3 = 0$ implies that $y = (\partial_3, i_Y)\tilde{f}_3$ satisfies

$$\begin{aligned} (\partial_Y, 1)_* y &= ((\partial_Y, 1)\partial_3, 1)\tilde{f}_3 \\ &= (0, 1)\tilde{f}_3 = 0 \end{aligned}$$

Hence we get for E_3 in (2.1)

$$d_3^* f_2 = E(y \odot u) = E_3(y) \tag{6}$$

Since $\gamma E_3 = 0$ we thus have $\gamma d_3^* f_2 = 0$. Next we have

$$\begin{aligned} d_3^* E(f'' \odot u) &= E(f'' \odot u) E(\partial_3 \odot 1) \\ &= E((f'', i_Y f)\partial_3 \odot u) \quad \text{where } u = \{f\}. \end{aligned} \tag{7}$$

Here $z = (f'', i_Y f)\partial_3$ satisfies

$$\begin{aligned} (\partial_Y, 1)_* z &= (\partial_Y, 1)(f'', i_Y f)\partial_3 \\ &= ((\partial_Y, 1) f'', f)\partial_3 \\ &= (f \partial_X, f)\partial_3 \\ &= f(\partial_X, 1)\partial_3 = 0 \end{aligned}$$

Here again we use the cocycle condition. Hence $z \in \Gamma''$ and therefore

$$d_3^* E(f'' \odot u) = E(y \odot u) = E_3(z) \tag{8}$$

satisfies $\gamma\, d_3^* E(f'' \odot u) = 0$. q.e.d.

(3.5) Lemma. *$\mathcal{O}(\tilde{f})$ is well defined by $\tilde{f}$.*

Proof. Let (g'', g) be a further map in **Twist** with $u = \{g\}$ and $f_1 = E(\nabla g \odot u)$. Since $u = \{g\} = \{f\}$ there is $\alpha \in \mathbf{T}(X', Y'' \vee Y)_2$ with

$$g = f + (\partial_Y, 1)\alpha. \tag{1}$$

This implies

$$E(\nabla g \odot u) = E(\nabla f \odot u) + d_Y\, E(\alpha \odot u)$$

and hence since $f_1 = E(\nabla g \odot u) = E(\nabla f \odot u)$ we get

$$d_Y\,E(\alpha \odot u) = 0 \tag{2}$$

so that $\gamma(E)(\alpha \odot u)$ is defined. Moreover we have

$$\begin{aligned}
&(\partial_Y, 1)_*(-(\alpha, i_Y f)(\nabla\partial_X) - f'' + g'')\\
&= -(\partial_Y, 1)(\alpha, i_Y f)(\nabla\partial_X) - f\partial_X + g\partial_X\\
&= -((\partial_Y, 1)\alpha, f)(\nabla\partial_X) - f\partial_X + (f + (\partial_Y, 1)\alpha)\partial_X\\
&= 0
\end{aligned} \tag{3}$$

In the last equation we use the equation in (I.3.2). Now (3) shows that

$$y = -(\alpha, i_Y f) \triangledown \partial_X - f'' + g'' \tag{4}$$

satisfies $(\partial_Y, 1)_* y = 0$ or $y \in \Gamma''$. Moreover we get

$$\begin{aligned}
E_3(y) = &- E((\alpha, i_Y f) \triangledown \partial_X \odot u)\\
&- E(f'' \odot u) + E(g'' \odot u)
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
E((\alpha, i_Y f) \triangledown \partial_X \odot u) &= E(\alpha \odot u) E(\nabla\partial_X \odot 1)\\
&= E(\alpha \odot u) d_X
\end{aligned} \tag{6}$$

Now (5) and (6) show that $\beta_f = \gamma(-f_2 + E(f'' \odot u))$ and $\beta_g = \gamma(-f_2 + E(g'' \odot u))$ satisfy

$$-\beta_f + \beta_g = d_X^* \gamma(E(\alpha \odot u)) \tag{7}$$

where we use (2). Hence β_f and β_g differ only by a cocycle so that the cohomology classes $\{\beta_f\} = \{\beta_g\}$ coincide. q.e.d.

(3.6) Proof of (3.2). If $\tilde{f}$ satisfies $K(\bar{f}) = \tilde{f}$ then one has by $\bar{f}$ a pair (f'', f) with $f_2 = E(f'' \odot u)$ and hence $\mathcal{O}(\tilde{f}) = 0$ by (3.3). Now assume $\tilde{f}$ satisfies $\mathcal{O}(\tilde{f}) = 0$. We have to construct $\bar{g} = (g_{\geq 3}, Eg'', g)$ in $\mathbf{TWIST}_2^c$ with $K(\bar{g}) = \tilde{f}$. We first choose (f'', f) as in (3.3). Then $\mathcal{O}(\tilde{f}) = 0$ implies that there is

$$\begin{cases} \beta \in \Gamma_1(\partial_Y)(X' \vee \partial_X)_u \quad \text{with} \\ (d_X)^* \beta = \gamma(-f_2 + E(f'' \odot u)) \end{cases} \tag{1}$$

Using diagram (2.1) we choose

$$\beta_2 \in \mathbf{mod}(X' \vee \partial_X, Y'' \vee \partial_Y)_u \tag{2}$$

with $(d_Y)_* \beta_2 = 0$ and $\gamma\beta_2 = \beta$ and we choose

$$\beta_1 \in \mathbf{T}(X', Y'' \vee Y)_2 \tag{3}$$

with $E_2\beta_1 = \beta_2$. We define

$$g = f + (\partial_Y, 1)_*\beta_1 \tag{4}$$

Then we have as in the proof of (1.5)

$$\begin{aligned} E(\nabla g \odot u) &= E(\nabla f \odot u) + d_Y E(\beta_1 \odot u) \\ &= f_1 + d_Y E_2 \beta_1 \\ &= f_1 + d_Y \beta_2 = f_1 \end{aligned} \tag{5}$$

Moreover we find g'' as follows. We have by (1) and (2)

$$\gamma(-\beta_2(\nabla\partial_X \odot 1) - E(f'' \odot u) + f_2) = 0 \tag{6}$$

In fact, $d_Y\beta_2 = 0$ and

$$d_Y f_2 = f_1 d_X = E(\nabla f \odot u) d_X = d_Y E(f'' \odot u)$$

so that the left hand side of (6) is well defined. Using (6) we can choose by the diagram in (3.3) an element

$$\delta \in \mathbf{T}(X'', Y'' \vee Y)_2 \tag{7}$$

with

$$\begin{cases} (\partial_Y, 1)_*(\delta) = 0 \\ E_3(\delta) = -\beta_2 E(\nabla\partial_X \odot 1) - E(f'' \odot u) + f_2 \end{cases}$$

Now we set

$$g'' = f'' + (\beta_1, i_Y f) \nabla \partial_X + \delta. \tag{8}$$

Then $(g'', g) : \partial_X \to \partial_Y$ is a map in **Twist** since we have

$$\begin{aligned} (\partial_Y, 1)_* g'' &= (\partial_Y, 1) f'' + (\partial_Y, 1)(\beta_1, i_Y f) \nabla \partial_X + (\partial_Y, 1)\delta \\ &= f\partial_X + ((\partial_Y, 1)\beta_1, f) \nabla \partial_X \\ &= (f + (\partial_Y, 1)\beta_1)\partial_X; \quad \text{see (3.3)} \\ &= g\,\partial_X \end{aligned}$$

Moreover we get by (8) and (3.3) and (7)

$$\begin{aligned} E(g'' \odot u) &= E(f'' \odot u) + E((\beta_1, i_Y f) \nabla \partial_X \odot u) + E_3(\delta) \\ &= E(\beta_1 \odot u) E(\nabla\partial_X \odot 1) + f_2 - \beta_2 E(\nabla\partial_X \odot 1) \\ &= f_2 \end{aligned} \tag{9}$$

since $E(\beta_1 \odot u) = E_2\beta_1 = \beta_2$. Now we set $g_i = f_i$ for $i \geq 3$. Then we have constructed by (5) and (9) a map $\bar{g}$ with $K(\bar{g}) = \tilde{f}$. q.e.d.

4 Twisted Homotopies

We introduce the notion of homotopy for twisted chain maps and ∂-compatible chain maps as defined in (1.7) and (1.9) and we compare the corresponding homotopy categories.

(4.1) Definition. Consider twisted chain maps $\bar{f}, \bar{g} : A|\partial_X \to B|\partial_Y$ in $\mathbf{TWIST}_2^c$ with $\bar{f} = (f_{\geq 1}, Ef'', f)$ and $\bar{g} = (g_{\geq 1}, Eg'', g)$. Then $\bar{f}$ is *twisted-homotopic* to $\bar{g}$, and we write $(\alpha_{\geq 1}, \alpha) : \bar{f} \simeq \bar{g}$, if there exist

$$\begin{cases} \alpha \in \mathbf{T}(X', Y'' \vee Y)_2 \\ \alpha_{\geq 1} : (f_{\geq 1}, \{f\}) \simeq (g_{\geq 1}, \{g\}) \end{cases}$$

where $\alpha_{\geq 1}$ is an u-equivariant homotopy in **chain** with $u = \{f\} = \{g\}$ such that

$$\begin{cases} \alpha_1 = E(\alpha \odot u) \quad \text{and} \\ g = f + (\partial_Y, 1)\alpha \,\text{in}\, \mathbf{T}(X, Y). \end{cases}$$

(4.2) Definition. Consider ∂-compatible chain maps $\tilde{f}, \tilde{g} : (A, \partial_X) \to (B, \partial_Y)$ in $\mathbf{TWIST}_1^c$ with $\tilde{f} = (f_{\geq 1}, u)$ and $\tilde{g} = (g_{\geq 1}, u)$. Then $\tilde{f}$ is *homotopic* to $\tilde{g}$ if there exists an u-equivariant homotopy $\alpha_{\geq 1} : \tilde{f} \simeq \tilde{g}$ in **chain**.

The homotopy categories for (4.1) and (4.2) are well defined and one obtains by K in (1.10) the induced functor

$$\mathbf{TWIST}_2^c/\simeq \xrightarrow{K} \mathbf{TWIST}_1^c/\simeq \tag{4.3}$$

where $\mathbf{TWIST}_1^c/\simeq$ is a subcategory of **chain**$/\simeq$. The morphisms in the image of the functor K in (4.3) can be characterized by the obstruction $\mathcal{O}$ in (3.2) since we have the following result.

(4.4) Theorem. *Let $\tilde{f}, \tilde{g} : (A, \partial_X) \to (B, \partial_X)$ be ∂-compatible chain maps in $\mathbf{TWIST}_1^c$. If there is a homotopy $\tilde{f} \simeq \tilde{g}$ in* **chain** *one has $\mathcal{O}(\tilde{f}) = \mathcal{O}(\tilde{g})$. That is $\mathcal{O}(\tilde{f})$ depends only on the homotopy class $\{\tilde{f}\}$ in $\mathbf{TWIST}_1^c/\simeq$ and there exists $\{\bar{g}\}$ in $\mathbf{TWIST}_2^c/\simeq$ with $K\{\bar{g}\} = \{\tilde{f}\}$ if and only if $\mathcal{O}\{\tilde{f}\} = 0$.*

Proof. Let $\tilde{f} = (f_{\geq 1}, u)$ and $\tilde{g} = (g_{\geq 1}, u)$ and let $\alpha_{\geq 1}$ be a homotopy $(f_{\geq 1}, u) \simeq (g_{\geq 1}, u)$ in **chain**. Hence we have

$$-f_1 + g_1 = d_Y \alpha_1 \tag{1}$$

$$-f_2 + g_2 = d_3 \alpha_2 + \alpha_1 d_X \tag{2}$$

where α_1, α_2 are u-equivariant in **mod**. Since E is full we choose for α_1 an element $\alpha \in \mathbf{T}(X', Y'' \vee Y)_2$ with $\alpha_1 = E(\alpha \odot u)$. Then we obtain with the choice (f'', f) for $(f_{\geq 1}, u)$ in (3.3) the following choice (g'', g) for $(g_{\geq 1}, u)$.

$$\begin{cases} g = f + (\partial_Y, 1)\alpha \\ g'' = f'' + (\alpha, i_Y f) \bigtriangledown \partial_X \end{cases} \tag{3}$$

In fact (g'', g) is a map in **TWIST** since

$$\begin{aligned}
(\partial_Y, 1)g'' &= (\partial_Y, 1)(f'' + (\alpha, i_Y f) \bigtriangledown \partial_X) \\
&= (\partial_Y, 1)f'' + (\partial_Y, 1)(\alpha, i_Y f) \bigtriangledown \partial_X \\
&= f\partial_X + ((\partial_Y, 1)\alpha, f) \bigtriangledown \partial_X \\
&= (f + (\partial_Y, 1)\alpha)\partial_X \\
&= g\,\partial_X
\end{aligned}$$

Moreover we know by (3) that $u = \{f\} = \{g\}$ and we get $g_1 = E(\bigtriangledown g \odot u)$ by the following argument. As in the proof of (1.5) we know

$$\begin{aligned}
E(\bigtriangledown g \odot u) &= E(\bigtriangledown f \odot u) + E(\bigtriangledown \partial_Y \odot 1)E(\alpha \odot u) \\
&= f_1 + d_Y \alpha_1 = g_1
\end{aligned} \tag{4}$$

where we use (1). This completes the proof that (g'', g) in (3) is a choice for $\tilde{g} = (g_{\geq 1}, u)$ defining the obstruction $\mathcal{O}(\tilde{g})$ as in (3.3). Now we get by (3) and (2) in (3.3)

$$\begin{aligned}
\mathcal{O}(\tilde{g}) &= \{\gamma(-g_2 + E(g'' \odot u))\} \\
&= \{\gamma(-f_2 - d_3\alpha_2 - \alpha_1 d_X + E(f'' \odot u) + \alpha_1 d_X)\} \\
&= \{\gamma(-f_2 + E(f'' \odot u))\} - \{\gamma(d_3\alpha_2)\} \\
&= \mathcal{O}(\tilde{f})
\end{aligned}$$

Here $\gamma(d_3\alpha_2) = 0$ follows from the cocycle condition for d_3 in (1.8) by the same argument as in (3.4) (5), (6). q.e.d.

The next lemma shows that the functor K from twisted chain complexes to chain complexes has the *homotopy lifting property*; see (VI.§ 3). We describe the lifting of homotopies in **chain** to obtain homotopies in $\mathbf{TWIST}_2^c$. Let $A|\partial_X, B|\partial_Y$ be objects in $\mathbf{TWIST}_2^c$ which are carried by the functor K to chain complexes (A, ∂_X) and (B, ∂_Y) respectively and consider maps

$$\begin{cases} \bar{f} : A|\partial_X \to B|\partial_Y \quad \text{in } \mathbf{TWIST}_2^c \\ \tilde{f}, \tilde{g} : (A, \partial_X) \to (B, \partial_Y) \quad \text{in } \mathbf{chain} \end{cases} \tag{4.5}$$

where $\tilde{f} = K(\bar{f})$ is the chain map induced by $\bar{f}$.

(4.6) Lemma on lifting homotopies. *If for the maps above there is a homotopy $\alpha_{\geq 1} : \tilde{f} \simeq \tilde{g}$ in* **chain** *then there exists a homotopy $(\alpha_{\geq 1}, \alpha) : \bar{f} \simeq \bar{g}$ in* $\mathbf{TWIST}_2^c$ *where $\bar{g}$ is a K-realization of $\tilde{g}$; that is $K\bar{g} = \tilde{g}$.*

Proof of (4.6). Let $\tilde{f} = (f_{\geq 1}, u)$ and $\tilde{g} = (g_{\geq 1}, u)$ and let $\alpha_{\geq 1} : \tilde{f} \simeq \tilde{g}$ be a homotopy. For α_1 and α_2 we choose elements

$$\begin{cases} \alpha \in \mathbf{T}(X', Y'' \vee Y)_2 \\ \alpha' \in \mathbf{T}(X'', B_3 \vee Y)_2 \end{cases} \tag{1}$$

with $\alpha_1 = E(\alpha \odot u)$ and $\alpha_2 = E(\alpha' \odot u)$. Moreover since (B, ∂_Y) satisfies the cocycle condition we choose ∂_3 for d_3 as in (1.8). Given $\bar{f} = (f_{\geq 1}, Ef'', f)$ we now define $\bar{g} = (g_{\geq 1}, Eg'', g)$ as follows: Let

$$g = f + (\partial_Y, 1)\alpha \tag{2}$$
$$g'' = f'' + (\alpha, i_Y f) \bigtriangledown \partial_X + (\partial_3, i_Y)\alpha' \tag{3}$$

Moreover $g_{\geq 1}$ in $\bar{g}$ coincides with $g_{\geq 1}$ in $\tilde{g}$. We have to check that $\bar{g}$ is a well defined twisted chain map. Using the cocycle condition (1.8) we know

$$(\partial_X, 1)(\partial_3, i_Y)\alpha' = ((\partial_X, 1)\partial_3, 1)\alpha' = (0, 1)\alpha' = 0$$

so that by the argument following (4.5) (3) the map (g'', g) is well defined in **Twist**. Moreover we see $g_1 = E(\bigtriangledown g \odot u)$ as in (4.5) (4) and we have $g_2 = E(g'' \odot u)$ since by (3)

$$\begin{aligned} E(g'' \odot u) &= E(f'' \odot u) + E(\alpha \odot u)d_X + E(\partial_3 \odot 1)E(\alpha' \odot u) \\ &= f_2 + \alpha_1 d_X + d_3 \alpha_2 \\ &= g_2, \quad \text{see (4.5) (2).} \end{aligned}$$

Hence $\bar{g}$ is a well defined twisted chain map. Moreover $(\alpha_{\geq 1}, \alpha) : \bar{f} \simeq \bar{g}$ is a twisted homotopy by (2) and the choice of α in (1); compare (4.1). q.e.d.

We now study the set of all K-realization of a given map $\tilde{f}$ in $\mathbf{TWIST}_1^c$.

(4.7) Definition. Given a ∂-compatible map $f : X \to Y$ in $\mathbf{T}$ representing $u = \{f\} : \partial_X \to \partial_Y$ in **Coef** we define the subset

$$\Gamma(\partial_X, \partial_Y)_f \subset \mathbf{T}(X', Y)$$

as follows. Here $\Gamma(\partial_x, \partial_Y)_f$ consists of all elements $\lambda \in \mathbf{T}(X', Y)$ for which there exist

$$\begin{cases} \alpha \in \mathbf{T}(X', Y'' \vee Y)_2 \\ \xi \in \mathbf{T}(X'', Y'' \vee Y)_2 \end{cases}$$

satisfying the equations

$$\begin{aligned} \lambda &= (\partial_Y, 1)\alpha \\ 0 &= d_Y\, E(\alpha \odot u) \\ 0 &= E(\xi \odot u) \\ 0 &= (\partial_Y, 1)(-\xi + (\alpha, i_X f) \bigtriangledown \partial_X) \end{aligned}$$

Here the last equation is equivalent to

$$(\lambda, f) \bigtriangledown \partial_X = (\partial_Y, 1)\xi.$$

Given a twisted chain map $\bar{f} = (f_{\geq 1}, Ef'', f) : A|\partial_X \to B|\partial_Y$ and $\lambda \in \Gamma_0(\partial_X, \partial_Y)_f$ we define the twisted chain map

$$\bar{f} + \lambda = (g_{\geq 1}, Eg'', g) : A|\partial_X \to B|\partial_Y$$

by $f_{\geq 1} = g_{\geq 1}$ and

$$\begin{cases} g = f + \lambda = f + (\partial_Y, 1)\alpha \\ g'' = f'' + \xi \end{cases}$$

where α and ξ are chosen for λ as above. As in (4.5) (3) we see that $\bar{f} + \lambda$ is a well defined map in $\mathbf{TWIST}_2^c$. Given $\lambda \in \Gamma(\partial_X, \partial_Y)_f$ one can check that $\lambda' \in \mathbf{T}(X', Y)$ satisfies $\lambda + \lambda' \in \Gamma(\partial_X, \partial_Y)_f$ if and only if $\lambda' \in \Gamma(\partial_X, \partial_Y)_{f+\lambda}$.

(4.8) Proposition. *Let $\bar{f}, \bar{g} : A|\partial_X \to B|\partial_Y$ be maps in $\mathbf{TWIST}_2^c$. Then we have $K\bar{f} = K\bar{g}$ if and only if there exists $\lambda \in \Gamma(\partial_X, \partial_Y)_f$ with $\bar{g} = \bar{f} + \lambda$. In fact, the function*

$$\Gamma(\partial_X, \partial_Y)_f \to \{\bar{g}; K\bar{g} = K\bar{f}\}$$

which carries λ to $\bar{f} + \lambda$ is a bijection.

Proof. By definition we have $K(\bar{f}+\lambda) = K(\bar{f})$. On the other hand since $K\bar{f} = K\bar{g}$ are the same chain maps we know that f and g represent the same morphism in u and therefore there is $\alpha \in \mathbf{T}(X', Y'' \vee Y)$ with $g = f + (\partial_Y, 1)\alpha$. Moreover there are $(g'', g), (f'', f) : \partial_X \to \partial_Y$ in **Twist** with

$$E(g'' \odot u) = g_2 = f_2 = E(f'' \odot u).$$

Hence we get

$$\begin{aligned} (\partial_Y, 1)g'' = g\partial_X &= (f + (\partial_Y, 1)\alpha)\partial_X \\ &= f\partial_X + ((\partial_Y, 1)\alpha, f) \bigtriangledown \partial_X \\ &= (\partial_Y, 1)f'' + (\partial_Y, 1)(\alpha, i_Y f) \bigtriangledown \partial_X. \end{aligned}$$

Therefore $\xi = -f'' + g''$ with $E(\xi \odot u) = 0$ satisfies

$$(\partial_Y, 1)\xi = (\partial_Y, 1)(\alpha, i_Y f) \bigtriangledown \partial_X.$$

Moreover

$$E(\bigtriangledown g \odot u) = g_1 = f_1 = E(\bigtriangledown f \odot u)$$

implies $d_Y E(\alpha \odot u) = 0$ since

$$E(\bigtriangledown g \odot u) = E(\bigtriangledown f \odot u) + d_Y\, E(\alpha \odot u)$$

by (1.5) (1). q.e.d.

(4.9) Definition. Given a map $u : \partial_X \to \partial_Y$ in **Coef** and chain complexes (A, ∂_X) and (B, ∂_Y) in $\mathbf{TWIST}_1^c$ we define the subgroup

$$H(A, \partial_X; B, \partial_Y)_u \subset \mathbf{T}(X', Y)$$

which consists of all elements $\lambda \in \mathbf{T}(X', Y)$ for which there exist $\alpha \in \mathbf{T}(X', Y'' \vee Y)_2$ and u-equivariant maps $(i \geq 1)$

$$\alpha_i : A_i \vee \partial_X \to B_{i+1} \vee \partial_Y$$

such that

$$\begin{cases} \lambda = (\partial_Y, 1)\alpha \\ \alpha_1 = E(\alpha \odot u) \\ 0 = d_{i+1}\alpha_i + \alpha_{i-1}d_i \quad \text{for } i \geq 1. \end{cases}$$

(4.10) Proposition. *Let $\bar{f} : A|\partial_X \to B|\partial_Y$ and let $\lambda \in \Gamma(\partial_X, \partial_Y)_f$. Then there exists a twisted homotopy $\bar{f} \simeq \bar{f} + \lambda$ if and only if $\lambda \in H(A, \partial_X; B, \partial_Y)_u$ where $u = \{f\}$ is represented by f.*

Proof. Let $(\alpha_{\geq 1}, \alpha)$ be a twisted homotopy $\bar{f} \sim \bar{f} + \lambda$. Then we have

$$f + \lambda = f + (\partial_Y, 1)\alpha$$

so that $\lambda = (\partial_Y, 1)\alpha$ by the affineness property of $\mathbf{T}$. Now it is clear that $\alpha \in H(A, \partial_X; B, \partial_Y)_u$ since $K\bar{f} = K(\bar{f} + \lambda)$. q.e.d.

We derive from (4.10), (4.8) and (4.6) the next result:

(4.11) Proposition. *Let $\bar{f}, \bar{g} : A|\partial_X \to B|\partial_Y$ be maps in $\mathbf{TWIST}_2^c$ and assume there exists a homotopy $K(\bar{g}) \simeq K(\bar{f})$ in $\mathbf{TWIST}_1^c$. Then there exists $\lambda \in \Gamma(\partial_X, \partial_Y)_f$ with $\bar{g} \simeq \bar{f} + \lambda$ in $\mathbf{TWIST}_1^c$. Moreover there is a bijection of sets*

$$\bar{f}_+ : \Gamma(\partial_X, \partial_Y)_f/\sim \xrightarrow{\approx} \{\{\bar{g}\};\ K\{\bar{g}\} = K\{\bar{f}\}\}$$

where $\{\bar{g}\}$ denotes the twisted homotopy class of $\bar{g}$ and where the equivalence relation $\sim$ is defined for $\lambda, \lambda' \in \Gamma(\partial_X, \partial_Y)_f$ by $\lambda \sim \lambda'$ if $\lambda - \lambda' \in H(A, \partial_X; B, \partial_Y)_u$ where u is represented by f. The bijection $\bar{f}_+$ carries the equivalence class $\{\lambda\}$ to $\{\bar{f} + \lambda\}$.

Proof. If $K(\bar{g}) \simeq K(\bar{f})$ we obtain by (4.6) a twisted homotopy $\bar{g} \simeq \bar{\bar{f}} = K\bar{f}$ so that $\bar{\bar{f}} = \bar{f} + \lambda$ with $\lambda \in \Gamma(\partial_X, \partial_Y)_f$ by (4.8). Now consider the function

$$\Gamma(\partial_X, \partial_Y)_f \to \{\{\bar{g}\};\ K\{\bar{g}\} = K\{\bar{f}\}\}$$

which carries λ to $\{\bar{f} + \lambda\}$. By the argument above this function is surjective. We claim that $\bar{f} + \lambda \simeq \bar{f} + \lambda'$ if and only of $\lambda \sim \lambda'$. In fact by (4.10) we have for $\lambda'' = -\lambda + \lambda'$ and $\bar{g} = \bar{f} + \lambda$ a homotopy $\bar{g} \simeq \bar{g} + \lambda''$ if and only if $\lambda'' \in H(A, \partial_X; B, \partial_Y)_u$ where u is represented by $g = f + \lambda$. Here u is also represented by f. q.e.d.

(4.12) Definition. Given a ∂-compatible map $f : X \to Y$ representing $u : \partial_X \to \partial_Y$ in **Coef** we obtain the canonical function

$$E_f : \Gamma(\partial_X, \partial_Y)_f \to \Gamma_1(\partial_Y)(X' \vee \partial_X)_u$$

which carries $\lambda = (\partial_Y, 1)\alpha$ to

$$E_f(\lambda) = \lambda\, E_2(\alpha) = \gamma E(\alpha \odot u)$$

Here we choose for λ the pair (α, ξ) as in (6.7) so that $d_Y\, E(\alpha \odot u) = 0$ and hence $E_f(\lambda)$ is well defined; compare the diagram in (2.1).

(4.13) Lemma. $\operatorname{kernel} E_f \subset H(A, \partial_X; B, \partial_Y)_u$.

Proof. Let $E_f(\lambda) = 0$. Then we have $\lambda = (\partial_Y, 1)\alpha$ with $E(\alpha \odot u) = E_3\delta$ with $\delta \in \mathbf{T}(X', Y'' \vee Y)_2$ and $(\partial_Y, 1)\delta = 0$. Hence we get $\lambda = (\partial_Y, 1)(\alpha - \delta)$ with $E(\alpha - \delta) = 0$ and we can choose all $\alpha_i = 0$ in (4.9).

Using (4.11), (4.13) and (4.4) we get

(4.14) Theorem. *Assume the function E_f in (4.12) is trivial for all f. Then the functor*

$$K : \mathbf{TWIST}_2^c/{\simeq} \to \mathbf{TWIST}_1^c/{\simeq}$$

is faithful. Moreover if $\Gamma_1(\partial_Y) = 0$ for all ∂_Y then this functor is full and faithful.

5 Twisted Homotopy Equivalences

A chain map $f : (A, \partial_X) \to (B, \partial_Y)$ in **chain** is a *homotopy equivalence* if there exists a chain map $g : (B, \partial_Y) \to (A, \partial_X)$ and homotopies of chain maps $gf \simeq 1$ and $fg \simeq 1$ where 1 denotes the identity of (A, ∂_X) and (B, ∂_Y) respectively. The homotopy class $\{f\}$ of f is then an equivalence in the homotopy category **chain**$/{\simeq}$.

A map $f : (A, \partial_X) \to (B, \partial_Y)$ is a homotopy equivalence in $\mathbf{TWIST}_1^c/{\simeq}$ if and only if f is a homotopy equivalence in **chain** such that a homotopy inverse g of f can be chosen to be a map in $\mathbf{TWIST}_1^c$.

A twisted chain map $f : A|\partial_X \to B|\partial_Y$ in $\mathbf{TWIST}_2^c$ is a *twisted homotopy equivalence* if there exist a map $g : B|\partial_Y \to A|\partial_X$ in $\mathbf{TWIST}_2^c$ and twisted homotopies $gf \simeq 1$ and $fg \simeq 1$ where 1 denotes the identity of the object $A|\partial_X$ and $B|\partial_Y$ respectively. Then the twisted homotopy class $\{f\}$ of f is an equivalence in the homotopy category $\mathbf{TWIST}_2^c/{\simeq}$.

We say that a functor $F : \mathbf{C} \to \mathbf{K}$ *reflects equivalences*(or satisfies the "sufficiency condition") if the following property holds. A map f in $\mathbf{C}$ is an equivalence in $\mathbf{C}$ if and only if the map $F(f)$ is an equivalence in $\mathbf{K}$.

(5.1) Theorem. *A twisted chain map $\bar{f} : A|\partial_X \to B|\partial_Y$ is a twisted homotopy equivalence if and only if the induced chain map $K\bar{f} : (A, \partial_X) \to (B, \partial_Y)$ is a homotopy equivalence in $\mathbf{TWIST}_1^c/\simeq$. Hence the functor*

$$K : \mathbf{TWIST}_2^c/\simeq \to \mathbf{TWIST}_1^c/\simeq$$

reflects equivalences.

For the proof of the theorem we use the following lemma.

(5.2) Lemma. *Let $\bar{h} : A|\partial_X \to A|\partial_X$ be a map in $\mathbf{TWIST}_2^c$ such that $K\bar{h} \simeq 1$ where 1 is the identity of (A, ∂_X). Then the function*

$$\Gamma(\partial_X, \partial_X)_1 \xrightarrow{h_*} \Gamma(\partial_X, \partial_X)_h/\sim$$

defined by $h_*(\lambda) = \{h\lambda\}$ is surjective. The equivalence relation $\sim$ on $\Gamma(\partial_X, \partial_X)_h$ is defined by $H(A, \partial_X; A, \partial_X)_u$ as in (4.11) where $u = \{h\} = 1$ is the identity of ∂_X in **Coef**.

Proof. One readily checks that h_* is a well defined function. In fact, this is also a consequence of (4.11). Let $\bar{h} = (h_{\geq 1}, Eh'', h)$. Since $K\bar{h} \simeq 1$ we know that $\{h\} = u$ is the identity in **Coef**. Hence there is $\delta \in \mathbf{T}(X', X'' \vee X)_2$ satisfying

$$h = 1 + \delta' \quad \text{with } \delta' = (\partial_X, 1)\delta. \tag{1}$$

We have by (I.3.3) (2)

$$\begin{aligned} (i_X\delta' + \alpha, i_X) \bigtriangledown \partial_X &= (i_X\delta', i_X) \bigtriangledown \partial_X + (\alpha, i_X + i_X\delta') \bigtriangledown \partial_X \\ &= (i_X\partial', i_X) \bigtriangledown \partial_X + (\alpha, i_X h) \bigtriangledown \partial_X \end{aligned} \tag{2}$$

Now we observe that

$$i_X\delta' \odot 1 = 0 \quad \text{in } \mathbf{premod} \quad \text{so that} \tag{3}$$

$$\begin{cases} E(i_X\delta' \odot 1 = 0 \quad \text{and} \\ E((i_X\delta', i_X) \bigtriangledown \partial_X \odot 1) = E(i_X\delta' \odot 1)d_X = 0 \end{cases}$$

Compare the definition of **premod** in I.§ 5.

Now let $\lambda \in \Gamma(\partial_X, \partial_X)_h$, that is

$$\lambda = (\partial_X, 1)\alpha \tag{4}$$

$$0 = d_X\, E(\alpha \odot 1) \tag{5}$$

$$0 = E(\xi \odot 1) \tag{6}$$

$$0 = (\partial_X, 1)(-\xi + (\alpha, i_X h) \bigtriangledown \partial_X) \tag{7}$$

Then we obtain $\lambda' \in \Gamma(\partial_X, \partial_X)_1$ satisfying

$$\lambda' = (\partial_X, 1)\alpha' \tag{8}$$
$$0 = d_X\, E(\alpha' \odot 1) \tag{9}$$
$$0 = E(\xi' \odot 1) \tag{10}$$
$$0 = (\partial_X, 1)(-\xi' + (\alpha', i_X) \bigtriangledown \partial_X) \tag{11}$$

as follows. Let

$$\alpha' = i_X\delta' + \alpha \tag{12}$$
$$\xi' = (i_X\delta', i_X) \bigtriangledown \partial_X + \xi \tag{13}$$

Then (8) defines λ' and we check (9) by

$$d_X E(i_X\delta' + \alpha) \odot 1 = d_X E(i_X\delta' \odot 1) + d_X E(\alpha \odot 1) = 0$$

where we use (3) and (5). Moreover we obtain (10) by (3) and (6). Finally we get

$$\begin{aligned} -\xi' + (\alpha', i_X) \bigtriangledown \partial_X &= -\xi - (i_X\delta', i_X) \bigtriangledown \partial_X \\ &\quad + (i_X\delta' + \alpha, i_X) \bigtriangledown \partial_X \\ &= -\xi + (\alpha, i_X h) \bigtriangledown \partial_X, \quad \text{see (2)} \end{aligned}$$

Therefore (11) is a consequence of (7). We now consider

$$\begin{aligned} h\lambda' - \lambda &= h(\partial_X, 1)(i_X\delta' + \alpha) - (\partial_X, 1)\alpha \\ &= (\partial_X, 1)\rho \quad \text{with} \\ \rho &= (h'', i_X)(i_X\delta' + \alpha) - \alpha \end{aligned} \tag{14}$$

Here we use the fact that $(h'', h) : \partial_X \to \partial_X$ is a map in **Twist**. We claim that:

$$(\partial_X, 1)\delta \in H(A, \partial_X; A, \partial_X)_1. \tag{15}$$

This shows by (14) that h_* in the lemma is surjective.

For the proof of (15) we use the assumption that there is a chain homotopy

$$\begin{aligned} &\gamma_{\geq 1} : K\bar{h} \simeq 1, \quad \text{that is} \\ &-h_1 + 1 = d_X\, \gamma_1 \\ &-h_2 + 1 = d_3\, \gamma_2 + \gamma_1\, d_X \\ &-h_i + 1 = d_{i+1}\gamma_i + \gamma_{i-1}d_i \quad \text{for } i \geq 2 \end{aligned} \tag{16}$$

where $h_1 = E(\bigtriangledown h \odot 1)$ and $h_2 = E(h'' \odot 1)$. Now we define the 1-equivariant maps

$$\rho_i : A_i \vee \partial_X \to A_{i+1} \vee \partial_X$$

with

$$\begin{aligned}
\rho_1 &= E(\rho \odot 1), \quad \text{see (14)} \\
&= E(h'' \odot 1)(E(i_X\delta' + \alpha) \odot 1) - E(\alpha \odot 1) \\
&= h_2 E(\alpha \odot 1) - E(\alpha \odot 1), \quad \text{see (16) and (3)} \\
&= (1 - d_3\gamma_2 - \gamma_1 d_X) E(\alpha \odot 1) - E(\alpha \odot 1), \quad \text{see (16)} \\
&= -d_3\gamma_2 E(\alpha \odot 1), \quad \text{see (5)}
\end{aligned} \tag{17}$$

Moreover let

$$\begin{aligned}
\rho_2 &= \gamma_2 E(\alpha \odot 1) d_X \quad \text{and} \\
\rho_n &= 0 \quad \text{for } n \geq 3.
\end{aligned} \tag{18}$$

Then we have

$$\begin{aligned}
d_X\rho_1 &= 0, \quad \text{see (17)} \\
d_3\rho_2 + \rho_1 d_X &= d_3\gamma_2 E(\alpha \odot 1) d_X - d_3\gamma_2 E(\alpha \odot 1) d_X = 0 \\
d_4\rho_3 + \rho_2 d_3 &= 0 + \gamma_2 E(\alpha \odot 1) d_3 d_X = 0
\end{aligned}$$

This completes the proof that (15) is satisfied. q.e.d.

Proof of (5.1). Let $\tilde{g}$ be a map in $\mathbf{TWIST}_1^c$ which is a homotopy inverse of $K\bar{f} = \tilde{f}$, that is $\tilde{f}\tilde{g} \simeq 1$ and $\tilde{g}\tilde{f} \simeq 1$ in **chain**. We have by (3.2) and (4.4)

$$0 = \mathcal{O}(1) = \mathcal{O}(\tilde{g}\tilde{f}) = (\tilde{f})^* \mathcal{O}(\tilde{g}) \tag{1}$$

where $\tilde{f}^*$ is an isomorphism. Hence $\mathcal{O}(\tilde{g}) = 0$ and therefore there exists $\bar{g} : B|\partial_Y \to A|\partial_X$ with $K\bar{g} = \tilde{g}$. By (4.11) we obtain the commutative diagram with $h = fg$

$$\begin{tikzcd}
\Gamma_0(\partial_Y, \partial_Y)_1 \arrow[d, "g_*"] \arrow[dr, "h_*"] & \\
\Gamma_0(\partial_Y, \partial_X)_g \arrow[r, "f_*"] \arrow[d, "\bar{g}_+"] & \Gamma_0(\partial_Y, \partial_Y)_h/\sim \arrow[d, "(\bar{f}\bar{g})_+", "\approx"'] \\
K^{-1}\{\tilde{g}\} \arrow[r, "\bar{f}_*"] & K^{-1}\{\tilde{f}\tilde{g}\}
\end{tikzcd}$$

where $(\bar{f}\bar{g})_+$ is a bijection. For $\bar{h} = \bar{f}\bar{g}$ we can use lemma (5.2) which shows that h_* in (2) is surjective. This implies that also f_* is surjective. Therefore there exists $\varepsilon \in \Gamma(\partial_Y, \partial_X)_g$ with

$$(\bar{f}\,\bar{g}) + f_*(\varepsilon) = \{1_B\} \in K^{-1}\{\tilde{f}\tilde{g}\} = K^{-1}\{1\} \tag{3}$$

where 1_B is the identity of $B|\partial_Y$. We define

$$\bar{\bar{g}} = \bar{g} + \varepsilon : B|\partial_Y \to A|\partial_X \tag{4}$$

so that by (2) and (3) we have a twisted homotopy

$$\bar{f}\bar{\bar{g}} \simeq 1_B \quad \text{and} \quad K\bar{\bar{g}} = K\bar{g} = \tilde{g}. \tag{5}$$

Let 1_A be the identity of $A|\partial_X$. Then we obtain by (4.11) an element λ and a twisted homotopy

$$\bar{\bar{g}}\bar{f} \simeq 1_A + \lambda \quad \text{with } \lambda \in \Gamma_0(\partial_X, \partial_X)_1 \tag{6}$$

since $\tilde{g}\tilde{f} \simeq 1$ in **chain**. Now we replace above $\bar{f}$ by $1_A + \lambda$. Then (5) shows that there exists $\bar{\bar{h}}$ with

$$(1_A + \lambda)\bar{\bar{h}} \simeq 1_A \tag{7}$$

Since $\simeq$ is a natural equivalence relation on $\mathbf{TWIST}_2^c$ we get by (5) and (6)

$$\begin{aligned} &\bar{\bar{f}} = \bar{f}\bar{\bar{h}} \quad \text{with} && (8)\\ &\bar{\bar{g}}\bar{\bar{f}} = \bar{\bar{g}}\bar{f}\bar{\bar{h}} \simeq (1_A + \lambda)\bar{\bar{h}} \simeq 1_A && \\ &\bar{\bar{f}} \simeq \bar{f}\bar{\bar{g}}\bar{\bar{f}} \simeq \bar{f} \quad \text{and hence} && \\ &\bar{\bar{g}}\bar{f} \simeq \bar{\bar{g}}\bar{\bar{f}} \simeq 1_A && (9) \end{aligned}$$

By (9) and (5) we see that $\bar{\bar{g}}$ is a twisted homotopy equivalence. Therefore by (5) also $\bar{f}$ is a twisted homotopy equivalence. q.e.d.

(5.3) Proposition. *The functor*

$$\mathbf{TWIST}_1^c/\simeq \;\to \mathbf{chain}_{\geq 1}/\simeq$$

reflects equivalences.

Combining (5.3) and (5.1) we thus get:

(5.4) Theorem. *A twisted chain map $\bar{f} : A \mid \partial_X \to B \mid \partial_Y$ is a twisted homotopy equivalence if and only if the induced chain map $K\bar{f} : (A, \partial_X) \to (B, \partial_Y)$ is a homotopy equivalence in* $\mathbf{chain}_{\geq 1}$. *Hence*

$$K : \mathbf{TWIST}_2^c/\simeq \;\to \mathbf{chain}_{\geq 1}/\simeq$$

reflects equivalences.

Proof of (5.3). Let $\bar{f} : (A, \partial_X) \to (B, \partial_Y)$ be a map in $\mathbf{TWIST}_1^c$ which is a homotopy equivalence in **chain**. Then we have

$$f_1 = (\triangledown f) \odot u \quad \text{with } u = f \tag{1}$$

where u is an isomorphism in **Coef** with inverse $v = \{h\}$. Moreover we have a v-equivariant map $\bar{g} : (B, \partial_Y) \to (A, \partial_X)$ and homotopies $\bar{g}\bar{f} \simeq 1$ and $\bar{f}\bar{g} \simeq 1$ in

chain. We have to show that there is a map $h : (B, \partial_Y) \to (A, \partial_X)$ in $\mathbf{TWIST}_1^c$ with $h \simeq g$ in **chain**. Since $vu = 1$ we know that $hf = 1 + (\partial_X, 1)\beta$. Moreover h is ∂-compatible so that we have $(h'', h) : \partial_Y \to \partial_X$ in **Twist**. Hence we get

$$\begin{aligned}(\triangledown h \odot v)f_1 &= (\triangledown h \odot v)(\triangledown f) \odot u \\ &= 1 + d(E\beta \odot 1)\end{aligned}$$

Moreover $fg \simeq 1$ implies that there is α_1 with $d\alpha_1 = -1 + f_1 g_1$. Hence we get by composing with $\triangledown h \odot v$ from the left

$$\begin{aligned}d(h'' \odot v)\alpha_1 &= (\triangledown h \odot v)d\alpha_1 \\ &= - \triangledown h \odot v + (\triangledown h \odot v)f_1 g_1 \\ &= - \triangledown h \odot v + (1 + d(E\beta \odot 1))g_1 \\ &= - \triangledown h \odot v + g_1 + d(E\beta \odot 1)g_1\end{aligned}$$

Hence we get

$$\begin{aligned}- \triangledown h \odot v + g_1 = d\gamma \quad &\text{with} \\ \gamma = (h'' \odot v)\alpha_1 &- (E\beta \odot 1)g_1\end{aligned}$$

We now define the map $\bar{h} : (B, \partial_Y) \to (A, \partial_X)$ in $\mathbf{TWIST}_1^c$ by $h_1 = \triangledown h \odot v$ and $h_2 = g_2 - \gamma d$ and $h_n = g_n$ for $n \geq 3$. Then clearly h is well defined in $\mathbf{TWIST}_1^c$ and γ yields a canonical homotopy $h \simeq g$.

6 The Augmentation Functor

All results in sections § 1 … § 5 above are obtained if a theory $\mathbf{T}$ of coactions is given. We now assume that $\mathbf{T}$ is augmented by Σ as in (I.7.2) or weakly augmented as in (I.7.11). Then we obtain a canonical *augmentation functor*

$$\mathrm{aug} : \mathbf{TWIST}_1^c \to \mathbf{chain}_{\geq 0} \tag{6.1}$$

which carries the chain complex $(A_{\geq 1}, \partial_X)$ in $\mathbf{TWIST}_1^c$ to the following chain complex $\mathrm{aug}(A_{\geq 1}, \partial_X) = (A_{\geq 0}, \partial_X)$ which coincides in degree ≥ 1 with (A, ∂_X) and which satisfies

$$A_0 = \Sigma, \quad A_1 = X', \quad A_2 = X'' \tag{1}$$

and for which the differentials in degree ≤ 2

$$X'' \vee \partial_X \xrightarrow{d_2} X' \vee \partial_X \xrightarrow{d_1} \Sigma \vee \partial_X \tag{2}$$

are given by the operators in (I.7.9), that is

$$\begin{cases} d_2 = E(\triangledown \partial_X \odot 1) \\ d_1 = E(\delta_X \odot 1) \end{cases} \tag{3}$$

The functor (6.1) carries a chain map

$$f : (A_{\geq 1}, \partial_X) \to (B_{\geq 1}, \partial_X) \in \mathbf{TWIST}_1^c$$

to the induced chain map

$$\mathrm{aug}(f) : (A_{\geq 0}, \partial_X) \to (B_{\geq 0}, \partial_X) \in \mathbf{chain}_{\geq 0}$$

which coincides in degree ≥ 1 with f and which is $1 \vee u$ in degree 0 where f is u-equivariant. By (I.7.9) and (I.7.10) or by (I.7.11) we see that the functor aug is well defined. Here the assumption $f_1 = E(\nabla f \odot u)$ on f in (1.9) is needed.

The augmentation functor is compatible with homotopies in (4.2) so that we obtain the induced functor

$$\mathrm{aug} : \mathbf{TWIST}_1^c/\simeq \to \mathbf{chain}_{\geq 0}/\simeq \tag{6.2}$$

While (6.2) is faithful the induced functor (6.2) needs not to be faithful. The functor (6.2) is well defined since a homotopy $\alpha_{\geq 1} : f \simeq g$ yields the homotopy $\alpha_{\geq 0} : \mathrm{aug}(f) \simeq \mathrm{aug}(g)$ which coincides with $\alpha_{\geq 1}$ in degree ≥ 1 and which satisfies $\alpha_0 = 0$.

7 Appendix: Homology of Coefficient Objects

We here introduce the cohomology and homology defined for objects in the category of coefficients **Coef** of a theory **T**. This covers many classical notions of (co-) homology in the literature. In special cases the cohomology and homology can be described by certain Ext and Tor groups respectively.

(7.1) Definition. We say that a sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

in an additive category **M** is *exact* if $\beta\alpha = 0$ in $\mathbf{M}(A, C)$ and if for all objects K in **M** and all morphisms $\xi : K \to B$ with $\beta\xi = 0$ there is $\eta : K \to A$ with $\alpha\eta = \xi$. Equivalently for all K the induced sequence of morphism sets

$$\mathbf{M}(K, A) \xrightarrow{\alpha_*} \mathbf{M}(K, B) \xrightarrow{\beta_*} \mathbf{M}(K, C)$$

is an exact sequence of abelian groups.

One readily checks that a chain complex (A, ∂_X) in **chain** is *exact in degree* n, that is

$$A_{n+1} \vee \partial_X \xrightarrow{d} A_n \vee \partial_X \xrightarrow{d} A_{n-1} \vee \partial_X$$

is exact in $\mathbf{mod}(\partial_X)$, if and only if the homology

$$H_n(A, \partial_X) = 0$$

is trivial; see (I.6.5).

(7.2) Definition. A *resolution* of the object $\partial_X \in$ **Coef** is a chain complex (A, ∂_X) which is exact in degree $n \geq 2$ with the property

$$\begin{cases} A_1 = X', \quad A_2 = X'' \\ d_2 = d_X = E(\nabla\partial_X \odot 1) : X'' \vee \partial_X \to X' \vee \partial_X \end{cases}$$

That is, a resolution of ∂_X is the same as a chain complex (A, ∂_X) in the category $\mathbf{TWIST}_1^c$ which is exact in degree ≥ 2. Compare the definition in (1.9). We do not assume exactness in degree $= 1$.

(7.3) Lemma. *Let (B, ∂_Y) be a chain complex in $\mathbf{TWIST}_1^c$ and let (A, ∂_X) be a resolution of $\partial_X \in$ **Coef**. Given a morphism $u : \partial_Y \to \partial_X \in$ **Coef** there exists a u-equivariant chain map*

$$\tilde{f} : (B, \partial_Y) \to (A, \partial_X) \in \mathbf{TWIST}_1^c$$

and two such u-equivariant chain maps are homotopic in $\mathbf{TWIST}_1^c$.

Proof. We know that $u = \{f\}$ is represented by a ∂-compatible map $f : Y \to X$ so that there exists $(f'', f) : \partial_Y \to \partial_X$ in **TWIST**. Hence we obtain the following commutative diagram in **mod**

$$\begin{tikzcd} B_1 \vee \partial_Y \arrow[d, "f_1"] & B_2 \vee \partial_Y \arrow[l, "d_Y"'] \arrow[d, "f_2"] & B_3 \vee \partial_Y \arrow[l, "d_3"'] \arrow[d, "f_3"] & \cdots \arrow[l] \\ A_1 \vee \partial_X & A_2 \vee \partial_X \arrow[l, "d_X"'] & A_3 \vee \partial_X \arrow[l, "d_3"'] & \arrow[l] \end{tikzcd}$$

where $f_1 = E(\nabla f \odot u)$ and $f_2 = E(f'' \odot u)$. Since d_X, d_3 is exact and since $d_X f_2 d_3 = f_1 d_Y d_3 = 0$ we can find f_3 with $d_3 f_3 = f_2 = d_3$. In the same way we can find inductively a u-equivariant map f_n defining a chain map $\tilde{f} = (f_{\geq 1}, u)$ in $\mathbf{TWIST}_1^c$; see (1.9). Now let $\tilde{g} = (g_{\geq 1}, u)$ be a further u-equivariant chain map

$$\tilde{g} : (B, \partial_Y) \to (A, \partial_X) \in \mathbf{TWIST}_1^c$$

Then we know that there is a ∂-compatible map $g : Y \to X$ representing $u = \{g\}$ with $g_1 = E(\nabla g \odot u)$. Since

$$\{f\} = u = \{g\}$$

we can find $\alpha : Y' \to X'' \vee X$ trivial on X with

$$g = f + (\partial_Y, 1)\alpha$$

Compare (I.4.1). Hence by the proof of (1.5) we obtain $\alpha_1 = E(\alpha \odot u)$ satisfying

$$-f_1 + g_1 = d\,\alpha_1$$

This implies

$$\begin{aligned} d(-\alpha_1 d - f_2 + g_2) &= \\ -d\alpha_1 d - df_2 + dg_2 &= \\ (-f_1 + g_1)d - df_2 + dg_2 &= 0 \end{aligned}$$

Therefore by exactness there exists α_2 with $-\alpha_1 d - f_2 + g_2 = \alpha_2 d$. Inductively we obtain this way the homotopy $(\alpha_{\geq 1}, u) : \tilde{f} \simeq \tilde{g}$. q.e.d.

(7.4) Corollary. *Let (A, ∂_X) and (B, ∂_X) be two resolutions of ∂_X. Then there exists a canonical ∂_X-equivariant homotopy equivalence*

$$(A, \partial_X) \simeq (B, \partial_X)$$

in $\mathbf{TWIST}_1^c/\simeq$.

Hence resolutions are up to canonical isomorphism unique. The following condition on $\mathbf{mod}(\partial_X)$ implies that resolutions always exist.

(7.5) Definition. We say that an additive category $\mathbf{M}$ has *enough exact sequences* if for each morphism $\beta : B \to C$ in $\mathbf{M}$ there is an exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

in $\mathbf{M}$; see (6.1). We say that $\mathbf{mod}$ has *enough exact sequences* if $\mathbf{mod}(\partial_X)$ has enough exact sequences for all ∂_X in $\mathbf{Coef}$.

(7.6) Proposition. *Assume* $\mathbf{mod}$ *has enough sequences. Then each object ∂_X in* $\mathbf{Coef}$ *has a resolution $R(\partial_X) = (A, \partial_X)$. By (7.3) the choice of such resolutions yields a functor*

$$R : \mathbf{Coef} \to \mathbf{TWIST}_1^c/\simeq$$

which splits the coefficient functor c. Two such functors obtained by resolutions are canonically isomorphic.

If $\mathbf{T}$ is an augmented theory of coactions we have in addition the composite of R with the augmentation functor (6.1)

$$R_{\mathrm{aug}} : \mathbf{Coef} \xrightarrow{R} \mathbf{TWIST}_1^c/\simeq \xrightarrow{\mathrm{aug}} \mathbf{chain}_{\geq 0}/\simeq$$

The functors R and R_{aug} respectively lead to the definition of (co-) homology of objects in $\mathbf{Coef}$; see (7.7) below.

Proof. Clearly ∂_X defines

$$E(\nabla \partial_X \odot 1) : X'' \vee \partial_X \to X' \vee \partial_X$$

and using the property in (7.5) we can choose inductively a sequence

$$X' \vee \partial_X \xleftarrow{d} X'' \vee \partial_X \xleftarrow{d} A_3 \vee \partial_X \longleftarrow \dots$$

which is exact in degree $n \geq 2$. Now the functor R is obtained by (7.3). q.e.d.

(7.7) Definition. Let $\mathbf{T}$ be a theory of coactions and assume that **mod** has enough exact sequences. Let N be a right (resp. left) $\mathbf{mod}(\partial_X)$-module as in (I.6.4). Then the $\mathbf{T}$ *-cohomology*, resp. the $\mathbf{T}$ *-homology*, of an object ∂_X in **Coef** is defined by a resolution of ∂_X; that is

$$\begin{aligned} H^n(\partial_X; N) &= H^n(R(\partial_X); N), \quad \text{resp.} \\ H_n(\partial_X; N) &= H_n(R(\partial_X); N). \end{aligned}$$

Using the results (7.3), (7.4) and (7.5) one readily checks that the $\mathbf{T}$-homology and $\mathbf{T}$-cohomology is well defined and natural in $\partial_X \in \mathbf{Coef}$ and in N. For $n = 0$ the $\mathbf{T}$-(co-)homology is trivial so that we obtain a *reduced* (co-)homology. Moreover using (I.6.5) we obtain the right $\mathbf{mod}(\partial_X)$-module $H_n(\partial_X) = H_n(R(\partial_X))$ and the left $\mathbf{mod}(\partial_X)$-module $H^n(\partial_X) = H^n(R(\partial_X))$.

In case $\mathbf{T}$ is an augmented theory of coactions we obtain the *non-reduced* (co-) homology by replacing R by R_{aug}; for example

$$\begin{aligned} \bar{H}^n(\partial_X; N) &= H^n(R_{\mathrm{aug}}(\partial_X); N) \\ \bar{H}_n(\partial_X) &= H_n(R_{\mathrm{aug}}(\partial_X)) \end{aligned}$$

Clearly in degree ≥ 2 the non-reduced (co-) homology coincides with the reduced (co-) homology.

(7.8) Example. Let $\mathbf{S}$ be a single sorted theory of cogroups and let $\mathbf{T} = \mathbf{free}(\mathbf{S})$. Then the category $\mathbf{mod} = \mathbf{mod}(U)$ is obtained by the enveloping functor U : $\mathbf{Coef} \to \mathbf{Rings}$. See (I.5.13). One readily checks that $\mathbf{mod}(U)$ has enough exact sequences in the sense of (7.5). Therefore the $\mathbf{T}$-(co-)homology of an object

$$G \in \mathbf{model}(\mathbf{S}) = \mathbf{Coef} \tag{1}$$

is defined. Here we use the equivalence of categories in (I.4.6). Hence the object G is given by a presentation ∂_X. The category $\mathbf{mod}(\partial_X)$ coincides with the category of free right $U(G)$-modules. Let M_G (resp. N_G) be a right (resp. left) $U(G)$-module. Then M_G defines the right $\mathbf{mod}(\partial_X)$-module

$$M = \mathrm{Hom}_{U(G)}(-, M_G) : \mathbf{mod}(\partial_X)^{\mathrm{op}} \to \mathbf{Ab} \tag{2}$$

which carries a free right $U(G)$-module A_G to the abelian group of $U(G)$-homomorphisms $A_G \to M_G$. Moreover N_G defines the left $\mathbf{mod}(\partial_X)$-module

$$N = - \otimes_{U(G)} N_G : \mathbf{mod}(\partial_X) \to \mathbf{Ab} \tag{3}$$

which carries A_G to the $U(G)$-tensor product $A_G \otimes_{U(G)} N_G$. Now we denote the $\mathbf{T}$-(co-)homology by

$$H^n(G, M_G) = H^n(\partial_X, M) \tag{4}$$

$$H_n(G, N_G) = H^n(\partial_X, N) \tag{5}$$

Here the right hand side is defined in (7.7). We define the functor (see (I.5.12))

$$D : \mathbf{model(S)} \to \mathbf{Mod}(U) \tag{6}$$

which carries G to right $U(G)$-module

$$D(G) = H_1(G, U(G)) = H_1(\partial_X)$$

This is the cokernel of d_X considered as a morphism in $\mathbf{mod}(U)$ where d_X is determined by the object $\partial_X \in \mathbf{Coef}$ representing G. It follows from (7.3) that D is a well defined functor. Now it is clear by definition of resolutions in (7.2) that one has for $n \in \mathbb{Z}$:

$$H^n(G, M_G) = \mathrm{Ext}^{n-1}_{U(G)}(D(G), M_G) \tag{7}$$

$$H_n(G, M_G) = \mathrm{Tor}^{U(G)}_{n-1}(D(G), M_G) \tag{8}$$

If $\mathbf{T} = \mathbf{gr}$ is the theory of free groups then

$$U(G) = \mathbb{Z}[G] \quad \text{and} \quad D(G) = I(G). \tag{9}$$

Here $I(G)$ is the augmentation ideal in $\mathbb{Z}[G]$ and hence (7), (8) is the classical (co-) homology of the group G with coefficients in the G-module M_G. If $\mathbf{T} = \mathbf{nil}_n$ then

$$U(G) = \mathbb{Z}[G]/I(G)^n \quad \text{and} \quad D(G) = I(G)/I(G)^{n+1}. \tag{10}$$

More generally if $\mathbf{T} = \mathbf{var}$ is given by a variety of groups $\mathbf{Var}$ then

$$U(G) = \mathbb{Z}[G]/\mathcal{V}(G) \quad \text{and} \quad D(G) = I(G) \otimes_{\mathbb{Z}[G]} U(G). \tag{11}$$

Here $U(G)$ is the factor ring of $\mathbb{Z}[G]$ described by Leedham-Green and the functor D in (11) coincides with the functor D considered in Leedham-Green [HV] page 2.

Finally let R be a commutative ring which is a principal ideal domain and let $\mathbf{alg}$ be the theory of free algebras over R. If $G \in \mathbf{Alg}$ is free as an R-module we have

$$U(G) = G^{\mathrm{op}} \otimes G \quad \text{and} \quad D(G) = \mathrm{kernel}(\mu : G \otimes_R G \to G) \tag{12}$$

where μ is the multiplication of G. In this case (7), (8) coincide with the classical *Hochschild-(co-)homology* of the algebra G, see Mac Lane [H].

8 Appendix: Twisted Homology of Coefficient Objects

We study a further concept of (co-) homology defined for objects in the category $\mathbf{Coef}$ of coefficients of a theory $\mathbf{T}$. We proceed similarly as in Appendix A where we used resolutions in the category $\mathbf{TWIST}^c_1$. We now define resolutions in the category $\mathbf{TWIST}_2^c$ termed twisted resolutions which yield canonically the twisted homology. Let $\mathbf{T}$ be a theory of coactions. We say that a sequence of morphisms

$$A \xrightarrow{\alpha} X'' \vee X \xrightarrow{(\partial_X,1)} X \tag{8.1}$$

is *exact* in $\mathbf{T}$ if A and X'' are cogroups and α is trivial on X such that for all cogroups B in $\mathbf{T}$ the induced sequence

$$\mathbf{T}(B, A \vee X)_2 \xrightarrow{(\alpha,1_X)_*} \mathbf{T}(B, X'' \vee X)_2 \xrightarrow{(\partial_X,1)_*} \mathbf{T}(B, X) \tag{8.2}$$

of group homomorphisms is exact.

(8.3) Definition. A *twisted resolution* of the object $\partial_X \in \mathbf{Coef}$ is an object $A|\partial_X$ in $\mathbf{TWIST}_2^c$ with the following properties. For $d_3 : A_3 \vee \partial_X \to X'' \vee \partial_X$ in $A|\partial_X$ there exists an exact sequence

$$A_3 \xrightarrow{\partial_3} X'' \vee X \xrightarrow{(\partial_X,1)} X \tag{1}$$

in $\mathbf{T}$ such that $d_3 = E(\partial_3 \odot 1)$. Moreover for $n \geq 3$ the sequences

$$A_{n+1} \vee \partial_X \xrightarrow{d_{n+1}} A_n \vee \partial_X \xrightarrow{d_n} A_{n-1} \vee \partial_X \tag{2}$$

are exact. The sequence (2) needs not to be exact for $n = 2$; the "exactness condition" for $n = 2$ is described by (1).

(8.4) Lemma. *Let $B|\partial_Y$ be an object in $\mathbf{TWIST}_2^c$ and let $A|\partial_X$ be a twisted resolution of $\partial_X \in \mathbf{Coef}$. Given a morphism $u : \partial_Y \to \partial_X \in \mathbf{Coef}$ there exists a u-equivariant twisted chain map*

$$\bar{f} : B|\partial_Y \to A|\partial_X \in \mathbf{TWIST}_2^c$$

and two such are homotopic in $\mathbf{TWIST}_2^c$.

Proof. We know that $u = \{f\}$ is represented by a ∂-compatible map f so that there is $(f'', f) : \partial_Y \to \partial_X$ in $\mathbf{Twist}$. Now consider the following diagram in $\mathbf{T}$.

$$\begin{array}{ccccc} B_3 & \xrightarrow{\partial_3} & Y'' \vee Y & \xrightarrow{(\partial_Y,1)} & Y \\ {\scriptstyle f'''}\downarrow & & \downarrow{\scriptstyle (f'',i_X f)} & & \downarrow{\scriptstyle f} \\ A_3 \vee X & \xrightarrow{(\partial_3,i_X)} & X'' \vee X & \xrightarrow{(\partial_X,1)} & X \end{array} \tag{1}$$

where ∂_3 in the top row is given by the cocycle condition for $B|\partial_Y$ in (1.8). Since (8.2) (1) is exact we can find f''' trivial on X such that the diagram commutes. We define the map $\bar{f} = (f_{\geq 1}, f)$ in the proposition by $f_1 = E(\nabla f \odot 1), f_2 = (f'' \odot 1)$ and $f_3 = E(f''' \odot 1)$. Then the diagram

$$\begin{array}{ccccc} B_4 \vee \partial_Y & \xrightarrow{d_4} & B_3 \vee \partial_Y & \xrightarrow{d_3} & B_2 \vee \partial_Y \\ {\scriptstyle f_4}\downarrow & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} \\ A_4 \vee \partial_X & \longrightarrow & A_3 \vee \partial_X & \longrightarrow & A_2 \vee \partial_X \end{array} \tag{2}$$

commutes so that by exactness of (8.2) (2) there exists f_4 extending the diagram commutatively. Inductively we obtain this way f_n for $n \geq 4$. Now let

$$\bar{g} = (g_{\geq 1}, g) : B|\partial_Y \to B|\partial_X$$

be a further u-equivariant twisted chain map. Then we know $\{g\} = u = \{f\}$ so that there exists $\alpha : Y'' \to X'' \vee X$ trivial on X with

$$g = f + (\partial_X, 1)\alpha \tag{3}$$

Compare (4). Moreover g is ∂-compatible with $(g'', g) : \partial_Y \to \partial_X \in \mathbf{Twist}$. Hence we get

$$(\partial_X, 1)(-f'' + g'' - (\alpha, i_X f) \bigtriangledown \partial_Y) = 0 \tag{4}$$

so that by exactness (8.1) (1) there is $\alpha' : Y'' \to A_3 \vee X$ trivial on X with

$$(\partial_3, i_X)\alpha' = -f'' + g'' - (\alpha, i_X f) \bigtriangledown \partial_Y \tag{5}$$

We now define a twisted homotopy (see (4.1)) of the form $(\alpha_{\geq 1}, \alpha) : \bar{f} \simeq \bar{g}$ as follows. Let $\alpha_1 = E(\alpha \odot 1)$ and $\alpha_2 = E(\alpha' \odot 1)$ then equation (5) shows

$$-f_2 + g_2 = d\,\alpha_2 + \alpha_1 d \tag{6}$$

Moreover we get

$$\begin{aligned} &d(-d_2 d - f_3 + g_3) = \\ &-d\alpha_2 d - f_2 d + g_2 d = 0 \end{aligned} \tag{7}$$

so that by exactness (8.2) (2) there exists α_3 with

$$d\alpha_3 = -\alpha_2 d - f_3 + g_3 \tag{8}$$

Inductively we obtain this way α_n for $n \geq 3$. q.e.d.

(8.5) Corollary. *Let $A|\partial_X$ and $B|\partial_X$ be two twisted resolutions of ∂_X. Then there exists a canonical ∂_X-equivariant twisted homology equivalence*

$$A|\partial_X \simeq B|\partial_X$$

in $\mathbf{TWIST}_2^c/\simeq$.

Hence twisted resolutions are unique up to canonical isomorphisms.

(8.6) Definition. We say that $\mathbf{T}$ has *enough exact sequences* if for all $\partial_X \in \mathbf{Coef}$ there exists an exact sequence

$$A \xrightarrow{\partial} X'' \vee X \xrightarrow{(\partial_X, 1)} X$$

in $\mathbf{T}$ and if $\mathbf{mod}$ has enough exact sequences in the sense of (7.5).

(8.7) Proposition. *Assume* $\mathbf{T}$ *has enough exact sequences. Then each object* $\partial_X \in \mathbf{Coef}$ *has a twisted resolution* $Q(\partial_X) = A|\partial_X$. *By (8.4) the choice of such twisted resolutions yields a functor*

$$Q : \mathbf{Coef} \to \mathbf{TWIST}_2^c/\simeq$$

which splits the coefficient functor c. Two such functors obtained by twisted resolutions are canonically isomorphic.

The composition of the functor Q and the chain functor K in (5.1) yields the notion of twisted (co-) homology below. Again if $\mathbf{T}$ is an augmented theory of coactions we also have the composite of functors

$$(KQ)_{\mathrm{aug}} : \mathbf{Coef} \xrightarrow{Q} \mathbf{TWIST}_2^c/\simeq \xrightarrow{K} \mathbf{TWIST}_1^c/\simeq \xrightarrow{\mathrm{aug}} \mathbf{chain}_{\geq 0}/\simeq$$

(8.8) Definition. Let $\mathbf{T}$ be a theory of coactions with enough exact sequences and let N be a right (resp. left) $\mathbf{mod}(\partial_X)$-module as in (I.6.4). Then the *twisted* $\mathbf{T}$ *-cohomology*, resp. the *twisted* $\mathbf{T}$ *-homology* is defined by a twisted resolution $A|\partial_X$ of ∂_X; that is

$$H^n_{\mathrm{twist}}(\partial_X; N) = H^n(KQ(\partial_X); N)$$
$$H_n^{\mathrm{twist}}(\partial_X; N) = H_n(KQ(\partial_X); N)$$

Moreover using (I.6.5) we then obtain the right $\mathbf{mod}(\partial_X)$-module $H_n^{\mathrm{twist}}(\partial_X) = H_n(KQ(\partial_X))$ and the left $\mathbf{mod}(\partial_X)$-module $H^n_{\mathrm{twist}}(\partial_X) = H^n(KQ(\partial_X))$.

In case $\mathbf{T}$ is an augmented theory of coactions we obtain the non-reduced twisted (co-) homology by replacing KQ by $(KQ)_{\mathrm{aug}}$; for example

$$\bar{H}^n_{\mathrm{twist}}(\partial_X; N) = H^n((KQ)_{\mathrm{aug}}(\partial_X); N)$$
$$\bar{H}_n^{\mathrm{twist}}(\partial_X) = H_n((KQ)_{\mathrm{aug}}(\partial_X))$$

In degree ≥ 2 this coincides with the corresponding reduced twisted (co-) homology above.

(8.9) Proposition. *Assume* $\mathbf{T}$ *has enough exact sequences. Using (7.3) we obtain a canonical natural transformation*

$$\theta : KQ(\partial_X) \to R(\partial_X)$$

in $\mathbf{chain}/\simeq$. *This shows that we have natural transformations*

$$\theta^* : H^n(\partial_X, N) \to H^n_{\mathrm{twist}}(\partial_X, N)$$
$$\theta_* : H_n^{\mathrm{twist}}(\partial_X, N) \to H_n(\partial_X, N)$$

which are isomorphisms for $n \leq 1$. *Moreover* θ_* *is surjective and* θ^* *is injective in degree 2.*

Proof. The two dimensional parts of $R(\partial_X)$ and $KQ(\partial_X)$ coincide. They are given by d_X. q.e.d.

The $\mathbf{mod}(\partial_Y)$-module $\Gamma_1(\partial_Y)$ in § 2 has a new interpretation by the next result.

(8.10) Proposition. *Assume* **T** *has enough exact sequences. Then we have the natural isomorphism of right* $\mathbf{mod}(\partial_Y)$*-modules*

$$H_2^{\text{twist}}(\partial_Y) = \Gamma_1(\partial_Y)$$

This follows readily from the definition of exactness in (8.1) and the definition of Γ_1 in § 2.

(8.11) Example. Let **S** be a single sorted theory of cogroups and let $\mathbf{T} = \mathbf{free}(\mathbf{S})$. Then it is easy to see that **T** has enough exact sequences. With the notation in (7.8) the twisted (co-) homology of $G \in \mathbf{model}(\mathbf{S}) = \mathbf{Coef}$ is defined by

$$\begin{cases} H^n_{\text{twist}}(G, M_G) = H^n_{\text{twist}}(\partial_X, M) \\ H_n^{\text{twist}}(G, N_G) = H_n^{\text{twist}}(\partial_X, N) \end{cases} \tag{1}$$

where ∂_X is a presentation of G. We also write $\Gamma_1(G) = \Gamma_1(\partial_X)$ so that by (8.10)

$$\Gamma_1(G) = H_2^{\text{twist}}(G, U(G)) \tag{2}$$

is a right $U(G)$-module. One readily obtains the exact sequences

$$H_3(G, M_G) \longrightarrow \Gamma_1(G) \otimes_{U(G)} M_G \longrightarrow H_2^{\text{twist}}(G, M_G) \xrightarrow{\theta} H_2(G, M_G) \longrightarrow 0 \tag{3}$$

$$H^3(G, M_G) \longleftarrow \mathrm{Hom}_{U(G)}(\Gamma_1(G), M_G) \longleftarrow H^2_{\text{twist}}(G, M_G) \xleftarrow{\theta} H^2(G, M_G) \longleftarrow 0 \tag{4}$$

Remark. If $S = \mathbf{var}^\sharp$ is a theory defining a variety of groups $\mathbf{model}(\mathbf{S}) = \mathbf{Var}$ then the exact sequences (3), (4) are exactly those of 3.2 II in Leedham-Green [HV]. For this use (8.8) (7), (8).

The Leedham-Green (co-) homology on the category **Var** is a special case of the *Quillen (co-) homology* which we denote by $H^n_Q(G, M_G)$ and $H_n^Q(G, M_G)$ respectively. Compare Quillen (2.1) [CR]. Here G is an object in $\mathbf{model}(\mathbf{S})$ and M_G is a right (resp. left) $U(G)$-module as above. We claim that there are natural transformations

$$\tau^* : H^n_{\text{twist}}(G, M_G) \to H_Q^{n-1}(G, M_G) \tag{5}$$

$$\tau_* : H^Q_{n-1}(G, M_G) \to H_n^{\text{twist}}(G, M_G) \tag{6}$$

which are isomorphisms in degree $n \leq 2$ and for which τ^* is injective and τ_* is surjective in degree 3. Compare (VI.12.10).

The module $\Gamma_1(G)$ in (2) for the variety of groups of nilpotency degree n can be described by the *relative dimension subgroup*, that is

$$\Gamma_1(G) = D_{n+1}(H, B) \cap B/B_2 \tag{7}$$

for $\mathbf{T} = \mathbf{nil}_n$ and $G \in \mathbf{Nil}_n$. Here H is an object in $\mathbf{nil}_n$ which surjects to G by $\beta : H \twoheadrightarrow G$ with kernel$(\beta) = B$ and B_2 the commutator subgroup of B. Moreover $D_{n+1}(H, B) = H \cap (1 + I(B) \cdot I(H) + I(H)^{n+1})$.

For $n = 2$ the group (7) was functorially computed by Hartl [H] namely

$$\Gamma_1(G) = G^{\mathrm{ab}} * G^{\mathrm{ab}}/\text{diagonal elements} = (L_1 S^2)(G^{\mathrm{ab}}) \tag{8}$$

for $\mathbf{T} = \mathbf{nil}_2$ and $G \in \mathbf{Nil}_2$. Here $*$ denotes the torsion product of abelian groups and G^{ab} is the abelianization of G. Moreover $S^2 : \mathbf{Ab} \to \mathbf{Ab}$ is the *symmetric square* functor with $S^2(A) = A \otimes A/(a \otimes b \sim b \otimes a)$ and $L_1 S^2$ is the first derived functor of S^2.

It is a classical result of J.H.C. Whitehead [CHII] that for the variety of groups we have

$$\Gamma_1(\pi) = 0 \tag{9}$$

for $\mathbf{T} = \mathbf{gr}$ and $\pi \in \mathbf{Gr}$. Moreover we prove in VI.4.11 Baues [AH] that one gets a non trivial Γ_1 for the theory $\mathbf{T} = \mathbf{gr}(G)$ where G is a group. In this case an object in $\mathbf{coef} = \mathbf{Gr}^G$ is a homomorphism $i : G \to \pi \in \mathbf{Gr}$ and we get

$$\Gamma_1(G \xrightarrow{i} \pi) = \mathbb{Z}[iG \setminus \pi] \otimes (\ker i)^{\mathrm{ab}} \tag{10}$$

Here $iG \setminus \pi$ denotes the set of left cosets of iG in the group π and $(\ker i)^{\mathrm{ab}}$ is the abelianization of the kernel $(i : G \to \pi)$. If $G = 0$ is trivial then (10) implies (9). The enveloping functor U on $\mathbf{Gr}^G$ is given by $U(G \to \pi) = \mathbb{Z}[\pi]$.

Chapter III: Basic Concepts of Homotopy Theory

In this chapter we describe most elementary properties of a homotopy theory. These properties are used to define the axioms of a cofibration category. We also describe basic results which can be deduced from these axioms and which are used in this book. We recall these results from Baues [AH]. In the applications we shall consider numerous different homotopy theories which satisfy the axioms of a cofibration category. This shows that all results in this chapter and in the following chapters can be applied in each of these examples of homotopy theories.

1 Cofibration Categories

Here we introduce the notion of a cofibration category; compare Baues [AH]. This is a category together with two classes of morphisms, called cofibrations and weak equivalences, such that four axioms $C1, \dots, C4$ are satisfied.

(1.1) Definition. A *cofibration category* is a category $\mathbf{C}$ with an additional structure

$$(\mathbf{C}, cof, we),$$

subject to axioms $C1, C2, C3$ and $C4$. Here *cof* and *we* are classes of morphisms in $\mathbf{C}$, called *cofibrations* and *weak equivalences* respectively.

Morphisms in $\mathbf{C}$ are also called *maps* in $\mathbf{C}$. We write $i : B \subset A$ or $B \rightarrowtail A$ for a cofibration and we call $u \mid B = ui : B \to U$ the *restriction* of $u : A \to U$. We write $X \xrightarrow{\sim} Y$ for a weak equivalence in $\mathbf{C}$. An isomorphism in $\mathbf{C}$ is denoted by $\cong$. The identity of the object X is $1 = 1_X = \mathrm{id}$. A map in $\mathbf{C}$ is a *trivial cofibration* if it is both a weak equivalence and a cofibration. An object R in a cofibration category $\mathbf{C}$ will be called a *fibrant model* (or simply *fibrant*) if each trivial cofibration $i : R \overset{\sim}{\rightarrowtail} Q$ in $\mathbf{C}$ admits a retraction $r : Q \to R, ri = 1_R$.

The axioms in question are:

(C1) *Composition axiom*: The isomorphisms in $\mathbf{C}$ are weak equivalences and are also cofibrations. For two maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

if any two of f, g and gf are weak equivalences, then so is the third. The composite of cofibrations is a cofibration.

(C2) *Push out axiom*: For a cofibration $i : B \rightarrowtail A$ and a map $f : B \to Y$ there exists the push out in $\mathbf{C}$

$$\begin{tikzcd} A \arrow[r, "\bar{f}"] & A \cup_B Y = A \cup_f Y \\ B \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "\bar{\imath}"] \end{tikzcd}$$

and $\bar{\imath}$ is a cofibration. Moreover:
(a) if f is a weak equivalence, so is $\bar{f}$,
(b) if i is a weak equivalence, so is $\bar{\imath}$.

(C3) *Factorization axiom*: For a map $f : B \to Y$ in $\mathbf{C}$ there exists a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, tail, "i"'] & & Y \\ & A \arrow[ur, "\sim", "g"'] & \end{tikzcd}$$

where i is a cofibration and g is a weak equivalence.

(C4) *Axiom on fibrant models*: For each object X in $\mathbf{C}$ there is a trivial cofibration $X \overset{\sim}{\rightarrowtail} RX$ where RX is fibrant in $\mathbf{C}$. We call $X \overset{\sim}{\rightarrowtail} RX$ a *fibrant model of* X.

In the book Baues [AH] we describe a rich homotopy theory which is available in any cofibration category. In the following sections we recall some basic definitions (like homotopy groups) and results from Baues [AH] needed in this book.

(1.2) Definition. Let $*$ be the initial object of the cofibration category $\mathbf{C}$. An object X in $\mathbf{C}$ is *cofibrant* if the unique morphism $* \to X$ is a cofibration. Let $\mathbf{C}_c$ be the full subcategory of $\mathbf{C}$ consisting of cofibrant objects. Then $\mathbf{C}_c$ with cofibrations and weak equivalences as in $\mathbf{C}$ is again a cofibration category. Let $\mathbf{C}_{cf}$ be the full subcategory of $\mathbf{C}$ consisting of *cofibrant and fibrant* objects in $\mathbf{C}$.

In the following we are mainly concerned with cofibration categories in which all objects are cofibrant. In this case (C2) (a) is equivalent to (C2) (b) by I.1.4 Baues [AH]. Moreover it is convenient to assume that all objects in $\mathbf{C}$ are fibrant. In this case the complication arising from choosing fibrant models is avoided. The theory in the following chapters is available also for cofibration categories in which not all objects are fibrant; for example in the spiral cofibration category $(\Delta\mathbf{Q})_s$ in Chapter D.

(1.3) Example. Let $\mathbf{M}$ be a *Quillen model category* and let $\mathbf{C}$ be the full subcategory of cofibrant objects in $\mathbf{M}$ with weak equivalences and cofibrations defined

by **M**. Then **C** is a cofibration category in which all objects are cofibrant; compare I.2.6 in Baues [AH]. This shows that all the results of the following chapters also hold in Quillen model categories. There are, however, many cofibration categories which have not the structure of a Quillen model category. For example the category **Topp** of compact maps or the category **End** of compactifications. This is the reason why we do not restrict to Quillen model categories.

(1.4) Definition. Let **C** be a cofibration category. Then **Pair**(**C**) is the following *category of pairs* in **C**. Objects are maps $i : B \to A$ in **C** also denoted by (A, B) and morphisms $(A, B) \to (X, Y)$ are commutative diagrams

$$\begin{tikzcd} A \arrow[r, "f"] & X \\ B \arrow[u, "i"] \arrow[r, "f'"] & Y \arrow[u, "i"'] \end{tikzcd}$$

in **C**. The morphism (f, f') is a weak equivalence if f and f' are weak equivalences in **C**. Moreover (f, f') is a cofibration if f' and

$$(f, i) : A \cup_B Y \to X$$

are cofibrations in **C**. Then **Pair**(**C**) is again a cofibration category. Compare II.1.5 in Baues [AH]. An object (A, B) is fibrant if and only if A and B are fibrant in **C**. Given an object Y in **C** we obtain the subcategory

$$\mathbf{C}^Y \subset \mathbf{Pair}(\mathbf{C}).$$

Objects in $\mathbf{C}^Y$ are maps $Y \to X$ and maps are maps under Y in **C**. Weak equivalences and cofibrations in **C** yield the structure of a cofibration category for $\mathbf{C}^Y$. The cofibrant objects in $\mathbf{C}^Y$ are the cofibrations $Y \rightarrowtail X$ in **C**.

(1.5) Definition. Let $f : Y \to B$ be a map in **C** then we obtain the *push foward functor*

$$(\mathbf{C}^Y)_c \xrightarrow{f_*} (\mathbf{C}^B)_c$$

which carries $Y \rightarrowtail X$ to the induced cofibration $B \rightarrowtail B \cup_f X$. If f is a weak equivalence then f_* induces an isomorphism of homotopy categories, see II.4.5 in Baues [AH]. For example let $i : * \xrightarrow{\sim} R*$ be a fibrant model of the initial object $*$ in **C** then the push forward functor

$$i_* : \mathbf{C}_c = (\mathbf{C}^*)_c \to (\mathbf{C}^{R*})_c = \mathbf{C}'$$

is an equivalence of homotopy categories. Here the initial object in $\mathbf{C}'$ is fibrant. For this reason we may assume below that the initial object in a cofibration category is always fibrant.

We now describe basic properties of a cofibration category used in this book. for cofibrant objects X, Y there exists the sum

$$X \vee Y = X \cup_* Y \tag{1.6}$$

in $\mathbf{C}$ which is the push out of $X \leftarrow * \rightarrow Y$. Let $(1,1) : X \vee X \rightarrow X$ be the folding map. A *cylinder* IX of X is obtained by choosing via axiom (C3) a factorization of the folding map

$$(1,1) : X \vee X \overset{(i_0,i_1)}{\rightarrowtail} IX \underset{\sim}{\xrightarrow{p}} X \tag{1.7}$$

where (i_0, i_1) is a cofibration and p is a weak equivalence. Two maps $\alpha, \beta : X \rightarrow U$ in $\mathbf{C}$ are homotopic if there exists a commutative diagram

$$\begin{tikzcd} X \vee X \arrow[dr, "(\alpha,\beta)"'] & \subset & IX \arrow[dl, "H"] \\ & U & \end{tikzcd}$$

Here $H : \alpha \simeq \beta$ is termed a *homotopy.* If X is cofibrant and U is fibrant then $\simeq$ is an equivalence relation. For a cofibrant object X in $\mathbf{C}$ and an object Y let

$$[X, Y] = \mathbf{C}(X, RY)/\simeq \tag{1.8}$$

be the *set of homotopy classes.* Here RY is a fibrant model chosen by (1.1) (C4). An element $g \in [X, Y]$ is represented by a map $g : X \rightarrow Y$ in $\mathbf{C}$ if Y is fibrant and is represented by a diagram $g : X \rightarrow RY \overset{\sim}{\leftarrowtail} Y$ if Y is not fibrant. The context will always make clear whether g denotes a map in $\mathbf{C}$ or a homotopy class. The set $[X, Y]$ is the set of morphisms $X \rightarrow Y$ in the *homotopy category* $\mathrm{Ho}(\mathbf{C})$ which is obtained from $\mathbf{C}$ by inverting weak equivalences; see II.3.6 in Baues [AH]. Homotopy $\simeq$ is a natural equivalence relation on $\mathbf{C}_{cf}$ in (1.2) so that the quotient category $\mathbf{C}_{cf}/\simeq$ is defined. The inclusion $\mathbf{C}_{cf} \subset \mathbf{C}$ induces the equivalence of homotopy categories

$$\mathbf{C}_{cf}/\simeq \overset{\sim}{\longrightarrow} \mathrm{Ho}(\mathbf{C})$$

We point out that the cylinder $I(X \vee Y)$ of a sum $X \vee Y$ of cofibrant objects can be chosen to be $IX \vee IY$. This shows that the sum $X \vee Y$ in $\mathbf{C}$ is also a sum in the homotopy category $\mathrm{Ho}(\mathbf{C})$.

We also need the *relative cylinder* $I_Y X$ of a cofibration $Y \rightarrowtail X$. Let $X \cup_Y X$ be the push out of $X \leftarrowtail Y \rightarrowtail X$ and let $(1,1) : X \cup_Y X \rightarrow X$ be the folding map. Then $I_Y X$ is obtained by a factorization

$$(1,1) : X \cup_Y X \overset{(i_0,i_1)}{\rightarrowtail} I_Y X \overset{\sim}{\longrightarrow} X \tag{1.9}$$

Two maps $f, g : X \to U$ with $f \mid Y = g \mid Y$ are *homotopic relative* Y if there exists a map $I_Y X \to U$ with $Hi_0 = f$ and $Hi_1 = g$. Clearly I_Y is a cylinder in the category $(\mathbf{C}^Y)_c$. Let $[X, U]^Y$ be the set of homotopy classes in $\mathrm{Ho}(\mathbf{C}^Y)$. If $X \rightarrowtail X$ is a cofibration and if $Y \to U$ is a map into a fibrant object U of $\mathbf{C}$ then

$$[X, U]^Y = \mathbf{C}^Y(X, U)/\simeq rel\, Y \tag{1.10}$$

where $\mathbf{C}^Y(X, U)$ is the set of maps $X \to U$ under Y in $\mathbf{C}^Y$; see (1.4).

2 Homotopy Groups

Let $\mathbf{C}$ be a cofibration category as in (1.1). A *based object* A in $\mathbf{C}$ is a cofibrant object A together with a map

$$A \xrightarrow{0} * \tag{2.1}$$

termed the *trivial* map on A. This defines the trivial map $0 : A \to * \to U$ for all objects U in $\mathbf{C}$ representing $0 \in [A, U]$. Given a based object we define the *cone* CA and the *suspension* ΣA by the push out diagrams

$$\begin{tikzcd} IA \arrow[r] & CA \arrow[r, "\pi_0"] & \Sigma A \\ A \vee A \arrow[u, "{(i_o,i_1)}"] \arrow[r, "{1,0}"'] & A \arrow[u, "i_0"'] \arrow[r, "0"'] & * \arrow[u] \end{tikzcd} \tag{2.2}$$

Here CA and ΣA are based objects by use of $0p : IA \to A \to *$. Hence the iterated suspensions $\Sigma^n A$, $n \geq 0$, are defined. We introduce the *homotopy group*

$$\pi_n^A(U) = [\Sigma^n A, U] \tag{2.3}$$

by use of the homotopy set (1.8). This is a pointed set for $n = 0$, a group for $n = 1$ and an abelian group for $n \geq 2$; see II.§ 6 Baues [AH]. Moreover, if A is a cogroup in $\mathrm{Ho}(\mathbf{C})$ then $\pi_n^A(U)$ is a group for $n = 0$ and an abelian group for $n \geq 1$. The pair (CA, A) is a based object in $\mathbf{Pair}(\mathbf{C})$ so that for an object (U, V) in $\mathbf{Pair}(\mathbf{C})$ also the *relative homotopy groups*

$$\pi_{n+1}^A(U, V) = [\Sigma^n(CA, A), (U, V)] \tag{2.4}$$

are defined. As usual one obtains the exact sequence $(n \geq 0)$

$$\cdots \longrightarrow \pi_{n+1}^A(U) \xrightarrow{j} \pi_{n+1}^A(U, V) \xrightarrow{\partial} \pi_n^A(V) \xrightarrow{i} \pi_n^A(U)$$

which is an exact sequence of groups if A is a cogroup in $\mathrm{Ho}(\mathbf{C})$. Compare II.7.8 Baues [AH].

Given a based object B and an object Y in $\mathbf{C}$ we consider the *retraction* $(0,1) : B \vee Y \to Y$ which defines

$$\pi_n^A(B \vee Y)_2 = [\Sigma^n A, B \vee Y]_2 = \operatorname{kernel}\{(0,1)_* : \pi_n^A(B \vee Y) \to \pi_n^A(Y)\} \tag{2.5}$$

If A is a cogroup in $\mathrm{Ho}(\mathbf{C})$ we see that the operators j and ∂ of the exact sequence above induce isomorphisms j and ∂ in the following diagram ($n \geq 1$):

$$\begin{tikzcd}
 & \pi_n^A(CB \vee Y, B \vee Y) \arrow[r, phantom, "\overset{\partial}{\cong}"] \arrow[d, "(\pi_0 \vee 1_Y)_*"] & \pi_{n-1}^A(B \vee Y)_2 \\
\pi_n^A(\Sigma B \vee Y)_2 \arrow[r, phantom, "\overset{j}{\cong}"] & \pi_n^A(\Sigma B \vee Y, Y) &
\end{tikzcd}$$

The *partial suspension*

$$E : \pi_{n-1}^A(B \vee Y)_2 \to \pi_n^A(\Sigma B \vee Y)_2 \tag{2.6}$$

is defined by the composite $E = j^{-1}(\pi_0 \vee 1_Y)_* \partial^{-1}$. Here π_0 is the map in (2.2). If $Y = *$ this is the suspension

$$\Sigma : \pi_{n-1}^A(B) \to \pi_n^A(\Sigma B)$$

Compare II.§ 11 in Baues [AH] where it is shown that E and Σ are homomorphisms of groups.

(2.7) Lemma. *If the based object A is a cogroup in* $\mathrm{Ho}(\mathbf{C})$ *then the group $[A, Y]$ acts from the right on the group $\pi_m^A(D \vee Y)_2$ for $m \geq 0$ and the partial suspension E in (2.6) is equivariant with respect to the action of $[A, Y]$.*

Proof. Since A is a cogroup we can define the element

$$\bar{\mu} = -i_2 + i_1 + i_2 \in [A, A \vee A]_2$$

The m-fold partial suspension

$$E^m \bar{\mu} \in [\Sigma^m A, \Sigma^m A \vee A]_2$$

defines the action of $\alpha \in [A, Y]$ on $\xi \in \pi_m^A(D \vee Y)_2$ by

$$\xi^\alpha = (E^m \bar{\mu})^*(\xi, \alpha)$$

Then clearly $E(\xi^\alpha) = (E\xi)^\alpha$; compare the properties of the partial suspension in II.§ 11 Baues [AH]. q.e.d.

3 Principal Cofibrations

We here describe principal cofibrations which are called principal since they are defined by an attaching map. The dual of a principal cofibration is a principal fibration obtained by a classifying map. In this sense an attaching map is a "co-classifying map".

Given a based object B and an object Y in $\mathbf{C}$ we define for a map $g : B \to Y$ the *mapping cone* C_g by the push out diagram

$$\begin{tikzcd} CB \vee Y \arrow[r, "(\pi_g,1)"] & C_g = CB \cup_g Y \\ B \vee Y \arrow[u, "i_0 \vee 1"] \arrow[r, "(g,1)"'] & Y \arrow[u, "i_g"'] \end{tikzcd} \tag{3.1}$$

where $i_0 \vee 1$ is a cofibration so that also i_g is a cofibration. We call a cofibration $Y \rightarrowtail \bar{Y}$ a *principal cofibration* with attaching map $g \in [B, Y]$ if there is a map $B \xrightarrow{g} RY \overset{\sim}{\leftarrowtail} Y$ in $\mathbf{C}$ representing g together with a weak equivalence

$$C_g = CB \cup_g RY \xrightarrow{\sim} R\bar{Y} \quad \text{under } Y. \tag{1}$$

At this point it is convenient to assume that all objects in $\mathbf{C}$ are fibrant so that we can choose $RY = Y$. In particular i_g in (3.1) is a principal cofibration. It is clear that for all $g \in [B, Y]$ there exists a principal cofibration with attaching map Y and up to equivalence in $\mathrm{Ho}(\mathbf{C}^Y)$ this cofibration in uniquely determined by $g \in [B, Y]$. The composite of pair maps

$$(CB, B) \to (C_g, Y) \sim (\bar{Y}, Y)$$

given by $(\pi_g, 1)$ in (3.1) and by (1) represents the *characteristic element*

$$\pi_g \in \pi_1^B(\bar{Y}, Y) \tag{2}$$

of the principal cofibration $(\bar{Y}, Y)$. Clearly the boundary operator $\partial : \pi_1^B(\bar{Y}, Y) \to \pi_0^B(Y)$ carries this element to the attaching map, that is $\partial \pi_g = g$.

By (3.1) we obtain the following commutative diagram of groups where A is a cogroup in $\mathrm{Ho}(\mathbf{C})$, $n \geq 1$.

$$\begin{tikzcd} \pi_n^A(CB \vee Y, B \vee Y) \arrow[r, "\partial", "\cong"'] \arrow[d, "(\pi_g,1)_*"] & \pi_{n-1}^A(B \vee Y)_2 \arrow[d, "(g,1)_*"] \\ \pi_n^A(C_g, Y) \arrow[r, "\partial"] \arrow[d, equal] & \pi_{n-1}^A(Y) \\ \pi_n^A(\bar{Y}, Y) & \end{tikzcd} \tag{3.2}$$

We define the operator $(n \geq 1)$

$$\begin{aligned} E_Y &: \pi_{n-1}^A(B \vee Y)_2 \to \pi_n^A(\bar{Y}, Y) \quad \text{by} \\ E_Y &= (\pi_g, 1)_* \partial^{-1} \end{aligned} \tag{1}$$

For an element $f \in \pi_n^A(C_g)$ we write

$$f \in E_g(\xi) \quad \text{with } \xi \in \pi_{n-1}^B(B \vee Y)_2 \tag{2}$$

if $j(f) = E_Y(\xi)$. Here $j : \pi_n^A(C_g) \to \pi_n^A(C_g, Y)$ is the operator from the exact sequence of a pair. Clearly $E_g(\xi) = j^{-1}E_Y(\xi)$ is non empty if and only if $(g,1)_*(\xi) = 0$. We also call f with $f \in E_g(\xi)$ a *functional suspension* of ξ. Compare II.§ 11 in Baues [AH]. If $g = 0$ then $C_g = \Sigma B \vee Y$ and $(\pi_g, 1)_*$ coincides with $(\pi \vee 1_Y)_*$ in (2.6). A principal cofibration has the following crucial property II.8.5 Baues [AH]:

(3.3) Lemma. *Let $(\bar{Y}, Y)$ be a principal cofibration with attaching map $g \in [B, Y]$ and let $u : Y \to U$ be a map in* **C** *where U is fibrant. Then there exists an extension $\bar{u}$ in* **C**

$$\begin{tikzcd} & \bar{Y} \arrow[dr, "\bar{u}"] & \\ B \arrow[r, "g"'] & Y \arrow[u, "i_g"] \arrow[r, "u"'] & U \end{tikzcd}$$

with $\bar{u}\, i_g = u$ if and only if the element $g^*\{u\} \in [B, U]$ is the trivial element 0 in the homotopy set $[B, U]$.

For each principal cofibration $(\bar{Y}, Y)$ with attaching map $g : B \to Y$ we have the *coaction* μ which is an element

$$\mu \in [C_g, C_g \vee \Sigma B]^Y = [\bar{Y}, \bar{Y} \vee \Sigma B]^Y \tag{3.4}$$

The coaction μ defines the *action* $+$ of $\beta \in [\Sigma B, U]$ on ξ with

$$\begin{cases} \xi, \xi + \beta \in [C_g, U]^Y = [\bar{Y}, U]^Y \\ \xi + \beta = \mu^*(\xi, \beta) \end{cases}$$

If U is fibrant and $\xi : \bar{Y} \to U, \beta : \Sigma B \to U$ are maps in **C** then

$$\xi + \beta : \bar{Y} \to U$$

denotes any map in **C** representing the element $\xi + \beta \in [\bar{Y}, U]^Y$. In particular the restrictions of $\xi + \beta$ and ξ to Y coincide, that is $(\xi + \beta) \mid Y = \xi \mid Y$.

(3.5) Lemma. *Assume the set $[\bar{Y}, U]^Y$ is non empty. Then the action $+$ of the group $[\Sigma B, U]$ on the set $[\bar{Y}, U]^Y$ is transitive and effective. That is for $\xi, \xi' \in [\bar{Y}, U]^Y$ there is a unique $\beta \in [\Sigma B, U]$ with $\xi' = \xi + \beta$.*

Compare II.8.9 in Baues [AH]. We use μ in (2.4) also for the definition of the action

$$[C_g, U] \times [\Sigma B, U] \xrightarrow{+} [C_g, U] \tag{3.6}$$

which carries (ξ, β) to $\xi + \beta = \mu^*(\xi, \beta)$. This action, however, is in general not transitive or effective. In topology the action $+$ is the well known action used in the Puppe sequence of a cofibration.

The element μ in (3.4) induces the map μ_* in the following diagram where i_1, resp. i_2, denote the inclusions of C_g, resp. ΣB, into the sum $C_g \vee \Sigma B$.

$$\begin{tikzcd}
\pi_1^B(C_g, Y) \arrow[r, "\mu_*"] & \pi_1^B(C_g \vee \Sigma B, Y) & \pi_1^B(C_g, Y) \arrow[l, "(i_1)_*"'] \\
& \pi_1^B(\Sigma B, *) \arrow[u, "(i_2)_*"'] &
\end{tikzcd}$$

(3.7) Lemma. *The characteristic element $\pi_g \in \pi_1^B(C_g, Y)$ satisfies the formula*

$$\mu_*(\pi_g) = (i_1)_*\pi_g + (i_2)_*\pi_0$$

where $\pi_0 \in \pi_1^B(\Sigma B, *)$ is represented by the quotient map $(CB, B) \to (\Sigma B, *)$.

Using the coaction (3.4) we define for a cogroup A in Ho(**C**) the *difference operator*

$$\begin{aligned} \nabla : \pi_1^A(C_g) &\to \pi_1^A(\Sigma B \vee C_g)_2 \\ \nabla(f) &= -i_2 f + (i_2 + i_1) f \end{aligned} \tag{3.8}$$

where $i_1 : \Sigma B \subset \Sigma B \vee C_g$ and $i_2 : C_g \subset \Sigma B \vee C_g$ are the inclusions; compare (I.3.2). Clearly $\nabla(f)$ is trivial on C_g. The difference operator ∇ is part of the following commutative diagram; compare II.§ 12 in Baues [AH].

$$\begin{tikzcd}
\pi_1^A(C_g) \arrow[r, "i"] \arrow[dr, "\nabla"'] & \pi_1^A(C_g, Y) \arrow[r, "(i_2+i_1)_*"] \arrow[d, "\nabla"] & \pi_1^A(\Sigma B \vee C_g, Y) \arrow[d, "\delta"] \\
& \pi_1^A(\Sigma B \vee C_g)_2 \arrow[r, "\cong", "j"'] & \pi_1^A(\Sigma B \vee C_g, C_g)
\end{tikzcd} \tag{3.9}$$

Here δ is induced by the inclusion of pairs given by $i : Y \subset C_g$ and j is the isomorphism already used in (2.6). Using the operators in (3.2) and (2.6) also the following diagram commutes.

$$\begin{tikzcd}
\pi_0^A(B \vee Y)_2 \arrow[r, "E"] \arrow[d, "E_Y"] & \pi_1^A(\Sigma B \vee Y)_2 \arrow[d, "(1 \vee i)_*"] \\
\pi_1^A(C_g, Y) \arrow[r, "\nabla"] & \pi_1^A(\Sigma B \vee C_g)_2
\end{tikzcd} \tag{3.10}$$

Compare II.12.5 in Baues [AH]. This diagram shows for a functional suspension f the following conclusion holds.

$$f \in E_g(\xi) \Rightarrow \nabla f = (1 \vee i)_* E(\xi)$$

Compare (3.2) (2).

(3.11) Definition. Let $(\bar{X}, X)$ and $(\bar{Y}, Y)$ be principal cofibrations with attaching maps $f : A \to X$ and $g : B \to Y$ where A is a cogroup in $\mathrm{Ho}(\mathbf{C})$. Then we consider the diagram

$$\begin{array}{ccc} & & \xi \in \pi_0^A(B \vee Y)_2 \xleftarrow{(i_1)_*} \pi_0^A(B) \ni \xi' \\ & & \downarrow E_Y \\ \pi_f \in \pi_1^A(\bar{X}, X) & \xrightarrow{F_*} & \pi_1^A(\bar{Y}, Y) \end{array}$$

where $F : (\bar{X}, X) \to (\bar{Y}, Y)$ is a pair map in $\mathbf{Pair}(\mathbf{C})$ and π_f is the characteristic element in (2.8) (2). We say that F is a *twisted map* associated to $\xi \in [A, B \vee Y]_2$ if the equation

$$F_*(\pi_f) = E_Y(\xi)$$

holds. Moreover F is a *principal map* associated to $\xi' \in [A, B]$ if

$$F_*(\pi_f) = E_Y(i_1)_*(\xi').$$

(3.12) Lemma. *Let $(\bar{X}, X)$ and $(\bar{Y}, Y)$ be principal cofibrations with attaching maps $f \in [A, X]$ and $g \in [B, Y]$ respectively. Given a map $\eta : X \to Y$ in $\mathbf{C}$ and an element $\xi \in [A, B \vee Y]_2$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\xi} & B \vee Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (g,1)} \\ X & \xrightarrow{\eta} & Y \end{array}$$

commutes in $\mathrm{Ho}(\mathbf{C})$, *that is* $(g, 1)_*\xi = \eta_*(f)$, *there exists a twisted map*

$$F : (\bar{X}, X) \to (\bar{Y}, Y) \quad \textit{in } \mathbf{Pair}(\mathbf{C})$$

extending η and associated to ξ.

For a proof see V.§ 2 in Baues [AH]. Even in topology this lemma is not so well known. The usual construction used in topology is described by the following special case of (3.12).

(3.13) Addendum. *Given $\eta : X \to Y$ in $\mathbf{C}$ and an element $\xi \in [A, B]$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\xi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\eta} & Y \end{array}$$

commutes in Ho(**C**)*, that is* $g_*\xi = \eta_* f$*, there exists a principal map*

$$F : (\bar{X}, X) \to (\bar{Y}, Y) \quad \text{in } \mathbf{Pair}(\mathbf{C})$$

extending η *and associated to* ξ.

(3.14) Definition. Let $(\bar{X}, X)$ and $(\bar{Y}, Y)$ be principal cofibrations with attaching maps $f \in [A, X]$ and $g \in [B, Y]$ respectively. Then any map

$$F : (\bar{X}, X) \to (\bar{Y}, Y) \quad \text{in } \mathbf{Pair}(\mathbf{C})$$

yields the *difference element*

$$\nabla(F) = \nabla F_*(\pi_f) \in [\Sigma A, \Sigma B \vee \bar{Y}]_2$$

by use of the operators

$$\pi_f \in \pi_1^A(\bar{X}, X) \xrightarrow{F_*} \pi_1^A(\bar{Y}, Y) \xrightarrow{\nabla} \pi_1^A(\Sigma B \vee \bar{Y})_2$$

Here π_f is the characteristic element (3.1) (2) and ∇ is the difference operator in (3.9).

In II.12.7 of Baues [AH] we show that the following diagram commutes

$$\begin{array}{ccc} \bar{X} & \xrightarrow{F} & \bar{Y} \\ {\scriptstyle i_2+i_1}\downarrow & & \downarrow{\scriptstyle i_2+i_1} \\ \Sigma A \vee \bar{X} & \xrightarrow{(\nabla(F), i_2 F)} & \Sigma B \vee \bar{Y} \end{array} \tag{3.15}$$

in the homotopy category Ho(**C**). This corresponds exactly to the diagram in (I.3.3). The map $i_2 + i_1$ is up to an interchange of summands the same as the coaction μ in (3.4). Using (3.10) and (3.11) we get

(3.16) Lemma. *Let* $F : (\bar{X}, X) \to (\bar{Y}, Y)$ *be a twisted map associated to* ξ *then*

$$\nabla(F) = (1 \vee i)_* E(\xi)$$

where $i : Y \subset \bar{Y}$ *is the inclusion.*

4 The Cylinder of Pairs

Let X be a cofibrant object in the cofibration category $\mathbf{C}$ and let

$$X \vee X \rightarrowtail IX \xrightarrow{\sim} X$$

be a cylinder as in (1.9). If $X \rightarrowtail \bar{X}$ is a cofibration then we can choose the cylinder IX and $I\bar{X}$ in such a way that we have cofibrations in $\mathbf{C}$

$$X \vee X \rightarrowtail IX \rightarrowtail \bar{X} \cup IX \cup \bar{X} \rightarrowtail I\bar{X} \tag{4.1}$$

Here $\bar{X} \cup IX \cup \bar{X}$ is the push out of $\bar{X} \vee \bar{X} \leftarrow X \vee X \rightarrow IX$. In fact, we know by (1.4) that **Pair** $(\mathbf{C})$ is again a cofibration category and that $(X, \bar{X})$ is a cofibrant object in **Pair** $(\mathbf{C})$. The cylinder $I(\bar{X}, X)$ in **Pair** $(\mathbf{C})$ yields the cofibration

$$(\bar{X}, X) \vee (\bar{X}, X) \rightarrowtail I(\bar{X}, X) = (I\bar{X}, IX)$$

with the properties in (4.1). For this compare the definition of cofibration in **Pair** $(\mathbf{C})$ in (1.4).

(4.2) Lemma. *Let $(\bar{X}, X)$ be a principal cofibration with attaching map $f \in [A, X]$. Then*

$$\bar{X} \cup IX \cup \bar{X} \rightarrowtail I\bar{X}$$

is a principal cofibration with attaching map $w_f \in [\Sigma A, \bar{X} \cup IX \cup \bar{X}]$. Moreover

$$i_0^{\bar{X}} + w_f = i_1^{\bar{X}} \in [\bar{X}, \bar{X} \cup IX \cup \bar{X}]$$

where $+$ is defined by (3.4). Here $i_0^{\bar{X}}$ and $i_1^{\bar{X}}$ are the two inclusions of $\bar{X}$ into $\bar{X} \cup IX \cup \bar{X}$.

Compare II.8.12 in Baues [AH].

We now consider a triple $(\bar{X}, X, T)$ where $(\bar{X}, X)$ and (X, T) are principal cofibrations with attaching maps $f \in [A, X]$ and $h \in [Q, T]$ respectively. Then we obtain via (4.1) cofibrations

$$X \cup IT \cup X \overset{j}{\rightarrowtail} \bar{X} \cup IX \cup \bar{X} \overset{i}{\rightarrowtail} I\bar{X} \tag{4.3}$$

Here (4.2) shows that i is a principal cofibration with attaching map $w_f \in [\Sigma A, \bar{X} \cup IX \cup \bar{X}]$. Moreover (4.2) shows also that j is a principal cofibration with attaching map

$$W = (i_0^X f, w_h, i_1^X f) \in [A \vee \Sigma Q \vee A, X \cup IT \cup X] \tag{4.4}$$

Here i_0^X, i_1^X are the two inclusions of X and w_h is obtained for $(IX, X \cup IT \cup X)$ as in (4.2). The next result shows the surprising fact that in this situation the map w_f is often a functional suspension; see (3.2).

(4.5) Theorem. *Let $A = \Sigma A'$ be a suspension in* **C**. *Then there is an element*

$$\xi \in [A, (A \vee \Sigma Q \vee A) \vee (X \cup IT \cup X)]_2$$

such that $w_f \in E_W(\xi)$. In fact, ξ can be chosen to be

$$\xi = -i_0^A + (\bar{\nabla} f)^*(-i_{\Sigma Q}, i_1^X) + i_1^A$$

Here the inclusions of the three summands in $A \vee \Sigma Q \vee A$ are termed $i_0^A, i_{\Sigma Q}, i_1^A$. Moreover

$$\bar{\nabla} f \in [A, \Sigma Q \vee X]_2$$

satisfies $\bar{\nabla} f = i_X f + \nabla(f) - i_X f$ where $\nabla(f)$ is the difference element in (3.8).

We point out that the inclusion $i : X \subset \bar{X}$ yields

$$(1 \vee i)_* \bar{\nabla}(f) = (1 \vee i)_* \nabla(f) \in [A, \Sigma Q \vee \bar{X}]_2 \tag{4.5}$$

Theorem (4.4) is proved in II.1.37 Baues [AH].

5 Homotopy Cogroups and Homotopy Coactions

Recall that we defined in (I.1.3) the notions of a cogroup and a coaction in a category. If **C** is a cofibration category we introduce the following additional concepts.

(5.1) Definition. Let **C** be a cofibration category with an initial object $*$. A *homotopy cogroup* in **C** is a cofibrant object A in **C** which is a cogroup $(A, 0, \mu, \nu)$ in homotopy category Ho(**C**) such that $0 : A \to *$ in Ho(**C**) can be represented by a map $A \to *$ in **C** and such that $\mu : A \to A \vee A$ can be represented by a map $\mu : A \to R(A \vee A)$ in **C** such that

$$(i_1, \mu) : A \vee A \xrightarrow{\sim} R(A \vee A)$$

is a weak equivalence in **C**. A *homotopy coaction* in **C** is a cofibrant object X in **C** which has the structure (X, A, μ) of a coaction in Ho(**C**). Here A is a homotopy cogroup and $\mu : X \to X \vee A \in$ Ho(**C**) can be represented by a map $\mu : X \to R(X \vee A)$ in **C** such that

$$(i_1, \mu) : X \vee X \xrightarrow{\sim} R(X \vee A)$$

is a weak equivalence in **C**.

We know by (I.1.12) that for any coaction (X, A, μ) in Ho(**C**) the map $(i_1, \mu) : X \vee X \to X \vee A \in$ Ho(**C**) is an isomorphism in Ho(**C**). For a homotopy coaction we require that this isomorphism is actually induced by a weak equivalence.

(5.2) Proposition. *Each suspension $A = \Sigma A'$ in $\mathbf{C}$ is a homotopy cogroup and each principal cofibration $(X, *)$ in $\mathbf{C}$ with attaching map $f \in [Q, *]$ yields a homotopy coaction $(X, \Sigma Q, \mu)$ where μ is the coaction in (3.4).*

Proof. We have $CQ \cup_f * = C_f \xrightarrow{\sim} RX$ and hence $C_f \vee C_f \xrightarrow{\sim} R(X \vee X)$. Since

$$Q \xrightarrow{f} * \rightarrowtail C_f$$

is null homotopic we obtain the weak equivalence

$$\Sigma Q \vee C_f \xrightarrow{\sim} R(C_f \vee C_f)$$

Therefore also $(i_1, \mu) : X \vee X \to R(X \vee \Sigma Q)$ is a weak equivalence. Compare also (3.5). q.e.d.

(5.3) Lemma. *Let (X, A, μ) be a homotopy coaction in $\mathbf{C}$. Then the cylinder $(IX, X \vee X)$ is a principal cofibration with attaching map $w_X \in [A, X \vee X]$. Moreover*

$$i_0^X + w_X = i_1^X \in [X, X \vee X]$$

where $+$ is defined by the coaction μ. Here i_0^X and i_1^X are the two inclusions of X into $X \vee X$.

This lemma is an analogue of lemma (4.2) above.

Proof of (5.3). Since $(i_X, \mu) : X \vee X \to R(X \vee A)$ is a weak equivalence there is a weak equivalence $\rho : R(X \vee A) \to R(X \vee X)$ such that $\rho(i_X, \mu)$ is homotopic to $X \vee X \overset{\sim}{\rightarrowtail} R(X \vee X)$. Hence we obtain the composite in $\mathbf{C}$

$$w : A \subset X \vee A \overset{\sim}{\rightarrowtail} R(X \vee A) \xrightarrow{\sim} R(X \vee X)$$

This map represents w_X in (5.2). Since $* \xrightarrow{\sim} CA$ is a weak equivalence we see that $j : X \to R(X \vee X) \cup_w CA$ is a weak equivalence. Moreover we choose h as in the commutative diagram

$$\begin{array}{ccc} X \vee X & \xrightarrow{(i_0, i_1)} & IX \\ \downarrow \wr & & \downarrow \wr \\ R(X \vee X) & \xrightarrow{h} & RIX \end{array}$$

Since $h_*(w) = 0 \in [A, IX]$ we see by (3.3) that there exists an extension

$$\bar{h} : R(X \vee X) \cup_w CA \to RIX$$

of h with $\bar{h}j = i_0$. Since j and i_0 both are weak equivalences also $\bar{\jmath}$ is one. This proves (5.3); compare the definition of principal cofibration in (3.1). q.e.d.

We now consider a principal cofibration $(\bar{X}, X)$ with attaching map $f \in [A, X]$ where (X, Q, μ) is a homotopy coaction. Then we obtain via (4.1) cofibrations

$$X \vee X \overset{j}{\rightarrowtail} \bar{X} \cup IX \cup \bar{X} \overset{i}{\rightarrowtail} I\bar{X} \tag{5.4}$$

Here (4.2) shows that i is a principal cofibration with attaching map w_f. Moreover (5.3) shows that j is a principal cofibration with attaching map

$$W = (i_0^X f, w_X, i_1^X f) \in [A \vee Q \vee A, X \vee X] \tag{5.5}$$

Here again i_0^X and i_1^X are the two inclusions of X in $X \vee X$. The map W is the analogue of W in (4.4). Also the following analogue of theorem (4.5) holds.

(5.6) Theorem. *Let A be a homotopy cogroup in* **C**. *Then there is an element*

$$\xi \in [A, (A \vee Q \vee A) \vee (X \vee X)]_2$$

such that $w_f \in E_W(\xi)$ is a functional suspension of ξ. In fact, ξ can be chosen to be

$$\xi = -i_0^A + (\bar{\nabla} f)^*(-i_Q, i_1^X) + i_1^A$$

Here the inclusion of the three summands in $A \vee Q \vee A$ are termed i_0^A, i_Q, i_1^A. Moreover

$$\bar{\nabla} f \in [A, Q \vee X]_2$$

satisfies $\bar{\nabla} f = i_X f + \nabla(f) + i_X f$ where $\nabla(f)$ is the difference element (I.3.2).

Again we have for the inclusion $i : X \subset \bar{X}$

$$(1 \vee i)_* \bar{\nabla} f = (1 \vee i)_* \nabla f \in [A, Q \cup \bar{X}]_2 \tag{5.7}$$

Theorem (5.6) is proved by a slight modification of the arguments in the proof of II.13.7 Baues [AH].

6 The Theories susp$(*)$ and cone$(*)$

Let **C** be a cofibration category with an initial object $*$. For each based object

$$B = \{* \rightarrowtail B \xrightarrow{0} *\} \tag{6.1}$$

in **C** we obtain the *suspension* ΣB which is a homotopy cogroup in $\mathbf{C}_c$ by (5.2). If B and B' are based objects then also the sum $B \vee B'$ is a based object and we have

$$\Sigma(B \vee B') = \Sigma B \vee \Sigma B' \tag{1}$$

Let $\mathbf{susp}(*)$ be the *homotopy category of suspensions* in $\mathbf{C}$, this is the full subcategory

$$\mathbf{susp}(*) \subset \mathrm{Ho}(\mathbf{C}_c) \tag{2}$$

consisting of all suspensions ΣB where B is a based object in $\mathbf{C}$. Then (1) and (5.2) show that $\mathbf{susp}(*)$ is a theory of cogroups; compare (I.1.9). Subtheories of $\mathbf{susp}(*)$ yield many examples of theories of cogroups. For example for $\mathbf{C} = \mathbf{Top}^*$ the theory $\mathbf{susp}(*, \mathcal{D})$ consisting of one point unions of 1-spheres in (I.2.4) is such a subtheory of $\mathbf{susp}(*)$. In $\mathbf{Top}^*$ the initial object $*$ is also the final object.

In general, however, we do not assume that $*$ is the final object of the cofibration category $\mathbf{C}$. For each based object B in (6.1) we therefore may have maps $g : B \to *$ which do not coincide with the trivial map 0 in (6.1). In this case we obtain the mapping cone, termed $*$-*cone*,

$$C_g = * \cup_g CB \tag{6.2}$$

where the cone CB is defined by the trivial map 0 in (6.1). If $g = 0$ is the trivial map then $C_g = \Sigma B$ is the suspension of B. Using a cylinder object IB of B we obtain C_g also by the push out diagram in $\mathbf{C}$

$$\begin{tikzcd} IB \arrow[r] & C_g \\ B \vee B \arrow[u, "(i_0,i_1)"] \arrow[r, "(g,0)"] & * \arrow[u] \end{tikzcd}$$

Hence C_g may also be considered to be a "double mapping cylinder" in $\mathbf{C}$ in which one of the glueing maps is specified to play the role of the trivial map. One needs this specification to define the coaction map

$$\mu : C_g \to C_g \vee \Sigma B \tag{1}$$

in $\mathrm{Ho}(\mathbf{C}_c)$; see (3.4). In fact, this is a homotopy coaction by (5.2). Moreover if $(g, g') : B \vee B' \to *$ is defined on a sum of based objects then

$$C_{(g,g')} = C_g \vee C_{g'} \tag{2}$$

This is a generalisation of (6.1) (1). Moreover (2) is compatible with the coaction maps at both sides of the equation. Let $\mathbf{cone}(*)$ be the *homotopy category of $*$-cones* in $\mathbf{C}$, this is the full subcategory in $\mathrm{Ho}(\mathbf{C}_c)$ consisting of $*$-cones as in (6.2). Then we have inclusions of full subcategories

$$\mathbf{susp}(*) \subset \mathbf{cone}(*) \subset \mathrm{Ho}(\mathbf{C}_c) \tag{6.3}$$

Using (6.2) (1), (2) we see that $\mathbf{cone}(*)$ is actually a theory of coactions. In fact, (3.5) shows that the affine property holds in $\mathbf{cone}(*)$; compare (I.1.11). The cogroups in $\mathbf{cone}(*)$ are exactly the suspensions in $\mathbf{susp}(*)$. Subtheories of $\mathbf{cone}(*)$

yield many examples of theories of coactions. Such subtheories $\mathbf{T} \subset \mathbf{cone}(*)$ yield examples of "cofibration categories $\mathbf{C}$ under $\mathbf{T}$" considered in (IV.2.1) below.

Now let D be a cofibrant object in $\mathbf{C}$. Then the category $\mathbf{C}^D$ of objects under D is again a cofibration category with initial object D. We denote by

$$\mathbf{susp}(D) \subset \mathbf{cone}(D) \subset \mathrm{Ho}(\mathbf{C}^D)_c \tag{6.4}$$

the full subcategories of suspensions and D-cones in $\mathbf{C}^D$ respectively. Hence (6.4) coincides with (6.3) if we replace $\mathbf{C}$ by $\mathbf{C}^D$. In $\mathbf{C}^D$ we have the special based object

$$\Sigma^0_D = (D \overset{i_1}{\rightarrowtail} D \vee D \xrightarrow{(1,1)} D) \tag{6.5}$$

so that the suspension $\Sigma(\Sigma^0_D) = \Sigma_*(D)$ is defined in $\mathbf{C}^D$. We also write $\Sigma^0_D = \Sigma^0_* D$. The based object

$$\Sigma_*(D) = (D \rightarrowtail \Sigma_* D \overset{p_1}{\rightarrowtail} D) \tag{1}$$

in $\mathbf{C}^D$ can be obtained more directly by the push out diagram

$$\begin{tikzcd} ID \arrow[r, "\pi"] & \Sigma_* D \arrow[r, "p^1"] & D \\ D \vee D \arrow[u, "{(i_0,i_1)}"] \arrow[r, "{(1,1)}"] & D \arrow[u, "i"'] & \end{tikzcd} \tag{2}$$

Here p^1 is defined by $p^1\pi = p$ where p is the projection of the cylinder and $p^1 i = 1_D$.

For example if $\mathbf{C} = \mathbf{Top}$ then we have $\Sigma_* D = S^1 \times D$ where the right hand side is the product of the 1-sphere S^1 and D.

The homotopy category $\mathrm{Ho}(\mathbf{C}^D)_c$ has sums which are given by

$$(X, D) \vee (Y, D) = (X \cup_D Y, D)$$

Here (X, D) denotes an object in $(\mathbf{C}^D)_c$ and $X \cup_D Y$ is a push out in$\mathbf{C}$. Using the homotopy extension property of the cofibration $D \rightarrowtail X$ (see II.2.17 and II.5.7 in Baues [AH]) we define the *augmentation map*

$$\varepsilon_X : (X, D) \longrightarrow (X, D) \vee (\Sigma_* D, D) \tag{6.6}$$

in $\mathrm{Ho}(\mathbf{C}^D)_c$ as follows. Let $I(X, D) = (IX, ID)$ be the cylinder of the pair (X, D) in (4.1). Then we have the cofibration

$$j : X \cup_D ID \longrightarrow IX$$

which is i_0 on X. Since j is a weak equivalence there exists a map r for which the following diagram commutes where $R(Y)$ is a fibrant model of Y in $\mathbf{C}$.

$$\begin{tikzcd} X \arrow[r, "i_1"] & IX \arrow[r, "r"] & R(X \cup_D ID) \arrow[r, "\bar{v}"] & R(X \cup_D \Sigma_* D) \\ & & X \cup ID \arrow[ul, "j"] \arrow[u] \arrow[r, "v"] & X \cup_D \Sigma_* D \arrow[u, "\wr"] \end{tikzcd}$$

Here $v = 1 \cup \pi$ is given by π in (6.5) (2) and $\bar{v}$ is an extension of V on fibrant models; see II.1.6 Baues [AH]. The composite $\bar{v}\, r\, i_1$ defines the map ε_X in $\mathrm{Ho}(\mathbf{C}^D)_c$ in (6.6).

(6.7) Lemma. *The map ε_X is a coaction and for all $f : (X,D) \to (Y,D)$ in $\mathrm{Ho}(\mathbf{C}^D)_c$ the diagram*

$$\begin{array}{ccc} (X,D) & \xrightarrow{\varepsilon_X} & (X,D) \vee (\Sigma_* D, D) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\vee 1} \\ (Y,D) & \xrightarrow{\varepsilon_Y} & (Y,D) \vee (\Sigma_* D, D) \end{array}$$

commutes in $\mathrm{Ho}(\mathbf{C}^D)_c$.

This is a consequence of II.5.9, II.5.10 in Baues [AH]. The lemma shows that the category $\mathrm{Ho}(\mathbf{C}^D)_c$ is a Σ-augmented theory where $\Sigma = (\Sigma_* D, D)$; see (I.7.2). Moreover we get the following result which yields many examples of augmented theories of coactions; compare the notation in (I.7.4).

(6.8) Proposition. *The category* **cone**(D) *is a theory of coactions which is augmented by* $\Sigma = (\Sigma_* D, D)$.

Proof. A cogroup in **cone**$(*)(D)$ is a suspension $\Sigma\bar{A}$ of a based object $\bar{A}$ in $(\mathbf{C}^D)_c$. Hence one has maps

$$\bar{A} = \{D \rightarrowtail A \xrightarrow{0} D\}$$

in $\mathbf{C}$. The trivial map 0 yields the suspension $\Sigma\bar{A}$ by the push out diagrams in $\mathbf{C}$

$$\begin{array}{ccccc} IA & \longrightarrow & I_D A & \longrightarrow & \Sigma\bar{A} \\ \uparrow & & \uparrow & & \uparrow \\ A \vee A & \longrightarrow & A \cup_D A & \xrightarrow{(0,0)} & D \end{array} \tag{1}$$

Here IA is the cylinder of A in $\mathbf{C}$ and $I_D A$ is the cylinder of $D \rightarrowtail A$ in $\mathbf{C}^D$. As in II.2.10 Baues [AH] we obtain for the map $0 : A \to D$ the map $I0$ between cylinder in $\mathbf{C}$ such that the following diagram commutes

$$\begin{array}{ccccc} IA & \longleftarrow & A \vee A & \xrightarrow{(0,0)} & D \\ {\scriptstyle I0}\downarrow & & \downarrow{\scriptstyle 0\vee 0} & & \downarrow{\scriptstyle 1} \\ ID & \longleftarrow & D \vee D & \xrightarrow{(1,1)} & D \end{array}$$

This diagram induces a linear map

$$\bar{\varepsilon}_{\Sigma(A,D)} : (\Sigma\bar{A}, D) \to (\Sigma_* D, D) \tag{2}$$

of cogroups in $\mathrm{Ho}(\mathbf{C}^D)_c$; for this use (1) and (6.5) (2). Now II.5.15 Baues [AH] shows that the maps in (2) and (6.6) satisfy a formula as in (I.7.4) (4). q.e.d.

Proposition (6.8) shows that each full subtheory of **cone**(D) which contains the D-torus $\Sigma_* D$ is actually an augmented theory of coactions.

(6.9) Example. Let $\mathbf{C} = \mathbf{Top}$ and let D be a discrete space. Then the D-torus $\Sigma = (S^1 \times D, D)$ is an object in the subtheory **cone**$(D, \mathcal{D})$ of **cone**(D); see (I.2.11). Here **cone**$(D, \mathcal{D})$ is an augmented theory of coactions by (6.8) above. In this case the augmentation in (6.6) and (6.8) (2) above coincides with the corresponding augmentation already described in (I.7.6).

7 Appendix: Categories with a Cylinder Functor

The cylinder IX of a cofibrant object X in a cofibration category is obtained by the factorization axiom C3; see (1.7). Therefore there are many choices of such cylinders of X and $X \mapsto IX$ is not a functor in X. There are, however, many examples of homotopy theories (in particular the homotopy theory of topological spaces) which are defined by a cylinder functor. In the following definition we describe the basic properties of a cylinder functor which are needed to obtain the structure of a cofibration category.

(7.1) Definition. An *I-category* is a category $\mathbf{C}$ with the structure $(\mathbf{C}, cof, I, \emptyset)$. Here *cof* is a class of morphisms in $\mathbf{C}$, called cofibrations, I is a functor $\mathbf{C} \to \mathbf{C}$ together with natural transformations i_0, i_1 and p, $\emptyset$ is the initial object in $\mathbf{C}$. The structure satisfies the following axioms (I1), ... , (I5).

(I1) Cylinder axiom: $I : \mathbf{C} \to \mathbf{C}$ is a functor together with natural transformations

$$i_0,\ i_1 : id_{\mathbf{C}} \to I, \quad p : I \to id_{\mathbf{C}},$$

such that for all objects X the composite $pi_\varepsilon : X \to IX \to X$ is the identity of X for $\varepsilon = 0$ and $\varepsilon = 1$.

(I2) Push out axiom: For a cofibration $i : B \to A$ and a map f there exists the push out

$$\begin{tikzcd} A \arrow[r] & A \cup_B X \\ B \arrow[u, "i"] \arrow[r, "f"'] & X \arrow[u, "\bar{\imath}"'] \end{tikzcd}$$

where $\bar{\imath}$ is also a cofibration. Morover, the functor I carries this push out diagram into a push out diagram, that is $I(A \cup_B X) = IA \cup_{IB} IX$. Moreover, $I\emptyset = \emptyset$.

(I3) Cofibration axiom: Each isomorphism is a cofibration and for each object X the map $\emptyset \to X$ is a cofibration. We thus have by (I2) the sum $X \cup_\emptyset Y = X \vee Y$. The composition of cofibrations is a cofibration. Moreover, a cofibration $i : B \rightarrowtail A$ has the following *homotopy extension property* in $\mathbf{C}$. Let $\varepsilon \in \{0, 1\}$. For each commutative diagram in $\mathbf{C}$

$$\begin{tikzcd} B \arrow[r, "i_\varepsilon"] \arrow[d, "i"'] & IB \arrow[d, "H"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

there is $E : IA \to X$ with $E(Ii) = H$ and $Ei_\varepsilon = f$.

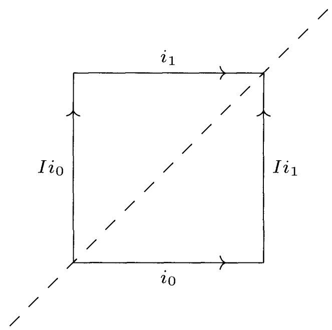
(I4) Relative cylinder axiom: For a cofibration $i : B \to A$ the map j defined by the following push out diagram is a cofibration:

$$\begin{tikzcd} B \vee B \arrow[r, "i \vee i"] \arrow[d, "(i_0, i_1)"'] \arrow[dr, phantom, "\text{push}"] & A \vee A \arrow[d, "\alpha"] & \\ IB \arrow[r, "\beta"'] & A \cup IB \cup A \arrow[r, "j"] & IA \end{tikzcd}$$

where $j\beta = Ii$ and $j\alpha = (i_0, i_1)$. Equivalently $(IB, B \vee B) \to (IA, A \vee A)$ is a cofibration in $\mathbf{Pair}(\mathbf{C})$, see (1.4).

(I5) The interchange axiom: For all objects X there exists a map $T : IIX \to IIX$ with $Ti_\varepsilon = I(i_\varepsilon)$ and $TI(i_\varepsilon) = i_\varepsilon$ for $\varepsilon = 0$ and $\varepsilon = 1$. We call T an interchange map.

We sketch the double cylinder IIX by



The interchange map T restricted to the boundary $\partial I^2 X$ is the reflection at the diagonal.

(7.2) Example. Let **Top** be the category of topological spaces and let $I = [0, 1] \subset \mathbb{R}$ be the unit interval. The *cylinder* of a space X is defined by the product $I(X) = I \times X$ with the product topology. Cofibrations in **Top** are the maps which have the homotopy extension property in **Top** and the interchange map

$$T : IIX = I \times I \times X \to I \times I \times X = IIX$$

carries (t_1, t_2, x) to (t_2, t_1, x) for t_1, $t_2 \in I$ and $x \in X$. Clearly this interchange map is natural in X. It is not hard to see that $(\mathbf{Top}, I, cof, \emptyset)$ satisfies all the axioms of an I-category. We prove this by (8.2) and (8.3) below. Compare I.4.2 in Baues [AH].

(7.3) Definition. Let **C** be an I-category. We say $f_0, f_1 : A \to X$ are *homotopic* if there is $G : IA \to X$ with $Gi_0 = f_0$ and $Gi_1 = f_1$. We call G a homotopy and we write $G : f_0 \simeq f_1$. A map $f : A \to X$ is a *homotopy equivalence* if there is $g : X \to A$ with $fg \simeq 1_X$ and $gf \simeq 1_A$.

The next result shows that actually all the properties and results for a cofibration category are available in an I-category.

(7.4) Theorem. *Let* $(\mathbf{C}, cof, I, \emptyset)$ *be an* I*-category. Then* **C** *is a cofibration category with the following structure. Cofibrations are those of* **C**, *weak equivalences are the homotopy equivalences and all objects are fibrant and cofibrant in* **C**.

This theorem was originally proved in I.3.3 of Baues [AH]. In an I-category **C** the projection $IX \to X$ is a homotopy equivalence (see I.3.13 Baues [AH]). Hence by (I4) with $B = *$ we see that IX is a cylinder in the sense of (1.7).

For a map $i : B \to A$ in an I-category **C** and $\varepsilon \in \{0, 1\}$ we obtain the push out diagram

$$\begin{tikzcd} IB \arrow[r] & IB \cup_\varepsilon A \arrow[r, "j_\varepsilon"] & IA \\ B \arrow[u, "i_\varepsilon"] \arrow[r, "i"'] \arrow[ur, phantom, "\text{push}"] & A \arrow[u] & \end{tikzcd}$$

with $j_\varepsilon \mid IB = Ii$ and $j_\varepsilon \mid A = i_\varepsilon$. Here i_ε is a cofibration by (I4).

(7.5) Lemma. *The map* $i : B \to A$ *satisfies the homotopy extension property in* **C** *if and only if there exist maps* $r_\varepsilon : IA \to IB \cup_\varepsilon A$ *with* $r_\varepsilon j_\varepsilon = 1$ *for* $\varepsilon \in \{0, 1\}$.

8 Appendix: Natural Cylinder Categories and Homotopy Theory of Diagrams

We first introduce the notion of a "natural I-category" and we then show that the category of diagrams in a natural I-category is again a natural I-category.

(8.1) Definition. A *natural* I*-category* $(\mathbf{C}, I, \emptyset)$ is a category **C** with an initial object $\emptyset$ and a cylinder functor I such that (I1) and (I2) in (7.1) hold. Here *cofibrations* are defined to be exactly the maps which satisfy the homotopy extension property in **C**; see (I3). Moreover (I4)$'$ and (I5)$'$ below are satisfied.

(I4)$'$ For $\varepsilon \in \{0, 1\}$ and $B \rightarrowtail A$ there exists a commutative diagram in **C** where j_ε is defined as in (7.5).

$$\begin{tikzcd} IA' \arrow[r, "\alpha_\varepsilon"] & IIA \\ IB' \cup_\varepsilon A' \arrow[u, "j_\varepsilon"] \arrow[r, "\beta_\varepsilon"] & I(IB \cup_\varepsilon A) \arrow[u, "I(j_\varepsilon)"'] \end{tikzcd}$$

Here $B' = A \cup IB \cup A \to IA = A'$ is the map j in (I4). Moreover α_ε and β_ε are natural in $(A, B) \in \mathbf{Pair}(\mathbf{C})_c$ and β_ε is an isomorphism in $\mathbf{C}$.

(I5)$'$ There exists an interchange map $T : IIX \to IIX$ as in (I5) which is natural in X.

(8.2) Proposition. *A natural I-category is an I-category.*

Proof. One readily checks (I3). Moreover (I4) holds by use of (7.5) since for a cofibration $B \rightarrowtail A$ with retraction r_ε the composite

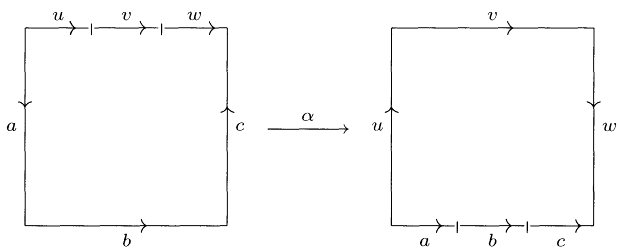
$$IIA \xrightarrow{\alpha_\varepsilon} IIA \xrightarrow{Ir_\varepsilon} I(IB \cup_\varepsilon A) \xrightarrow{\beta_\varepsilon^{-1}} IB' \cup_\varepsilon A'$$

is a retraction for j'_ε. Clearly (I5) is satisfied by (I5)$'$. q.e.d.

(8.3) Example. The category **Top** of topological spaces with the cylinder in (7.2) is a natural I-category. We define the natural homeomorphism

$$\alpha_\varepsilon = T(\alpha \times 1) : I \times I \times X \longrightarrow I \times I \times X$$

by T in (7.2) and a homeomorphism $\alpha : I \times I \to I \times I$ which is given on a boundary by the sketches



Compare I.4.2 in Baues [AH]. One can check that the restriction β_ε is well defined.

(8.4) Definition. Let $\mathbf{J}$ be a small category and let $\mathbf{C}$ be a category. A $\mathbf{J}$ *-diagram in* $\mathbf{C}$ is a functor

$$X : \mathbf{J} \to \mathbf{C}$$

A morphism between such $\mathbf{J}$-diagrams X, Y is a natural transformation $f : X \to Y$ in $\mathbf{C}$. Hence f is given by a collection of maps

$$f : X_j \to Y_j \quad \text{in } \mathbf{C}$$

where $j \in \mathrm{Ob}(\mathbf{J})$ is an object in $\mathbf{J}$. Here we write $X(j) = X_j$. Let $\mathbf{C}^\mathbf{J}$ be the category of such $\mathbf{J}$-diagrams and morphisms.

(8.5) Theorem. *Let $\mathbf{C}$ be a natural I-category and let $\mathbf{J}$ be a small category. Then also the category $\mathbf{C}^\mathbf{J}$ of $\mathbf{J}$-diagrams in $\mathbf{C}$ is a natural I-category.*

Proof. The cylinder of a diagram X is defined by the composite of functors

$$IX : \mathbf{J} \xrightarrow{X} \mathbf{C} \xrightarrow{I} \mathbf{C} \tag{1}$$

where I is the cylinder in $\mathbf{C}$. Hence we have $(IX)_j = I(X_j)$ for $j \in \mathrm{Ob}(\mathbf{J})$. The natural transformation i_0, i_1, p for $I(X_j)$ in $\mathbf{C}$ yield such transformation for IX. Hence (I1) is satisfied. Now cofibrations $B \rightarrowtail A$ are defined by the homotopy extension property in $\mathbf{C}^{\mathbf{J}}$. Hence we have by (7.5) the retraction

$$r_\varepsilon : IA \longrightarrow IB \cup_\varepsilon A \tag{2}$$

in the category of diagrams. Push outs of diagrams are obtained by push outs in $\mathbf{C}$. Hence by (2) we obtain for each $j \in \mathrm{Ob}(\mathbf{J})$ the retraction in $\mathbf{C}$

$$r_\varepsilon : IA_j \longrightarrow IB_j \cup_\varepsilon A_j \tag{3}$$

showing that $B_j \to A_j$ is a cofibration in $\mathbf{C}$ and hence the push outs in (I2) for diagrams exist. Since I is compatible with push outs in $\mathbf{C}$ by (I2) we see that I is also compatible with push outs of diagrams. Hence (I2) holds for the category of diagrams; (it is easily seen that i has again the homotopy extension property). Now (I4)$'$ and (I5)$'$ are clearly satisfied for diagrams since $\alpha_\varepsilon, \beta_\varepsilon$ and T are natural in $\mathbf{C}$. q.e.d.

9 Appendix: Homotopy Theory of Chain Complexes

Let $\mathbf{A}$ be an additive category with sums $A \oplus B$. A *chain complex* in $\mathbf{A}$ is a graded object $V = \{V_i,\ i \in \mathbb{Z}\}$ in $\mathbf{A}$ together with a map $d : V \to V$ of degree -1 satisfying $dd = 0$. A chain map $f : V \to W$ is a map of degree 0 with $df = fd$ and a homotopy $\alpha : f \simeq g$ between chain maps is a map $\alpha : V \to W$ of degree 1 with $-f+g = d\alpha + \alpha d$. A chain map $i : V \to W$ is a *cofibration* if the underlying graded object of W is a direct sum $W = V \oplus \bar{W}$ and $i : V \to V \oplus \bar{W}$ is the inclusion; then V is a subcomplex of W; see (I.§ 6).

(9.1) Definition. For a graded object V let sV be the *suspension* of V with $(sV)_n = V_{n-1}$ and let $s : V \to sV$ be the map of degree $+1$ given by the identity. We define the *cylinder* $I(V)$ of a chain complex V by the graded object

$$I(V) = V' \oplus V'' \oplus sV$$

Here $V' = V'' = V$ are two copies of the graded object V. Let $i_0 : V \to I(V)$, $i_1 : V \to I(V)$ be the inclusions given by $V = V'$ and $V = V''$ respectively. Then the differential d of $I(V)$ is defined by $di_0 = i_0 d, di_1 = i_1 d$ and

$$ds = -i_0 + i_1 - sd.$$

One readily checks that a homotopy α corresponds to the chain map $H : I(V) \to V'$ with $Hs = \alpha, Hi_0 = f, Hi_1 = g$. A chain map $f : V \to W$ induces the chain map $If : IV \to IW$ given by $f \oplus f \oplus sf$. Let $p : I(V) \to V$ be the chain map with $pi_0 = pi_1 = 1$ and $p \mid sV = 0$.

(9.2) Proposition. *The category* $\mathbf{chain}_{\mathbf{A}}^{+}$ *of chain complexes in* $\mathbf{A}$ *which are bounded below and the cylinder* I *and cofibrations as above form an* I*-category in the sense of (7.1).*

This is an example of an I-category which is not a natural I-category. For the proof of (9.2) we use the *tensor product* functor

$$\otimes : \mathbf{ab}^{\sharp} \times \mathbf{A} \to \mathbf{A} \tag{9.3}$$

where $\mathbf{ab}^{\sharp}$ is the category of finitely generated free abelian groups denoted by $\bigoplus_E \mathbb{Z}$, E a finite set. For $A \in \mathbf{A}$ the tensor product is defined by

$$\left(\bigoplus_E \mathbb{Z}\right) \otimes A = \bigoplus_E A.$$

There is an obvious way to define $\otimes$ on morphisms. If V is a chain complex in $\mathbf{A}$ and C is a chain complex in $\mathbf{ab}^{\sharp}$ then $C \otimes V$ is the chain complex in $\mathbf{A}$ given by

$$(C \otimes V)_n = \bigoplus_{i+j=n} C_i \otimes V_j. \tag{9.4}$$

The differential on $C_i \otimes V_j$ is given by $d \otimes V_j + (-1)^i C_i \otimes d$. Let I be the chain complex in $\mathbf{ab}^{\sharp}$ (concentrated in degree 0 and 1) which is generated by $\{0\}$ and $\{1\}$ in degree 0 and by s in degree 1 with the differential $d\{s\} = -\{0\} + \{1\}$. Then we identify the chain complexes

$$IV = I \otimes V \tag{9.5}$$

by $V' = \{0\} \otimes V, V'' = \{1\} \otimes V$ and $sV = \{s\} \otimes V$. Using (9.1) we see that this is an isomorphism of chain complexes.

Proof of (9.2). It is easy to see that (I1) and (I2) and (I4) are satisfied. For the proof of (I3) we have to check that a cofibration $V \rightarrowtail W$ has the homotopy extension property, $\varepsilon = 0$. For $W = V \oplus \bar{W}$ we define the subchain complex $W_{(n)} = V \oplus \bar{W}_{\leq n}$. We define inductively the homotopy extension

$$E_{(n)} : IW_{(n)} \cup W \longrightarrow X$$

by $E_{(n+1)} \mid s\bar{W}_{n+1} = 0$ and

$$E_{(n+1)} i_1 \mid \bar{W}_{n+1} = f \mid \bar{W}_{n+1} + E_{(n)} sd \mid \bar{W}_{n+1}.$$

One readily checks that $E_{(n+1)}$ is a well defined chain map; see the proof of I.6.10 in Baues [AH]. Since we assume that W is bounded below we can start the induction. Hence $E = \lim E_{(n)}$ is defined and therefore (I3) holds. Finally we obtain a natural interchange map

$$T = t \otimes 1 : I \otimes I \otimes V \longrightarrow I \otimes I \otimes V$$

where $t : I \otimes I \to I \otimes I$ is defined by $t(x \otimes y) = (-1)^{|x|\,|y|} y \otimes x$. q.e.d.

Recall that the homology and cohomology of chain complexes is defined as in (I.§6).

(9.6) Theorem. *Let V, W be chain complexes in $\mathbf{A}$ which are bounded below and let $f : V \to W$ be a chain map. Then (a) and (b) are equivalent.*

(a) f is a homotopy equivalence.
(b) f induces an isomorphism

$$f_* : H_n(V, M) \to H_n(W, M)$$

for all left $\mathbf{A}$-modules $M = \mathrm{Hom}_{\mathbf{A}}(A, \quad)$ where A is an object in $\mathbf{A}$ and $n \in \mathbb{Z}$.

Moreover if $\mathbf{A}$ is the additive subcategory of an abelian category $\mathbf{B}$ such that all objects of $\mathbf{A}$ are projective in $\mathbf{B}$ then (a) is equivalent to (c).

(c) f induces an isomorphism

$$f^* : H^n(W, N) \to H^n(V, N)$$

for all right $\mathbf{A}$-modules $N = \mathrm{Hom}_{\mathbf{B}}(\quad , B)$ where B is an object in $\mathbf{B}$ and $n \in \mathbb{Z}$.

Theorem (I.6.6) is an easy consequence of this result. The equivalence (a) $\Leftrightarrow$ (b) in the theorem is a special case of the general Whitehead theorem in the next chapter.

Proof of (9.6). By (9.2) we know that $\mathbf{chain}^+_{\mathbf{A}}$ is a cofibration category; see (7.4). Hence we can use the Puppe or cofiber sequence of f. Since f is a homotopy equivalence if and only if the suspension Σf is a homotopy equivalence we see via the Puppe sequence that this is the case if and only if the mapping cone $U = C_f$ is contractible. On the other hand the (co-) homology of U with coefficients M (resp. N) is trivial if and only if the maps f_* in (b) (resp. f^* in (c)) are isomorphisms for all n. This shows that it suffices to prove (9.6) if W is the trivial chain complex. It is clear that $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ holds. We now show $(b) \Rightarrow (a)$. For this we need the assumption that v is bounded below with $V_i = *$ for $i < n_0$. Since $H_{n_0}(V, M) = 0$ with $M = \mathrm{Hom}(V_{n_0}, -)$ we see that there exists $\alpha_{n_0} : V_{n_0} \to V_{n_0+1}$ with $d\alpha_{n_0} = 1$. Now assume $\alpha_n : V_n \to V_{n+1}$ is constructed with $1 = d\alpha_n + \alpha_{n-1}d$. Then $d(1 - \alpha_n d) = d - d\alpha_n d = \alpha_{n-1}dd = 0$ and hence $1 - \alpha_n d$ represents an element in $H_{n+1}(V, M) = 0$ with $M = \mathrm{Hom}(V_{n+1}, -)$. Hence there exists α_{n+1} with $d\alpha_{n+1} = 1 - \alpha_n d$. This completes the proof that V is contractible.

Next we show $(c) \Rightarrow (b)$. For this let $H^n(V, N) = 0$ for $n \in \mathbb{Z}$ and all N as in (c). In (c) we have $\mathbf{A} \subset \mathbf{B}$ where $\mathbf{B}$ is an abelian category. (We may assume that $\mathbf{A}$ and $\mathbf{B}$ are small and that $\mathbf{B}$ is a subcategory of the category of modules over some ring; see Borceaux [CA]). Consider in $\mathbf{B}$ the diagram

$$V_{n+1} \xrightarrow{d_{n+1}} V_n \overset{q}{\twoheadrightarrow} \mathrm{cok}(d_{n-1}) \overset{j}{\twoheadrightarrow} \mathrm{im}(d_n) \overset{i}{\subset} V_{n-1}$$

where cok = cokernel and im = image.

Since $qd_{n+1} = 0$ we see that q represents an element in $H^n(V, N) = 0$ with $N = \mathrm{Hom}(-, \operatorname{cok} d_{n+1})$. Hence there exists $p : V_{n-1} \to \operatorname{cok} d_{n+1}$ with $pd_n = q$ or $pijq = q$ and hence $pij = 1$. Hence j is a monomorphism. Since j is also an epimorphism we see that j is an isomorphism. This implies $\operatorname{im} d_{n+1} = \ker d_n$ so that V is exact. Since the objects of **A** are projective in **B** we see that also $\mathrm{Hom}_{\mathbf{A}}(A, V)$ is exact for all objects A in **A** and therefore (b) holds. This completes the proof of (9.6). q.e.d.

Chapter IV: Complexes in Cofibration Categories

We introduce "complexes" and "cellular objects" in cofibration categories which correspond to CW-complexes in algebraic topology. We prove a general Whitehead theorem for complexes and for cellular objects. This theorem yields as specialization most of the various Whitehead theorems proved independently in different fields of the literature. We also study the general concepts in cofibration categories which in algebraic topology correspond to the "cellular approximation theorem" and the "Blakers-Massey theorem".

1 Filtered Objects

We consider filtered objects in a cofibration category $\mathbf{C}$ and we define 0-homotopies and 1-homotopies for maps between filtered objects. In the next section we introduce complexes in a cofibration category which are examples of filtered objects.

(1.1) Definition. Let $\mathbf{C}$ be a cofibration category. Then $\mathbf{Fil}_0(\mathbf{C})$ is the following *category of filtered objects* in $\mathbf{C}$. Objects are diagrams

$$A_{\geq 0} = (A_0 \to A_1 \to \dots A_n \to A_{n+1} \to \dots)$$

of maps $i : A_n \to A_{n+1}$ in $\mathbf{C}$, $n \geq 0$. A morphism $f : A_{\geq 0} \to B_{\geq 0}$ is a sequence of maps $f_n : A_n \to B_n$, with $if_n = f_{n+1}i$. We say that f is a *weak equivalence* if each f_n is a weak equivalence in $\mathbf{C}$. Moreover, f is a *cofibration* if each map

$$(f_{n+1}, f_n) : (A_{n+1}, A_n) \to (B_{n+1}, B_n)$$

is a cofibration in $\mathbf{Pair}(\mathbf{C})$; see (III.1.4). We have the full inclusion of categories

$$\mathbf{C} \subset \mathbf{Fil}_0(\mathbf{C})$$

which carries $A \in \mathbf{C}$ to the *constant filtered object* with $A_n = A$ for $n \geq 0$ and $i = 1_A$. The initial object of $\mathbf{Fil}_0(\mathbf{C})$ is the constant filtered object given by $*$ in $\mathbf{C}$. Moreover we say that $X_{\geq 0}$ is of *dimension* $\leq n$ if $X_m = X_n$ for $m \geq n$ and if $i : X_m \to X_{m+1}$ is the identity. Let

$$\mathbf{Fil}_1(\mathbf{C}) \subset \mathbf{Fil}_0(\mathbf{C})$$

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be the full subcategory of objects $X_{\geq 0}$ with $X_0 =$ the initial object of $\mathbf{C}$. We also write $X_{\geq 1} \in \mathbf{Fil}_1(\mathbf{C})$ where $X_{\geq 1} = (X_1 \to X_2 \to \dots)$ is given by $X_{\geq 0} = (* \to X_1 \to X_2 \to \dots)$.

(1.2) Lemma. *The category* $\mathbf{Fil}_0(\mathbf{C})$ *with weak equivalences and cofibrations in (1.1) is a cofibration category. An object* $A_{\geq 0}$ *is fibrant if and only if all objects* A_i*,* $i \geq 0$*, are fibrant in* $\mathbf{C}$*. Moreover* $A_{\geq 0}$ *is cofibrant if* A_0 *is cofibrant in* $\mathbf{C}$ *and all* $i : A_n \to A_{n+1}$ *are cofibrations in* $\mathbf{C}$*.*

Compare III.1.2 in Baues [AH].

(1.3) Definition. In the category $\mathbf{Fil}_0(\mathbf{C})$ we consider two notions of homotopies. Given a cofibrant object $A_{\geq 0}$ the cylinder

$$IA_{\geq 0} = (IA_0 \subset IA_1 \subset \dots) \tag{1}$$

consists of a sequence of cylinders IA_n in $\mathbf{C}$, $n \geq 0$. Two maps $f, g : A_{\geq 0} \to U_{\geq 0}$ are homotopic if there exists a map $H : IA_{\geq 0} \to U_{\geq 0}$ in $\mathbf{Fil}_0(\mathbf{C})$ with $Hi_0 = f$, $Hi_1 = g$. We call such a homotopy a *0-homotopy* $H : f \overset{0}{\simeq} g$. Let

$$i : U_{\geq 0} \to s^{-1}U_{\geq 0} \tag{2}$$

be the canonical *shift map* in $\mathbf{Fil}_0(\mathbf{C})$. Here we set $(s^{-1}U_{\geq 0})_n = U_{n+1}$ for $n \geq 0$ so that $s^{-1}U_{\geq 0} = (U_1 \to U_2 \to \dots)$. Then (2) in degree n is the map $i : U_n \to U_{n+1}$. The maps f, g are *1-homotopic*, $f \overset{1}{\simeq} g$; if there exists a 0-homotopy $if \overset{0}{\simeq} ig$. We define the *cylinder object for 1-homotopies* $\bar{I}A_{\geq 0} \in \mathbf{Fil}_0(\mathbf{C})$ by

$$\begin{cases} (\bar{I}A_{\geq 0})_0 = A_0 \vee A_0 \\ (\bar{I}A_{\geq 0})_n = A_n \cup IA_{n-1} \cup A_n \quad \text{for } n \geq 1 \end{cases} \tag{3}$$

where the right hand side is the push out of $A_n \vee A_n \leftarrow A_{n-1} \vee A_{n-1} \to IA_{n-1}$. Hence we have the cofibration $A_{\geq 0} \vee A_{\geq 0} \rightarrowtail \bar{I}A_{\geq 0}$ and a 1-homotopy $H : f \overset{1}{\simeq} g$ is the same as a map $H : \bar{I}A_{\geq 0} \to U_{\geq 0}$ with $Hi_0 = f$ and $Hi_1 = g$. Compare also (III.4.3).

Example. If $\mathbf{C} = \mathbf{Top}$ and if $A_{\geq 0}$ is the filtration of skeleta of the CW-complex A in $\mathbf{C}$ then $\bar{I}A_{\geq 0}$ is the filtration of skeleta of the cylinder $I \times A$. Moreover if $\mathbf{C} = \mathbf{Top}^*$ is the category of pointed spaces and if $A_{\geq 1}$ is the filtration of skeleta $A^1 \subset A^2 \subset \dots$ of a reduced CW-complex A with $A^0 = *$ then $\bar{I}A_{\geq 1}$ is the filtration of skeleta of the reduced cylinder $I \times A / I \times \{*\}$.

Let $\mathbf{Fil}_0(\mathbf{C})_{cf}$ be the full subcategory of cofibrant and fibrant objects in $\mathbf{Fil}_0(\mathbf{C})$. For objects $X_{\geq 0}$, $U_{\geq 0}$ in $\mathbf{Fil}_0(\mathbf{C})_{cf}$ we have the quotient map

$$[X_{\geq 0}, U_{\geq 0}]/\overset{0}{\simeq} \longrightarrow [X_{\geq 0}, U_{\geq 0}]/\overset{1}{\simeq} \tag{1.4}$$

where the left hand side is the set of 0-homotopy classes and the right hand side is the set of 1-homotopy classes. Accordingly one has the quotient functor

$$\mathbf{Fil}_0(\mathbf{C})_{cf}/\overset{0}{\simeq} \longrightarrow \mathbf{Fil}_0(\mathbf{C})_{cf}/\overset{1}{\simeq}$$

of homotopy categories. Compare also Baues III.1.5 [AH].

In general we do not assume that for an object $X_{\geq 0}$ the *direct limit* (also termed *colimit*) $\lim(X_{\geq 0})$ exists in $\mathbf{C}$. Later we shall, however, use the following property of an object in $\mathbf{Fil}_0(\mathbf{C})_c$.

(1.5) Definition. We say that a cofibrant object $A_{\geq 0}$ in $\mathbf{Fil}_0(\mathbf{C})$ has the *limit property* if the direct limits $A = \lim(A_{\geq 0})$ and $IA = \lim(IA_{\geq 0})$ exist and if IA is a cylinder object for A. That is, A is a cofibrant object in $\mathbf{C}$ and the maps

$$A_{\geq 0} \vee A_{\geq 0} \xrightarrow{(i_0,i_1)} IA_{\geq 0} \xrightarrow{\ p\ } A_{\geq 0}$$

in $\mathbf{Fil}_0(\mathbf{C})$ induce maps on direct limits

$$A \vee A \xrightarrow{(i_0,i_1)} IA \xrightarrow{\ p\ } A$$

where (i_0, i_1) is a cofibration and p is a weak equivalence in $\mathbf{C}$. It is clear that each finite dimensional object $A_{\geq j}$ in $\mathbf{Fil}_0(\mathbf{C})$ has this limit property. Moreover the object $\bar{I}A_{\geq 0}$ in (1.3) (3) satisfies

$$\lim(IA_{\geq 0}) = \lim(\bar{I}A_{\geq 0}).$$

2 Complexes Associated to Theories of Coactions

Let $\mathbf{T}$ be a theory of coactions as in (I.§ 1). Given $\mathbf{T}$ we have all the notation and results in chapter I and chapter II at hand. We now combine $\mathbf{T}$ with the homotopy category of a cofibration category $\mathbf{C}$ with initial object $*$. Recall that this homotopy category is given by $\mathrm{Ho}(\mathbf{C}_c)$ or by $\mathbf{C}_{cf}/\simeq$ and that we have the equivalence of categories

$$R : \mathrm{Ho}(\mathbf{C}_c) \xrightarrow{\sim} \mathbf{C}_{cf}/\simeq$$

which carries X to a fibrant model RX with $X \overset{\sim}{\rightarrowtail} RX$. Sums $X \vee Y$ exist in $\mathbf{C}_c$ and $X \vee Y$ is also a sum in $\mathrm{Ho}(\mathbf{C}_c)$ so that $R(X \vee Y)$ is the sum of RX and RY in $\mathbf{C}_{cf}/\simeq$.

(2.1) Definition. Let $\mathbf{T}$ be a theory of coactions. A *cofibration category under* $\mathbf{T}$ is a cofibration category $\mathbf{C}$ together with a full embedding of categories

$$\mathbf{T} \subset \mathrm{Ho}(\mathbf{C}_c) \sim \mathbf{C}_{cf}/\simeq \tag{1}$$

which carries sums in $\mathbf{T}$ to sums in $\mathrm{Ho}(\mathbf{C}_c)$ such that objects in $\mathbf{T}$ are homotopy coactions in $\mathbf{C}$; see (III.5.1). Given an object X in $\mathbf{T}$ we denote the corresponding object in $\mathbf{C}_{cf}$ as well by X.

For each cofibration category $\mathbf{C}$ with initial object $*$ we obtain canonically full subcategories

$$\mathbf{susp}(*) \subset \mathbf{cone}(*) \subset \mathrm{Ho}(\mathbf{C}_c) \tag{2}$$

as described in (III.§6). Here the objects of $\mathbf{cone}(*)$ are the $*$-*cones* of the form

$$C_g = * \cup_g CB \tag{3}$$

where $* \rightarrowtail B \xrightarrow{0} *$ is a based object in $\mathbf{C}$ and where $g : B \to *$ is a map in $\mathbf{C}$. If $g = 0$ is the trivial map then $C_g = \Sigma B$ is the suspension of B. Such *suspensions* ΣB of based objects B in $\mathbf{C}$ are the objects of $\mathbf{susp}(*)$. By (III.5.2) we see that $\mathbf{C}$ is always a cofibration category under the theory of coactions $\mathbf{cone}(*)$.

In many examples the theory $\mathbf{T}$ of coactions in (2.1) will be a subtheory of $\mathbf{susp}(*)$ or $\mathbf{cone}(*)$ above. There are however examples (like the categories of differential algebras or simplicial groups) where $\mathbf{T}$ is not a subcategory of $\mathbf{cone}(*)$.

(2.2) Definition. Let $\mathbf{C}$ be a cofibration category under $\mathbf{T}$. A *complex* or more precisely a $\mathbf{T}$ *-complex* in $\mathbf{C}$ is a cofibrant object

$$X_{\geq 1} = (X_1 \subset X_2 \subset \dots)$$

in $\mathbf{Fil}_1(\mathbf{C})$ with the following properties. The object X_1 is an object in $\mathbf{T}$ and the pair (X_{n+1}, X_n), $n \geq 1$, is a principal cofibration (see (III.3.1)) with attaching map

$$\partial_{n+1} \in [\Sigma^{n-1} A_{n+1}, X_n].$$

Here A_{n+1} is a cogroup in $\mathbf{T}$ for $n \geq 1$. In particular $\partial_X = \partial_2 \in [A_2, X_1]$ is given by a map in $\mathbf{T}$ which represents an object ∂_X in $\mathbf{Coef}$ and X_2 is given by the mapping cone of ∂_X. Let

$$\mathbf{Complex} \subset \mathbf{Fil}_1(\mathbf{C})_c$$

be the full subcategory consisting of $\mathbf{T}$-complexes $X_{\geq 1} = (X_{\geq 1}, A_{\geq 1}, \partial_{\geq 2})$. Here A_1 is the cogroup associated to the coaction on X_1. We write $\mathbf{Complex} = \mathbf{Complex}(\mathbf{T})$. We also call a $\mathbf{T}$-complex a *reduced* complex. In chapter VIII we shall discuss "non-reduced" complexes in a cofibration category.

(2.3) Remark. If $\mathbf{C}$ is a cofibration category under $\mathbf{T}$ then $\mathbf{Pair}(\mathbf{C})$ is a cofibration category under $\mathbf{T}(2)$. Here $\mathbf{T}(2)$ is the following theory of coactions. Objects (X_1, Y_1) in $\mathbf{T}(2)$ are inclusions

$$Y_1 \to Y_1 \vee \bar{X}_1 = X_1$$

given by the sum of objects $Y_1, \bar{X}_1$ in $\mathbf{T}$. Morphisms in $\mathbf{T}(2)$ are commutative diagrams

$$\begin{array}{ccc} Y_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ Y_1 \vee \bar{X}_1 = X_1 & \longrightarrow & A_1 = B_1 \vee \bar{A}_1 \end{array}$$

in $\mathbf{T}$. The coaction for Y_1 and $\bar{X}_1$ in $\mathbf{T}$ yields the coaction for (X_1, Y_1) in $\mathbf{T}(2)$ in the obvious way.

(2.4) Definition. We say that a $\mathbf{T}$-complex $Y_{\geq 1}$ is a *subcomplex* of the $\mathbf{T}$-complex $X_{\geq 1}$ if a $\mathbf{T}(2)$-complex

$$(X_{\geq 1}, Y_{\geq 1}) = \big((X_1, Y_1) \subset (X_2, Y_2) \subset \dots\big)$$

in $\mathbf{Pair(C)}$ is given with the following properties. The pair $(X_1, Y_1) \in \mathbf{T}(2)$ is obtained by an inclusion

$$Y_1 \subset X_1 = Y_1 \vee \bar{X}_1 \tag{1}$$

where Y_1 and $\bar{X}_1$ are objects in $\mathbf{T}$; see (2.3). Moreover the attaching maps are of the form

$$\partial_{n+1} : \Sigma^{n-1}(A_{n+1}, B_{n+1}) \longrightarrow (X_n, Y_n).$$

Here (A_{n+1}, B_{n+1}) is a cogroup in $\mathbf{T}(2)$ given by an inclusion

$$B_{n+1} \subset A_{n+1} = B_{n+1} \vee \bar{A}_{n+1} \tag{2}$$

where B_{n+1} and $\bar{A}_{n+1}$ are cogroups in $\mathbf{T}$. As a special case one obtains for $n \geq 1$ the *n-skeleton* $X^n \subset X_{\geq 1}$ which is a subcomplex of $X_{\geq 1}$. Here $(X^n)_i = X_i$ for $i \leq n$ and $(X^n)_j = X^n$ for $j \geq n$. We define the *dimension* of a complex $X_{\geq 1}$ by $\dim X_{\geq 1} \leq n$ if $X_{\geq 1} = X^n$ is an n-skeleton. Then $A_i = *$ for $i \geq n+1$.

(2.5) Proposition. *Let $X_{\geq 1}$ be a complex and let $\bar{I}X_{\geq 1}$ be the cylinder object for 1-homotopies in (1.3) with*

$$(\bar{I}X_{\geq 1})_n = X_n \cup IX_{n-1} \cup X_n$$

Then $\bar{I}X_{\geq 1}$ is a complex and $X_{\geq 1} \vee X_{\geq 1}$ is a subcomplex of $\bar{I}X_{\geq 1}$.

Proof. If $X_{\geq 1}$ has attaching maps $\partial_{n+1} : \Sigma^{n-1}A_{n+1} \to X_n$ then $\bar{I}X_{\geq 1}$ has the attaching maps $(n \geq 2)$

$$\bar{\partial}_{n+1} : \Sigma^{n-1}A_{n+1} \vee \Sigma^{n-1}A_n \vee \Sigma^{n-1}A_{n+1} \longrightarrow (\bar{I}X_{\geq 1})_n$$

obtained by (III.4.4) and (III.5.5). q.e.d.

(2.6) Example. Let $\mathbf{C} = \mathbf{Top}^*$ be the cofibration category of pointed topological spaces. Weak equivalences are homotopy equivalences in $\mathbf{Top}$ and cofibrations are defined by the homotopy extension property in $\mathbf{Top}$. All objects in $\mathbf{C}$ are fibrant. The cofibrant objects in $\mathbf{C}$ are also termed "well pointed" spaces. Let

$$\mathbf{T} = \mathbf{susp}(*, \mathcal{D}) \subset (\mathbf{Top}^*)_c/\simeq$$

be the full subcategory consisting of one point unions $A = \vee_E S^1$ of 1-spheres S^1 where E is an index set. Then $\mathbf{T}$ is a theory of cogroups isomorphic to the category $\mathbf{gr}$ of free groups and $\mathbf{Top}^*$ is a cofibration category under $\mathbf{T}$. A CW-complex X is *reduced* if the 0-skeleton $X^0 = *$ is the base point. Each reduced CW-complex X yields a filtered object $X^{\geq 1} = (X^1 \subset X^2 \subset \dots)$ given by the skeleta X^n of X. This filtered object is a $\mathbf{T}$-complex in the sense of (2.2). In fact X^1 is a one point union of 1-spheres so that $X^1 \in \mathbf{T}$ and there exists a map $g : \Sigma^{n-1}A \to X^n$ in $\mathbf{Top}^*$ with $A \in \mathbf{T}$ such that X^{n+1} is homotopy equivalent under X^n to the mapping cone C_g. The homotopy class of g in $\mathbf{Top}^*$ is determined up to the action of $\pi_1(X^n)$ by the CW-complex X. If the reduced CW-complex X is *normalized* (in the sense that all attaching maps $\alpha : S^n \to X^n$ of $(n+1)$-cells in X carry the basepoint of S^n to $\{*\} = X^0$) then the structure of X as a $\mathbf{T}$-complex is well defined. A reduced CW-complex X has the additional property that

$$X = \lim(X^{\geq 1})$$

is the direct limit of the filtered object $X^{\geq 1}$ in the category $\mathbf{Top}^*$. We point out that in the general definition of a $\mathbf{T}$-complex in (2.2) we do not assume that the direct limit $\lim(X_{\geq 1})$ of the complex $X_{\geq 1}$ exists in $\mathbf{C}$. One readily checks that a subcomplex Y of a reduced CW-complex is also a subcomplex in the sense of (2.4). Moreover $\bar{I}X^{\geq 1}$ in (2.5) corresponds to the skeletal filtration of the cylinder $I \times X / I \times \{*\}$.

3 The Whitehead Theorem

The classical Whitehead theorem shows that a weak equivalence between CW-complexes is also a homotopy equivalence in the category $\mathbf{Top}$. We here study the analogue of this theorem for $\mathbf{T}$-complexes in a cofibration category $\mathbf{C}$ under $\mathbf{T}$. This leads to the following notions of lifting map, elementary lifting map, $\mathbf{T}$-equivalence and weak $\mathbf{T}$-equivalence.

(3.1) Definition. We say that a map $f : Y_{\geq 1} \to X_{\geq 1}$ in $\mathbf{Fil}_1(\mathbf{C})_{cf}$ is a *lifting map* if for all $\mathbf{T}$-complexes $K_{\geq 1}$ with subcomplex $L_{\geq 1}$ and commutative diagrams

$$\begin{array}{ccc} L_{\geq 1} & \xrightarrow{b} & Y_{\geq 1} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ K_{\geq 1} & \xrightarrow{a} & X_{\geq 1} \end{array}$$

in $\mathbf{Fil}_1(\mathbf{C})_c$ (where j is the inclusion) there exist a map

$$d : K_{\geq 1} \to Y_{\geq 1} \in \mathbf{Fil}_1(\mathbf{C})_c$$

with $dj = b$ and a 1-homotopy $fd \overset{1}{\simeq} a \, rel \, L_{\geq 1}$. This map d is termed a *lift* of the diagram.

(3.2) Definition. We say that a map $f : Y_{\geq 1} \to X_{\geq 1}$ in $\mathbf{Fil}_1(\mathbf{C})_{cf}$ is an *elementary lifting map* if for all cogroups A and objects Z in $\mathbf{T}$ the following properties hold, $n \geq 1$.

$$(if_{n+1})_* : \pi_{n-1}^A(Y_{n+1}) \longrightarrow \pi_{n-1}^A(X_{n+2}) \quad \text{is injective.} \tag{i}$$

$$(if_{n+1})_* : \pi_n^A(Y_{n+1}) \longrightarrow \pi_n^A(X_{n+2}) \quad \text{is surjective.} \tag{ii}$$

(iii) For the maps between homotopy sets

$$[Z, Y_1] \xrightarrow{(if_1)_*} [Z, X_2] \xleftarrow{i_*} [Z, X_1]$$

we have $\text{image}(if_1)_* = \text{image}(i_*)$.

(3.3) Proposition. *An elementary lifting map f is also a lifting map.*

Proof. We first show that it is sufficient to prove (3.3) for the case that f is a cofibration in $\mathbf{Fil}_1(\mathbf{C})$. To see this we choose a factorization

$$f : Y_{\geq 1} \overset{f'}{\rightarrowtail} \bar{X}_{\geq 1} \underset{p}{\xrightarrow{\sim}} X_{\geq 1}$$

by (C3) in $\mathbf{Fil}_1(\mathbf{C})$ where $\bar{X}_{\geq 1}$ is fibrant. Then we consider the diagram in $\mathbf{Fil}_1(\mathbf{C})_c$

$$\begin{array}{ccc} L_{\geq 1} & \xrightarrow{f'b} & \bar{X}_{\geq 1} \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle p}\big\downarrow{\scriptstyle \wr} \\ K_{\geq 1} & \xrightarrow{a} & X_{\geq 1} \end{array}$$

By use of II.1.11 (b) in Baues [AH] there exists $a' : K_{\geq 1} \to \bar{X}_{\geq 1}$ with $a'j = f'b$ and $pa' \overset{0}{\simeq} a$ relative $L_{\geq 1}$. Since f is an elementary lifting map one readily checks that also f' is an elementary lifting map. Below we show that f' is a lifting map so that there is a lift d for the diagram

$$\begin{array}{ccc} L_{\geq 1} & \xrightarrow{b} & Y_{\geq 1} \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ K_{\geq 1} & \xrightarrow{a'} & \bar{X}_{\geq 1} \end{array}$$

with $dj = b$ and $f'd \overset{1}{\simeq} a' rel \, L_{\geq 1}$. Hence $fd = pf'd \overset{1}{\simeq} pa' \overset{0}{\simeq} a$. This shows that also f is a lifting map.

Now let f in (3.3) be a cofibration in $\mathbf{Fil}_1(\mathbf{C})$. We observe that the assumption on f in (3.2) imply that for any cogroup A in $\mathbf{T}$ the relative homotopy groups

$$\pi_n^A(X_{n+2}, Y_{n+1}) = 0 \tag{1}$$

are trivial for $n \geq 1$. To see this we consider the following exact sequence:

$$\pi_n^A(Y_{n+1}) \to \pi_n^A(X_{n+2}) \to \pi_n^A(X_{n+2}, Y_{n+1}) \to \pi_{n-1}^A(Y_{n+1}) \to \pi_{n-1}^A(X_{n+2})$$

Here the right hand side is injective by (3.2) (i) and the left hand side is surjective by (3.2) (ii). Hence exactness implies (1).

We now construct the lift d and the 1-homotopy $H : fd \overset{1}{\simeq} a\, rel\, L_{\geq 1}$ inductively. To start the induction we use (3.2) (iii) for $Z = \bar{K}_1$. For this recall that for $L_{\geq 1} \subset K_{\geq 1}$ we have by (2.4) (i) in degree 1 the inclusion $L_1 \subset L_1 \vee \bar{K}_1 = K_1$. Now (3.2) (iii) shows for $\{\bar{a}_1 = a_1 \mid \bar{K}_1\} \in [\bar{K}_1, X_1]$ that there exists $\{\bar{d}_1\} \in [\bar{K}_1, Y_1]$ with $(if_1)_*\{\bar{d}_1\} = \{\bar{a}_1\}$. Hence we have a map in $\mathbf{C}$

$$\bar{d}_1 : \bar{K}_1 \longrightarrow Y_1 \tag{2}$$

and a homotopy in $\mathbf{C}$

$$\bar{H}_1 : if_1 d_1 \simeq i\bar{a}_1, \quad \bar{H}_1 : I\bar{K}_1 \to X_2,$$

where $i : X_1 \to X_2$ is the inclusion. We define

$$\begin{cases} d_1 = (b_1, \bar{d}_1) : K_1 = L_1 \vee \bar{K}_1 \longrightarrow Y_1 \\ H_1 = (ib_1, \bar{H}_1) : I_{L_1}(K_1) = L_1 \vee I\bar{K}_1 \longrightarrow X_2 \end{cases} \tag{3}$$

Now let $n \geq 1$ and assume that

$$\begin{cases} d_n : K_n \to Y_n \\ H_n : I_{L_n} K_n \to X_{n+1} \\ H_n : if_n d_n \simeq ia_n \quad \text{relative} \quad L_n \end{cases} \tag{4}$$

are defined. Let A' be a cogroup in $\mathbf{T}$ and let

$$g : A = \Sigma^{n-1} A' \to K_n$$

be the attaching map of the principal cofibration (K'_{n+1}, K_n) for which

$$\begin{tikzcd} L_n \arrow[r, "j"] \arrow[d] & K_n \arrow[r] & K'_{n+1} \arrow[d] \\ L_{n+1} \arrow[rr] & & K_{n+1} \end{tikzcd} \tag{5}$$

is a push out diagram; see (2.4). Then we get the commutative diagram

$$\begin{tikzcd}
CA \arrow[r] & K'_{n+1} \arrow[r] & K_{n+1} \arrow[r, "ia_{n+1}"] & X_{n+2} \\
A \arrow[u] \arrow[r, "g"] \arrow[d, "i_1"'] & K_n \arrow[u] \arrow[rr, "ia_n"] \arrow[d] & & X_{n+1} \arrow[u, "i"'] \arrow[d, equal] \\
IA \arrow[r, "Ig"] & I_{L_n}K_n \arrow[rr, "H_n"] & & X_{n+1} \\
A \arrow[u, "i_0"] \arrow[r, "g"] & K_n \arrow[u] \arrow[rr, "\mathrm{id}_n"] & & Y_{n+1} \arrow[u, "f_{n+1}"']
\end{tikzcd} \tag{6}$$

Diagram (6) yields a map of pairs

$$(\alpha, \mathrm{id}_n g) : (CA \vee_{i_1} IA, A) \longrightarrow (X_{n+2}, Y_{n+1}) \tag{7}$$

and hence an element in the group

$$\pi_1^A(X_{n+2}, Y_{n+1}) = \pi_n^{A'}(X_{n+2}, Y_{n+1}) = 0$$

which is trivial by (1). Therefore the map (7) admits an extension

$$(\beta', \beta) : (C(CA \cup_{i_1} IA), CA) \longrightarrow (X_{n+2}, Y_{n+1}) \tag{8}$$

where the left hand side is the cone in $\mathbf{Pair}(\mathbf{C})$. We now obtain the diagram

$$\begin{tikzcd}
C_g = K_n \cup_g CA \arrow[r, "(\mathrm{id}_n, \beta)"] \arrow[d, "\wr"'] & Y_{n+1} \arrow[d, equal] \\
K'_{n+1} \arrow[r, "d'_{n+1}"] & Y_{n+1}
\end{tikzcd}$$

where the extension d'_{n+1} exists since Y_{n+1} is fibrant. Let

$$d_{n+1} = (d'_{n+1}, b_{n+1}) : K_{n+1} \longrightarrow Y_{n+1} \tag{9}$$

be obtained by the push out (5). Then (8) shows that one gets a homotopy $H_{n+1} : if_{n+1}d_{n+1} \simeq ia_{n+1}$ relative L_{n+1} extending H_n. Here we use (III.4.2). This completes the proof of (3.3). q.e.d.

(3.4) Definition. A map $f : Y_{\geq 1} \to X_{\geq 1}$ in $\mathbf{Fil}_1(\mathbf{C})_{cf}$ is a $\mathbf{T}$ *-equivalence* if for all $\mathbf{T}$-complexes $K_{\geq 1}$ the induced map

$$f_* : [K_{\geq 1}, Y_{\geq 1}]/\overset{1}{\simeq} \longrightarrow [K_{\geq 1}, X_{\geq 1}]/\overset{1}{\simeq} \tag{1}$$

is a bijection. Here we use the sets of 1-homotopy classes in (1.4). Moreover f is a *weak* $\mathbf{T}$ *-equivalence* if for all cogroups A and objects Z in $\mathbf{T}$ and $n \geq 1$ the induced maps f_* below are bijections, where im = image.

$$\operatorname{im}\{[Z, Y_1] \xrightarrow{i_*} [Z, Y_2]\} \xrightarrow{f_*} \operatorname{im}\{[Z, X_1] \xrightarrow{i_*} [Z, X_2]\} \tag{2}$$

$$\operatorname{im}\{\pi_n^A Y_{n+1} \to \pi_n^A Y_{n+2}\} \xrightarrow{f_*} \operatorname{im}\{\pi_n^A X_{n+1} \to \pi_n^A X_{n+2}\} \tag{3}$$

One readily checks that a **T**-equivalence is also a weak **T**-equivalence. For this let $K_{\geq 1}$ be the constant object given by Z. Then (1) implies (2). Moreover choosing for $K_{\geq 1}$ the complex with trivial n-skeleton of dimension n given by $\Sigma^n A$ shows that (1) implies (3).

(3.5) Proposition. *A lifting map f is a* **T**-*equivalence.*

Proof. Using $L = *$ in (3.1) we see that f_* in (3.4) (1) is surjective. Moreover if $a, b : K_{\geq 1} \to Y_{\geq 1}$ are maps for which $H : fa \overset{1}{\simeq} fb$ then one gets the commutative diagram

$$\begin{tikzcd} K_{\geq 1} \vee K_{\geq 1} \arrow[r, "{(a,b)}"] \arrow[d] & Y_{\geq 1} \arrow[d, "f"] \\ \bar{I}K_{\geq 1} \arrow[r, "H"] & X_{\geq 1} \end{tikzcd}$$

A lift of this diagram yields a 1-homotopy $a \overset{1}{\simeq} b$. This shows that f_* in (3.4) (1) is also injective. q.e.d.

Using (3.3), (3.4) and (3.5) we have the following implications in a cofibration category under **T**:

$$\begin{array}{c} \text{elementary lifting map} \\ \Downarrow \\ \text{lifting map} \\ \Downarrow \\ \mathbf{T}\text{-equivalence} \\ \Downarrow \\ \text{weak } \mathbf{T}\text{-equivalence} \end{array} \tag{3.6}$$

The converse of these implications are true if we assume that $X_{\geq 1}$, $Y_{\geq 1}$ have an additional property as follows.

(3.7) Definition. We say that $X_{\geq 1} \in \mathbf{Fil}_1(\mathbf{C})_{cf}$ is **T** *-good* if for all cogroups A in **T** and $n \geq 1$ the groups $\pi_n^A(X_{n+2}, X_{n+1}) = 0$ are trivial and $\pi_0^A(X_1) \to \pi_0^A(X_2)$ is surjective.

(3.8) Proposition. *Assume that $Y_{\geq 1}$ and $X_{\geq 1}$ are* **T**-*good. Then a weak* **T**-*equivalence $f : Y_{\geq 1} \to X_{\geq 1}$ is also an elementary lifting map.*

Proof. First we observe that (3.4) (2) implies (3.2) (iii). Moreover since $Y_{\geq 1}, X_{\geq 1}$ are $\mathbf{T}$-good we see that (3.4) (3), (2) imply

$$f_* : \pi_n^A Y_{n+2} \xrightarrow{\approx} \pi_n^A X_{n+2}$$

is an isomorphism for $n \geq 0$ and all cogroups A in $\mathbf{T}$. Here we use the exact homotopy sequence. Moreover the diagram ($n \geq 1$)

$$\begin{tikzcd} \pi_n^A Y_{n+1} \arrow[r, "i_*"] & \pi_n^A Y_{n+2} \arrow[r] \arrow[d, "f_*", "\approx"'] & 0 \\ & \pi_n^A X_{n+2} & \end{tikzcd}$$

with exact row shows that (3.2) (ii) holds. Finally the diagram ($n \geq 1$)

$$\begin{tikzcd} & \pi_{n-1}^A Y_{n+1} \arrow[d, "f_*", "\approx"'] & \\ 0 \arrow[r] & \pi_{n-1}^A X_{n+1} \arrow[r] & \pi_{n-1}^A X_{n+2} \end{tikzcd}$$

with exact row shows that (3.2) (i) holds. q.e.d.

Now (3.8) and (3.6) imply the following result

(3.9) Theorem. *Let* $\mathbf{C}$ *be a cofibration category under* $\mathbf{T}$ *and let* $f : Y_{\geq 1} \to X_{\geq 1}$ *be a map in* $\mathbf{Fil}_1(\mathbf{C})_{cf}$ *between* $\mathbf{T}$*-good objects. Then we have the following equivalences:*

$$\begin{array}{c} f \text{ is elementary lifting map} \\ \Updownarrow \\ f \text{ is lifting map} \\ \Updownarrow \\ f \text{ is } \mathbf{T}\text{-equivalence} \\ \Updownarrow \\ f \text{ is weak } \mathbf{T}\text{-equivalence} \end{array}$$

(3.10) Proposition. *Let* $f : X_{\geq 1} \to Y_{\geq 1}$ *be a map between* $\mathbf{T}$*-complexes which is a* $\mathbf{T}$*-equivalence. Then* f *is a 1-homotopy equivalence (that is an isomorphism in the category* $\mathbf{Complex}(\mathbf{T})/\overset{1}{\simeq}$*).*

Proof. Since

$$f_* : [Y_{\geq 1}, X_{\geq 1}]/\overset{1}{\simeq} \longrightarrow [Y_{\geq 1}, Y_{\geq 1}]/\overset{1}{\simeq}$$

is bijective we see that there is $g : Y_{\geq 1} \to X_{\geq 1}$ with $fg \overset{1}{\simeq} 1$. Since $f_*(gf) = f_*(1)$ the injectivity of f_* shows $gf \overset{1}{\simeq} 1$. q.e.d.

As a corollary of (3.10) and (3.9) we get the following generalization of the classical Whitehead theorem.

(3.11) General Whitehead Theorem (I). *Let $X_{\geq 1}, Y_{\geq 1}$ be* $\mathbf{T}$*-complexes which are* $\mathbf{T}$*-good. Then a map $f : X_{\geq 1} \to Y_{\geq 1}$ is a weak* $\mathbf{T}$*-equivalence if and only if f is a 1-homotopy equivalence.*

(3.12) Example. Let $\mathbf{A}$ be an additive category and let $\mathbf{C}$ be the full subcategory of the I-category $\mathbf{chain}_{\mathbf{A}}^{+}$ in (III.9.2) consisting of chain complexes concentrated in degree ≥ 1. Then $\mathbf{C}$ is a cofibration category under $\mathbf{A}$ where $\mathbf{A} \subset \mathrm{Ho}(\mathbf{C})$ carries the object $A \in \mathbf{A}$ to the corresponding chain complex concentrated in degree 1. One readily checks that

$$\mathrm{Ho}(\mathbf{C}) = \mathbf{Complex}(\mathbf{A})/\overset{1}{\simeq}$$

Moreover the Whitehead theorem (3.11) yields for chain complexes in $\mathbf{C}$ the equivalence (a) $\Leftrightarrow$ (b) in theorem (II.6.6) or theorem (III.9.6).

There are many generalizations of the Whitehead theorem in various homotopy theories. A general form of this theorem is (3.11) above which by specialization yields most of the Whitehead theorems in the literature. Using the cellular approximation we see in the next section that the classical Whitehead theorem is a consequence of (3.11).

4 Cellular Approximation

We first consider the classical cellular approximation theorem for reduced CW-complexes in $\mathbf{C} = \mathbf{Top}^*$. Compare example (2.6). A map $f : K \to X$ between reduced CW-complexes is termed *cellular* if f carries the n-skeleton K^n of K to the n-skeleton X^n of X. Hence a cellular map f is equivalent to a filtered map $f^{\geq 1} : K^{\geq 1} \to X^{\geq 1}$ with $\lim(f^{\geq 1}) = f$. The classical cellular approximation theorem shows that each map $g : K \to X$ in $\mathbf{C}$ which restricted to a subcomplex L of K is cellular is homotopic relative L to a cellular map $f : K \to X$. We can reformulate this by considering the following commutative diagram of filtered objects

$$\begin{array}{ccc} L^{\geq 1} & \xrightarrow{g|L} & X^{\geq 1} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ K^{\geq 1} & \xrightarrow[g]{} & X \end{array} \tag{4.1}$$

Here X is the constant filtered object and p is the canonical map given in degree n by the inclusion $X^n \subset X$. The filtered map g in the diagram is given in degree n by the composite $K^n \subset K \xrightarrow{g} X$. Now the cellular approximation theorem is equivalent to the existence of a lift f of diagram (4.1) with $fj = g \mid L$ and

$pf \overset{1}{\simeq} g\, rel\, L^{\geq 1}$; compare (3.1). Hence using the notion of lifting map in (3.1) we get:

(4.2) Cellular approximation theorem. *Let X be a reduced CW-complex. Then $p : X^{\geq 1} \to X$ in (4.1) is a lifting map.*

Here a lifting map is defined as in (3.1) for the cofibration category $\mathbf{C} = \mathbf{Top}^*$ under $\mathbf{T} = \mathbf{susp}(*, \mathcal{D})$; compare (2.6). Theorem (4.2) leads to the following definition.

(4.3) Definition. Let $\mathbf{C}$ be a cofibration category under $\mathbf{T}$. We call a cofibrant and fibrant object X in $\mathbf{C}$ *weakly cellular* if there exists a $\mathbf{T}$-complex $X_{\geq 1}$ and a lifting map

$$p : X_{\geq 1} \longrightarrow X \tag{1}$$

where X is the constant filtered object given by X. Moreover X is *cellular* if there exists a lifting map as in (1) for which $X_{\geq 1}$ has the limit property in (1.5) and the induced map

$$p : \lim(X_{\geq 1}) \xrightarrow{\sim} X \tag{2}$$

is a weak equivalence in $\mathbf{C}$. Let

$$\mathbf{Cell} \subset \mathbf{Well} \subset \mathbf{C}_{cf} \tag{3}$$

be full subcategories where **Cell** consists of cellular objects and **Well** consists of weakly cellular objects.

Remark. Consider the cofibration category $\mathbf{C} = \mathbf{Top}^*$ under $\mathbf{T} = \mathbf{susp}(*, \mathcal{D})$ as in (2.6). Then each reduced CW-complex X is cellular since $X^{\geq 1}$ has the limit property and since the cellular approximation theorem (4.2) holds. In fact the notion of "cellular object in a cofibration category $\mathbf{C}$ under $\mathbf{T}$" is the appropriate generalization of the classical notion of CW-complex in algebraic topology. Each path connected space U in $\mathbf{Top}^*$ is weakly cellular since there exists a reduced CW-complex X and a map $p : X \to U$ inducing isomorphisms $p_* : \pi_n X \cong \pi_n U$ for $n \geq 1$. Such a map is by (3.9) a lifting map. The CW-complex X is termed a *CW-approximation* of U; for example we can choose X to be the realization of the reduced singular set of Y; see Fritsch-Piccinini [CW]. Homology of U is defined by the homology of the CW-approximation of U.

(4.4) Proposition. *There is a canonical functor*

$$\varphi : \mathbf{Well}/\simeq \longrightarrow \mathbf{Complex}/\overset{1}{\simeq}$$

which carries a weakly cellular object X to the $\mathbf{T}$-complex $X_{\geq 1}$ chosen for X as in (4.3). Moreover $X_{\geq 1}$ is $\mathbf{T}$-good in the sense of (3.7).

Proof. Given $f : Y \to X$ in **Well** we obtain

$$\begin{tikzcd} * \arrow[rr] \arrow[d] & & X_{\geq 1} \arrow[d] \\ Y_{\geq 1} \arrow[r] & Y \arrow[r, "f"'] & X \end{tikzcd} \tag{1}$$

and a lift $\bar{f} : Y_{\geq 1} \to X_{\geq 1}$. The functor φ carries the homotopy class of f to the 1-homotopy class of $\bar{f}$. Let $H : IY \to X$ be a homotopy $f \simeq g$ in $\mathbf{C}_{cf}$. Then we obtain the commutative diagram

$$\begin{tikzcd} Y_{\geq 1} \vee Y_{\geq 1} \arrow[r, "(\bar{f},\bar{g})"] \arrow[d, "(i_0,i_1)"'] & X_{\geq 1} \arrow[d, "p"] \\ \bar{I}Y_{\geq 1} \arrow[r, "\bar{H}"] & X \end{tikzcd} \tag{2}$$

in $\mathbf{Fil}_1(\mathbf{C})_{cf}$ as follows. Let $i : Y_{\geq 1} \to s^{-1}Y_{\geq 1}$ be the shift map with $(s^{-1}Y_{\geq 1})_n = Y_{n+1}$ so that i in degree n is the inclusion $Y_n \subset Y_{n+1}$; see (1.3).

Then we get $\bar{p} : s^{-1}Y_{\geq 1} \to Y$ with $\bar{p}i = p$. Hence we get the composite

$$H_1 : \bar{I}Y_{\geq 1} \xrightarrow{j} Is^{-1}Y_{\geq 1} \xrightarrow{I\bar{p}} IY \xrightarrow{H} X \tag{3}$$

where j is the inclusion and $I\bar{p}$ is obtained in the cofibration category $\mathbf{Fil}_1(\mathbf{C})$ as in (II.2.10) of Baues [AH]. Clearly $H_1 i_0 = fp$ and $H_1 i_1 = gp$. Moreover we have 1-homotopies $H_0 : p\bar{f} \overset{1}{\simeq} fp$ and $H_2 : gp \overset{1}{\simeq} p\bar{g}$. We can add the homotopies H_0, H_1 and H_2 and get a map $\bar{H}$ representing $H_0 + H_1 + H_2$ such that (2) commutes. A lift of (2) gives us a 1-homotpy $\bar{f} \overset{1}{\simeq} \bar{g}$. This shows that the functor φ is well defined.

We still have to show that $X_{\geq 1}$ is $\mathbf{T}$-good. For this we use the lifting map $p : X_{\geq 1} \to X$ which by (3.6) is a weak $\mathbf{T}$-equivalence. Hence the maps

$$\begin{cases} \operatorname{im}\big([Z, X_1] \to [Z, X_2]\big) \xrightarrow{p_*} [Z, X] \\ \operatorname{im}\big(\pi_n^A X_{n+1} \to \pi_n^A X_{n+2}\big) \xrightarrow{p_*} \pi_n^A X \end{cases} \tag{4}$$

are bijections. We now show that

$$\begin{cases} [Z, X_2] \to [Z, X] \\ \pi_n^A X_{n+2} \to \pi_n^A X \end{cases} \tag{5}$$

are injective and hence by (4) bijective. Assume we have maps $f,\ g : Z \to X_2$ and a homotopy $H : if \simeq ig$ where $i : X_2 \subset X$ is given by p. Then we have the commutative diagram in $\mathbf{Fil}_1(\mathbf{C})$

$$\begin{tikzcd}
Z \vee Z \arrow[r, "{(f,g)}"] \arrow[d] & X_1 \arrow[r] & X_{\geq 1} \arrow[d, "p"] \\
\bar{I}Z \arrow[rr, "H"] & & X
\end{tikzcd}$$

and a lift of this diagram yields a homotopy $f \simeq g$. Thus the first map in (5) is injective. A similar argument replacing Z by $\Sigma^n A$ shows that the second map in (5) is injetive. Using (4) and (5) we see that $\pi_0^A(X_1) \to \pi_0^A(X_2)$ is surjective. Moreover for $n \geq 1$ we have the exact sequence

$$\pi_n^A(X_{n+1}) \xrightarrow{j} \pi_n^A(X_{n+2}) \longrightarrow \pi_n^A(X_{n+2}, X_{n+1}) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \pi_{n-1}^A(X_{n+1}) \xrightarrow{i} \pi_{n-1}^A(X_{n+2})$$

where j is surjective and i is injective. In fact, j is surjective since the composite

$$q : \pi_n^A(X_{n+1}) \xrightarrow{j} \pi_n^A(X_{n+2}) \xrightarrow{\approx} \pi_n^A(X)$$

is surjective. Moreover i is injective since the composite

$$\pi_{n-1}^A(X_{n+1}) \xrightarrow{i} \pi_{n-1}^A(X_{n+2}) \longrightarrow \pi_{n-1}^A(X)$$

is a bijection. q.e.d.

(4.5) Theorem. *The restriction*

$$\varphi : \mathbf{Cell}/\simeq \longrightarrow \mathbf{Complex}/\overset{1}{\simeq}$$

of φ in (4.4) is a full and faithful functor.

Proof. The functor φ is full since $\varphi(\lim f_{\geq 1}) = f_{\geq 1}$. The functor φ is faithful since a 1-homotopy $H : f_{\geq 1} \overset{1}{\simeq} g_{\geq 1}$ yields by (1.5) a homotopy $\lim H : \lim f_{\geq 1} \simeq \lim g_{\geq 1}$. q.e.d.

Using theorem (4.5) we obtain the following result which is a more direct analogue of the cassical Whitehead theorem.

(4.6) General Whitehead theorem (II). *Let $\mathbf{C}$ be a cofibration category under $\mathbf{T}$ and let X and Y be cellular objects in $\mathbf{C}$. Then $f : Y \to X$ is a homotopy equivalence in $\mathbf{C}_{cf}/\simeq$ if and only if for all cogroups A and objects Z in $\mathbf{T}$ and $n \geq 1$ the induced maps*

$$f_* : \pi_n^A(Y) \to \pi_n^A(X)$$
$$f_* : [Z, Y] \to [Z, X]$$

are bijections.

Proof. We choose $X_{\geq 1} \to X$ and $Y_{\geq 1} \to Y$ as in (4.3) where $X_{\geq 1}$ and $Y_{\geq 1}$ are $\mathbf{T}$-good by (4.4). Since φ in (4.5) is full and faithful we have to show that $f_{\geq 1} : Y_{\geq 1} \to X_{\geq 1}$ with $\varphi(f_{\geq 1}) = f$ is a 1-homotopy equivalence. By (3.9) it suffices to show that $f_{\geq 1}$ is an elementary lifting map. Now (3.2) (i), (ii) hold since we have the following commutative diagrams where i is injective and q is surjective; compare (4.4) (4), (5).

$$\begin{tikzcd}
\pi^A_{n-1}(Y_{n+1}) \arrow[rr, "i", "\approx"'] \arrow[d, "f_*"'] & & \pi^A_{n-1}(Y) \arrow[d, "\cong"] \\
\pi^A_{n-1}(X_{n+1}) \arrow[rr, "i", "\approx"'] \arrow[dr, "i"'] & & \pi^A_{n-1}(X) \\
& \pi^A_{n-1}X_{n+2} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\pi^A_{n}(Y_{n+1}) \arrow[rr, "q"] \arrow[d, "f_*"'] & & \pi^A_{n}(Y) \arrow[d, "\cong"] \\
\pi^A_{n}(X_{n+1}) \arrow[rr, "q"] \arrow[dr] & & \pi^A_{n}(X) \\
& \pi^A_{n}(X_{n+2}) \arrow[ur, "\cong", "i"'] &
\end{tikzcd}$$

Moreover (3.2) (iii) is a consequence of the following diagram where q is surjective and i is bijective.

$$\begin{tikzcd}
{[Z, Y_1]} \arrow[rr, "q"] \arrow[d, "f_*"'] & & {[Z, Y]} \arrow[d, "\cong"] \\
{[Z, X_1]} \arrow[rr, "q"] \arrow[dr, "q"'] & & {[Z, X]} \\
& {[Z, X_2]} \arrow[ur, "\approx", "i"'] &
\end{tikzcd}$$

q.e.d.

(4.7) Example. Let $\mathbf{C} = \mathbf{Top}^*$ and $\mathbf{T} = \mathbf{susp}(*, \mathcal{D})$ as in (2.6). Then we obtain as a specialization of (4.6) the following *classical Whitehead theorem*: Let $f : X \to Y$ be a map between reduced CW-complexes in $\mathbf{Top}^*$. Then f is a homotopy equivalence if and only if f induces isomorphisms

$$f_* : \pi_n X \xrightarrow{\approx} \pi_n Y \tag{*}$$

for $n \geq 1$. Since all objects of $\mathbf{susp}(*, \mathcal{D})$ are one point unions of 1-spheres we see that (*) is equivalent to the corresponding condition in (4.6). Clearly all reduced CW-complexes are cellular by (4.2).

5 The Blakers-Massey Property

Let $\mathbf{T}$ be a theory of coactions and let $\mathbf{C}$ be a cofibration category under $\mathbf{T}$. Using the theory $\mathbf{T}$ we define below the notion of m-connected maps in $\mathbf{C}$ where $m \geq 1$. We then describe the Blakers-Massey property of a cofibration category $\mathbf{C}$ which for $\mathbf{C} = \mathbf{Top}^*$ is equivalent to the classical Blakers-Massey theorem.

(5.1) Remark. We are not able to define a 0-connected map in $\mathbf{C}$ since we do not have the analogue of a discrete set in the cofibration category $\mathbf{C}$. In most examples of the cofibration category $\mathbf{C}$ the objects of $\mathbf{T}$ are actually "0-connected"; consider in particular $\mathbf{C} = \mathbf{Top}^*$ and $\mathbf{T} = \mathbf{susp}(*, \mathcal{D})$.

(5.2) Definition. A cofibration $i : L \rightarrowtail K$ in $\mathbf{C}_c$ is 1-*connected* if for all objects Z in $\mathbf{T}$ the induced map between homotopy sets in $\mathrm{Ho}(\mathbf{C})$

$$i_X : [Z, L] \longrightarrow [Z, K]$$

is surjective. Moreover $i : L \rightarrowtail K$ is m-*connected* with $m \geq 1$ if i is 1-connected and if the relative homotopy groups

$$\pi_r^A(K, L) = 0$$

are trivial for all cogroups A in $\mathbf{T}$ and $1 \leq r \leq m-1$. Here we use $r \leq m-1$ since a cogroup A in $\mathbf{T}$ has dimension 1. A cofibrant object K is m-*connected* if $* \to K$ is m-connected; this implies that the induced map $\pi_r^A(*) \to \pi_r^A(K)$ is surjective for $r \leq m-1$ and bijective for $0 \leq r \leq m-2$. We do not assume that $\pi_r^A(*)$ is trivial.

(5.3) Definition. We say that the cofibration category $\mathbf{C}$ under $\mathbf{T}$ has the *Blakers-Massey property* if (a) and (b) are satisfied.

(a) For all cogroups A in $\mathbf{T}$ the n-fold suspension $\Sigma^n A$ is n-connected.
(b) Consider finite dimensional $\mathbf{T}$-complexes

$$K_{\geq 1} \xleftarrow{i} L_{\geq 1} \xrightarrow{j} Y_{\geq 1}$$

where i and j are inclusions of a subcomplex. By applying the direct limit we obtain the induced cofibrations

$$K \xleftarrow{i} L \xrightarrow{j} Y$$

for which the push out diagram

$$\begin{array}{ccc} K & \xrightarrow{\bar{\jmath}} & K \cup_L Y \\ \uparrow & & \uparrow \\ L & \xrightarrow{\jmath} & Y \end{array}$$

is defined in $\mathbf{C}$. Let $m,\ n \geq 1$ and assume (K, L) is m-connected and (Y, L) is n-connected. Then $(K \cup_L Y, Y)$ is m-connected and for all cogroups A in $\mathbf{T}$ the induced map

$$\bar{\jmath}_* : \pi_r^A(K, L) \longrightarrow \pi_r^A(K \cup_L Y, Y)$$

is surjective for $1 \leq r \leq n + m - 1$ and bijective for $1 \leq r \leq n + m - 2$.

(5.4) Example. Let $\mathbf{C} = \mathbf{Top}^*$ and $\mathbf{T} = \mathbf{susp}(*, \mathcal{D})$. Then $(\mathbf{C}, \mathbf{T})$ has the Blakers-Massey property as follows from the *Blakers-Massey theorem* in algebraic topology; see for example tom Dieck-Kamps-Puppe [HT] and Gray [HT].

In the next theorem we consider the map $(\pi_g, 1)_*$ in (III.2.9). This result is used in chapter V to show that the Blakers-Massey property implies that $(\mathbf{C}, \mathbf{T})$ is "homological".

(5.5) Theorem. *Assume that $(\mathbf{C}, \mathbf{T})$ satisfies the Blakers-Massey property. Let A, D be cogroups in $\mathbf{T}$ and $B = \Sigma^n D$ with $n \geq 0$ and let $Y_{\geq 1}$ be a finite dimensional $\mathbf{T}$-complex with $Y = \lim(Y_{\geq 1})$. Moreover let $g : B \to Y_{n+1} \subset Y$ be a map in $\mathbf{C}$. Then the induced map*

$$(\pi_g, 1)_* : \pi_r^A(CB \vee Y, B \vee Y) \longrightarrow \pi_r^A(C_g, Y)$$

is surjective for $r \leq 2n + 1$ and bijective for $r \leq 2n$. Moreover (C_g, Y) is $(n+1)$-connected.

Using the trivial map $g = 0 : B \to * \to Y$ in (5.5) we obtain by the definition of the partial suspension E in (III.2.6) the next result.

(5.6) Theorem. *Assume that $(\mathbf{C}, \mathbf{T})$ satisfies the Blakers-Massey property. Let A, D be cogroups in $\mathbf{T}$ and let $Y_{\geq 1}$ be a finite dimensional $\mathbf{T}$-complex with $Y = \lim(Y_{\geq 1})$. Then the partial suspension*

$$E : \pi_{r-1}^A(\Sigma^n D \vee Y)_2 \longrightarrow \pi_r^A(\Sigma^{n+1} D \vee Y)_2$$

is surjective for $r \leq 2n + 1$ and bijective for $r \leq 2n$. Using $Y = *$ we see that the suspension

$$\Sigma : \pi_{r-1}^A(\Sigma^n D) \longrightarrow \pi_r^A(\Sigma^{n+1} D)$$

is surjective for $r \leq 2n + 1$ and bijective for $r \leq 2n$.

The second part of this theorem is the analogue of the Freudenthal suspension theorem.

Proof of (5.5). We consider the mapping cylinder Z_g obtained by the push out

$$\begin{tikzcd} IB \arrow[r, "\bar{g}"] & Z_g \arrow[r, "\sim", "p"'] & Y \\ B \arrow[u, "i_0"] \arrow[r, "g"] & Y \arrow[u, "\bar{\imath}_0"'] & \end{tikzcd} \tag{1}$$

Here p is defined by $p\bar{g} = p$ and $p\bar{\imath}_0 = 1$. Hence

$$j : B \vee Y \xrightarrow{(\bar{g}i_1, \bar{\imath}_0)} Z_g \tag{2}$$

is a cofibration and we obtain the push out

$$\begin{tikzcd} CB \vee Y \arrow[r, "\bar{\jmath}"] & CB \cup_B Z_g \arrow[r, "1 \cup p", "\sim"'] & CB \cup_g Y = C_g \\ B \vee Y \arrow[u] \arrow[r, "j"'] \arrow[ur, phantom, "\text{push}"] & Z_g \arrow[u] \arrow[r, "p"'] & Y \arrow[u] \end{tikzcd} \tag{3}$$

The composite of the top row is $(\pi_g, 1) = (1 \cup p)\bar{\jmath}$. We can apply the Blakers-Massey property to the push out diagram in (3) since j is obtained by the inclusion of a subcomplex; see (2.5). This yields the result in (5.5) since we show that $(CB \vee Y, B \vee Y)$ is $(n+1)$-connected and that $(Z_g, B \vee Y)$ is $(n+1)$-connected: We have the maps

$$(0, 1) : B \vee Y \longrightarrow CB \vee Y \xrightarrow{\sim} Y$$
$$(g, 1) : B \vee Y \longrightarrow Z_g \xrightarrow{\sim} Y$$

which both admit the splitting $i_2 : Y \to B \vee Y$. Hence $(0, 1)$ and $(g, 1)$ are 1-connected; see (5.2). Moreover we have the push out diagram

$$\begin{tikzcd} B \arrow[r] & B \vee Y \\ * \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

Since $B = \Sigma^n D$ the Blakers-Massey property shows that $Y \to B \vee Y$ is n-connected so that $\pi_i^A(Y) \to \pi_i^A(B \vee Y)$ is surjective and hence bijective for $i \leq n-1$. Therefore the short exact sequences

$$0 \longrightarrow \pi_i^A(CB \vee Y, B \vee Y) \longrightarrow \pi_{i-1}^A(B \vee Y) \longrightarrow \pi_{i-1}^A(Y) \longrightarrow 0$$
$$0 \longrightarrow \pi_i^A(Z_g, B \vee Y) \longrightarrow \pi_{i-1}^A(B \vee Y) \longrightarrow \pi_{i-1}^A(Y) \longrightarrow 0$$

show that $\pi_i^A(CB \cup Y, B \cup Y) = 0 = \pi_i^A(Z_g, B \vee Y)$ for $i \leq n$. q.e.d.

(5.7) Proposition. *Assume* $(\mathbf{C}, \mathbf{T})$ *satisfies the Blakers-Massey property and let* $Y_{\geq 1}$ *be a* $\mathbf{T}$*-complex. Then* (Y_k, Y_n) *is* n*-connected for* $k \geq n \geq 1$ *and thus* $Y_{\geq 1}$ *is* $\mathbf{T}$*-good.*

Proof. By (5.5) we see that for $n \geq 1$ the pair (Y_{n+1}, Y_n) is n-conneted. Hence (Y_{n+1}, Y_n) is 1-connected and

$$\pi_r^A(Y_n) \longrightarrow \pi_r^A(Y_{n+1})$$

is surjective for $r \leq n-1$ and bijective for $r \leq n-2$. This shows that for $k \geq n \geq 1$ also (Y_k, Y_n) is 1-connected and

$$\pi_r^A(Y_n) \longrightarrow \pi_r^A(Y_k)$$

is surjective for $r \leq n-1$ and bijective for $r \leq n-2$. Thus (Y_k, Y_n) is n-connected. q.e.d.

(5.8) Proposition. *Assume* $(\mathbf{C}, \mathbf{T})$ *satisfies the Blakers-Massey property and let* $Y_{\geq 1}$ *be a finite dimensional* $\mathbf{T}$*-complex with* $Y = \lim(Y_{\geq 1})$*. Then*

$$p : Y_{\geq 1} \longrightarrow Y$$

is a lifting map so that Y *is cellular.*

Proof. It suffices to show that p is an elementary lifting map; see (3.3). Now (3.2) (i), (ii) are satisfied for p by (5.7). Moreover (3.2) (iii) is satisfied since (Y, Y_1) is 1-connected by (5.7). q.e.d.

Proposition (5.8) shows that the Blakers-Massey property implies the cellular approximation theorem for finite dimensional complexes; see (4.2). In the case of infinite dimensional complexes we need the following property of $(\mathbf{C}, \mathbf{T})$.

(5.9) Definition. Let $\mathbf{C}$ be a cofibration category under $\mathbf{T}$. Then we say that $(\mathbf{C}, \mathbf{T})$ has *good limits* if all $\mathbf{T}$-complexes $Y_{\geq 1}$ have the limit property (1.5) and if for all cogroups A and objets Z in $\mathbf{T}$ we have

$$[Z, \lim Y_{\geq 1}] = \lim_k [Z, Y_k]$$
$$\pi_n^A(\lim Y_{\geq 1}) = \lim_k \pi_n^A Y_k$$

(5.10) Theorem. *Assume* $(\mathbf{C}, \mathbf{T})$ *satisfies the Blakers-Massey property and has good limits. Then the functor*

$$\varphi : \mathbf{Cell}/\simeq \longrightarrow \mathbf{Complex}/\overset{1}{\simeq}$$

in (4.5) is an equivalence of categories.

Proof. We have to show that φ is representative. For this let $Y_{\geq 1}$ be a $\mathbf{T}$-complex with $Y = \lim(Y_{\geq 1})$ and let $Y \rightarrowtail Y'$ be a fibrant model of Y. We show that Y' is cellular with $\varphi(Y') = Y_{\geq 1}$. In fact $Y_{\geq 1}$ has the limit property since $(\mathbf{C}, \mathbf{T})$ has good limits. Moreover the induced map

$$p : Y_{\geq 1} \longrightarrow Y \xrightarrow{\sim} Y'$$

is an elementary lifting map by the argument in (5.7) and (5.9). Hence p is a lifting map so that Y' is cellular with $\varphi(Y_{\geq 1}) = Y'$. q.e.d.

Chapter V: Homology of Complexes

Given a theory of coactions $\mathbf{T}$ and a cofibration category $\mathbf{C}$ under $\mathbf{T}$ we introduced in chapter IV the notion of a $\mathbf{T}$-complex $X_{\geq 1}$ in $\mathbf{C}$ which is the analogue of the classical notion of CW-complex in algebraic topology. We now describe the properties of $\mathbf{C}$ and $\mathbf{T}$ which are needed for the definition of homology and cohomology of a $\mathbf{T}$-complex. These properties lead to the definition of a "homological cofibration category under $\mathbf{T}$" in § 1. We define homology and cohomology by a "chain functor"

$$C_* : \mathbf{Complex} \to \mathbf{chain}$$

where **Complex** is the category of $\mathbf{T}$-complexes and where **chain** is the category of chain complexes associated to the theory $\mathbf{T}$ in chapter I. The cohomology obtained is sufficiently powerful to describe an obstruction theory for extension problems on $\mathbf{T}$-complexes which specializes to the classical obstruction theory for CW-complexes. Moreover we obtain for $\mathbf{T}$-complexes a generalization of the Hurewicz homomorphism and we are able to embed this Hurewicz homomorphism into an exact sequence which specializes to J.H.C. Whitehead's certain exact sequence [CE] for CW-complexes.

1 Homological Cofibration Categories

In the following definition we describe the appropriate conditions on a cofibration category $\mathbf{C}$ which will be used in the next section to obtain the chain complex $C_*(X_{\geq 1})$ of a complex $X_{\geq 1}$ in $\mathbf{C}$. This chain complex yields the notion of homology and cohomology of $X_{\geq 1}$.

(1.1) Definition. Let $\mathbf{T}$ be a theory of coactions and let $\mathbf{C}$ be a cofibration category under $\mathbf{T}$ as in (IV.2.1). Then we say that $\mathbf{C}$ is a *homological cofibration category under* $\mathbf{T}$ or that $(\mathbf{C}, \mathbf{T})$ is homological if all $\mathbf{T}$-complexes are $\mathbf{T}$-good as in (IV.3.7) and if the following conditions (a) and (b) are satisfied for all cogroups A, B in $\mathbf{T}$ and $\mathbf{T}$-complexes $X_{\geq 1}$.

(a) For $i \geq 0$ the inclusion $X_n \subset X_{n+1}$ induces the map

$$\pi_i^A(\Sigma^i B \vee X_n)_2 \longrightarrow \pi_i^A(\Sigma^i B \vee X_{n+1})_2$$

which is surjective for $n = 1$ and bijective for $n \geq 2$; see (III.2.5).

(b) Let $g : D = \Sigma^{n-1}A_{n+1} \to X_n$ be the attaching map of the principal cofibration (X_{n+1}, X_n) where A_{n+1} is a cogroup in $\mathbf{T}$. Then

$$(\pi_g, 1)_* : \pi_n^A(CD \vee X_n, D \vee X_n) \longrightarrow \pi_n^A(X_{n+1}, X_n)$$

is surjective for $n = 1$ and bijective for $n \geq 2$; see (III.3.2). The same holds for the composite

$$E_X : \pi_{n-1}^A(D \vee X_n)_2 \xrightarrow[\cong]{\partial^{-1}} \pi_n^A(CD \vee X_n, D \vee X_n) \xrightarrow{(\pi_g,1)_*} \pi_n^A(X_{n+1}, X_n)$$

where ∂ is the isomorphism in (III.3.2). We derive from (a) and (b) the following property (c).

(c) For $m \geq 1$ the partial suspension

$$E : \pi_{n-1}^A(\Sigma^{n-1}B \vee X_m)_2 \longrightarrow \pi_n^A(\Sigma^n B \vee X_m)_2$$

is surjective for $n = 1$ and bijective for $n \geq 2$.

Proof of (c). To derive (c) from (a) and (b) we first choose in (b) the trivial attaching map $g = 0$ with $D = \Sigma^{n-1}B$ and $X_n = X_m$ for $m \leq n$. This shows that (c) holds for $m \leq n$; see (III.3.2). Now (a) implies that (c) also holds for $m > n$. q.e.d.

Remark. The assumption above that all $\mathbf{T}$-complexes are $\mathbf{T}$-good is not needed in chapter V and chapter IV if one considers only finite dimensional $\mathbf{T}$-complexes. For infinite dimensional $\mathbf{T}$-complexes we apply in (VI.7.1) and (VI.8.4) the general Whitehead theorem (IV.3.11) and this theorem requires that $\mathbf{T}$-complexes are $\mathbf{T}$-good.

(1.2) Proposition. *Assume* $(\mathbf{C}, \mathbf{T})$ *has the Blakers-Massey property. Then* $(\mathbf{C}, \mathbf{T})$ *is homological.*

This result yields many examples of homological cofibration categories under $\mathbf{T}$.

Proof of (1.2). Theorem (IV.5.5) shows that (1.1) (b) holds. Moreover we obtain (1.1) (a) by the following push out diagram where $\bar{B} = \Sigma^i B$.

$$\begin{tikzcd} C\bar{B} \vee X_n \arrow[r] & C\bar{B} \vee X_{n+1} \\ \bar{B} \vee X_n \arrow[u, "j_1"] \arrow[r, "j_2"] & \bar{B} \vee X_{n+1} \arrow[u] \end{tikzcd}$$

Here j_2 is n-connected since (X_{n+1}, X_n) is n-connected by (IV.5.7) and since we can use the Blakers-Massey property for $X_{n+1} \leftarrow X_n \to \bar{B} \vee X_n$. Moreover j_1 is $(i+1)$-connected; compare the proof of (IV.5.5). Thus the Blakers-Massey property shows that

$$\begin{array}{c} \pi_r^A(C\bar{B} \vee X_n, \bar{B} \vee X_n) = \pi_{r-1}^A(\bar{B} \vee X_n)_2 \\ \downarrow \\ \pi_r^A(C\bar{B} \vee X_{n+1}, \bar{B} \vee X_{n+1}) = \pi_{r-1}^A(\bar{B} \vee X_{n+1})_2 \end{array}$$

is surjective for $1 \leq r \leq i+n$ and bijective for $r \leq i+n$. Choosing $r-1=i$ we see that (1.1) (a) holds. Moreover all **T**-complexes are **T**-good by (IV.5.7). q.e.d.

(1.3) Proposition. *Assume* $(\mathbf{C}, \mathbf{T})$ *is homological. Then one has the functor*

$$c : \mathbf{Complex}/\overset{1}{\simeq} \longrightarrow \mathbf{Coef}$$

which carries $X_{\geq 1}$ *to the attaching map* ∂_X *of* (X_2, X_1) *and which carries* $f_{\geq 1}$ *to* $\{f_1\}$. *Moreover the restriction of* c *to the subcategory of 2-dimensional complexes is a full functor.*

Proof. We have to show that c is well defined for morphisms. Now (1.1) (b) implies that for a map $f_{\geq 1}$ the induced map

$$(f_2, f_1) : (X_2, X_1) \longrightarrow (Y_2, Y_1)$$

is a twisted map; see (III.3.11). Hence f_1 is ∂-compatible. Similarly each map

$$(IX_1, X_1 \vee X_1) \longrightarrow (Y_2, Y_1)$$

is a twisted map by (1.1) (b). This shows that $f_{\geq 1} \overset{1}{\simeq} g_{\geq 1}$ implies that f_1 is ∂-equivalent to g_1. Thus the ∂-equivalence class $\{f_1\}$ is well defined by the 1-homotopy class of $f_{\geq 1}$. Given a ∂-compatible map $\{f_1\}$ in **Coef** one obtains an associated twisted map (f_2, f_1) as above. Hence c is full if restricted to 2-dimensional complexes. q.e.d.

(1.4) Proposition. *Let* $(\mathbf{C}, \mathbf{T})$ *be homological and let* A, B, A_2 *be cogroups and* Z, X_1 *be objects in* $\mathbf{T}$. *Moreover let* (X_2, X_1) *be a principal cofibration with attaching map* $\partial_X : A_2 \to X_1$. *Then we have canonical bijections*

$$\begin{aligned} \operatorname{im}\{[Z, X_1] \to [Z, X_2]\} &= \mathbf{Coef}(Z, \partial_X) \\ [A, B \vee X_2]_2 &= \mathbf{Coef}(A, B \vee \partial_X)_2 \end{aligned}$$

Recall that $\mathbf{Coef}(A, B \vee \partial_X)_2$ is used for the definition of morphisms in the category **premod**. Hence the second bijection in (1.4) yields a new interpretation of such morphisms.

Proof of (1.4). The first bijection is obtained by the same arguments as in the proof of (1.3). The second bijection is a consequence of the first bijection and of (1.1) (a). q.e.d.

Using (I.4.7) we derive from (1.4) the following corollary.

(1.5) Corollary. *Let* $(\mathbf{C}, \mathbf{T})$ *be homological and let* $f : X_{\geq 1} \to Y_{\geq 1}$ *be a map in* $\mathbf{Complex}/\overset{1}{\simeq}$. *Then* $c(f) : \partial_X \to \partial_Y$ *is an isomorphism in* **Coef** *if and only if the induced map*

$$f_* : \mathrm{im}\{[Z, X_1] \to [Z, X_2]\} \longrightarrow \mathrm{im}\{[Z, Y_1] \to [Z, Y_2]\}$$

is a bijection for all objects Z *in* $\mathbf{T}$.

The corollary describes exactly condition (IV.3.4) (2) for a weak T-equivalence.

Recall that a category of modules for $\mathbf{T}$ is a quotient category of the category **premod** with certain additive properties.

(1.6) Definition. Let $(\mathbf{C}, \mathbf{T})$ be homological. Then we define the *category of modules* $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$ as follows: Objects are $A \vee \partial_X$ where A is a cogroup in $\mathbf{T}$ and $\partial_X \in \mathbf{Coef}$ and a morphism

$$\alpha \odot u : A \vee \partial_X \longrightarrow B \vee \partial_Y$$

is given by a morphism $u : \partial_X \to \partial_Y$ in **Coef** and by an element

$$\alpha \in [\Sigma A, \Sigma B \vee Y_2]_2$$

where (Y_2, Y_1) is a principal cofibration with attaching map ∂_Y. Composition is defined by

$$(\beta \odot v)(\alpha \odot u) = (\beta, \bar{v})\alpha \odot (vu)$$

where $\bar{v}$ is a map $(Y_2, Y_1) \to (Z_2, Z_1)$ in **Complex** with $c(\bar{v}) = v$ by (1.3). Now (1.1) (a) shows that the composition is well defined.

(1.7) Lemma. *The category* $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$ *is a category of modules for the theory* $\mathbf{T}$ *with the properties in (I.5.6). In fact, the functor* E *is given by the following commutative diagram:*

$$\begin{array}{ccc} [A, B \vee Y_2]_2 & = & \mathbf{premod}(A \vee \partial_X, B \vee \partial_Y)_u \\ \big\downarrow{\scriptstyle E} & & \big\downarrow{\scriptstyle E} \\ [\Sigma A, \Sigma B \vee Y_2]_2 & = & \mathbf{mod}(A \vee \partial_X, B \vee \partial_Y)_u \end{array}$$

Here the left hand side is the partial suspension which is surjective by (1.1) (c). The identification in the top row of the diagram is obtained by (1.4).

We have the commutative diagram of functors

$$\begin{array}{ccc} & \mathbf{premod} & \\ \swarrow & & \searrow{\scriptstyle E} \\ \mathbf{premod}^{\mathrm{ad}} & \xrightarrow{E_\sharp} & \mathbf{mod}(\mathbf{C}) \end{array} \tag{1.8}$$

where $E_\sharp$ is a full functor which is the identity on objects induced by E; see (I.5.8). For various examples of homological cofibration categories $(\mathbf{C}, \mathbf{T})$ the functor $E_\sharp$ is actually an isomorphism. In general we do not assume that this is the case. This is the reason for the definition of "categories of modules for $\mathbf{T}$" in (I.5.8) which are not uniquely determined by $\mathbf{T}$. In the following we always use the category $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$ for the definition of categories like **chain** or $\mathbf{TWIST}_2^c$ and $\mathbf{TWIST}_1^c$, etc.

Proof of (1.7). One readily checks that the isomorphism in the top row is compatible with the action of the group

$$\mathrm{im}\{[A, Y_1] \to [A, Y_2]\} = \mathbf{Coef}(A, \partial_Y)$$

and (III.2.7) shows that E is equivariant with respect to this action. Hence (I.5.6) (ii) is satisfied. Next the first equation of (I.5.6) (iii) is obviously satisfied; we obtain the second equation as follows. Consider the composite

$$\begin{tikzcd}
CB_1 \vee CB_2 \arrow[r, "\pi \vee 1"] \arrow[d, equal] & \Sigma B_1 \vee CB_2 \arrow[r, "1 \vee \pi"] & \Sigma B_1 \vee \Sigma B_2 \arrow[d, equal] \\
C(B_1 \vee B_2) \arrow[rr, "\pi"] & & \Sigma(B_1 \vee B_2)
\end{tikzcd}$$

Using the definition of the partial suspension we get the following diagram in which the row is split exact with $B = B_1 \vee B_2$

$$\begin{tikzcd}
 & & \pi_1^A(CB \vee X, B \vee X) \cong \pi_0^A(B \vee X)_2 \arrow[d] & & \\
0 \arrow[r] & \pi_1^A(\Sigma B_1 \vee X)_2 \arrow[r] & \pi_1^A(\Sigma B_1 \vee CB_2 \vee X, B_2 \vee X) \arrow[r] \arrow[d] & \pi_0^A(B_2 \vee X)_2 \arrow[r] & 0 \\
 & & \pi_1^A(\Sigma B \vee X)_2 \cong \pi_1^A(\Sigma B \vee X, X) & &
\end{tikzcd}$$

The exactness of the row is readily obtained by the exact homotopy sequence of a pair. The composite of the morphisms in the column yields the partial suspension which is surjective by (1.1) (c). Hence since the row is split exact we get

$$\pi_1^A(\Sigma B \vee X)_2 = \pi_1^A(\Sigma B_1 \vee X)_2 \oplus \pi_1^A(\Sigma B_2 \vee X)_2$$

This shows that also the second equation of (I.5.6) (iii) holds. q.e.d.

(1.9) Definition. For cogroups A, B in $\mathbf{T}$ and for a $\mathbf{T}$-complex $Y_{\geq 1}$ we can form the double colimit of groups termed *stabilization*

$$\{A, B \vee Y_{\geq 1}\}_2 = \lim_{i \geq 0} \lim_{j \geq 1} [\Sigma^i A, \Sigma^i B \vee Y_j]_2. \tag{1}$$

Here the limit over $j \geq 1$ is induced by the inclusions $Y_j \subset Y_{j+1}$ and the limit over $i \geq 0$ is induced by the partial suspension E. Hence we have canonical maps

$$\tau_{i,j} : [\Sigma^i A, \Sigma^i B \vee Y_j]_2 \longrightarrow \{A, B \vee Y_{\geq 1}\}_2$$

which by the conditions in (1.1) are surjective for $i = 0$ or $j = 1$ and are bijective otherwise. In particular $\tau_{1,2}$ is an isomorphism so that a morphism $\alpha \odot u$ in $\mathbf{mod}(\mathbf{C})$ with $\alpha \in [\Sigma A, \Sigma B \vee Y_2]_2$ is determined by its stabilization $\tau_{1,2}(\alpha) \in \{A, B \vee Y_{\geq 1}\}_2$; see (1.6). Given elements

$$\begin{aligned} \alpha &\in [\Sigma^i A, \Sigma^i B \vee Y_j]_2 \\ \beta &\in [\Sigma^r A, \Sigma^r B \vee Y_t]_2 \end{aligned}$$

we write $\alpha \equiv \beta$ if α and β have the same stabilization, that is

$$\alpha \equiv \beta \Longleftrightarrow \tau_{i,j}(\alpha) = \tau_{r,t}(\beta) \tag{2}$$

Here α is uniquely determined by β if $i > 0$ and $j > 1$.

2 The Chains of a Complex

Assuming that the cofibration category $\mathbf{C}$ under $\mathbf{T}$ is homological we are able to define the chain complex $C_* X_{\geq 1}$ of a $\mathbf{T}$-complex $X_{\geq 1}$. Here a chain complex is defined by the category $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$ in (1.7).

Given a $\mathbf{T}$-complex $X_{\geq 1} = (X_{\geq 1}, A_{\geq 1}, \partial_{\geq 2})$ we have the attaching map $\partial_{n+1} \in [\Sigma^{n-1} A_{n+1}, X_n]$ of the principal cofibration (X_{n+1}, X_n). Then we obtain the *difference element*

$$\nabla \partial_{n+1} \in [\Sigma^{n-1} A_{n+1}, \Sigma^{n-1} A_n \vee X_n]_2 \tag{2.1}$$

for $n \geq 1$. For $n = 1$ this element is defined by the coaction $\mu : X_1 \to X_1 \vee A_1$ in $\mathbf{T}$ and for $n \geq 2$ we use the coaction $\mu : X_n \to X_n \vee \Sigma^{n-1} A_n$ of the principal cofibration (X_n, X_{n-1}) so that

$$\nabla \partial_{n+1} = -i_2 \partial_{n+1} + (i_2 + i_1) \partial_{n+1}$$

Here $i_2 + i_1$ is defined by μ; compare (III.3.8).

Moreover given a map $f_{\geq 1} : X_{\geq 1} \to Y_{\geq 1}$ between $\mathbf{T}$-complexes we obtain for $n \geq 1$ the *difference element*

$$\nabla f_n \in [\Sigma^{n-1} A_n, \Sigma^{n-1} B_n \vee Y_n]_2 \tag{2.2}$$

as follows. For $n = 1$ this is the difference element defined by the map $f_1 : X_1 \to Y_1 \in \mathbf{T}$; compare (I.3.3). For $n \geq 2$ we have the map

$$f_n : (X_n, X_{n-1}) \longrightarrow (Y_n, Y_{n-1})$$

between principal cofibrations which induces the map $(f_n)_*$ in the diagram

$$\begin{array}{ccc} \pi_n^X \in \pi_{n-1}^{A_n}(X_n, X_{n-1}) & \xrightarrow{(f_n)_*} & \pi_{n-1}^{A_n}(Y_n, Y_{n-1}) \\ & & \downarrow \triangledown \\ & & \pi_{n-1}^{A_n}(\Sigma^{n-1}B_n \vee Y_n)_2 \end{array}$$

Here π_n^X is the characteristic element of (X_n, X_{n-1}) and $\triangledown$ is the difference operator; see (III.3.1) (2) and (III.3.9). We now define $\triangledown f_n$ for $n \geq 2$ as in (III.3.14) by

$$\triangledown f_n = \triangledown (f_n)_*(\pi_n^X).$$

(2.3) Definition. Let $(\mathbf{C}, \mathbf{T})$ be homological. Then we define the *chain functor* C_* for which the following diagram of functors commutes

$$\begin{array}{ccc} \mathbf{Complex}/\overset{0}{\simeq} & \xrightarrow{C_*} & \mathbf{chain}_{\geq 1} \\ & {}_{c}\searrow \quad \swarrow_{c} & \\ & \mathbf{Coef} & \end{array}$$

Here c is the coefficient functor in (1.3) and (I.6.3). The functor C_* carries a $\mathbf{T}$-complex $X_{\geq 1} = (X_{\geq 1}, A_{\geq 1}, \partial_{\geq 2})$ to the chain complex

$$C_*(X_{\geq 1}) = (A_{\geq 1}, \partial_X). \tag{1}$$

Here $\partial_X = \partial_2 \in \mathbf{Coef}$ is the attaching map of (X_2, X_1) and the cogroup $A_{n+1} \in \mathbf{T}$ for $A_{\geq 1}$ is given by the attaching map $\partial_{n+1} : \Sigma^{n-1}A_{n+1} \to X_n$ of (X_{n+1}, X_n) for $n \geq 1$. Let A_1 be the cogroup associated to $X_1 \in \mathbf{T}$. The differential in the chain complex (A, ∂_X)

$$\begin{cases} d_{n+1} \odot 1 : A_{n+1} \vee \partial_X \to A_n \vee \partial_X \in \mathbf{mod} \\ d_{n+1} \in [\Sigma A_{n+1}, \Sigma A_n \vee X_2]_2 \end{cases}$$

is defined by

$$d_{n+1} \equiv \triangledown \partial_{n+1} \tag{2}$$

for $n \geq 1$; see (2.1) and (1.9). A map $f : X_{\geq 1} \to Y_{\geq 1}$ between $\mathbf{T}$-complexes induces the chain map

$$C_*(f) : C_*(X_{\geq 1}) = (A, \partial_X) \longrightarrow C_*(Y_{\geq 1}) = (B, \partial_Y) \tag{3}$$

which is u-equivariant. Here $u = c(f) = \{f_1\}$ is given by the coefficient functor c in (1.3). In degree n the chain map

$$\begin{cases} C_n(f) = \xi_n \odot u : A_n \vee \partial_X \longrightarrow B_n \vee \partial_Y \in \mathbf{mod} \\ \xi_n \in [\Sigma A_n, \Sigma B_n \vee Y_2]_2 \end{cases}$$

is defined by

$$\xi_n \equiv \nabla f_n \tag{4}$$

for $n \geq 1$; see (2.2) and (1.9). One can check that C_* is a well defined functor. In fact C_* coincides with the functor $\check{K}$ in III.4.2 Baues [AH]. If $\mathbf{T}$ is an augmented theory of coactions, then also the functor

$$\operatorname{aug} C_* : \mathbf{Complex}/\overset{0}{\simeq} \longrightarrow \mathbf{chain}_{\geq 0} \tag{5}$$

is defined since C_* above carries a $\mathbf{T}$-complex to an object in the subcategory $\mathbf{TWIST}_1^c$ of $\mathbf{chain}_{\geq 1}$; see (II. § 6). We call $\operatorname{aug} C_*(X_{\geq 1})$ the *non-reduced* chain complex of $X_{\geq 1}$.

(2.4) Proposition. *The functor C_* in (2.3) carries subcomplexes of $\mathbf{T}$-complexes to subcomplexes of chain complexes. Moreover C_* is compatible with cylinders; that is there is a canonical isomorphism of chain complexes*

$$C_* \bar{I}(X_{\geq 1}) \cong IC_*(X_{\geq 1})$$

Here $\bar{I}(X_{\geq 1})$ is the cylinder for 1-homotopies in (IV.1.3) and $IC_*(X_{\geq 1})$ is the cylinder of a chain complex in $\mathbf{mod}(\partial_X)$ defined in (III.9.1).

Proof of (2.4). The 2-skeleton of $\bar{I}(X_{\geq 1})$ is a principal cofibration with attaching map

$$\begin{cases} w_X : A_2 \vee A_1 \vee A_2 \longrightarrow X \vee X \\ w_X = (i_0^X \partial_X, w_X, i_1^X \partial_X) \end{cases}$$

Here $w_X = W$ coincides with the attaching map W in (III.5.5). We have the canonical map

$$\partial_X \xrightarrow{i_1^X} w_X \xrightarrow{p} \partial_X$$

in the category $\mathbf{Coef}$ where i_1^X is given by the second inclusion $X \to X \vee X$ and where p is given by $(1,1) : X \vee X \to X$. We claim that i_1^X is an isomorphism in $\mathbf{Coef}$ with inverse p. In fact $pi_1^X = 1_X$. On the other hand we have

$$i_1^X p = 1_{X \vee X} + (w_X, 1)\alpha \tag{1}$$

where $\alpha : A_1 \vee A_1 \to (A_2 \vee A_1 \vee A_2) \vee (X \vee X)$ is the inclusion of A_1 on the first summand of $A_1 \vee A_1$ and is trivial on the second summand of $A_1 \vee A_1$. Then (1) is a consequence of $i_0^X + w_X = i_1^X$; compare (III.5.3). By (1) we see that $i_1^X p$ is

∂-equivalent to $1_{X\vee X}$. Hence p is an isomorphism in **Coef**. Using this isomorphism we obtain the p-equivariant isomorphism

$$\bar{p} : C_*\bar{I}(X_{\geq 1}) \cong IC_*(X_{\geq 1}) \tag{2}$$

as follows. Here $C_*\bar{I}(X_{\geq 1})$ is a chain complex $(\bar{A}, w_X)$ with $\bar{A}_n \in T$ given by

$$\bar{A}_n = \begin{cases} A_1 \vee A_1 & \text{for } n = 1 \\ A_n \vee A_{n-1} \vee A_n & \text{for } n \geq 2 \end{cases}$$

Compare (IV.2.5). Now $\bar{p}$ is the p-equivariant chain map which is the identity on $\bar{A}_n$, that is

$$\bar{p} : C_n\bar{I}(X_{\geq 1}) = \bar{A}_n \vee w_X \xrightarrow{1\odot p} \bar{A}_n \vee \partial_X = (IC_*(X_{\geq 1}))_n$$

It is a consequence of (III.4.5) and (III.5.6) that $\bar{p}$ is a well defined chain map. Moreover $\bar{p}$ is an isomorphism since p is an isomorphism in **Coef**. q.e.d.

Proposition (2.4) implies that C_* induces a functor

$$C_* : \mathbf{Complex}/\overset{1}{\simeq} \longrightarrow \mathbf{chain}/\simeq \tag{2.5}$$

between homotopy categories.

Using the category **Well** of weakly cellular objects in **C** we obtain the composite chain functor

$$C_* : \mathbf{Well}/\simeq \xrightarrow{\varphi} \mathbf{Complex}/\overset{1}{\simeq} \xrightarrow{C_*} \mathbf{chain}/\simeq \tag{2.6}$$

as well denoted by C_*; see (IV.4.4). This yields the chain complex of a weakly cellular object in the cofibration category **C**. There are similar properties of the non-reduced chain complex $\operatorname{aug} C_*$ if **T** is augmented; see (2.3) (5) and (II.§ 6).

3 The Homology of a Complex

Let **C** be a homological cofibration category under **T**. Then we can use the chain complex in § 2 to define the homology and cohomology of a complex in **C**. We also obtain this way the homology of a weakly cellular object in **C**.

(3.1) Definition. Let $X_{\geq 1}$ be a **T**-complex in **C** with coefficients $\partial_X = c(X_{\geq 1}) \in$ **Coef**. Moreover let M (resp. N) be a left (resp. right) $\mathbf{mod}(\partial_X)$-module. Then the *homology* and *cohomology*

$$H_n(X_{\geq 1}; M) = H_n(C_*(X_{\geq 1}); M)$$
$$H^n(X_{\geq 1}; N) = H^n(C_*(X_{\geq 1}); N)$$

is defined as in (I.§6). Here $C_*(X_{\geq 1})$ is the chain complex in §2. If $Y_{\geq 1}$ is a subcomplex of $X_{\geq 1}$ then $C_*Y_{\geq 1}$ is a subcomplex of the chain complex $C_*X_{\geq 1}$ so that similarly the relative (co-) homology groups $H_n(X_{\geq 1}, Y_{\geq 1}; M)$ are defined by (I.§6). If $\mathbf{T}$ is augmented we define accordingly the *non-reduced (co-) homology* by the non-reduced chain complex $\operatorname{aug} C_*$ in (2.3) (5).

We also have the left $\mathbf{mod}(\partial_X)$-module $H^n(X_{\geq 1}) = H^n(C_*(X_{\geq 1}))$ with

$$H^n(X_{\geq 1})(D) = H^n(X_{\geq 1}; \operatorname{Hom}_{\partial_X}(-, D)) \tag{3.2}$$

where D is a cogroup in $\mathbf{T}$ corresponding to the object $D \vee \partial_X$ in $\mathbf{mod}(\partial_X)$. Dually one obtains the right $\mathbf{mod}(\partial_X)$-module $H_n(X_{\geq 1}) = H_n(C_*(X_{\geq 1}))$ with

$$H_n(X_{\geq 1})(D) = H_n(X_{\geq 1}; \operatorname{Hom}_{\partial_X}(D, -)) \tag{3.3}$$

Compare (I.§6). More explicitly $H_n(X_{\geq 1})(D)$ can be obtained by the following chain complexes of abelian groups. Consider

$$[\Sigma D, \Sigma A_{n+1} \vee X_2]_2 \xrightarrow{(d_{n+1},1)_*} [\Sigma D, \Sigma A_n \vee X_2]_2 \xrightarrow{(d_n,1)_*} [\Sigma D, \Sigma A_{n-1} \vee X_2]_2$$

Here d_n is defined as in (2.3) and we set $(d_n, 1) = (d_n, i_2)$ where i_2 is the inclusion of X_2.

(3.4) Lemma. $H_n(X_{\geq 1})(D) = \operatorname{kernel}(d_n, 1)_* / \operatorname{image}(d_{n+1}, 1)_*$

This follows readily from the definition of $H_n(X_{\geq 1})(D)$.

The homology $H_n(X_{\geq 1})(D)$ can also be described for $n \geq 3$ by the formula in the following lemma. For this we define the operators δ_n, $n \geq 3$, by the composites

$$\begin{cases} \delta_3 : \pi_2^D(X_3, X_2) \xrightarrow{\partial} \pi_1^D(X_2) \xrightarrow{\nabla} \pi_1^D(\Sigma A_2 \vee X_2)_2 \\ \delta_{n+1} : \pi_n^D(X_{n+1}, X_n) \xrightarrow{\partial} \pi_{n-1}^D(X_n) \xrightarrow{j} \pi_{n-1}^D(X_n, X_{n-1}) \end{cases} \tag{3.5}$$

where ∂ and j are the maps in the homotopy exact sequence of pairs and where ∇ is the difference operator in (III.3.8).

(3.6) Lemma. *We have $\delta_n \delta_{n+1} = 0$ and for $n \geq 3$ there is a natural isomorphism*

$$H_n(X_{\geq 1})(D) \cong \operatorname{kernel}(\delta_n) / \operatorname{image}(\delta_{n+1}).$$

Proof. The isomorphism is induced by the composite $(n \geq 3)$

$$\lambda_n : \pi_1^D(\Sigma A_n \vee X_2)_2 \longrightarrow \pi_{n-2}^D(\Sigma^{n-2} A_n \vee X_{n-1})_2 \longrightarrow \pi_{n-1}^D(X_n, X_{n-1})$$

of maps $\lambda_n = E_X(1 \vee i)_* E^{n-3}$ where $i : X_2 \subset X_n$ is the inclusion and E_X is the map in (1.1) (b). Using (1.1) (a), (b), and (c) we see that λ_n is an isomorphism. Using (III.3.2), (III.3.9) and (III.3.10) and (I.3.4) we show that the diagram $(n \geq 3)$

$$\begin{array}{ccc} \pi_n^D(X_{n+1}, X_n) & \xrightarrow{\delta_{n+1}} & \pi_{n-1}^D(X_n, X_{n-1}) \\ \lambda_{n+1}\Big\uparrow\cong & & \cong\Big\uparrow\lambda_n \\ \pi_1^D(\Sigma A_{n+1} \vee X_2)_2 & \xrightarrow{(d_{n+1},1)_*} & \pi_1^D(\Sigma A_n \vee X_2)_2 \end{array} \tag{1}$$

commutes. Moreover for $n = 2$ the diagram

$$\begin{array}{ccc} \pi_2^D(X_3; X_2) & \xrightarrow{\delta_3} & \pi_1^D(\Sigma A_2 \vee X_2)_2 \\ \lambda_3\Big\uparrow\cong & & \Big\| \\ \pi_1^D(\Sigma A_3 \vee X_2)_2 & \xrightarrow{(d_3,1)_*} & \pi_1^D(\Sigma A_2 \vee X_2)_2 \end{array} \tag{2}$$

commutes. This proves (3.6) by use of (3.4). For the proof of (1) and (2) we first observe that by (III.3.10) the diagram with $n \geq 2$

$$\begin{array}{ccc} \pi_{n-2}^D(\Sigma^{n-2} A_n \vee X_{n-1})_2 & \xrightarrow{E_X} & \pi_{n-1}^D(X_n, X_{n-1}) \\ E\Big\downarrow & & \Big\downarrow\nabla \\ \pi_{n-1}^D(\Sigma^{n-1} A_n \vee X_{n-1})_2 & \xrightarrow{(1\vee i)_*} & \pi_{n-1}^D(\Sigma^{n-1} A_n \vee X_n)_2 \end{array} \tag{3}$$

commutes. Here E, E_X and $(1 \vee i)_*$ are isomorphisms for $n \geq 3$ by (1.1) so that also ∇ is an isomorphism. We now show (1) by proving $\nabla\delta_{n+1}\lambda_{n+1} = \nabla\lambda_n(d_{n+1}, 1)_*$. In fact we have

$$\begin{aligned} \nabla\delta_{n+1}\lambda_{n+1}(\xi) &= \nabla j\partial E_X(1 \vee i)_* E^{n-2}\xi \\ &= \nabla\left((\partial_{n+1}, 1)(1 \vee i)E^{n-2}\xi\right), \quad \text{see (III.3.9) and (III.3.2)} \\ &= (\nabla\partial_{n+1}, i_2)(1 \vee i)E^{n-2}\xi, \quad \text{see (4) below} \\ &= ((1 \vee i)E^{n-2}d_{n+2}, i)E^{n-2}\xi, \quad \text{see (2.1) (4).} \end{aligned}$$

$$\begin{aligned} \nabla\lambda_n(d_{n+1}, 1)_*(\xi) &= \nabla E_X(1 \vee i)E^{n-3}(d_{n+1}, 1)_*\xi \\ &= (1 \vee i)E(1 \vee i)E^{n-3}((d_{n+1}, 1)\xi), \quad \text{see (3)} \\ &= ((1 \vee i)E^{n-2}d_{n+1}, i)E^{n-2}\xi. \end{aligned}$$

Let $\eta = E^{n-2}\xi$. Since $\eta = E\eta'$ we have for $\bar{\eta} = (1 \vee i)\eta$ by II.11.17 in Baues [AH] the equations

$$\begin{aligned} \nabla((\partial_{n+1}, 1)\bar{\eta}) &= -i_2(\partial_{n+1}, 1)\bar{\eta} + (i_2 + i_1)(\partial_{n+1}, 1)\bar{\eta} \\ &= (-i_2\partial_{n+1}, i_2)\bar{\eta} + ((i_2 + i_1)\partial_{n+1}, i_2 + i_1)\bar{\eta} \\ &= (-i_2\partial_{n+1}, 1)\bar{\eta} + ((i_2 + i_1)\partial_{n+1}, 1)\bar{\eta} \qquad (*) \\ &= (\nabla\partial_{n+1}, 1)\bar{\eta} \end{aligned} \tag{4}$$

Here we get $(*)$ since $(i_2 + i_1)i = i_2 i$.

Finally we prove (2) by

$$\begin{aligned}\delta_2\lambda_3(\xi) &= \nabla\partial E_X(1\vee i)_*(\xi)\\ &= \nabla((\partial_3,1)\xi), \quad \text{since } i=1\\ &= (\nabla\partial_3,1)\xi, \quad \text{by (4) above}\\ &= (d_3,1)\xi, \quad \text{see (2.1) (4).}\end{aligned}$$

Here we use the fact that there exists an element ξ' with $\xi = (1\vee i)E\xi'$ by (1.1) (a), (c).

4 The Obstruction Cocycle

We first define homotopy groups for filtered objects and then we show that an extension problem is related to a cohomology class with coefficients in such homotopy groups. For CW-complexes this is a classical result of obstruction theory.

(4.1) Definition. Let $\mathbf{C}$ be a cofibration category under $\mathbf{T}$ and let $U_{\geq 1}$ be a filtered object in $\mathbf{Fil}_1(\mathbf{C})_{cf}$. For each cogroup D in $\mathbf{T}$ and $n\geq 0$ we define the *homotopy group*

$$\pi_n^D(U_{\geq 1}) = \operatorname{image}\{i_* : \pi_n^D(U_{n+1}) \to \pi_n^D(U_{n+2})\} \tag{1}$$

Here $i: U_{n+1}\subset U_{n+2}$ is the cofibration in $\mathbf{C}$ given by $U_{\geq 1}$. It is easy to see that homotopy groups yield a functor

$$\pi_n^D : \mathbf{Fil}(\mathbf{C})_{cf}/\overset{1}{\simeq} \longrightarrow \mathbf{Gr} \tag{2}$$

where on the left hand side we use the quotient category defined by 1-homotopies. For $n\geq 1$ the groups $\pi_n^D(U_{\geq 1})$ are abelian. If $(\mathbf{C},\mathbf{T})$ is homological then the collection of homotopy groups $\pi_{n+1}^D(U_{\geq 1})$ for all D has the following additional structure.

(4.2) Definition. Let $\mathbf{C}$ be a homological cofibration category under $\mathbf{T}$. Then $\mathbf{Coef}$ and $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$ are defined and for $\partial_X \in \mathbf{Coef}$ we have the additive category $\mathbf{mod}(\partial_X)$ of 1-equivariant maps $\alpha\odot 1 : A\vee\partial_X \to B\vee\partial_X$. Here A and B are cogroups in $\mathbf{T}$ and α is an element $\alpha\in[\Sigma A, \Sigma B\vee X_2]_2$ where (X_2,X_1) is a principal cofibration with attaching map $\partial_X : A_2\to X_1\in\mathbf{T}$. Given an object $U_{\geq 1}\in\mathbf{Fil}(\mathbf{C})_{cf}$ we say that

$$v : X_1 \longrightarrow U_1 \in \mathbf{C} \tag{1}$$

is ∂ *-compatible* if $iv\partial_X\simeq 0$ where $i: U_1\to U_2$. Equivalently v is ∂-compatible if and only if there exists a map $\bar v : (X_2,X_1)\to(U_2,U_1)$ between pairs extending v. We define for the pair $(v,U_{\geq 1})$ and $n\geq 1$ the right $\mathbf{mod}(\partial_X)$-module $v^*\pi_{n+1}(U_{\geq 1})$ termed the *(total) homotopy group* of $(v,U_{\geq 1})$. The functor

$$v^*\pi_{n+1}(U_{\geq 1}) : \mathbf{mod}(\partial_X)^{\mathrm{op}} \longrightarrow \mathbf{Ab} \tag{2}$$

carries $A \vee \partial_X$ to the homotopy group

$$v^*\pi_{n+1}(U_{\geq 1})(A) = \pi_n^A(U_{\geq 1})$$

defined in (4.1). Here we use a shift in degree since A has dimension 1. Hence on objects the functor (2) does not depend on v. On morphisms we define the functor (2) by the induced map

$$(\alpha \odot 1)^* : \pi_n^B(U_{\geq 1}) \longrightarrow \pi_n^A(U_{\geq 1}) \tag{3}$$

which carries $\xi \in \pi_n^B(U_{\geq 1})$ to the composite

$$(\alpha \odot 1)^*(\xi) = (\xi, i\bar{v})(E^{n-1}\alpha) : \Sigma^n A \longrightarrow \Sigma^n B \vee X_2 \to U_{n+2} \tag{4}$$

Here $i : U_2 \to U_{n+2}$ is the inclusion and $\bar{v}$ is chosen for v as in (1). By (1.1) (a) α factors through $B \vee X_1$. This implies that (4) depends only on v and hence (3) is well defined. Using II.11.16 in Baues [AH] we see that (2) is a well defined functor. Clearly a map $f : U_{\geq 1} \to V_{\geq 1}$ in $\mathbf{Fil}_1(\mathbf{C})_{cf}/\overset{1}{\simeq}$ induces a morphism of right $\mathbf{mod}(\partial_X)$-modules

$$f_* : v^*\pi_{n+1}(U_{\geq 1}) \longrightarrow (f_1 v)^*\pi_{n+1}(V_{\geq 1}) \tag{5}$$

If $X_{\geq 1}$ is a $\mathbf{T}$-complex we obtain as a special case the right $\mathbf{mod}(\partial_X)$-module $\pi_{n+1}(X_{\geq 1})$ which is the total homotopy group of $(v, X_{\geq 1})$ where $v = 1 : X_1 \to X_1$ is the identity.

We now study the following *extension problem* in a homological cofibration category $\mathbf{C}$ under $\mathbf{T}$. Let $L_{\geq 1}$ be a subcomplex of the $\mathbf{T}$-complex $K_{\geq 1}$ and consider the diagram in $\mathbf{Fil}_1(\mathbf{C})_{cf}$.

$$\begin{tikzcd} K_{\geq 1} \arrow[dr, "g"] & \\ L_{\geq 1} \arrow[u, "i"] \arrow[r, "f"'] & U_{\geq 1} \end{tikzcd} \tag{4.3}$$

where i is the inclusion. Given f and i does there exist a map g such the diagram commutes? If $U_{\geq 1} = U$ is the constant filtered object this corresponds to the classical extension problem of algebraic topology. For this problem one considers inductively extensions g_n of f where

$$g_n : L_{\geq 1} \cup K^n \longrightarrow U_{\geq 1} \in \mathbf{Fil}_1(\mathbf{C})_{cf}$$

with $n \geq 1$. Here K^n is the n-skeleton of $K_{\geq 1}$ and $L_{\geq 1} \cup K^n$ is the subcomplex of $K_{\geq 1}$ given by the union of $L_{\geq 1}$ and K^n.

(4.4) Theorem. *Let $n \geq 2$ and assume an extension g_n of f exists so that we obtain the ∂-compatible map $v = g_n \mid K_1 : K_1 \to U_1$ as a restriction of g_n. Then a relative cocycle $\delta(g_n)$ of the cochain complex $C^*(K_{\geq 1}, L_{\geq 1})$ with coefficients in the right $\mathbf{mod}(\partial_K)$-module $v^*\pi_n(U_{\geq 1})$ is defined. This cocycle has the property $\delta(g_n) = 0$ if and only if an extension g_{n+1} of g_n exists. Moreover an extension g_{n+1} of $g_{n-1} = g_n \mid L_{\geq 1} \cup K^{n-1}$ exists if and only if the cohomology class*

$$\{\delta(g_n)\} \in H^{n+1}(K_{\geq 1}, L_{\geq 1}; v^*\pi_n(U_{\geq 1}))$$

is trivial.

The cocycle $\delta(g_n)$ is termed the *obstruction cocycle.*

Proof. Since $L_{\geq 1} \subset K_{\geq 1}$ is a subcomplex we see that $(K_{n+1}, L_{n+1} \cup K_n)$ is a principal cofibration with attaching map

$$\bar{\partial}_{n+1} : \Sigma^{n-1}\bar{A}_{n+1} \longrightarrow L_{n+1} \cup K_n$$

Now the cocycle is given by the composite

$$\delta(g_n) = g_n\bar{\partial}_{n+1} \in \pi^A_{n-1}(U_{\geq 1})$$

with $A = \bar{A}_{n+1}$. Here we identify $\pi^A_{n-1}(U_{\geq 1})$ with the group of relative cochains in degree $n+1$ of $C_*(L_{\geq 1}) \subset C_*(K_{\geq 1})$ with coefficients in $v^*\pi_n(U_{\geq 1})$. Now we can use (III.3.3) to see that g_{n+1} exists if and only if $\delta(g_n) = 0$. Moreover by (III.3.5) we know that two extensions g_n, g'_n of $g_n \mid L_{\geq 1} \cup K^{n-1}$ differ only by an element $\alpha \in \pi^B_n(U_{\geq 1})$ with $B = \bar{A}_n$, that is $g'_n = g_n + \alpha$. By definition of d_{n+1} we get

$$\begin{aligned}\delta(g'_n) &= (g_n + \alpha)\bar{\partial}_{n+1} \\ &= g_n\bar{\partial}_{n+1} + (g_n, \alpha) \bigtriangledown \bar{\partial}_{n+1} \\ &= \delta(g_n) + d^*_{n+1}(\alpha)\end{aligned}$$

This implies the property of the cohomology class $\{\delta(g_n)\}$ in (4.4). q.e.d.

5 The Hurewicz Homomorphism and Whitehead's Exact Sequence

Let $\mathbf{C}$ be a homological cofibration category under $\mathbf{T}$. Then we obtain for each $\mathbf{T}$-complex $X_{\geq 1}$ the right $\mathbf{mod}(\partial_X)$-modules $H_n(X_{\geq 1})$ and $\pi_n(X_{\geq 1})$ for $n \geq 2$; see (4.2) and (3.3). The general *Hurewicz homomorphism* is a homomorphism of right $\mathbf{mod}(\partial_X)$-modules

$$h : \pi_n(X_{\geq 1}) \longrightarrow H_n(X_{\geq 1}) \tag{5.1}$$

which is natural in $X_{\geq 1} \in \mathbf{Complex}/\overset{1}{\simeq}$, $n \geq 2$. For the definition of h we observe that for a cogroup B in T the $(n-1)$-fold suspension $\Sigma^{n-1}B$ can be considered to be an n-dimensional $\mathbf{T}$-complex with trivial $(n-1)$-skeleton. Then we can identify

$$\pi_n(X_{\geq 1})(B) = [\Sigma^{n-1}B, X_{\geq 1}]/\overset{1}{\simeq}$$

where the right hand side is a set of morphisms in the category $\mathbf{Complex}/\overset{1}{\simeq}$. Therefore we can apply the functor C_* in (2.2) which defines h by the commutative diagram

$$\begin{tikzcd} \pi_n(X_{\geq 1})(B) \arrow[r, "h"] \arrow[d, equal] & H_n(X_{\geq 1})(B) \arrow[d, equal] \\ {[\Sigma^{n-1}B, X_{\geq 1}]/\overset{1}{\simeq}} \arrow[r, "C_*"] & {[C_*\Sigma^{n-1}B, C_*X_{\geq 1}]} \end{tikzcd}$$

The isomorphism on the right hand side is given by (3.3) and (I.6.5). By applying C_* to the composite

$$\Sigma^{n-1}A \xrightarrow{\ \alpha\ } \Sigma^{n-1}B \vee X_2 \xrightarrow{(\xi,i)} X_{\geq 1}$$

in (4.2) (4) we see that $h(\alpha \odot 1)^*\xi = (\alpha \odot 1)^*(\xi)$. Hence h is a well defined homomorphism between right $\mathbf{mod}(\partial_X)$-modules.

(5.2) Remark. For $\mathbf{C} = \mathbf{Top}^*$ and $\mathbf{T} = \mathbf{susp}(*, \mathcal{D})$ the Hurewicz homomorphism (5.1) specializes to the classical Hurewicz homomorphism

$$h : \pi_n(X) \cong \pi_n(\tilde{X}) \to H_n(\tilde{X})$$

where $\tilde{X}$ is the universal covering of the reduced CW-complex X. Here h is a natural homomorphism of right $\mathbb{Z}[\pi_1(X)]$-modules.

(5.3) Definition. Let $(\mathbf{C}, \mathbf{T})$ be homological. Then we define for a $\mathbf{T}$-complex $X_{\geq 1}$ and $n \geq 1$ the right $\mathbf{mod}(\partial_X)$-module $\Gamma_n(X_{\geq 1})$ as follows. For $n = 1$ let

$$\Gamma_1(X_{\geq 1}) = \Gamma_1(\partial_X) \tag{1}$$

be defined by the module in (II.§ 2). For $n \geq 1$ and a cogroup B in $\mathbf{T}$ we define

$$\Gamma_{n+1}(X_{\geq 1}) : \mathbf{mod}(\partial_X)^{\mathrm{op}} \to \mathbf{Ab} \tag{2}$$

by

$$\Gamma_{n+1}(X_{\geq 1})(B) = \begin{cases} \text{image}\{\pi_n^B X_n \to \pi_n^B X_{n+1}\} & \text{for } n \geq 2 \\ \text{kernel}\{\bigtriangledown : \pi_1^B X_2 \to \pi_1^B(\Sigma A_2 \vee X_2)_2\} & \text{for } n = 1 \end{cases}$$

For $n \geq 2$ this definition is up to a shift in degree similar to the definition of $\pi_n^D(U_{\geq 1})$ in (4.1) (1). For $n = 1$ we use the difference operator $\bigtriangledown$ in (III.3.9). We define the structure of a right $\mathbf{mod}(\partial_X)$-module similarly as in (4.2). That is for $\xi \in \Gamma_{n+1}(X_{\geq 1})(B)$ we define $(\alpha \odot 1)^*(\xi)$ by the composite

$$(\alpha \odot 1)^*(\xi) = (\xi, i)(E^{n-1}\alpha) : \Sigma^n A \longrightarrow \Sigma^n B \vee X_2 \longrightarrow X_{n+1}$$

Here $i : X_2 \to X_{n+1}$ is the inclusion which is well defined for $n \geq 1$. For $n = 1$ this is again an element in kernel $\bigtriangledown$ since $\xi \in \text{kernel}(\bigtriangledown)$ and since α is trivial on X_2. This shows that the functor (2) is well defined.

We are now ready to describe the following ultimate generalization of J.H.C. Whitehead's [CE] certain exact sequence

(5.4) Theorem. *Let* $(\mathbf{C}, \mathbf{T})$ *be homological. Then one has for each* $\mathbf{T}$*-complex* $X_{\geq 1}$ *the following long exact sequence of right* $\mathbf{mod}(\partial_X)$*-modules,* $n \geq 2$,

$$\begin{aligned} &\longrightarrow \Gamma_n(X_{\geq 1}) \xrightarrow{i} \pi_n(X_{\geq 1}) \xrightarrow{h} H_n(X_{\geq 1}) \xrightarrow{b} \Gamma_{n-1}(X_{\geq 1}) \xrightarrow{i} \dots \\ &\longrightarrow \Gamma_2(X_{\geq 1}) \xrightarrow{i} \pi_2(X_{\geq 1}) \xrightarrow{h} H_2(X_{\geq 1}) \xrightarrow{b} \Gamma_1(X_{\geq 1}) \longrightarrow 0 \end{aligned}$$

This sequence is natural in $X_{\geq 1} \in \mathbf{Complex}/\overset{1}{\simeq}$. *The operator* h *is the Hurewicz homomorphism.*

The theorem yields a new interpretation of the module Γ_1 in (II.§ 2) as the cokernel of the Hurewicz map in degree 2. Various explicit computations of Γ_1 are described in (II.8.11).

Proof of (5.4). We define the operator i for $n \geq 2$ by the inclusion $X_{n+1} \subset X_{n+2}$; compare (5.8) and (4.1) (1). It is clear that i is a homomorphism of $\mathbf{mod}(\partial_X)$-modules. We define a sequence of abelian groups and homomorphisms, $n \in \mathbb{Z}$,

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\beta} A_n \xrightarrow{j} C_n \xrightarrow{\beta} A_{n-1} \xrightarrow{j} \dots \tag{1}$$

with the property image(j) = kernel(β). Then one obtains the exact sequence

$$\cdots \longrightarrow H_{n+1} \xrightarrow{b} \Gamma_n \xrightarrow{i} \pi_n \xrightarrow{h} H_n \xrightarrow{b} \dots \tag{2}$$

as follows:

$$\begin{aligned} \Gamma_n &= \operatorname{kernel}\{j : A_n \to C_n\} \\ \pi_n &= A_n / \beta C_{n+1} \\ H_n &= \operatorname{kernel}(d_n) / \operatorname{image}(d_{n+1}) \end{aligned}$$

Here $d = j\beta : C_n \to C_{n-1}$ satisfies $dd = 0$. The operator i is given by $\Gamma_n \subset A_n \twoheadrightarrow \pi_n$. Moreover h is induced by h' with $j : A_n{}' \xrightarrow{h} \operatorname{kernel}(d_n) \subset C_n$. Finally b is induced by $b' : \operatorname{kernel}(d_n) \to \Gamma_{n-1}$ which is the restriction of $\beta : C_n \to A_{n-1}$. A diagram chase shows that (2) is exact; compare J.H.C. Whitehead [CE]. We define (1) by

$$A_n = \begin{cases} \pi_{n-1}^B X_n & \text{for } n \geq 2 \\ \Gamma_1(\partial_X)(B) & \text{for } n = 1 \\ 0 & \text{for } n \leq 2 \end{cases}$$

$$C_n = \begin{cases} \pi_{n-1}^B(X_n, X_{n-1}) & \text{for } n \geq 3 \\ \operatorname{kernel}(d_2, 1)_* & \text{for } n = 2 \\ 0 & \text{for } n \leq 1 \end{cases}$$

For the definition of C_2 we use the homomorphism

$$(d_2,1)_*:\pi_1^B(\Sigma A_2\vee X_2)_2\longrightarrow\pi_1^B(\Sigma A_1\vee X_2)_2,$$

compare (3.4). For $n\geq 2$ we have the exact sequence $A_{n+1}\to C_{n+1}\to A_n$, namely

$$\pi_n^B(X_{n+1})\xrightarrow{j}\pi_n^B(X_{n+1},X_n)\xrightarrow{\beta}\pi_{n-1}^B(X_n)$$

is given by the homotopy exact sequence. For $n=1$ we obtain the exact sequence $A_2\to C_2$ by the exact top row in the following diagram

$$\begin{tikzcd}
 & \pi_1^B(X_2) \arrow[r, "j"] \arrow[d] & \mathrm{kernel}(d_2,1)_* \arrow[r, "\beta"] \arrow[d, phantom, "\cap"] & \Gamma_1(\partial_X)(B) \arrow[r] & 0 \\
\pi_0^B(A_2\vee X_1)_2 \arrow[r, "E_X"] & \pi_1^B(X_2,X_1) \arrow[r, "\nabla"] \arrow[d, "\partial"'] & \pi_1^B(\Sigma A_2\vee X_2)_2 \arrow[d, "(d_2{,}1)_*"] & & \\
 & \pi_0^B(X_1) & \pi_1^B(\Sigma A_1\vee X_2)_2 & &
\end{tikzcd}$$

Here E_X is surjective and the columns are exact. Moreover $\partial E_X=(\partial_X,1)_*$. Hence this diagram corresponds exactly to the definition of $\Gamma_1(\partial_X)(B)$ in (II.2.1) since $\nabla E_X=(1\vee i)_*E$ is given by the partial suspension E. By (3.6) we see that $H_n=H_n(X_{\geq 1})(B)$ and one readily checks that $\pi_n=\pi_n(X_{\geq 1})(B)$ and $\Gamma_n=\Gamma_n(X_{\geq 1})(B)$. This completes by (2) the proof that the sequence in (5.4) is exact. One can check that h in (2) coincides with h in (5.1). Moreover b in (2) is a homomorphism of right $\mathbf{mod}(\partial_X)$-modules. q.e.d.

(5.5) Lemma. *We have for $n\geq 1$ the equation*

$$\Gamma_{n+1}(X_{\geq 1})(B)=\mathrm{kernel}\{\nabla:\pi_n^B X_{n+1}\to\pi_n^B(\Sigma^n A_{n+1}\vee X_{n+1})_2\}$$

For $n=1$ this is exactly the definition in (5.3).

Proof. We observe that for $n\geq 1$ the following diagram commutes; see (III.3.10) and (III.3.9).

$$\begin{tikzcd}
\pi_n^B(X_{n+1}) \arrow[r, "\nabla"] \arrow[d, "j"'] & \pi_n^B(\Sigma^n A_{n+1}\vee X_{n+1})_2 \\
\pi_n^B(X_{n+1},X_n) & \pi_{n-1}^B(\Sigma^{n-1}A_{n+1}\vee X_n)_2 \arrow[l, "E_X"] \arrow[u, "(1\vee i)_*E"']
\end{tikzcd}$$

Here E_X and $(1\vee i)_*E$ are both isomorphisms for $n\geq 2$ by (V.1.1). Hence we get for $n\geq 2$

$$\mathrm{image}\{\pi_n^B X_n\to\pi_n^B X_{n+1}\}=\mathrm{kernel}(j)=\mathrm{kernel}(\nabla)$$

q.e.d.

Chapter VI: Realization of Chain Maps

In this chapter we consider fundamental properties of the chain functor C_* which carries a $\mathbf{T}$-complex $X_{\geq 1}$ to a chain complex $A = C_* X_{\geq 1}$. Then $X_{\geq 1}$ is termed a realization of the chain complex A. We introduce partial realizations of a chain complex which are termed "twisted homotopy systems"; this generalizes the notion of a twisted chain complex in chapter II. Using twisted homotopy systems we study partial realizations of chain maps. This leads to an obstruction theory both for the realization of a chain complex and for the realization of chain maps. To discuss these properties we introduce some useful language on "linear extensions of categories", "exact sequences for functors" and "towers of categories"; see § 5. The homological tower of categories in § 6 is a first main result which is needed to prove the homological Whitehead theorem in § 7 and the "model lifting property" of the twisted chain functor in § 8. The model lifting property is a key point in the proof of the Hurewicz theorem in § 10 and in the proof of the finiteness obstruction theorem in the next chapter VII.

1 Twisted Homotopy Systems of Order n

Let $\mathbf{C}$ be a homological cofibration category under a theory $\mathbf{T}$ of coactions. Then $\mathbf{T}$-complexes $X_{\geq 1}$ and the associated chain complex $C_* X_{\geq 1}$ are defined. In order to construct a $\mathbf{T}$-complex $X_{\geq 1}$ which realizes a given chain complex (A, ∂_X) we consider inductively twisted homotopy systems of order $n \geq 2$ as follows.

(1.1) Definition. Let $\mathbf{C}$ be a homological cofibration category under $\mathbf{T}$ and let $n \geq 1$. A *twisted homotopy system of order* $(n+1)$ or equivalently an $(n+1)$ *-system* for short is a triple

$$X = (A, \partial_{n+1}, X^n) \tag{1}$$

Here $X^n = (X_1 \subset X_2 \subset \cdots \subset X_n)$ is an n-dimensional $\mathbf{T}$-complex, ∂_{n+1} is an element $\partial_{n+1} \in [\Sigma^{n-1} A_{n+1}, X_n]$ and $A = (A, \partial_X)$ is a chain complex in **chain**. For $n = 1$ we have $\partial_X = \partial_2$ and for $n \geq 2$ we obtain ∂_X by the attaching map of (X_2, X_1). Moreover for $n \geq 2$ the chain complex A coincides in degree $\leq n$ with $C_*(X^n)$; see (V.2.3), and

$$d_{n+1} \equiv \nabla \partial_{n+1} \tag{2}$$

holds as in (V.2.3) (2). Here $d_{n+1} \odot 1$ is the differential of A. Hence X is a "partial realization" of A; in fact, the part of degree $\leq n$ in A is realized by X^n and in addition the differential $d_{n+1} \odot 1$ is realized by an attaching map ∂_{n+1}. We say that X in (1) satisfies the *cocycle condition* if there exists $\bar{d}_{n+2} \in [\Sigma^{n-1}A_{n+2}, \Sigma^{n-1}A_{n+1} \vee X_1]_2$ with

$$\begin{cases} \bar{d}_{n+2} \equiv d_{n+2}, & \text{see (V.1.9)}, \\ (\partial_{n+1}, i)_* \bar{d}_{n+2} = 0 & \text{in } [\Sigma^{n-1}A_{n+2}, X_n]. \end{cases} \tag{3}$$

A map between $(n+1)$-systems is a pair (ξ, η) which we write

$$(\xi, \eta) : X = (A, \partial_{n+1}, X^n) \longrightarrow Y = (B, \partial_{n+1}, Y^n) \tag{4}$$

Here $\eta : X^n \to Y^n$ is a map in **Complex** and $\xi : A \to B$ is an η_1-equivariant chain map where $\eta_1 : \partial_X \to \partial_Y$ is defined by the restriction $\eta_1 : X_1 \to Y_1$ of η. Moreover ξ coincides in degree $\leq n$ with $C_*\eta$ and there exists an element $\bar{\xi}_{n+1} \in [\Sigma^{n-1}A_{n+1}, \Sigma^{n-1}B_{n+1} \vee Y_1]_2$ with

$$\bar{\xi}_{n+1} \equiv \xi_{n+1}, \quad \text{see (V.1.9)}, \tag{5}$$

such that the following diagram commutes in $\mathrm{Ho}(\mathbf{C})$,

$$\begin{array}{ccc} \Sigma^{n-1}A_{n+1} & \xrightarrow{\bar{\xi}_{n+1}} & \Sigma^{n-1}B_{n+1} \vee Y_1 \\ {\scriptstyle \partial_{n+1}}\downarrow & & \downarrow{\scriptstyle (\partial_{n+1}, i)} \\ X_n & \xrightarrow[\eta_n]{} & Y_n \end{array}$$

that is:

$$(\partial_{n+1}, i)_* \bar{\xi}_{n+1} = (\eta_n)_* \partial_{n+1} \quad \text{in } [\Sigma^{n-1}A_{n+1}, Y_n]. \tag{6}$$

We say that (ξ, η) in (4) is the inclusion of a *subcomplex* if both $\xi : A \to B$ and $\eta : X^n \to Y^n$ are inclusions of subcomplexes in **chain** and **Complex** respectively; see (IV.2.4).

Two maps (ξ, η) and (ξ', η') as in (4) are 0 *-homotopic* if $\xi = \xi'$ and if there exists a 0-homotopy $\eta \overset{0}{\simeq} \eta'$ in **Complex**; see (IV.1.3).

Let $\mathbf{TWIST}^c_{n+1}$ be the following category. Objects are $(n+1)$-systems which satisfy the cocycle condition and morphisms are maps (ξ, η) as in (4) above. There is an obvious composition of such morphisms. Let $\mathbf{TWIST}^c_{n+1}/\overset{0}{\simeq}$ be the quotient category obtained by 0-homotopies.

(1.2) Remark. Let $\mathbf{TWIST}^c_2(\mathbf{T})$ be the category of twisted chain complexes in chapter II defined by a theory $\mathbf{T}$ of coactions and by a category **mod** of modules for $\mathbf{T}$. If $(\mathbf{C}, \mathbf{T})$ is a homological cofibration category and $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$ then one has an isomorphism of categories

$$\psi : \mathbf{TWIST}_2^c(\mathbf{T}) \cong \mathbf{TWIST}_2^c/\overset{0}{\simeq}$$

where the right hand side is the category defined in (1.1) above for $n = 1$. From this point of view twisted homotopy systems of order n are the canonical higher order analogues of twisted chain complexes. The isomorphism ψ of categories carries the twisted chain complex $A \mid \partial_X$ to the 2-system (A, ∂_X, X) with $X = X^1 \in \mathbf{T} \subset \mathbf{C}_{cf}/\simeq$ and carries the twisted chain map $\bar{f} = (f_{\geq 1}, Ef'', f)$ to the 0-homotopy class of (ξ, η) with $\xi = f_{\geq 1}$ and $f = \{\eta\}$. The map f'' corresponds to $\bar{\xi}_2$ in (1.1) (6) where $n = 1$.

We now define a canonical commutative diagram of functors ($n \geq 2$)

$$\begin{array}{ccc} \mathbf{Complex} & \xrightarrow{r_{n+1}} & \mathbf{TWIST}_{n+1}^c \\ {\scriptstyle C_*}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ \mathbf{chain} & \xleftarrow{K_n} & \mathbf{TWIST}_n^c \end{array} \tag{1.3}$$

with $K_n \lambda = K_{n+1}$ and $\lambda r_{n+1} = r_n$. We obtain the *restriction functor* r_{n+1} by

$$r_{n+1}(X_{\geq 1}) = (A, \partial_{n+1}, X^n) \tag{1}$$

where X^n is the n-skeleton of X, ∂_{n+1} is the attaching map of (X_{n+1}, X_n) and $A = C_* X$ is the chain complex of X. Similarly we get

$$\lambda(A, \partial_{n+1}, X^n) = (A, \partial_n, X^{n-1}) \tag{2}$$

where ∂_n is the attaching map of (X_n, X_{n-1}) and X^{n-1} is the $(n-1)$-skeleton of X^n. Moreover K_n is the forgetful functor with

$$K_n(A, \partial_n, X^{n-1}) = A. \tag{3}$$

(1.4) Lemma. *The functors in (1.3) are well defined.*

Proof. It suffices to consider r_{n+1}. In fact, by (V.1.1) (b) we see that the attaching map $\Sigma^n A_{n+2} \to X_{n+1}$ of $X_{\geq 1}$ is a functional suspension. Hence there exists $\bar{d}_{n+2}$ so that the cocycle condition is satisfied. Moreover

$$f_{n+1} : (X_{n+1}, X_n) \longrightarrow (Y_{n+1}, Y_n)$$

is a twisted map by (V.1.1) (b). Hence there exists $\bar{\xi}_{n+1}$ satisfying (1.1) (b). q.e.d.

One readily checks that the functors (1.3) are compatible with 0-homotopies. We therefore obtain the infinite sequence of functors, $n \geq 1$,

$$\mathbf{Complex}/\overset{0}{\simeq} \longrightarrow \cdots \longrightarrow \mathbf{TWIST}_{n+1}^c/\overset{0}{\simeq} \xrightarrow{\lambda} \mathbf{TWIST}_n^c/\overset{0}{\simeq} \longrightarrow \cdots \longrightarrow \mathbf{chain} \tag{1.5}$$

where the functor λ for $n = 1$ coincides with the functor K in (II.1.10). The sequence (1.5) is a factorization of the chain functor C_*. Next we define the notion of 1-homotopy for maps between twisted homotopy systems; compare (IV.1.3).

(1.6) Definition. Let $(\xi, \eta), (\xi', \eta') : X \to Y$ be maps in $\mathbf{TWIST}^c_{n+1}$, $n \geq 1$. Then (ξ, η) and (ξ', η') are 1-*homotopic* and we write $(\xi, \eta) \overset{1}{\simeq} (\xi', \eta')$ if (a) and (b) hold.

(a) There is $\bar{\alpha}_n \in [\Sigma^{n-1}A_n, \Sigma^{n-1}B_n \vee Y_1]_2$ and there is a 1-homotopy

$$H : \eta + (\partial_{n+1}, i)\bar{\alpha}_n \overset{1}{\simeq} \eta'$$

of maps in $\mathbf{Fil}(\mathbf{C})$. Here the action $+$ for maps in $\mathbf{C}$ is defined as in (III.3.4). We define $\alpha_n \in [\Sigma A_n, \Sigma B_n \vee Y_2]_2$ by $\alpha_n \equiv \bar{\alpha}_n$; compare (V.1.9).

(b) There are $\alpha_m \in [\Sigma A_m, \Sigma B_{m+1} \vee Y_2]_2$ for $m \geq n+1$ such that

$$\xi_m - \xi'_m = (\alpha_{m-1} \odot u)d_m + d_{m+1}(\alpha_m \odot u)$$

Here $u : \partial_X \to \partial_Y \in \mathbf{Coef}$ is the map given by $u = \{\eta_1\} = \{\eta'_1\}$. The equation $\{\eta_1\} = \{\eta'_1\}$ is a consequence of (a).

(1.7) Lemma. *Given a $(n+1)$-system X in $\mathbf{TWIST}^c_{n+1}$ there is a cylinder object $\bar{I}X$ in $\mathbf{TWIST}^c_{n+1}$ such that the functor r_{n+1} satisfies*

$$r_{n+1}(\bar{I}X_{\geq 1}) = \bar{I}(X)$$

for $X = r_{n+1}(X_{\geq 1})$. Here $\bar{I}X_{\geq 1}$ is the cylinder object for 1-homotopies in (IV.1.3). Moreover the homotopy relation associated to the cylinder object $\bar{I}(X)$ coincides with the relation of 1-homotopy in (1.6).

Proof. We have the isomorphism of chain complexes

$$1 \odot p : C_*(\bar{I}X_{\geq 1}) \cong IC_*X_{\geq 1} = IA$$

defined in (V.2.4). Hence we can define for $X = (A, \partial_{n+1}, X^n)$ the cylinder object

$$\begin{aligned} &\bar{I}X = (IA, \bar{\partial}_{n+1}, (\bar{I}X^n)^n) \quad \text{with} \\ &\bar{\partial}_{n+1} \in [\Sigma^{n-1}A_{n+1} \vee \Sigma^{n-1}A_n \vee \Sigma^{n-1}A_{n+1}, X_n \cup IX_{n-1} \cup X_n] \\ &\bar{\partial}_{n+1} = (i_1\partial_{n+1}, w, i_2\partial_{n+1}) \end{aligned}$$

Here $w = w_f$ for $f = \partial_n$ as in (III.4.2). By (III.4.5) and (III.5.6) we see that ∂_{n+1} satisfies the cocycle condition. As in VII.2.6 Baues [AH] we see that 1-homotopies in (1.6) correspond to homotopies defined by $\bar{I}X$. q.e.d.

One can check that 1-homotopy is a natural equivalence relation on $\mathbf{TWIST}^c_{n+1}$ and by (1.7) the functors in (1.3) induce functors between homotopy categories

$$\begin{array}{ccc} \mathbf{Complex}/\overset{1}{\simeq} & \xrightarrow{r_{n+1}} & \mathbf{TWIST}^c_{n+1}/\overset{1}{\simeq} \\ {\scriptstyle C_*}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \mathbf{chain}/\simeq & \xleftarrow{K_n} & \mathbf{TWIST}^c_n/\overset{1}{\simeq} \end{array} \tag{1.8}$$

Given an n-system $X = (A, \partial_n, X^{n-1})$ in $\mathbf{TWIST}_n$ with $n \geq 2$ and a right $\mathbf{mod}(\partial_X)$-module M we obtain the *cohomology*

$$H^m(X; M) = H^m(K_n X; M) = H^m(A; M) \tag{1.9}$$

for $m \in \mathbb{Z}$. We shall use the right $\mathbf{mod}(\partial_X)$-modules $\Gamma_n X$ of an n-system X, $n \geq 2$, defined as follows. We choose a principal cofibration (X_n, X_{n-1}) with attaching map ∂_n so that X^{n-1} is the $(n-1)$-skeleton of the $\mathbf{T}$-complex $X^n = (X_1 \subset X_2 \subset \cdots \subset X_n)$. Then we set

$$\Gamma_n X = \Gamma_n(X^n) \tag{1.10}$$

where the right hand side is given by (V.5.3).

(1.11) Lemma. *Let $n \geq 2$. A map $f : X \to Y$ in $\mathbf{TWIST}_n^c/\overset{1}{\simeq}$ with $c(f) = u$ induces a well defined homomorphism of right $\mathbf{mod}(\partial_X)$-modules*

$$f_* = \Gamma_n(f) : \Gamma_n(X) \longrightarrow u^* \Gamma_n(Y).$$

That is $f_0 \overset{1}{\simeq} f_1$ implies $\Gamma_n(f_0) = \Gamma_n(f_1)$.

Proof. The result is clear for $n \geq 3$ by the definition of Γ_n in (V.5.3). For $n = 2$ we use the following argument. Let $\bar{I}X_{\geq 1}$ be the cylinder for 1-homotopies and let $i : X_{\geq 1} \to \bar{I}X_{\geq 1}$ be an inclusion. Then we obtain by the exact Γ-sequence the commutative diagram

$$\begin{array}{ccccccccc}
\pi_3 X_{\geq 1} & \longrightarrow & H_3 X_{\geq 1} & \longrightarrow & \Gamma_2 X_{\geq 1} & \longrightarrow & \pi_2 X_{\geq 1} & \longrightarrow & H_2 X_{\geq 1} \\
\downarrow{\scriptstyle i} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_3 \bar{I}X_{\geq 1} & \longrightarrow & H_3 \bar{I}X_{\geq 1} & \longrightarrow & \Gamma_2 \bar{I}X_{\geq 1} & \longrightarrow & \pi_2 \bar{I}X_{\geq 1} & \longrightarrow & H_2 \bar{I}X_{\geq 1}
\end{array}$$

Here the two arrows on the left hand side and also the two arrows on the right hand side are easily seen to be isomorphisms. Hence the 5-lemma shows that also the arrow in the middle is an isomorphism. q.e.d.

2 Obstructions for the Realizability of Chain Maps

We consider the following problem. Given $\mathbf{T}$-complexes $X_{\geq 1}$ and $Y_{\geq 1}$ and a u-equivariant chain map $\xi : C_* X_{\geq 1} \to C_* Y_{\geq 1}$ is there a map $f_{\geq 1} : X_{\geq 1} \to Y_{\geq 1}$ with $C_* f_{\geq 1} = \xi$? We call $f_{\geq 1}$ a *realization* of ξ. Using the categories $\mathbf{TWIST}_n^c$ we describe a sequence of obstructions for the realizability of ξ. In fact, if ξ is a map in the subcategory $\mathbf{TWIST}_1^c$ then we have seen in (II.3.2) that there is a map $\bar{\xi} : r_2 X_{\geq 1} \to r_2 Y_{\geq 1}$ in $\mathbf{TWIST}_2^c$ with $K_2(\bar{\xi}) = \xi$ if and only if an obstruction element

$$\mathcal{O}_1(\xi) \in H^2(X_{\geq 1}, u^*\Gamma_1 Y_{\geq 1}) \tag{2.1}$$

vanishes. We now obtain more generally the following result describing higher order obstructions for the realizability of ξ. For this we use the functors r_n, r_{n+1} and λ in (1.3).

(2.2) Theorem. *Assume* **C** *is a homological cofibration category under* **T**. *Let* $X_{\geq 1}, Y_{\geq 1}$ *be* **T***-complexes and let*

$$f : r_n X_{\geq 1} \longrightarrow r_n Y_{\geq 1}$$

be a map in $\mathbf{TWIST}_n^c$, $n \geq 1$. *Then there exists a map*

$$\bar{f} : r_{n+1} X_{\geq 1} \longrightarrow r_{n+1} Y_{\geq 1}$$

in $\mathbf{TWIST}_{n+1}^c$ *with* $\lambda(\bar{f}) = f$ *if and only if an obstruction element*

$$\mathcal{O}_n(f) \in H^{n+1}(X_{\geq 1}, u^*\Gamma_n Y_{\geq 1})$$

vanishes. Here $u = c(f) : \partial_X \to \partial_Y$ *is induced by* f.

For $n = 1$ this is a consequence of (II.3.2). For $n \geq 2$ we obtain (2.2) by (2.3) below. It is clear that theorem (2.2) yields a sequence of obstructions for the realizability of a chain map ξ. The obstruction $\mathcal{O}_n(\xi)$ is the subset $\{\mathcal{O}_n(f); K_n(f) = \xi\}$ of $H^{n+1}(X_{\geq 1}, u^*\Gamma_n(Y_{\geq 1}))$. If all $\mathcal{O}_n(\xi)$ are trivial, that is, if $0 \in \mathcal{O}_n(\xi)$ for all $n \geq 1$, then ξ is realizable.

(2.3) Obstruction theorem. *Let* **C** *be a homological cofibration category under* **T** *and let* X, Y *be objects in* $\mathbf{TWIST}_{n+1}^c$ *where* $n \geq 2$. *For* $f : \lambda X \to \lambda Y$ *in* $\mathbf{TWIST}_n^c$ *there exists* $\bar{f} : X \to Y$ *in* $\mathbf{TWIST}_{n+1}^c$ *with* $\lambda \bar{f} = f$ *if and only if an obstruction element*

$$\mathcal{O}_{X,Y}(f) \in H^{n+1}(X, u^*\Gamma_n(Y))$$

vanishes. Here $u = c(f) : \partial_X \to \partial_Y$ *is induced by* f.

Proof. We first choose for $f = (\xi, \eta^{n-1})$ a map $F : X^n \to Y^n$ in **Complex** extending $\eta^{n-1} : X^{n-1} \to Y^{n-1}$. We obtain

$$F : (X_n, X_{n-1}) \longrightarrow (Y_n, Y_{n-1}) \tag{1}$$

as a twisted map associated to $\bar{\xi}_n$ in (1.1) (6); compare (III.3.12). Then we know by (III.3.16) that the chain map C_*F coincides with ξ in degree $\leq n$. For the map F we obtain the following diagram (2) where $\bar{\xi}_{n+1} \equiv \xi_{n+1}$. This diagram corresponds to (1.1) (6) so that (ξ, F) is a map in $\mathbf{TWIST}_{n+1}^c$ if and only if (2) commutes in $\mathrm{Ho}(\mathbf{C})$.

$$\begin{array}{ccc} \Sigma^{n-1}A_{n+1} & \xrightarrow{\xi_{n+1}} & \Sigma^{n-1}B_{n+1} \vee Y_1 \\ {\scriptstyle \partial_{n+1}} \downarrow & & \downarrow {\scriptstyle (\partial_{n+1}, i)} \\ X_n & \xrightarrow{F} & Y_n \end{array} \tag{2}$$

This diagram, however, needs not to commute in Ho(**C**). Hence we obtain the *obstruction*

$$\mathcal{O}(F) = -(\partial_{n+1}, i)\bar{\xi}_{n+1} + F\partial_{n+1}. \tag{3}$$

This is actually an element in

$$\mathcal{O}(F) \in \Gamma_n(Y)(A_{n+1}) \subset [\Sigma^{n-1}A_{n+1}, Y_n]. \tag{4}$$

Moreover $\mathcal{O}(F)$ is a cocycle representing the cohomology class $\mathcal{O}_{X,Y}(f)$ in (2.3). By (V.5.5) we know that

$$\Gamma_n(Y)(B) = \text{kernel}(\triangledown) \tag{5}$$

where $\triangledown : \pi^B_{n-1}Y_n \to \pi^B_{n-1}(\Sigma^{n-1}B_n \vee Y_n)_2$. Hence (4) is a consequence of $\triangledown\mathcal{O}(F) = 0$ where we set $B = A_{n+1}$. We check this by

$$\begin{aligned}
\triangledown(F\partial_{n+1}) &= -i_2F\partial_{n+1} + (i_2 + i_1)F\partial_{n+1} \\
&= -i_2F\partial_{n+1} + (\triangledown(F), i_2F)(i_2 + i_1)\partial_{n+1} && (6)\\
&= (\triangledown(F), i_2F) \triangledown \partial_{n+1} \\
&= ((1 \vee i)_* E\bar{\xi}_n, i\eta_2)E^{n-2}d_{n+1} && (7)\\
&= (1 \vee i)_* E^{n-2}((\xi_n \odot u)d_{n+1}) \\
&= (1 \vee i)_* E^{n-2}((d_{n+1} \odot 1)\xi_{n+1}) && (8)\\
&= (\triangledown\partial_{n+1}, i)E^{n-2}\xi_{n+1} && (9)\\
&= \triangledown((\partial_{n+1}, i)\bar{\xi}_{n+1}) && (10)
\end{aligned}$$

For (6) we use (III.3.16). For (7) we use (III.3.16) and (V.2.3)(4). Moreover (8) holds since ξ is a chain map; see (1.1) (4). For (9) we use again (V.2.3) (4). Moreover (10) is a consequence of the fact that $E^{n-2}\xi_{n+1}$ is a partial suspension also for $n = 2$ by (V.1.1). Since $\triangledown$ is a homomorphism we see that (6) ... (10) imply $\triangledown\mathcal{O}(F) = 0$ and hence (4) holds.

Next we check that $\mathcal{O}(F)$ is a cocycle, that is

$$d^*_{n+2}\mathcal{O}(F) = 0 \tag{11}$$

Here $d^*_{n+2}\mathcal{O}(F) \in \Gamma_n(Y_{\geq 1})(A_{n+2})$ is represented by the composite

$$\Sigma^{n-1}A_{n+2} \xrightarrow{E^{n-2}d_{n+2}} \Sigma^{n-1}A_{n+1} \vee X_2 \xrightarrow{(\mathcal{O}(F), i\eta_2)} Y_n.$$

We consider the two summands of $\mathcal{O}(F)$ and get accordingly the composites (12) and (13).

$$(F\partial_{n+1}, i\eta_2)E^{n-2}d_{n+2} = F(\partial_{n+1}, i)\bar{d}_{n+2} = 0 \tag{12}$$

Here we use the cocyle condition (1.1) (7). On the other hand we get

$$\begin{aligned}
&((\partial_{n+1}, i)E^{n-2}\xi_{n+1}, i\eta_2)E^{n-2}d_{n+2} \\
&= (\partial_{n+1}, i)E^{n-2}((\xi_{n+1} \odot u)d_{n+2}) \\
&= (\partial_{n+1}, i)E^{n-2}(d_{n+2}(\xi_{n+2} \odot u)) \\
&= (\partial_{n+1}, i)E^{n-2}d_{n+2}(E^{n-2}\xi_{n+2}, i\eta_2) = 0
\end{aligned} \tag{13}$$

Here we again use the cocycle condition (1.1) (7). By (12) and (13) we get (11).

We can alter F in (2) by $\alpha \in [\Sigma^{n-1}A_n, Y_n]$ so that we obtain the map $F+\alpha \in$ **Complex** as in (III.3.4). The restriction of $F+\alpha$ to X_{n-1} coincides with η_{n-1} and we choose α such that $C_*F = C_*(F+\alpha) = \xi$. This is the case iff $\alpha \in \Gamma_n(Y_{\geq 1})(A_n)$ by (5) and (III.3.7); see (V.2.3) (4). We claim that

$$\mathcal{O}(F+\alpha) - \mathcal{O}(F) = (d_{n+1})^*(\alpha) \tag{14}$$

is a coboundary. In fact by (3) we have

$$\begin{aligned}
\mathcal{O}(F+\alpha) &= -(\partial_{n+1}, i)E^{n-2}\xi_{n+1} + (F+\alpha)\partial_{n+1} \\
&= -(\partial_{n+1}, i)E^{n-2}\xi_{n+1} + F\partial_{n+1} + (\alpha, F) \bigtriangledown \partial_{n+1} \\
&= \mathcal{O}(F) + (\alpha, \eta_2)E^{n-2}d_{n+1}.
\end{aligned}$$

Here we use (V.2.3) (4). We now are ready to prove (2.3). If $\bar{f}$ exists then $\mathcal{O}_{X,Y}(f)$ is trivial since for $F = \eta_n$ diagram (2) commutes in $\mathrm{Ho}(\mathbf{C})$ by the assumption on $\bar{f} = (\xi, \eta_n)$. On the other hand if $\mathcal{O}_{X,Y}(f) = 0$. Then by (14) there exists α such that $\mathcal{O}(F+\alpha) = 0$ and hence we can choose $\bar{f} = (\xi, F+\alpha)$ with $\lambda(\bar{f}) = f$. q.e.d.

In the following proposition we use the fact that the cohomology group in (2.3) actually depends only on λX and λY, that is

$$H^{n+1}(X, u^*\Gamma_n(Y)) = H^{n+1}(\lambda X, u^*\Gamma_n(\lambda Y)) \tag{2.4}$$

Hence the maps g_* and f^* are well defined.

(2.5) Proposition. *Let $n \geq 2$. The obstruction element in (2.3) has the derivation property. That is, for objects X, Y, Z in $\mathbf{TWIST}^c_{n+1}$ and maps $gf : \lambda X \to \lambda Y \to \lambda Z$ in $\mathbf{TWIST}^c_n$ we have the formula*

$$\mathcal{O}_{X,Z}(gf) = g_*\mathcal{O}_{X,Y}(f) + f^*\mathcal{O}_{Y,Z}(g).$$

This is an easy consequence of the definition of $\mathcal{O}(F)$ in (2.3) (3).

(2.6) Proposition. *Let $n \geq 2$. The obstruction element (2.3) depends only on the homotopy class of f in $\mathbf{TWIST}^c_n/\overset{1}{\simeq}$. That is for X, Y and f, $g : \lambda X \to \lambda Y$ with $f \overset{1}{\simeq} g$ we have*

$$\mathcal{O}_{X,Y}(f) = \mathcal{O}_{X,Y}(g).$$

Proof. Let $\bar{I}X$ be the cylinder object for X with $\lambda(\bar{I}X) = \bar{I}(\lambda X)$; see (1.7). A homotopy $H : f \overset{1}{\simeq} g$ is given by a map $H : \bar{I}(\lambda X) \to \lambda Y$ with $Hi_1 = f$ and $Hi_2 = g$. Hence we obtain by the derivation property

$$\mathcal{O}_{X,Y}(f) = i_1^* \mathcal{O}_{\bar{I}X,Y}(H) \tag{1}$$

$$\mathcal{O}_{X,Y}(g) = i_2^* \mathcal{O}_{\bar{I}X,Y}(H) \tag{2}$$

since $\mathcal{O}_{X,\bar{I}X}(i_1) = 0 = \mathcal{O}_{X,\bar{I}X}(i_2)$. Since the projection $p : \bar{I}X \to X$ induces an isomorphism in cohomology and since $pi_1 = 1 = pi_2$ we see that (1) and (2) imply the equation in (2.6). q.e.d.

The next result can be used for counting possible realizations of a chain complex.

(2.7) Proposition. *Let X be an object in* $\mathbf{TWIST}^c_{n+1}$, *$n \geq 1$, and let $\alpha \in H^{n+1}(X, \Gamma_n X)$. Then there exists an object Y in* $\mathbf{TWIST}^c_{n+1}$ *with $\lambda X = \lambda Y$ such that*

$$\mathcal{O}_{X,Y}(1) = \alpha.$$

Here 1 is the identity of $\lambda X = \lambda Y$ and $\mathcal{O}_{X,Y}$ is the obstruction operator in (2.3).

Proof. Let $X = (A, \partial_{n+1}, X^n)$ and let $\bar{\alpha} \in \Gamma_n(X)(A_{n+1})$ be a cocycle representing $-\alpha$ and assume $\bar{\alpha}$ is given by a map $\bar{\alpha} : \Sigma^{n-1} A_n \to X_n$. Then we obtain the object

$$Y = (A, \partial_{n+1} + \bar{\alpha}, X^n)$$

which is well defined in $\mathbf{TWIST}^c_{n+1}$. Using the definition of $\mathcal{O}_{X,Y}(1)$ in (2.3) one readily checks that $\mathcal{O}_{X,Y}(1) = \alpha$. q.e.d.

There is also a relative form of the obstruction theorem (2.3). For this we use the notion of subcomplex in a twisted homotopy system of order n in (1.1).

(2.8) Obstruction theorem (relative form). *Let $n \geq 2$, let X, Y be objects in* $\mathbf{TWIST}^c_{n+1}$ *and let $X' \subset X$ be a subcomplex. Let $f' : X' \to Y$ be a map in* $\mathbf{TWIST}^c_{n+1}$ *and let $f : \lambda X \to \lambda Y$ be a map in* $\mathbf{TWIST}^c_n$ *which extends $\lambda f'$ and which induces u in* **Coef**. *Then there exists a map $\bar{f} : X \to Y$ extending f' and satisfying $\lambda(\bar{f}) = f$ if and only if an obstruction*

$$\mathcal{O}_{X,Y}(f, f') \in H^{n+1}(X, X'; u^* \Gamma_n Y)$$

vanishes.

The proof of (2.8) is an easy modification of the proof of (2.3).

3 The Homotopy Lifting Property of the Chain Functor

Let **C** be a homological cofibration category under **T**. Then we show that the chain functor

$$C_* : \mathbf{Complex} \longrightarrow \mathbf{chain}$$

defined in (V.2.3) has the following *homotopy lifting property.*

(3.1) Theorem. *Let $\bar{f} : X_{\geq 1} \to Y_{\geq 1}$ be a map in* **Complex** *with $f = C_* \bar{f}$ and let $\alpha_{\geq 1} : f \simeq g$ be a homotopy in* **chain**. *Then there exists a 1-homotopy $H : \bar{f} \overset{1}{\simeq} \bar{g}$ satisfying*

$$\begin{cases} C_*(\bar{g}) = g & \text{and} \\ C_*(H) = \alpha_{\geq 1} \end{cases}$$

Here we use (2.4) so that H and $\alpha_{\geq 1}$ can be considered to be maps in **Complex** and **chain** respectively. The theorem describes a homotopy lifting property analogous to the "homotopy lifting property" of a Hurewicz fibration in topology. In lemma (II.4.6) we have seen that the functor

$$K_2 : \mathbf{TWIST}_2^c \longrightarrow \mathbf{chain}$$

has the homotopy lifting property. We now show that also for $n \geq 2$ the functor

$$\lambda : \mathbf{TWIST}_{n+1}^c \longrightarrow \mathbf{TWIST}_n^c$$

has the homotopy lifting property in the following lemma.

(3.2) Lemma. *Let $\bar{f} : X \to Y$ be a map in* $\mathbf{TWIST}_{n+1}^c$ *and let $H : f \overset{1}{\simeq} g$ be a 1-homotopy in* $\mathbf{TWIST}_n^c$ *with $f = \lambda \bar{f}$. Then there exists a 1-homotopy $\bar{H} : \bar{f} \overset{1}{\simeq} \bar{g}$ with*

$$\begin{cases} \lambda \bar{g} = g & \text{and} \\ \lambda \bar{H} = H \end{cases}$$

Proof. Let $\bar{I}X$ be the cylinder of X in $\mathbf{TWIST}_{n+1}^c$. Then $i_0 : X \to \bar{I}X$ is a subcomplex and we can use the relative obstruction theorem (2.8) which yields the obstruction

$$\mathcal{O}_{\bar{I}X,Y}(H, \bar{f}) \in H^{n+1}(\bar{I}X, X; u^* \Gamma_n Y) = 0$$

Here the cohomology group vanishes. Hence (3.2) is a consequence of (2.8). q.e.d.

Proof of (3.1). We construct inductively maps $\bar{g}_n \in \mathbf{TWIST}_n^c$ and homotopies $H_n : r_n \bar{f} \overset{1}{\simeq} g_n$ with $K_n g_n = g$ and $K_n H_n = \alpha_{\geq 1}$. For $n = 2$ we use (II.4.6) and we use (3.2) for $n \geq 3$. The sequence of maps H_n, g_n defines $\bar{H}, \bar{g}$ in (3.1). q.e.d.

4 Counting Realization of Chain Maps

Let **C** be a homological cofibration category under **T**. We can realize a chain map inductively by use of the tower of categories in (1.5). Obstructions for this realization are described in (2.3). We now describe the number of realizations obtained in each step; see theorem (4.2).

Let X, Y be objects in $\mathbf{TWIST}^c_{n+1}$ with $n \geq 2$ and let $u : \partial_X \to \partial_Y$ be a morphism in **Coef**. Then $[X, Y]_u$ denotes the set of all 1-homotopy classes of maps $f : X \to Y$ in $\mathbf{TWIST}^c_{n+1}$ inducing $c(f) = u$. The functor λ yields the function

$$[X, Y]_u \xrightarrow{\lambda} [\lambda X, \lambda Y]_u. \tag{4.1}$$

The image of this function is the kernel of the obstruction operator $\mathcal{O}_{X,Y}$ in (2.3). In the next theorem we consider inverse images $\lambda^{-1}\lambda(f)$ which are subsets of $[X, Y]_u$.

(4.2) Theorem. *There is an action of the group $H^n(X, u^*\Gamma_n Y)$ on the set $[X, Y]_u$ such that the orbits of this action coincide with the subsets $\lambda^{-1}\lambda(f)$.*

We denote the action of $\alpha \in H^n(X, u^*\Gamma_n Y)$ on $f \in [X, Y]_u$ by $f + \alpha$. Then (4.2) shows that maps $f, g \in [X, Y]_u$ satisfy $\lambda(f) = \lambda(g)$ if and only if there exists α with $g = f + \alpha$. Hence we obtain the *exact sequence* of sets

$$H^n(X, u^*\Gamma_n Y) \xrightarrow{+} [X, Y]_u \xrightarrow{\lambda} [\lambda X, \lambda Y]_u \xrightarrow{\mathcal{O}_{X,Y}} H^{n+1}(X, u^*\Gamma_n Y) \tag{4.3}$$

Here the arrow $\xrightarrow{+}$ denotes the action in (4.2) and $\mathcal{O}_{X,Y}$ is the obstruction operator in (2.3). We define the action in (4.2) as follows.

(4.4) Definition. Let $X = (A, \partial_{n+1}, X^n)$ and $Y = (B, \partial_{n+1}, Y^n)$ and let

$$f = (\xi, \eta) : X \to Y \tag{1}$$

be a map in $\mathbf{TWIST}^c_{n+1}$ inducing $u = \{\eta_1\}$. Here $\eta : X^n \to Y^n$ is a filtered map in **Complex**. Given $\alpha \in H^n(X, u^*\Gamma_n Y)$ we choose a cocycle

$$\bar{\alpha} \in \Gamma_n(Y)(A_n) \tag{2}$$

which is represented by a map $\bar{\alpha} : \Sigma^{n-1} A_n \to Y_n$. Using the action in (III.3.4) we can alter η by $\bar{\alpha}$ so that we obtain a map $\eta + \bar{\alpha} : X^n \to Y^n$ in **Complex** extending $\eta \mid X^{n-1}$. Then $f + \bar{\alpha} = (\xi, \eta + \bar{\alpha})$ is a map in $\mathbf{TWIST}^c_{n+1}$ and for the 1-homotopy class $\{f\} \in [X, Y]_u$ we define the action in (4.2) by

$$\{f\} + \alpha = \{f + \bar{\alpha}\} \tag{3}$$

Proof of (4.2). We first check that $(\xi, \eta + \bar{\alpha})$ is a well defined map in $\mathbf{TWIST}^c_{n+1}$. Since $n \geq 2$ and since $(\eta + \bar{\alpha}) \mid X_{n-1} = \eta \mid X_{n-1}$ we see that $\eta + \bar{\alpha}$ induces $u = \{\eta_1\}$ in **Coef** and that

$$\lambda(\xi, \eta + \bar{\alpha}) = \lambda(\xi, \eta) \tag{4}$$

We show that

$$C_*(\eta + \alpha) = C_*(\eta) = \xi \quad \text{in degree} \leq n \tag{5}$$

and that for $\bar{\xi}_{n+1}$ in (1.1) (6) we have

$$(\eta_n + \bar{\alpha})\partial_{n+1} = (\partial_{n+1}, i)\bar{\xi}_{n+1} \quad \text{in Ho}(\mathbf{C}). \tag{6}$$

Then (2) and (3) imply that $(\xi, \eta + \bar{\alpha})$ is a well defined map in $\mathbf{TWIST}^c_{n+1}$. Now (5) is clear in degree $\leq n-1$ and in degree n we have $C_*(\eta + \alpha) = C_*\eta$ by (2.3) (5), (III.3.4) (2); compare (V.2.3) (4). This argument was already used in (2.3) (14). Next we have

$$(\eta_n + \bar{\alpha})\partial_{n+1} = \eta_n\partial_{n+1} + (\bar{\alpha}, \eta_n) \bigtriangledown \partial_{n+1}.$$

Here $(\bar{\alpha}, \eta_n) \bigtriangledown \partial_{n+1}$ is trivial since $\bar{\alpha}$ is a cocycle. Hence (6) holds since (1.1) (6) is satisfied for (ξ, η). This completes the proof of (5) and (6). A homotopy $H : f \overset{1}{\simeq} g$ yields a homotopy $f + \bar{\alpha} \simeq g + \bar{\alpha}$ since $+$ is defined by the coaction map μ and the cylinder of a sum is a sum of cylinders.

Now assume that the cocycle $\bar{\alpha}$ is a coboundary. This is the case if and only if $\bar{\alpha}$ in (2) admits a factorization

$$\bar{\alpha} : \Sigma^{n-1}A_n \xrightarrow{\bar{d}_n} \Sigma^{n-1}A_{n-1} \vee Y_1 \xrightarrow{(\beta, i)} Y_n$$

where $\bar{d}_n \equiv d_n$ and $\beta \in \Gamma_n(Y)(A_{n-1})$. Using the argument in VII.2.12 (3) (4) (5) Baues [AH] we see that in this case $f + \bar{\alpha} \overset{1}{\simeq} f$. This completes the proof that the action in (3) above is well defined.

Finally we have to show

$$\lambda\{f\} = \lambda\{g\} \iff \exists \alpha \text{ with } \{g\} = \{f\} + \alpha. \tag{7}$$

The direction $\Leftarrow$ is clear by (4). Now assume that we have a homotopy $H : \lambda f \overset{1}{\simeq} \lambda g$. Then by the homotopy lifting property of λ in (3.2) we see that we obtain a 1-homotopy $f \overset{1}{\simeq} f'$ with $\lambda f' = \lambda g$. Hence (III.3.5) shows that there exists $\beta : \Sigma^{n-1}A_n \to Y_n$ with $\eta' + \beta \simeq \eta''$ rel X_{n-1} where $f' = (\xi', \eta')$ and $g = (\xi', \eta'')$. Hence $f' + \beta$ is a well defined map in $\mathbf{TWIST}^c_{n+1}$ which is 0-homotopic to g. This implies that $\beta \in \Gamma_n(Y)(A_n)$ and that β is a cocycle by arguments as in (5) and (6). Hence β represents α in (7). q.e.d.

(4.5) Proposition. *The action + in (4.4) has the linear distributivity law, that is for $f \in [X, Y]_u, g \in [Y, Z]_v$ we have*

$$(g + \beta)(f + \alpha) = gf + g_*\alpha + f^*\beta$$
$$f^* : H^{n+1}(Y, v^*\Gamma_n Z) \longrightarrow H^{n+1}(X, (vu)^*\Gamma_n Z)$$
$$g_* : H^{n+1}(X, u^*\Gamma_n Y) \longrightarrow H^{n+1}(X, (vu)^*\Gamma_n Z)$$

We point out that f^* and g_* in (4.5) actually depend only on λf and λg respectivly, that is $f^* = (\lambda f)^*$ and $g_* = (\lambda g)_*$ by (2.4).

Proof. This is a consequence of V.3.4 Baues [AH] since all maps $\eta : (X_n, X_{n-1}) \to (Y_n, Y_{n-1})$ are twisted. q.e.d.

5 Linear Extensions and Towers of Categories

The results on the obstruction operator in § 2 and the action in § 4 lead to certain concepts in category theory like "linear extensions of categories" and "exact sequence for a functor". In the next section we apply these concepts to the categories $\mathbf{TWIST}_n^c$ of twisted homotopy systems of order n.

The concept of exact sequences for groups is fundamental in algebraic topology. We can consider a group to be a category with a single object in which all morphisms are equivalences. Therefore we might try to find a more general notion of an exact sequence for categories and functors. In this section we introduce for a functor λ exact sequences of the form

$$D+ \longrightarrow \mathbf{A} \xrightarrow{\lambda} \mathbf{B} \xrightarrow{\mathcal{O}} H.$$

Here, however, D and H are not categories but "natural systems" of abelian groups on $\mathbf{B}$. Special exact sequences are the linear extensions of $\mathbf{B}$ by D denoted by

$$D+ \longrightarrow \mathbf{A} \xrightarrow{\lambda} \mathbf{B}$$

the equivalence classes of which are classified by the cohomology group $H^2(\mathbf{B}, D)$. See Baues [AH]. This fact generalizes the classical result on extensions of a group B by a B-module D which are classified by the cohomology $H^2(B, D)$.

Exact sequences for a functor λ and linear extensions arise frequently in algebraic topology and in many other fields of mathematics. In fact, once the reader learned about these concepts he shall recognize many examples himself and soon the usefulness and naturality of such notions will become apparent.

(5.1) Definition. Let $\mathbf{C}$ be a category. The *category of factorizations* in $\mathbf{C}$, denoted by $F\mathbf{C}$, is given as follows. Objects are morphisms f, g, ... in $\mathbf{C}$ and morphisms $f \to g$ are pairs (α, β) for which

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A' \\ B \arrow[u, "f"] & B' \arrow[l, "\beta"'] \arrow[u, "g"'] \end{tikzcd}$$

commutes in $\mathbf{C}$. Here $\alpha f \beta$ is a factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$.

A *natural system* (of abelian groups) on $\mathbf{C}$ is a functor $D : F\mathbf{C} \to \mathbf{Ab}$. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ to the induced homomorphism

$$D(\alpha,\beta) = \alpha_* \beta^* : D_f \longrightarrow D_{\alpha f \beta} = D_g.$$

Here we set $D(\alpha, 1) = \alpha_*, D(1, \beta) = \beta^*$.

We have a canonical forgetful functor $\pi : F\mathbf{C} \to \mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ so that each *bifunctor* $D : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$ yields a natural system $D\pi$, as well denoted by D. Such a bifunctor is also called a $\mathbf{C}$-*bimodule*.

We now describe examples of natural systems used below. For $m \in \mathbb{Z}$ and $n \geq 2$ we have a well defined natural system

$$H^m \Gamma_n : F(\mathbf{TWIST}_n^c / \overset{1}{\simeq}) \longrightarrow \mathbf{Ab} \tag{5.2}$$

which carries the morphism $f : X \to Y$ in $\mathbf{TWIST}_n^c / \overset{1}{\simeq}$ to the cohomology group

$$H^m \Gamma_n(f) = H^m(X, c(f)^* \Gamma_n Y).$$

Here $c : \mathbf{TWIST}_n^c / \overset{1}{\simeq} \to \mathbf{Coef}$ is the coefficient functor which coincides with cK_n. Compare (1.9) and (1.10).

(5.3) Definition. Let D be a natural system on $\mathbf{C}$. We say that

$$D+ \longrightarrow \mathbf{E} \xrightarrow{p} \mathbf{C}$$

is a *linear extension of the category* $\mathbf{C}$ by D if (a), (b) and (c) hold.

(a) $\mathbf{E}$ and $\mathbf{C}$ have the same objects and p is a full functor which is the identity on objects.
(b) For each $f : A \to B$ in $\mathbf{C}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\mathbf{E}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.
(c) The action suffices the *linear distributivity law*:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.$$

Two linear extensions $\mathbf{E}$ and $\mathbf{E}'$ are *equivalent* if there is an isomorphism of categories $\varepsilon : \mathbf{E} \cong \mathbf{E}'$ with $p'\varepsilon = p$ and with $\varepsilon(f_0 + \alpha) = \varepsilon(f_0) + \alpha$ for $f_0 \in \mathrm{Mor}(\mathbf{E})$, $\alpha \in D_{pf_0}$. The extension $\mathbf{E}$ is *split* if there is a functor $s : \mathbf{C} \to \mathbf{E}$ with $ps = 1$.

Let $\mathbf{C}$ be a small category and let $M(\mathbf{C}, D)$ be the set of equivalence classes of linear extensions of $\mathbf{C}$ by $\mathbf{D}$. Then there is a canonical bijection

$$\psi : M(\mathbf{C}, D) \cong H^2(\mathbf{C}, D) \tag{d}$$

which maps the split extension to the zero element, see IV § 5 in Baues [AH].

Next we consider exact sequences for functors which were introduced in IV.4.10 Baues [AH].

(5.4) Definition. Let $\lambda : \mathbf{A} \to \mathbf{B}$ be a functor and let D and H be natural systems of abelian groups on $\mathbf{B}$. We call the sequence

$$D \xrightarrow{+} \mathbf{A} \xrightarrow{\lambda} \mathbf{B} \xrightarrow{\mathcal{O}} H$$

an *exact sequence* for λif the following properties (a), ... , (e) are satisfied.

(a) For each morphism $f_0 : X \to Y$ in $\mathbf{A}$ the abelian group $D_f, f = \lambda f_0$, acts transitively on the set of morphisms $\lambda^{-1}(f) \subset \mathbf{A}(X,Y)$. Let $\mathbf{I}_{f_0} = \{\alpha \in D_f, f_0 + \alpha = f_0\}$ be the isotropy group.
(b) The linear distributivity law (5.3) (c) is satisfied.
(c) For all objects X, Y in $\mathbf{A}$ and for all morphisms $f : \lambda X \to \lambda Y$ in $\mathbf{B}$ an *obstruction element* $\mathcal{O}_{X,Y}(f) \in H(f)$ is given such that $\mathcal{O}_{X,Y}(f) = 0$ if and only if there is a morphism $f_0 : X \to Y$ with $\lambda f_0 = f$.
(d) $\mathcal{O}$ is a *derivation*, that is $\mathcal{O}_{X,Z}(gf) = g_*\mathcal{O}_{X,Y}(f) + f^*\mathcal{O}_{Y,Z}(g)$ for $f : \lambda X \to \lambda Y, g : \lambda Y \to \lambda Z$.
(e) For all objects X in $\mathbf{A}$ and for all $\alpha \in H(1_{\lambda X})$ there is an object Y in $\mathbf{A}$ with $\lambda Y = \lambda X$ and $\mathcal{O}_{X,Y}(1_{\lambda X}) = \alpha$, we write $X = Y + \alpha$ in this case.
(f) A *tower of categories* is a diagram ($i \in \mathbb{Z}$)

$$\begin{array}{ccccc} & & \downarrow & & \\ D_i & \longrightarrow & \mathbf{H}_i & \longrightarrow & \Gamma_{i+1} \\ & & \downarrow{\scriptstyle\lambda} & & \\ D_{i-1} & \longrightarrow & \mathbf{H}_{i-1} & \longrightarrow & \Gamma_i \\ & & \downarrow & & \end{array}$$

where $D_i \longrightarrow \mathbf{H}_i \longrightarrow \mathbf{H}_{i-1} \longrightarrow \Gamma_i$ is an exact sequence.

We say that D *acts on* λ if (a) above is satisfied. Moreover, D *acts linearly* on λ if (a) and (b) are satisfied. We say that D *acts effectively* if all isotropy groups in (a) are trivial. A linear extension as in (5.3) yields an exact sequence

$$D \xrightarrow{+} \mathbf{E} \longrightarrow \mathbf{C} \xrightarrow{\mathcal{O}} 0$$

where 0 is the trivial natural system. On the other hand each exact sequence as in (5.4) yields a linear extension of categories

$$D/I \xrightarrow{+} \mathbf{A} \longrightarrow \lambda\mathbf{A}. \tag{5.5}$$

Here $\lambda\mathbf{A}$ is the image category of $\lambda : \mathbf{A} \to \mathbf{B}$. Objects in $\lambda\mathbf{A}$ are the same as in $\mathbf{A}$ and morphisms $X \to Y$ in $\lambda\mathbf{A}$ are the maps $f : \lambda X \to \lambda Y$ in the image set $\lambda\mathbf{A}(X,Y)$. Clearly λ induces functors

$$\mathbf{A} \xrightarrow{\lambda} \lambda\mathbf{A} \xrightarrow{i} \mathbf{B}$$

where λ is full and where i is faithful. We say that λ is a *quotient functor* if i is an isomorphism of categories. The natural system D/I on $\lambda\mathbf{A}$ in (5.5) is given by $(D/I)(f) = D_f/I_{f_0}$, $f_0 \in \lambda^{-1}(f)$, see (5.4) (a).

An equivalence or isomorphism in a category $\mathbf{A}$ is written $f : A \cong B$. This is an automorphism of A if $A = B$. Such automorphisms form the group $\mathrm{Aut}_{\mathbf{A}}(A)$. For the groups of automorphisms in an exact sequence (5.4) we obtain the exact sequence of sets

$$D(1_{\lambda A}) \xrightarrow{1^+} \mathrm{Aut}_{\mathbf{A}}(A) \xrightarrow{\lambda} \mathrm{Aut}_{\mathbf{B}}(\lambda A) \xrightarrow{\bar{\mathcal{O}}} H(1_{\lambda A}). \tag{5.6}$$

Here λ is the homomorphism of groups induced by λ and 1^+ is the homomorphism of groups given by $1^+(\alpha) = 1_{\lambda A} + \alpha$. Moreover, the function $\bar{\mathcal{O}}$ is defined by $\bar{\mathcal{O}}(f) = (f^{-1})_*\mathcal{O}_{A,A}(f)$. In fact, $\bar{\mathcal{O}}$ is a derivation of groups with $\bar{\mathcal{O}}(fg) = \bar{\mathcal{O}}(f)^g + \bar{\mathcal{O}}(f)$. Here we set $x^g = g^*(g^{-1})_*(x)$ for $x \in H(1_{\lambda A})$. Compare (IV.4.11) Baues [AH].

(5.7) Lemma. *A functor λ in an exact sequence reflects equivalences.*

Proof. Let $\bar{f} : A \to B$ be a morphism in $\mathbf{A}$ such that $f = \lambda\bar{f} : \lambda A \to \lambda B$ is an equivalence in $\mathbf{B}$. Then we can choose $g : \lambda B \to \lambda A$ with $gf = 1$ and $fg = 1$. Now we get

$$f^*\mathcal{O}_{B,A}(g) = \mathcal{O}_{A,A}(gf) = \mathcal{O}_{A,A}(1) = 0.$$

Here f^* is injective since f is an equivalence. Hence $\mathcal{O}_{B,A}(g) = 0$ and therefore there exists a morphism $g' : B \to A$ with $\lambda(g') = g$. Moreover there exist α, β with $g'\bar{f} = 1_A + \alpha$ and $\bar{f}g' = 1_B + \beta$ since $\lambda(g'f) = 1$ and $\lambda(fg') = 1$ are the identity morphisms. We can alter g' by $\delta \in D(g)$ so that we get $g' + \delta$ satisfying

$$(g' + \delta)\bar{f} = g'\bar{f} + f^*\delta = 1_A + \alpha + f^*\delta.$$

Here f^* is surjective so that we can find δ_0 with $\alpha + f^*\delta_0 = 0$. Hence $\bar{g} = g' + \delta_0$ satisfies $\bar{g}\bar{f} = 1_A$. On the other hand we have

$$\bar{f}\bar{g} = \bar{f}(g' + \delta_0) = \bar{f}g' + f_*\delta_0 = 1_B + \beta + f_*\delta_0.$$

Hence we get for $\xi = \{\beta + f_*\delta_0\} \in (D/I)(1_{\lambda B})$ the equation $\bar{f}\bar{g} = 1_B + \xi$ and

$$\bar{f} = \bar{f}\bar{g}\bar{f} = (1_B + \xi)\bar{f} = \bar{f} + f^*\xi$$

where $f^* : (D/I)(1_{\lambda B}) \to (D/I)(f)$ is injective. Since $(D/I)(f)$ acts effectively we see that $f^*\xi = 0$ and hence $\xi = 0$. This implies $\bar{f}\bar{g} = 1_B$. q.e.d.

(5.8) Definition. Let $\lambda : \mathbf{A} \to \mathbf{B}$ be a functor. A *realization* of an object B in $\mathbf{B}$ is a pair (A, b) where A is an object in $\mathbf{A}$ and $b : \lambda A \cong B$ is an isomorphism in $\mathbf{B}$. We have an equivalence relation on such pairs by $(A, b) \sim (A', b')$ if there is an equivalence $g : A' \cong A$ in $\mathbf{A}$ with $\lambda(g) = b^{-1}b'$. The equivalence classes form

$$\mathrm{Real}_\lambda(B) = \big\{(A, b);\ b : \lambda A \cong B\big\}/\!\sim$$

Let $\{A, b\}$ be the equivalence class of (A, b). Let $\{A\}$ be the equivalence class of $A \in \mathbf{A}$. Then

$$\text{types}_\lambda(B) = \big\{\{A\};\ \exists b : \lambda A \cong B\big\}$$

We have a surjective function

$$\text{Real}_\lambda(B) \twoheadrightarrow \text{types}_\lambda(B)$$

which carries $\{A, b\}$ to $\{A\}$.

Now we consider again a functor λ which is part of an exact sequence (5.4). Then we obtain for $\{A, b\} \in \text{Real}_\lambda(B)$ the derivation

$$\bar{\mathcal{O}} = \bar{\mathcal{O}}\{A, b\} : \text{Aut}_B(B) \longrightarrow H(1_B) \tag{5.9}$$

similarly as in (5.6). Here $H(1_B\}$ is a right $\text{Aut}_B(B)$-module by $x^g = g^*(g^{-1})_*(x)$, $x \in H(1_B)$, $g \in \text{Aut}_B(B)$. The derivation $\bar{\mathcal{O}}$ is defined by the obstruction operator $\mathcal{O}$ in the exact sequence (5.4), namely

$$\bar{\mathcal{O}}(f) = (f^{-1}b)_*(b^{-1})^*\mathcal{O}_{A,A}(b^{-1}fb).$$

Let

$$\Delta_B = \{\bar{\mathcal{O}}\} \in H^1(\text{Aut}_B(B), H(1_B)) \tag{5.10}$$

be the cohomology class represented by $\bar{\mathcal{O}}$. This class does not depend on the choice of $\{A, b\} \in \text{Real}_\lambda(B)$, in fact, the set $\{\bar{\mathcal{O}}\{A, b\};\ \{A, b\} \in \text{Real}_\lambda(B)\}$ coincides with the full cohomology class Δ_B. This follows from (5.4) (e).

(5.11) Proposition. *Assume λ is a functor in an exact sequence (5.4). Let B be an object in $\mathbf{B}$ and let* $\text{Real}_\lambda(B)$ *be non empty. Then the group $H(1_B)$ acts transitively and effectively on* $\text{Real}_\lambda(B)$. *In particular* $\text{Real}_\lambda(B)$ *is a set. Moreover the cohomology class Δ_B determines the number of elements in the set* $\text{types}_\lambda(B)$. *In fact, let Δ be a derivation which represents Δ_B. Then there is a bijection*

$$\text{types}_\lambda(B) \approx H(1_B)/\!\sim$$

where the equivalence relation $\sim$ on $H(1_B)$ is defined by Δ, that is $\alpha \sim \beta$ if and only if there exists $f \in \text{Aut}_B(B)$ with $\Delta(f) = f^*(\beta) - f_*(\alpha)$.

The first part of (5.11) is proved in IV.4.12 Baues [AH] and the second part is proved in II.1.14 Baues [CH].

6 The Homological Tower of Categories

Let $\mathbf{C}$ be a homological cofibration category under $\mathbf{T}$. Then the results in § 2 and § 4 show that one has for $n \geq 2$ an exact sequence

$$H^n\Gamma_n \xrightarrow{+} \mathbf{TWIST}^c_{n+1}/\overset{1}{\simeq} \xrightarrow{\lambda} \mathbf{TWIST}^c_n/\overset{1}{\simeq} \xrightarrow{\mathcal{O}} H^{n+1}\Gamma_n. \tag{6.1}$$

Here $H^m\Gamma_n$ is the natural system in (5.2). Hence (1.8) shows that we get the following *homological tower of categories* approximating the homotopy category of **T**-complexes.

$$\begin{tikzcd}
 & \mathbf{Complex}/\overset{1}{\simeq} \arrow[d, "r"] & \\
 & \vdots & \\
H^n\Gamma_n \arrow[r, "+"] & \mathbf{TWIST}^c_{n+1}/\overset{1}{\simeq} \arrow[d, "\lambda"] & \\
 & \mathbf{TWIST}^c_n/\overset{1}{\simeq} \arrow[r, "\mathcal{O}"] \arrow[d, "\lambda"] & H^{n+1}\Gamma_n \\
 & \vdots & \\
H^2\Gamma_2 \arrow[r, "+"] & \mathbf{TWIST}^c_3/\overset{1}{\simeq} \arrow[d, "\lambda"] & \\
 & \mathbf{TWIST}^c_2/\overset{1}{\simeq} \arrow[r, "\mathcal{O}"] & H^3\Gamma_2
\end{tikzcd}$$

(6.2) Remark. Using (1.2) the twisted tower (6.1) has the prolongation given by

$$\mathbf{TWIST}^c_2/\overset{1}{\simeq} \xrightarrow{K} \mathbf{TWIST}^c_1/\simeq \xrightarrow{\mathcal{O}} H^2\Gamma_1.$$

Here $K = \lambda$ is the functor in (II.4.3) and $\mathcal{O}$ is the obstruction in (II.3.2). The pair $(K, \mathcal{O})$ has only partially the properties of an exact sequence by the results in (II.§ 3). In particular the action on K is more complicated by (II.4.11).

We can apply all results in section § 5 to the twisted tower in (6.1). For example by (5.7) we get

(6.3) Proposition. *The functor λ in (6.1) reflects equivalences. That is, a map $f : X \to Y$ in $\mathbf{TWIST}^c_{n+1}$ is a 1-homotopy equivalence if and only if the induced map $\lambda f : \lambda X \to \lambda Y$ is a 1-homotopy equivalence.*

Inductively we get by (6.2) and (II.5.3) the corollary:

(6.4) Corollary. *For $n \geq 2$ the functor*

$$K_n : \mathbf{TWIST}^c_n/\overset{1}{\simeq} \longrightarrow \mathbf{chain}/\simeq$$

reflects equivalences.

This corollary is the essential step for the proof of the "homological Whitehead theorem" in the next section.

Now let $X_{\geq 1}$ and $Y_{\geq 1}$ be **T**-complexes and let $[X_{\geq 1}, Y_{\geq 1}]_u$ be the set of 1-homotopy classes $f : X_{\geq 1} \to Y_{\geq 1}$ inducing $u = c(f) : \partial_X \to \partial_Y$ in **Coef**. Similarly let $[X_{\geq 1}, Y_{\geq 1}]_u^n$ be the set of 1-homotopy classes $f : r_n X_{\geq 1} \to r_n Y_{\geq 1}$ inducing $u = c(f)$. Then we obtain as in (4.3) the following diagram of exact sequences of sets.

$$\begin{array}{ccccc}
 & & [X_{\geq 1}, Y_{\geq 1}]_u & & \\
 & & \downarrow r & & \\
 & & \vdots & & \\
H^n(X_{\geq 1}, u^* \Gamma_n Y_{\geq 1}) & \xrightarrow{+} & [X_{\geq 1}, Y_{\geq 1}]_u^{n+1} & & \\
 & & \downarrow \lambda & & \\
 & & [X_{\geq 1}, Y_{\geq 1}]_u^n & \xrightarrow{\mathcal{O}} & H^{n+1}(X_{\geq 1}, u^* \Gamma_n Y_{\geq 1}) \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^3(X_{\geq 1}, u^* \Gamma_3 Y_{\geq 1}) & \xrightarrow{+} & [X_{\geq 1}, Y_{\geq 1}]_u^3 & & \\
 & & \downarrow \lambda & & \\
 & & [X_{\geq 1}, Y_{\geq 1}]_u^2 & \xrightarrow{\mathcal{O}} & H^3(X_{\geq 1}, u^* \Gamma_2 Y_{\geq 1})
\end{array}$$

Here we have kernel($\mathcal{O}$) = image(λ) and we have $\lambda(f) = \lambda(g)$ if and only if there exists α with $g = f + \alpha$. Moreover, for an N-dimensional complex $X = X^N$ the map

$$r_n : [X_{\geq 1}, Y_{\geq 1}]_u \longrightarrow [X_{\geq 1}, Y_{\geq 1}]_u^n \tag{6.6}$$

is bijective for $n = N + 1$ and surjective for $n = N$. This follows readily from the definitions.

Next we derive from the twisted tower a structure theorem for the *group* $\mathrm{Aut}(X_{\geq 1})$ *of 1-homotopy equivalences of* $X_{\geq 1}$ in **Complex**$/\overset{1}{\simeq}$. Let $E_n(X_{\geq 1})$ be the corresponding group of 1-homotopy equivalences of $r_n X_{\geq 1}$ in $\mathbf{TWIST}_n^c/\overset{1}{\simeq}$. Then we obtain the following diagram of exact sequences; compare (5.6).

$$
\begin{array}{ccccc}
 & & \mathrm{Aut}(X_{\geq 1}) & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^n(X_{\geq 1}, \Gamma_n X_{\geq 1}) & \xrightarrow{1^+} & E_{n+1}(X_{\geq 1}) & & \\
 & & \downarrow \lambda & & \\
 & & E_n(X_{\geq 1}) & \xrightarrow{\bar{\mathcal{O}}} & H^{n+1}(X_{\geq 1}, \Gamma_n X_{\geq 1}) \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^2(X_{\geq 1}, \Gamma_2 X_{\geq 1}) & \xrightarrow{1^+} & E_3(X_{\geq 1}) & & \\
 & & \downarrow \lambda & & \\
 & & E_2(X_{\geq 1}) & \xrightarrow{\bar{\mathcal{O}}} & H^3(X_{\geq 1}, \Gamma_2 X_{\geq 1})
\end{array}
\tag{6.7}
$$

Here 1^+ and λ are homomorphisms of groups and $\mathcal{O}$ is a derivation as in (5.6). Moreover $\mathrm{kernel}(\bar{\mathcal{O}}) = \mathrm{image}(\lambda)$ and $\mathrm{kernel}(\lambda) = \mathrm{image}(1^+)$. If $X_{\geq 1} = X^N$ is finite dimensional the homomorphism

$$r_n : \mathrm{Aut}(X_{\geq 1}) \longrightarrow E_n(X) \tag{6.8}$$

is an isomorphism for $n = N + 1$ and surjective for $n = N$.

The isotropy group of the actions in (6.5) and the kernel of 1^+ in (6.7) can be described by use of a spectral sequence; see Baues [AH].

7 The Homological Whitehead Theorem

In addition to the Whitehead theorems in chapter IV we obtain the following homological Whitehead theorem.

(7.1) Theorem. *Let* $\mathbf{C}$ *be a homological cofibration category under* $\mathbf{T}$ *and let* $f : X_{\geq 1} \to Y_{\geq 1}$ *be a map between* $\mathbf{T}$*-complexes. Then* f *is a 1-homotopy equivalence, i.e. an isomorphism in the category* $\mathbf{Complex}/\overset{1}{\simeq}$*, if and only if the induced chain map*

$$C_* f : C_*(X_{\geq 1}) \longrightarrow C_*(Y_{\geq 1})$$

is a homotopy equivalence in $\mathbf{chain}/\simeq$.

Using theorem (I.6.6) we derive from (7.1) the following two additional results.

(7.2) Addendum. *With the assumption in (7.1) the map f is a 1-homotopy equivalence if and only if the induced map $u = c(f)$ is an isomorphism in* **Coef** *and*

$$f_* : H_n(X_{\geq 1}) \longrightarrow u^* H_n(Y_{\geq 1})$$

is an isomorphism of right $\mathbf{mod}(\partial_X)$*-modules for $n \in \mathbb{Z}$.*

(7.3) Addendum. *Assume* $\mathbf{mod}(\partial_X)$ *is an additive subcategory of an abelian category* $\mathbf{M}$ *such that all objects of* $\mathbf{mod}(\partial_X)$ *are projective in* $\mathbf{M}$. *If the assumptions on f in (7.1) hold then f is a 1-homotopy equivalence if and only if the induced map $u = c(f)$ is an isomorphism in* **Coef** *and*

$$f^* : H^n(Y_{\geq 1}; N) \longrightarrow H^n(X_{\geq 1}, u^* N)$$

is an isomorphism of abelian groups for all right $\mathbf{mod}(\partial_X)$*-modules N of the form $N = \mathrm{Hom}_{\mathbf{M}}(-, M)$ where M is an object in* $\mathbf{M}$ *and $n \in \mathbb{Z}$.*

Proof of (7.1). If $C_* f$ is a homotopy equivalence in **chain** then (6.4) shows that $r_n f$ is a 1-homotopy equivalence in $\mathbf{TWIST}_n^c$ for all $n \geq 2$. If $X_{\geq 1}$ and $Y'_{\geq 1}$ are finite dimensional this implies that f is a 1-homotopy equivalence by (6.6). If $X_{\geq 1}$ or $Y_{\geq 1}$ are infinite dimensional then the 1-homotopy equivalence $r_n f$, $n \geq 2$, shows that f is a weak $\mathbf{T}$-equivalence in the sense of (IV.3.4). Then f is a 1-homotopy equivalence by the general Whitehead theorem (IV.3.11) since we assume that all $\mathbf{T}$-complexes are $\mathbf{T}$-good; see (V.1.1). Compare also (V.1.4). q.e.d.

8 The Model Lifting Property of the Twisted Chain Functor

Let Y be a simply connected topological space for which the total homology $H_2(Y) \oplus H_3(Y) \oplus \dots$ is a finitely generated abelian group. Then it is well known that there exists a finite CW-complex X and a weak equivalence $X \to Y$. The CW-complex X is termed a "finite model" of Y. Similarly "minimal models" play an important role in rational homotopy theory. We describe the underlying general concept of models as follows.

(8.1) Definition. Let $(\mathbf{A}, \simeq)$ and $(\mathbf{B}, \simeq)$ be categories with natural equivalence relations, $\simeq$, which we call homotopy. Moreover let $\lambda : \mathbf{A} \to \mathbf{B}$ be a functor which induces a functor $\lambda : \mathbf{A}/\simeq \to \mathbf{B}/\simeq$ between homotopy categories. Suppose we have an object Y in $\mathbf{A}$ and a morphism

$$\beta : B \to \lambda Y$$

in $\mathbf{B}$ which is a homotopy equivalence, that is, an isomorphism in $\mathbf{B}/\simeq$. Then β is termed a *model of* λY. We say that the functor λ has the *model lifting property* if for all Y in $\mathbf{A}$ and all models β of λY in $\mathbf{B}$ the following holds. There is a morphism

$$\alpha : A \to Y$$

in **A** which is a homotopy equivalence in **A** and there is an isomorphism $i : \lambda(A) \cong B$ in **B** with

$$\lambda(\alpha) = \beta i : \lambda(A) \cong B \longrightarrow \lambda Y.$$

Then we say that α is a model of Y which is a *lifting* of the model β of λY.

Now let **C** be a homological cofibration category under **T**. Then we have the chain functor

$$C_* : \mathbf{Complex}/\overset{1}{\simeq} \longrightarrow \mathbf{chain}/\overset{1}{\simeq} \tag{8.2}$$

defined in (V.2.3). Unfortunately in general C_* has not the model lifting property. But if we replace chain complexes by twisted chain complexes then liftings of models exist. This is an important advantage of the category of twisted chain complexes in chapter II. We therefore consider the *twisted chain functor*

$$r_2 : \mathbf{Complex}/\overset{1}{\simeq} \longrightarrow \mathbf{TWIST}_2^c(\mathbf{T})/\overset{1}{\simeq} \tag{8.3}$$

Here $\mathbf{TWIST}_2^c(\mathbf{T})$ is the category of twisted chain complexes in chapter II defined by **T** and $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$. As observed in (1.2) this category coincides with the 1-homotopy category $\mathbf{TWIST}_2^c/\overset{1}{\simeq}$ of the category of 2-systems in **C**. The twisted chain functor r_2 is defined as in (1.3).

(8.4) Theorem. *The twisted chain functor r_2 has the model lifting property.*

Theorem (8.4) is a consequence of the following result using the homological tower of categories.

(8.5) Proposition. *For $n \geq 2$ the functor*

$$\lambda : \mathbf{TWIST}_{n+1}^c/\overset{1}{\simeq} \longrightarrow \mathbf{TWIST}_n^c/\overset{1}{\simeq}$$

has the model lifting property.

Proof. Let $Y = (B, \partial'_{n+1}, Y^n)$ be an $(n+1)$-system with $\lambda Y = (B, \partial'_n, Y^{n-1})$ and let

$$G = (\xi, \eta) : X = (A, \partial_n, X^{n-1}) \longrightarrow \lambda Y \tag{1}$$

be a 1-homotopy equivalence in $\mathbf{TWIST}_n^c$ with $u = c(G) = \{\eta_1\}$. We construct an object $\bar{X} = (A, \partial_{n+1}, X^n)$ with $\lambda\bar{X} = X$ and a map

$$\bar{G} = (\xi, \bar{\eta}) : \bar{X} \to Y \quad \text{with } \lambda\bar{G} = G. \tag{2}$$

Then $\bar{G}$ is a 1-homotopy equivalence since λ in (8.5) reflects equivalences by (6.3). Hence $\bar{G}$ is a lifting of the model G. Since G is a 1-homotopy equivalence we see by (1.11) that G induces an isomorphism

$$G_* : \Gamma_n(X) \xrightarrow{\cong} \Gamma_n(\lambda Y) = \Gamma_n(Y). \tag{3}$$

For the construction of $\bar{X}$ and $\bar{G}$ we first choose a principal cofibration (X_n, X_{n-1}) with attaching map ∂_n in (1). Then $X^n = (X_1 \subset \cdots \subset X_n)$ is given by X^{n-1} in (1) and (X_n, X_{n-1}). Moreover since ∂_n in (1) satisfies the cocycle condition we can choose a functional suspension $\partial^\sharp_{n+1}$ of $\bar{d}_{n+1}$ in (1.1) (3), that is

$$\partial^\sharp_{n+1} \in E_{\partial_n}(\bar{d}_{n+1}) \subset [\Sigma^{n-1} A_{n+1}, X_n]. \tag{4}$$

Then we obtain the $(n+1)$-dimensional object

$$X^\sharp = (A_{\leq n+1}, \partial^\sharp_{n+1}, X^n) \tag{5}$$

which satisfies the cocycle condition since $A_{\leq n+1}$ is trivial in degree $n+2$. Moreover (ξ, η) in (1) yields a map

$$G' = (\xi_{\leq n+1}, \eta) : \lambda(X^\sharp) \longrightarrow \lambda(Y) \tag{6}$$

in $\mathbf{TWIST}^c_n$ so that the obstruction

$$-\alpha = \mathcal{O}_{X^\sharp, Y}(G') \in H^{n+1}(X^\sharp, u^* \Gamma_n Y) \tag{7}$$

is defined. The map (3) induces an isomorphism

$$G_* : H^{n+1}(X^\sharp, \Gamma_n X) \xrightarrow{\cong} H^{n+1}(X^\sharp, u^* \Gamma_n Y).$$

We now choose a cocycle $\bar{\alpha} \in \Gamma_n(X)(A_{n+1})$ representing the cohomology class $(G_*)^{-1}(\alpha)$. Here $\bar{\alpha}$ is given by a map $\bar{\alpha} : \Sigma^{n-1} A_{n+1} \to X_n$ and we can alter $\partial^\sharp_{n+1}$ by $\bar{\alpha}$ to obtain

$$\partial_{n+1} = \partial^\sharp_{n+1} + \alpha. \tag{8}$$

This yields the $(n+1)$-dimensional object

$$X^\alpha = (A_{\leq n+1}, \partial_{n+1}, X^n) \tag{9}$$

in $\mathbf{TWIST}^c_{n+1}$ with $\lambda(X^\alpha) = \lambda(X^\sharp)$ and by (2.3) (3) we see that

$$\mathcal{O}_{X^\alpha, Y}(G') = 0. \tag{10}$$

Hence (2.3) shows that there exists a map

$$G'' = (\xi_{\leq n}, \bar{\eta}) : X^\alpha \to Y \quad \text{with } \lambda(G'') = G'. \tag{11}$$

We now define $\bar{X}$ in (2) by

$$\bar{X} = (A, \partial_{n+1}, X^n) \tag{12}$$

so that X^α is the $(n+1)$-dimensional part of $\bar{X}$. We have to show that $\bar{X}$ satisfies the cocycle condition and that

$$\bar{G} = (\xi, \bar{\eta}) : \bar{X} \to Y \tag{13}$$

is a well defined map in $\mathbf{TWIST}^c_{n+1}$. Clearly $\bar{G}$ is an extension of G'' in (11) and therefore $\bar{G}$ satisfies the properties in (1.1) (5) (6) and hence is well defined. We therefore have the following commutative diagram in $\mathrm{Ho}(\mathbf{C})$.

$$\begin{array}{ccc}
\Sigma^{n-1}A_{n+2} \vee X_2 & \xrightarrow{(\bar{\xi}_{n+2},\bar{\eta}_2)} & \Sigma^{n-1}B_{n+2} \vee Y_2 \\
{\scriptstyle (\bar{d}_{n+2},1)}\downarrow & & \downarrow{\scriptstyle (\bar{d}'_{n+2},1)} \\
\Sigma^{n-1}A_{n+1} \vee X_2 & \xrightarrow{(\bar{\xi}_{n+2},\bar{\eta}_2)} & \Sigma^{n-1}B_{n+1} \vee Y_2 \\
{\scriptstyle (\partial_{n+1},i)}\downarrow & & \downarrow{\scriptstyle (\partial'_{n+1},i)} \\
X_n & \xrightarrow[\bar{\eta}_n]{} & Y_n
\end{array}$$

Here $\bar{\xi}_{n+2}$ and $\bar{\xi}_{n+1}$ are defined by $\bar{\xi}_{n+2} \equiv \xi_{n+2}$ and $\bar{\xi}_{n+1} \equiv \xi_{n+1}$. Therefore the top square of the diagram commutes since ξ is a chain map. Moreover the bottom square of the diagram commutes since G'' in (11) is a well defined map in $\mathbf{TWIST}^c_n$. By the cocycle condition for Y we know that on the right hand side of the diagram we have $(\partial'_{n+1}, i)\bar{d}'_{n+2} = 0$. This implies that

$$\bar{\eta}_n\beta = 0 \quad \text{with } \beta = (\partial_{n+1}, i)\bar{d}_{n+2}. \tag{14}$$

Here, however, we have $\bigtriangledown(\beta) = 0$ since A is a chain complex. Hence we know by (V.5.5) that

$$\beta \in \Gamma_n(X)(A_{n+2}). \tag{15}$$

The following diagram commutes where G_* is an isomorphism by (3).

$$\begin{array}{ccc}
\Gamma_n(X)(A_{n+2}) & \subset & [\Sigma^{n-1}A_{n+2}, X_n] \\
{\scriptstyle G_*}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle (\bar{\eta}_n)_*} \\
u^*\Gamma_n(X)(A_{n+2}) & \subset & [\Sigma^{n-1}A_{n+2}, Y_n]
\end{array}$$

Hence by (15) and (14) we see

$$G_*\beta = (\bar{\eta}_n)_*\beta = 0 \tag{16}$$

and therefore $\beta = 0$. This shows that $\bar{X}$ satisfies the cocycle condition and the proof of (8.5) is complete. q.e.d.

Proof of (8.4). Let $Y_{\geq 1}$ be a $\mathbf{T}$-complex and let

$$\alpha_2 : X \to r_2 Y_{\geq 1}$$

be a model in $\mathbf{TWIST}_2^c$. Then we can choose inductively models in $\mathbf{TWIST}_n^c$

$$\alpha_n : X_{(n)} \to r_n Y_{\geq 1}$$

with $n \geq 3$, $\lambda X_{(n)} = X_{(n-1)}$, $\lambda(\alpha_n) = \alpha_{n-1}$. Here we use the construction in the proof of (8.5). The sequence of objects $X_{(n)}$, $n \geq 2$ defines a $\mathbf{T}$-complex $X_{\geq 1}$ and the sequence of maps α_n defines a map

$$\beta : X_{\geq 1} \to Y_{\geq 1}$$

with $r_2(X_{\geq 1}) = X$ and $r_2(\beta) = \alpha_2$. Since α_2 is a 1-homotopy equivalence we see that $C_*\beta$ is a homotopy equivalence and hence β is a 1-homotopy equivalence by the homological Whitehead theorem (7.1). This shows that β is a lifting of the model α. q.e.d.

Remark. The model lifting property of the twisted chain functor in (8.4) is the ultimate generalization of theorem 17 in the classical paper on simple homotopy types of J.H.C. Whitehead [SH]. This theorem shows that the functor which carries a reduced CW-complex X to its crossed chain complex has the model lifting property. On the other hand Wall [FC] § 4 reproved Whitehead's result by showing that the functor which carries X to the "admissible chain complex $(C_*\tilde{X}, \partial_X)$" has the model lifting property. Our proof of the model lifting property does not rely on the proofs of Whitehead and Wall since it uses only properties of the homological tower of categories. Compare also VI.7.5 Baues [AH] where this result was proved for relative CW-complexes (X, D) under a path connected space D; this is a special case of (8.4) if one considers the category $\mathbf{C} = \mathbf{Top}^D$ of spaces under D.

9 Obstructions for the Realizability of Twisted Chain Complexes

We consider the following problem. Given a twisted chain complex $A \mid \partial_X$ is there a $\mathbf{T}$-complex $X_{\geq 1}$ such that $r_2(X_{\geq 1})$ and $A \mid \partial_X$ are isomorphic in $\mathbf{TWIST}_2^c$? Then $X_{\geq 1}$ is termed a *realization* of $A \mid \partial_X$. Using the categories $\mathbf{TWIST}_n^c$ we now describe a sequence of obstruction for the realizability of $A \mid \partial_X$.

(9.1) Theorem. *Assume* $\mathbf{C}$ *is a homological cofibration category under* $\mathbf{T}$. *Let* $n \geq 2$ *and let* X *be an* n*-system in* $\mathbf{TWIST}_n^c$. *Then there exists an* $(n+1)$*-system* $\bar{X}$ *in* $\mathbf{TWIST}_{n+1}^c$ *with* $\lambda\bar{X} \cong X$ *in* $\mathbf{TWIST}_n^c$ *if and only if an obstruction*

$$\mathcal{O}_n(X) \in H^{n+2}(X, \Gamma_n X)$$

vanishes.

Proof. For $X = (A, \partial_n, X^{n-1})$ we can choose a principal cofibration (X_n, X_{n-1}) with attaching map ∂_n. Moreover by the cocycle condition (1.1) (3) we can choose

$$\begin{aligned} \partial_{n+1} &\in [\Sigma^{n-1}A_{n+1}, X_n] \\ \partial_{n+1} &\in E_{\partial_n}((1 \vee i)\bar{d}_{n+1}) \end{aligned} \tag{1}$$

Here ∂_{n+1} is a functional suspension of $(1 \vee i)\bar{d}_{n+1}$ in (1.1) (3). The element ∂_{n+1} yields

$$\mathcal{O}(X) = (\partial_{n+1}, i)\bar{d}_{n+2} \in [\Sigma^{n-1}A_{n+2}, X_n] \tag{2}$$

where $\bar{d}_{n+2} \equiv d_{n+2}$. We have $\mathcal{O}(X) = 0$ if and only if ∂_{n+1} satisfies the cocycle condition. Hence for $\mathcal{O}(X) = 0$ the object $\bar{X}^0 = (A, \partial_{n+1}, X^n)$ is well defined in $\mathbf{TWIST}^c_{n+1}$ and satisfies $\lambda(\bar{X}^0) = X$. We observe that (1) and (2) imply $\nabla\mathcal{O}(X) = 0$. Therefore by (V.5.5) we see

$$\mathcal{O}(X) \in \Gamma_n(X)(A_{n+2}). \tag{3}$$

Moreover $\mathcal{O}(X)$ is a cocycle since $\bar{d}_{n+2} \equiv d_{n+2}$. Therefore $\mathcal{O}(X)$ represents the cohomology class $\mathcal{O}_n(X)$ in (9.1). Now assume $\mathcal{O}_n(X) = 0$. Then $\mathcal{O}(X)$ is a coboundary. Hence there is $\delta \in \Gamma_n(X)(A_{n+1})$ such that

$$d^*_{n+2} : \Gamma_n(X)(A_{n+1}) \longrightarrow \Gamma_n(X)(A_{n+2}) \tag{4}$$

satisfies $d^*_{n+2}(\delta) = -\mathcal{O}(X)$. Now δ is represented by a map

$$\delta : \Sigma^{n-1}A_{n+1} \longrightarrow X_n$$

and we can alter ∂_{n+1} by δ. Since $\nabla\delta = 0$ we still have by (1) and (4)

$$\begin{gathered} \nabla(\partial_{n+1} + \delta) \equiv d_{n+1} \\ (\partial_{n+1} + \delta, i)\bar{d}_{n+2} = 0 \end{gathered}$$

Hence $\bar{X} = (A, \partial_{n+1} + \delta, X^n)$ is a well defined object in $\mathbf{TWIST}^c_{n+1}$ satisfying $\lambda(\bar{X}) = X$. q.e.d.

(9.2) Proposition. *Let $f : X \to Y$ be a map in $\mathbf{TWIST}^c_n$. Then we have the formula*

$$f_*\mathcal{O}_n(X) = f^*\mathcal{O}_n(Y) \in H^{n+2}(X, u^*\Gamma_n Y)$$

where $u = c(f) \in \mathbf{Coef}$ is induced by f.

Proof. Let

$$f = (\xi, \eta) : X = (A, \partial_n, X^{n-1}) \longrightarrow Y = (B, \partial_n, Y^{n-1})$$

Then we have the following diagram in $\mathrm{Ho}(\mathbf{C})$

$$\begin{tikzcd}
\Sigma^{n-1}A_{n+2} \vee X_2 \arrow[r, "(\bar{\xi}_{n+2},\eta_2)"] \arrow[d, "(\bar{d}_{n+2},1)"'] & \Sigma^{n-1}B_{n+2} \vee Y_2 \arrow[d, "(\bar{d}_{n+2},1)"] \\
\Sigma^{n-1}A_{n+1} \vee X_2 \arrow[r, "(\bar{\xi}_{n+1},\eta_2)"] \arrow[d, "(\partial_{n+1},i)"'] & \Sigma^{n-1}B_{n+1} \vee Y_2 \arrow[d, "(\partial_{n+1},i)"] \\
X_n \arrow[r, "\bar{\eta}"] & Y_n
\end{tikzcd}$$

Here the upper square commutes since ξ is a chain map. The map $\bar{\eta}$ is a twisted map associated to $\bar{\xi}_n$; for this we use (1.1) (6) with $(n+1)$ replaced by n. This implies that

$$\delta = \bar{\eta}\,\partial_{n+1} - (\partial_{n+1}, i)\bar{\xi}_{n+1}$$

satisfies $\bigtriangledown(\delta) = 0$. Hence we get

$$\begin{aligned}
\eta_*\mathcal{O}(X) &= \bar{\eta}(\partial_{n+1}, i)\bar{d}_{n+2} \\
&= ((\partial_{n+1}, i)\bar{\xi}_{n+1} + \delta, \eta_2)\bar{d}_{n+2} \\
\xi^*\mathcal{O}(Y) &= (\partial_{n+1}, i)(\bar{d}_{n+2}, 1)\bar{\xi}_{n+2} \\
&= (\partial_{n+1}, i)(\bar{\xi}_{n+1}, \eta_2)\bar{d}_{n+2}
\end{aligned}$$

This shows that $\eta_*\mathcal{O}(X)$ and $\xi^*\mathcal{O}(Y)$ differ only by the coboundary $d^*_{n+2}(\delta)$ and therefore (9.2) holds. q.e.d.

We now consider the functor

$$\lambda : \mathbf{TWIST}^c_{n+1}/\overset{1}{\simeq} \longrightarrow \mathbf{TWIST}^c_n/\overset{1}{\simeq} \tag{9.3}$$

Then we have by (5.8)

$$\mathrm{Real}_\lambda(X) = \big\{(\bar{X}, b);\; b : \lambda\bar{X} \overset{1}{\simeq} X\big\}/\sim$$

with $(\bar{X}, b) \sim (\bar{Y}, b')$ if there is $g : \bar{Y} \overset{1}{\simeq} \bar{X}$ with $b\lambda(g) \overset{1}{\simeq} b'$. Here b and g denote maps which are 1-homotopy equivalences.

(9.4) Proposition. *Let $n \geq 2$. Then* $\mathrm{Real}_\lambda(X)$ *is non empty if and only if $\mathcal{O}_n(X) = 0$. In this case $H^{n+1}(X, \Gamma_n X)$ acts transitively and effectively on the set* $\mathrm{Real}_\lambda(X)$.

Proof. The second part is a consequence of (5.11) and (6.1). If $\mathrm{Real}_\lambda(X)$ is non empty we have $b : \lambda\bar{X} \simeq X$ and the model lifting property (8.5) shows that there exists $\bar{\bar{X}}$ with $\lambda\bar{\bar{X}} = X$ so that $\mathcal{O}_n(X) = 0$ by (9.1). Conversely if $\mathcal{O}_n(X) = 0$ then there exists $\bar{X}$ with $\lambda\bar{X} = X$ by (9.1) and hence $\mathrm{Real}_\lambda(X)$ is not empty. q.e.d.

The results above yield an obstruction theory for the realization of a twisted chain complex $A \mid \partial_X$. The first obstruction is

$$\mathcal{O}_2(A \mid \partial_X) \in H^4(A, \partial_X; \Gamma_2(A \mid \partial_X)) \tag{9.5}$$

where $\Gamma_2(A \mid \partial_X)$ is defined by (1.10) and (1.2). If (9.5) is trivial we get the set $\mathrm{Real}_\lambda(A \mid \partial_X)$ by (9.4) and for each class $\{\bar{X}, b\}$ in this set we get

$$\mathcal{O}\{\bar{X}, b\} \in \mathbb{Z}/2 \tag{9.6}$$

with $\mathcal{O}\{\bar{X}, b\} = 0$ if $\mathcal{O}(\bar{X}) = 0$ and $\mathcal{O}\{\bar{X}, b\} = 1$ otherwise. The element (9.6) is well defined by (9.2). The collection of all elements (9.6) forms the secondary obstruction.

10 The Hurewicz Theorem

Let $\mathbf{C}$ be a homological cofibration category under $\mathbf{T}$ and let $X_{\geq 1}$ be a $\mathbf{T}$-complex with coefficient object $\partial_X \in \mathbf{Coef}$ given by the attaching map of the principal cofibration (X_2, X_1)

(10.1) Definition. Let 1_* be the initial object of $\mathbf{Coef}$ given by the identity $1_* : * \to *$ of $* \in \mathbf{T}$. We say that $X_{\geq 1}$ is 1-*connected* if the map

$$1_* \to \partial_X$$

is an isomorphism in $\mathbf{Coef}$. Moreover $X_{\geq 1}$ is n-*connected* with $n \geq 1$ if $X_{\geq 1}$ is 1-connected and if the right mod(∂_X)-module $\pi_i(X_{\geq 1})$ is trivial for $2 \leq i \leq n$.

(10.2) Lemma. *Let the complex $X_{\geq 1}$ be 1-connected. Then the right* $\mathbf{mod}(\partial_X)$-*module $H_1(X_{\geq 1})$ is trivial.*

Proof. Clearly $H_1(*) = 0$ where $*$ is the trivial $\mathbf{T}$-complex $Y_{\geq 1}$ with $Y_n = *$ for all n. Here we assume that $*$ is fibrant in $\mathbf{C}$. Moreover we have maps

$$\partial_X \xrightarrow{u} 1_* \xrightarrow{0} \partial_X \tag{1}$$

in $\mathbf{TWIST}$ which induce isomorphisms in $\mathbf{Coef}$. In particular $0u \sim 1$. The maps (1) induce via the functor K maps between chain complexes of dimension ≤ 2 such that

$$H_1(d_X) \xrightarrow{u_*} H_1(d_*) \xrightarrow{0} H_1(d_X) \tag{2}$$

is the identity of $H_1(d_X) = H_1(X_{\geq 1})$. Since $H_1(d_*) = H_1(*) = 0$ this implies $H_1(X_{\geq 1}) = 0$. q.e.d.

(10.3) Hurewicz theorem. *Assume that* $\mathbf{mod}(1_*)$ *has enough exact sequences as in (II.7.5). Let* $n \geq 2$. *Then a* $\mathbf{T}$*-complex* $X_{\geq 1}$ *is* n*-connected if and only if* $X_{\geq 1}$ *is 1-connected and the homology modules* $H_i(X_{\geq 1}) = 0$ *are trivial for* $2 \leq i \leq n$. *Moreover if* $X_{\geq 1}$ *is* n*-connected then the Hurewicz homomorphism*

$$h : \pi_i(X_{\geq 1}) \longrightarrow H_i(X_{\geq 1})$$

between right $\mathbf{mod}(1_*)$*-modules is an isomorphism for* $i = n$ *and is a surjection for* $i = n + 1$.

For the proof we shall use the following lemma on additive categories.

(10.4) Lemma. *Let* $\mathbf{A}$ *be an additive category with enough exact sequences as in (II.7.5). Let* D_* *be a chain complex in* $\mathbf{A}$ *which is bounded below such that for* $i \leq n$ *and all objects* A *in* $\mathbf{A}$ *the homology* $H_i(D_*, \mathrm{Hom}_A(A, -)) = 0$ *is trivial. Then there exists a chain complex* B_* *which is trivial in degree* $\leq n$ *and a homotopy equivalence* $B_* \to D_*$ *of chain complexes.*

Proof. We work on the I-category $\mathbf{chain}^+_{\mathbf{A}}$ in (III.C.2). We can find a map

$$f : D_* \to K_*$$

of chain complexes where K_* has trivial homology and hence is contractible. Here K_* in degree $\leq n+1$ coincides with D_* and f in degree $\leq n+1$ is the identity. We obtain the exact prolongation K_* of $D_{\leq n+1}$ since $\mathbf{A}$ has enough exact sequences. Now we desuspend f and obtain

$$g : D'_* \to K'_*$$

with $\Sigma D'_* = D_*, \Sigma K'_* = K_*$ and $\Sigma g = f$. Then the cofiber sequence yields the homotopy equivalence

$$C_g = K'_* \cup_g C(D'_*) \simeq \Sigma D'_* = D_*$$

since K'_* and $\Sigma K'_*$ are contractible. Since g in degree $\leq n$ is the identity we see that we have the cofibration

$$C(K'_{\leq n}) \subset C_g.$$

Here the cone $C(K'_{\leq n})$ is contractible. Hence we obtain the homotopy equivalence

$$B_* = C_g / C(K'_{\leq n}) \simeq C_g \simeq D_*$$

where clearly $B_{\leq n}$ is trivial. q.e.d.

Proof of (10.3). Let $X_{\geq 1}$ be n-connected, $n \geq 2$. Then ∂_X is isomorphic to 1_* in **Coef** and hence we get by the isomorphism $u : \partial_X \to \partial_*$ the isomorphism of categories $u_* : \mathbf{mod}(\partial_X) \to \mathbf{mod}(1_*)$ which maps the chain complex $C_*(X_{\geq 1}) = (A, \partial_X)$ to the chain complex $(A, 1_*) = A$ in $\mathbf{mod}(1_*)$. Assume now that (A, ∂_X)

and hence A have trivial homology in degree $\leq n-1$. This is true for $n=1$ by (10.2). Then we obtain by (10.4) a chain complex B in $\mathbf{mod}(1_*)$ which is trivial in degree $\leq n-1$ and a map $A \to B$ which is a homotopy equivalence of chain complexes. Now let $\partial_0 : B_2 \to *$ be given by the 0-map of the cogroup B_2. Then we have the isomorphism $\partial_0 \cong 1_*$ in **Coef** and hence we get the homotopy equivalence in **chain**

$$(B, \partial_0) \to (A, \partial_0) \to C_*(X_{\geq 1}) = (A, \partial_X) \tag{1}$$

Here (A, ∂_X) is the chain complex of an object $A \mid \partial_X$ in $\mathbf{TWIST}_2^c$. Moreover (B, ∂_0) is the chain complex of the object $B \mid \partial_0$ in $\mathbf{TWIST}_2^c$. Since E in (I.5.6) and j in (I.5.5) are surjective we see that $B \mid \partial_0$ satisfies the cocycle condition. Again since E and j in (I.5.6) and (I.5.5) are surjective we see that (1) is induced by a map

$$B \mid \partial_0 \longrightarrow A \mid \partial_X \quad \text{in } \mathbf{TWIST}_2^c \tag{2}$$

which is a twisted homotopy equivalence by (II.5.4). Hence we obtain by the model lifting property of r_2 a $\mathbf{T}$-complex $Y_{\geq 1}$ with $C_*(Y_{\geq 1}) = (B, \partial_0)$ and a 1-homotopy equivalence

$$Y_{\geq 1} \to X_{\geq 1}. \tag{3}$$

Here the $(n-1)$-skeleton of $Y_{\geq 1}$ is trivial and hence $\Gamma_i Y_{\geq 1} = 0$ for $1 \leq i \leq n$. Therefore the exact Γ-sequence (V.5.4) yields the result in (10.3). q.e.d.

11 Appendix: Eilenberg-Mac Lane Complexes and $(\mathbf{C}, \mathbf{T})$-Homology of Coefficient Objects

We describe the analogue of an Eilenberg-Mac Lane space $K(\pi, 1)$ in a cofibration category. This leads to the $(\mathbf{C}, \mathbf{T})$-homology of a coefficient object ∂_X which as specialization yields the Quillen homology of ∂_X.

Let $\mathbf{C}$ be a homological cofibration category under $\mathbf{T}$. Since $\mathbf{T}$-complexes $X_{\geq 1}$ are $\mathbf{T}$-good by the assumptions in (V.1.1) we know that the homotopy groups are given by

$$\begin{aligned} \pi_n(X_{\geq 1})(D) &= \operatorname{image}\{\pi_{n-1}^D(X_n) \to \pi_{n-1}^D(X_{n+1})\} \\ &= \pi_{n-1}^D(X_{n+1}) \end{aligned} \tag{11.1}$$

for $n \geq 1$. Here $\pi_n(X_{\geq 1})$ is a right $\mathbf{mod}(\partial_X)$-module.

(11.2) Definition. A *resolution* of $\partial_X \in \mathbf{Coef}$ in $\mathbf{C}$ is a $\mathbf{T}$-complex $X_{\geq 1}$ with coefficient object $c(X_{\geq 1}) = \partial_X$ satisfying

$$\pi_n(X_{\geq 1}) = 0 \quad \text{for } n > 1$$

We also call $X_{\geq 1} = K(\partial_X, 1)$ an *Eilenberg-Mac Lane complex for ∂_X*.

(11.3) Lemma. *Let K be a* **T***-complex with subcomplex $L \subset K$ and let $f : L \to K(\partial_X, 1)$ be a map in* **Complex** *and let $u : \partial_K \to \partial_X$ be a map in* **Coef** *with $u \mid \partial_L = c(f)$. Then there exists a map $g : K \to K(\partial_X, 1)$ with $g \mid L = f$ and $c(g) = u$.*

Proof. Let $X_{\geq 1} = K(\partial_X, 1)$. We know that K_2 is given by an attaching map $\partial_K = (\partial_L, \bar{\partial}) : B_2 \vee \bar{B}_2 \to L_1 \vee K_1$ so that $K_2 = L_2 \cup C\bar{B}_2$. Since u is ∂-compatible and $u \mid \partial_L = c(f)$ there exists a map $g_2 : K_2 \to X_2$ with $g_2 \mid L_2 = f \mid L_2$ and $c(g_2) = u$. Now we can apply the obstruction theorem (V.4.4) and we get inductively a map g with the properties in (11.3). q.e.d.

(11.4) Corollary. *Given $u : \partial_Y \to \partial_X$ in* **Coef** *there exists a map*

$$f : K(\partial_Y, 1) \longrightarrow K(\partial_X, 1)$$

with $c(f) = u$ and two such maps are 1-homotopic.

Proof. Use (11.3) and the cylinder $\bar{I}X_{\geq 1}$ in (IV.2.5). q.e.d.

(11.5) Corollary. *Let $K(\partial_X, 1)$ and $K'(\partial_X, 1)$ be both Eilenberg-Mac Lane complexes for ∂_X. Then there exists a 1-homotopy equivalence $f : K(\partial_X, 1) \to K'(\partial_X, 1)$ with $c(f) = 1$ which is unique up to 1-homotopy.*

Hence Eilenberg-Mac Lane complexes are unique up to canonical isomorphism in $\mathbf{Complex}/\overset{1}{\simeq}$. We now consider the existence of such resolutions.

(11.6) Definition. We say that an additive category $\mathbf{M}$ has *enough presentations* if for each right $\mathbf{M}$-module M there exists an object A in $\mathbf{M}$ together with a surjective map

$$\mathrm{Hom}_{\mathbf{M}}(-, A) \twoheadrightarrow M$$

of right $\mathbf{M}$-modules. We say that $\mathbf{mod} = \mathbf{mod}(\mathbf{C})$ has enough presentations if for all $\partial_X \in \mathbf{Coef}$ the category $\mathbf{mod}(\partial_X)$ has enough presentations. One readily checks that this implies that **mod** has enough exact sequences, see (II.A.5).

(11.7) Proposition. *Assume* **C** *is a homological cofibration category under* **T** *and suppose that* **mod** *has enough presentations. Then an Eilenberg-Mac Lane complex $K(\partial_X, 1)$ exists for each object ∂_X in* **Coef** *and choosing such complexes yields a functor*

$$K(-, 1) : \mathbf{Coef} \longrightarrow \mathbf{Complex}/\overset{1}{\simeq}$$

Proof. It suffices to check the existence of $K(\partial_X, 1)$. The functorial property is a consequence of (11.4). We construct $X_{\geq 1} = K(\partial_X, 1)$ inductively as follows. Let (X_2, X_1) be a principal cofibration with attaching map ∂_X. Then we get the $\mathbf{mod}(\partial_X)$-module $\pi_2(X_2)$ and by (11.6) we can choose a surjection

$$\varphi : \mathrm{Hom}(-, A_3 \vee \partial_X)_1 \twoheadrightarrow \pi_2(X_2) \tag{1}$$

of $\mathbf{mod}(\partial_X)$-modules so that for all D

$$\varphi_D : \mathrm{Hom}(D \vee \partial_X, A_3 \vee \partial_X)_1 \to \pi_2(X_2)(D) = \pi_1^D(X_2) \tag{2}$$

is surjective. For $D = A_3$ let

$$\partial_3 = \varphi_{A_3}(1) \in [\Sigma A_3, X_2]. \tag{3}$$

We choose a principal cofibration (X_3, X_2) with attaching map ∂_3. Then we have the commutative diagram

$$\begin{tikzcd}
\pi_2^D(X_3, X_2) \arrow[r, "\partial"] \arrow[d, equal] & \pi_1^D(X_2) \arrow[r] & \pi_1^D(X_{\geq 1}) \arrow[r] & 0 \\
\pi_1^D(A_3 \vee X_2)_2 \arrow[r, equal] & \mathrm{Hom}(D \vee \partial_X, A_3 \vee \partial_X)_1 \arrow[u, "\varphi_{A_3}"'] & &
\end{tikzcd}$$

Hence ∂ is surjective and since the row is exact we get $\pi_1^D(X_{\geq 1}) = 0$. Therefore $\pi_2(X_{\geq 1}) = 0$. Inductively we get this way the Eilenberg-Mac Lane complex $K(\partial_X, 1)$. q.e.d.

(11.8) Definition. Let $\mathbf{C}$ be a homological cofibration category under $\mathbf{T}$ and assume $\mathbf{mod}(\mathbf{C})$ has enough presentations. Then the $(\mathbf{C}, \mathbf{T})$ *-homology* of an object ∂_X in $\mathbf{Coef}$ is defined by

$$H_n^{\mathbf{C}}(\partial_X; N) = H_n(K(\partial_X, 1); N)$$
$$H_{\mathbf{C}}^n(\partial_X, M) = H^n(K(\partial_X, 1); M)$$

Here N is a left and M is a right $\mathbf{mod}(\partial_X)$-module. Moreover using (I.6.5) we obtain the right $\mathbf{mod}(\partial_X)$-module

$$H_n^{\mathbf{C}}(\partial_X) = H_n(K(\partial_X, 1))$$

and the left $\mathbf{mod}(\partial_X)$-module

$$H_{\mathbf{C}}^n(\partial_X) = H^n(K(\partial_X, 1)).$$

(11.9) Lemma. *There is a natural isomorphism of right* $\mathbf{mod}(\partial_X)$*-modules,* $n \geq 2$*,*

$$H_n^{\mathbf{C}}(\partial_X) \cong \Gamma_{n-1}(K(\partial_X, 1)).$$

This follows readily from Whitehead's exact Γ-sequence (V.5.4). We now compare the $(\mathbf{C}, \mathbf{T})$-homology with the twisted homology in (II.§ 8). For this we assume that $(\mathbf{C}, \mathbf{T})$ has the properties in (11.8) and that $\mathbf{T}$ has enough exact sequences.

(11.10) Proposition. *There is a* ∂_X*-equivariant map*

$$\tau : C_* K(\partial_X, 1) \longrightarrow KQ(\partial_X) \in \mathbf{chain}/\simeq$$

which is natural in $\partial_X \in \mathbf{Coef}$. *Hence we get natural transformations*

$$\tau_* : H_m^{\mathbf{C}}(\partial_X; N) \longrightarrow H_m^{\text{twist}}(\partial_X; N)$$
$$\tau^* : H_{\mathbf{C}}^m(\partial_X; N) \longrightarrow H^m_{\text{twist}}(\partial_X; N)$$

Here τ_* *and* τ^* *are isomorphisms in degree* ≤ 2 *and* τ_* *is surjective and* τ^* *is injective in degree 3.*

This is an analogue of proposition (II.8.9).

Proof. The property of $Q(\partial_X)$ in (II.8.4) yields a unique map

$$\bar{\tau} : r_2 K(\partial_X, 1) \longrightarrow Q(\partial_X) \in \mathbf{TWIST}_2^c/\overset{1}{\simeq}$$

with $c(\bar{\tau})$ = identity of ∂_X. Now let τ be the chain map induced by $\bar{\tau}$. The 3-dimensional part of $Q(\partial_X)$ is realizable by a T-complex Y^3. Now (11.3) shows that one gets a map $g : Y^3 \to K(\partial_X, 1)$ with $c(g)$ = identity of ∂_X. Then g induces the map $\bar{g} = r_2(g)$ such that

$$\bar{\tau}\bar{g} : r_2 Y^3 = Q(\partial_x)^3 \longrightarrow C_* K(\partial_X, 1) \to Q(\partial_X)$$

is a ∂_X-equivaraint map in $\mathbf{TWIST}_2^c$. By uniqueness in (II.8.4) this map is 1-homotopic to the inclusion. This implies the result. q.e.d.

Proposition (11.10) can be generalized by using resolutions in $\mathbf{TWIST}_n^c$, $n \geq 2$.

(11.11) Proposition. *Assume* $(\mathbf{C}, \mathbf{T})$ *has the properties in (11.8) and that* $\mathbf{T}$ *has enough exact sequences. Then one has for* $n \geq 2$ *the functor*

$$Q_n : \mathbf{Coef} \longrightarrow \mathbf{TWIST}_n^c/\overset{1}{\simeq}$$

which for $n = 2$ *coincides with* Q *in (II.§ 8). Moreover there are natural maps in* $\mathbf{chain}/\simeq$

$$C_* K(\partial_X, 1) \xrightarrow{\tau_{n+1}} K_{n+2} Q_{n+1}(\partial_X) \xrightarrow{\lambda} K_n Q_n(\partial_X)$$

with $\lambda \tau_{n+1} = \tau_n$.

Proof. Let $n \geq 3$. We define $Q_n(\partial_X) = (A, \partial_n, X^{n-1}) = X$ by choosing an n-system with the following properties termed a *resolution* in $\mathbf{TWIST}_n^c$. Let (X_n, X_{n-1}) be a principal cofibration with attaching map ∂_n so that the $\mathbf{T}$-complex X^n is defined. The properties in question are

$$\pi_j(X^n) = 0 \quad \text{for } 2 \leq j \leq n-1. \tag{1}$$

Moreover for all cogroups D in $\mathbf{T}$ the sequence

$$\pi_{n-2}^D(\Sigma^{n-2} A_{n+1} \vee X^2) \xrightarrow{(\bar{d}_{n+1}, 1)_*} \pi_{n-2}^D(\Sigma^{n-2} A_n \vee X^2) \xrightarrow{(\partial_n, i)_*} \pi_{n-2}^D(X_{n-1}) \tag{2}$$

is exact. Here $\bar{d}_{n+1} \equiv d_{n+1}$ is given by (A, ∂_X). Finally for $j \geq n+1$ the sequence

$$A_{j+1} \vee \partial_X \longrightarrow A_j \vee \partial_X \longrightarrow A_{j-1} \vee \partial_X \tag{3}$$

given by the chain complex (A, ∂_X) is exact in $\mathbf{mod}(\partial_X)$; see (II.7.1). The sequence (3) needs not to be exact in degree $j < n+1$. One readily checks with arguments as in the proof of (11.7) and (II.8.7) that such resolutions in $\mathbf{TWIST}_2^c$ exist and have the properties in (11.11). q.e.d.

Using $Q_n(\partial_X)$ we can define the *n-th order twisted homology* by the chain complex $K_n Q_n(\partial_X)$. Then one obtains a similar result as in (11.10) with 2 replaced by n.

Chapter VII: Finiteness Obstructions

We prove in this chapter a generalization of Wall's finiteness obstruction theorem. This theorem was originally formulated for CW-complexes. We show that such a result actually holds for **T**-complexes in any homological cofibration category. The basic ingredient in the proof are the model lifting property of the twisted chain functor in (VI.§ 8) and the translation of Ranicki's "instant finiteness obstructions" to the language of twisted chain complexes in § 3.

1 The Reduced Projective Class Group

The finiteness obstructions which we shall consider are elements in the reduced projective class group $\tilde{K}_0(\mathbf{R})$ of a small ringoid $\mathbf{R}$. We need the following notation.

Let $\mathbf{R}$ be a ringoid. Recall that a right $\mathbf{R}$-module M is an additive functor

$$M : \mathbf{R}^{\mathrm{op}} \to \mathbf{Ab}.$$

Let $\mathbf{Mod}(\mathbf{R})$ be the category of right $\mathbf{R}$-modules; morphisms are natural transformations. We have the Yoneda inclusion

$$\mathbf{R} \subset \mathbf{Mod}(\mathbf{R}) \tag{1.1}$$

which carries $A \in \mathbf{R}$ to Mor_A with $\mathrm{Mor}_A(B) = \mathbf{R}(B, A)$ for $B \in \mathbf{R}$. A module F is called *free* if it is a direct sum of such presented modules Mor_A. Moreover F is *finitely generated and free* if there exist objects $A_1, \dots, A_n$ in $\mathbf{R}$ with

$$F \cong \mathrm{Mor}_{A_1} \oplus \cdots \oplus \mathrm{Mor}_{A_n}.$$

Let $\mathbf{mod}(\mathbf{R})^\sharp$ be the full subcategory of $\mathbf{Mod}(\mathbf{R})$ consisting of finitely generated free objects. We call a module M *finitely generated projective* if there exists $F \in \mathbf{mod}(\mathbf{R})^\sharp$ together with an idempotent map $p : F \to F$ such that $M = \mathrm{image}\, p$, that is M is a direct summand of F. Here idempotent means that $pp = p$ holds. Let $\mathbf{proj}(\mathbf{R})$ be the full subcategory of $\mathbf{Mod}(\mathbf{R})$ consisting of finitely generated projective $\mathbf{R}$-modules. Then $\mathbf{mod}(\mathbf{R})^\sharp$ and $\mathbf{proj}(\mathbf{R})$ are additive categories.

For any small additive category $\mathbf{A}$ let $K_0(\mathbf{A})$ be the *isomorphism class group*, i.e. the abelian group with one generator [A] for each isomorphism class of objects $A \in \mathbf{A}$ with relations $[A] + [B] = [A \oplus B]$. This is just the Grothendieck group

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of $\mathbf{A}$ as defined by Bass [AK]. A typical element of $K_0(\mathbf{A})$ is a formal difference $[A]-[B]$ with

$$[A]-[B]=[A']-[B']$$

if and only if there exists an isomorphism in $\mathbf{A}$

$$A\oplus B'\oplus C\cong A'\oplus B\oplus C$$

for some object C in $\mathbf{A}$. For the ringoid $\mathbf{R}$ we obtain the induced map

$$K_0(\mathbf{mod}(\mathbf{R})^\sharp)\longrightarrow K_0(\mathbf{proj}(\mathbf{R})) \tag{1.2}$$

the cokernel of which is termed $\tilde{K}_0(\mathbf{R})$ the *reduced projective class group* of $\mathbf{R}$. Clearly an additive functor $\varphi:\mathbf{R}\to\mathbf{S}$ between ringoids induces $\varphi_*:\tilde{K}_0(\mathbf{R})\to\tilde{K}_0(\mathbf{S})$. Here φ_* carries an element in $K_0(\mathbf{R})$ represented by $p:F\to F$ to $\varphi p:\varphi F\to\varphi F$ where φF is readily seen to be an idempotent map in $\mathbf{mod}(\mathbf{S})^\sharp$. We point out that $\mathbf{proj}(\mathbf{R})$ coincides with the "idempotent completion" of $\mathbf{mod}(\mathbf{R})^\sharp$ in the sense of Ranicki [FO].

Let **Ringoids** be the category of small ringoids and additive functors. Then the reduced projective class group yields the functor

$$\tilde{K}_0:\mathbf{Ringoids}\to\mathbf{Ab} \tag{1.3}$$

which carries $\mathbf{R}$ to $\tilde{K}_0(\mathbf{R})$. Now let $\mathbf{T}$ be a theory of coactions and let $\mathbf{mod}$ be a category of modules for $\mathbf{T}$. For a set $\mathcal{A}$ of cogroups in $\mathbf{T}$ we obtain as in (I.5.11) the $\mathcal{A}$ *-enveloping functor*

$$U_{\mathcal{A}}:\mathbf{Coef}\to\mathbf{Ringoids} \tag{1.4}$$

which for $\mathcal{A}=\{A\}$ coincides with the enveloping functor U_A. The functor $U_{\mathcal{A}}$ carries $\partial_X\in\mathbf{Coef}$ to the full subcategory $U_{\mathcal{A}}(\partial_X)$ in $\mathbf{mod}(\partial_X)$ consisting of all objects $A\vee\partial_X$ with $A\in\mathcal{A}$. Since $\mathbf{mod}(\partial_X)$ is an additive category we see that $U_{\mathcal{A}}(\partial_X)$ is a ringoid. For $u:\partial_X\to\partial_Y$ in $\mathbf{Coef}$ the induced morphism

$$u_*:U_{\mathcal{A}}(\partial_X)\to U_{\mathcal{A}}(\partial_Y)$$

between ringoids carries $A\vee\partial_X$ to $A\vee\partial_Y$ and carries $\xi\odot 1\in\mathbf{mod}(B\vee\partial_X,A\vee\partial_X)_1$ with A, $B\in\mathcal{A}$ to $((1_A\vee u)\xi)\odot 1$. Using (1.3) and (1.4) we obtain the composite functor

$$\tilde{K}_0U_{\mathcal{A}}:\mathbf{Coef}\to\mathbf{Ab} \tag{1.5}$$

which will be used in the finiteness obstruction theorem.

2 The Finiteness Obstruction Theorem

We now describe the finiteness obstruction theorem for complexes in a cofibration category which yields as a specialization the classical result of Wall [FC] for reduced CW-complexes.

Let $\mathbf{C}$ be a homological cofibration category under the theory $\mathbf{T}$ of coactions and let $\mathcal{A}$ be a *set of cogroups* in $\mathbf{T}$.

(2.1) Definition. A cogroup A in $\mathbf{T}$ is *$\mathcal{A}$-finite* if there exist objects $A_1, \dots, A_n \in \mathcal{A}$ with $n < \infty$ such that $A = A_1 \vee \cdots \vee A_n$. Moreover an object $X \in \mathbf{T}$ which has a coaction $\mu : X \to X \vee X'$ is $\mathcal{A}$-finite if the cogroup X' associated to X by the coaction μ is $\mathcal{A}$-finite.

(2.2) Definition. A $\mathbf{T}$-complex

$$X_{\geq 1} = (X_{\geq 1}, A_{\geq 1}, \partial_{\geq 2})$$

as in (IV.2.2) is *$\mathcal{A}$-finite* if $X_{\geq 1}$ is finite dimensional and if all A_i, $i \geq 1$, are $\mathcal{A}$-finite.

(2.3) Definition. Let $Y_{\geq 1}$ be a $\mathbf{T}$-complex. A *domination* $(X_{\geq 1}, f, g, H)$ of $Y_{\geq 1}$ is a $\mathbf{T}$-complex $X_{\geq 1}$ together with maps

$$Y_{\geq 1} \xrightarrow{f} X_{\geq 1} \xrightarrow{g} Y_{\geq 1}$$

in **Complex** and a 1-homotopy $H : gf \overset{1}{\simeq} 1$. The domination has *dimension* $\leq n$ if $\dim(X_{\geq 1}) \leq n$ and the domination is $\mathcal{A}$ *-finite* if $X_{\geq 1}$ is $\mathcal{A}$-finite.

In the next result we use the functor $\tilde{K}_0 U_{\mathcal{A}}$ in (1.5).

(2.4) Theorem. *Let $Y_{\geq 1}$ be a $\mathbf{T}$-complex in $\mathbf{C}$ which admits an $\mathcal{A}$-finite domination. Then a finiteness obstruction*

$$[Y_{\geq 1}] = [C_*(Y_{\geq 1})] \in \tilde{K}_0(U_{\mathcal{A}}(\partial_Y))$$

is defined where ∂_Y is the attaching map of (Y_2, Y_1). Moreover $[Y_{\geq 1}] = 0$ if and only if there exists an $\mathcal{A}$-finite complex $X_{\geq 1}$ and a 1-homotopy equivalence $X_{\geq 1} \to Y_{\geq 1}$ in **Complex**$/\overset{1}{\simeq}$.

Addendum. *The finiteness obstruction depends only on the homotopy type of $Y_{\geq 1}$. More precisely let $Y_{\geq 1} \to Z_{\geq 1}$ be a 1-homotopy equivalence in* **Complex**$/\overset{1}{\simeq}$ *which induces the isomorphism $u : \partial_Y \to \partial_Z$ in* **Coef**. *Then $[Z_{\geq 1}]$ is defined and the equation $[Z_{\geq 1}] = u_*[Y_{\geq 1}]$ holds.*

We call (2.4) the "weak form" of the finiteness obstruction theorem. This is a consequence of the following "delicate form" which allows weaker assumptions and takes care of dimensions.

(2.5) Theorem on finiteness obstructions. *Let* $\mathbf{C}$ *be a homological cofibration category under* $\mathbf{T}$*. Let* $Y_{\geq 1}$ *be a* $\mathbf{T}$*-complex which admits an* $\mathcal{A}$*-finite domination of dimension* $\leq n$ *or more generally assume that* $Y_{\geq 1}$ *is a* $\mathbf{T}$*-complex for which the twisted chain complex* $r_2(Y_{\geq 1})$ *admits an* $\mathcal{A}$*-finite twisted domination of dimension* $\leq n$*; see § 3 below. Then a finiteness obstruction*

$$[Y_{\geq 1}] = [C_*(Y_{\geq 1})] \in \tilde{K}_0(U_{\mathcal{A}}(\partial_Y))$$

is defined where ∂_Y *is the attaching map of* (Y_2, Y_1)*. Moreover* $[Y_{\geq 1}] = 0$ *if and only if there exists an* $\mathcal{A}$*-finite* $\mathbf{T}$ *complex* $X_{\geq 1}$ *of dimension* $\leq \max(3,n)$ *and a 1-homotopy equivalence* $X_{\geq 1} \to Y_{\geq 1}$ *in* $\mathbf{Complex}/\overset{1}{\simeq}$*.*

We prove this result in § 4 below.

Remark. Wall's original result in [FC], [FCII] on finiteness obstructions for CW-complexes is a special case of (2.5). For this let $\mathbf{C}$ be the cofibration category $\mathbf{Top}^*$ of pointed spaces and let $\mathbf{T}$ be the theory of cogroups in $\mathrm{Ho}(\mathbf{C})$ consisting of one point unions of circles S^1. Moreover let $\mathcal{A} = \{S^1\}$ be the set which contains only the cogroup S^1. In this case ∂_X is given by the fundamental group $\pi_1(X)$ and the finiteness obstruction is an element in $\tilde{K}_0(\mathbb{Z}[\pi_1 X])$ where $\mathbb{Z}[\pi_1 X]$ is the group ring.

The range of applications of theorem (2.5) is remarkable since only the assumptions (V.1.1) on a homological cofibration category make this result available. Therefore we get the theorem in many topological and algebraic contexts. We discuss some of these contexts in the applications of the introductory chapters A, ..., D.

3 Finiteness Obstructions for Twisted Chain Complexes

All results in this section are available if a theory $\mathbf{T}$ of coactions is given. We here deal only with chain complexes and twisted chain complexes as defined in chapter I, II.

Let $\mathcal{A}$ be a set of cogroups in $\mathbf{T}$. Then $\mathcal{A}$-finite objects in $\mathbf{T}$ are defined as in (2.1).

(3.1) Definition. A chain complex (A, ∂_X) in **chain** or a twisted chain complex $A|\partial_X$ in $\mathbf{TWIST}_2^c$ is of *dimension* $\leq n$ if $A_i = *$ for $i > n$. Moreover (A, ∂_X), resp. $A|\partial_X$, is $\mathcal{A}$ *-finite* if A_i is $\mathcal{A}$-finite for all $i \geq 1$ and if there exists n with $A_i = *$ for $i > n$. Hence $\mathcal{A}$-finite implies finite dimensional.

(3.2) Definition. Let $B|\partial_Y$ be a twisted chain complex. A *twisted domination* $(D|\bar{\partial}_X, \bar{f}, \bar{g}, \bar{\alpha})$ of $B|\partial_Y$ is a twisted chain complex $D|\bar{\partial}_X$ together with twisted chain maps

$$B|\partial_Y \xrightarrow{\bar{f}} D|\bar{\partial}_X \xrightarrow{\bar{g}} B|\partial_Y$$

in $\mathbf{TWIST}_2^c$ and a twisted homotopy $\bar{\alpha} : \bar{g}\bar{f} \simeq 1$; see (II.4.1). The domination is $\mathcal{A}$ *-finite* if $D|\bar{\partial}_X$ is $\mathcal{A}$-finite. Moreover the domination has dimension $\leq n$ if $D|\bar{\partial}_X$ is of dimension $\leq n$.

(3.3) Theorem. *Let $B|\partial_Y$ be a twisted chain complex which admits an $\mathcal{A}$-finite twisted domination. Then a finiteness obstruction*

$$[B, \partial_Y] \in \tilde{K}_0(U_{\mathcal{A}}(\partial_Y))$$

is defined. Moreover $[B, \partial_Y] = 0$ if and only if there exists an $\mathcal{A}$-finite twisted chain complex $\bar{C}|\partial_X$ and a twisted homotopy equivalence $B|\partial_Y \simeq \bar{C}|\partial_X$ in $\mathbf{TWIST}_2^c/\simeq$.

Addendum. *The finiteness obstruction depends only on the homotopy type of $B|\partial_Y$ in $\mathbf{TWIST}_2^c/\simeq$. More precisely let $B|\partial_Y \simeq E|\partial_V$ be a twisted homotopy equivalence which is u-equivariant with $u : \partial_Y \cong \partial_V \in$ **Coef**. Then $[E, \partial_V]$ is defined and the equation $[E, \partial_V] = u_*[B, \partial_Y]$ holds where u_* is given by the functor (1.5).*

The proposition is a consequence of the following "delicate form" which also takes care of dimensions.

(3.4) Theorem. *Let $B|\partial_Y$ be a twisted chain complex which admits an $\mathcal{A}$-finite twisted domination of dimension $\leq n$. Then a finiteness obstruction*

$$[B, \partial_Y] \in \tilde{K}_0(U_{\mathcal{A}}(\partial_Y))$$

is defined. Moreover $[B, \partial_Y] = 0$ if and only if there exists an $\mathcal{A}$-finite twisted chain complex $\bar{C}|\partial_X$ of dimension $\leq \max(3, n)$ and a twisted homotopy equivalence $B|\partial_Y \simeq \bar{C}|\partial_X$ in $\mathbf{TWIST}_2^c/\simeq$.

According to (II.1.7) and (II.4.1) we fix the notation for the twisted domination

$$B|\partial_Y \xrightarrow{\bar{f}} D|\bar{\partial}_X \xrightarrow{\bar{g}} B|\partial_Y \tag{3.5}$$

$$\begin{cases} \bar{f} = (f_{\geq 1}, Ef'', f) \\ \bar{g} = (g_{\geq 1}, Eg'', g) \\ \bar{\alpha} = (\alpha_{\geq 1}, \alpha) : \bar{g}\bar{f} \simeq 1 \end{cases}$$

Here the composite

$$Y \xrightarrow{f} X \xrightarrow{g} Y \tag{1}$$

satisfies $1_Y = gf + (\partial_Y, 1)\alpha$ and for the composite

$$X \xrightarrow{g} Y \xrightarrow{f} X \tag{2}$$

there exists a unique $\varphi : X' \to X$ with $fg + \varphi = 1_X$; see the affine property in (I.1.11). Using $\bar{\partial}_X$ in (3.5) and φ we define

$$\partial_X = (\varphi, \bar{\partial}_X) : X' \vee X'' \longrightarrow X \tag{3}$$

where $X'' = D_2$ and $X' = D_1$.

(3.6) Lemma. *The objects ∂_Y and ∂_X above are isomorphic in* **Coef**. *The isomorphism $\partial_Y \cong \partial_X$ is given by the ∂-compatible map f, the inverse of this isomorphism is given by the ∂-compatible map g.*

Proof. We have to show that f and g are ∂-compatible. For this we use the fact that by (3.5) we have maps in **Twist**

$$(f'', f) : \partial_Y \to \bar{\partial}_X$$
$$(g'', g) : \bar{\partial}_X \to \partial_Y$$

Therefore we get the following maps in **Twist**

$$\begin{tikzcd} Y'' \arrow[r, "f''"] \arrow[d, "\partial_Y"'] & X'' \vee X \arrow[d, "(\bar{\partial}_X, 1)"] & \subset & (X' \vee X'') \vee X \arrow[d, "(\partial_X, 1)"] \\ Y \arrow[r, "f"] & X & = & X \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} X' \vee X'' \arrow[r, "(g', g'')"] \arrow[d, "(\varphi, \bar{\partial}_X) = \partial_X"'] & Y'' \vee Y \arrow[d, "(\partial_Y, 1)"] \\ X \arrow[r, "g"] & Y \end{tikzcd} \tag{2}$$

Here we define g' by $g' = -(-\alpha, 1_Y) \bigtriangledown g$. We show that (2) commutes. For this we have

$$\begin{aligned} g &= g(fg + \varphi) \\ &= gfg + g\varphi \\ &= (1_Y - (\partial_Y, 1)\alpha)g + g\varphi \\ &= g + (-(\partial_Y, 1)\alpha, 1_Y) \bigtriangledown g + g\varphi. \end{aligned}$$

This implies by the affineness property

$$\begin{aligned} g\varphi &= -(-(\partial_Y, 1)\alpha, 1_Y) \bigtriangledown g \\ &= (\partial_Y, 1)_*(-(-\alpha, 1_Y) \bigtriangledown g) = (\partial_Y, 1)_* g'. \end{aligned}$$

Hence (2) commutes. On the other hand the equations

$$\begin{aligned} fg &= 1_X - (\partial_X, 1)\beta \quad \text{with } \beta = i_{X'} \\ gf &= 1_Y - (\partial_Y, 1)\alpha \end{aligned}$$

show that in **Coef** we have $\{f\}\{g\} = 1$ and $\{g\}\{f\} = 1$. q.e.d.

(3.7) Lemma. *For $D_1 = X'$ and $D_2 = X''$ let*

$$d_2 : D_1 \vee D_2 \vee \partial_X \longrightarrow D_1 \vee \partial_X$$

be the ∂_X-equivariant map in **mod** *which is the differential associated to ∂_X; that is $d = E(\nabla\partial_X \odot 1)$. Then we have*

$$d_2|D_1 \vee \partial_X = 1 - f_1 g_1$$
$$d_2|D_2 \vee \partial_X = E(\nabla\bar{\partial}_X \odot 1)$$

Here f_1, g_1 are given by (3.5) and $E(\nabla\bar{\partial}_X \odot 1)$ is the differential in the chain complex $(D, \bar{\partial}_X)$.

Proof. Since $fg + \varphi = 1_X$ we have by (I.3.4)

$$i_{X'} = \nabla 1_X = \nabla(fg + \varphi) = -i_X\varphi + \nabla(fg) + i_X\varphi + \nabla\varphi. \tag{1}$$

Since $\varphi = (\bar{\partial}_X, 1)\beta, \beta = i_{X'}$, we see by (I.5.4) (1) that (1) implies

$$-E(\nabla(fg) \odot 1) + 1 = E(\nabla\varphi \odot 1) \tag{2}$$

where the left hand side coincides with $1 - f_1 g_1$. Hence (2) implies the equation for $d|D_1 \vee \partial_X$ in (3.7). q.e.d.

(3.8) Definition. We define the infinite dimensional twisted chain complex $C|\partial_X$ *associated* to the domination (3.5) as follows. Let ∂_X be given as in (3.5) (3). Moreover let

$$C_i = D_1 \vee D_2 \vee \cdots \vee D_i$$

be the sum of the objects $D_1, \ldots, D_i$ given by $D|\bar{\partial}_X$ in (3.5). The differential

$$d_i : C_i \vee \partial_X \longrightarrow C_{i-1} \vee \partial_X$$

in the chain complex $(C, \partial_X) = K(C|\partial_X)$ is given by the coordinates

$$d_i^{jk} : D_j \vee \partial_X \longrightarrow D_k \vee \partial_X$$

with $i \leq j \leq i$ and $1 \leq k \leq i-1$. These coordinates are the unique ∂_X-equivariant maps for which the following diagram commutes.

$$\begin{array}{ccc} D_j \vee \partial_X & \xrightarrow{d_i^{jk}} & D_k \vee \partial_X \\ {\scriptstyle 1\vee i}\uparrow & & \uparrow{\scriptstyle 1\vee i} \\ D_j \vee \bar{\partial}_X & \xrightarrow{\bar{d}_i^{jk}} & D_k \vee \bar{\partial}_X \end{array}$$

Here $\bar{\partial}_X \subset \partial_X$ is the inclusion; see (3.5) (3). Moreover using the notation in (3.5) we set:

$$\bar{d}_i^{jk} = \begin{cases} 0 & \text{if } j \geq k+2 \\ (-1)^{i+k+1}\bar{d}_{k+1} & \text{if } j = k+1 \\ 1 - f_j g_j & \text{if } j = k,\ j \equiv i+1 \pmod 2 \\ f_j g_j & \text{if } j = k,\ j \equiv i \pmod 2 \\ (-1)^{i+k} f_k \alpha^{k-j} g_j & \text{if } j \leq k-1 \end{cases}$$

Here $\bar{d}_k$ is the differential of $(D, \bar{\partial}_X)$ and α^{k-j} is the composite

$$B_j \vee \partial_Y \xrightarrow{\alpha_j} B_{j+1} \vee \partial_Y \xrightarrow{\alpha_{j+1}} \dots \xrightarrow{\alpha_{k-1}} B_k \vee \partial_Y$$

given by $\alpha_{\geq 1}$ in (3.5). The definition of $C|\partial_X$ is essentially due to Ranicki [FO]; compare Ranicki-Yamasaki [CK]. The refinement here is the fact that Ranicki's construction for chain complexes actually can be achieved in the category of twisted chain complexes as is shown in the next lemma. If $\dim(D|\partial_X) \leq n$ then $D_i = *$ for $i > n$ and hence $C_n = C_{n+1} = \dots$. Moreover we define in this case the ∂_X-equivariant map

$$p : C_n \vee \partial_X \to C_n \vee \partial_X \in \mathbf{mod}(\partial_X)$$

by $p = d_{n+1}$ for $n+1 \equiv 0 \pmod 2$ and $p = 1 - d_{n+1}$ for $n+1 \equiv 1 \pmod 2$. One can check that the map p is idempotent, that is $pp = p$; see Ranicki [FO].

(3.9) Lemma. *The twisted chain complex $C|\partial_X$ associated to a domination in (3.8) is well defined.*

Proof. We first observe that d_2 in (C, ∂_X) coincides with d_2 in (3.7) so that (3.7) shows that $d_2 = E(\bigtriangledown\partial_X \odot 1)$. See (II.1.7) (1). Hence it remains to check the cocycle condition in (II.1.8). We have to find

$$\partial_3 \in \mathbf{T}(C_3, C_2 \vee X)_2 \quad \text{with} \quad \begin{cases} C_2 = D_1 \vee D_2 = X' \vee X'' \\ C_3 = D_1 \vee D_2 \vee D_3 \end{cases} \tag{1}$$

with $(\partial_X, 1)\partial_3 = 0$ in $\mathbf{T}(C_3, X)$ and $d_3 = E(\partial_3 \odot 1)$. Here

$$d_3 : D_1 \vee D_2 \vee D_3 \vee \partial_X \longrightarrow D_1 \vee D_2 \vee \partial_X \tag{2}$$

is given by the coordinates d_3^{jk} with $1 \leq j \leq 3$ and $1 \leq k \leq 2$ satisfying (see (3.8))

$$\bar{d}_3^{11} = f_1 g_1 \tag{3}$$
$$\bar{d}_3^{12} = -f_2\,\alpha_1 g_1 \tag{4}$$
$$\bar{d}_3^{21} = -\bar{d}_2 \tag{5}$$
$$\bar{d}_3^{22} = 1 - f_2 g_2 \tag{6}$$
$$\bar{d}_3^{31} = 0 \tag{7}$$
$$\bar{d}_3^{32} = \bar{d}_3 \tag{8}$$

We define ∂_3 by coordinates

$$\partial_3^{jk} : D_j \longrightarrow D_k \vee X \tag{9}$$

which are trivial on X and which satisfy $\partial_3|D_j = \partial_3^{j1} + \partial_3^{j2}$ and

$$\bar{d}_3^{jk} = E(\partial_3^{jk} \odot 1). \tag{10}$$

Here we set

$$\partial_3^{11} = ((-1 \vee 1)(\nabla f)(-1), i_X f) \nabla g \tag{11}$$
$$\partial_3^{12} = -(f'', i_X f)(\alpha, i_Y) \nabla g \tag{12}$$
$$\partial_3^{21} = (-1 \vee 1) \nabla \bar{\partial}_X \tag{13}$$
$$\partial_3^{22} = -i_{X''} + (f'', i_X f) g'' \tag{14}$$
$$\partial_3^{31} = 0 \tag{15}$$
$$\partial_3^{32} = \bar{\partial}_3 \tag{16}$$

Here $\bar{\partial}_3$ is given by the cocycle condition for the twisted chain complex $D|\bar{\partial}_X$. One now readily checks that (10) holds and this implies $d_3 = E(\partial_3 \odot 1)$. It remains to check that $(\partial_X, 1)\partial_3 = 0$ with $\partial_X = (\varphi, \bar{\partial}_X)$. This, in fact, is a consequence of the following equations:

$$(\varphi, 1)\partial_3^{11} = -(\bar{\partial}_X, 1)\partial_3^{12} \tag{17}$$
$$(\varphi, 1)\partial_3^{21} = -(\bar{\partial}_X, 1)\partial_3^{22} \tag{18}$$
$$(\varphi, 1)\partial_3^{31} = 0 = -(\bar{\partial}_X, 1)\partial_3^{32} \tag{19}$$

Here (19) holds by the cocycle condition for $D|\bar{\partial}_X$. Moreover we prove (17) and (18) as follows.

Since $fg + \varphi = 1$ and $gf + (\partial_Y, 1)\alpha = 1$ we have

$$\begin{aligned} f &= (fg + \varphi)f = fgf + (\varphi, fg) \nabla f \quad \text{by (I.3.3)} \\ &= f(1 - (\partial_Y, 1)\alpha) + (\varphi, fg) \nabla f \\ &= f - f(\partial_Y, 1)\alpha + (\varphi, fg) \nabla f. \end{aligned} \tag{20}$$

Hence the affineness property yields

$$\begin{aligned} f(\partial_Y, 1) &= (\varphi, fg) \nabla f \\ &= (\varphi, 1 - \varphi) \nabla f \\ &= -(-\varphi, 1) \nabla f \quad \text{by (I.3.3) (2)} \\ &= (\varphi, 1)(-1 \vee 1)(\nabla f)(-1). \end{aligned}$$

Hence we get (17) by

$$\begin{aligned} (\varphi, 1)\partial_3^{11} &= (\varphi, 1)((-1 \vee 1)(\nabla f)(-1), i_X f) \nabla g \\ &= (f(\partial_Y, 1)\alpha, f) \nabla g \\ &= f(\partial_Y, 1)(\alpha, i_Y) \nabla g \\ &= (\bar{\partial}_X, 1)(f'', i_X f)(\alpha, i_Y) \nabla g \\ &= -(\bar{\partial}_X, 1)\partial_3^{12}. \end{aligned}$$

Next we obtain

$$\begin{aligned}(\varphi, fg) \bigtriangledown \bar{\partial}_X &= (\varphi, 1-\varphi) \bigtriangledown \bar{\partial}_X \\ &= -(-\varphi, 1) \bigtriangledown \bar{\partial}_X \quad \text{by (I.3.3) (2).}\end{aligned} \tag{21}$$

On the other hand

$$\begin{aligned}(\varphi, fg) \bigtriangledown \bar{\partial}_X &= -fg\bar{\partial}_X + (fg+\varphi)\bar{\partial}_X \quad \text{by (I.3.3)} \\ &= -fg\,\bar{\partial}_X + \bar{\partial}_X \\ &= -(\bar{\partial}_X, 1)(f'', i_X f)g'' + \bar{\partial}_X \\ &= -(\bar{\partial}_X, 1)(-i_{X''} + (f'', i_X f)g'').\end{aligned} \tag{22}$$

This implies (18) since we get

$$\begin{aligned}(\varphi, 1)\partial_3^{21} &= (\varphi, 1)(-1 \vee 1) \bigtriangledown \bar{\partial}_X \\ &= -(\varphi, fg) \bigtriangledown \bar{\partial}_X \quad \text{by (21)} \\ &= -(\bar{\partial}_X, 1)\partial_3^{22} \quad \text{by (22).}\end{aligned}$$

Hence the proof of (3.9) is complete. q.e.d.

(3.10) Theorem. *Let $(D|\bar{\partial}_X, \bar{f}, \bar{g}, \bar{\alpha})$ be a twisted domination of $B|\partial_Y$ and let $C|\partial_X$ be associated to this domination as in (3.8). Then there is a twisted homotopy equivalence*

$$\bar{h} : B|\partial_Y \xrightarrow{\simeq} C|\partial_X$$

in $\mathbf{TWIST}_2^c/\simeq$.

Proof. We define the map $\bar{h} = (h_{\geq 1}, Eh'', f)$ by $f : Y \to X$ and the composite $(n \geq 1)$

$$h_n : B_n \vee \partial_Y \xrightarrow{f_n} D_n \vee \bar{\partial}_X \subset C_n \vee \partial_X$$

where we use the canonical inclusions $D_n \subset C_n$ and $i : \bar{\partial}_X \subset \partial_X$; see (3.8). Moreover h'' is the composite

$$h'' : Y'' \xrightarrow{f''} X'' \vee X \subset C_2 \vee X$$

Using (3.6) (1) we see that $\bar{h}$ is a well defined map in $\mathbf{TWIST}_2^c$. Using theorem (II.5.1) we see that $\bar{h}$ is a twisted homotopy equivalence if and only if the induced chain map $K\,\bar{h}$ is a homotopy equivalence in $\mathbf{TWIST}_1^c$. We describe a homotopy inverse $(k_{\geq 1}, \{g\})$ of $K\,\bar{h} = (h_{\geq 1}, \{f\})$ by the $\{g\}$-equivariant maps

$$k_n : C_n \vee \partial_X = D_1 \vee \cdots \vee D_n \vee \partial_X \longrightarrow B_n \vee \partial_Y$$

which are determined by the coordinates k_n^i.

$$\begin{tikzcd} D_i \vee \bar{\partial}_X \arrow[r, "\alpha_i"] \arrow[d, "1\vee i"'] & D_{i+1} \vee \bar{\partial}_X \arrow[r] & \cdots \arrow[r, "\alpha_{n-1}"] & D_n \vee \bar{\partial}_X \arrow[d, "g^n"] \\ D_i \vee \partial_X \arrow[rrr, "k_n^i"] & & & B_n \vee \partial_Y \end{tikzcd}$$

The commutativity of this diagram determines k_n^i for $i = 1, \ldots, n$. The homotopy

$$(k_{\geq 1}, \{g\})(h_{\geq 1}, \{f\}) = \bar{g}\,\bar{f} \simeq 1$$

is given by $\bar{\alpha}$ in (3.5). Moreover the homotopy

$$\bar{\beta} : (h_{\geq 1}, \{f\})(k_{\geq 1}, \{g\}) \simeq 1$$

with $\bar{\beta} = (\beta_{\geq 1}, 1)$ is given by the inclusion

$$\beta_n : C_n \vee \partial_X \subset C_{n+1} \vee \partial_X .$$

One can check that these maps and homotopies are well defined in $\mathbf{TWIST}_1^c$. Compare Ranicki [FO]. q.e.d.

(3.11) Definition. Let $B|\partial_Y$ be a twisted chain complex which admits a domination as in (3.5) where $D|\bar{\partial}_X$ is $\mathcal{A}$-finite and $\dim(D|\bar{\partial}_X) = n$. Then $C|\partial_X$ is defined as in (3.8) such that

$$p : C_n \vee \partial_X \longrightarrow C_n \vee \partial_X$$

is an idempotent map where C_n is $\mathcal{A}$-finite. Hence p defines by the Yoneda embedding an idempotent map p in $\mathbf{mod}(U_{\mathcal{A}}(\partial_X))^\sharp$ representing the element

$$[C, \partial_X] \in \tilde{K}_0(U_{\mathcal{A}}(\partial_X))$$

which is termed the *instant finiteness obstruction.* Using the isomorphism $v = \{g\} : \partial_X \cong \partial_Y \in \mathbf{Coef}$ in (3.6) let

$$[B, \partial_Y] = v_*[C, \partial_X]$$

where v_* is induced by the functor (1.5). Ranicki [FO] shows that $[C, \partial_X]$ is well defined by the homotopy type of (C, ∂_X) in $\mathbf{chain}(\partial_X)$.

Proof of (3.4). Assume first that there is an $\mathcal{A}$-finite twisted chain complex $A|\partial_X$ which is homotopy equivalent to $B|\partial_Y$ in $\mathbf{TWIST}_2^c$. Then $[B, \partial_Y] = v_*[A, \partial_X]$ for $v : \partial_X \cong \partial_Y$ with $[A, \partial_X] = 0$ since $[A, \partial_X]$ is $\mathcal{A}$-finite, On the other hand assume $[B, \partial_Y] = 0$, that is $[C, \partial_X] = 0$ for the element in (3.11). The image $\mathrm{im}(d_{n+1}) \subset C_n \vee \partial_X$ where d_{n+1} is considered as a morphism in $\mathbf{Mod}(U_{\mathcal{A}}(\partial_X))$ is a direct summand. Let $\mathrm{im}(d_{n+1})^\perp$ be a complement of this summand. Then there exists $F \in \mathbf{mod}(U_{\mathcal{A}}(\partial_X))^\sharp$ such that

$$\mathrm{im}(d_{n+1})^\perp \oplus F \cong F' \tag{1}$$

where F' is finitely generated and free; see (3.8). Hence we obtain homotopy equivalences of chain complexes in $\mathbf{Mod}(U_{\mathcal{A}}(\partial_X))$ as follows where we omit the Yoneda embedding from the notation.

$$\begin{tikzcd}
(C,\partial_X): \dots & C_{n-2}\vee\partial_X \arrow[l] & C_{n-1}\vee\partial_X \arrow[l] & C_n\vee\partial_X \arrow[l] & \dots \arrow[l, "d_{n+1}"'] \\
\overline{\overline{C}}: \dots \arrow[u, "i"] \arrow[d, "j"'] & C_{n-2}\vee\partial_X \arrow[l] \arrow[u, equal] \arrow[d, equal] & C_{n-1}\vee\partial_X \arrow[l] \arrow[u, equal] \arrow[d, "j_{n-1}"] & \mathrm{im}(d_{n+1})^{\perp} \arrow[l] \arrow[u, "i_n"'] \arrow[d, "j_n"] & \\
(\bar{C},\partial_X): \dots & C_{n-2}\vee\partial_X \arrow[l] & C_{n-1}\vee A\vee\partial_X \arrow[l, "e_{n-1}"'] & A'\vee\partial_X \arrow[l, "e_n"'] &
\end{tikzcd}$$

Here F is given by $A_0, \dots, A_n$ as in (1.1) with $A = A_0 \vee \dots \vee A_n$. Similarly F' is given by $A'_0, \dots, A'_m$ with $A' = A'_0 \vee \dots \vee A'_m$. The middle and the bottom row are chain complexes of dimensions $\leq n$. The map i_n is the inclusion. Moreover j_{n-1} is the obvious inclusion and the inclusion j_n is obtained by the isomorphism (1). The map e_{n-1} restricted to A is trivial and the map e_n is obtained by the direct sum of F and the composite

$$\mathrm{im}(d_{n+1})^{\perp} \subset C_n \vee \partial_X \xrightarrow{d_{n-1}} C_{n-1} \vee \partial_X$$

using (1). There is the canonical retraction r of j such that we obtain the homotopy equivalence

$$ir : (\bar{C}, \partial_X) \xrightarrow{r} \overline{\overline{C}} \xrightarrow{i} (C, \partial_X)$$

which is the identity in degree $\leq n-2$. If $n \geq 3$ then this map corresponds to a twisted chain map

$$\overline{ir} : \bar{C}|\partial_X \longrightarrow C|\partial_X$$

which is again the identity in degree $\leq n-2$. One readily checks that $\overline{ir}$ is actually a map in $\mathbf{TWIST}_2^c$. Now $\overline{ir}$ is a twisted homotopy equivalence by (II.5.1). Since $\bar{C}|\partial_X$ is $\mathcal{A}$-finite the proof is complete by use of (3.10). q.e.d.

4 Proof of the Finiteness Obstruction Theorem

We are now eady to prove theorem (2.5). As basic facts we need the finiteness obstruction theorem for twisted chain complexes in (3.4) and the model lifting property of the twisted chain functor in (VI.8.4).

Let $Y_{\geq 1}$ be given as in (2.5). Using (VI.1.7) it is clear that the twisted chain functor r_2 carries a domination of $Y_{\geq 1}$ to a twisted domination of $r_2(Y_{\geq 1})$. Now we can apply theorem (3.4) which shows that

$$[C_* Y_{\geq 1}] \in \tilde{K}_0(U_{\mathcal{A}}(\partial_Y))$$

satisfies $[C_* Y_{\geq 1}] = 0$ if and only if there exists a model

$$\alpha : \bar{C} \mid \partial_X \longrightarrow r_2(Y_{\geq 1}) = B \mid \partial_Y$$

in $\mathbf{TWIST}_2^c/\overset{1}{\simeq}$ where $\bar{C} \mid \partial_X$ is $\mathcal{A}$-finite of dimension $\leq \max(3, n)$. Hence by the model lifting property (VI.8.4) we obtain a model

$$\beta : \bar{X}_{\geq 1} \longrightarrow Y_{\geq 1}$$

in $\mathbf{Complex}/\overset{1}{\simeq}$ with $r_2(\bar{X}_{\geq 1}) = \bar{C} \mid \partial_X$. Hence $\bar{X}_{\geq 1}$ is of dimension $\leq \max(3, n)$ and $\mathcal{A}$-finite. q.e.d.

Chapter VIII: Non-Reduced Complexes and Whitehead Torsion

In dealing with the general concept of Whitehead torsion in cofibration categories we have to introduce "non-reduced complexes" in cofibration categories which generalize CW-complexes with arbitrary 0-skeleton. We compare such non-reduced complexes termed $\mathcal{D}$-complexes with the **T**-complexes studied in the chapters above. In fact "normalized" $\mathcal{D}$-complexes are special **T**-complexes. Using $\mathcal{D}$-complexes we define the Whitehead group $\mathrm{Wh}(L)$ and we show that this group coincides with the algebraic Whitehead group $\mathrm{Wh}(\partial_L)$. Hence the material in this chapter covers most of the results of J.H.C. Whitehead on simple homotopy types.

1 Classes of Discrete Objects

The definition of a non-reduced complex in § 3 relies on the choice of a class $\mathcal{D}$ of discrete objects in a cofibration category $\mathbf{C}$. These discrete objects will serve as the 0-skeleta.

(1.1) Definition. A *class (resp. set) of discrete objects* $\mathcal{D}$ is a class (resp. set) of objects in $\mathbf{C}_c$ which is closed under the formation of sums; that is, if X, $Y \in \mathcal{D}$ then also $X \vee Y \in \mathcal{D}$. In particular the empty sum $*$ is an element in $\mathcal{D}$.

For example if $\mathbf{C} = \mathbf{Top}$ we can choose for $\mathcal{D}$ the class of all discrete spaces in **Top**. Moreover if $\mathbf{C} = \mathbf{Top}^*$ we can choose for $\mathcal{D}$ the class of all discrete spaces with base point. In **Top** the initial object is the empty set and sums in **Top** are given by the disjoint union. While in $\mathbf{Top}^*$ the point $*$ is the initial object and the sum in $\mathbf{Top}^*$ is the one-point union. The next definition generalizes the definition in (I.2.8).

(1.2) Definition. Let D be an object in $\mathbf{C}_c$ and let $\mathcal{D}$ be a class of discrete objects in $\mathbf{C}$. Then we define the subcategory

$$\mathbf{cone}(D, \mathcal{D}) \subset \mathrm{Ho}(\mathbf{C}^D)_c = (\mathbf{C}^D)_c/\simeq \mathrm{rel}\, D$$

This is the full subcategory consisting of the objects $C_{\alpha,\beta}$ obtained by push out diagrams in $\mathbf{C}$

H.-J. Baues, *Combinatorial Foundation of Homology and Homotopy*

$$\begin{tikzcd} IE \arrow[r] & C_{\alpha,\beta} \\ E \vee E \arrow[u, "(i_0,i_1)"] \arrow[r, "(\alpha,\beta)"] & D \arrow[u] \end{tikzcd}$$

Here $E \in \mathcal{D}$ and α, β are maps in $\mathbf{C}$. The cofibration (i_0, i_1) is part of the structure of the cylinder IE of E in $\mathbf{C}$. If $\alpha = \beta$ then we call $C_{\alpha,\alpha}$ a suspension. This yields the full subcategory

$$\mathbf{susp}(D, \mathcal{D}) \subset \mathbf{cone}(D, \mathcal{D})$$

consisting of suspensions.

Recall that we have defined the category of D-cones termed $\mathbf{cone}(D)$ in (III.6.4) and that $\mathbf{cone}(D)$ is a theory of coactions.

(1.3) Proposition. *One has a full inclusion of theories*

$$\mathbf{cone}(D, \mathcal{D}) \subset \mathbf{cone}(D)$$

so that $\mathbf{cone}(D, \mathcal{D})$ *is a theory of coactions. The suspensions in* $\mathbf{susp}(D, \mathcal{D})$ *are the cogroups in* $\mathbf{cone}(D, \mathcal{D})$*. Moreover* $\mathbf{cone}(D, \mathcal{D})$ *is augmented by the* D*-torus* $\Sigma = \Sigma_* D$ *if* $D \in \mathcal{D}$.

Proof. An object $C_f \in \mathbf{cone}(D)$ is given by a based object

$$A = (D \rightarrowtail A \xrightarrow{0} D) \in \mathbf{C}^D \tag{1}$$

and a map $f : A \to D$ in $\mathbf{C}^D$. Then C_f is given by the push out diagrams

$$\begin{tikzcd} I_D A \arrow[r] & CA \arrow[r] & C_f \\ A \cup_D A \arrow[u, "(i_o,i_1)"] \arrow[r, "(1,0)"] & A \arrow[u] \arrow[r, "f"] & D \arrow[u] \end{tikzcd}$$

Here $I_D A$ is the cylinder of (A, D) in $\mathbf{C}^D$ which is obtained by the push out

$$\begin{tikzcd} IA \arrow[r] & I_D A \\ ID \arrow[u] \arrow[r, "p"] & D \arrow[u] \end{tikzcd}$$

in $\mathbf{C}$. Here (IA, ID) is the cylinder of the pair (A, D); see (III.1.9) and (III.4.1). Given E, α, β as in (1.2) we define $A = D \vee E$ so that

$$A = (D \rightarrowtail D \vee E \xrightarrow{(1,\beta)} D) \tag{2}$$

is a based object as in (1). Moreover we define

$$f : A = D \vee E \xrightarrow{(1,\alpha)} D \tag{3}$$

by the map α. Then one readily checks that one can identify

$$C_f = C_{\alpha,\beta}. \tag{4}$$

For this we only have to use the fact that the cylinder of a sum $D \vee E$ can be chosen to be $I(D \vee E) = ID \vee IE$. This implies $I_D(D \vee E) = D \vee IE$. Clearly if $\alpha = \beta$ then $f = (1, \alpha)$ coincides with the trivial map $0 = (1, \beta)$ so that in this case $C_{\alpha,\alpha} = \Sigma A$ is the suspension of the based object (2). If $\alpha = 1$ is the identity of $D \in \mathcal{D}$ we obtain the suspension $C_{1,1}$ obtained by the pushout

$$\begin{tikzcd} ID \arrow[r] & C_{1,1} \\ D \vee D \arrow[u] \arrow[r, "(1,1)"] & D \arrow[u] \end{tikzcd}$$

This is the D-torus $C_{1,1} = \Sigma_D^1$ in (III.6.5) (2) and in the next section. q.e.d.

2 Cells in a Cofibration Category

CW-complexes in **Top** are obtained by attaching cells. In a cofibration category the corresponding notion of cell is given by the notion of n-dimensional D-torus and $(n+1)$-dimensional D-disk defined as follows.

Let $\mathbf{C}$ be a cofibration category with initial object $*$ and let D be a cofibrant object in $\mathbf{C}$. As in (II.6.5) we have the based object

$$\Sigma_D^0 = (D \xrightarrow{i_1} D \vee D \xrightarrow{(1,1)} D) \tag{2.1}$$

in $\mathbf{C}^D$. Hence the n-fold suspension of Σ_D^0 in $\mathbf{C}^D$ is defined. We call

$$\Sigma_D^n = \Sigma^n(\Sigma_D^0) \in \mathbf{susp}(D) \tag{1}$$

the *n-dimensional D-torus.* This is again a based object in $\mathbf{C}^D$ so that we have

$$D \overset{i}{\rightarrowtail} \Sigma_D^n \xrightarrow{p^n} D \quad \text{with } pi = 1_D \tag{2}$$

where the trivial map $p^n = 0$ is termed the projection. For a cofibrant pair (X, A) we obtain the *relative D-torus* $\Sigma_A(X)$ by the push out diagram (see (III.6.5))

$$\begin{tikzcd} I_A X \arrow[r, "\pi"] & \Sigma_A(X) \arrow[r, "p^1"] & X \\ X \cup_Y X \arrow[u, "(i_0,i_1)"] \arrow[r, "(1,1)"] & X \arrow[u, "i"] & \end{tikzcd} \tag{3}$$

Here the projection p^1 is given by the projection $p^1\pi = p$ of the relative cylinder $I_A X$ and by $p^1 i = 1_X$. Now we have

$$\Sigma_D^1 = \Sigma_* D$$

and inductively we obtain the n-fold D-torus by the following push out diagram $(n \geq 1)$

$$\begin{tikzcd} \Sigma_D(\Sigma_D^n) \arrow[r, "\pi"] & \Sigma_D^{n+1} \arrow[r, "p^{n+1}"] & D \\ \Sigma_D^n \arrow[u] \arrow[r, "p^n"] & D \arrow[u, "i"] & \end{tikzcd} \tag{4}$$

Here p^{n+1} is defined by $p^{n+1}\pi = p^n p^1$ and $p^{n+1} i = 1_D$. Compare also (II.6.3) and (II.§ 10) in Baues [AH]. We define the $(n+1)$ *dimensional D-ball* or *D-disk* by

$$\Delta_D^{n+1} = C\Sigma_D^n \tag{5}$$

where the right hand side is the cone in $\mathbf{C}^D$ of the based object (2). Equivalently we get the D-ball by the push out

$$\begin{tikzcd} I_D\Sigma_D^n \arrow[r, "\pi"] & \Delta_D^{n+1} \arrow[r, "q"] & D \\ \Sigma_D^n \cup_D \Sigma_D^n \arrow[u, "(i_0, i_1)"] \arrow[r, "(1, ip^n)"] & \Sigma_D^n \arrow[u, "i"] & \end{tikzcd} \tag{6}$$

Here q is given by $q\pi = p^n\pi$ and $qi = p^n$; this is the trivial map of the cone in (5) so that q is a weak equivalence in $\mathbf{C}$ and

$$p^n : \Sigma_D^n \overset{i}{\rightarrowtail} \Delta_D^{n+1} \underset{\sim}{\xrightarrow{q}} D \tag{7}$$

is a factorization of the projection p^n in the sense of the factorization axiom. We can obtain the $(n+1)$-dimensional D-torus Σ_D^{n+1} by the push out diagram

$$\begin{tikzcd} \Delta_D^{n+1} \arrow[r, "\pi"] & \Sigma_D^{n+1} \\ \Sigma_D^n \arrow[u, "i"] \arrow[r, "p^n"] & D \arrow[u, "i"] \end{tikzcd} \tag{8}$$

This also makes sense for $n = 0$; in this case we use $p^1 = (1, 1)$ on Σ_D^0 in (2.1) and $\Delta_*^1 D = ID$ is the cylinder on D. Hence for $n = 0$ diagram (8) coincides with diagram (3). The pair

$$(\Delta_D^{n+1}, \Sigma_D^n) \tag{9}$$

plays the role of an $(n+1)$*-dimensional closed cell* in the cofibration category $\mathbf{C}$. Compare example (2.3) below. We also call $\Sigma_D^n = \partial\Delta_D^{n+1}$ the *boundary* of Δ_D^{n+1}.

(2.2) Remark. If $\mathbf{C}$ is an I-category then the cylinder $I(D)$ is a functor in D and accordingly also all constructions in (2.1) (1) ... (8) are natural in $D \in \mathbf{C}_c$. Moreover the functors

$$\mathbf{C}_c \to \mathbf{C}_c, \quad D \mapsto \Sigma_D^n, \quad D \mapsto \Delta_D^n$$

carry push out diagrams to push out diagrams as follows from the push out axiom (III.7.1) (I2). These functors carry $*$ to $*$ and hence sums to sums, that is $\Sigma_{D\vee E}^n = \Sigma_D^n \vee \Sigma_E^n$ and $\Delta_{D\vee E}^n = \Delta_D^n \vee \Delta_E^n$.

(2.3) Example. We first consider the case $\mathbf{C} = \mathbf{Top}$. Let S^n be the *n-sphere* which is the boundary of the $(n+1)$-*ball* D^{n+1} in $\mathbf{Top}$. Then we have for D in $\mathbf{Top}$ and $n \geq 0$ the isomorphism of pairs

$$(D^{n+1} \times D, S^n \times D) = (\Delta_D^{n+1}, \Sigma_D^n) \tag{1}$$

Here $S^n \times D$ and $D^{n+1} \times D$ are the corresponding product spaces. The inclusion $D \rightarrowtail S^n \times D$ is given by the base point $* \in S^n$ and the projection $p^n : S^n \times D \to D$ is the projection of the product. For the isomorphism (1) we use the fact that the initial object of $\mathbf{Top}$ is the empty set.

For $\mathbf{C} = \mathbf{Top}^*$ the initial object is the point $*$. In this case we get for the cofibrant space D in $\mathbf{Top}^*$ the isomorphism of pairs

$$(D^{n+1} \wedge D, S^n \wedge D) = (\Delta_D^{n+1}, \Sigma_D^n) \tag{2}$$

Here the left hand side uses the smash product of pointed spaces defined by the quotient space $A \wedge B = A \times B / A \times \{*\} \cup \{*\} \times B$.

3 Non-Reduced Complexes

Let $\mathbf{C}$ be a cofibration category with initial object $*$ and let $\mathcal{D}$ be a class of discrete objects in $\mathbf{C}$. For $D \in \mathcal{D}$ we obtain as in § 2 the n-dimensional D-ball $(\Delta_D^n, \Sigma_D^{n-1})$ which we also call a $\mathcal{D}$-*disk* of dimension n with $n \geq 1$. We say that X is obtained from Y by *attaching a $\mathcal{D}$-cell* e_D^n and we write

$$X = Y \cup e_D^n = Y \cup_f e_D^n \tag{3.1}$$

if a push out diagram

$$\begin{tikzcd} \Delta_D^n \arrow[r, "\pi"] & X \\ \Sigma_D^{n-1} \arrow[u] \arrow[r, "f"] & Y \arrow[u] \end{tikzcd} \tag{2}$$

is given in $\mathbf{C}$. We call f the *attaching map* of the $\mathcal{D}$-cell e_D^n and we call π the *characteristic map* of this cell. We call a pair map

$$(\bar{h}, h) : (Y \cup_f e_D^n, Y) \longrightarrow (Z \cup_g e_D^n, Z) \tag{2}$$

a *cell map* if the induced map

$$\bar{h} : Z \cup_{hf} e_D^n \longrightarrow Z \cup_g e_D^n$$

which extends the identity on Z is an isomorphism in $\mathbf{C}$. Here $Z \cup_{hf} e_D^n$ is also the push out of

$$Y \cup_f e_D^n \xleftarrow{i} Y \xrightarrow{h} Z.$$

(3.2) Definition. We say that an object

$$X_{\geq 0} = (X_0 \subset X_1 \subset X_2 \subset \dots) \in \mathbf{Fil}_0(\mathbf{C})_c \tag{1}$$

is a *$\mathcal{D}$-complex* if $X_0 \in \mathcal{D}$ and if X_n is obtained from X_{n-1} by attaching an n-dimensional $\mathcal{D}$-cell, $n \geq 1$. Such $\mathcal{D}$-complexes are termed *non-reduced* complexes in $\mathbf{C}$; they are *reduced* if $X_0 = * \in \mathcal{D}$ is the initial object. If the direct limit

$$X = \lim(X_{\geq 0}) \tag{2}$$

exists in $\mathbf{C}$ then we also call the object X in $\mathbf{C}$ a $\mathcal{D}$-complex with *skeletal filtration* $X_{\geq 0}$. Clearly the direct limit X exists if $X_{\geq 0}$ is finite dimensional. A map $f : X \to Y$ in $\mathbf{C}$ between $\mathcal{D}$-complexes is *cellular* if $f = \lim f_{\geq 0}$ where $f_{\geq 0} : X_{\geq 0} \to Y_{\geq 0}$ is a map in $\mathbf{Fil}_0(\mathbf{C})_c$. Such a cellular map f is a *$\mathcal{D}$-isomorphism* if f_0 is an isomorphism in $\mathbf{C}$ and if

$$(f_n, f_{n-1}) : (X_n, X_{n-1}) \longrightarrow (Y_n, Y_{n-1}) \tag{3}$$

is a cell map or equivalently if $f_{\geq 0}$ is an isomorphism in $\mathbf{Fil}(\mathbf{C})_c$.

Remark. If $X_{\geq 0}$ is a $\mathcal{D}$-complex and if $f_{\geq 0} : X_{\geq 0} \to Y_{\geq 0}$ is an isomorphism in $\mathbf{Fil}(\mathbf{C})_c$ with $Y_0 \in \mathcal{D}$ then also $Y_{\geq 0}$ is a $\mathcal{D}$-complex. In fact, consider the diagram, $n \geq 1$,

$$\begin{array}{ccccc} \Delta_D^n & \xrightarrow{\pi} & X_n & \xrightarrow{f_n} & Y_n \\ \uparrow & & \uparrow & & \uparrow \\ \Sigma_D^{n-1} & \xrightarrow{\partial_n} & X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array} \tag{4}$$

Here the left hand square is a push out and since f_n and f_{n-1} are isomorphisms also the right hand square is a push out. Hence the outer square is a push out. A $\mathcal{D}$-complex $X_{\geq 0}$ has the structure

$$X_{\geq 0} = (X_{\geq 0}, D_{\geq 0}, \partial_{\geq 1}) \tag{3.3}$$

where D_0, D_1, D_2, $\dots$ is a sequence of objects in $\mathcal{D}$ with $X_0 = D_0$ and where

$$X_n = X_{n-1} \cup e^n_{D_n}$$

is obtained by attaching the cell $e^n_{D_n}$ via an attaching map ∂_n, $n \geq 1$. We say that $X_{\geq 0}$ is *normalized* if one has a commutative diagram in $\mathbf{C}$ with $n \geq 1$

$$\begin{array}{ccc} \Sigma^{n-1}_{D_n} & \xrightarrow{\partial_n} & X_{n-1} \\ \uparrow{\scriptstyle i} & & {\scriptstyle i_0}\uparrow \\ D_n & \xrightarrow{\mathcal{O}_n} & X_0 \end{array} \tag{3.4}$$

A *normalization* of the $\mathcal{D}$-complex $X_{\geq 0}$ is a family H of tracks, $n \geq 1$,

$$H_n : \partial_n i \simeq i_0 \mathcal{O}_n$$

where $\mathcal{O}_n$ is a map as in (3.4). Here we choose H_n to be the trivial track if diagram (3.4) commutes. Recall that a *track* $H : f \simeq g$ with $f, g : X \to Y \in \mathbf{C}$ is a homotopy class $\mathrm{rel}\, X \vee X$ of homotopies $H : IX \to Y, Hi_0 = f$ and $Hi_1 = g$. In (5.5) we need the existence of such normalization. If $L_{\geq 0} \subset X_{\geq 0}$ is a subcomplex such that $L_{\geq 0}$ is normalized then a normalization of $X_{\geq 0}$ rel $L_{\geq 0}$ is given by tracks as above defined for cells in $X - L$.

(3.5) Definition. Let $\mathcal{D}$ be a class of discrete objects in the cofibration category $\mathbf{C}$. Then we define the class $\mathcal{D}(2)$ of discrete objects in $\mathbf{Pair}(\mathbf{C})$ as follows. Objects in $\mathcal{D}(2)$ are the pairs $(Y \vee \bar{Y}, Y)$ where $Y \rightarrowtail Y \vee \bar{Y}$ is the inclusion of the first summand with $Y, \bar{Y} \in \mathcal{D}$. We say that a $\mathcal{D}$-complex $L_{\geq 0}$ is a *subcomplex* of the $\mathcal{D}$-complex $K_{\geq 0}$ if $(K_{\geq 0}, L_{\geq 0})$ is a $\mathcal{D}(2)$-complex in $\mathbf{Pair}(\mathbf{C})$. Similarly we say that a map $i : L \to K$ in $\mathbf{C}$ is the inclusion of a subcomplex and we write $L < K$ if $i = \lim i_{\geq 0}$ where $i_{\geq 0} : L_{\geq 0} \to K_{\geq 0}$ is given by a subcomplex $L_{\geq 0}$ in the $\mathcal{D}$-complex $K_{\geq 0}$.

We write $\dim(K - L) \leq n$ if K_m, $m \geq n$, is the push out of $L_m \leftarrow L_n \to K_n$. Now consider the following push out diagram in $\mathbf{C}$

$$\begin{array}{ccc} K & \longrightarrow & X \cup_L K \\ {\scriptstyle i}\uparrow & & \uparrow \\ L & \xrightarrow{f} & X \end{array}$$

(3.6) Lemma. *Let K and X be $\mathcal{D}$-complexes and let L be a subcomplex of K and let $i : L \to K$ be the inclusion. Moreover let f be a cellular map. Then the push out $X \cup_L K$ is again a $\mathcal{D}$-complex.*

The lemma is readily proved by going back to the definitions. For $X_0 \in \mathcal{D}$ we have by (1.2) and (1.3) the theory of coactions

$$\mathbf{T} = \mathbf{cone}(X_0, \mathcal{D}) \subset \mathrm{Ho}(\mathbf{C}^{X_0})_c$$

so that also $\mathbf{T}$-complexes are defined in $\mathbf{Fil}_1(\mathbf{C}^{X_0})_c$.

(3.7) Proposition. *Let $X_{\geq 0}$ be a $\mathcal{D}$-complex. Then $X_{\geq 0}$ defines the object $X_{\geq 1}$ in $\mathbf{Fil}_1(\mathbf{C}^{X_0})_c$. If $X_{\geq 0}$ is normalized then $X_{\geq 1}$ is a $\mathbf{cone}(X_0, \mathcal{D})$-complex.*

Proof. We define the structure

$$X_{\geq 1} = (X_{\geq 1}, A_{\geq 1}, \bar{\partial}_{\geq 2}) \tag{1}$$

of the $\mathbf{cone}(X_0, \mathcal{D})$-complex $X_{\geq 1}$ as follows. Clearly X_1 is obtained by attaching $e^1_{D_1}$ to X_0 so that for

$$\partial_1 = (\bar{\partial}_1, 0_1) : D_1 \vee D_1 = \Sigma^0_* D_1 \longrightarrow X_0 \tag{2}$$

we have $X_1 = C_{\bar{\partial}_1, 0_1} \in \mathbf{cone}(X_0, \mathcal{D})$; compare (1.2). Moreover we define the based object $(n \geq 1)$

$$A'_n = (X_0 \rightarrowtail X_0 \vee D_n \xrightarrow{1,0_n} X_0) \tag{3}$$

in $\mathbf{C}^{X_0}$ by (3.4). Then the suspension

$$A_n = \Sigma A'_n \tag{4}$$

is defined in $\mathbf{C}^{X_0}$. We have

$$\Sigma^{n-1} A_{n+1} = \Sigma^n A'_{n+1} = X_0 \vee \Sigma^n_{D_{n+1}} \tag{5}$$

as follows from (3) and (2.1) (1). Now let

$$\bar{\partial}_{n+1} : \Sigma^{n-1} A_{n+1} = X_0 \vee \Sigma^n_{D_{n+1}} \longrightarrow X_n \tag{6}$$

be the map which is the inclusion on X_0 and which is ∂_{n+1} on $\Sigma^n_{D_{n+1}}$. We also have

$$C\Sigma^{n-1} A_{n+1} = X_0 \vee \Delta^{n+1}_{D_{n+1}} \tag{7}$$

where the left hand side is the cone in $\mathbf{C}^{X_0}$. Hence comparing the definition of a $\mathcal{D}$-complex and a $\mathbf{T}$-complex with $\mathbf{T} = \mathbf{cone}(X^0, \mathcal{D})$ shows that $X_{\geq 1}$ in (1) is a well defined $\mathbf{T}$-complex. q.e.d.

(3.8) Remark. We can study normalized $\mathcal{D}$-complexes along similar lines as $\mathbf{T}$-complexes. In particular we can apply all the results on $\mathbf{T}$-complexes in the chapters above since each normalized $\mathcal{D}$-complex $X_{\geq 0}$ yields by (3.6) a $\mathbf{T}$-complex in $\mathbf{C}^{X_0}$. For example if $\mathbf{C}^{X_0}$ is a homological cofibration category under $\mathbf{T} = \mathbf{cone}(X_0, \mathcal{D})$ then the augmented chain complex

$$C_* X_{\geq 0} = \operatorname{aug} C_* X_{\geq 1}$$

is defined which leads to the notion of homology and cohomology for $X_{\geq 0}$.

(3.9) Example. Let $\mathbf{C} = \mathbf{Top}$ and let $\mathcal{D}$ be the class of discrete spaces in **Top**. Then (2.3) (1) shows that a $\mathcal{D}$-complex is the same as a CW-complex in **Top**. In fact we obtain for a CW-complex X the corresponding $\mathcal{D}$-complex $X_{\geq 0}$ as follows. Here $X_n = X^n$ is the n-skeleton of X and D_n is the set of n-cells in X with the discrete topology. Moreover the attaching map

$$\partial_{n+1} : S^n \times D_n \longrightarrow X_n$$

of the $\mathcal{D}$-complex is obtained by $\partial_{n+1} \mid S^n \times \{e\} = \alpha_e$ for $e \in D_n$ where α_e is the attaching map of the cell e in the CW-complex X. It is clear that each CW-complex X is 0-homotopy equivalent to a *normalized CW-complex.* Here we say that X is normalized if each attaching map $\alpha_e : S^n \to X^n$ of an $(n+1)$-cell e in X with $n \geq 0$ carries the basepoint $*$ of S^n to a point in the 0-skeleton X^0. A $\mathcal{D}$-isomorphism between CW-complexes is the same as a *CW-isomorphism* (i.e. a cellular homeomorphism for which the image of every cell is a cell).

4 The Ball Pair Axiom

Let $\mathbf{C} = (\mathbf{C}, cof, I, *)$ be an I-category as defined in (III.7.1) with initial object $*$. Then we have for each object X in $\mathbf{C}$ the cylinder

$$X \vee X \overset{i_0,i_1}{\rightarrowtail} IX \xrightarrow{p} X \tag{4.1}$$

in $\mathbf{C}$ which is natural in X and $pi_0 = 1$, $pi_1 = 1$. This cylinder defines homotopies and homotopy equivalences in $\mathbf{C}$. We know by (III.7.4) that $\mathbf{C}$ is a cofibration category is which the weak equivalences are the homotopy equivalences. Moreover all objects in $\mathbf{C}$ are fibrant and cofibrant. We shall use the push out diagram in $\mathbf{C}$

$$\begin{array}{ccc} X & \xrightarrow{i_0} & IX \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle j_1} \\ IX & \xrightarrow{j_0} & IX \cup_X IX \end{array} \tag{4.2}$$

Moreover we have by (2.1) (9) the $(n+1)$-dimensional closed cell $(\Delta_X^{n+1}, \Sigma_X^n)$ in $\mathbf{C}$ which for $n \geq 0$ is a functor in X by (2.2). Here $\Sigma_X^n = \partial\Delta_X^{n+1} = \partial$ is the boundary of the cell. For $n = 0$ the pair

$$(\Delta_X^1, \Sigma_X^0) = (IX, X \vee X)$$

is given by (4.1). We also need the following push out diagram in $\mathbf{C}$, $n \geq 1$ and $\varepsilon \in \{0, 1\}$,

$$\begin{array}{ccccc} \Sigma_X^{n-1} & \xrightarrow{i_\varepsilon} & I\Sigma_X^{n-1} & & \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \bar{\imath}} & & \\ \Delta_X^n & \xrightarrow{\bar{\imath}_\varepsilon} & \Delta_X^n \cup_{\partial_\varepsilon} I\Sigma_X^{n-1} & \xrightarrow{j_\varepsilon} & I\Delta_X^n \end{array} \tag{4.3}$$

Here j_ε is defined by $j_\varepsilon \bar{\imath} = I(i)$ and $j_\varepsilon \bar{\imath}_\varepsilon = i_\varepsilon$. All maps in (4.2) and (4.3) are cofibrations in $\mathbf{C}$; compare the relative cylinder axiom I4 in (III.7.1). Moreover we choose two copies P_X^n, Q_X^n of the ball Δ_X^n; that is $P_X^n = Q_X^n = \Delta_X^n$ with boundary $\partial = \Sigma_X^{n-1}$ and we form the push out in $\mathbf{C}$:

$$\begin{tikzcd} \Sigma_X^{n-1} \arrow[r, "i"] \arrow[d, "i"'] & Q_X^n \arrow[d, "i_Q"] \\ P_X^n \arrow[r, "i_P"] & P_X^n \cup_\partial Q_X^n \end{tikzcd} \tag{4.4}$$

For $n = 0$ let $\Sigma_X^{-1} = *$ be the initial object and let $P_X^0 = Q_X^0 = X$ so that $P_X^0 \cup_\partial Q_X^0 = X \vee X$.

(4.5) Definition. We say that the I-category $\mathbf{C}$ satisfies the *ball-pair axiom* if the following properties hold. For all X in $\mathbf{C}$ there are isomorphisms in $\mathbf{C}$

$$\begin{cases} n : IX \cong IX \\ m : IX \cong IX \cup_X IX \\ T : IIX \cong IIX \end{cases} \tag{1}$$

satisfying the following equations, $\varepsilon \in \{0, 1\}$,

$$\begin{gathered} ni_0 = i_1, \quad ni_1 = i_0, \quad pn = p, \\ mi_\varepsilon = j_\varepsilon i_\varepsilon, \quad (p,p)m = p, \quad (j_1 n, j_0 n)m = mn, \\ Ti_\varepsilon = I(i_\varepsilon), \quad TIi_\varepsilon = i_\varepsilon. \end{gathered}$$

Moreover n, m and T are natural in $X \in \mathbf{C}$. We show in (4.7) below that n and m induce an isomorphism κ as in the following push out diagram which defines $\square_X^{n+1}$.

$$\begin{tikzcd} \Delta_X^{n-1} \arrow[r, "\bar{\kappa}", "\cong"'] & \square_X^{n+1} \\ \Sigma_X^n \arrow[u] \arrow[r, "\kappa", "\cong"'] & P_X^n \cup_\partial Q_X^n \arrow[u] \end{tikzcd} \tag{2}$$

For $n = 0$ the isomorphism κ is the identity of $X \vee X$. We now require that in addition to the isomorphisms in (1) there exist isomorphisms α and β as in the following commutative diagram in $\mathbf{C}$.

$$\begin{tikzcd} \Sigma_X^{n-1} \arrow[r, "\bar{\imath} i_1"] & \Delta_X^n \cup_{\partial_0} I\Sigma_X^{n-1} \arrow[r, "j_0"] & I\Delta_X^n & \Delta_X^n \arrow[l, "i_1"'] \\ \Sigma_X^{n-1} \arrow[u, "1"] \arrow[r] & P_X^n \arrow[u, "\alpha"', "\cong"] \arrow[r] & \square_X^{n+1} \arrow[u, "\beta"', "\cong"] & Q_X^n \arrow[l] \arrow[u, "1"'] \end{tikzcd} \tag{3}$$

The bottom row is given by (2) and the top row is defined by (4.3) above. We call $(\square_X^{n+1}, Q_X^n)$ a *ball pair* in $\mathbf{C}$ and we call P_X^n the *complement of* Q_X^n *in the boundary*.

The map n in (1) above reverses the direction of the cylinder and $Hn = -H$ defines the negative of the homotopy H. Moreover m in (2) is a homotopy addition map which yields the addition of homotopies $H_1 : f \simeq g$, $H_2 : g \simeq h$ by $H_1 + H_2 = (H_1, H_2)m$. Finally T is an interchange map as in the interchange axiom (III.7.1) (I5). Maps n, m, T exist in each I-category $\mathbf{C}$ but in general they are not natural isomorphisms in $\mathbf{C}$; for this compare II.2.4, II.2.5 in Baues [AH].

(4.6) Example. It is well known that the I-category **Top** of topological spaces satisfies the ball pair axiom above. In fact in **Top** all isomorphisms $n, m, T, \kappa, \alpha, \beta$ in (4.5) can be defined to be natural in $X \in$ **Top** by choosing such isomorphisms for $X =$ point.

(4.7) Lemma. *The isomorphisms m, n induce a natural isomorphism $\kappa : \Sigma_X^k \cong P_X^k \cup_\partial Q_X^k$ for $k \geq 1$.*

Proof. Let (Y, B) be a cofibrant pair in $\mathbf{C}$ and let $I_B Y$ be the relative cylinder. Then $(p,p)m = p$ and naturality of m yield the isomorphism

$$m : I_B Y \cong I_B Y \cup_Y I_B Y.$$

Hence we get for the relative torus $\Sigma_B Y$ the isomorphism

$$\bar{m} : \Sigma_B Y = I_B Y \cup_\partial Y \xrightarrow{m \cup 1} (I_B Y \cup_Y I_B Y) \cup_\partial Y = I_B Y \cup_\partial I_B Y.$$

Here $\partial = \partial_0 \cup \partial_1 = Y \cup_B Y$ is the boundary of $I_B Y$. If we set $(Y, B) = (*, *)$ then $\bar{m}$ yields the isomorphism κ for $k = 1$. Moreover we obtain for $(Y, B) = (\Sigma_X^k, X)$ by (2.1) (4), (6) the following isomorphisms, $k \geq 1$.

$$\begin{aligned}
\Sigma_X^{k+1} &= \Sigma_B(Y) \cup_Y X \cong (I_B Y \cup_\partial I_B Y) \cup_{\partial_1} X \\
&= (I_B Y \cup_{\partial_1} X) \cup_X (I_B Y \cup_{\partial_0} X) \\
&\cong (I_B Y \cup_{\partial_1} X) \cup_X (I_B Y \cup_{\partial_1} X), \quad \text{induced by } n \\
&= P_X^{k+1} \cup_\partial Q_X^{k+1}, \quad \text{see (2.1) (6).}
\end{aligned}$$

q.e.d.

(4.8) Remark. Let $\mathcal{D}$ be a class of discrete objects in $\mathbf{C}$ and let $X \in \mathcal{D}$. Then we have for the ball pair $(\square_X^{n+1}, Q_X^n)$ a canonical structure as a $\mathcal{D}$-complex $\square_X^{n+1}$ with

$$\begin{gathered}
0\text{-skeleton} = (n-2)\text{-skeleton} = X \\
(n-1)\text{-skeleton} = \Sigma_X^{n-1} = \partial \\
n\text{-skeleton} = P_X^n \cup_\partial Q_X^n \\
(n+1)\text{-skeleton} = \square_X^{n+1}
\end{gathered}$$

Moreover P_X^n and Q_X^n are subcomplexes of $\square_X^{n+1}$.

(4.9) Lemma. *For $k \geq 1$ there is an isomorphism $n : \Sigma_X^k \cong \Sigma_X^k$ such that the composite*

$$\tau = \kappa n \kappa^{-1} : P_X^k \cup_\partial Q_X^k \cong \Sigma_X^k \cong \Sigma_X^k \cong P_X^k \cup_\partial Q_X^k$$

interchanges the role of P and Q that is $\tau i_P = i_Q$ and $\tau i_Q = i_P$; see (4.4). This implies that one has an isomorphism τ' for which the following diagram commutes

$$\begin{array}{ccc} \square_X^{k+1} & \xrightarrow{\tau'} & \square_X^{k+1} \\ \uparrow & & \uparrow \\ P_X^k \cup_\partial Q_X^k & \xrightarrow{\tau} & P_X^k \cup_\partial Q_X^k \end{array}$$

Proof. The lemma is a consequence of the assumption $(j_0 n, j_1 n)m = mn$ in (4.5) and the definition of κ in (4.7). Clearly n on Σ_X^k is induced by n in (4.7) in the obvious way and by (2.1) (6) we see that n extends to an isomorphism $\Delta_X^{k+1} \cong \Delta_X^{k+1}$ which yields τ'. q.e.d.

(4.10) Lemma. *Assume the ball pair axiom holds in* **C** *and let X be a finite dimensional $\mathcal{D}$-complex in* **C**. *Then the cylinder IX in* **C** *is again a finite dimensional $\mathcal{D}$-complex with $X \vee X$ as a subcomplex. The skeletal filtration of IX is given by $\bar{I}X_{\geq 0}$ where $\bar{I}$ is defined in (IV.1.3) (3).*

Proof. This is a consequence of the push out axiom in (III.7.1) (I2) and (3.6) and attaching maps defined by α in (4.5) (3).

(4.11) Remark. We can use also attaching maps of cells in IX defined as follows. Apply the isomorphism $n : I\Delta_X^n \to I\Delta_X^n$ to the top row of diagram (4.5) (3). Then we obtain an attaching map defined by the composite $n\alpha$ which does not agree with the attaching map chosen in the proof above. Hence IX has two different cell structures. The identity of IX however is easily seen to be a $\mathcal{D}$-isomorphism of $\mathcal{D}$-complexes.

We define the *mapping cylinder* M_f of a map $f : X \to Y$ in **C** by the push out diagram

$$\begin{array}{ccc} IX & \xrightarrow{\bar{f}} & M_f \\ \uparrow{\scriptstyle i_1} & & {\scriptstyle \bar{i}_1}\uparrow \\ X & \xrightarrow{f} & Y \end{array} \tag{4.12}$$

If X and Y are finite dimensional $\mathcal{D}$-complexes and if f is cellular then (3.6) and (4.10) show that also M_f is a finite dimensional $\mathcal{D}$-complex and $\bar{i}_1 : Y \to M_f$ and $\bar{f} i_1 : X \to M_f$ are inclusions of subcomplexes. We also denote Y by ∂_1 and X by ∂_0 where ∂_0 and ∂_1 are boundary components of M_f. The projection $p : IX \to X$ induces the natural *retraction* $p : M_f \to Y$.

5 Cellular *I*-Categories

We now describe cellular I-categories. They have exactly the properties which we need to develop a theory of Whitehead torsion.

(5.1) Definition. A *cellular I-category* $(\mathbf{C}, \mathcal{D})$ is an I-category $\mathbf{C} = (\mathbf{C}, cof, I, *)$ as defined in (III.A.1) together with a set $\mathcal{D}$ of discrete objects in $\mathbf{C}$ as in (1.1) such that

(a) the ball pair axiom (4.5) holds and
(b) the following *cellular approximation property* is satisfied. Let X and K be finite dimensional $\mathcal{D}$-complexes and let L be a subcomplex of K. Moreover let $g : K \to X$ be a map in $\mathbf{C}$ such that the restriction $g \mid L : L \to X$ is cellular. Then there exists a cellular map $f : K \to X$ extending $g \mid L$ and a homotopy $f \simeq g \operatorname{rel} L$.

(5.2) Example. Let $\mathbf{C} = \mathbf{Top}$ be the category of topological spaces and let $\mathcal{D}_f$ be the class of all discrete spaces which are finite sets. Then a finite dimensional $\mathcal{D}_f$-complex is the same as a *finite CW-complex.* Moreover it is well known that $(\mathbf{Top}, \mathcal{D}_f)$ is a cellular I-category. Clearly also $(\mathbf{Top}, \mathcal{D})$ is a cellular I-category where $\mathcal{D}$ is the class of all discrete spaces in $\mathbf{Top}$. The cellular approximation property is a consequence of the classical cellular approximation theorem.

Given a cellular I-category $(\mathbf{C}, \mathcal{D})$ we have the functor

$$\lim : \mathcal{D}\text{-}\mathbf{Complex} \longrightarrow \mathcal{D}\text{-}\mathbf{cell} \tag{5.3}$$

Here $\mathcal{D}$-**Complex** is the full subcategory of $\mathbf{Fil}_0(\mathbf{C})_c$ consisting of finite dimensional $\mathcal{D}$-complexes and $\mathcal{D}$-**cell** is the full subcategory of $\mathbf{C}$ consisting of finite dimensional $\mathcal{D}$-complexes. Moreover the functor lim carries $X_{\geq 0}$ to the direct limit $\lim(X_{\geq 0})$.

(5.4) Properties. *Let* $(\mathbf{C}, \mathcal{D})$ *be a cellular I-category. Then the functor* lim *induces a functor*

$$\operatorname{Holim} : \mathcal{D}\text{-}\mathbf{Complex}/\overset{1}{\simeq} \longrightarrow \mathcal{D}\text{-}\mathbf{cell}/\simeq$$

between homotopy categories which is an equivalence of categories.

Proof. The cellular approximation property shows that Holim is full. Moreover if $f = \lim f_{\geq 0} \simeq \lim g_{\geq 0} = g$ there exists by (4.10) and the cellular approximation property a homotopy $H : f \simeq g$ where H is cellular and hence a 1-homotopy. q.e.d.

(5.5) Proposition. *Let* $(\mathbf{C}, \mathcal{D})$ *be a cellular I-category. Then each finite dimensional* $\mathcal{D}$*-complex* $X_{\geq 0}$ *has a normalization* $H = \{H_n,\, n \geq 0\}$ *as defined in (3.4). Moreover for a normalization* H *we can choose a normalized* $\mathcal{D}$*-complex* $X^H_{\geq 0}$ *and a homotopy equivalence*

$$\varphi^H : X^H_{\geq 0} \longrightarrow X_{\geq 0}$$

in $\mathcal{D}$-**Complex**$/\overset{0}{\simeq}$. *If* $K_{\geq 0} \subset X_{\geq 0}$ *is a subcomplex which is normalized then* $K_{\geq 0} \subset X^H_{\geq 0}$ *is a subcomplex and* φ_H *is the identity on* $K_{\geq 0}$.

Proof. The cellular approximation property readily shows that a normalization H as in (3.4) exists. We now choose the n-skeleton Y_n of $Y_{\geq 0} = X^H_{\geq 0}$ inductively as follows. Let $Y_0 = X_0$ and let $\varphi^H_0 : Y_0 \to X_0$ be the identity. Assume $\varphi^H_n : Y_n \to X_n$ is already constructed $n \geq 0$. Then we have the attaching map

$$\partial_{n+1} : \Sigma^n_D \longrightarrow X_n \quad \text{with } D = D_{n+1}$$

and a normalization $H_{n+1} : \partial_{n+1} \mid D \simeq i_0 0_n$ where $0_n : D \to X_0$. We now can choose by the cellular approximation theorem a homotopy

$$G : \varphi^H_n \partial_{n+1} \simeq \partial^H_{n+1}$$

such that $G \mid ID$ represents $\varphi^H_n H_{n+1}$. Then ∂^H_{n+1} is a cellular map and the homotopy class

$$\partial^H_{n+1} \in [\Sigma^n_D, Y_n]^D$$

is *well defined* by the choice of H. Now let $Y_{n+1} = Y_n \cup e^n_D$ be defined by the attaching map ∂^H_{n+1}. Then Y_{n+1} is normalized and the homotopy G yields an extension $\varphi^{n+1}_H : Y_{n+1} \to X_{n+1}$ of φ^n_H which is well defined up to homotopy rel Y_n and φ^{n+1}_H is a homotopy equivalence. q.e.d.

6 Elementary Expansions

The classical notion of elementary expansion for CW-complexes is defined by attaching ball pairs. Since we have such ball pairs also in a cellular I-category (see § 4) we obtain the following definition which describes the obvious generalization of Whitehead's definition of elementary expansion and elementary collapse.

(6.1) Definition. Let L be a finite dimensional $\mathcal{D}$-complex in the cellular I-category $(\mathbf{C}, \mathcal{D})$ and let $A \in \mathcal{D}$. Then we have by (4.5) for $n \geq 0$ the ball pair $(\square^{n+1}_A, Q^n_A)$ with the complement P^n_A in the boundary. Consider the push out diagram in $\mathbf{C}$.

$$\begin{array}{ccc} \square^{n+1}_A & \longrightarrow & K \\ {\scriptstyle i}\uparrow & & \uparrow \\ P^n_A & \xrightarrow{f} & L \end{array} \tag{1}$$

where f is given by a pair map

$$f : (P^n_A, \Sigma^{n-1}_A) \longrightarrow (L_n, L_{n-1}). \tag{2}$$

Here $\Sigma_A^{-1} = * = L_{-1}$ is the initial object for $n = 0$. The map f needs not to be cellular since we do not require that the 0-skeleton A of P_A^n is mapped by f to the 0-skeleton L_0. Still one readily checks that K is again in the obvious way a $\mathcal{D}$-complex with subcomplex L. Note that we have

$$K = L \cup e_A^n \cup e_A^{n+1} \tag{3}$$

where e_A^n is the n-cell of $Q_A^n = \Sigma_A^{n-1} \cup e_A^n$ and where e_A^{n+1} is the $(n+1)$-cell of the ball $\square_A^{n+1}$. Now let $\bar{K}$ be a $\mathcal{D}$-complex with subcomplex L and let $\varphi : \bar{K} \to K$ be an $\mathcal{D}$-isomorphism under L. Then we call $\bar{K}$ an *elementary expansion* of L. Since i in (1) is a weak equivalence we see that also $\bar{\imath} : L \to \bar{K}$ is a weak equivalence and hence by II.1.11 and II.1.12 in Baues [AH] there exists a retraction $r : \bar{K} \to L$ with $r\bar{\imath} = 1$ and $\bar{\imath}r \simeq 1 \operatorname{rel} L$. Here r is unique up to homotopy relative L. We call any such map $r : \bar{K} \to L$ an *elementary collapse* and we say that r "removes" the cells e_A^n and e_A^{n+1} of $\bar{K}$.

(6.2) Lemma. *Let $L \to K$ be an elementary expansion and let $h : (K, L) \cong (K', L')$ be a $\mathcal{D}$-isomorphism. Then also $L' \to K'$ is an elementary expansion.*

This is an immediate application of the definition of $\mathcal{D}$-isomorphism. Now

$$P_X^n \longrightarrow \square_X^{n+1} \tag{6.3}$$

$$Q_X^n \longrightarrow \square_X^{n+1} \tag{6.4}$$

are both elementary expansions. For (6.3) this is a consequence of the definition and for (6.4) we use (6.2) and (4.9).

(6.5) Example. For $\varepsilon \in \{0, 1\}$, $n \geq 0$ and $A \in \mathcal{D}$ the map

$$j_\varepsilon : \Delta_A^n \cup_{\partial_\varepsilon} I\Sigma_A^{n-1} \longrightarrow I\Delta_A^n$$

is an elementary expansion. This is clear for $\varepsilon = 0$ by (4.5) (3). For $\varepsilon = 1$ we apply the isomorphism n to the upper sequence in (4.5) (3) and we obtain isomorphisms $(1 \cup n)\alpha$ and $n\beta$ which show that j_1 is also an elementary expansion; compare (4.11).

(6.6) Lemma. *For $A \in \mathcal{D}$ and $n \geq 0$ the map*

$$j : \square_A^{n+1} \cup IP_A^n \cup \square_A^{n+1} \longrightarrow I\square_A^{n+1}$$

is an elementary expansion.

Here j is defined as in the relative cylinder axiom (III.7.1) (I4).

Proof. For $n = 0$ this is a special case of (6.5) since $P_A^0 \to \square_A^1$ coincides with $i_0 : A \to IA$. For $n \geq 1$ we have the following isomorphisms of pairs where we set $I_1 = I_2 = I$ and $\square = \square_A^{n+1}$, $P = P_A^n$, $Q = Q_A^n$, $\Delta = \Delta_A^n$, $\Sigma = \Sigma_A^{n-1}$.

$$\begin{array}{ccc}
(I_1\Box, \Box \cup I_1 P \cup \Box) & \xrightarrow{I\beta} & (I_1 I_2 \Delta, I_2\Delta \cup I_1(\Delta \cup I_2\Sigma) \cup I_2\Delta) \\
\uparrow & & \uparrow T \\
(I_2\Box, \Box \cup I_2(P \cup Q)) & \xrightarrow{I\beta} & (I_2 I_1 \Delta, I_1\Delta \cup I_2(\Delta \cup I_1\Sigma) \cup I_1\Delta) \\
\uparrow I\bar{\kappa} & & \\
(I\Delta_A^{n+1}, \Delta_A^{n+1} \cup I\Sigma_A^n) = j_0 & &
\end{array}$$

Here j_0 is an elementary expansion by the example (6.5). Hence the isomorphism $(I\beta^{-1})T(I\beta)(I\bar{\kappa})$ shows that also j in (6.6) is an elementary expansion. q.e.d.

7 Formal Deformations and Simple Homotopy Equivalences

Let $(\mathbf{C}, \mathcal{D})$ be a cellular I-category. Let K and L be finite dimensional $\mathcal{D}$-complexes in $\mathbf{C}$ and let Y be a subcomplex of L and K. We say that a map $j : L \to K$ under Y is an *expansion relative* Y if j is the finite composition

$$L = K_{(0)} \to K_{(1)} \to \cdots \to K_{(r)} = K$$

of elementary expansions $K_{(i)} \to K_{(i+1)}$ with $0 \le i < r$. In this case we write $L \nearrow K \operatorname{rel} Y$. On the other hand a map $r : K \to L$ under Y is a *collapse relative* Y if r is the finite composition

$$K = L_{(0)} \to L_{(1)} \to \cdots \to L_{(r)} = L$$

of elementary collapses $L_{(i)} \to L_{(i+1)}$ which do not remove cells in Y. Then we write $K \searrow L \operatorname{rel} Y$. A finite composition of expansions and collapses relative Y

$$L = X_{(0)} \nearrow X_{(1)} \searrow X_{(2)} \nearrow \cdots \searrow X_{(r)} = K \tag{7.1}$$

is called a *formal deformation relative* Y. Hence a formal deformation relative Y is a map $L \to K$ under Y in $\mathbf{C}$ which is obtained by a finite composition of a sequence consisting of elementary expansions and elementary collapses respectively. A formal deformation relative Y is symbolized by $L \curvearrowright K \operatorname{rel} Y$. A map $f : L \to K$ under Y in $\mathbf{C}$ is a *simple homotopy equivalence* $\operatorname{rel} Y$ if f is homotopic $\operatorname{rel} Y$ to a formal deformation $L \curvearrowright K \operatorname{rel} Y$. If $Y = *$ is the initial object then we say that f is a simple homotopy equivalence in the cellular I-category $(\mathbf{C}, \mathcal{D})$.

We now follow the book of Marshall Cohen [SH] in describing properties of formal deformations; in fact this section contains all the results of § 5 in Cohen [SH].

Let X, Y, Z, K, L, J ... be finite dimensional $\mathcal{D}$-complexes in $\mathbf{C}$. We write $Y < K$ or $K > Y$ if Y is a subcomplex of the $\mathcal{D}$-complex K.

(7.2) Lemma. *Let $f : K \to L$ be cellular and let $Y < K$ be a subcomplex. Then $M_{f|Y} \nearrow M_f$.*

This is an inductive application of (6.5). The lemma implies for $Y = *$ that $L \nearrow M_f$. Moreover we get by (6.5) that for $\varepsilon \in \{0,1\}$

$$IY \cup_{\partial_\varepsilon} K \nearrow IK \tag{7.3}$$

where the left hand side is the push out of $IY \xleftarrow{i_\varepsilon} Y \to K$. For $\varepsilon = 1$ we use (4.11) and (6.2) so that (7.3) is true for both cell structures of IK.

(7.4) Lemma. *(a) Let $K > Y < L$ be $\mathcal{D}$-isomorphic to $K' > Y' < L'$ and let $K \curvearrowright L \operatorname{rel} Y$. Then $K' \curvearrowright L' \operatorname{rel} Y'$.*

(b) Let $K > Y < L$ be given with $K \curvearrowright L \operatorname{rel} Y$ and let $f : Y \to Y''$ be cellular. Then $K \cup_f Y'' \curvearrowright L \cup_f Y''$.

This follows from (6.2) and properties of push outs.

(7.5) Lemma. *Let $L > Y < K$ and let $h : L \to K$ be a $\mathcal{D}$-isomorphism under Y. Then $L \curvearrowright K \operatorname{rel} Y$.*

Proof. We have $IY \cup_{\partial_1} K < M_h$ since h is the identity on Y so that $IY \cup_{\partial_1} K \nearrow M_h \operatorname{rel} IY$ by (7.2). Moreover Ih induces a $\mathcal{D}$-isomorphism $M_h \cong IL$ and this shows that also $IY \cup_{\partial_0} L \nearrow M_h \operatorname{rel} IY$ by (7.4). Now let

$$\bar{M}_h = M_h \cup_{IY} Y$$

where we use the projection $p : IY \to Y$. Then (7.4) shows

$$K \nearrow \bar{M}_h \nwarrow L \operatorname{rel} Y.$$

q.e.d.

(7.6) Lemma. *Let $f : K \to L$ be cellular and $Y \nearrow K$ then $K \cup_Y M_{f|Y} \nearrow M_f$.*

Proof. This is an inductive application of (6.6). q.e.d.

(7.7) Lemma. *Let $f, g : K \to L$ be homotopic cellular maps then $M_f \curvearrowright M_g \operatorname{rel} L \cup K$.*

Proof. Let $H : IK \to L$ be a homotopy $f \simeq g$ which we may assume to be cellular. Then (7.6) shows

$$M_f \cup IK \nearrow M_H \nwarrow M_g \cup IK \operatorname{rel} L \cup IK$$

since $i_0 : K \nearrow IK$ and $i_1 : K \nearrow IK$ by (7.3). Let $p : IK \to K$ the projection and $M = M_H \cup_p K$. Then (7.4) shows

$$M_f \nearrow M \nwarrow M_g \operatorname{rel} L \cup K.$$

q.e.d.

(7.8) Lemma. *Let $f : L \to K, g : K \to J$ be cellular and let $M_f \cup_K M_g$ be the push out of $M_f \leftarrow K \to M_g$. Then*

$$M_{gf} \curvearrowright M_f \cup_K M_g \ \mathrm{rel}\, L \cup I.$$

Proof. Let $p : M_f \to K$ be the natural retraction. Then $gp : M_f \to J$ is cellular with $gp \mid K = g$ and $gp \mid L = gf$. Since $M_f \searrow K$ it follows from (7.6) that $M_{gp} \searrow M_f \cup M_g$. On the other hand, since $L < M_f$ we see by (7.2) that $M_{gp} \searrow M_{gf}$. q.e.d.

More generally one gets for a sequence of cellular maps

$$K_{(1)} \xrightarrow{f_1} K_{(2)} \longrightarrow \dots \xrightarrow{f_{q-1}} K_{(q)}$$

with $f = f_{q-1} \dots f_1$ the deformation

$$M_f \curvearrowright M_{f_1} \cup M_{f_2} \cup \dots \cup M_{f_{q-1}} \ \mathrm{rel}\, K_{(1)} \cup K_{(q)}. \tag{7.9}$$

(7.10) Lemma. *Given a map $f : K \to L$ the following are equivalent statements:*

(a) f is a simple homotopy equivalence.
(b) There exists a cellular approximation g to f such that $M_g \curvearrowright K$ rel K.
(c) For any cellular approximation g to f one has $M_g \curvearrowright K$ rel K.

Proof. (a) $\Rightarrow$ (b): We know that f is homotopic to a formal deformation g with

$$g : K = K_{(0)} \to K_{(1)} \to \dots \to K_{(q)} = L$$

and $g_i : K_{(i)} \to K_{(i+1)}$. Note that $M_{g_i} \searrow K_{(i)}$ for all i. In fact, if $K_{(i)} \nearrow K_{(i+1)}$ is an elementary expansion then

$$M_{g_i} = IK_{(i)} \cup_{g_i} K_{(i+1)} \searrow IK_{(i)} \searrow K_{(i)}$$

and if $K_{(i)} \searrow K_{(i+1)}$ is an elementary collapse then by (7.6)

$$M_{g_i} \searrow M_{g_i|K_{(i+1)}} \cup K_{(i)} = IK_{(i+1)} \cup_{\partial_0} K_{(i)} \searrow K_{(i)}.$$

Hence (a) $\Rightarrow$ (b) follows from (7.9). Now (b) $\Rightarrow$ (c) follows from (7.7). Moreover (c) $\Rightarrow$ (a) is an exercise; see 5.8 Cohen [SH]. q.e.d.

(7.11) Lemma. *Let $U < Y < K$ and $U < X$ and let $f : Y \to X$ be cellular under U and consider the push out diagram*

$$\begin{array}{ccc} K & \xrightarrow{g} & K \cup_f X \\ \uparrow & & \uparrow \\ Y & \xrightarrow{f} & X \end{array}$$

If f is a simple homotopy equivalence rel U *then also g is a simple homotopy equivalence* rel U.

This is a *simple analogue of the push out axiom* (C2) (a) in a cofibration category; see (III.1.1).

Proof. The mapping cylinder M_g satisfies $M_g = IK \cup_q M_f$ where $q : IY \to M_f$ is $\bar{f}$ in (4.12). But $IY \cup_{\partial_0} K \nearrow IK$ so that by (7.4)

$$M_f \cup_{\partial_0} K \nearrow M_g.$$

Moreover $M_f \cup_{\partial_0} K \curvearrowright K$ rel K by (7.10) and (7.4) (b). Clearly $g \mid Y = f$ and by (7.10) g is a simple homotopy equivalence. q.e.d.

(7.12) Lemma. *Let $A \in \mathcal{D}$ and let*

$$f,\ g : \Sigma_A^n \longrightarrow Y_n$$

be maps which are homotopic in Y; that is, there is $H : if \simeq ig$ with $i : Y_n \subset Y$ where $n \geq 0$. Then there is a formal deformation

$$Y \cup_f e_A^{n+1} \curvearrowright Y \cup_g e_A^{n+1} \ \mathrm{rel}\, Y.$$

Proof. First we observe that there is a homotopy $H_1 : f \simeq f'$ in Y_n where f' is cellular. Similarly there is a homotopy $H_2 : g \simeq g'$ in Y_n where g' is cellular. Now H shows that $H' : f' \simeq g'$ in Y and cellular approximation allows to choose the homotopy $H' : I\Sigma_A^n \to Y$ to be cellular. Hence we may assume that H is a homotopy $jf \simeq jg$ where $j : Y_n \subset Y_{n+1}$. We consider the space

$$K = Y \cup_f e \cup_g e' \quad \text{with } e = e' = e_A^{n+1}.$$

Using H we obtain the following commutative diagram where $\Delta = \Delta_A^{n+1}$ and $\Sigma = \Sigma_A^n$.

$$\begin{array}{ccc} \Sigma & \xrightarrow{i_1} & \Delta \cup_{\partial_0} I\Sigma \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varphi_g H} \\ \Delta & \xrightarrow{\varphi'} & K_{n+1} \subset K \end{array}$$

where φ' and φ are the characteristic maps of e' and e respectively with $\varphi i = f$ and $\varphi' i = g$. Hence we obtain the push out

$$\begin{array}{ccc} (\Delta \cup_{\partial_0} I\Sigma) \cup_{\partial_1} \Delta & \overset{j}{\rightarrowtail} & I\Delta \\ \downarrow{\scriptstyle (\varphi, H, \varphi')} & & \downarrow \\ K & \longrightarrow & \bar{K} \end{array}$$

Here j is given by the maps in the top row of (4.5) (3) and hence j is isomorphic to $P_A^n \cup Q_A^n \rightarrowtail \square_A^{n+1}$. This shows that

$$Y \cup_f e \nearrow \bar{K}$$

since $P_A^n \nearrow \square_A^{n+1}$. Since also $Q_A^n \nearrow \square_A^{n+1}$ by (6.4) we see that also

$$Y \cup_g e' \nearrow \bar{K}$$

Hence we obtain the proposition. q.e.d.

(7.13) Proposition. *Let* $(\mathbf{C}, \mathcal{D})$ *be a cellular* I*-category and let* X *be a finite dimensional* $\mathcal{D}$*-complex. Then there exists a normalized finite dimensional* $\mathcal{D}$*-complex* Y *together with a simple homotopy equivalence* $Y \to X$.

Proof. We use the same method as in the proof of (5.5). Then inductive application of (7.12) and (7.11) yields the result. q.e.d.

(7.14) Lemma. *Let* $L < K$ *and let* $f, g : L \to J$ *be cellular maps which are homotopic* $f \simeq g$. *Then* $K \cup_f J \curvearrowright K \cup_g J \operatorname{rel} J$.

We point out that (7.12) is not directly a consequence of (7.14) since in (7.12) we do not assume that f, g are cellular.

Proof. Let $H : f \simeq g$ be a homotopy which we can choose to be a cellular map $H : IL \to J$. Let $j_0 : K \cup_{\partial_0} IL \to IK$ and $j_1 : K \cup_{\partial_1} IL \to IK$ be the inclusions in (7.3). Then we have by (7.4) (b)

$$\begin{aligned} K \cup_f J &= (K \cup_{\partial_0} IL) \cup_H J \nearrow IK \cup_H J \operatorname{rel} J, \\ K \cup_g J &= (K \cup_{\partial_1} IL) \cup_H J \nearrow IK \cup_H J \operatorname{rel} J. \end{aligned}$$

q.e.d.

(7.15) Lemma. *Let* $L < K < J$ *and let* $i : L \to K$ *be a homotopy equivalence with retraction* r. *Then* $J \curvearrowright K \cup_L (J \cup_r L)$.

Proof. We have $ir \simeq 1$ so that by (7.14)

$$J = J \cup_1 K \curvearrowright J \cup_{ir} K = K \cup_L (J \cup_r L).$$

q.e.d.

8 The Whitehead Group and Whitehead Torsion

Let $(\mathbf{C}, \mathcal{D})$ be a cellular I-category. In this section we construct a functor

$$\mathrm{Wh} : \mathcal{D}\text{-}\mathbf{cell}/{\simeq} \longrightarrow \mathbf{Ab} \tag{8.1}$$

which carries a finite dimensional $\mathcal{D}$-complex L in $\mathbf{C}$ to the *Whitehead group* $\mathrm{Wh}(L)$. Moreover we obtain a function assigning to any homotopy equivalence $f : Y \to L \in \mathbf{C}/{\simeq}$ between finite dimensional $\mathcal{D}$-complexes Y, L an element

$$\tau(f) \in \mathrm{Wh}(L) \tag{8.2}$$

termed the *Whitehead torsion* of f. The Whitehead torsion detects simple homotopy equivalences in the following sense.

(8.3) Theorem. *A homotopy equivalence $f : Y \to L$ between finite dimensional $\mathcal{D}$-complexes is a simple homotopy equivalence if and only if $\tau(f) = 0$.*

Moreover the following formulas hold:

(8.4) Addendum. τ is a *derivation* on the subcategory of $\mathcal{D}$-**cell**$/\simeq$ consisting of homotopy equivalences; that is

$$\tau(gf) = \tau(g) + g_* \tau(f).$$

Here $gf : L \to K \to J$ is the composite of homotopy equivalences g and f between finite dimensional $\mathcal{D}$-complexes. This is also called the "logarithmic property" of τ.

(8.5) Addendum. Consider the commutative diagram

$$\begin{array}{ccccc} K' & > & Y' & < & L' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ K & > & Y & < & L \end{array}$$

where K, L, K', L' are finite dimensional $\mathcal{D}$-complexes with subcomplexes Y and Y' respectively as indicated. Then we obtain the induced map

$$f \cup g : K' \cup_{Y'} L' \longrightarrow K \cup_Y L.$$

If f, h and g are homotopy equivalences in $\mathbf{C}$ then II.1.2 in Baues [AH] shows that also $f \cup g$ is a homotopy equivalence and one obtains the following *additivity formula.*

$$\tau(f \cup g) = j_*^K \tau(f) + j_*^L \tau(g) - j_*^Y \tau(h).$$

Here j^K, j^L, j^Y are the canonical maps from K, L, resp. Y to $K \cup_Y L$.

(8.6) Definition of the Whitehead group. We call (K, L) a $\mathcal{D}$- *pair* if L is a subcomplex of the finite dimensional $\mathcal{D}$-complex K such that the inclusion $L \to K$ is a homotopy equivalence in $\mathbf{C}$. Two $\mathcal{D}$-pairs (K, L) and (K', L) are equivalent if $K \curvearrowright K' \operatorname{rel} L$. This is an equivalence relation and we let $[K, L]$ denote the equivalence class of (K, L). An addition of equivalence classes is defined by

$$[K, L] + [K', L] = [K \cup_L K', L].$$

Since we assume $\mathcal{D}$ to be a set we see by (7.5) that the equivalence classes $[K, L]$ of $\mathcal{D}$-pairs (K, L) form a set $\mathrm{Wh}(L)$ which is termed the *Whitehead group* of L.

(8.7) Lemma. *The set* $\mathrm{Wh}(L)$ *with the addition* $+$ *is a well defined abelian group.*

Proof. Clearly if (K, L) and (K', L) are $\mathcal{D}$-pairs then also $(K \cup_L K', L)$ is a $\mathcal{D}$-pair. Moreover if $[K, L] = [J, L]$, then

$$K \cup_L K' \curvearrowright J \cup_L K' \operatorname{rel} L$$

by (7.4) (b). Hence $+$ is well defined. It is clear that $+$ is associative and commutative. The element $[L, L]$ is a neutral element, denoted by 0. We have to show that each element $[K, L]$ admits an inverse $[\bar{K}, L]$ with

$$[\bar{K}, L] + [K, L] = [L, L]. \tag{1}$$

That is $[\bar{K}, L] = -[K, L]$. We construct $\bar{K}$ as follows. Let $f : K \to L$ be a retraction of the inclusion $L \to K$. Such a retraction exists by II.1.12 Baues [AH]. Moreover by the cellular approximation property we may assume that f is cellular. Now let

$$\bar{K} = M_f \cup_K M'_f \tag{2}$$

where $M_f = M'_f$ is the mapping cylinder of f. We have for $L = L'$ the inclusions $L \subset M_f \subset \bar{K}$ and $L' \subset M'_f \subset \bar{K}$ of subcomplexes. We then get the following equations with $K'' = K$,

$$\begin{aligned} [\bar{K}, L] + [K'', L] &= [\bar{K} \cup_L K'', L] \\ &= [(M_f \cup_L K'') \cup_K M'_f, L] && (3) \\ &= [M_{if} \cup_K M'_f, L], \quad \text{where } i : L < K. && (4) \end{aligned}$$

Since $if \simeq 1_K$ we see by (7.7) that

$$M_{if} \curvearrowright M_{1_K} = IK \operatorname{rel} \partial_0 \cup \partial_1 = K \vee K.$$

Hence by (7.4) (b) we have

$$\left.\begin{aligned} &= [IK \cup_K M'_f, L] \\ &= [IL \cup_L M'_f, L] \quad \text{since } IK \searrow IL \cup_{\partial_0} K \text{ by (7.3)} \\ &= [IL \cup_L IL, L] \quad \text{since } M'_f \searrow IL \text{ by (7.2)} \\ &= [L, L] \quad \text{since } IL \searrow L \text{ by (7.3).} \end{aligned}\right. \tag{4}$$

q.e.d.

(8.8) Definition of induced maps. Let $f : L \to J$ be a cellular map between finite dimensional $\mathcal{D}$-complexes. Then the induced map

$$f_* : \mathrm{Wh}(L) \to \mathrm{Wh}(J)$$

between Whitehead groups is defined by

$$f_*[K, L] = [K \cup_L J, J] = [K \cup_L M_f, J].$$

The equation on the right hand side holds since the projection $p : M_f \to J$ satisfies $p \mid J = 1$ so that by (7.11) we have

$$M_f \cup_L K \curvearrowright (M_f \cup_L K) \cup_p J = K \cup_L J \operatorname{rel} J.$$

Now it is easy to see that f_* is a group homomorphism and that $g_* f_* = (gf)_*$. If $f \simeq g$ then $f_* = g_*$ as follows from (7.7).

(8.9) Definition of the Whitehead torsion. Let $f : K \to L$ be a cellular homotopy equivalence in $\mathbf{C}$ between finite dimensional $\mathcal{D}$-complexes. Then we define

$$\begin{aligned} \tau(f) &= f_*[M_f, K] \\ &= [M_f \cup_K L, L] \in \mathrm{Wh}(L). \end{aligned}$$

By (7.7) we see that $f \simeq g$ implies $\tau(f) = \tau(g)$.

(8.10) Proof of (8.3). If f is a simple homotopy equivalence then (7.10) shows that $M_f \curvearrowright K$ rel K and hence $[M_f \cup_K L, L] = [L, L] = 0$ by (7.4) (b) so that $\tau(f) = 0$. On the other hand assume now that $\tau(f) = f_*[M_f, K] = 0$. Since Wh is a functor on the homotopy category $\mathcal{D}$-**cell**$/\simeq$ we know that

$$f_* : \mathrm{Wh}(K) \longrightarrow \mathrm{Wh}(L)$$

is an isomorphism. Hence $\tau(f) = 0$ if and only if $[M_f, K] = 0$ and this implies by (7.10) that f is a simple homotopy equivalence. q.e.d.

(8.11) Proof of (8.4). For $L \xrightarrow{f} K \xrightarrow{g} J$ we have to show

$$(gf)_*[M_{gf}, L] = g_*[M_g, K] + g_* f_*[M_f, L]$$

or equivalently since g_* and f_* are isomorphisms

$$[M_{gf}, L] = \bar{f}_*[M_g, K] + [M_f, L]$$

where $\bar{f} : K \to L$ is a cellular homotopy inverse to f. This is equivalent by (7.8) to

$$M_g \cup_K M_f \curvearrowright M_g \cup_K L \cup_L M_f \ \mathrm{rel}\, L.$$

But this follows by (7.14) since the following diagram homotopy commutes

$$\begin{tikzcd} & K \arrow[dr] & \\ K \arrow[ur, "1"] \arrow[r, "\bar{f}"] & L \arrow[u, "f"] \arrow[r] & M_f \end{tikzcd}$$

(8.12) Proof of (8.5). Let $X = K \cup_Y L$ and $X' = K' \cup_{Y'} L'$ and $d = f \cup g : X \to X'$. Since $d_* : \mathrm{Wh}(X) \to \mathrm{Wh}(X')$ is an isomorphism it suffices to prove in $\mathrm{Wh}(X')$

$$[M_d, X'] = [M_f \cup_{K'} X', X'] + [M_g \cup_{L'} X', X'] - [M_h \cup_{Y'} X', X'].$$

Using properties of push outs we obtain the following cubical diagram in which all squares are push outs.

$$\begin{tikzcd}
M_h \arrow[rrr, "k"] \arrow[ddd] & & & M_g \arrow[ddd] \\
& Y' \arrow[r] \arrow[d] \arrow[ul] & L' \arrow[d] \arrow[ur] & \\
& K' \arrow[r] \arrow[dl] & X' \arrow[dr] & \\
M_f \arrow[rrr] & & & M_d
\end{tikzcd}$$

By (7.2) we have $M_g \nearrow M_k$ and by (7.4) (b) we get $M_d \nearrow M_d \cup_{M_g} M_k$. Hence we obtain by (7.4) (b) in $\mathrm{Wh}(X')$ the equations

$$[X' \cup_{L'} M_g, X'] = [X' \cup_{L'} M_k, X']$$
$$[M_d, X'] = [M_d \cup_{M_g} M_k, X']$$

Write $X'' = M_d \cup_{M_g} M_k$, $L'' = X' \cup_{L'} M_k$, $K'' = X' \cup_{K'} M_f$, $Y'' = X' \cup_{Y'} M_k$. Then we have to show in $\mathrm{Wh}(X')$

$$[X'', X'] = [K'', X'] + [L'', X'] - [Y'', X'].$$

Here $L'' > Y'' < K''$ are subcomplexes of X'' where X'' is the push out of $L'' \leftarrow Y'' \rightarrow K''$. Let $r : Y'' \rightarrow X'$ be a retraction. Then (7.15) shows in $\mathrm{Wh}(X')$

$$[X'', X'] = r_*[X'', Y''] + [Y'', X']$$
$$[K'', X'] = r_*[K'', Y''] + [Y'', X']$$
$$[L'', X'] = r_*[L'', Y''] + [Y'', X']$$

On the other hand we have in $\mathrm{Wh}(Y'')$

$$[X'', Y''] = [K'', Y''] + [L'', Y''].$$

These equations imply in $\mathrm{Wh}(X')$

$$[X'', X'] - [Y'', X'] = [K'', X'] - [Y'', X'] + [L'', X'] - [Y'', X'].$$

q.e.d.

(8.13) Remark. Following Eckmann-Maumary [GS] and Siebenmann [S] the book of Kamps-Porter [AH] describes a different axiomatic approach concerning the geometric Whitehead group. The axioms they use do not rely on cells as in our approach above and assume a priori that all isomorphisms are simple. This is actually a great disadvantage concerning the applicability of the axioms. Moreover the axioms of Kamps-Porter are by far not sufficient to prove a result like in § 12 below on the isomorphism between geometric Whitehead torsion and algebraic Whitehead torsion which is the main result of simple homotopy theory.

9 Simplified Form of Elements in the Whitehead Group

Let $(\mathbf{C}, \mathcal{D})$ be a cellular I-category and let L be a finite dimensional $\mathcal{D}$-complex. Then the Whitehead group $\mathrm{Wh}(L)$ is defined. The elements $[K, L] \in \mathrm{Wh}(L)$ are represented by $\mathcal{D}$-pairs (K, L) as defined in (8.6).

(9.1) Proposition. *Let (K, L) be a $\mathcal{D}$-pair with $\dim(K - L) \leq n$ and let $r \geq n-1$. Then there exists a $\mathcal{D}$-pair (K', L) such that*

$$K' = L \cup_f e^r_D \cup e^{r+1}_{D'}$$

with $D, D' \in \mathcal{D}$ and $[K, L] = [K', L]$. Moreover the attaching map f of e^r_D is a composite

$$f : \Sigma^{r-1}_D \xrightarrow{p^{r-1}} D \longrightarrow L_0 \subset L$$

where p^{r-1} is the projection in (2.1) (2). In addition if L is normalized then K' can be chosen to be normalized.

The pair (K', L) with the properties in the proposition is termed a $\mathcal{D}$ *-pair in simplified form.*

For the proof of (9.1) we consider the following diagram which is given by (4.5) (3) with $X \in \mathcal{D}$ and $\Sigma^r_X = \Sigma^r$, $P^r = P^r_X$, ... (i.e., we omit the index X for simplicity).

$$\begin{tikzcd}
\Sigma^r \arrow[r, "i"] \arrow[d, "\kappa"] & \Delta^{r+1} \arrow[d, "\bar{\kappa}"] \\
P^r \cup_\partial Q^r \arrow[r] \arrow[d, "\alpha \cup 1"] & \square^{r+1} \arrow[d, "\beta"] \\
\partial I\Delta^r = \Delta^r \cup I\Sigma^{r-1} \cup \Delta^r \arrow[r, "j"] & I\Delta^r
\end{tikzcd}$$

Here we set $j = (i_0, Ii, i_1)$. The vertical arrows are isomorphisms as defined in (4.5) (3) and (4.5) (2). We now apply the cylinder functor I to this diagram and we obtain

$$\begin{tikzcd}
P^{r+1} \arrow[r] \arrow[d, "\alpha"] & \square^{r+2} \arrow[d, "\beta"] & Q^{r+1} \arrow[l] \arrow[d, "1"] \\
\Delta^{r+1} \cup I\Sigma^r \arrow[r] \arrow[d] & I\Delta^{r+1} \arrow[d, "I(\beta\bar{\kappa})"] & \Delta^{r+1} \arrow[l] \arrow[d] \\
\Delta^{r+1} \cup I\partial I\Delta^r \arrow[r, "(i_0, Ij)"] & II\Delta^r & I\Delta^r \arrow[l, "i_1"']
\end{tikzcd} \tag{9.3}$$

Here again the vertical arrows are isomorphisms. This shows that $i_1 : I\Delta^r \to II\Delta^r$ is an expansion and that also the composite

$$\Delta^r = Q^r \subset \partial \xrightarrow{i_1} I\partial \subset \Delta^{r+1} \cup I\partial \tag{9.4}$$

is an expansion with $\partial = \partial I \Delta^r$.

(9.5) Lemma. *Let (K, L) be a $\mathcal{D}$-pair with*

$$K = L \cup e^r_{D_r} \cup e^{r+1}_{D_{r+1}} \cup \cdots \cup e^n_{D_n}$$

where $0 \le r \le n$ and $D_i \in \mathcal{D}$ for $r \le i \le n$. Then there exists a $\mathcal{D}$-pair (K', L) with $[K, L] = [K', L]$ where

$$K' = L \cup e^{r+1}_{D_{r+1}} \cup e^{r+2}_{D_{r+2} \vee D_r} \cup e^{r+3}_{D_{r+3}} \cup \cdots \cup e^n_{D_n}.$$

Proof. Since (K, L) is a $\mathcal{D}$-pair there exists a retraction $q : K \to L$ of $i : L \to K$ with $qi = 1$ and

$$H : 1 \simeq iq \operatorname{rel} L. \tag{1}$$

Moreover we may assume that q and the homotopy H are cellular. We now consider for $X = D_r$ the composite

$$\varphi : \Delta^r \longrightarrow L \cup e^r_{D_r} \subset K \tag{2}$$

where the left hand map is the characteristic map of the r-cell e^r_D. By applying the homotopy (1) we get a homotopy rel Σ^{r-1}

$$F : I\Delta^r \to K, \quad F : \varphi \simeq \Psi \tag{3}$$

with $\Psi = iq\varphi : \Delta^r \to L$. Moreover F maps $I\Delta^r$ to K_{r+1} and maps $\partial(I\Delta^r)$ to K_r. For $i_0 : I\Delta^r \to II\Delta^r$ let

$$Y = K \cup_F II\Delta^r. \tag{4}$$

Then clearly $K \nearrow P$. Moreover let Y_0 be the subcomplex of Y given by

$$Y_0 = L \cup e^r_{D_r} \cup_{F_0} E^{r+1} \quad \text{with} \tag{5}$$

$$\begin{cases} E^{r+1} = I\Delta^r \cup_{\partial_1} I(\partial I \Delta^r) \xrightarrow{(i_1, I_i)} II\Delta^r \\ F_0 = F \mid \partial I \Delta^r \end{cases}$$

Then (9.4) shows that $L \nearrow Y_0$. Let $g : Y_0 \to L$ be a retraction which is cellular and let

$$K' = Y \cup_g L. \tag{6}$$

Then (7.11) shows that $K \curvearrowright K' \operatorname{rel} L$ and hence $[K, L] = [K', L]$. Moreover by (9.3) we see that K' has the cell structure in (9.5). q.e.d.

(9.6) Proof of (9.1). We can apply (9.5) inductively and get a $\mathcal{D}$-pair $(\hat{K}, L)$ with $\hat{K}_{r-1} = L_{r-1}$. Since $r \geq n-1$ we do not obtain any cells in $\hat{K}$ of dimension greater $r+1$. Hence we get

$$\hat{K} = L \cup_g e_D^r \cup e_{D'}^{r+1}.$$

We claim that there exists a homotopy $g \simeq f$ where f is a map as in (9.1). Then (7.12) yields the result in (9.1). As in (1) we choose a retraction $\hat{q} : \hat{K} \to L$. Then the composite

$$\Delta_D^r \longrightarrow L \cup_g e_D^r \longrightarrow \hat{K} \xrightarrow{\hat{q}} L$$

is an extension of $g : \Sigma_D^{r-1} \to L$. This shows by (2.1) (6) that one has a homotopy $g \simeq \beta p^{r-1}$ in L where $\beta : D \to L$. Moreover by cellular approximation β is homotopic to a map $\alpha : D \to L_0 \to L$. This yields the homotopy $g \simeq \alpha p^{r-1} = f$. As in (7.13) we may choose the attaching map of the $(r+1)$-cell $e_{D'}^{r+1}$ to be normalized. q.e.d.

We say that an elementary expansion $K \to K' = K \cup e_D^r \cup e_D^{r+1}$ has *order* $r+1$, also the collapse $K' \to K$ has order $r+1$.

(9.7) Proposition. *Let (K, L) and (K', L) be $\mathcal{D}$-pairs and assume $K_{r-1} = L_{r-1} = K'_{r-1}$ for some $r \geq 0$. If $D : K \curvearrowright K'$ $\mathrm{rel}\, L$ yields a simple homotopy equivalence $\varphi : K \to K'$ under L then there is a sequence of elementary deformations $D' : K \curvearrowright K'$ all of which have orders $\geq r+1$, such that D' yields a map $\varphi' : K \to K'$ under L with $\varphi \simeq \varphi'$ $\mathrm{rel}\, L$.*

Proof. The result is true for $r = 0$. Assume now the result holds for $r = n-1 \geq 0$. Then we prove the case $r = n$ with $K_{n-1} = L_{n-1} = K'_{n-1}$ as follows. The inductive assumption shows that we can find $D : K \curvearrowright K'$ $\mathrm{rel}\, L$ where D is a sequence of order $\geq n$. By reordering we get sequences of elementary expansions $K \nearrow X$, $K' \nearrow Y$ of order $\geq n$ and a $\mathcal{D}$-isomorphism $\alpha : X \cong Y$ under L such that

$$D : K \nearrow X \cong Y \searrow K' \ \mathrm{rel}\, L. \tag{1}$$

By the assumption we have

$$K = L \cup \{\text{cells of dimension} \geq n\} \tag{2}$$

$$K' = L \cup \{\text{cells of dimension} \geq n\} \tag{3}$$

$$X = K \cup e_U^{n-1} \cup e_U^n \cup \{\text{cells of dimension} \geq n\} \tag{4}$$

$$Y = K' \cup e_V^{n-1} \cup e_V^n \cup \{\text{cells of dimension} \geq n\} \tag{5}$$

Here $U, V \in \mathcal{D}$ and $e_U^{n-1} \cup e_U^n$ and $e_V^{n-1} \cup e_V^n$ denote elementary expansions which are attached to L by (2) and (3) respectively. Let $I_L X$ be the relative cylinder and let Z be the push out of $r_1 : X \to I_L X$ and $\alpha : X \to Y$. Then we also have by (1)

$$K \nearrow X \nearrow Z \nwarrow Y \nwarrow K' \ \mathrm{rel}\, L \tag{6}$$

where all expansions are of order $\geq n$. We now "kill" the $(n-1)$ cells e_U^{n-1} and e_V^{n-1} by the procedure in the proof of (9.5) such that we obtain new $(n+1)$-cells e_U^{n+1} and e_V^{n+1} for which $e_U^n \cup e_U^{n+1}$ and $e_V^n \cup e_V^{n+1}$ describes expansions of order $n+1$. Moreover by (7.6) we see that

$$X \cup_L Y \nearrow X \cup_L Y \cup Ie_U^{n-1} \cup Ie_U^n \subset Z \tag{7}$$

is an elementary expansion of order $n+1$. This completes the proof. q.e.d.

10 The Torsion Group K_1

Following Ranicki [AT] we define for a small additive category $\mathbf{A}$ the torsion groups $K_1^{\mathrm{iso}}(\mathbf{A})$ and $K_1^{\mathrm{aut}}(\mathbf{A})$ respectively.

(10.1) Definition. Let $\mathbf{A}$ be a small additive category with direct sum $\oplus$.

(a) The *isomorphism torsion group* $K_1^{\mathrm{iso}}(\mathbf{A})$ is the abelian group with one generator $\tau(f)$ for each isomorphism $f : M \to N$ in $\mathbf{A}$, subject to the relations

$$\tau(gf : M \to N \to P) = \tau(f : M \to N) + \tau(g : N \to P)$$
$$\tau(f \oplus f' : M \oplus M' \to N \oplus N') = \tau(f : M \to N) + \tau(f' : M' \to N')$$

(b) The *reduced* isomorphism torsion group $\tilde{K}_1^{\mathrm{iso}}(\mathbf{A})$ is the quotient of $K_1^{\mathrm{iso}}(\mathbf{A})$ defined by the additional relation $\tau(\varepsilon_{M,N}) = 0$ where

$$\varepsilon_{M,N} : M \oplus N \to N \oplus M, \quad \varepsilon_{M,N} = \begin{pmatrix} 0 & 1_N \\ 1_M & 0 \end{pmatrix}$$

is the *interchange isomorphism* for $M,\ N \in \mathbf{A}$. Let $K_1^{\mathrm{iso}}(\mathbf{A}) \to \tilde{K}_1^{\mathrm{iso}}(\mathbf{A})$, $\tau(f) \mapsto \tilde{\tau}(f)$ be the quotient map.

(c) The *automorphism torsion group* $K_1^{\mathrm{aut}}(\mathbf{A})$ $K_1^{\mathrm{aut}}(\mathbf{A})$ is the abelian group with one generator $\tau(f)$ for each automorphism $f : M \to M$ in $\mathbf{A}$, subject to the relations

$$\tau(gf : M \to M \to M) = \tau(f) + \tau(g)$$
$$\tau(ifi^{-1} : M' \to M \to M \to M') = \tau(f)$$
$$\tau(f \oplus f' : M \oplus M' \to M \oplus M') = \tau(f) + \tau(f')$$

Here $i : M' \to M$ is an isomorphism in $\mathbf{A}$.

Clearly an additive functor $\alpha : \mathbf{A} \to \mathbf{B}$ between small additive categories induces homomorphisms

$$\alpha_* : K_1^{\mathrm{iso}}(\mathbf{A}) \to K_1^{\mathrm{iso}}(\mathbf{B})$$
$$\alpha_* : \tilde{K}_1^{\mathrm{iso}}(\mathbf{A}) \to \tilde{K}_1^{\mathrm{iso}}(\mathbf{B})$$
$$\alpha_* : K_1^{\mathrm{aut}}(\mathbf{A}) \to K_1^{\mathrm{aut}}(\mathbf{B})$$

which carry $\tau(f)$ to $\tau(\alpha f)$. The automorphism torsion group $K_1^{\mathrm{aut}}(\mathbf{A})$ is just the "Whitehead group" of $\mathbf{A}$ in the sense of Bass [AK] p. 348 and p. 397 also denoted by

$$K_1(\mathbf{A}) = K_1^{\mathrm{aut}}(\mathbf{A}). \tag{10.2}$$

We have the forgetful homomorphism

$$\varphi : K_1^{\mathrm{aut}}(\mathbf{A}) \longrightarrow K_1^{\mathrm{iso}}(\mathbf{A}) \tag{10.3}$$

which carries $\tau(f)$ to $\tau(f)$. We refer the reader to Ranicki [AT] for a careful study of these torsion groups. Here we only recall some results of Ranicki [AT] needed below.

(10.4) Definition. Let $\mathbf{chain}^\sharp_{\mathbf{A}}$ be the category of finite chain complexes

$$C : \cdots \longrightarrow 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0$$

in $\mathbf{A}$ and chain maps. The *torsion of an isomorphism* $f : C \to D$ in $\mathbf{chain}^\sharp_{\mathbf{A}}$ is defined by

$$\tau(f) = \sum_{r=0}^{\infty}(-1)^r \tau\big(f : C_r \to D_r\big) \in K_1^{\mathrm{iso}}(\mathbf{A}). \tag{1}$$

Moreover the *torsion of a contractible finite* chain complex C with contraction homotopy $\alpha : 0 \simeq 1 : C \to C$ is defined by

$$\tau(C) = \tau(d + \Gamma) \in K_1^{\mathrm{iso}}(\mathbf{A}) \tag{2}$$

where

$$d + \Gamma : C_{\mathrm{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \cdots \longrightarrow C_{\mathrm{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \dots$$

is an isomorphism in $\mathbf{A}$ defined by the matrix

$$d + \Gamma = \begin{pmatrix} d & 0 & 0 & 0 & \dots \\ \Gamma & d & 0 & 0 & \dots \\ 0 & \Gamma & d & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Ranicki shows that $d + \Gamma$ is actually an isomorphism and that $\tau(d + \Gamma)$ does not depend on the choice of Γ. Moreover the following result holds. Recall that a sequence $M \to N \to K$ in $\mathbf{A}$ is *exact* if for all X in $\mathbf{A}$ the sequence of abelian groups

$$\mathbf{A}(X, M) \longrightarrow \mathbf{A}(X, N) \longrightarrow \mathbf{A}(X, K)$$

is exact. Moreover a sequence $C \to D \to E$ of maps between chain complexes in $\mathbf{A}$ is exact if for all $r \in \mathbb{Z}$ the sequence $C_r \to D_r \to E_r$ is exact.

(10.5) Proposition. *Let*

$$0 \longrightarrow C \xrightarrow{i} C'' \xrightarrow{j} C' \longrightarrow 0$$

be a short exact sequence of contractible chain complexes in $\mathbf{chain}^\sharp_{\mathbf{A}}$. *Then the reduced torsion satisfies the sum formula in* $\tilde{K}_1^{\mathrm{iso}}(\mathbf{A})$

$$\tilde{\tau}(C'') = \tilde{\tau}(C) + \tilde{\tau}(C') + \sum_{r=0}^{\infty}(-1)^r \tilde{\tau}\big((i,k) : C_r \oplus C'_r \to C''_r\big)$$

with $\{k : C'_r \to C''_r,\ r \geq 0\}$ *any sequence of splitting morphisms such that* $jk = 1 : C'_r \to C'_r$ $(r \geq 0)$ *and each* $(i,k) : C_r \oplus C'_r \to C''_r$ $(r \geq 0)$ *is an isomorphism.*

(10.6) Definition. A *canonical structure* φ on an additive category $\mathbf{A}$ is a collection of isomorphisms $\{\varphi_{M,N} : M \to N\}$, one for each ordered pair (M, N) of isomorphic objects in $\mathbf{A}$, such that

$$\varphi_{M,M} = 1 : M \to M, \tag{i}$$
$$\varphi_{M,P} = \varphi_{N,P}\varphi_{M,N} : M \to N \to P, \tag{ii}$$
$$\varphi_{M\oplus M',N\oplus N'} = \varphi_{M,N} \oplus \varphi_{M',N'} : M \oplus M' \to N \oplus N' \tag{iii}$$

(10.7) Lemma. *A canonical structure* φ *on an additive category* $\mathbf{A}$ *determines a retraction*

$$\varphi_* : K_1^{\mathrm{iso}}(\mathbf{A}) \longrightarrow K_1^{\mathrm{aut}}(\mathbf{A})$$

of the forgetful homomorphism φ *in (10.3), that is* $\varphi_*\varphi = 1$. *Here* φ_* *carries* $\tau(f : M \to N)$ *to* $\tau(\varphi_{N,M} f : M \to N \to M)$. *The kernel of* φ_* *is generated by the elements* $\tau(\varphi_{N,M})$.

11 The Algebraic Whitehead Group

Recall the definition of an enveloping functor U in (I.§ 5). We now enrich the structure of U as follows.

(11.1) Definition. Let $\mathbf{C}$ be a category and let $\mathcal{A}$ be a set and let

$$U : \mathbf{C} \to \mathbf{Ringoids}(\mathcal{A})$$

be an enveloping functor as in (I.5.11). Moreover let $U(\partial)$ be an additive category for all objects ∂ in $\mathbf{C}$. We say that J is an *isomorphism structure* for U if for each ∂ a subcategory $J(\partial) \subset U(\partial)$ is given where $J(\partial)$ has the same objects as $U(\partial)$ and all morphisms of $J(\partial)$ are isomorphisms. Moreover a map $u : \partial \to \partial'$ in $\mathbf{C}$ induces a functor $u_* : U(\partial) \to U(\partial')$ which carries $J(\partial)$ to $J(\partial')$. Hence J is a functor from $\mathbf{C}$ to the category of groupoids which is a subfunctor of U. In addition we assume that all interchange isomorphism $\varepsilon_{M,N}$ in $U(\partial)$ are morphisms in $J(\partial)$; see (10.1) (b).

For an enveloping functor U with isomorphism structure J we define the functor

$$\mathrm{Wh} = \mathrm{Wh}_J(U) : \mathbf{C} \to \mathbf{Ab} \tag{11.2}$$

termed the *algebraic Whitehead group*. We obtain $\mathrm{Wh}(\partial)$ by the quotient group

$$\mathrm{Wh}(\partial) = K_1^{\mathrm{iso}}(U(\partial))/\sim \tag{1}$$

defined by the relations

$$\tau(j) \sim 0 \quad \text{for } j \in J(\partial). \tag{2}$$

Since we assume that the interchange isomorphisms are in $J(\partial)$ one gets the natural surjective homomorphism

$$\tilde{K}_1^{\mathrm{iso}}(U(\partial)) \twoheadrightarrow \mathrm{Wh}(\partial) \tag{3}$$

which carries $\tilde{\tau}(f)$ to the equivalence class of $\tau(f)$.

(11.3) Lemma. *Let $\partial \in \mathbf{C}$ and assume that the category $J(\partial)$ contains a canonical structure φ on the additive category $U(\partial)$; see (10.6). Then one has a natural surjective homomorphism*

$$K_1^{\mathrm{aut}}(U(\partial)) \twoheadrightarrow \mathrm{Wh}(\partial)$$

which carries $\tau(f)$ to the equivalence class of $\tau(f)$.

Proof. We have the commutative diagram

$$\begin{tikzcd}
K_1^{\mathrm{aut}}(U(\partial)) \arrow[r, "\varphi"] & K_1^{\mathrm{iso}}(U(\partial)) \arrow[r, "\varphi_*"] \arrow[d, "q"] & K_1^{\mathrm{aut}}(U(\partial)) \arrow[dl, "q'"] \\
 & \mathrm{Wh}(\partial) &
\end{tikzcd}$$

where q admits a factorization $q = q'\varphi_*$ by (10.7) so that q' is surjective. Moreover by (10.7) we have $\varphi_*\varphi = 1$ so that $q' = q'\varphi_*\varphi = q\varphi$ is surjective. Here $q\varphi$ is clearly natural. q.e.d.

12 The Isomorphism Between the Geometric and Algebraic Whitehead Group

Let $(\mathbf{K}, \mathcal{D})$ be a cellular I-category with initial object $\emptyset$ and let L be a finite dimensional $\mathcal{D}$-complex in $\mathbf{K}$. Then we obtain as in § 8 the Whitehead group $\mathrm{Wh}(L)$ which we also call the *geometric Whitehead group*. We want to compare the group $\mathrm{Wh}(L)$ with an algebraic Whitehead group $\mathrm{Wh}(\partial)$ defined in § 11. For this we consider the cofibration category

$$\mathbf{C} = \mathbf{K}^{L_0} \tag{12.1}$$

with initial object $* = L_0$ where L_0 is the 0-skeleton of L. By (1.3) we see that $\mathbf{C}$ is a cofibration category under the theory of coactions

$$\mathbf{T} = \mathbf{cone}(L_0, \mathcal{D}). \tag{12.2}$$

Let $\mathcal{D}(L_0)$ be the set of all maps $\beta : E \to L_0$ in $\mathbf{K}$ with $E \in \mathcal{D}$. Such a map β defines a based object Σ^0_β in $\mathbf{C}$ by

$$\Sigma^0_\beta = (L_0 \rightarrowtail L_0 \cup_\emptyset E \xrightarrow{(1,\beta)} L_0). \tag{1}$$

Here $\cup_\emptyset$ is the sum in $\mathbf{K}$ since $\emptyset$ is the initial object. The n-fold suspension of Σ^0_β in $\mathbf{C}$ yields the *n-sphere*

$$\Sigma^n_\beta = \Sigma^n(\Sigma^0_\beta). \tag{2}$$

One can check that with the notation (2.1) (1) we have the push out diagram in $\mathbf{K}$

$$\begin{tikzcd} \Sigma^n_E \arrow[r] & \Sigma^n_\beta \\ E \arrow[u] \arrow[r, "\beta"] & L_0 \arrow[u] \end{tikzcd} \tag{3}$$

This fact is used in (3.7) to show that for $\mathbf{T}$ above a $\mathbf{T}$-complex in $\mathbf{C}$ is the same as a normalized $\mathcal{D}$-complex in $\mathbf{K}$.

The cogroups in $\mathbf{T}$ are the 1-spheres Σ^1_β. More generally all objects of $\mathbf{T}$ are of the form $X_1 = C_{\alpha,\beta}$ as in (1.2) such that for $\alpha = \beta$ we have $\Sigma^1_\beta = C_{\beta,\beta}$.

Let $\mathcal{A}$ be the set of all cogroups in $\mathbf{T}$. Then one has the bijection

$$\mathcal{D}(L_0) = \mathcal{A}, \quad \lambda \mapsto \Sigma^1_\lambda, \tag{4}$$

which we may use as an identification.

The theory $\mathbf{T}$ yields the category $\mathbf{Coef}$ as in chapter I. The objects $\partial \in \mathbf{Coef}$ are elements

$$\partial \in [\Sigma^1_\gamma, X_1] \quad \text{with } X_1 = C_{\alpha,\beta} \tag{5}$$

and α, β, $\gamma \in \mathcal{D}(L_0)$. Here $[A, B]$ denotes the set of homotopy classes in $\mathbf{C}/\simeq$. Using ∂ we can choose a principal cofibration

$$X_2 = X_1 \cup_f C\Sigma^1_\gamma = C_f \tag{6}$$

which is given by a mapping cone C_f in $\mathbf{C}$ of a map f representing ∂.

(12.3) Definition. We say that $(\mathbf{K}, \mathcal{D})$ is a *homological cellular I-category* if $(\mathbf{K}; \mathcal{D})$ is a cellular I-category and if for $L_0 \in \mathcal{D}$ and

$$\begin{aligned} \mathbf{C} &= \mathbf{K}^{L_0} \\ \mathbf{T} &= \mathbf{cone}(L_0, \mathcal{D}) \end{aligned}$$

the pair $(\mathbf{C}, \mathbf{T})$ is a homological cofibration category under $\mathbf{T}$; see (V.1.1).

Now assume that $(\mathbf{K}, \mathcal{D})$ is a homological cellular I-category. We fix $L_0 \in \mathcal{D}$ so that $(\mathbf{C}, \mathbf{T})$ is defined as above. As in (V.1.6) we obtain the category

$$\mathbf{mod} = \mathbf{mod}(\mathbf{C}) \tag{12.4}$$

of modules associated to $\mathbf{T}$. An object in $\mathbf{mod}$ is given by $\Sigma^1_\gamma \vee \partial$ with $\gamma \in \mathcal{D}(L_0)$ and $\partial \in \mathbf{Coef}$ and a morphism

$$g \odot u : \Sigma^1_\gamma \vee \partial' \longrightarrow \Sigma^1_\beta \vee \partial \tag{1}$$

is given by $u : \partial \to \partial'$ in $\mathbf{Coef}$ and

$$g \in [\Sigma^2_\gamma, \Sigma^2_\beta \vee X_2]_2 \quad \text{in } \mathbf{C}/\simeq. \tag{2}$$

Here X_2 is defined by ∂ as in (12.2) (6). For the set $\mathcal{A}$ of cogroups in $\mathbf{T}$ in (12.2) (4) we obtain by (I.5.11) the enveloping functor

$$U = U_{\mathcal{A}} : \mathbf{Coef} \longrightarrow \mathbf{Ringoids}(\mathcal{A}) \tag{12.5}$$

Here $U(\partial) = U_{\mathcal{A}}(\partial) \subset \mathbf{mod}$ is the following subcategory. Objects are all $\Sigma^1_\alpha \vee \partial$ with $\alpha \in \mathcal{D}(L_0) = \mathcal{A}$ and morphisms are all $g \odot 1$ as above where 1 is the identity of ∂. This shows that $U_{\mathcal{A}}(\partial)$ is an additive category with the sum

$$(\Sigma^1_\alpha \vee \partial) \oplus (\Sigma^1_\beta \vee \partial) = \Sigma^1_{\alpha \oplus \beta} \vee \partial \tag{1}$$

where $\alpha \oplus \beta = (\alpha, \beta) : E \cup_\emptyset E' \to L_0$. We have for $n \geq 0$

$$\Sigma^n_{\alpha \oplus \beta} = \Sigma^n_\alpha \vee \Sigma^n_\beta \tag{2}$$

where the right hand side is a sum of spheres in $\mathbf{C}$.

(12.6) Definition. We say that an *isomorphism structure J for $U_{\mathcal{A}}$ is cellular* if the following property holds where $r \geq 3$. Let L be a finite dimensional $\mathcal{D}$-complex which is normalized with the 0-skeleton L_0 chosen above and let (K, L) and (M, L) be $\mathcal{D}$-pairs satisfying $K_{r-1} = L_{r-1} = M_{r-1}$ and let

$$\varphi : K \cong M \tag{1}$$

be a $\mathcal{D}$-isomorphism under L. We choose normalizations H and G of K and M respectively and we choose φ^H and φ^G as in (5.5) so that we get cellular maps

$$K^H \xrightarrow{\varphi^H} K \xrightarrow{\varphi} M \xrightarrow{\bar{\varphi}^G} M^G \tag{2}$$

under L where $\bar{\varphi}^G$ is a 0-homotopy inverse of φ^G. Here K^H and M^G are normalized with 2-skeleton L_2. Let ∂_L be the attaching map of $L_1 \subset L_2$. Then the chain functor (V.2.3) yields the induced isomorphism

$$f_{\geq 0} = (\bar{\varphi}^G \varphi \varphi^H)_* : C_*(K^H, L) \xrightarrow{\cong} C_*(M^G, L) \tag{3}$$

of (relative) chain complexes in $U_{\mathcal{A}}(\partial_L)$. For the isomorphism structure J we have $J(\partial_L) \subset U_{\mathcal{A}}(\partial_L)$ and we require that all chain isomorphisms $f_{\geq 0}$ as constructed in (3) satisfy

$$f_n \in J(\partial_L) \tag{4}$$

for all $n \geq r$. Moreover for normalized $\mathcal{D}$-pairs $(\bar{K}, L)$ and $(\bar{M}, L)$ and for an isomorphism

$$f_{\geq 0} : C_*(\bar{K}, L) \cong C_*(\bar{M}, L) \tag{5}$$

satisfying (4) there exists an isomorphism $\varphi : K \to M$ as in (1) and normalizations H, G such that $K^H = \bar{K}$ and $M^G = \bar{M}$ and $f_{\geq 0} = C_*(\bar{\varphi}^G \varphi \varphi^H)$; (this condition is actually only needed for very special $\bar{K}$ and $\bar{M}$ as in the proof of (12.7) (29) below).

(12.7) Theorem. *Let $(\mathbf{K}, \mathcal{D})$ be a homological cellular I-category and $L_0 \in \mathcal{D}$. Moreover let J be a cellular isomorphism structure for the enveloping functor $U_{\mathcal{A}}$ where $\mathcal{A} = \mathcal{D}(L_0)$. Then one has for each normalized finite dimensional $\mathcal{D}$-complex L with 0-skeleton L_0 an isomorphism*

$$\tau : \mathrm{Wh}(L) \cong \mathrm{Wh}(\partial_L)$$

of abelian groups. Here $\mathrm{Wh}(L)$ is the geometric Whitehead group and $\mathrm{Wh}(\partial_L)$ is the algebraic Whitehead group defined by $(U_{\mathcal{A}}, J)$ in (11.2). Moreover τ is natural for maps $f : L \to L'$ which extend the identity $L_0 = L'_0$. In fact τ carries $[K, L] \in \mathrm{Wh}(L)$ to the equivalence class of $\tau(C_*(K^H, L))$. Here H is a normalization of $K \operatorname{rel} L$ and $C = C_*(K^H, L)$ is a contractible chain complex with $\tau(C)$ defined in (10.4).

Proof. First we check that the map τ is a well defined homomorphism. Using (9.1) and (9.7) we have to show that an elementary expansion $K \nearrow M$ of order $\geq r + 1$ satisfies $\tau(C_*(K^H, L)) \sim \tau(C_*(M^G, L))$ where G is a normalization of $M \operatorname{rel} L$. We first choose a normalization F of $N = K \cup e^n_A \cup e^{n+1}_A$ extending the normalization H of K where N is $\mathcal{D}$-isomorphic to M under K. Then we have the short exact sequence of contractible chain complexes

$$0 \longrightarrow C_*(K^H, L) \xrightarrow{i} C_*(N^F, L) \xrightarrow{j} C_*(N^F, K^H) \longrightarrow 0 \tag{1}$$

and we can apply (10.5). The cell structure of N shows that we have canonical splitting morphisms k for which (i,k) in (10.5) is the identity. Hence (10.5) implies

$$\tau(C_*(K^H,L)) = \tau(C_*(N^F,L)) \quad \text{since} \tag{2}$$
$$\tau(C_*(N^F,K^H)) = 0. \tag{3}$$

Here (3) is the consequence of the definition of an elementary expansion which shows that $C_*(N^F,K^H)$ is concentrated in degree n and $n+1$ and that the differential is the identity. Hence (10.4) (2) and (10.1) (a) yield (3) since $\tau(\mathrm{id}) = 0$. Next we have as in (12.6) (2) the cellular map

$$N^F \to N \cong M \to M^G \tag{4}$$

which induces the isomorphism

$$f_{\geq 0} : C_*(N^F,L) \longrightarrow C_*(M^G,L) \tag{5}$$

with $f_n \in J(\partial_L)$ since J is cellular; see (12.6). Again (10.5) with $C = 0$ and $j = f_{\geq 0}$ shows that

$$C_*(N^F,L) \sim C_*(M^G,L) \tag{6}$$

where we use the equivalence relation $\sim$ defined by J; see (11.2) (2). Hence the functor τ in (12.7) is well defined. Moreover τ is a homomorphism since

$$C_*(K^H \cup_L M^G, L) = C_*(K^H,L) \oplus C_*(M^G,L). \tag{7}$$

Here we use again (10.5). We now define a homomorphism λ for which the composite

$$K_1^{\mathrm{iso}}(U_{\mathcal{A}}(\partial_L)) \xrightarrow{\lambda} \mathrm{Wh}(L) \xrightarrow{\tau} \mathrm{Wh}(\partial_L) \tag{8}$$

is the quotient map in (11.2) (1). Let g be an isomorphism in $U_{\mathcal{A}}(\partial_L)$. Hence g is given by an element

$$g \in [\Sigma^2_\lambda, \Sigma^2_\beta \vee L_2]_2 \quad \text{in } \mathbf{C}/\simeq. \tag{9}$$

See (12.4) (2). Since $(\mathbf{C},\mathbf{T})$ is homological we have the isomorphism, $r \geq 3$,

$$[\Sigma^2_\lambda, \Sigma^2_\beta \vee L_2]_2 \cong [\Sigma^r_\lambda, \Sigma^r_\beta \vee L_2]_2 \tag{10}$$

which carries g to the $(r-2)$-fold partial suspension $E^{r-2}(g)$; see (V.1.1). We define

$$K_g = L \cup_{L_2} C_{g_0} \quad \text{with } g_0 \in E^{r-2}(g). \tag{11}$$

Here C_{g_0} is the mapping cone of g_0 in $\mathbf{C}$. For $\beta : D \to L_0$ and $\lambda : E \to L_0$ in $\mathcal{D}(L_0)$ we have the cell structure

$$K_g = L \cup e^r_D \cup e^{r+1}_E. \tag{12}$$

Moreover the chain complex $C_*(K_g, L)$ is concentrated in degree r and $r+1$ and the differential of $C_*(K_g, L)$ coincides with g. Since g is an isomorphism we see by the homological Whitehead theorem (VI.7.1) that $L \to K_g$ is a homotopy equivalence in **C**. Hence the element $[K_g, L] \in \mathrm{Wh}(L)$ is defined and we set

$$\lambda(g) = [K_g, L]. \tag{13}$$

In fact the right hand side does not depend on the choice of g_0 in (11) since we have (7.12). Now (13) defines a homomorphism λ in (8) if the following equations (14) and (15) hold

$$\lambda(g \oplus g') = \lambda(g) + \lambda(g'), \tag{14}$$
$$\lambda(gf) = \lambda(g) + \lambda(f). \tag{15}$$

Compare (10.1) (a). For elements g as in (9) and $g' \in [\Sigma^2_{\lambda'}, \Sigma^2_{\beta'} \vee L_2]_2$ the sum $g \oplus g'$ is given by

$$g \oplus g' = (i_1 g, i_2 g') \in [\Sigma^2_\lambda \vee \Sigma^2_{\lambda'}, \Sigma^2_\beta \vee \Sigma^2_{\beta'} \vee L_2] \tag{16}$$

where i_1 and i_2 are the obvious inclusions. Now it is clear that

$$K_{g \oplus g'} = K_g \cup_L K_{g'} \tag{19}$$

and hence (14) holds; see (8.6). The proof of (15) is more complicated. We write $D = \Sigma^r_\delta, G = \Sigma^r_\lambda$ and $B = \Sigma^r_\beta$ and we consider $(gf)_0$ defined by the composite

$$(gf)_0 : D \xrightarrow{f_0} G \vee L_2 \xrightarrow{(g_0, i_2)} B \vee L_2 \quad \text{in } \mathbf{C}. \tag{20}$$

Here g_0 and f_0 are trivial on L_2. We have to show

$$K_{gf} \simeq K_g \cup_L K_f \ \mathrm{rel}\, L. \tag{21}$$

Then (15) is a consequence of (21) and (8.6). Now (21) is a consequence of (11) and

$$C_{(gf)_0} \simeq C_{g_0} \cup_{L_2} C_{f_0} \ \mathrm{rel}\, L_2. \tag{22}$$

Let N_{f_0} be the mapping cylinder of f_0 rel L_0, that is N_{f_0} is defined by the push out

$$\begin{array}{ccccc} j_1 : D & \xrightarrow{i_1} & I_{L_0} D & \longrightarrow & N_{p_0} \\ & & \uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle j_0} \\ & & D & \xrightarrow{f_0} & G \vee L_2 \end{array}$$

Moreover let CD be the cone of D in **C**. Then we readily get by (7.3) and (7.11)

$$C_{f_0} \curvearrowright N_{f_0} \cup_D CD \operatorname{rel} L_2. \tag{23}$$

Moreover as in (7.8) we have

$$N_{(fg)_0} \curvearrowright N_{(g_0,i_2)} \cup_{G \vee L_2} N_{f_0}$$

where $N_{(g_0,i_2)} = N_{g_0} \cup_{L_2} I_{L_0} L_2$ in $\mathbf{C}$ so that

$$N_{(fg)_0} \curvearrowright N_{g_0} \cup_{G \vee L_2} N_{f_0} = R. \tag{24}$$

Here R is the push out of $j_0 : G \vee L_2 \to N_{f_0}$ and $(j_1, j_0 i_2) : G \vee L_2 \to N_{g_0}$. Hence we get

$$C_{(fg)_0} \curvearrowright N_{(fg)_0} \cup_D CD \curvearrowright R \cup_D CD = S. \tag{25}$$

We now choose an inverse h of f in $U_{\mathcal{A}}(\partial_L)$. Then h_0 is given by a map in $\mathbf{C}$

$$h_0 : G \longrightarrow D \vee L_2 \tag{26}$$

and we obtain a homotopy commutative diagram in $\mathbf{C}$ as follows.

$$\begin{tikzcd}
 & L_2 \arrow[r, "i"] & N_{f_0} \cup CD \\
G \arrow[ur, "0_G"] \arrow[r, "h_0"] \arrow[dr, "i_1"'] & D \vee L_2 \arrow[u, "{(0_D,1)}"] \arrow[r, "{(j_1,j_0 i_2)}"] \arrow[d, "{(f_0,i_2)}"] & N_{f_0} \arrow[u, "j"'] \\
 & G \vee L_2 \arrow[ur, "j_0"'] &
\end{tikzcd}$$

Here 0_G is the composite $G \xrightarrow{0} L_0 \xrightarrow{i} L_2$ where 0 is the trivial map on G. In the same way we define 0_D. The inclusion $j_0 i_1 : G \to N_{f_0}$ is homotopic to $d = (j_1, j_0 i_2) h_0$ so that by (7.14)

$$R \curvearrowright N_{(g_0,i_2)} \cup_{(d, j_0 i_2)} N_{f_0} = T. \tag{27}$$

Moreover jd is homotopic to $i0_G$ so that we get again by (7.14)

$$\begin{aligned}
S = R \cup_D CD &\curvearrowright T \cup_D CD \\
&\curvearrowright N_{g_0} \cup_{(i0_G, j_0 i_2)} (N_{f_0} \cup_D CD) \\
&= C_{g_0} \cup_{L_2} (N_{f_0}) \cup_D CD) \\
&\curvearrowright C_{g_0} \cup_{L_2} C_{f_0}.
\end{aligned} \tag{28}$$

This completes the proof of (22) and hence of (21) and therefore λ in (8) is a well defined homomorphism for which $\tau\lambda$ is the quotient map. In fact the homomorphism λ induces a splitting of τ. For this we have to show $\lambda(g) = 0$ for $g \in J(\partial_L)$ or equivalently

$$[K_g, L] = [L, L] \quad \text{for } g \in J(\partial_L). \tag{29}$$

For this let $\bar{K} = C_{g_0} \cup_{L_2} L$ and let $\bar{M} = C_b \cup_{L_2} L = CB \vee L$ where $b : B \to B \vee L_2$ is the inclusion. Then we have an isomorphism

$$f_{\geq 0} : C_*(\bar{K}, L) \longrightarrow C_*(\bar{M}, L)$$

with $f_{r+1} = g$ and $f_r =$ identity. Hence $f_{\geq 0}$ satisfies (12.6) (5) and therefore $f_{\geq 0} = C_*(\varphi^H \varphi \bar{\varphi}^G)$ as in (12.6) (2). This shows by (7.13) and (7.5) that $\bar{K} \curvearrowright \bar{M}$ rel L. Clearly $[\bar{M}, L] = 0$ since $L \curvearrowright \bar{M}$ rel L by (7.3). This completes the proof of (29) and hence λ in (8) induces a splitting

$$\lambda : \mathrm{Wh}(\partial_L) \longrightarrow \mathrm{Wh}(L) \tag{30}$$

of τ in (8). Hence τ and λ are isomorphisms since we show in (12.9) below that λ is surjective. For this we need the following lemma. q.e.d.

(12.8) Lemma. *Let L be a finite dimensional $\mathcal{D}$-complex in* $\mathbf{K}$ *and let $A = \Sigma^n_\alpha$ with $n \geq 1$ be an n-sphere in* $\mathbf{C}$ *given by $\alpha : D \to L_0$ with $D \in \mathcal{D}$. Hence A is an n-fold suspension in* $\mathbf{C}$ *and we can consider the map*

$$f = (i_1 + i_2 a, i_2) : A \vee L \longrightarrow A \vee L \quad \textit{in } \mathbf{C}$$

where $a : A \to L$ is in $\mathbf{C}$. *Then f is a simple homotopy equivalence in* $\mathbf{K}$ rel L.

Proof. We have $A = \Sigma^n_D \cup_\alpha L_0$ and

$$A \vee L = \Sigma^n_D \cup_\alpha L. \tag{1}$$

We consider the mapping cylinder N_f given by the push out

$$\begin{array}{ccccc} j_1 : A \vee L & \xrightarrow{i_1} & I_{L_0}(A \vee L) = I_{L_0}A \vee I_{L_0}L & \longrightarrow & N_f \\ & & {\scriptstyle i_0}\uparrow & & \uparrow{\scriptstyle j_0} \\ & & A \vee L & \xrightarrow{f} & A \vee L \end{array} \tag{2}$$

We want to show that the inclusion $j_1 : A \vee L \to N_f$ is given by a deformation

$$j_1 : A \vee L \curvearrowright N_f \text{ rel } L. \tag{3}$$

This yields the proposition.

Let g be the restriction of f to $B = \Sigma^n_D$ and let $P^n_D = P$ and $Q^n_D = Q$ as in (4.5). Then g is given by the composite

$$g : B \overset{\kappa}{\cong} P \cup_\partial Q \xrightarrow{\pi \cup \pi} B \cup_D B \xrightarrow{i_1 \cup b} A \vee L. \tag{4}$$

Here π is the quotient map in (2.1) (8) and b is the restriction of a. We consider the push out

$$\begin{tikzcd} j_1 : B \arrow[r, "i_1"] & I_D B \arrow[r] & N \\ & B \arrow[u, "i_0"] \arrow[r, "g"] & A \vee L \arrow[u, "j_0"'] \end{tikzcd} \tag{5}$$

Then we have by diagram (2)

$$N_f = N \cup_L I_{L_0} L. \tag{6}$$

Since the projection $I_{L_0} L \to L$ is a collapse rel $i_1 L$ we see that (3) follows from an expansion

$$(j_1, j_0 i_2) : B \cup_\alpha L \nearrow N. \tag{7}$$

Using (4) we see that N is also given by the push out

$$\begin{tikzcd} I_D(B) \cong I_D(P \cup_\partial Q) \arrow[r] & N \\ Q \arrow[u] \arrow[r, "b\pi"] & L \arrow[u] \end{tikzcd} \tag{8}$$

Hence (7) follows from an expansion

$$i_1(P \cup_\partial Q) \cup_D i_0 Q \nearrow I_D(P \cup_\partial Q). \tag{9}$$

Using κ in (4.5) we see that $\partial \subset Q$ is an expansion. Hence by (7.6) or (6.6) we see that

$$i_1 Q \cup_D i_0 Q \nearrow I_D Q \tag{10}$$

is an expansion. Moreover by (7.2) or by (6.5) we see that

$$i_1 P \cup_\partial I_D \partial \nearrow I_D P \tag{11}$$

is an expansion. Now (10) and (11) together yield (9) and hence (7) and (3) are proved. q.e.d.

(12.9) Lemma. *λ in (12.7) (8) is surjective.*

Proof. Each element $[K, L] \in \mathrm{Wh}(L)$ is represented via (9.1) by a $\mathcal{D}$-pair (K, L) which is normalized and in simplified form. We show that there is a $\mathcal{D}$-pair (K_h, L) with

$$\lambda(h) = [K_h, L] = [K, L]. \tag{1}$$

Since (K, L) is in simplified form and normalized we have spheres $A = \Sigma^r_\alpha$ and $B = \Sigma^r_\beta$ in $\mathbf{C}$ and a map

$$g : A \to B \vee L \quad \text{in } \mathbf{C} \tag{2}$$

such that $K = C_g$ is the mapping cone of g in $\mathbf{C}$. Since $L \to K$ is a homotopy equivalence in $\mathbf{K}$ there exists a retraction $r : K \to L$ which shows that the composite

$$A \xrightarrow{g} B \vee L \xrightarrow{(b,1)} L \tag{3}$$

is homotopic to 0. Here $b = r \mid B$ is the restriction of the retraction r. We have the sum decomposition

$$g = h + i_L a \tag{4}$$

where $a \in [A, L]$ and $h \in [A, B \vee L]_2$. This is clear since we have the split short exact sequence

$$0 \longrightarrow [A, B \vee L]_2 \longrightarrow [A, B \vee L] \longrightarrow [A, L] \longrightarrow 0.$$

By the assumption on a homological cofibration category in (V.1.1) we know that h is a partial suspension. Now (3) and (4) show

$$0 = (b, 1)_* g = (b, 1)_*(h + i_L a) = (b, 1)_* h + a$$

so that

$$a = -(b, 1)_*(h) \tag{5}$$

and hence

$$g = h - i_L (b, 1)_*(h). \tag{6}$$

On the other hand since h is a partial suspension we have

$$\begin{aligned} (i_B - i_L b, i_L) h &= (i_B, i_L) h - (i_L b, i_L) h \\ &= h - i_L (b, 1) h. \end{aligned} \tag{7}$$

This shows that

$$g = (i_B - i_L b, i_L) h \tag{8}$$

and hence we have the push out diagram

$$\begin{array}{ccc} K_h = C_h & \longrightarrow & C_g = K \\ \uparrow & & \uparrow \\ B \vee L & \xrightarrow{\bar{b}} & B \vee L \end{array} \tag{9}$$

where $\bar{b} = (i_B - i_L b, i_L)$. Here it is sufficient to consider homotopy classes h, g since we can use (7.12). Now lemma (12.8) above shows that $\bar{b}$ is a simple homotopy equivalence under L. Hence by (7.11) also the map $K_h \to K$ in (9) is a simple homotopy equivalence rel L. Therefore (1) holds and the proof of (12.9) is complete. q.e.d.

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Errata

As a result of a technical error the title of the sixth chapter appears incorrectly in the Table of Contents. On page IX, line 7 should read:

The following References should be inserted after page 354.

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