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Homotopy Type and Homology

HANS-JOACHIM BAUES

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Homotopy Type and Homology

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Für Barbara
und für unsere Kinder Charis und Sarah

PREFACE

The main problem and the hard core of algebraic topology is the classification of homotopy types of polyhedra. Here the general idea of *classification* is to attach to each polyhedron invariants, which may be numbers, or objects endowed with algebraic structures (such as groups, rings, modules, etc.) in such a way that homotopy equivalent polyhedra have the same invariants (up to isomorphism in the case of algebraic structures). Such invariants are called *homotopy invariants*. The ideal would be to have an algebraic invariant which actually characterizes a homotopy type completely.

This book represents a new attempt to classify simply connected homotopy types in terms of homology and homotopy groups and additional algebraic structure on these groups. The main new result and our principal objective is the *classification theorem* in Chapter 3 on 'k-invariants' and 'boundary invariants' which supplements considerably the classical picture of homology and homotopy groups in the literature.

The second part of the book (Chapters 6–12) displays a number of explicit computations of homotopy types which are obtained by applying the classification theorem. In particular J.H.C. Whitehead's classical theorem on 1-connected 4-dimensional homotopy types follows immediately. The old challenging problem of extending Whitehead's classification for *1-connected 5-dimensional homotopy types* is solved in Chapter 12. We also classify *2-connected 6-dimensional homotopy types* and *$(n-1)$-connected $(n+3)$-dimensional homotopy types*, $n \geq 4$, which are in the stable range (so that the classification does not depend on the choice of n). A complete list of all finite stable $(n-1)$-connected $(n+3)$-dimensional homotopy types is described in Chapter 10. For example, there are exactly 24 simply connected homotopy types X with homology groups

$$H_4(X) = \mathbb{Z}/6, \qquad H_5(X) = \mathbb{Z}/2, \qquad H_6(X) = \mathbb{Z}/2, \qquad H_7(X) = \mathbb{Z}$$

and $H_i(X) = 0$ otherwise. For $n \geq 2$ we also compute the homotopy types Y with at most three non-trivial homotopy groups π_n, π_{n+1} and π_{n+2}. For example, there exist exactly seven homotopy types Y with

$$\pi_4(Y) = \mathbb{Z}/6, \qquad \pi_5(Y) = \mathbb{Z}/2, \qquad \pi_6(Y) = \mathbb{Z}/2$$

and $\pi_i(Y) = 0$ otherwise. We point out that our results for the first time provide methods to compute such homotopy types with three non-trivial homology groups or homotopy groups.

Such classification results involve computations of low-dimensional homotopy groups, Chapter 11. For this the results on homotopy groups of mapping cones in the Appendix are needed. We obtain complete information on the *fourth homotopy group* $\pi_4(X)$ of a simply connected space X and more generally on the homotopy group π_{n+2} of an $(n-1)$-connected space, $n \geq 2$. This continues the classical programme of Hurewicz, resp. J.H.C. Whitehead, who achieved such results for the homotopy groups π_n, resp. π_{n+1}, of an $(n-1)$-connected space, $n \geq 2$.

The book is essentially self-contained; prerequisites are elementary topology and elementary algebra and some basic notions of category theory. It can be used as an introduction to the subject and as a basis of further research. Moreover it provides methods and examples of explicit homotopy classification for those who would like to use such results in other fields, for example for the classification of manifolds.

We refer the reader to the survey article (Baues [HT]) for a general outline of the theory of homotopy types in the literature.

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Bonn H.-J.B.
June 1995

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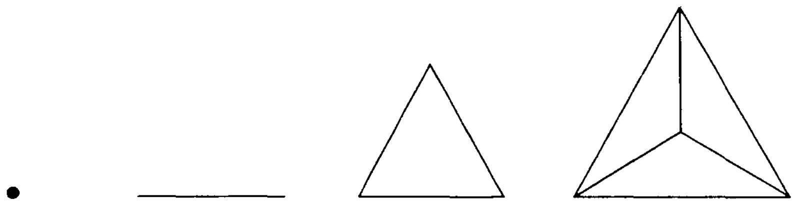
INTRODUCTION

For each number $n = 0, 1, 2, \ldots$ one has the *simplex* Δ^n which is the convex hull of the unit vectors $e_0, e_1, \ldots e_n$ in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$. Hence Δ^0 is a point, Δ^1 an interval, Δ^2 a triangle, Δ^3 a tetrahedron, and so on:



Δ^0 Δ^1 Δ^2 Δ^3

The dimension of Δ^n is n. The name simplex describes an object which is supposed to be very simple; indeed, natural numbers and simplexes both have the same kind of innocence. Yet once the simplex was created, algebraic topology had to emerge.

For each subset $a \subset \{0, 1, \ldots, n\}$ with $a = \{a_0 < \cdots < a_r\}$ one has the r-dimensional *face* $\Delta_a \subset \Delta^n$ which is the convex hull of the set of vertices $e_{a_0}, \ldots, e_{a_r}$. Hence the set of all subsets of the set $[n] = \{0, 1, \ldots, n\}$ can be identified with the set of faces of the simplex Δ^n. There are 'substructures' S of the simplex obtained by the union of several faces, that is,

$$S = \Delta_{a_1} \cup \Delta_{a_2} \cup \cdots \cup \Delta_{a_k} \subset \Delta^n.$$

Finite polyhedra are topological spaces X homeomorphic to such substructures S of simplexes Δ^n, $n \geq 0$. A homeomorphism $S \approx X$ is called a *triangulation* of X. Hence a polyhedron X is just a topological space in which we do not see any simplexes. We can introduce simplexes via a triangulation, but this must be seen as an artifact similar to the choice of coordinates in a vector space or manifold.† Finite polyhedra form a large universe of objects. One is not interested in a particular individual object of the universe but in the classification of species. A system of such species and subspecies is obtained by the equivalence classes

homotopy types and homeomorphism types.

Recall that two spaces X, Y are *homeomorphic*, $X \approx Y$, if there are continuous maps $f: X \to Y$ and $g: Y \to X$ such that the composites $fg = 1_Y$ and

† Compare H. Weyl, *Philosopy of Mathematics and Natural Science*, 1949: 'The introduction of numbers as coordinates...is an act of violence...'

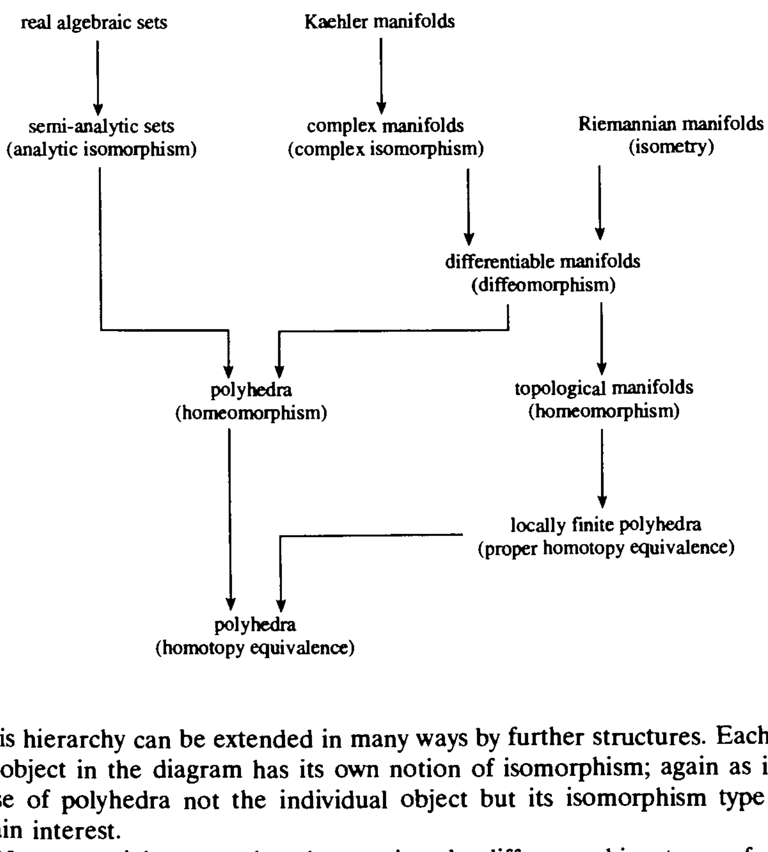
$gf = 1_X$ are the identity maps. A class of homeomorphic spaces is called a *homeomorphism type*. The initial problem of algebraic topology—Seifert and Threlfall called it the main problem—was the classification of homeomorphism types of finite polyhedra. Up to now such a classification was possible only in a very small number of special cases. One might compare this problem with the problem of classifying all *knots* and *links*. Indeed the initial datum for a finite polyhedron is just a set $\{a_1, \dots, a_k\}$ of subsets $a_i \subset [n]$ as above, and the initial datum to describe a link, namely a finite sequence of neighbouring pairs $(i, i+1)$ or $(i+1, i)$ in $[n]$, (specifying the crossings of $n+1$ strands) is of similar or even higher complexity. But we must emphasize that such a description of an object like a polyhedron or a link cannot be identified with the object itself: there are in general many different ways to describe the same object, and we care only about the equivalence classes of objects, not about the choice of description.

Homotopy types are equivalence classes of spaces which are considerably larger than homeomorphism types. To this end we use the notion of deformation or homotopy. The principal idea is to consider 'nearby' objects (that is, objects, which are 'deformed' or 'perturbed' continuously a little bit) as being similar. This idea of perturbation is a common one in mathematics and science; properties which remain valid under small perturbations are considered to be the stable and essential features of an object. The equivalence relation generated by 'slight continuous perturbations' has its precise definition by the notion of homotopy equivalence: two spaces X and Y are *homotopy equivalent*, $X \simeq Y$, if there are continuous maps $f: X \to Y$ and $g: Y \to X$ such that the composites fg and gf are homotopic to the identity maps, $fg \simeq 1_Y$ and $gf \simeq 1_X$. (Two maps $f, g: X \to Y$ are *homotopic*, $f \simeq g$, if there is a family of maps $f_t: X \to Y$, $0 \le t \le 1$, with $f_0 = f$, $f_1 = g$ such that the map $(x, t) \mapsto f_t(x)$ is continuous as a function of two variables.) A class of homotopy equivalent spaces is called a *homotopy type*.

Using a category $\mathbf{C}$ in the sense of S. Eilenberg and Saunders Mac Lane [GT] one has the general notion of isomorphism type. Two objects X, Y in $\mathbf{C}$ are called equivalent or *isomorphic* is there are morphisms $f: X \to Y$, $g: Y \to X$ in $\mathbf{C}$ such that $fg = 1_Y$ and $gf = 1_X$. An *isomorphism type* is a class of isomorphic objects in $\mathbf{C}$. We may consider isomorphism types as being special entities: for example, the isomorphism types in the category of finite sets are the *numbers*. A homeomorphism type is then an isomorphism type in the category $\mathbf{Top}$ of topological spaces and continuous maps, whereas a homotopy type is an isomorphism type in the homotopy category $\mathbf{Top}/\simeq$ in which the objects are topological spaces and the morphisms are not individual maps but homotopy classes of ordinary continuous maps.

The Euclidean spaces $\mathbb{R}^n$ and the simplexes Δ^n, $n \ge 1$, all represent different homeomorphism types but they are *contractible*, i.e. homotopy equivalent to a point. As a further example, the homeomorphism types of connected 1-dimensional polyhedra are the *graphs* which form a world of their own, but the homotopy types of such polyhedra correspond only to numbers since each graph is homotopy equivalent to the one-point union of a certain number of circles S^1.

Homotopy types of polyhedra are archetypes underlying most geometric structures. This is demonstrated by the following diagram which describes a hierarchy of structures based on homotopy types of polyhedra. The arrows indicate the forgetful functors.



This hierarchy can be extended in many ways by further structures. Each kind of object in the diagram has its own notion of isomorphism; again as in the case of polyhedra not the individual object but its isomorphism type is of main interest.

Now one might argue that the set given by diffeomorphism types of closed differentiable manifolds is more suitable and restricted than the vast variety of homotopy types of finite polyhedra. This, however, turned out not to be true. Surgery theory showed that homotopy types of arbitrary simply connected finite polyhedra play an essential role for the understanding of differentiable manifolds. In particular, one has the following embedding of a set of homotopy types into the set of diffeomorphism types. Let X be a finite simply connected n-dimensional polyhedron, $n > 2$. Embed X into a Euclidean space $\mathbb{R}^{k+1}$, $k \geq 2n$, and let $N(X)$ be the boundary of a regular neighbourhood of $X \subset \mathbb{R}^{k+1}$. This construction yields a well-defined function $\{X\} \mapsto \{N(X)\}$ which carries homotopy types of simply connected n-dimensional finite polyhedra to diffeomorphism types of k-dimensional manifolds. Moreover for $k = 2n + 1$ this function is injective. Hence the set of simply connected diffeomorphism types is at least as complicated as the set of homotopy types of simply connected finite polyhedra.

In dimension ≥ 5 the classification of simply connected diffeomorphism types (up to connected sum with homotopy spheres) is reduced via surgery to problems in homotopy theory which form the unsolved hard core of the question. This kind of reduction of geometric questions to problems in homotopy theory is an old and standard operating procedure. Further examples are the classification of fibre bundles and the determination of the ring of cobordism classes of manifolds.

All this underlines the fundamental importance of homotopy types of polyhedra. There is no good intuition of what they actually are, but they appear to be entities as genuine and basic as numbers or knots. In my book [AH] I suggested an axiomatic background for the theory of homotopy types; A. Grothendieck [PS] commented:

> 'Such suggestion was of course quite interesting for my present reflections, as I do have the hope indeed that there exists a 'universe' of schematic homotopy types...'.

Moreover J.H.C. Whitehead [AH] in his talk at the International Congress of Mathematicians, 1950, in Harvard said with respect to homotopy types and the homotopy category of polyhedra:

> 'The ultimate object of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry'.

Polyhedra are of combinatorial nature, but they often can only be described by an enormous number of simplexes even in the case of simple spaces like products of spheres. J.H.C. Whitehead observed that for many purposes only the 'cell structure' of spaces is needed. In some sense 'cells' play a role in topology which is similar to the role of 'generators' in algbera. Let

$$D^n = \{x \in \mathbb{R}^n, \|x\| \leq 1\}$$

$$\mathring{D}^n = \{x \in \mathbb{R}^n, \|x\| < 1\},$$

$$\partial D^n = D^n - \mathring{D}^n = S^{n-1}$$

be the closed and open n-dimensional disk and the $(n-1)$-dimensional sphere. An (open) *n-cell*, $n \geq 1$, in a space X is a homeomorphic image of the open disk $\mathring{D}^n$ in X; a 0-cell is a point in X. As a set a CW-complex is the disjoint union of such cells. A CW-complex is not just a combinatorial affair since the 'attaching maps' in general may have very complicated topological descriptions.

Definition A *CW-complex* X with skeleta $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$ is a topological space constructed inductively as follows:

(a) X^0 is a discrete space whose elements are the 0-cells of X.

(b) X^n is obtained by attaching to X^{n-1} a disjoint union of n-disks D_i^n via continuous functions $\varphi_i: \partial(D_i^n) \to X^{n-1}$, i.e. take the disjoint union $X^{n-1} \cup \bigcup D_i^n$ and pass to the quotient space given by the identifications $x \sim \varphi_i(x)$, $x \in \partial D_i^n$. Each $\mathring{D}_i^n$ then projects homeomorphically to an n-cell e_i^n of X. The map φ_i is called the attaching map of e_i^n.

(c) X has the weak topology with respect to the filtration of skeleta.

A *CW-space* is a space homotopy equivalent to a CW-complex. Homotopy types of polyhedra are the same as homotopy types of CW-spaces. The main numerical invariants of a homotopy type are 'dimension' and 'degree of connectedness'.

Definition The *dimension* $\mathrm{Dim}(X) \leq \infty$ of a CW-complex is defined by $\mathrm{Dim}(X) \leq n$ if $X = X^n$ is the n-skeleton. The dimension $\dim(X)$ of the homotopy type $\{X\}$ is defined by $\dim(X) \leq \mathrm{Dim}(Y)$ for all CW-complexes Y homotopy equivalent to X.

Definition A space X is *(path) connected* or 0-connected if any two points in X can be joined by a path in X; this is the same as saying that any map $\partial D^1 \to X$ can be extended to a map $D^1 \to X$ where D^1 is the 1-dimensional disk. This notion has an obvious generalization: a space X is *k-connected* if, for all $n \leq k+1$, any map $\partial D^n \to X$ can be extended to a map $D^n \to X$ where D^n is the n-dimensional disk. The 1-connected spaces are also called *simply connected.*

The dimension is related to homology since all homology groups above the dimension are trivial, whereas the degree of connectedness is related to homotopy since below this degree all homotopy groups vanish. It took a long time in the development of algebraic topology to establish homology and homotopy groups as the main invariants of a homotopy type. The crucial importance of homotopy groups and homology groups relies on the following results due to J.H.C. Whitehead.

Theorem

(*A*) *A connected CW-space X is contractible if and only if all homotopy groups $\pi_n(X)$, $n \geq 1$, are trivial.*

(*B*) *A simply connected CW-space X is contractible if and only if all homology groups $H_n(X)$, $n \geq 2$, are trivial.*

The theorem shows that homotopy groups, and in the simply connected case also homology groups, are able to detect the trivial homotopy type. In fact, homotopy groups and homology groups are able to decide whether two spaces have the same homotopy type:

Whitehead theorem Let X and Y be simply connected CW-complexes and let $f: X \to Y$ be a map. Then f is a homotopy equivalence if and only if condition (A) or equivalently (B) holds:

(A) the map f induces an isomorphism of homotopy groups, $f_*: \pi_n X \cong \pi_n Y$ for $n > 2$;

(B) the map f induces an isomorphism of homology groups, $f_*: H_n X \cong H_n Y$ for $n \geq 2$.

Hence both homology groups and homotopy groups constitute systems of algebraic invariants which, in a certain sense, are sufficiently powerful to characterize simply connected homotopy types. This does not mean that there is a homotopy equivalence, $X \simeq Y$, between simply connected CW-spaces just because there exist isomorphisms of abelian groups $H_n X \cong H_n Y$ (or $\pi_n X \cong \pi_n Y$) for all n. The crux of the matter is not merely that $H_n X \cong H_n Y$, but that a certain family of isomorphisms, $\phi_n: H_n X \cong H_n Y$, has a *geometrical realization* $f: X \to Y$. That is to say, the latter map f induces all isomorphisms ϕ_n via the functor H_n, namely $\phi_n = H_n(f)$ for all n. Therefore the emphasis is shifted to the following question (pointed out by Whitehead [AH] at the International Congress in Harvard (1950)).

Realization problem of J. H. C. Whitehead

(A) Find necessary and sufficient conditions in order that a given set of isomorphisms or, more generally, homomorphisims $\phi_n: \pi_n X \to \pi_n Y$, have a geometrical realization $X \to Y$.

(B) Find necessary and sufficient conditions in order that a given set of homomorphisms, $\phi_n: H_n X \to H_n Y$, have a geometrical realization $X \to Y$.

This realization problem is far from being solved. Only for simply connected *rational* CW-complexes does there exist satisfying solutions by the minimal models of Quillen and Sullivan; compare Baues and Lemaire [MM]. In this book we describe new solutions for special classes of CW-complexes. As a fundamental tool we obtain the *classification theorem* in Chapter 3 which shows that a homotopy type of a simply connected $(n+1)$-dimensional CW-space X is determined in two different ways, either by the invariants

$$P_{n-1}X, \qquad \pi_n X, \qquad H_{n+1}X, \qquad b_{n+1}X, \qquad k_n X \tag{$*$}$$

or by the invariants

$$P_{n-1}X, \qquad H_n X, \qquad H_{n+1}X, \qquad b_{n+1}X, \qquad \beta_n X. \tag{$**$}$$

Here $P_{n-1}X$ is the $(n-1)$-type of X and $b_{n+1}X$ is the secondary boundary operator in the 'certain exact sequence' of J.H.C. Whitehead [CE]. Moreover $k_n X$ is the *Postnikov invariant* of X, while $\beta_n X$ is a new invariant which we call the *boundary invariant* of X. The classification theorem also describes all invariants $(*)$, resp. $(**)$, which are realizable by spaces. All morphisms between such invariants which are realizable by maps are specified.

The classification theorem yields insight into how homology groups and homotopy groups depend on each other. In fact, we classify all possible abstract homomorphisms between abelian groups,

$$h_n\colon \pi_n \to H_n,$$

which can be realized as the *Hurewicz homomorphism* of a space with a given $(n-1)$-type; compare Theorem 3.4.7. We also show that the Hurewicz homomorphism h_n can be deduced from either the Postnikov invariant $k_n X$ or the boundary invariant $\beta_n X$ (see Sections 2.5 and 2.6). The following result makes clear that the Hurewicz homomorphism

$$h_n X\colon \pi_n X \to H_n X$$

has indeed a strong impact on homotopy types; compare Propositions 2.5.20 and 2.6.15.

Proposition *Let X be a simply connected CW-space. Then (A) and (B) hold:*

(A) *the Hurewicz homomorphism $h_n X$ is split injective for all n if and only if X has the homotopy type of a product of Eilenberg–Mac Lane spaces;*

(B) *moreover $h_n X$ is split surjective for all n if and only if X has the homotopy type of a one-point union of Moore spaces.*

This result and the Whitehead theorem show that homotopy groups and homology groups are indeed the basic invariants of a homotopy type. Moreover a classifying invariant of a simply connected homotopy type should determine the Hurewicz homomorphism $\pi_n \to H_n$ for each non-trivial homology group H_n.

1

LINEAR EXTENSIONS AND MOORE SPACES

In this chapter we describe some basic concepts used in this book. We first introduce purely categorical notions like detecting functor, linear extension of categories, and the cohomology of categories. Then we describe the properties of Whitehead's Γ-functor which we shall need for the description of homotopy classes of maps between Moore spaces in degree 2. In fact, the homotopy category $\mathbf{M}^2$ of such Moore spaces is canonically embedded in a linear extension of categories

$$\mathrm{Ext}(-,\Gamma) \rightarrowtail \mathbf{M}^2 \twoheadrightarrow \mathbf{Ab}$$

which represents a non-trivial cohomology class in $H^2(\mathbf{Ab}, \mathrm{Hom}(-,\Gamma))$. We use Moore spaces for the definition of homotopy groups with coefficients.

1.1 Detecting functors, linear extensions, and the cohomology of categories

A letter like $\mathbf{C}$ denotes a category, $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$ are the classes of objects and of maps (morphisms) respectively. The identity of an object A is $1_A = 1 = \mathrm{id}$ and $\mathbf{C}(A,B)$ is the set of morphisms $A \to B$. The group of automorphisms of A is $\mathrm{Aut}_{\mathbf{C}}(A) = \mathrm{Aut}(A)$. An isomorphism in C is written $f\colon A \cong B$. An isomorphism is also called an equivalence. A *natural equivalence relation* $\sim$ on the category $\mathbf{C}$ is given by an equivalence relation $\sim$ on each morphism set $\mathbf{C}(A,B)$ such that for $f,g \in \mathbf{C}(A,B)$ and $a,b \in \mathbf{C}(B,C)$ the relations $f \sim g$, $a \sim b$ imply the relation $af \sim bg$. In this case we obtain the *quotient category* $\mathbf{C}/\!\sim$ which has the same objects as $\mathbf{C}$ and for which a set of morphisms is the set $\mathbf{C}(A,B)/\!\sim$ of equivalence classes. Hence a morphism $\{f\}\colon A \to B$ in $\mathbf{C}/\!\sim$ is the equivalence class of a morphism $f\colon A \to B$ in $\mathbf{C}$. Now let $\simeq$ be a natural equivalence relation on a category $\mathbf{A}$ which is called *homotopy*. Then a *homotopy equivalence* $f\colon A \simeq B$ is the same as an isomorphism in the quotient category $\mathbf{A}/\!\simeq$. The *homotopy type* $\{B\}$ of B is the class of all objects A homotopy equivalent to B.

Surjective maps, resp. *injective maps*, between sets are denoted by

(1.1.1) $$A \twoheadrightarrow B, \quad \text{resp.} \quad A \rightarrowtail B.$$

(The arrow $\rightarrowtail$ also describes a cofibration in a cofibration category, but the meaning of $\rightarrowtail$ will always be clear from the context.) A functor $\lambda: \mathbf{A} \to \mathbf{B}$ is *full*, resp. *faithful* if the induced maps $\lambda: \mathbf{A}(X,Y) \to \mathbf{B}(\lambda X, \lambda Y)$ are surjective, resp. injective for all objects $X, Y \in \mathbf{A}$; we also write $\lambda: \mathbf{A} \twoheadrightarrow \mathbf{B}$, resp. $\lambda: \mathbf{A} \rightarrowtail \mathbf{B}$ in this case. An *equivalence between categories* is denoted by $\mathbf{A} \xrightarrow{\sim} \mathbf{B}$. For a functor $\lambda: \mathbf{A} \to \mathbf{B}$ let $\lambda\mathbf{A}$ be the *image category* of λ. Objects in $\lambda\mathbf{A}$ are the same as in $\mathbf{A}$ and morphisms $X \to Y$ in $\lambda\mathbf{A}$ are the maps $f: \lambda X \to \lambda Y$ in the image set $\lambda\mathbf{A}(X,Y)$. Clearly λ induces functors

$$\mathbf{A} \overset{\lambda}{\twoheadrightarrow} \lambda\mathbf{A} \overset{i}{\rightarrowtail} \mathbf{B} \tag{1.1.2}$$

where λ is full and where i is faithful. We say that λ is a *quotient functor* if i is an isomorphism of categories. The following notation was introduced by J.H.C. Whitehead, see for example §14 of Whitehead [CE].

(1.1.3) Definition Let $\lambda: \mathbf{A} \to \mathbf{B}$ be a functor. By the *sufficiency* and the *realizability conditions*, with respect to λ, we mean the following:

(a) *Sufficiency:* If $\lambda(f)$ is an isomorphism, so is f, where f is a morphism in $\mathbf{A}$. That is, the functor λ *reflects* isomorphisms.

(b) *Realizability:* The functor $i: \lambda\mathbf{A} \to \mathbf{B}$ in (1.1.2) is an equivalence of categories. This is equivalent to the following two conditions (b1) and (b2).

(b1) The functor λ is *representative*, that is, for each object B in $\mathbf{B}$ there is an object A in $\mathbf{A}$ such that λA is isomorphic to B. In this case we say that B is *λ-realizable.*

(b2) The functor λ is *full*, that is, for objects X, Y in $\mathbf{A}$ and for each morphism $f: \lambda X \to \lambda Y$ in $\mathbf{B}$ there is a morphism $f_0: X \to Y$ in $\mathbf{A}$ with $\lambda f_0 = f$. In this case we also say that f is *λ-realizable.*

In this book the 'Whitehead theorem' is often used for checking that a functor satisfies the sufficiency condition. The proof of realizability conditions is then the hard part in classification problems. Since the sufficiency and realizability conditions appear frequently it is convenient to condense these conditions in the following definition.

(1.1.4) Definition We call $\lambda: \mathbf{A} \to \mathbf{B}$ a *detecting functor* if λ satisfies both the sufficiency and the realizable conditions, or equivalently if λ refects isomorphisms, is representative and full.

Clearly, a faithful detecting functor is the same as an equivalence of categories. By a $1-1$ *correspondence* we always mean a function which is injective and surjective.

(1.1.5) Lemma *A detecting functor $\lambda: \mathbf{A} \to \mathbf{B}$ induces a $1-1$ correspondence between isomorphism classes of objects in $\mathbf{A}$ and isomorphism classes of objects in $\mathbf{B}$.*

Next we consider pairs (A,b) where A is an object in $\mathbf{A}$ and where $b\colon \lambda A \cong B$ is an equivalence in $\mathbf{B}$. We have an equivalence relation on such pairs given by $(A,b) \sim (A',b')$ if and only if there is an equivalence $g\colon A' \cong A$ in $\mathbf{A}$ with $\lambda(g) = b^{-1}b'$. The equivalence classes form the *class of realizations* of B denoted by

$$\text{(1.1.6)} \qquad \mathrm{Real}_\lambda(B) = \{(A,b) \mid b\colon \lambda A \cong B\}/\sim .$$

Let $\{A,b\}$ be the equivalence class of (A,b).

We now consider functors which are embedded into linear extensions of categories. Such linear extensions arise frequently in algebraic topology and in many other fields of mathematics. In fact, once the reader has learnt about this concept he will recognize many examples himself and soon the usefulness and naturalness of the notion will become apparent. We first consider the classical notion of an extension of groups by modules. An extension of a group G by a G-module A is a short exact sequence of groups

$$\text{(1.1.7)} \qquad 0 \to A \xrightarrow[i]{} E \xrightarrow[p]{} G \to 0$$

where i is compatible with the action of G, that is $i(g\cdot a) = x(ia)x^{-1}$ for $x \in p^{-1}(g)$. Two such extensions E and E' are equivalent if there is an isomorphism $\varepsilon\colon E \cong E'$ of groups with $p'\varepsilon = p$ and $\varepsilon i = i'$. It is well known that the equivalence classes of extensions are classified by the cohomology $H^2(G,A)$. Linear extensions of a small category $\mathbf{C}$ by a 'natural system' D generalize such extensions of groups. We show that the equivalence classes of linear extensions are equally classified by the cohomology $H^2(\mathbf{C},D)$. A natural system D on a category $\mathbf{C}$ is the appropriate generalization of a G-module. Recall that $\mathbf{Ab}$ denotes the category of abelian groups.

(1.1.8) Definition Let $\mathbf{C}$ be a category. The *category of factorizations* in $\mathbf{C}$, denoted by $F\mathbf{C}$, is given as follows. Objects are morphisms $f, g, \ldots$ in $\mathbf{C}$ and morphisms $f \to g$ are pairs (α, β) for which

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A' \\ B \arrow[u, "f"] & B' \arrow[l, "\beta"] \arrow[u, "g"'] \end{tikzcd}$$

commutes in $\mathbf{C}$. Here $\alpha f \beta$ is a factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha,\beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$. A *natural system* (of abelian groups) on $\mathbf{C}$ is a functor $D\colon F\mathbf{C} \to \mathbf{Ab}$. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha,\beta)\colon f \to g$ to the induced homomorphism

$$D(\alpha,\beta) = \alpha_* \beta^*\colon D_f \to D_{\alpha f \beta} = D_g.$$

Here we set $D(\alpha,1) = \alpha_*$ and $D(1,\beta) = \beta^*$.

We have a canonical forgetful functor $\pi: F\mathbf{C} \to \mathbf{C}^{op} \times \mathbf{C}$ so that each *bifunctor* $D: \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}$ yields a natural system $D\pi$, also denoted by D. Such a bifunctor is also called a $\mathbf{C}$-*bimodule*. In this case $D_f = D(B, A)$ depends only on the objects A, B for all $f \in \mathbf{C}(B, A)$. As an example we have for functors $F, G: \mathbf{Ab} \to \mathbf{Ab}$ the $\mathbf{Ab}$-bimodule

$$\mathrm{Hom}(F, G): \mathbf{Ab}^{op} \times \mathbf{Ab} \to \mathbf{Ab}$$

which carries (A, B) to the group of homomorphisms $\mathrm{Hom}(FA, GB)$. If F is the identity functor we write $\mathrm{Hom}(-, G)$ for $\mathrm{Hom}(F, G)$. For a group G and a G-module A the corresponding natural system D on the group G, considered as a category, is given by $D_g = A$ for $g \in G$ and $g_* a = g \cdot a$ for $a \in A$, $g^* a = a$. If we restrict the following notion of a 'linear extension' to the case $\mathbf{C} = G$ and $D = A$ we obtain the notion of a group extension above.

(1.1.9) Definition Let D be a natural system on $\mathbf{C}$. We say that

$$D + \rightarrowtail E \overset{p}{\twoheadrightarrow} \mathbf{C}$$

is a *linear extension of the category* $\mathbf{C}$ by D if (a), (b) and (c) hold.

(a) $\mathbf{E}$ and $\mathbf{C}$ have the same objects and p is a full functor which is the identity on objects.

(b) For each $f: A \to B$ in $\mathbf{C}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\mathbf{E}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.

(c) The action satisfies the *linear distributivity law:*

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.$$

Two linear extensions $\mathbf{E}$ and $\mathbf{E}'$ are *equivalent* if there is an isomorphism of categories $\varepsilon: \mathbf{E} \cong \mathbf{E}'$ with $p'\varepsilon = p$ and with $\varepsilon(f_0 + \alpha) = \varepsilon(f_0) + \alpha$ for $f_0 \in \mathrm{Mor}(\mathbf{E})$, $\alpha \in D_{pf_0}$. The extension $\mathbf{E}$ is *split* if there is a functor $s: \mathbf{C} \to \mathbf{E}$ with $ps = 1$. We obtain the *canonical split linear extension*

(d) $$D + \rightarrowtail \mathbf{C} \times D \twoheadrightarrow \mathbf{C}$$

as follows. Objects in $\mathbf{C} \times D$ are the same as in $\mathbf{C}$ and morphisms $X \to Y$ in $\mathbf{C} \times D$ are pairs (f, α) where $f: X \to Y \in \mathbf{C}$ and $\alpha \in D(f)$. The composition law is given by

(e) $$(f, \alpha)(g, \beta) = (fg, f_* \beta + g^* \alpha).$$

Clearly the projection $\mathbf{C} \times D \to \mathbf{C}$ carries (f, α) to f and the action $D+$ is given by $(f, \alpha) + \alpha' = (f, \alpha + \alpha')$ for $\alpha' \in D(f)$. A splitting functor s yields the equivalence of linear extensions

(f) $$\varepsilon: \mathbf{C} \times D \cong \mathbf{E}$$

given by $\varepsilon(f, \alpha) = s(f) + \alpha$. We say that $D+ \rightarrowtail \mathbf{E} \xrightarrow{\lambda} \mathbf{F}$ is a weak linear extension if $\lambda\mathbf{E} \to \mathbf{F}$ is an equivalence of categories and if $D+ \rightarrowtail \mathbf{E} \to \lambda\mathbf{E}$ is a linear extension. In this case λ is not the identity on objects, but it is easy to replace the objects in $\mathbf{E}$ by objects in $\mathbf{F}$: for this we choose for each object X in $\mathbf{F}$ a realization $M(X)$ in $\mathbf{E}$ so that the functor λ, carrying $M(X)$ to X, is considered as a functor which is the identity on objects. In a weak linear extension the functor λ is always a detecting functor. We also consider the following *maps between linear extensions*

(1.1.10)
$$\begin{tikzcd} D+ \arrow[r, tail] \arrow[d, "d"] & \mathbf{E} \arrow[r, two heads, "p"] \arrow[d, "\varepsilon"] & \mathbf{F} \arrow[d, "\varphi"] \\ D'+ \arrow[r, tail] & \mathbf{E}' \arrow[r, two heads, "p'"] & \mathbf{F}' \end{tikzcd}$$

Here ε and φ are functors with $p'\varepsilon = \varphi p$ and $d: D_f \to D'_{\varphi f}$ is a natural transformation compatible with the action $+$, that is

$$\varepsilon(f_0 + \alpha) = \varepsilon(f_0) + d(\alpha)$$

for $\alpha \in D_f$.

(1.1.11) Lemma *If φ is an equivalence of categories and if d is a natural isomorphism then also ε is an equivalence of categories.*

Let $\mathbf{C}$ be a small category and let $M(\mathbf{C}, D)$ be the set of equivalence classes of linear extensions of $\mathbf{C}$ by D. Then there is a canonical bijection

(1.1.12) $$\psi: M(\mathbf{C}, D) \cong H^2(\mathbf{C}, D)$$

which maps the split extension to the zero element, see IV §6 in Baues [AH]. Here $H^n(\mathbf{C}, D)$ denotes the *cohomology* of $\mathbf{C}$ with coefficients in D which is defined by the nerve of $\mathbf{C}$, see Definition 1.1.15 below. We obtain a *representing cocycle* Δ_t of the cohomology class $\{\mathbf{E}\} = \psi(\mathbf{E}) \in H^2(\mathbf{C}, D)$ as follows. Let t be a 'splitting' function for p which associates with each morphism $f: A \to B$ in $\mathbf{C}$ a morphism $f_0 = t(f)$ in $\mathbf{E}$ with $pf_0 = f$. Then t yields a cocycle Δ_t by the formula

(1.1.13) $$t(gf) = t(g)t(f) + \Delta_t(g, f)$$

with $\Delta_t(g, f) \in D(g, f)$. The cohomology class $\{\mathbf{E}\} = \{\Delta_t\}$ is trivial if and only if $\mathbf{E}$ is a split extension.

(1.1.14) Remark For a linear extension

$$D+ \rightarrowtail \mathbf{E} \twoheadrightarrow \mathbf{C} \tag{1}$$

the corresponding cohomology class $\{\mathbf{E}\} = \Psi(\mathbf{E}) \in H^2(\mathbf{C}, D)$ has the following *classifying property* with respect to the groups of automorphisms in $\mathbf{E}$: for an object A in $\mathbf{E}$ the extension (1) yields the group extension

$$0 \to \bar{A} \to \mathrm{Aut}_{\mathbf{E}}(A) \to \mathrm{Aut}_{\mathbf{C}}(A) \to 0 \tag{2}$$

by restriction. Here $\alpha \in \mathrm{Aut}_{\mathbf{C}}(A)$ acts on $x \in \bar{A} = D(1_A)$ by $\alpha \cdot x = (\alpha^{-1})^* \alpha_*(x)$. The cohomology class corresponding to the extension (2) is given by the image of the class $\Psi\{\mathbf{E}\}$ under the homomorphism

$$H^2(\mathbf{C}, D) \xrightarrow{t_* i^*} H^2\big(\mathrm{Aut}_{\mathbf{C}}(A), \bar{A}\big).$$

Here i is the inclusion functor $\mathrm{Aut}_{\mathbf{C}}(A) \hookrightarrow \mathbf{C}$ and $t: i^*D \to \bar{A}$ is the isomorphism of natural systems with $t = (\alpha^{-1})^*: D_\alpha \to D(1_A) = \bar{A}$. Further results on linear extension of categories can be found in Baues [AH], Baues and Wirsching [CS] and Baues and Dreckmann [GL].

Next we define the cohomology of a category $\mathbf{C}$ with coefficients in a natural system D on $\mathbf{C}$. In order to get cohomology groups which are actually sets we have to assume that $\mathbf{C}$ is a small category; by change of universe it is also possible to define this cohomology in case $\mathbf{C}$ is not small.

(1.1.15) Definition Let $\mathbf{C}$ be a small category and let $N_n(\mathbf{C})$ be the set of sequences $(\lambda_1, \ldots, \lambda_n)$ of n composable morphisms in $\mathbf{C}$ (which are the n-simplices of the *nerve* of $\mathbf{C}$). For $n = 0$ let $N_0(C) = \mathrm{Ob}(C)$ be the set of objects in $\mathbf{C}$. The cochain group $F^n = F^n(\mathbf{C}, D)$ is the abelian group of all functions

$$c: N_n(\mathbf{C}) \to \Big(\bigcup_{g \in \mathrm{Mor}(\mathbf{C})} D_g \Big) = D \tag{1}$$

with $c(\lambda_1, \ldots, \lambda_n) \in D_{\lambda_1 \circ \cdots \circ \lambda_n}$. Addition in F^n is given by adding pointwise in the abelian groups D_g. The coboundary $\delta: F^{n-1} \to F^n$ is defined by the formula

$$\begin{aligned}(\delta c)(\lambda_1, \ldots, \lambda_n) = {} & (\lambda_1)_* c(\lambda_2, \ldots, \lambda_n) \\ & + \sum_{i=1}^{n-1} (-1)^i c(\lambda_1, \ldots, \lambda_i \lambda_{i+1}, \ldots, \lambda_n) \\ & + (-1)^n (\lambda_n)^* c(\lambda_1, \ldots, \lambda_{n-1}). \end{aligned} \tag{2}$$

For $n = 1$ we have $(\delta c)(\lambda) = \lambda_* c(A) - \lambda^* c(B)$ for $\lambda: A \to B \in N_1(\mathbf{C})$. One can check that $\delta c \in F^n$ for $c \in F^{n-1}$ and that $\delta\delta = 0$. Hence the *cohomology groups*

$$H^n(\mathbf{C}, D) = H^n(F^*(\mathbf{C}, D), \delta) \tag{3}$$

are defined, $n \geq 0$. These groups are discussed in Baues [AH]. A functor $\phi: \mathbf{C}' \to \mathbf{C}$ induces the homomorphism

(1.1.16) $$\phi^*: H^n(\mathbf{C}, D) \to H^n(\mathbf{C}', \phi^* D)$$

where $\phi^* D$ is the natural system given by $(\phi^* D)_f = D_{\phi(f)}$. On cochains the map ϕ^* is given by the formula

$$(\phi^* f)(\lambda'_1, \ldots, \lambda'_n) = f(\phi\lambda'_1, \ldots, \phi\lambda'_n)$$

where $(\lambda', \ldots, \lambda'_n) \in N_n(\mathbf{C}')$. In (IV.5.8) of Baues [AH] we show:

(1.1.17) Proposition *Let $\phi: \mathbf{C} \to \mathbf{C}'$ be the equivalence of categories. Then ϕ^* is an isomorphism of groups.*

A natural transformation $\tau: D \to D'$ between natural systems induces a homomorphism

(1.1.18) $$\tau_*: H^n(\mathbf{C}, D) \to H^n(\mathbf{C}, D')$$

by $(\tau_* f)(\lambda_1, \ldots, \lambda_n) = \tau_\lambda f(\lambda_1, \ldots, \lambda_n)$ where $\tau_\lambda: D_\lambda \to D'_\lambda$ with $\lambda = \lambda_1 \circ \cdots \circ \lambda_n$ is given by the transformation τ. Now let

$$D'' \overset{\iota}{\rightarrowtail} D \overset{\tau}{\twoheadrightarrow} D'$$

be a short extract sequence of natural systems on $\mathbf{C}$. Then we obtain as usual the natural long exact sequence

(1.1.19)

$$\to H^n(\mathbf{C}, D') \xrightarrow{\iota_*} H^n(\mathbf{C}, D) \xrightarrow{\tau_*} H^n(\mathbf{C}, D'') \xrightarrow{\beta} H^{n+1}(\mathbf{C}, D') \to$$

where β is the Bockstein homomorphism. For a cochain c'' representing a class (c'') in $H^n(\mathbf{C}, D'')$ we obtain $\beta(c'')$ by choosing a cochain c as in (1) of Definition 1.1.15 with $\tau c = c''$. This is possible since τ is surjective. Then $\iota^{-1}\delta c$ is a cocycle which represents $\beta\{c''\}$.

(1.1.20) Remark The cohomology of Definition 1.1.15 generalizes the *cohomology of a group*. In fact, let G be a group and let $\mathbf{G}$ be the corresponding category with a single object and with morphisms given by the elements in G. A G-module D yields a natural system $\bar{D}: F\mathbf{G} \to \mathbf{Ab}$ by $\bar{D}_g = D$ for $g \in G$.

The induced maps are given by $f^*(x)=x$ and $h_*(y)=h\cdot y$, $f,h\in G$. Then the classical definition of the cohomology $H^n(G,D)$ coincides with the definition of $H^n(\mathbf{G},\overline{D})=H^n(G,D)$ given by Definition 1.1.15.

1.2 Whitehead's quadratic functor Γ

We describe the universal quadratic functor $\Gamma\colon \mathbf{Ab}\to\mathbf{Ab}$ which was introduced by J.H.C. Whitehead [CE] and which also was considered by Eilenberg and Mac Lane [II]. The functor Γ is characterized by the following property: a function $\eta\colon A\to B$ between abelian groups is called a *quadratic map* if $\eta(-a)=\eta(a)$ and if the function $A\times A\to B$ with $(a,b)\mapsto\eta(a+b)-\eta(a)-\eta(b)$ is bilinear. For each abelian group A there is a universal quadratic map

(1.2.1) $$\gamma\colon A\to\Gamma(A)$$

with the property that for all B and all quadratic maps $\eta\colon A\to B$ there is a unique homomorphism $\eta^{\square}\colon\Gamma(A)\to B$ with $\eta^{\square}\gamma=f$. Now Γ is a functor since a homomorphism $\varphi\colon A\to B$ yields the quadratic map $\gamma\varphi$ which induces a unique homomorphism $\Gamma(\varphi)=(\gamma\varphi)^{\square}$ such that the diagram

$$\begin{tikzcd}
\Gamma(A) \arrow[r, dashed, "\Gamma(\varphi)"] & \Gamma(B) \\
A \arrow[u, "\gamma"] \arrow[r, "\varphi"'] & B \arrow[u, "\gamma"']
\end{tikzcd}$$

commutes.

We have the following examples of quadratic maps. Let $\sigma_0\colon A\to A\otimes\mathbb{Z}/2$ be given by $\sigma_0(a)=a\otimes 1$. Then it is easily checked that σ_0 is quadratic. Therefore we obtain the canonical surjective homomorphism

(1.2.2) $$\sigma\colon\Gamma(A)\to A\otimes\mathbb{Z}/2 \quad\text{with}\quad \sigma\gamma=\sigma_0.$$

We consider the function $H_0\colon A\to A\otimes A$ with $H_0(a)=a\otimes a$. Clearly, H_0 is quadratic and yields the canonical homomorphism

(1.2.3) $$H\colon\Gamma(A)\to A\otimes A \quad\text{with}\quad H\gamma=H_0.$$

The cokernel of H is the *exterior square* $A\wedge A=A\otimes A/\{a\otimes a\sim 0\}$. Next we obtain by the quadratic map γ the bilinear pairing

(1.2.4)
$$[\ ,\ ]=[1,1]\colon A\otimes A\to\Gamma(A) \quad\text{with}\quad [a,b]=\gamma(a+b)-\gamma(a)-\gamma(b).$$

We write $[f,g]=[1,1](f\otimes g)\colon X\otimes Y\to A\otimes A\to\Gamma(A)$ where $f\colon X\to A$, $g\colon Y\to A$ are homomorphisms. Clearly, we have $[a,b]=[b,a]$ and

$$\text{(1.2.5)}\qquad \sigma[a,b]=0 \quad\text{and}\quad H[a,b]=a\otimes b+b\otimes a.$$

Moreover, the sequence

$$\text{(1.2.6)}\qquad A\otimes A\xrightarrow{[1,1]}\Gamma(A)\xrightarrow{\sigma}A\otimes\mathbb{Z}/2\to 0$$

is exact and natural in A. By considering the equation $[a+b,c]=[a,c]+[b,c]$ we see that the following relations are satisfied in $\Gamma(A)$:

$$\left\{\begin{array}{l}\gamma(-a)=\gamma(a)\\ \gamma(a+b+c)-\gamma(a+b)-\gamma(b+c)-\gamma(a+c)+\gamma(a)+\gamma(b)+\gamma(c)=0\end{array}\right\}. \tag{$*$}$$

We can construct the group $\Gamma(A)$ as follows. Consider the map $\bar\gamma\colon A\to\bar A$ where $\bar A$ is the free abelian group generated by the underlying set of A. The map $\bar\gamma$ is the inclusion of generators. We set $\Gamma(A)=\bar A/R$ where R denotes the relations $(*)$ with γ replaced by $\bar\gamma$. Now γ is the composite $A\to\bar A\to\bar A/R$ of $\bar\gamma$ and of the quotient map. One easily checks that this composition has the universal property in (1.2.1).

For a direct sum $A\oplus A'$ we have the isomorphism

$$\text{(1.2.7)}\qquad \Gamma(A\otimes A')=\Gamma(A)\oplus A\otimes A'\oplus\Gamma(A')$$

which is given by $\Gamma(i)$, $\Gamma(i')$, and $[i,i']$, where i,i' are the inclusions of A and A' into $A\oplus A'$ respectively . A similar result is true for an arbitrary direct sum where I is an ordered set:

$$\Gamma\Big(\bigoplus_I A_i\Big)=\bigoplus_I\Gamma(A_i)\oplus\bigoplus_{i<j}A_i\otimes A_j.$$

Moreover, Γ commutes with direct limits of abelian groups. If $A=\mathbb{Z}$ then $\Gamma(A)=\mathbb{Z}$ is generated by $\gamma 1$. This shows that for a free abelian group A, also $\Gamma(A)$ is free abelian. If B is an ordered basis of A then $\{\gamma(b)\mid b\in B\}\cup\{[b,b']\mid b<b';b,b'\in B\}$ is a *basis* of $\Gamma(A)$. For an arbitrary abelian group we obtain a presentation of $\Gamma(A)$ by the following crucial result:

(1.2.8) Lemma *Let $C\xrightarrow{d}D\to A\to 0$ be an exact sequence of abelian groups. Then the sequence*

$$\Gamma(C)\oplus C\otimes D\xrightarrow{\bar d}\Gamma(D)\to\Gamma(A)\to 0$$

is exact. Here $\bar d$ is defined by $\bar d=(\Gamma(d),[d,1])$ where 1 is the identity of D.

If we set $a = b = -c$ in $(*)$ above we get $\gamma(a) - \gamma(2a) + 3\gamma(a) = 0$ and therefore $\gamma(2a) = 4\gamma(a)$. More generally we get $\gamma(na) = n^2\gamma(a)$. By definition we see $2\gamma(a) = [a, a]$. Using these equations we derive from the lemma:

(1.2.9) Corollary *For a cyclic group $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$ we have*

$$\Gamma(\mathbb{Z}/n) = \mathbb{Z}/(n^2, 2n)$$

where $(n^2, 2n)$ is the greatest common divisor. The group is generated by $\gamma(1)$ where 1 is a generator of $\mathbb{Z}/n$, $1 = 1 + n\mathbb{Z}$.

Hence the surjective homomorphism

$$H: \Gamma(\mathbb{Z}/n) \to \mathbb{Z}/n \otimes \mathbb{Z}/n = \mathbb{Z}/n \tag{1.2.10}$$

has kernel $\mathbb{Z}/2$ if n is even and is an isomorphism if n is odd. Clearly, H is surjective since $H\gamma(1) = 1 \otimes 1$ is a generator. We derive from the universal property of γ that the following diagram commutes since $[1,1]H\gamma(a) = [a, a] = 2\gamma(a)$.

$$\begin{tikzcd} & A \otimes A \arrow[dr, "{[1,1]}"] & \\ \Gamma A \arrow[ur, "H"] \arrow[rr, "\cdot 2"'] & & \Gamma A \end{tikzcd}$$

Here $\cdot 2$ is the multiplication by 2. This shows:

(1.2.11) Proposition *Let A be an abelian group such that multiplication by 2 is an isomorphism on A. Then multiplication by 2 on ΓA is an isomorphism and $H: \Gamma A \to A \otimes A$ is injective and admits a natural retraction, namely $(1/2)\,[1,1]$.*

Proof Since $\cdot 2$ is an isomorphism, $\Gamma(\cdot 2) = \cdot 4: \Gamma A \to \Gamma A$ is also an isomorphism. Since $\cdot 4 = (\cdot 2)(\cdot 2)$, $\cdot 2$ is also an isomorphism on ΓA. $\square$

We finally describe an important example in topology. It is a classical result of J.H.C Whitehead [CES] that the *Hopf map* $\eta_2: S^3 \to S^2$ induces a quadratic function

$$\eta_2^*: \pi_2 X \to \pi_3 X, \qquad \eta_2^*(\alpha) = \alpha\eta_2, \tag{1.2.12}$$

between homotopy groups of a space X. This function satisfies the *left distributivity law*

$$\eta_2^*(\alpha + \beta) - \eta_2^*(\alpha) - \eta_2^*(\beta) = [\alpha, \beta]$$

where $[\alpha, \beta] \in \pi_3 X$ is the Whitehead product of $\alpha, \beta \in \pi_2 X$. Clearly the induced homomorphism $\eta = (\eta_2^*)^\square$,

$$\eta: \Gamma\pi_2 X \to \pi_3 X \quad \text{with} \quad \eta(\gamma\alpha) = \eta_2^*(\alpha), \tag{2.1.13}$$

carries the bracket $[\alpha, \beta] \in \Gamma\pi_2 X$ to the Whitehead product $[\alpha, \beta] \in \pi_3 X$.

1.3 Moore spaces and homotopy groups with coefficients

We start with the definition of a Moore space. Let $n \geq 2$. A *Moore space of degree n* is a simply connected CW-space X with a single non-vanishing homology group of degree n, that is $\tilde{H}_i(X,\mathbb{Z}) = 0$ for $i \neq n$. We write $X = M(A,n)$ if an isomorphism $A \cong H_n(X,\mathbb{Z})$ is fixed. Let $\mathbf{M}^n$ be the full *homotopy category of all Moore spaces of degree n*. We have the homology functor $H_n: \mathbf{M}^n \to \mathbf{Ab}$ which carries $M(A,n)$ to the abelian group A. The $(n-1)$-fold suspension of a *pseudo-projective plane* $P_q = S^1 \cup_q e^2$ is a Moore space of the cyclic group $\mathbb{Z}/q = \mathbb{Z}/q\mathbb{Z}$, that is

$$\Sigma^{n-1} P_q = M(\mathbb{Z}/q, n).$$

Clearly a sphere S^n is a Moore space $S^n = M(\mathbb{Z}, n)$.

(1.3.1) Lemma *The functor $\mathbf{M}^n \to \mathbf{Ab}$ is a detecting functor, that is, for each abelian group A there is a Moore space $M(A,n)$, $n \geq 2$, the homotopy type of which is well defined by (A,n). Moreover, for each homomorphism $\varphi: A \to B$ there is a map $\bar{\varphi}: M(A,n) \to M(B,n)$ with $H_n\bar{\varphi} = \varphi$.*

One has to be careful since for two Moore spaces X, Y of type $M(A,n)$ there is no 'canonical' choice for the homotopy equivalence $X \simeq Y$, since the homotopy class $\bar{\varphi}$ is not uniquely determined by φ. We can easily construct $M(A,n)$ as follows. Choose for A a short exact sequence $C \overset{d}{\rightarrowtail} D \twoheadrightarrow A$ where C and D are free abelian. Then d yields, up to homotopy, a unique map $d: M(C,n) \to M(D,n)$ the mapping cone of which is $M(A,n)$. For $M(C,n)$ and $M(D,n)$ we can take one-point unions of n-spheres. This shows that $M(A,n)$ can be represented by a CW-complex with cells only in dimension n and $n+1$. The suspension of a Moore space of degree n is a Moore space of degree $(n+1)$, that is $\Sigma M(A,n) = M(A, n+1)$. Also a Moore space of degree 2 has the homotopy type of suspension $M(A,2) = \Sigma M_A$ where M_A for example can be chosen to be the mapping cone of a map $d': M_C \to M_D$ where M_C and M_D are one-point unions of 1-spheres and $\Sigma d' \simeq d$. The homotopy type of M_A is not determined by A.

(1.3.2) Definition Let U be a space with base point $*$. The homotopy set of base-point preserving maps

$$\pi_n(A;U) = [M(A,n), U] \quad (n \geq 2) \tag{1.3.3}$$

is called a *homotopy group with coefficients in A*. For $n \geq 3$ this is an abelian group. Also $\pi_2(A;U)$ has a group structure which, however, depends on the choice of M_A. These homotopy groups are covariant functors in U. They are not contravariant functors in A; but they are contravariant functors on the homotopy category $\mathbf{M}^n$ of Moore spaces of degree n. The following proposition is the '*universal coefficient theorem*' for homotopy groups.

(1.3.4) Proposition *For $n \geq 2$ there is the central extension of groups*

$$\mathrm{Ext}(A, \pi_{n+1}U) \overset{\Delta}{\rightarrowtail} \pi_n(A;U) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A, \pi_n U)$$

which is natural in U and which is natural in A in the following sense. Let $\overline{\varphi}: M(A,n) \to M(B,n)$ be a map with $H_n\overline{\varphi} = \varphi: A \to B$. Then we have $\Delta\varphi^* = \overline{\varphi}^*\Delta$ and $\mu\overline{\varphi}^* = \varphi^*\mu$.

Proof For the mapping cone $M(A,n)$ of the map d above we have the exact cofibre sequence of groups

$$\mathrm{Hom}(D, \pi_{n+1}) \xrightarrow{d^*} \mathrm{Hom}(C, \pi_{n+1}) \to [M(A,n), U]$$
$$\to \mathrm{Hom}(D, \pi_n) \xrightarrow{d^*} \mathrm{Hom}(C, \pi_n),$$

where $\pi_n = \pi_n(U)$. For $n = 2$ the extension is central by (II.8.26) Baues [AH]. □

The universal coefficient sequence is compatible with the suspension functor Σ. In fact we have the following commutative diagram of homomorphisms between groups, $n \geq 2$.

$$(1.3.5) \quad \begin{array}{ccccc} \mathrm{Ext}(A, \pi_{n+1}U) & \rightarrowtail & \pi_n(A;U) & \twoheadrightarrow & \mathrm{Hom}(A, \pi_n U) \\ \downarrow{\scriptstyle \Sigma_*} & & \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma_*} \\ \mathrm{Ext}(A, \pi_{n+2}\Sigma U) & \rightarrowtail & \pi_{n+1}(A;\Sigma U) & \twoheadrightarrow & \mathrm{Hom}(A, \pi_{n+1}\Sigma U). \end{array}$$

This follows easily by the naturality in U if we consider $U \to \Omega\Sigma U$ where $\Omega\Sigma U$ is the loop space of ΣU. We now consider the categories $\mathbf{M}^n$ of Moore spaces. The suspension functor Σ yields the sequence of functors

$$(1.3.6) \qquad \mathbf{M}^2 \overset{\Sigma}{\twoheadrightarrow} \mathbf{M}^3 \underset{\sim}{\xrightarrow{\Sigma}} \mathbf{M}^4 \underset{\sim}{\longrightarrow} \cdots$$

which commute with the homology functor, that is $H_{n+1}\Sigma = H_n: \mathbf{M}^n \to \mathbf{Ab}$. The Freudenthal suspension theorem shows that Σ is full on $\mathbf{M}^2$ and that the functor $\Sigma: \mathbf{M}^n \to \mathbf{M}^{n+1}$ is an equivalence of categories for $n \geq 3$. Therefore it is enough to compute $\mathbf{M}^2$ and $\mathbf{M}^3$. Let Γ be the quadratic functor of J.H.C. Whitehead and let $\gamma: A \to \Gamma(A)$ be the universal quadratic function. Then we have the suspension map $\sigma: \Gamma(A) \to A \otimes \mathbb{Z}/2$ with $\sigma(\gamma a) = a \otimes 1$. We define for $n \geq 2$ the functor $\Gamma_n^1: \mathbf{Ab} \to \mathbf{Ab}$ by

$$(1.3.7) \qquad \Gamma_n^1(A) = \begin{cases} \Gamma(A) & \text{for } n = 2 \\ A \otimes \mathbb{Z}/2 & \text{for } n \geq 3. \end{cases}$$

We have a natural isomorphism

$$\eta: \Gamma_n^1(A) \cong \pi_{n+1}M(A,n)$$

which, for $a \in A = \pi_n M(A,n)$, is defined as follows. For $n=2$ the isomorphism $\gamma(a)$ to $\eta_2^*(a)$ where $\eta_2 : S^3 \to S^2$ is the Hopf map; in this way we identify η_2^* with the universal quadratic map

(1.3.8) $$\gamma : A = \pi_2 M(a,2) \xrightarrow{\eta_2^*} \pi_3 M(A,2) = \Gamma(A).$$

Moreover, for $n \geq 3$ the isomorphism carries $a \otimes 1$ to $\eta_n^* a$ where $\eta_n = \Sigma^{n-2}\eta_2 : S^{n+1} \to S^n$ is the suspended Hopf map. The suspension Σ on $\pi_3 M(A,2)$ is now identified with

(1.3.9) $$\sigma : \Gamma(A) = \pi_3 M(A,2) \xrightarrow{\Sigma} \pi_4 M(A,3) = A \otimes \mathbb{Z}/2.$$

As a corollary of the universal coefficient theorem we get

(1.3.10) Corollary *For $n \geq 2$ one has the binatural central extension of groups*

$$\mathrm{Ext}(A, \Gamma_n^1 B) \overset{\Delta}{\rightarrowtail} [M(A,n), M(B,n)] \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A,B)$$

where $\mu = H_n$ is given by the homology functor. We set $\overline{\varphi} + \alpha = \overline{\varphi} + \Delta(\alpha)$ for $\varphi : A \to B$. Then the 'linear distributivity law'

$$(\overline{\psi} + \beta) \circ (\overline{\varphi} + \alpha) = \overline{\psi}\overline{\varphi} + \psi_*(\alpha) + \varphi^*(\beta)$$

*is satisfied with $\psi : B \to C \in$ **Ab** and $\alpha \in \mathrm{Ext}(A, \Gamma_n^1 B)$, $\beta \in \mathrm{Ext}(B, \Gamma_n^1 C)$.*

The corollary shows that we have a linear extension of categories ($n \geq 2$)

(1.3.11) $$\mathrm{Ext}(-, \Gamma_n^1) \rightarrowtail \mathbf{M}^n \xrightarrow{H_n} \mathbf{Ab};$$

compare also (V.3a) in Baues [AH]. This implies that the group of homotopy equivalences $\mathfrak{E}(X)$ for $X = M(A,n)$ is embedded in the extension of groups

(1.3.12) $$\mathrm{Ext}(A, \Gamma_n^1 A) \overset{1^+}{\rightarrowtail} \mathfrak{E}(X) \twoheadrightarrow \mathrm{Aut}(A).$$

Here $\mathrm{Ext}(A, \Gamma_n^1 A)$ is an $\mathrm{Aut}(A)$-module by $\varphi \cdot \alpha = \varphi_*(\varphi^{-1})^*(\alpha)$. The inclusion homomorphism 1^+ is defined by $1^+(\alpha) = 1 + \alpha$ where 1 is the identity of $M(A,n)$. The linear distributivity law shows that 1^+ is actually a homomorphism. The extensions (1.3.11) and (1.3.12) in general are not split, see Baues [AH].

Next we obtain, for $\varphi : A \to B$ and $\overline{\varphi} + \alpha \in [M(A,n), M(B,n)]$, $\alpha \in \mathrm{Ext}(A, \Gamma_n^1 B)$, the induced function

(1.3.13) $$(\overline{\varphi} + \alpha)^* : \pi_n(B,U) \to \pi_n(A,U)$$

which satisfies the formula, $x \in \pi_n(B,U)$,

$$(\overline{\varphi} + \alpha)^*(x) = \overline{\varphi}(x) + \Delta\alpha^*\mu(x).$$

Here $\alpha^*: \mathrm{Hom}(B, \pi_n U) \to \mathrm{Ext}(A, \pi_{n+1} U)$ with $\alpha^*(b) = (\eta_n^* b)^\square_*(\alpha)$ is defined by the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "\gamma_n"'] & \pi_n(U) \arrow[d, "\eta_n^*"] \\
\Gamma_n^1 B \arrow[r, "(\eta_n^* b)^\square"] & \pi_{n+1}(U)
\end{tikzcd}$$

where γ_n is the universal quadratic function for $n = 2$ and is $\sigma\gamma$ for $n \geq 3$.

(1.3.14) Definition For $n \geq 3$ let $\pi_n'(A, X)$ be the kernel of the homomorphism

$$\pi_n(A, X) \xrightarrow{\mu} \mathrm{Hom}(A, \pi_n X) \xrightarrow{(\eta_n^*)_*} \mathrm{Hom}(A, \pi_{n+1} X).$$

Then (1.3.13) shows that π_n' is actually a well-defined bifunctor

$$\pi_n': \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Top}/\simeq \to \mathbf{Ab}$$

together with a natural short exact sequence

$$\mathrm{Ext}(A, \pi_{n+1} X) \overset{\Delta}{\rightarrowtail} \pi_n'(A, X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A, \pi_n' X).$$

Here $\pi_n'(X)$ denotes the kernel of $\eta_n^*: \pi_n(X) \to \pi_{n+1}(X)$.

1.4 Suspended pseudo-projective planes

Pseudo-projective planes, P_f, are the most elementary 2-dimensional CW-complexes. They are obtained by attaching a 2-cell e^2 to a 1-sphere S^1 by an attaching map $f: S^1 \to S^1$ of degree $f \geq 1$, that is

(1.4.1) $$P_f = S^1 \cup_f e^2 = D/\sim .$$

Here D is the unit disk of complex numbers with boundary $S^1 = \partial D$ and with base point $* = 1$. The equivalence relation $\sim$ on D is generated by the relations $x \sim y \Leftrightarrow x^f = y^f$ with $x, y \in S^1$. Clearly $P_2 = \mathbb{R}P_2$ is the real projective plane. We obtain, for each pair (ξ, η) of natural numbers with $g\xi = \eta f$, a map

(1.4.2) $$\tau_\xi : P_f \to P_g \quad \text{by} \quad \tau_\xi\{x\} = \{x^\xi\} \quad \text{for} \quad x \in D.$$

The induced map $\pi_1 \tau_\xi : \mathbb{Z}/f = \pi_1 P_f \to \pi_1 P_g = \mathbb{Z}/g$ on fundamental groups is given by the number $\eta = g\xi/f$ which carries the generator $1 \in \mathbb{Z}/f$ with $1 = 1 + f\mathbb{Z}$ to $\eta \cdot 1 \in \mathbb{Z}_g$. We call the homotopy class of τ_ξ in $\mathbf{Top}^*$ a *principal* *map* between pseudo-projective planes. We point out that the homotopy class of τ_ξ is not determined by $\pi_1(\tau_\xi)$; see III Appendix B in Baues [CH] where we compute the set $[P_f, P_g]$. We consider the suspensions

$$\Sigma^{n-1}P_f = S^n \cup_f e^{n+1} = M(\mathbb{Z}/f, n) \tag{1.4.3}$$

of pseudo-projective planes, $n \geq 2$, which are Moore spaces of cyclic groups.

(1.4.4) Theorem *For $\varphi \in \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g)$ there is a unique element $\overline{\varphi} = B_2(\varphi) \in [\Sigma P_f, \Sigma P_g]$ which induces $\varphi = H_2\overline{\varphi}$ and which is the suspension of a principal map $P_f \to P_g$. Moreover for $n \geq 3$ there is a unique element $\overline{\varphi} = B_n(\varphi) \in [\Sigma^{n-1}P_f, \Sigma^{n-1}P_g]$ which induces $\varphi = H_n\overline{\varphi}$ and which is the $(n-1)$-fold suspension of a map $P_f \to P_g$.*

This crucial fact is proved in III Appendix D of Baues [CH]. Let $\mathbf{P}^n$ be the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of the sphere S^n and the spaces $\Sigma^{n-1}P_f$, $f \geq 1$. Let

$$H_n : \mathbf{P}^n \to \mathbf{Cyc} \tag{1.4.5}$$

be the homology functor where **Cyc** is the full subcategory of cyclic groups $\mathbb{Z}/n$, $n \geq 0$, in **Ab** with $\mathbb{Z}/0 = \mathbb{Z}$.

(1.4.6) Corollary *For $n \geq 2$ the homology functor in* (1.4.5) *admits a splitting functor*

$$B_n : \mathbf{Cyc} \to \mathbf{P}^n$$

with $H_n B_n = 1$.

For the proof of the corollary we only observe that the composition of principal maps between pseudo-projective planes is principal. Hence the corollary is an immediate consequence of the theorem. For $\Sigma^{n-1}P_f = S^n \cup_f e^{n+1}$ we have the *inclusion of the bottom sphere* i and the *pinch map* q such that

$$S^n \xrightarrow{i} \Sigma^{n-1}P_f \xrightarrow{q} S^{n+1} \tag{1.4.7}$$

is a cofibre sequence. The function $B_n : \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g) \to [\Sigma^{n-1}P_f, \Sigma^{n-1}P_g]$ in Theorem 1.4.4 is not additive. But we have the following rule:

(1.4.8) Theorem *For $\varphi, \varphi' \in \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g)$ we have*

$$B_n(\varphi + \varphi') = B_n(\varphi) + B_n(\varphi') + \Delta(\varphi, \varphi')$$

$$\Delta(\varphi, \varphi') = (f(f-1)/2) \cdot \varphi_1' \cdot \varphi_1' \cdot i\eta_n q.$$

Here φ_1, φ_1' are numbers with $\varphi(1) = \varphi_1 1$ and $\varphi'(1) = \varphi_1' 1$ and $\eta_n : S^{n+1} \to S^n$ is the Hopf map. In particular we get $B_n(r\varphi) = rB_n(\varphi) + \frac{1}{2}r(r-1)\Delta(\varphi, \varphi)$.

This result, together with the central extension of groups (see Proposition 1.3.4 and 1.3.7)

(1.4.9) $\mathrm{Ext}(\mathbb{Z}/f, \Gamma_n^1\mathbb{Z}/g) \rightarrowtail [\Sigma^{n-1}P_f, \Sigma^{n-1}P_g] \xrightarrow{H_n} \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g)$

determines completely the group structure of $[\Sigma^{n-1}P_f, \Sigma^{n-1}P_g]$. Here the kernel of H_n is a cyclic group generated by the element $i\eta_n q$; recall that for the cyclic groups $\mathbb{Z}/g$ we have $\Gamma(\mathbb{Z}/g) = \mathbb{Z}/(g^2, 2g)$ so that $\mathrm{Ext}(\mathbb{Z}/f, \Gamma(\mathbb{Z}/g)) = \mathbb{Z}/(f, g^2, 2g)$. Here $(g^2, 2g)$ and $(f, g^2, 2g)$ denote the greatest common divisors.

(1.4.10) Corollary *For all f, g the group $[\Sigma P_f, \Sigma P_g]$ is abelian. Let a, b be defined by $f = 2^a f_0$ and $g = 2^b g_0$ where f_0 and g_0 are odd. Then the homomorphism $(n \geq 2)$*

$$H_n\colon [\Sigma^{n-1}P_f, \Sigma^{n-1}P_g] \twoheadrightarrow \mathrm{Hom}(\mathbb{Z}/f, \mathbb{Z}/g) = \mathbb{Z}/d$$

has an additive splitting of abelian groups if and only if $(a, b) \neq (1, 1)$. Moreover for $d = (f, g)$, $c = (f, g^2, 2g)$, and $e = (f, g, 2)$ we have

$$[\Sigma P_f, \Sigma P_g] = \begin{cases} \mathbb{Z}/d \oplus \mathbb{Z}/c & \text{for } (a, b) \neq (1, 1) \\ \mathbb{Z}/2d \oplus \mathbb{Z}/(c/2) & \text{for } (a, b) = (1, 1) \end{cases}$$

$$[\Sigma^2 P_f, \Sigma^2 P_g] = \begin{cases} \mathbb{Z}/d \oplus \mathbb{Z}/e & \text{for } (a, b) \neq (1, 1) \\ \mathbb{Z}/2d & \text{for } (a, b) = (1, 1). \end{cases}$$

Compare also III Appendix D in Baues [CH] and Barratt [TG]. A restriction of the linear extension (1.3.11) for $\mathbf{M}^n$ yields the linear extension

$$\mathrm{Ext}(-, \Gamma_n^1) + \rightarrowtail \mathbf{P}^n \twoheadrightarrow \mathbf{Cyc}$$

which is split by B_n for $n \geq 2$. Hence Definition 1.1.9(f) yields the equivalence

(1.4.11) $$\mathbf{P}^n \cong \mathbf{Cyc} \times \mathrm{Ext}(-, \Gamma_n^1).$$

Moreover for $n \geq 3$ Theorem 1.4.8 determines $\mathbf{P}^n$ as a pre-additive category. We use this fact in the following section.

1.5 The homotopy category of Moore spaces $\mathbf{M}^n$, $n \geq 3$

In Section 1.4 we computed completely the homotopy category $\mathbf{P}^n$ of suspended pseudo-projective planes. Here we use this result for the computation of the homotopy category $\mathbf{M}^n$ of Moore spaces. We recall the following notation:

(1.5.1) Definition A category $\mathbf{P}$ is *pre-additive* or a *ringoid* if all morphism sets $\mathbf{P}(A,B)$ are abelian groups such that the composition is bilinear. The *category of matrices over* $\mathbf{P}$ is given as follows. Objects are tuples $(A_1,\dots,A_n)$ of objects in $\mathbf{P}$ and morphisms from $(A_1,\dots,A_n)$ to $(B_1,\dots,B_m)$ are matrices

$$M = \left(M_{ji} \in \mathbf{P}(A_i,B_j) \mid i=1,\dots,n; j=1,\dots,m\right).$$

Composition is defined for $N:(B_1,\dots,B_m) \to (C_1,\dots,C_s)$ by

$$(NM)_{ki} = \sum_{j=1}^{m} N_{kj} \circ M_{ji} \in \mathbf{P}(A_i,C_k).$$

The category of matrices over $\mathbf{P}$ is also called the additive completion $\mathbf{Add(P)}$. A category $\mathbf{A}$ is called *additive* if $\mathbf{A}$ is pre-additive and if finite sums exist in $\mathbf{A}$. Each additive functor $F:\mathbf{P} \to \mathbf{A}$ has a unique additive extension $\bar{F}:\mathbf{Add(P)} \to \mathbf{A}$ which carries $(A_1,\dots,A_n)$ to the sum of $FA_1,\dots,FA_n$ in $\mathbf{A}$. Recall that a functor between ringoids is additive if it is a homomorphism on morphism sets.

We consider the full homotopy category $\mathbf{FM}^n$ $(n \geq 2)$ which consists of Moore spaces $M(A,n)$ where A is a finitely generated abelian group. For each such group we have a direct sum decomposition

(1.5.2) $$A \cong \mathbb{Z}/a_1 \oplus \cdots \oplus \mathbb{Z}/a_r, \quad a_i \geq 0,$$

of cyclic groups. Associated with this isomorphism there is a homotopy equivalence

$$M(A,n) \simeq \Sigma^{n-1}(P_{a_1} \vee \cdots \vee P_{a_r})$$

where $P_n = S^1 \cup_n e^2$ is a pseudo-projective plane if $n > 0$ and where $P_n = S^1$ for $n = 0$. This leads to the following result:

(1.5.3) Proposition *For $n \geq 3$ the category $\mathbf{FM}^n$ is equivalent to the category of matrices over the pre-additive category $\mathbf{P}^n$.*

Proof The inclusion $\mathbf{P}^n \subset \mathbf{FM}^n$ yields the additive extension $\mathbf{Add(P}^n) \to \mathbf{FM}^n$ which is an equivalence of categories. □

There are certain subcategories of $\mathbf{M}^n$ which have a simple algebraic characterization. We use the ring

$$R_t = \left[\Sigma^{n-1}P_t, \Sigma^{n-1}P_t\right], \quad n \geq 3.$$

The ring structure is given by composition and addition of maps. In Section 1.4 we computed the rings R_t.

(1.5.4) Corollary *The full subcategory in* $\mathbf{M}^n$ $(n \geq 3)$ *consisting of Moore spaces* $M(A,n)$ *for which* A *is a finitely generated free* $\mathbb{Z}/t$*-module is equivalent to the category of finitely generated free* R_t*-modules.*

(1.5.5) Remark Let $\mathbf{FAb}$ be the full subcategory of $\mathbf{Ab}$ consisting of finitely generated abelian groups. Then we obtain the non-split linear extension $(n \geq 2)$

$$\mathrm{Ext}(-,\Gamma_n^1) + \rightarrowtail \mathbf{FM}^n \twoheadrightarrow \mathbf{FAb} \tag{1}$$

which is the restriction of (1.3.11). Using (1.1.12) we thus have the non-trivial element

$$0 \neq \{\mathbf{FM}^n\} \in H^2(\mathbf{FAb}, \mathrm{Ext}(-,\Gamma_n^1)) \tag{2}$$

which, however, restricted to the subcategory $\mathbf{Cyc} \subset \mathbf{FAb}$ is trivial. On the other hand M. Hartl has shown that the cohomology

$$H^2(\mathbf{FAb}, \mathrm{Ext}(-,\Gamma_n^1)) \cong \mathbb{Z}/2 \tag{3}$$

is a cyclic group of order 2. Hence $\{\mathbf{FM}^n\}$ is the generator of the group so that the linear extension (1) is, up to equivalence, the unique extension of $\mathbf{FAb}$ by $\mathrm{Ext}(-,\Gamma_n^1)$ which is not split. For $n \geq 2$ this is a kind of fancy characterization of the category $\mathbf{FM}^n$. Using Proposition 1.5.3 one can compute a cocycle representing the cohomology class $\{\mathbf{FM}^n\}$, $n \geq 3$; for $n = 2$ we shall compute such a cocycle below.

1.6 Moore spaces and the category G

In this section we describe an algebraic representation of the homotopy category of Moore spaces $\mathbf{M}^n$, $n \geq 3$. Using homomorphisms between certain abelian groups $G(A)$ we define an algebraic category $\mathbf{G}$ and we describe an equivalence of additive categories $\mathbf{M}^n \simeq \mathbf{G}$ for $n \geq 3$. This then leads to the computation of the homotopy group $\pi_n(A,X)$, $n \geq 3$, with coefficients in A in terms of the operator $\eta_n^*: \pi_n(X) \to \pi_{n+1}(X)$ induced by the Hopf map η_n. We identify

(1.6.1)
$$\begin{aligned} \mathrm{Ext}(\mathbb{Z}/q, A) &= A \otimes \mathbb{Z}/q = A/qA \\ \mathrm{Hom}(\mathbb{Z}/q, A) &= A * \mathbb{Z}/q = \mathrm{Ker}(A \xrightarrow{\cdot q} A). \end{aligned}$$

The identification is natural in A, but clearly not natural in $\mathbb{Z}/q$. Moreover we shall use the following natural transformation of functors

(1.6.2)
$$\begin{cases} g = g_q : \mathrm{Ext}(A,B) \to \mathrm{Hom}(A * \mathbb{Z}/q, B \otimes \mathbb{Z}/q) \\ g(\alpha)(x) = x^*(\alpha). \end{cases}$$

Here an element $x \in A * \mathbb{Z}/q = \mathrm{Hom}(\mathbb{Z}/q, A)$ induces the homomorphism $x^*: \mathrm{Ext}(A, B) \to \mathrm{Ext}(\mathbb{Z}/q, B) = B \otimes \mathbb{Z}/q$.

(1.6.3) Lemma *If A is a free $\mathbb{Z}/q$-module then g_q above is an isomorphism.*

Compare (IX.52.2) in Fuch [I]. We now consider, for $n \geq 3$, the abelian group

(1.6.4) $$G(A) = \pi_n(\mathbb{Z}/2, M(A, n))$$

which does not depend on $n \geq 3$ since we have the suspension isomorphism. The extension

$$A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \xrightarrow{\mu} A * \mathbb{Z}/2, \tag{1}$$

given by (1.6.1) and Corollary 1.3.10, thus yields an extension element

$$\{G(A)\} \in \mathrm{Ext}(A * \mathbb{Z}/2, A \otimes \mathbb{Z}/2) \overset{g}{=} \mathrm{Hom}(A * \mathbb{Z}/2, A \otimes \mathbb{Z}/2). \tag{2}$$

The corresponding homomorphism

$$G_A = g\{G(A)\}: A * \mathbb{Z}/2 \to A \otimes \mathbb{Z}/2 \tag{3}$$

carries x to $\Delta^{-1}(2\bar{x})$ where $\bar{x} \in G(A)$ is an element with $\mu\bar{x} = x$.

(1.6.5) Proposition *The homomorphism G_A coincides with composition*

$$G_A: A * \mathbb{Z}/2 \subset A \twoheadrightarrow A \otimes \mathbb{Z}/2,$$

of the inclusion and projection.

In particular we have $G(A \oplus B) \cong G(A) \oplus G(B)$ and

$$G(\mathbb{Z}/q) \cong \begin{cases} \mathbb{Z}/4 & \text{if } q \equiv 2 \bmod 4, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } q \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

This yields $G(A)$ for each finitely generated abelian group A.

Proof of Proposition 1.6.5 For $\bar{x} \in G(A)$ we have $2\bar{x} = (2\,\mathrm{id})^*\bar{x}$ where $2\,\mathrm{id}$ is given by the homotopy commutative diagram

$$\begin{tikzcd} \Sigma^{n-1}P_2 \arrow[r, "2\,\mathrm{id}"] \arrow[d, "q"] & \Sigma^{n-1}P_2 \\ S^{n+1} \arrow[r, "\eta_n"'] & S^n. \arrow[u] \end{tikzcd}$$

Here η_n is the Hopf map. Now the definition of Δ via q and η_n in (1.3.7), and (1.6.4)(3) yields the result. □

We use the extension for the group $G(A)$ above in the definition of the following category.

(1.6.6) Definition of the category G Objects are abelian groups. A map from A to B is a pair (φ, ψ) with $\varphi \in \mathrm{Hom}(A, B)$ and $\psi \in \mathrm{Hom}(GA, GB)$ such that the following diagram commutes:

$$\begin{tikzcd} A \otimes \mathbb{Z}/2 \arrow[r, tail] \arrow[d, "\varphi \otimes 1"] & G(A) \arrow[r, two heads] \arrow[d, "\psi"] & A * \mathbb{Z}/2 \arrow[d, "\varphi * 1"] \\ B \otimes \mathbb{Z}/2 \arrow[r, tail] & G(B) \arrow[r, two heads] & B * \mathbb{Z}/2 \end{tikzcd}$$

We call (φ, ψ) a *proper map* $G(A) \to G(B)$. Let $\mathbf{G}(A, B)$ be the set of all proper maps from $G(A)$ to $G(B)$. Then the naturality of the universal coefficient sequence yields a function

$$\begin{cases} G: [M(A,n), M(B,N)] \to \mathbf{G}(A,B) \\ G(\overline{\varphi}) = (\varphi, \overline{\varphi}_*) \end{cases}$$

where $\varphi = H_n \overline{\varphi}$ and $\overline{\varphi}_* = \pi_n(\mathbb{Z}/2, \overline{\varphi})$. Clearly, G is a functor; in fact:

(1.6.7) Theorem *The functor $G: \mathbf{M}^n \to \mathbf{G}$ is an equivalence of additive categories $(n \geq 3)$ and there is a commutative diagram of linear extensions of categories*

$$\begin{tikzcd} E^n \arrow[r, tail, "+"] \arrow[d, "g"', "\cong"] & \mathbf{M}^n \arrow[r, two heads, "H_n"] \arrow[d, "G"', "\sim"] & \mathbf{Ab} \arrow[d, equal] \\ F \arrow[r, tail, "+"] & \mathbf{G} \arrow[r, two heads, "pr"] & \mathbf{Ab} \end{tikzcd}$$

Here g is the isomorphism

$$g: E^n(A, B) = \mathrm{Ext}(A, B \otimes \mathbb{Z}/2) = \mathrm{Hom}(A * \mathbb{Z}/2, B \otimes \mathbb{Z}/2) = F(A, B)$$

given by g_2 above. The functor $pr: \mathbf{G} \to \mathbf{Ab}$ is the projection $(\varphi, \psi) \mapsto \varphi$ and the action $F \overset{+}{\rightarrowtail} \mathbf{G}$ is given by $(\varphi, \psi) + \beta = (\varphi, \psi + \Delta \beta \mu)$ for $\beta \in F(A, B)$.

(1.6.8) Corollary *Let $n \geq 3$ and let A be any abelian group. Then the group of homotopy equivalences of the Moore space $M(A, n)$ is the group of proper automorphisms of $G(A)$.*

Proof of Theorem 1.6.7 It is enough to show that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Ext}(A, B \otimes \mathbb{Z}/2) \arrow[r, "g", "\cong"'] \arrow[d, "\bar{\varphi}^{+}"] & \mathrm{Hom}(A * \mathbb{Z}/2, B \otimes \mathbb{Z}/2) \arrow[d, "\bar{\varphi}_{*}^{+}"] \\
{[M(A,n), M(B,n)]} \arrow[r, "G"'] & \mathrm{Hom}(G(A), G(B))
\end{tikzcd}$$

where $\bar{\varphi}^{+}(\alpha) = \bar{\varphi} + \alpha$ is given by the universal coefficient theorem and where we set $\bar{\varphi}_{*}^{+}(\beta) = \bar{\varphi}_{*} + \Delta\beta\mu$. Now we have, by the distributivity law for $\mathbf{M}^n$, the following equation, where $x \in [\Sigma^{n-1}P_2, M(A,n)] = G(A)$:

$$G(\bar{\varphi}^{+}(\alpha))(x) = (\bar{\varphi} + \alpha)x = \bar{\varphi}x + x^{*}(\alpha) = \bar{\varphi}_{*}x + \Delta g(\alpha)\mu. \qquad \square$$

Theorem 1.6.7 yields a complete algebraic description of the additive category $\mathbf{M}^n$, $n \geq 3$, as an extension of $\mathbf{Ab}$. Such a simple description is unfortunately not available for the category $\mathbf{M}^2$ of Moore spaces of degree 2.

We now consider, for $n \geq 3$, the functorial properties in A of the homotopy groups

$$\pi_n(A, X) = [M(A,n), X]$$

with coefficient in an abelian group. Using the equivalence of categories G in Theorem 1.6.7 we obtain, for each pointed space X, a functor

$$\mathbf{G}^{\mathrm{op}} = (\mathbf{M}^n)^{\mathrm{op}} \to \mathbf{Ab} \tag{1.6.9}$$

which carries $A \in \mathbf{G}$ to the group $\pi_n(A, X)$. For the computation of this functor we introduce the following notation.

(1.6.10) Definition Let A, π, π' be abelian groups and let $\eta\colon \pi \otimes \mathbb{Z}/2 \to \pi'$ be a homomorphism. We define the abelian group $G(A, \eta)$ by the following push-out diagram in $\mathbf{Ab}$ in which rows are short exact.

$$\begin{tikzcd}
\mathrm{Hom}(A * \mathbb{Z}/2, \pi \otimes \mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, equal] & \mathbf{G}(A, \pi) \arrow[r, two heads, "\mu"] \arrow[dd, "\text{push}"'] & \mathrm{Hom}(A, \pi) \arrow[dd, equal] \\
\mathrm{Ext}(A, \pi \otimes \mathbb{Z}/2) \arrow[d, "\eta_{*}"] & & \\
\mathrm{Ext}(A, \pi') \arrow[r, tail, "\Delta"] & G(A, \eta) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, \pi)
\end{tikzcd}$$

The top row is given by the abelian group $\mathbf{G}(A, \pi)$ of morphisms $A \to \pi$ in the category $\mathbf{G}$ (see Definition 1.6.6), with $\mu(\varphi, \psi) = \varphi$ and $\Delta(\beta) = (0, \Delta\beta\mu)$ (see (1.6.4)(1)). The isomorphism on the left-hand side is defined in (1.6.4)(2). The diagram is in the obvious way functorial in $A \in \mathbf{G}$ and hence we obtain by $G(A,\eta)$ a functor $G(-,\eta)\colon \mathbf{G}^{\mathrm{op}} \to \mathbf{Ab}$. This functor is used in the next result for the computation of the functor (1.6.9).

(1.6.11) Theorem *Let $n \geq 3$ and let X be a pointed space and let*

$$\eta = \eta_n^* : \pi_n(X) \otimes \mathbb{Z}/2 \to \pi_{n+1}(X)$$

be induced by the Hopf map η_n. Then one has an isomorphism of groups

$$\pi_n(A,X) \cong G(A,\eta)$$

which is natural in $A \in \mathbf{G}$ and for which the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Ext}(A,\pi_{n+1}X) & \overset{\Delta}{\rightarrowtail} & \pi_n(A,X) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A,\pi_n X) \\
\| & & \| & & \| \\
\mathrm{Ext}(A,\pi_{n+1}X) & \overset{\Delta}{\rightarrowtail} & G(A,\eta) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A,\pi_n X)
\end{array}$$

Here the top row is the universal coefficient sequence in Proposition 1.3.4 and the bottom row is given by the diagram in Definition 1.6.10. We point out that $G(A,\eta)$ is not functorial in η and that the isomorphism of Theorem 1.6.11 is not natural in X.

(1.6.12) Corollary *Let $n \geq 3$ and assume that $\pi_n(X)$ and A are finitely generated abelian groups. Then the (Δ,μ)-extension for $\pi_n(A,X)$ is split if and only if one of the following three conditions is satisfied:*

(a) *A has no direct summand $\mathbb{Z}/2$;*

(b) *$\pi_n(X)$ has no direct summand $\mathbb{Z}/2$;*

(c) *each element $\alpha \in \pi_n(X)$ generating a direct summand $\mathbb{Z}/2$ satisfies $\eta(\alpha) = \alpha \circ \eta_n = 2\alpha'$ for some $\alpha' \in \pi_{n+1}(X)$.*

Hence, if (a), (b), or (c) hold, one has an isomorphism of abelian groups

$$\pi_n(A,X) \cong \mathrm{Hom}(A,\pi_n X) \oplus \mathrm{Ext}(A,\pi_{n+1}X)$$

which, however, is not natural in A or in X.

Proof of Corollary 1.6.12 If (a) or (b) hold the top row in the diagram of Theorem 1.6.11 is split; if (c) holds the bottom row in the diagram of Theorem 1.6.11 is still split. $\square$

Proof of Theorem 1.6.11 We may assume that X is a connected CW-space. Let $f: Y \to X$ be the $(n-1)$-connected cover of X; this is the fibre of the Postnikov map $X \to P_{n-1}X$, see Section 2.6. Then f induces isomorphisms $\pi_i(f)$ for $i \geq n$. For $\pi = \pi_n Y \cong \pi_n X$ we can choose a map $g: M(\pi, n) \to Y$ which induces the identity $H_n g: \pi = \pi_n Y = H_n Y$ in homology. Such a map g exists since Y is $(n-1)$-connected, $n \geq 3$. The map f induces an isomorphism

$$f_* : \pi_n(A, Y) \cong \pi_n(A, X)$$

and g induces the commutative diagram

$$\begin{tikzcd} \mathrm{Ext}(A, \pi \otimes \mathbb{Z}/2) \arrow[r, tail] \arrow[d, "\mathrm{Ext}(A,\eta)"] & \pi_n(A, M(\pi, n)) \arrow[r, two heads] \arrow[d, "g_*"] & \mathrm{Hom}(A, \pi) \arrow[d, equal] \\ \mathrm{Ext}(A, \pi_{n+1}X) \arrow[r, tail] & \pi_n(A, Y) \arrow[r, two heads] & \mathrm{Hom}(A, \pi) \end{tikzcd}$$

where $\eta = \eta_n^*$ is induced by the Hopf map. Since the rows are short exact this is a push-out diagram of abelian groups. Therefore Theorem 1.6.11 is a consequence of the isomorphism G in Theorem 1.6.7. □

2

INVARIANTS OF HOMOTOPY TYPES

The classical algebraic invariants of a simply connected homotopy type $\{X\}$ are the *homotopy groups* $\pi_n(X)$ and the *homology groups* $H_n(X)$, $n \geq 2$. They are connected by the Hurewicz homomorphism

$$h_n X : \pi_n(X) \to H_n(X) \tag{1}$$

which is embedded in Whitehead's certain exact sequence

$$\to H_{n+1}(X) \xrightarrow{b_{n+1}X} \Gamma_n(X) \xrightarrow{i_n X} \pi_n X \xrightarrow{h_n X} H_n X \xrightarrow{b_n X} \Gamma_{n-1} X \to . \tag{2}$$

It is well known that neither homotopy groups $\pi_*(X)$ nor homology groups $H_*(X)$ suffice to determine the homotopy type of X. However a simply connected space X is contractible if and only if all homotopy groups $\pi_n X$ vanish, or equivalently if all homology groups $H_n X$ vanish. Moreover one has the following facts which show that the Hurewicz homomorphism is indeed significant for the characterization of homotopy types.

Proposition 2.5.20 *A simply connected space X is homotopy equivalent to a product of Eilenberg–Mac Lane spaces if and only if $h_n(X)$ is split injective for all n.*

Proposition 2.6.15 *A simply connected space X is homotopy equivalent to a one-point union of Moore spaces if and only if $h_n(X)$ is split surjective for all n.*

In addition to the Hurewicz homomorphism (1) and Whitehead's exact sequence (2) we have to study deeper invariants of a homotopy type. There are, on the one hand, *Postnikov invariants* or k-invariants which are related to homotopy groups; they are nowadays explained in many textbooks on homotopy theory. On the other hand, we introduce new invariants of a simply connected homotopy type which we call *boundary invariants*. They are related to homology groups similarly to the way Postnikov invariants are related to homotopy groups. The duality between Postnikov invariants and boundary invariants is striking. The main results in this chapter describe properties of Postnikov invariants and boundary invariants respectively. Our results on boundary invariants are new and also some of the properties of k-invariants described in this chapter seem to be new. We use Postnikov invariants and boundary invariants for the classification of homotopy types. The fundamental classification theorem based on these invariants is obtained in Section 3.4. We also consider *unitary invariants* of a homotopy type which are introduced by G.W. Whitehead [RA].

2.1 The Hurewicz homomorphism and Whitehead's certain exact sequence

We consider the Hurewicz homomorphism which is embedded in Whitehead's certain exact sequence and we define the operators in the sequence. We also fix some notation for homotopy groups, (co)homology groups, and chain maps. Let **Top*** be the category of topological spaces with base point $*$ and of base-point preserving maps. A *homotopy* $H: f \simeq g$ in **Top*** is given by a map $H: I_* X \to Y$ with $Hi_0 = f$, $Hi_1 = g$. Here $I_* X = I \times X / I \times *$ is the reduced cylinder on X (given by the unit interval $I = [0,1]$) and $i_t: X \to I_* X$ is the inclusion with $i_t(x) = (t, x)$ for $t \in I$, $x \in X$. Let

$$[X, Y] = \mathbf{Top}^*(X, Y)/\simeq \tag{2.1.1}$$

be the *set of homotopy classes* of maps $f: X \to Y$ in **Top***. This is the set of morphisms $\{f\}: X \to Y$ in the quotient category $\mathbf{Top}^*/\simeq$. We often denote the homotopy class $\{f\}$ simply by f. We have the trivial map $0: X \to * \to Y$ which represents $0 \in [X, Y]$. The *cone* of X is $CX = I_* X / i_1 X$ and the *suspension* of X is $\Sigma X = CX / i_0 X$. The *n-sphere* S^n satisfies $\Sigma S^n = S^{n+1}$ and *homotopy groups* are given by $(n \geq 0)$

$$\begin{cases} \pi_n(X) = [S^n, X] \\ \pi_{n+1}(Y, X) = [(CS^n, S^n), (Y, X)] \end{cases} \tag{2.1.2}$$

where (Y, X) is a pair in **Top***. We have the long exact sequence of homotopy groups $(n \geq 0)$

$$\pi_{n+1} X \xrightarrow{i} \pi_{n+1} Y \xrightarrow{j} \pi_{n+1}(Y, X) \xrightarrow{\partial} \pi_n X \xrightarrow{i} \pi_n Y. \tag{2.1.3}$$

Here ∂ is the restriction and j is induced by the quotient map $(CS^n, S^n) \to (S^{n+1}, *)$. Clearly i is induced by the inclusion $X \subset Y$. The exact sequence is natural with respect to pair maps $(X, Y) \to (X', Y')$ in **Top***. We are mainly interested in *CW-spaces* also termed *spaces*. These are spaces which have the homotopy type of a CW-complex in $\mathbf{Top}^*/\simeq$. Let X be a CW-complex with skeleta X^n. A map $F: X \to Y$ between CW-complexes is *cellular* if $F(X^n) \subset Y^n$. Let **CW** be the following category. Objects are CW-complexes X with trivial 0-skeleton $X^0 = *$ and morphisms are cellular maps $F: X \to Y$. The objects of **CW** are also called reduced CW-complexes. The cylinder $I_* X$ of a CW-complex X is again a CW-complex with skeleta

$$(I_* X)^n = X^n \cup I_* X^{n-1} \cup X^n. \tag{2.1.4}$$

We call a cellular map $H: I_* X \to Y$ a *1-homotopy* and we write $H: f \overset{1}{\simeq} g$.

Moreover we call H a *0-homotopy* if Hi_t is cellular for all $t \in I$; in this case we write $H: f \overset{0}{\simeq} g$. The natural equivalence relations $\overset{0}{\simeq}$ and $\overset{1}{\simeq}$ yield the quotient functors

(2.1.5) $$\mathbf{CW} \to \mathbf{CW}/\overset{0}{\simeq} \to \mathbf{CW}/\overset{1}{\simeq} = \mathbf{CW}/\simeq$$

between the corresponding homotopy categories. The following cellular approximation theorem shows that actually $\mathbf{CW}/\simeq = \mathbf{CW}/\overset{1}{\simeq}$; this is a full subcategory of $\mathbf{Top}^*/\simeq$.

(2.1.6) Cellular approximation theorem *Let X_0 be a subcomplex of the CW-complex X and let $f: X \to Y$ be a map such that $f \mid X_0$ is cellular. Then there exists a cellular map $g: X \to Y$ with $g \mid X_0 = f \mid X_0$ and with $f \simeq g$* rel *X_0.*

For this recall that a homotopy $H: f \simeq g$ is a *homotopy rel X_0* (resp. *under X_0*) if $f \mid X_0 = (Hi_t) \mid X_0$ for all $t \in I$. We now associate with a CW-complex X the *cellular chain complex* $C_* X$; this is the chain complex defined by the relative homology groups

(2.1.7) $$C_n X = H_n(X^n, X^{n-1})$$

with the boundary

$$d = d_n: H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{j} H_{n-1}(X^{n-1}, X^{n-2})$$

given by the triple (X^n, X^{n-1}, X^{n-2}). Let $\mathbf{Chain}_{\mathbb{Z}}$ be the following category. Objects are chain complexes $C = (C_n, d_n, n \in \mathbb{Z})$ of abelian groups and morphisms $F: C' \to C$ are chain maps. Two such chain maps are homotopic, $F \simeq G$, if there exists a homomorphism $\alpha: C' \to C$ of degree $+1$ with $d\alpha + \alpha d = -F + G$. The chain map F is a *homology isomorphism* if F induces an isomorphism $F_*: H_* C' \cong H_* C$. Here the *homology* of C is defined by the quotient group

(2.1.8) $$H_n C = Z_n / B_n$$

where $Z_n = \text{kernel}\{d_n: C_n \to C_{n-1}\}$ is the group of *cycles* and where $B_n = \text{image}\{d_{n+1}: C_{n+1} \to C_n\}$ is the group of *boundaries*. The cellular chain complex above yields a functor

(2.1.9) $$C_*: \mathbf{CW}/\overset{0}{\simeq} \to \mathbf{Chain}_{\mathbb{Z}}$$

which determines the functor between homotopy categories $C_*: \mathbf{CW}/\simeq \to \mathbf{Chain}_{\mathbb{Z}}/\simeq$. For an abelian group A the homology of $C_* X \otimes A$ is the usual *homology with coefficients* in A,

(2.1.10) $$H_n(X, A) = H_n(C_*(X) \otimes A).$$

Similarly the cohomology of the cochain complex $\mathrm{Hom}(C_* X, A)$ is the *cohomology with coefficients* in A,

$$H^n(X, A) = H^n(\mathrm{Hom}(C_* X, A)). \tag{2.1.11}$$

For $A = \mathbb{Z}$ we also write $H_n(X, \mathbb{Z}) = H_n(X)$ and $H^n(X, \mathbb{Z}) = H^n(X)$.

The *Hurewicz homomorphism h* is a natural homomorphism ($n \geq 1$)

$$\begin{cases} h\colon \pi_n(X) \to H_n(X), \\ h\colon \pi_{n+1}(Y, X) \to H_{n+1}(Y, X). \end{cases} \tag{2.1.12}$$

We define h by $h(\alpha) = \alpha_* e_n$ where e_n is an appropriate generator in $H_n(S^n) \cong \mathbb{Z}$ or $H_{n+1}(CS^n, S^n) \cong \mathbb{Z}$ such that h is compatible with the exact sequences of the pair (Y, X); see (2.1.3). Now let X be a CW-complex with $* \in X^0$. We obtain, for $n \geq 1$, Whitehead's *Γ-groups* by the image group

$$\Gamma_n(X) = \mathrm{image}(i_* : \pi_n X^{n-1} \to \pi_n X^n). \tag{2.1.13}$$

Here $i\colon X^{n-1} \subset X^n$ is the inclusion of the $(n-1)$-skeleton into the n-skeleton of X. Clearly $\Gamma_n(X) = 0$ for $n = 1, 2$ since $\pi_n(X^{n-1}) = 0$ in this case. Hence $\Gamma_n(X)$ is an abelian group for all n. A cellular map $F\colon X \to Y$ induces a homomorphism $\Gamma_n(F)\colon \Gamma_n(X) \to \Gamma_n(Y)$. The cellular approximation theorem shows that $\Gamma_n(F)$ depends only on the homotopy class of F so that we obtain a well-defined functor

$$\Gamma_n : \mathbf{CW}/\simeq \; \to \mathbf{Ab}. \tag{2.1.14}$$

Recall that a space X is *n-connected* if $\pi_i(X) = 0$ for $i \leq n$. The following lemma is well known.

(2.1.15) Lemma *Let X be an n-connected CW-complex. Then there exists a homotopy equivalence $Y \simeq X$ where Y is a CW-complex with a trivial n-skeleton $Y^n = *$.*

The lemma implies for the Γ-groups above the

(2.1.16) Corollary *Let X be an $(n-1)$-connected CW-complex. Then $\Gamma_i(X) = 0$ for $i \leq n$.*

Now let X be a simply connected CW-complex. Then the Hurewicz map $h = h_n$ is embedded in the following long exact sequence which is the *certain exact sequence* of J.H.C. Whitehead, $n \geq 2$,

$$\cdots \to H_{n+1} X \xrightarrow{b_{n+1}} \Gamma_n X \xrightarrow{i_n} \pi_n X \xrightarrow{h_n} H_n X \xrightarrow{b_n} \Gamma_{n-1} X. \tag{2.1.17}$$

The sequence is natural with respect to maps in $\mathbf{CW}/\simeq$. The operator i_n is induced by the inclusion $X^n \subset X$. Moreover, the *secondary boundary* b_n is defined by the following commutative diagram, where h is an isomorphism for $n \geq 3$.

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) & \pi_n(X^n, X^{n-1}) \arrow[l, "h"', "\cong"] \arrow[r, "\partial"] & \pi_{n-1}(X^{n-1}) \\
H_n(X^n) \arrow[u, phantom, "\cup"] \arrow[rr, dashed, "\partial_0"] \arrow[dr] & & \Gamma_{n-1}(X) \arrow[u, phantom, "\cup"] \\
& H_n(X) \arrow[ur, "b_n"'] &
\end{tikzcd}$$

One readily checks that ∂h^{-1} induces maps ∂_0 and b_n such that the diagram commutes. Let Z_n be the set of n-cells of $X \in \mathbf{CW}$. Then

(2.1.18)
$$C_n(X) = \mathbb{Z}[Z_n]$$

is the *free* abelian group generated by the set Z_n. We obtain the basis $Z_n \subset C_n(X)$ by the Hurewicz map

$$h: \pi_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1}) = C_n(X)$$

which carries the element $c_e \in \pi_n(X^n, X^{n-1})$, given by the characteristic map of the cell e, to the generator $e \in C_n(X)$. We can choose a map

(2.1.19)
$$f: A_n = \bigvee_{Z_n} S^{n-1} \to X^{n-1}$$

and a homotopy equivalence $c: C_f \simeq X^n$ under X^{n-1}. Here A_n is a one-point union of $(n-1)$-spheres S_e^{n-1} corresponding to n-cells $e \in Z_n$ and C_f is the mapping cone $C_f = CA_n \cup_f X^{n-1}$. We call f the *attaching map* of n-cells in X. The induced map

$$C_n(X) = \mathbb{Z}[Z_n] = \pi_{n-1}(A_n) \xrightarrow{f_*} \pi_{n-1}(X^{n-1})$$

coincides with the boundary map ∂h^{-1} in the diagram above. This yields a direct connection between the attaching map f and the secondary boundary b_n. The exactness of the sequence in (2.1.17) and Corollary 2.1.16 imply the following result.

(2.1.20) Hurewicz theorem *Let X be an $(n-1)$-connected CW-complex, $n \geq 2$. Then*

$$h: \pi_n(X) \cong H_n(X)$$

is an isomorphism and $h: \pi_{n+1}(X) \twoheadrightarrow H_{n+1}(X)$ is surjective.

For an $(n-1)$-connected space X we use the isomorphism $h: \pi_n(X) \cong H_n(X)$ as an identification. In addition to the Hurewicz theorem we have the following result due to J.H.C. Whitehead. For this we consider the functor

$$\Gamma_n^1 : \mathbf{Ab} \to \mathbf{Ab} \tag{2.1.21}$$

given by $\Gamma_2^1(A) = \Gamma(A)$ and $\Gamma_n^1(A) = A \otimes \mathbb{Z}/2$ for $n \geq 3$. Here Γ denotes Whitehead's quadratic functor and $\gamma: A \to \Gamma_n^1(A)$ is the universal quadratic map for $n = 2$ and the quotient map for $n \geq 3$; see (1.2.1).

(2.1.22) Theorem *Let $n \geq 2$ and let X be an $(n-1)$-connected CW-complex. Then there is a natural isomorphism*

$$\bar{\eta}: \Gamma_n^1(H_n X) \cong \Gamma_{n+1}(X).$$

Hence the certain exact sequence yields the natural exact sequence

$$H_{n+2}(X) \xrightarrow{b} \Gamma_n^1(H_n X) \xrightarrow{\eta} \pi_{n+1}(X) \xrightarrow{h} H_{n+1}(X) \to 0.$$

The homomorphism η is induced by the *Hopf map* η_n which is a generator of $\pi_{n+1}(S^n)$, $n \geq 2$. In fact, for an element $\alpha \in H_n X$ representing $\alpha: S^n \xrightarrow{\bar{\alpha}} X^n \subset X$ the isomorphism $\bar{\eta}$ carries $\gamma\alpha$ to the composite $\{\bar{\alpha}\eta_n\}$. Moreover the homomorphism η carries $\gamma\alpha$ to $\alpha\eta_n$. We shall use the exact sequence of Theorem 2.1.22 for the classification of $(n-1)$-connected $(n+2)$-dimensional homotopy types. One readily derives from the theorem the following isomorphisms for Eilenberg–Mac Lane spaces $K(A,n)$ and Moore spaces $M(A,n)$.

(2.1.23) Corollary *There are natural isomorphisms*

$$\theta: \pi_3 M(A,2) = \Gamma(A) = H_4 K(A,2),$$

$$\theta: \pi_{n+1} M(A,n) = A \otimes \mathbb{Z}/2 = H_{n+2} K(A,n), \quad n \geq 3.$$

Clearly $H_n K(A,n) = A = \pi_n M(A,n)$ and $H_{n+1} K(A,n) = 0$ by the Hurewicz theorem. We also use the operators in Whitehead's exact sequence for the definition of the natural transformation $(m > n)$

$$\theta: \pi_m M(A,n) \to H_{m+1} K(A,n). \tag{2.1.24}$$

For this let $k: M(A,n) \to K(A,n)$ be the map which induces in homology the identity $H_n(k) = 1_A$ of A. Then θ is the composite $\theta = b_{m+1}^{-1} \Gamma_m(k) i_m^{-1}$:

$$\pi_m M(A,n) \cong \Gamma_m M(A,n) \xrightarrow{k_*} \Gamma_m K(A,n) \cong H_{m+1} K(A,n).$$

Obstruction theory shows that the cohomology of $X \in \mathbf{CW}$ can be described by the natural isomorphism ($n \geq 1$)

$$H^n(X, A) = [X, K(A, n)], \tag{2.1.25}$$

where $[X, K(A, n)]$ is the set of homotopy classes in $\mathbf{Top}^*/\simeq$. Let $k_A \in H^n(K(A, n), A) = \mathrm{Hom}(A, A)$ be given by the identity of A. Then the isomorphism carries the homotopy class $\xi \in [X, K(A, n)]$ to the cohomology $\xi^* k_A$. We also have the natural isomorphism

$$H^n(X, A) = [C_* X, C_* M(A, n)]. \tag{2.1.26}$$

Here the right-hand side is the set of homotopy classes in $\mathbf{Chain}_{\mathbb{Z}}/\simeq$. Dually we define the *pseudo-homology*

$$H_n(A; X) = [C_* M(A, n), C_* X] \tag{2.1.27}$$

with coefficients in A, not to be confused with $H_n(X, A)$ above. One has the following homotopy classification of chain maps; see for example 10.13 in Dold [AT].

(2.1.28) Theorem *Let C and D be chain complexes in $\mathbf{Chain}_{\mathbb{Z}}$ and assume C_n is free abelian for all n. Then there is a natural short exact sequence*

$$\mathrm{Ext}(H_{*-1}C, H_* D) \overset{\Delta}{\rightarrowtail} [C, D] \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H_* C, H_* D)$$

of abelian groups. Here $[C, D]$ is the set of homotopy classes of chain maps and $\mathrm{Hom}(H_* C, H_* D)$ is the group of degree 0 homomorphisms $H_* C \to H_* D$, that is the product of all groups $\mathrm{Hom}(H_n C, H_n D)$ for $n \in \mathbb{Z}$. Moreover $\mathrm{Ext}(H_{*-1}C, H_* D)$ is the product of all groups $\mathrm{Ext}(H_{n-1}C, H_n D)$ for $n \in \mathbb{Z}$. The map μ carries a chain map $F: C \to D$ to the induced homomorphism $\mu\{F\} = F_*$. The exact sequence is split (unnaturally).

As a special case of the theorem we get, for the cohomology $H^n(X, A)$ and pseudo-homology $H_n(A, X)$, the following *universal coefficient formulas.*

(2.1.29) Corollary *For $X \in \mathbf{CW}$ and $n \geq 1$ there are natural short exact sequences*

$$\mathrm{Ext}(H_{n-1}X, A) \overset{\Delta}{\rightarrowtail} H^n(X, A) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H_n X, A)$$

$$\mathrm{Ext}(A, H_{n+1}X) \overset{\Delta}{\rightarrowtail} H_n(A, X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A, H_n X).$$

The sequences are split (unnaturally).

There are the following two ways of representing elements in the group $\operatorname{Ext}(A, B)$ where A and B are abelian groups. On the one hand each short exact sequence

$$B \rightarrowtail E \twoheadrightarrow A$$

of abelian groups represents an element $\{E\} \in \operatorname{Ext}(A, B)$. On the other hand, we use a *free resolution* of A; this is a short exact sequence

$$A_1 \overset{d}{\rightarrowtail} A_0 \twoheadrightarrow A$$

where A_0, A_1 are free abelian groups. Then d induces a homomorphism

$$d^* = \operatorname{Hom}(d, 1): \operatorname{Hom}(A_0, B) \to \operatorname{Hom}(A_1, B)$$

and we may define $\operatorname{Ext}(A, B)$ to be the cokernel of d^*. Thus homomorphisms $g_1 \in \operatorname{Hom}(A_1, B)$ represents elements $\{g_1\} \in \operatorname{Ext}(A, B)$. We have the well-known

(2.1.30) Lemma *The elements* $\{E\}$ *and* $\{g_1\}$ *in* $\operatorname{Ext}(A, B)$ *above coincide, that is* $\{E\} = \{g_1\}$ *if and only if there is a commutative diagram*

$$\begin{array}{ccccc} A_1 & \rightarrowtail & A_0 & \twoheadrightarrow & A \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \| \\ B & \rightarrowtail & E & \twoheadrightarrow & A \end{array}$$

We shall also use the following facts about chain complexes; for a more detailed discussion we refer the reader to XII §4 in G.W. Whitehead [EH]. Let C be a chain complex of free abelian groups. Then the *universal coefficient theorem* asserts that there is a short exact sequence (see Theorem 2.1.28)

(2.1.31) $$\operatorname{Ext}(H_{n-1}C, A) \overset{\Delta}{\rightarrowtail} H^n(C, A) \overset{\mu}{\twoheadrightarrow} \operatorname{Hom}(H_nC, A)$$

which is natural with respect to both chain maps $F: C \to C'$ and homomorphisms $f: A \to A'$. The sequence admits a splitting (unnaturally). Let $u \in H^n(C, A)$. If $a: C_n \to A$ is a cocycle representing u, then the restriction $a \mid Z_n: Z_n \to A$ maps the group B_n of bounding cycles into zero and thereby induces a homomorphism

$$u_* = \mu(u): H_n(C) \to A \tag{1}$$

with $u_*\{x\} = a(x)$ for $x \in Z_n$. This defines μ in (2.1.31). Now let $e \in \operatorname{Ext}(H_{n-1}C, A)$. The group $H_{n-1}C$ has the convenient free resolution

$$B_{n-1} \rightarrowtail Z_{n-1} \twoheadrightarrow H_{n-1}C. \tag{2}$$

Let $b: B_{n-1} \to A$ be a homomorphism representing e, see Lemma 2.1.30. Let $d: C_n \twoheadrightarrow B_{n-1}$ be defined by the boundary in C. Then $bd: C_n \to A$ is a cocycle, representing the element $\Delta(e) = \{bd\}$. This defines Δ in (2.1.31). For $u \in H^n(C, A)$ we have by (1) the exact sequence

$$H_n(C) \xrightarrow{u_*} A \xrightarrow{q} \operatorname{cok} u_* \to 0 \tag{3}$$

where q is the quotient map. Clearly $qu_* = 0$. In view of the naturality of the exact sequence (2.1.31) there is a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ext}(H_{n-1}C, A) & \overset{\Delta}{\rightarrowtail} & H^n(C, A) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(H_nC, A) \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\
\operatorname{Ext}(H_{n-1}C, \operatorname{cok} u_*) & \overset{\Delta}{\rightarrowtail} & H^n(C, \operatorname{cok} u_*) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(H_nC, \operatorname{cok} u_*)
\end{array}$$

Since $q_* \mu(u) = q_* u_* = qu_* = 0$ we see that the element

$$u_\dagger = \Delta^{-1} q_*(u) \in \operatorname{Ext}(H_{n-1}C, \operatorname{cok} u_*) \tag{4}$$

is well defined. Thus each element $u \in H^n(C, A)$ determines canonically a pair of elements $(u_*, u_\dagger)$. The operations $u \mapsto u_*$, $u \mapsto u_\dagger$ have naturality properties which are explained in G.W. Whitehead [EH].

We need the following property of the element $u_\dagger$ with respect to mapping cones in the category of chain complexes. Let $f: C \to D$ be a chain map between chain complexes of free abelian groups. The *mapping cone* of f is the free chain complex C_f with

(2.1.32)
$$\begin{cases} (C_f)_n = C_{n-1} \oplus D_n \\ d(x, y) = (dx, f(x) - dy). \end{cases}$$

We obtain the short exact sequence of chain complexes (cofibre sequence)

$$D \overset{i}{\rightarrowtail} C_f \overset{\pi}{\twoheadrightarrow} sC. \tag{1}$$

Here i is the obvious inclusion and sC is the *suspension* of C, that is the mapping cone of $C \to 0$. Then π is defined by $\pi(x, y) = x$. We clearly have $H_n sC = H_{n-1}C$. The short exact sequence (1) yields the *exact homology sequence*

$$H_nC \xrightarrow{H_nf} H_nD \to H_nC_f \to H_{n-1}C \xrightarrow{H_{n-1}f} H_{n-1}D \tag{2}$$

which in turn gives rise, for each n, to a short exact sequence

$$\operatorname{cok}(H_nf) \rightarrowtail H_nC_f \twoheadrightarrow \ker(H_{n-1}f) \tag{3}$$

representing an element

$$\{H_nC_f\} \in \operatorname{Ext}(\ker H_{n-1}f, \operatorname{cok} H_nf). \tag{4}$$

We proceed to explain how the operator $u \mapsto u_\dagger$ above leads to a description of this extension element. A cohomology class $u \in H^n(D, H_n D)$ is said to be *unitary* if and only if the homomorphism u_* is the identity of $H_n D$. Let u be such a unitary class and consider the element $f^*u \in H^n(C, H_n D)$. Then $(f^*u)_* = H_n f$ so that

$$(f^*u)_\dagger \in \mathrm{Ext}(H_{n-1}C, \mathrm{cok}\, H_n f).$$

For the inclusion $j\colon \ker(H_{n-1}f) \subset H_{n-1}C$ inducing

$$j^*\colon \mathrm{Ext}(H_{n-1}C, \mathrm{cok}\, H_n f) \to \mathrm{Ext}(\ker H_{n-1}f, \mathrm{cok}\, H_n f)$$

we get the equation:

(2.1.33) Theorem $\{H_n C_f\} = -j^*(f^*u)_\dagger$.

For a proof see XII.4.9 in G.W. Whitehead [EH]. We point out that the theorem can be applied to *topological mapping cones*. In fact let $f\colon X \to Y$ be a cellular map. Then the mapping cone C_f is a CW-complex for which $C_*(C_f)$ is the mapping cone of the chain map $C_*F\colon C_*X \to C_*Y$.

2.2 Γ-Groups with coefficients

Using a Moore space $M(A,n)$ we obtain the homotopy groups of a pointed CW-complex X with coefficients in A by the group of homotopy classes $(n \geq 2)$ $\pi_n(A,X) = [M(A,n), X]$. Here $M(A,n) = \Sigma^{n-1} M_A$ is an $(n-1)$-fold suspension. For the pseudo-homology $H_n(A,x) = [C_* M(A,n), C_* X]$ we thus have the *Hurewicz homomorphism*

(2.2.1) $$h_A\colon \pi_n(A,X) \to H_n(A,X)$$

which carries the homotopy class of a cellular map $x\colon M(A,n) \to X$ to the homotopy class of the induced chain map $C_*(x)\colon C_*M(A,n) \to C_*X$. For $A = \mathbb{Z}$ this Hurewicz homomorphism coincides with the classical homorphism in (2.1.12) above. Moreover the universal coefficient sequences yield the commutative diagram

(2.2.2) $$\begin{tikzcd} \mathrm{Ext}(A, \pi_{n+1}X) \arrow[r, tail, "\Delta"] \arrow[d, "h_*"] & \pi_n(A,X) \arrow[r, two heads, "\mu"] \arrow[d, "h_A"] & \mathrm{Hom}(A, \pi_n X) \arrow[d, "h_*"] \\ \mathrm{Ext}(A, H_{n+1}X) \arrow[r, tail, "\Delta"] & H_n(A,X) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, H_n X) \end{tikzcd}$$

The classical Hurewicz homomorphism $h = h_{\mathbb{Z}}$ is embedded in Whitehead's exact sequence. We want to show that also the homomorphism h_A is part of such an exact sequence. For this we introduce new 'Γ-groups with coefficients'. As the classical Γ-groups $\Gamma_n(X)$ of J.H.C. Whitehead these new groups are derived from the homotopy groups of the skeleta of a CW-complex X. Recall that $\Gamma_n(X)$ is defined by the image

$$\Gamma_n = \Gamma_n X = \mathrm{image}\{i_* : \pi_n X^{n-1} \to \pi_n X^n\}$$

where $i: X^{n-1} \subset X^n$ is the inclusion.

(2.2.3) Definition Let A be an abelian group and let X be a 1-connected CW-complex. We have the canonical inclusion and projection respectively

$$\begin{cases} i: \Gamma_n X \rightarrowtail \pi_n X^n, \\ p: \pi_{n+1} X^n \twoheadrightarrow \Gamma_{n+1} X. \end{cases} \tag{1}$$

We use these for the definition of the groups $\Gamma_n(A;X)$ for $n \geq 3$ as follows. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}(A,\Gamma_{n+1}X) \arrow[d, tail, "\Delta"'] \arrow[dr, phantom, "\text{push}"] & \mathrm{Ext}(A,\pi_{n+1}X^n) \arrow[l, two heads, "p_*"'] \arrow[d, tail] \arrow[dr, tail, bend left, "\Delta"] & \\
\Gamma_n(A;X) \arrow[dr, two heads, bend right, "\mu"'] & P \arrow[l, two heads] \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & \pi_n(A;X^n) \arrow[d, "\mu"] \\
 & \mathrm{Hom}(A,\Gamma_n X) \arrow[r, tail, "i_*"'] & \mathrm{Hom}(A,\pi_n X^n)
\end{tikzcd} \tag{2}$$

Here 'pull' and 'push' denote a pull-back diagram and a push-out diagram respectively in the category of abelian groups. Thus the *Γ-groups with coefficients* $\Gamma_n(A;X)$ are embedded in the short exact sequence

$$\mathrm{Ext}(A,\Gamma_{n+1}X) \overset{\Delta}{\rightarrowtail} \Gamma_n(A;X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A,\Gamma_n X). \tag{3}$$

Clearly, this sequence is natural with respect to cellular maps $X \to Y$ since, for the restriction $X^n \to Y^n$ of such a map, all arrows are natural. Below we show that cellular maps $X \to Y$ which are homotopic induce the same homomorphism $\Gamma_n(A;X) \to \Gamma_n(A;Y)$. This shows that (3) is actually a homotopy invariant of X. The group $\Gamma_n(A;X)$ is an abelian group for all $n \in \mathbb{Z}$. For $n \leq 1$ we set $\Gamma_n(A;X) = 0$ and we set

$$\Gamma_2(A;X) = \mathrm{Ext}(A,\Gamma_3 X) \tag{4}$$

since $\Gamma_2 X = 0$. The group $\Gamma_n(A;X)$ generalizes the Γ-group of J.H.C. Whitehead since we clearly have $\Gamma_n(\mathbb{Z};X) = \mathrm{Hom}(\mathbb{Z}, \Gamma_n X) = \Gamma_n X$. If X is a pointed CW-complex which is not simply connected we set

$$\Gamma_n(A,X) = \Gamma_n(A,\hat{X}) \tag{5}$$

where $\hat{X}$ is the universal covering of X with a base point $*$ mapping to the base point of X. If X is just a pointed space we define

$$\Gamma_n(A,X) = \Gamma_n(A,|SX|) \tag{6}$$

where $|SX|$ is the realization of the singular set of X. In this book, however, we consider mainly 1-connected CW-complexes.

(2.2.4) Remark For an n-dimensional CW-complex X^n with $\pi_1 X^n = 0$ we have Whitehead's exact sequence

$$\Gamma_n = \Gamma_n X^n \overset{i}{\rightarrowtail} \pi_n X^n \xrightarrow{h} H_n X^n \to \Gamma_{n-1}$$

where $Z_n = H_n X^n$ is free. Therefore $im(h)$ is free and thus $i\colon \Gamma_n \rightarrowtail \pi_n X^n$ admits a retraction. This shows that i_* in (2) above is injective.

(2.2.5) Proposition *Let $F, G\colon X \to Y$ be cellular maps between simply connected CW-complexes. If F and G are homotopic we have*

$$F_* = G_* : \Gamma_n(A;X) \to \Gamma_n(A;Y).$$

This shows that $\Gamma_n(A;.)$ is a well-defined *homotopy functor* on the category of simply connected spaces or more generally on $\mathbf{Top}^*/\simeq$.

Proof of Proposition 2.2.5 Let $y\colon M(A,n) \to X^n$ be a map with $\mu\{y\} \in im(i^*)$. Then we know that the restriction of y to the n-skeleton of $M(A,n)$ actually factors up to homotopy over X^{n-1}. Thus we can assume that y is a pair map

$$y\colon (M(A,n), M(A,n)^n) \to (X^n, X^{n-1}). \tag{1}$$

Since X is simply connected this map is a twisted map between mapping cones; compare V.7.8 in Baues [AH]. We point out that y shifts the dimension; the n-skeleton of $M(A,n)$ is mapped to the $(n-1)$-skeleton of X. Next consider the restrictions $F^n, G^n\colon X^n \to Y^n$ of the cellular maps F and G. Since $F \simeq G$ there is

$$\alpha : M(C_n X, n) \to M(C_{n+1} Y, n) = M \tag{2}$$

with

$$G^n = F^n + (g_{n+1})_* \{\alpha\} \quad \text{in} \quad [X^n, Y^n]. \tag{3}$$

Here $C_* X$, $C_* Y$ are the cellular chain complexes and $g_{n+1}: M \to Y^n$ is the attaching map of $(n+1)$-cells in Y. By (3) we get in $\pi_n(A, Y^n)$ the equation

$$G_*^n\{y\} = y^*(F^n + g_{n+1}\{\alpha\}) \tag{4}$$

$$= y^* F^n + \nabla_y^*(g_{n+1}\{\alpha\}, F^n) \tag{5}$$

where ∇_y is $E\xi$ with ξ associated with the twisted map y; see V.3.12 in Baues [AH]. In (5) the addition is given by the mapping cone structure of $M(A, n)$. Therefore

$$\beta = \left\{\nabla_y^*(g_{n+1}\{\alpha\}, F^n)\right\} \in \operatorname{Ext}(A, \pi_{n+1} Y^n)$$

and (5) is equivalent to

$$G_*^n\{y\} = F_*^n\{y\} + \Delta(\beta) \tag{6}$$

with Δ defined by the universal coefficient sequence; here $+$ is the group operation in $\pi_n(A, Y^n)$. We claim that for $p: \pi_{n+1} Y \to \Gamma_{n+1} Y$ we have

$$p_* \beta = 0 \quad \text{in} \quad \operatorname{Ext}(A, \Gamma_{n+1} Y). \tag{7}$$

This shows that the term β vanishes in $\Gamma_n(A; Y)$; see (2) in Definition 2.2.3. Thus Proposition 2.2.5 is a consequence of (7) and (6). We check (7) as follows. We have the commutative diagram

$$\begin{tikzcd} \pi_{n+1}(M \vee Y^n)_2 \arrow[r, "(g_{n+1}, 1)_*"'] & \pi_{n+1}(Y^n) \arrow[r, "i_*"'] \arrow[dr, "p"'] & \pi_{n+1}(Y^{n-1}) \\ & & \Gamma_{n+1} Y \arrow[u, tail] \end{tikzcd}$$

where $i_*(g_{n+1}, 1)_* = 0$ since $i g_{n+1} \simeq 0$. This shows that $p(g_{n+1}, 1)_* = 0$. Since by definition of β above $\beta \in \operatorname{image} \operatorname{Ext}(A, (g_{n+1}, 1)_*)$ we get $p_* \beta = 0$. This completes the proof of Proposition 2.2.5. □

The Γ-group $\Gamma_n(A, X)$ is natural in A in the following sense. Let $\varphi: A \to B$ be a homomorphism between abelian groups and let $\bar{\varphi}: M(A, n) \to M(B, n)$ be a map between Moore spaces which induces φ in homology. Then $\bar{\varphi}$ induces a homomorphism

(2.2.6) $$\bar{\varphi}^*: \Gamma_n(B, X) \to \Gamma_n(A, X)$$

since all arrows in Definition 2.2.3 are natural with respect to $\bar{\varphi}$. In particular the following diagram commutes

(2.2.7) $$\begin{tikzcd} \operatorname{Ext}(B, \Gamma_{n+1} X) \arrow[r, tail, "\Delta"] \arrow[d, "\varphi^*"] & \Gamma_n(B, X) \arrow[r, two heads, "\mu"] \arrow[d, "\bar{\varphi}^*"] & \operatorname{Hom}(B, \Gamma_n X) \arrow[d, "\varphi^*"] \\ \operatorname{Ext}(A, \Gamma_{n+1} X) \arrow[r, tail, "\Delta"] & \Gamma_n(A, X) \arrow[r, two heads, "\mu"] & \operatorname{Hom}(A, \Gamma_n X) \end{tikzcd}$$

For $\alpha \in \mathrm{Ext}(A, B \otimes \mathbb{Z}/2)$ and $n \geq 3$ we obtain a map $\overline{\varphi} + \Delta(\alpha): M(A,n) \to M(B,n)$ which induces φ in homology; here we identify $B \otimes \mathbb{Z}/2 = \pi_{n+1} M(B,n)$. Now we get the formula

$$(\overline{\varphi} + \Delta(\alpha))^* = \overline{\varphi}^* + \Delta \alpha^* \mu \tag{2.2.8}$$

where $\alpha^*: \mathrm{Hom}(B, \Gamma_n X) \to \mathrm{Ext}(A, \Gamma_{n+1} X)$ is defined by

$$\alpha^*(y) = (\eta(y \otimes \mathbb{Z}/2))_*(\alpha).$$

Here the homomorphism $\eta: \Gamma_n(X) \otimes \mathbb{Z}/2 \to \Gamma_{n+1}(X)$ is induced by the Hopf map $\eta_n: S^{n+1} \to S^n$, that is, for $\xi: S^n \to X^{n-1}$ with $\{\xi\} \in \Gamma_n(X)$ we set $\eta(\{\xi\} \otimes 1) = \{i \xi \eta_n\}$ where $i: X^{n-1} \subset X^n$ is the inclusion. The formula for $(\overline{\varphi} + \Delta(a))^*$ above is obtained in the same way as in (1.3.13).

(2.2.9) Definition We here define two natural subgroups $\Gamma_n''(A, X)$ and $\Gamma_n'(A, X)$ of the group $\Gamma_n(A, X)$. For $n \geq 2$ let

$$\Gamma_n'(X) = \mathrm{kernel}(\eta: \Gamma_n(X) \to \Gamma_{n+1}(X))$$

where η is induced by the Hopf map. Moreover let

$$\begin{aligned} \Gamma_n''(X) &= \mathrm{image}(b_{n+1} X: H_{n+1} X \to \Gamma_n X) \\ &= \mathrm{kernel}(i_n X: \Gamma_n X \to \pi_n X). \end{aligned}$$

In Lemma 2.3.4 below we show that we have the natural inclusions

$$\Gamma_n''(X) \subset \Gamma_n'(X) \subset \Gamma_n(X).$$

These groups are trivial for $n = 2$. We obtain the corresponding binatural inclusions

$$\Gamma_n''(A, X) \subset \Gamma_n'(A, X) \subset \Gamma_n(A, X)$$

by the following pull-back diagrams in which the rows are short exact.

$$\begin{tikzcd}
\mathrm{Ext}(A, \Gamma_{n+1} X) \arrow[r, tail, "\Delta"] & \Gamma_n(A, X) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, \Gamma_n X) \\
\mathrm{Ext}(A, \Gamma_{n+1} X) \arrow[r, tail] \arrow[u, equal] & \Gamma_n'(A, X) \arrow[r, two heads] \arrow[u] & \mathrm{Hom}(A, \Gamma_n' X) \arrow[u] \\
\mathrm{Ext}(A, \Gamma_{n+1} X) \arrow[r, tail] \arrow[u, equal] & \Gamma_n''(A, X) \arrow[r, two heads] \arrow[u] & \mathrm{Hom}(A, \Gamma_n'' X) \arrow[u]
\end{tikzcd}$$

All vertical arrows are inclusions. The advantage of Γ'_n is that a homomorphism $\varphi: A \to B$ induces a homomorphism

$$\varphi^* = \bar{\varphi}^*: \Gamma'_n(B, X) \to \Gamma'_n(A, X),$$

which by (2.2.8) above does not depend on the choice of $\bar{\varphi}$; compare the definition of $\pi'_n(A, X)$ in Definition 1.3.14. Hence we get well-defined bifunctors, $n \geq 2$,

$$\Gamma'_n, \Gamma''_n : \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Top}^*/\simeq \to \mathbf{Ab}$$

together with the natural short exact (Δ, μ)-sequences above. Clearly if $A = \mathbb{Z}$ we have $\Gamma'_n(\mathbb{Z}, X) = \Gamma'_n(X)$ and $\Gamma''_n(\mathbb{Z}, X) = \Gamma''_n(X)$. Below we shall see that $\Gamma''_n(A, X)$ is naturally isomorphic to a pseudo-homology group; see Theorem 2.6.14(4).

2.3 An exact sequence for the Hurewicz homomorphism with coefficients

We generalize Whitehead's certain exact sequence by introducing coefficients in abelian groups. We describe the operators of the sequence explicitly in terms of the CW-structure of a CW-complex X; in the next section we give an alternative construction by a fibre sequence. Let X be a simply connected CW-complex and let A be an abelian group. Then there is the long exact sequence $(x \in \mathbb{Z})$

(2.3.1)

$$\to H_{n+1}(A, X) \xrightarrow{b_{n+1}} \Gamma_n(A, X) \xrightarrow{i_n} \pi_n(A, X) \xrightarrow{h_n} H_n(A, X) \xrightarrow{b_n} .$$

Here $h_n = h_n^A$ is the Hurewicz map for homotopy groups with coefficients in A, see (2.2.1), and $i_n = i_n^A$ is induced by the inclusion $X^n \subset X$, see Definition 2.2.3. The boundary operator $b_n = b_n^A$ is explicitly constructed in Definition 2.3.5 below. The universal coefficient sequences yield the commutative diagram

(2.3.2)

$$\begin{tikzcd}
\mathrm{Ext}(A, H_{n+2}X) \arrow[r, "b_*"] \arrow[d, tail] & \mathrm{Ext}(A, \Gamma_{n+1}X) \arrow[r, "i_*"] \arrow[d, tail] & \mathrm{Ext}(A, \pi_{n+1}X) \arrow[r, "h_*"] \arrow[d, tail] & \mathrm{Ext}(A, H_{n+1}X) \arrow[d, tail] \\
H_{n+1}(A, X) \arrow[r, "b^A"] \arrow[d, two heads] & \Gamma_n(A, X) \arrow[r, "i^A"] \arrow[d, two heads] & \pi_n(A, X) \arrow[r, "h^A"] \arrow[d, two heads] & H_n(A, X) \arrow[d, two heads] \\
\mathrm{Hom}(A, H_{n+1}X) \arrow[r, "b_*"] & \mathrm{Hom}(A, \Gamma_n X) \arrow[r, "i_*"] & \mathrm{Hom}(A, \pi_n X) \arrow[r, "h_*"] & \mathrm{Hom}(A, H_n X)
\end{tikzcd}$$

Here the top row and the bottom row are induced by the classical certain exact sequence of J.H.C. Whitehead.

(2.3.3) Theorem *The sequence* (2.3.1) *is exact and natural with respect to maps* $\bar{\varphi}: M(A,n) \to M(B,n)$ *and* $X \to Y$ *in* **Top**$^*/\simeq$. *Moreover diagram* (2.3.2) *commutes.*

Clearly for $A = \mathbb{Z}$ the exact sequence (2.3.1) coincides with Whitehead's exact sequence. For the definition of the boundary operator b_A we need the

(2.3.4) Lemma *Let* $\eta: \Gamma_{n-1}(X) \to \Gamma_n(X)$ *be induced by the Hopf map* $\eta_{n-1}: S^n \to S^{n-1}$. *Then the composition*

$$0 = \eta b_n: H_n(X) \to \Gamma_{n-1}(X) \to \Gamma_n(X)$$

is trivial.

Proof We consider the following commutative diagram

$$\begin{tikzcd}
H_n X^n \arrow[r, two heads] & H_n X \arrow[r, "b_n"] & \Gamma_{n-1} X \arrow[rr, "\eta"] \arrow[d, phantom, "\cap"] & & \Gamma_n X \arrow[d, phantom, "\cap"] \\
C_n X \arrow[u, phantom, "\cap"] \arrow[rr, "f_n"'] & & \pi_{n-1} X^{n-1} \arrow[r, "j"'] & \pi_{n-1} X^n \arrow[r, "\eta_{n-1}^*"'] & \pi_n X^n
\end{tikzcd}$$

where $jf_n = 0$ and therefore $\eta b_n = 0$. □

(2.3.5) Definition For a 1-connected CW-complex X we define the *boundary operator*

$$b_n^A = b_n: H_n(A,X) \to \Gamma_{n-1}(A,X)$$

as follows, $n \geq 3$. We may assume that $X^1 = *$. Let $C = C_* X$ be the cellular chain complex of X and let $Z_n = \ker d_n$ and $B_n = \operatorname{im} d_{n+1}$ be the groups of cycles and of boundaries in C respectively. Then

$$B_n \overset{d}{\rightarrowtail} Z_n \twoheadrightarrow H_n \tag{1}$$

is a free presentation of H_n and we can choose the Moore space $M(H_n, n-1)$ to be the mapping cone of

$$d: M(B_n, n-1) \to M(Z_n, n-1). \tag{2}$$

Here d is given by (1) and by the canonical bijection

$$[M(G,n), X] = \operatorname{Hom}(G, \pi_n X) \tag{3}$$

which exists whenever G is a free abelian group. We choose for the exact sequence

$$\begin{tikzcd} Z_n \arrow[r, tail] & C_n \arrow[r, two heads] & B_{n-1} \arrow[l, dashed, bend right, "t"'] \end{tikzcd} \tag{4}$$

a splitting t so that $C_n = Z_n \oplus tB_{n-1}$. Let

$$f_{n+1}: C_{n+1} \to \pi_n X^n \tag{5}$$

be the attaching map of $(n+1)$-cells in X. Since the diagram

$$\begin{tikzcd} C_{n+1} \arrow[rr, "f_{n+1}"'] & & \pi_n X^n \\ \cup & & \cup \\ Z_{n+1} \arrow[r, two heads, "p"'] \arrow[rrd, "b"'] & H_{n+1} \arrow[r, "b_{n+1}"'] & \Gamma_n \\ & & \pi_n X^{n-1} \arrow[u, two heads] \end{tikzcd} \tag{6}$$

commutes (where b is a lift of $b_{n+1}p$) we can assume that the attaching map f_{n+1} yields the commutative diagram

$$\begin{tikzcd} M(C_{n+1}, n) \arrow[r, "f_{n+1}"'] & X^n \\ \cup & \cup \\ M(Z_{n+1}, n) \arrow[r, "b"'] & X^{n-1} \end{tikzcd} \tag{7}$$

Here we have by (4)

$$M(C_{n+1}, n) = M(Z_{n+1}, n) \vee M(tB_n, n). \tag{8}$$

A crucial step for the construction of b_n^A is the 'principal reduction' of the CW-complex X as described in (3.5.13) of Baues [OT], compare also VII.3.3 in Baues [AH]. The principal reduction yields a map

$$v: V \to X^{n-1} \quad (n \geq 3) \tag{9}$$

with the following properties (10) and (11): V is a CW-complex with cells in dimension $n-1$ and n together with a cellular homotopy equivalence

$$\Sigma V \simeq X^{n+1}/X^{n-1} \tag{10}$$

for the suspension ΣV. There is a cellular homotopy equivalence

$$C_v \simeq X^{n+1} \tag{11}$$

under X^{n-1} which induces the identity on $C_* X^{n+1}$; C_υ denotes the mapping cone of υ in (9).

Now (2), (4), and (8) imply

$$\begin{cases} V = M(Z_{n+1}, n) \vee M(H_n, n-1) \vee M(tB_{n-1}, n-1) \\ V^{n-1} = M(C_n, n-1) = M(Z_n, n-1) \vee M(tB_{n-1}, n-1). \end{cases} \tag{12}$$

Here the inclusion $j: V^{n-1} \subset V$ yields the identity on $M(tB_{n-1}, n-1)$ and yields the canonical inclusion $M(Z_n, n-1) \subset C_d = M(H_n, n-1)$ given by (2). The restriction of the map υ in (9) to the subspaces of V in (12) have the following properties:

$$\upsilon|_{M(Z_{n+1}, n)} = b, \tag{13}$$

see (7), and

$$\upsilon|_{M(tB_{n-1}, n-1)} = f_n|_{M(tB_{n-1}, n-1)}. \tag{14}$$

Moreover, for $\beta = \upsilon|_{M(H_n, n-1)}$ the diagram

$$\begin{array}{ccc} M(H_n, n-1) & \xrightarrow{\beta} & X^{n-1} \\ \cup & & \cup \\ M(Z_n, n-1) & \xrightarrow[b]{} & X^{n-2} \end{array} \tag{15}$$

commutes. This shows that the element β represents an element

$$\{\beta\} \in \Gamma_{n-1}(H_n, X) \quad \text{with} \quad \mu\{\beta\} = b_n. \tag{16}$$

We use the element $\{\beta\}$ for the definition of the boundary operator b_n^A above. Let $\{a\} \in H_n(A, X)$, that is

$$a: C_* M(A, n) \to C_* X \tag{17}$$

is a chain map. Using a splitting $C_{n+1} = Z_{n+1} \oplus tB_n$ as in (4) we obtain from a the commutative diagram

$$\begin{array}{ccccc} C_{n+1}M(A,n) & \xrightarrow{a_{n+1}} & Z_{n+1} \oplus tB_n & = & C_{n+1} \\ \downarrow & & \downarrow{\scriptstyle (0,d)} & & \downarrow{\scriptstyle d} \\ C_n M(A,n) & \xrightarrow[a_n]{} & Z_n & \subset & C_n \end{array} \tag{18}$$

Here a_{n+1} has the coordinates $a_{n+1} = (\psi^0, a_n^0)$ where a_n^0 is the restriction of a_n and where ψ^0 represents an element $\psi_a \in \operatorname{Ext}(A, H_{n+1})$. The chain map a

induces $a_* = \varphi_a \in \operatorname{Hom}(A, H_n)$ in homology. We choose a realization $\overline{\varphi}_a : M(A,n) \to M(H_n, n)$ of φ_a which induces

$$\overline{\varphi}_a^* : \Gamma_{n-1}(H_n, X) \to \Gamma_{n-1}(A, X). \tag{19}$$

Now we define the boundary operator b_n^A above by the formula

$$b_n^A\{a\} = \overline{\varphi}_a^*\{\beta\} + \Delta(b_{n+1})_* \psi_a. \tag{20}$$

Here we use $\{\beta\}$ in (16) and $(b_{n+1})_* : \operatorname{Ext}(A, H_{n-1}) \to \operatorname{Ext}(A, \Gamma_n)$. The element (20) is well defined. In fact, for a different choice $\overline{\varphi}_a + a$ which realizes the homomorphism φ, we have by (2.2.8)

$$(\overline{\varphi}_a + \alpha)^*\{\beta\} = \overline{\varphi}_a^*\{\beta\} + \Delta\alpha^*\mu\{\beta\} = \overline{\varphi}_a^*\{\beta\} \tag{21}$$

where $\alpha^*\mu\{\beta\} = \alpha^* b_n = (\eta(b_n \otimes \mathbb{Z}/2))_*(\alpha)$ with $\eta b_n = 0$ by Lemma 2.3.4. Similarly we see that $b_n^A\{\alpha\}$ depends only on the homotopy class $\{a\}$ of the chain map a.

(2.3.6) Lemma *The boundary operator b_n^A is natural in X.*

Proof Let $F: X \to Y$ be a cellular map between CW-complexes with $X^1 = * = Y^1$. Let

$$w: W \to Y^{n-1}$$

be chosen for Y (with $C_w = Y^{n+1}$) in the same way as v in Definition 2.3.5 is chosen for X. The cellular map F yields the map

$$F: (C_v, X^{n-1}) \to (C_w, Y^{n-1})$$

between mapping cones which, by V.3.12 in Baues [AH], is a *twisted map* since $n \geq 3$. Therefore, there is a homotopy commutative diagram

$$\begin{tikzcd}
 & Y^{n-1} \\
V \arrow[ur, "0"] \arrow[r, "\xi_0"'] \arrow[d, "v"] & W \vee Y^{n-1} \arrow[u, "{(0,1)}"'] \arrow[d, "{(w,1)}"] \\
X^{n-1} \arrow[r, "\eta_0"'] & Y^{n-1}
\end{tikzcd} \tag{1}$$

which is associated with F, see V.3.12 in Baues [AH]. Here η_0 is the restriction of the cellular map F and ξ_0 is a map with the following properties. Let

$$q: M(H_n, n-1) = C_d \to M(B_n, n)$$

be the quotient map. Clearly, q^* induces the inclusion Δ in the universal coefficient sequence. Now the restriction

$$\xi = \xi_{0|M(H_n, n-1)} \tag{2}$$

is the sum of the following four maps:

$$\begin{aligned}
&\xi_1: M(H_n, n-1) \xrightarrow{q} M(B_n, n) \to M(Z'_{n+1}, n) \subset W\\
&\xi_2: M(H_n, n-1) \xrightarrow{\overline{\varphi}_n} M(H'_n, n-1) \subset W\\
&\xi_3: M(H_n, n-1) \xrightarrow{q} M(B_n, n) \to M(B_{n-1}, n-1) \subset W\\
&\xi_4: M(H_n, m-1) \xrightarrow{q} M(B_n, n) \to M(C_n, n-1) \vee Y^2.
\end{aligned}$$

Here ξ_3 factors over q since $V \to W \vee Y^{n-1} \to W$ induces the chain map $C_* F$ on cellular chains. Moreover, ξ_1 factors over q for dimensional reasons, since ξ_4 is trivial on $M(H_n, n-1)$ and on Y^{n-1}; also ξ_4 factors over q and maps to the subspace $M(C_n, n-1) \vee Y^2 \subset W \vee Y^{n-1}$ for dimensional reasons (compare the Hilton–Milnor theorem). The map $\varphi_n: H_n \to H'_n$ is induced by F in homology, $\varphi_n = H_n(F)$.

Since X^n is the mapping cone of f_n we know that for $i: X^{n-1} \subset X^n$ the composition $if_n \simeq 0$ is trivial. Therefore also the projection p with

$$\begin{array}{ccc}
\pi_n Y^{n-1} & \overset{p}{\twoheadrightarrow} & \Gamma_n Y \\
& {\scriptstyle i_*}\searrow & \cap \\
& & \pi_n Y^n
\end{array}$$

satisfies $p(f_n)_* = 0$. This shows that $(w,1)_* \xi_3$ and $(w,1)_* \xi_4$ vanish in $\Gamma_{n-1}(H_n X, Y)$; see Definition 2.3.5(2). Moreover ξ_1 represents an element $\{\xi_1\} \in \operatorname{Ext}(H_n X, H_{n+1} Y)$ such that

$$\Delta(b_{n+1})_* \{\xi_1\} \in \Gamma_{n-1}(H_n X, Y) \tag{3}$$

is represented by $(w,1)_* \xi_1$; here we use Definition 2.3.5(6) for Y. By definition of the boundary b_n^A we now have, on the one hand, (see Definition 2.3.5(20))

$$F_* b_n^A\{a\} = \overline{\varphi}_a^* F_* \{\beta\} + \Delta(b_{n+1})_* F_*(\psi_a). \tag{4}$$

On the other hand, we get for $b = (C_* F)a$, resp. $\{b\} = F_*\{a\}$,

$$b_n^A\{b\} = \overline{\varphi}_b^* \{\beta'\} + \Delta(b_{n+1})_* \psi'_b. \tag{5}$$

Here β' and ψ'_b are chosen for Y as in Definition 2.3.5 and $\varphi_b = \varphi_n \varphi: A \to H_n X \to H_n Y$. Now we have, by the commutativity of (1) and by (2), (3),

$$F_*\{\beta\} = \overline{\varphi}_n^* \{\beta'\} + \Delta(b_{n+1})_* \{\bar{\xi}_1\}. \tag{6}$$

Thus (4) = (5) since we get

$$F_*(\psi_a) + \varphi_a^*\{\xi_1\} = \psi_b' \tag{7}$$

by definition of ψ_a, ψ_b' and ξ_1; see Definition 2.3.5(18). □

Proof of Theorem 2.3.3 The naturality of the sequence (2.3.1) with respect to maps $X \to Y$ is obtained by Lemma 2.3.6 since one readily checks that also i^A and h^A are natural with respect to such maps. Moreover it is easy to derive the naturality of (2.3.1) with respect to maps $\overline{\varphi}: M(A,n) \to M(B,n)$ from the definition of the operators b^A, i^A, h^A. Also the definitions show readily that $b^A h^A = 0$, $h^A i^A = 0$, and $i^A b^A = 0$ and that diagram (2.3.2) commutes. Thus it remains to check exactness. Since we give an alternative proof in the next section we leave this to the reader. □

2.4 Infinite symmetric products and Kan loop groups

In this section we compare the Γ-groups of J.H.C. Whitehead with the homotopy groups of a certain space ΓX. This is related to results of Dold and Thom on the infinite symmetric product of X and to results of Kan on the loop group of X. The results here are useful background knowledge on the Γ-groups and on Whitehead's exact sequence. In our proofs, however, we will always use the more direct definition of the Γ-groups in terms of the skeleta of a CW-complex; see Sections 2.1 and 2.2. In particular our explicit construction of the secondary boundary b_n^A in Definition 2.3.5 is an essential step for the proof of the boundary classification theorem below; a definition of b_n^A as given here by a fibre sequence is not appropriate for this proof.

Let X be a simply connected CW-complex with base point and let $SP^\infty X$ be the *infinite symmetric product* of X. This is the limit of the inclusion maps

(2.4.1) $$X = SP^1 X \subset SP^2 X \subset SP^3 X \subset \cdots.$$

Here $SP^n X = (X \times \cdots \times X)/S(n)$ is the quotient space of the n-fold product $X \times \cdots \times X$ by the action of the symmetric group $S(n)$ which permutes the coordinates. By the result of Dold and Thom [SP] we have the natural isomorphism

(2.4.2) $$\pi_n SP^\infty X = H_n(X)$$

where H_n is the integral homology of X. Moreover, the inclusion $X \to SP^\infty X$ induces the Hurewicz homomorphism

(2.4.3) $$h: \pi_n X \to \pi_n SP^\infty X = H_n(X).$$

Let ΓX be the homotopy theoretic *fibre* of the inclusion $i: X \subset SP^\infty X$. Then the fibre sequence

(2.4.4) $$\Gamma X \to X \to SP^\infty X$$

yields an exact sequence of homotopy groups which by (2.4.3) has the following form

$$\cdots \xrightarrow{h} H_{n+1}X \to \pi_n \Gamma X \to \pi_n X \xrightarrow{h} H_n X \to \cdots.$$

This sequence is similar to Whitehead's exact sequence in Section 2.1. In fact, since X is 1-*connected* we have a natural isomorphism

(2.4.5) $$\pi_n \Gamma X \cong \Gamma_n X$$

such that the diagram

(2.4.6) $$\begin{array}{ccccccccc} \cdots \to & H_{n+1}X & \to & \pi_n\Gamma X & \to & \pi_n X & \to & H_n C & \to \cdots \\ & \| & & \| & & \| & & \| & \\ \cdots \to & H_{n+1}X & \to & \Gamma_n X & \to & \pi_n X & \to & H_n X & \to \cdots \end{array}$$

commutes. Here the bottom row is Whitehead's exact sequence and the top row is induced by (2.4.4) and (2.4.3).

A different approach is due to Kan [CW]. This result of Kan can be used for the proof of (2.4.5) and (2.4.6). Let Y be a reduced simplicial set, for example let $Y = SX$ be the reduced singular set of the space X. Then Kan defines the *loop group* GY which is a free simplicial group. The *realization* $|GY|$ is a topological group which is equivalent to the loop space $\Omega|Y|$. For $F = GY$ denote by $[F, F] \subset F$ the *commutator subgroup*, i.e. the simplicial subgroup such that $[F, F]_n = [F_n, F_n]$ for all n. Then Kan proves that there is a natural equivalence

(2.4.7) $$\pi_{n-1}[GY, GY] = \Gamma_n|Y|$$

if $|Y|$ is simply connected. Now consider the fibre sequence

(2.4.8) $$[GY, GY] \xrightarrow{i} GY \xrightarrow{p} AY$$

with $AY = GY/[GY, GY]$. Here AY is the simplicial group for which $(AY)_n$ is the group $(GY)_n$ made abelian. The map i is the inclusion and p denotes the projection. Kan proves that there is a natural equivalence $\pi_{n-1}AY = H_nY$ and that p induces the Hurewicz homomorphism. Therefore the homotopy sequence of the fibre sequence (2.4.8) is the top row in the following commutative diagram

(2.4.9) $$\begin{array}{ccccccc} H_{n+1}Y & \to & \pi_{n-1}[GY, GY] & \to & \pi_{n-1}GY & \to & H_nY \\ \| & & \| & & \| & & \| \\ H_{n+1}|Y| & \longrightarrow & \Gamma_n|Y| & \longrightarrow & \pi_n|Y| & \to & H_n|Y|. \end{array}$$

The bottom row is Whitehead's exact sequence. Commutativity of (2.4.9) was proved by Kan. Now (2.4.7) yields (2.4.6) since there is the equivalence of fibre sequences:

$$\text{(2.4.10)}\qquad \begin{array}{ccccc} \|[GY,GY]\| & \to & |GY| & \to & |AY| \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ \Omega\Gamma X & \to & \Omega X & \to & \Omega SP^{\infty}X. \end{array}$$

The vertical rows in the commutative diagram are homotopy equivalences (X is simply connected and $Y = SX$ is the reduced singular set of X as above).

Next we consider homotopy groups of the space ΓX with coefficients in an abelian group A. The natural equation $\pi_n \Gamma X = \Gamma_n X$ leads to the short exact coefficient sequence (see (2.2.2))

$$\text{(2.4.11)}\qquad \operatorname{Ext}(A, \Gamma_{n+1}X) \overset{\Delta}{\rightarrowtail} \pi_n(A, \Gamma X) \xrightarrow{\mu} \operatorname{Hom}(A, \Gamma_n X)$$

which by a construction similar to the one of Kan shows that we have a natural isomorphism

$$\text{(2.4.12)}\qquad \pi_n(A, \Gamma X) \cong \Gamma_n(A, X).$$

Here the right-hand side is the Γ-group with coefficients constructed in Definition 2.2.3; the isomorphism is compatible with Δ and μ in (4.1.11) and Definition 2.2.3(3) respectively. For $A = \mathbb{Z}$ this is just the same isomorphism as in (2.4.5). For our purposes the definition of $\Gamma_n(A, X)$ by the skeleta of X is more appropriate.

On the other hand we have the homotopy groups of the spaces $SP^{\infty}X$ with coefficients in an abelian group A. The result of Dold and Thom (2.4.2) yields the short exact coefficient sequence

$$\text{(2.4.13)}\qquad \operatorname{Ext}(A, H_{n+1}X) \overset{\Delta}{\rightarrowtail} \pi_n(A, SP^{\infty}X) \xrightarrow{\mu} \operatorname{Hom}(A, H_n X)$$

as in (2.2.2) which, by the equivalence $|AY| \simeq \Omega SP^{\infty}X$ in (2.4.10), shows that we have a natural isomorphism

$$\text{(2.4.14)}\qquad \pi_n(A, SP^{\infty}X) \cong H_n(A, X).$$

Here $H_n(A, X) = [C_* M(A, n), C_* X]$ is the pseudo-homology defined by homotopy classes of chain maps; see (2.1.27). The isomorphism (2.4.14) is compatible with Δ and μ in (2.4.13) and Corollary 2.1.29 respectively. For the proof of (2.4.14) we also use the Dold–Kan theorem (see for example Dold and Puppe [HN]) which shows that the simplicial abelian group AY in (2.4.10) is completely determined by the singular chain complex $C_*|Y|$ which is equivalent to the chain complex $C_* X$. The pseudo-homology groups

$\pi_n(A, SP^\infty X)$ were also considered by Hilton [HT] (Chapter 5) who showed that (2.4.13) is always split. We here point out that these groups can easily be described by chain maps as in (2.4.14).

The fibre sequence (2.4.4) yields for homotopy groups with coefficients in A an exact sequence which, by (2.4.12) and (2.4.14), has the following form

(2.4.15)

$$\begin{array}{ccccccc}
\pi_{n+1}(A, SP^\infty X) & \xrightarrow{\partial} & \pi_n(A, \Gamma X) & \longrightarrow & \pi_n(A, X) & \longrightarrow & \pi_n(A, SP^\infty X) \\
\| & & \| & & \| & & \| \\
H_{n+1}(A, X) & \xrightarrow{b} & \Gamma_n(A, X) & \xrightarrow{i} & \pi_n(A, X) & \xrightarrow{h} & H_n(A, X)
\end{array}$$

Here the top row is the usual fibre sequence given by (2.4.4) and the bottom row is our exact sequence constructed in (2.3.1). Using the construction of the isomorphisms (2.4.12) and (2.4.14) one can check that the diagram commutes; see for example (2.4.3) and (2.4.8).

2.5 Postnikov invariants of a homotopy type

We describe in this section the classical 'k-invariants' which were introduced by Postnikov in 1951; they were also studied by J.H.C. Whitehead [GD]. We show that Postnikov's invariants of a homotopy type have properties which are exactly analogous to the properties of the boundary invariants in Section 2.6 below. As pointed out by J.H.C. Whitehead [I] one has to consider the hierarchy of categories and functors

(2.5.1) $$\mathbf{1\text{-}types} \xleftarrow{P} \mathbf{2\text{-}types} \xleftarrow{P} \mathbf{3\text{-}types} \leftarrow \cdots$$

where n-types are connected CW-spaces Y with $\pi_i Y = 0$ for $i > n$ and where n-**types** is the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of n-types. The functor P which carries $(n+1)$-types to n-types is given by the *Postnikov functor*

(2.5.2) $$P_n : \mathbf{CW}/\simeq \to n\text{-}\mathbf{types}.$$

Here $\mathbf{CW}/\simeq$ is the full homotopy category of CW-complexes X with $X^0 = *$. For X in $\mathbf{CW}$ we obtain $P_n X$ by 'killing homotopy groups'; that is, we choose a CW-complex $P_n X$ with $(n+1)$-skeleton

$$\begin{cases} (P_n X)^{n+1} = X^{n+1} & \text{and} \\ \pi_i(P_n X) = 0 & \text{for} \quad i > n. \end{cases}$$

For a cellular map $F: X \to Y$ in $\mathbf{CW}$ we choose a map $PF^{n+1}: P_n X \to P_n Y$ which extends the restriction $F^{n+1}: X^{n+1} \to Y^{n+1}$ of F. This is possible by usual arguments of obstruction theory since $\pi_i(P_n X) = 0$ for $i > n$. The functor P_n in (2.5.2) carries X to $P_n X$ and carries F to the homotopy class of PF^{n+1}. Different choices for $P_n X$ yield canonically isomorphic functors.

Since 1-types are the same as Eilenberg–Mac Lane spaces $K(\pi, 1)$ we can identify a 1-type with an abstract group. In fact, let **Gr** be the category of groups. Then the fundamental group π_1 gives us an equivalence of categories

$$\pi_1 : 1\text{-}\mathbf{types} \xrightarrow{\sim} \mathbf{Gr} \tag{2.5.3}$$

together with a natural isomorphism $\pi_1(P_1 X) \cong \pi_1(X)$ where $P_1 X = K(\pi_1 X, 1)$. From this point of view n-types are natural objects of higher complexity extending abstract groups. Following up this idea Whitehead looked for a purely algebraic equivalent of an n-type, $n \geq 2$. An important requirement for such an algebraic system is *realizability*, in three senses. In the first instance this means that there is an n-type which is in the appropriate relation to a given one of these algebraic systems, just as there is a 1-type whose fundamental group is isomorphic to a given group. The second kind is the 'realizability' of 'homomorphisms' between such algebraic systems by maps of the corresponding n-types. The third kind is a 1–1 correspondence of such homomorphisms and the homotopy classes of maps between n-types. For example the functor π_1 in (2.5.3), which carries a 1-type to its fundamental group, satisfies these three properties. We have further examples for certain subcategories of the category of n-types. Let

$$\mathbf{types}_m^{n-m} \tag{2.5.4}$$

be the full subcategory of n-**types** consisting of $(m-1)$-connected n-types. Then we have for $m \geq 2$ the equivalence of categories

$$\pi_m : \mathbf{types}_m^0 \xrightarrow{\sim} \mathbf{Ab} \tag{2.5.5}$$

where **Ab** is the category of abelian groups. This equivalence is the higher-dimensional analogue of the equivalence (2.5.3). An object in the category $\mathbf{types}_m^0$, that is an $(m-1)$-connected m-type, is the same as an Eilenberg–Mac Lane space $K(A, m)$ which is determined by an abelian group A.

(2.5.6) Definition A map

$$p_n : X \to P_n X \tag{1}$$

which extends the inclusion $X^{n+1} \subset P_n X$ in (5.2) is called the *n-type* or a *Postnikov section* of X. Clearly p_n induces isomorphisms of homotopy groups

$$(p_n)_* : \pi_i X \cong \pi_i P_n X \quad \text{for} \quad i \leq n. \tag{2}$$

The map p_n is natural with respect to the functor in (2.5.2). The *Postnikov tower* $\{q_n\}$ is given by maps

$$q_n: P_n X \to P_{n-1} X \tag{3}$$

with $q_n p_n \simeq p_{n-1}$. Such maps are well defined up to homotopy. For further properties of the Postnikov tower we refer the reader to Baues [OT] and Baues [AH]; compare also G.W. Whitehead [EH], Spanier [AT], Mosher and Tangora [CO], and many other books on homotopy theory.

(2.5.7) Definition Let X be a simply connected CW-complex and let A be an abelian group. We define the abelian group

$$\mathfrak{P}_{n-1}(X, A) = H^{n+1}(P_{n-1}X, A) \tag{1}$$

by use of the Postnikov functor P_{n-1}, see (2.5.2), and by the cohomology with coefficients in A. Thus $\mathfrak{P}_{n-1}$ is a bifunctor

$$\mathfrak{P}_{n-1}: \mathbf{spaces}_2^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab} \tag{2}$$

where $\mathbf{spaces}_2$ is the full homotopy category of simply connected CW-spaces. The $(n-1)$-Postnikov section $p_{n-1}: X \to P_{n-1}X$ induces the following commutative diagram for Whitehead's exact sequence (see also II.4.8 in Baues [CH])

$$\begin{tikzcd}
H_{n+1}P_{n-1}X \cong \Gamma_n P_{n-1}X \arrow[r] & 0 \arrow[r] & H_n P_{n-1}X \arrow[r, tail] & \Gamma_{n-1}P_{n-1}X \\
H_{n+1}X \arrow[u] \arrow[r] & \Gamma_n X \arrow[u, "\cong"'] \arrow[r] & \pi_n X \arrow[u] \arrow[r] & H_n X \arrow[u] \arrow[r] & \Gamma_{n-1}X \arrow[ul, "\cong"']
\end{tikzcd} \tag{3}$$

Here the vertical arrows are induced by p_{n-1}. Thus we have natural isomorphisms $\Theta = (p_{n-1})_*^{-1} b$

$$\begin{cases} \Theta: H_n P_{n-1} X \cong \Gamma''_{n-1} X \\ \Theta: H_{n+1} P_{n-1} X \cong \Gamma_n X \end{cases} \tag{4}$$

where $\Gamma''_{n-1}X = \ker(i_{n-1}X)$ is the image of the operator $b_n X: H_n X \to \Gamma_{n-1}X$. We use these isomorphisms as identification. Thus the universal coefficient formula for the cohomology group (1) leads to the commutative diagram of short exact sequences

$$\begin{tikzcd}
\mathrm{Ext}(H_n P_{n-1}X, A) \arrow[r, tail, "\Delta"] \arrow[d, equal] & H^{n+1}(P_{n-1}X, A) \arrow[r, two heads, "\mu"] \arrow[d, equal] & \mathrm{Hom}(H_{n+1}P_{n-1}X, A) \arrow[d, equal] \\
\mathrm{Ext}(\Gamma''_{n-1}X, A) \arrow[r, tail, "\Delta"] & \mathfrak{P}_{n-1}(X, A) \arrow[r, "\mu"] & \mathrm{Hom}(\Gamma_n X, A)
\end{tikzcd} \tag{5}$$

The diagram is natural in X and A. Given an element $k \in \mathfrak{P}_{n-1}(X, A)$ we obtain elements (see (1.2.31))

$$\begin{cases} k_* = \mu k \in \operatorname{Hom}(\Gamma_n X, A) \\ k_\dagger = \Delta^{-1} q_*(k) \in \operatorname{Ext}(\Gamma''_{n-1} X, \operatorname{cok} k_*). \end{cases} \tag{6}$$

Here $q\colon A \to \operatorname{cok} k_*$ is the projection for the cokernel of k_* and

$$q_* : \mathfrak{P}_{n-1}(X, A) \to \mathfrak{P}_{n-1}(X, \operatorname{cok} k_*) \tag{7}$$

is given by the functor $\mathfrak{P}_{n-1}$ in (1). Since clearly $\mu q_*(k) = q_* \mu k = q_* k_* = qk_* = 0$ we see that $k_\dagger$ in (6) is well defined. We use the functor $\mathfrak{P}_{n-1}$ for the following definition of Postnikov invariants of X.

(2.5.8) Definition Let X be a simply connected CW-complex. Then we obtain the *Postnikov invariant* or *k-invariant* ($n \geq 3$)

$$k_n = k_n X \in \mathfrak{P}_{n-1}(X, \pi_n X) = H^{n+1}(P_{n-1} X, \pi_n X) \tag{1}$$

as follows. For $A = \pi_n X$ we have the fibre sequence

$$K(A, n) \xrightarrow{j_n} P_n X \xrightarrow{q_n} P_{n-1} X \xrightarrow{k_n} K(A, n+1) \tag{2}$$

where q_n is defined by the Postnikov tower of X, see Definition 2.5.6(3). It is clear that the fibre of q_n is an Eilenberg–Mac Lane space $K(A, n)$. We thus obtain the classifying map k_n in (2) which represents k_n in (1) by the *transgression element*

$$k_n = (q_n^*)^{-1} \partial \mu^{-1}(1_{\pi_n} X) \tag{3}$$

where $1_{\pi_n} X \in \operatorname{Hom}(A, A)$ is the identity of $A = \pi_n X$. Here we use the homomorphisms

$$\begin{tikzcd} \operatorname{Hom}(A, A) & H^n(K(A, n), A) \arrow[l, "\mu"', "\cong"] \arrow[d, "\partial"] & & H^{n+1}(P_{n-1} X, A) \\ & H^{n+1}(P_n X, K(A, n), A) & & H^{n+1}(P_{n-1} X, *, A) \arrow[ll, "q_n^*"', "\cong"] \arrow[u, "\cong"'] \end{tikzcd} \tag{4}$$

where ∂ is the boundary for the pair $P_n X$, $K(A, n)$ given by j_n in (2). The map q_n^*, induced by q_n in (2), is an isomorphism since X is simply connected. Compare (5.2.9) and (5.3.2) in Baues [OT]. Using the notation in Definition 2.5.7(6) we obtain by $k_n X$ the homomorphism

$$(k_n X)_* : \Gamma_n X \to \pi_n X \tag{5}$$

which, as we shall see in Theorem 2.5.10, coincides with the operator $i_n X$ in Whitehead's exact sequence. We thus get by Definition 2.5.7(6) the element

$$(k_n X)_\dagger \in \mathrm{Ext}(\Gamma''_{n-1}X, \mathrm{cok}\, i_n X) \tag{6}$$

where $\Gamma''_{n-1}X = \ker(i_{n-1}X)$. On the other hand, the exact sequence

$$\Gamma_n X \xrightarrow{i_n X} \pi_n X \to H_n X \to \Gamma_{n-1} X \xrightarrow{i_{n-1}X} \pi_{n-1} X$$

yields the short exact sequence of abelian groups

$$\mathrm{cok}(i_n X) \rightarrowtail H_n X \twoheadrightarrow \ker(i_{n-1} X) \tag{7}$$

which also represents an element

$$\{H_n X\} \in \mathrm{Ext}(\ker(i_{n-1}X), \mathrm{cok}(i_n X)). \tag{8}$$

Theorem 2.5.10 below shows that the elements in (6) and (8) coincide.

(2.5.9) Addendum Let X be a CW-complex with $\pi_1 X = 0$. Using the CW-structure we obtain the following alternative definition of the Postnikov invariant $k_n X$; compare VI.8.13 in Baues [AH]. We can choose $P_{n-1}X$ with $X^n = (P_{n-1}X)^n$ in such a way that the attaching map of $(n+1)$-cells in $P_{n-1}X$, given by a homomorphism

$$f^0_{n+1}: C_{n+1}(P_{n-1}X) \to \pi_n X^n$$

as in (2.1.19), is surjective. Then the composition $i_* f^0_{n+1}$,

$$C_{n+1}(P_{n-1}X) \twoheadrightarrow \pi_n X^n \twoheadrightarrow \pi_n X,$$

is a cocycle which represents the Postnikov invariant

$$k_n X = \{i_* f^0_{n+1}\} \in H^{n+1}(P_{n-1}X, \pi_n X).$$

Here i_* is induced by the inclusion $X^n \subset X$.

(2.5.10) Theorem on Postnikov invariants *With each* 1-*connected CW-complex X there is canonically associated a sequence of elements $(k_3, k_4, \dots)$ with*

$$k_n = k_n X \in \mathfrak{P}_{n-1}(X, \pi_n X)$$

such that the following properties are satisfied:

(*a*) *Naturality: for a map $F: X \to Y$ we have*

$$(\pi_n F)_*(k_n X) = F^*(k_n Y) \in \mathfrak{P}_{n-1}(X, \pi_n Y).$$

(*b*) *Compatibility with* $i_n X$:

$$(k_n X)_* = i_n X \in \mathrm{Hom}(\Gamma_n X, \pi_n X).$$

(*c*) *Compatibility with* $\{H_n X\}$:

$$(k_n X)_\dagger = \{H_n X\} \in \mathrm{Ext}(\ker i_{n-1} X, \operatorname{cok} i_n X).$$

(*d*) *Vanishing condition: all Postnikov invariants are trivial, that is* $k_n = 0$ *for* $n \geq 3$, *if and only if X has the homotopy type of a product of Eilenberg–Mac Lane spaces* $K(\pi_n, n)$, $n \geq 2$.

This theorem is the precise analogue of the corresponding theorem for boundary invariants; see Theorem 2.6.9 below.

Remark Postnikov invariants k_n (also called k-invariants) were invented by Postnikov in 1951. From a different point of view J.H.C. Whitehead [GD] studied them in the context of the certain exact sequence. Nowadays Postnikov invariants are discussed in many textbooks on algebraic topology and homotopy theory. While the naturality of the invariants is a classical result which appears in many textbooks I could not find the compatibility properties of Theorem 2.5.10(b), (c) in the literature. In fact, G.W. Whitehead [EH] in 1978 discussed Postnikov invariants and also the elements u_* and $u_\dagger$ associated with a cohomology class u, see (2.1.31); the compatibility properties directly related to these elements are not explained although the certain exact sequence is described. Compare also the remark following Theorem 4.4.4.

Proof of Theorem 2.5.10 The naturality is an easy consequence of the description of $k_n X$ as a transgression element; see Definition 2.5.8(3). The naturality is also proved in IX.2.6 of G.W. Whitehead [EH]. The vanishing condition (d) is well known; it is an easy consequence of the fibre sequence of Definition 2.5.8(2). We now prove the compatibility with $i_n X$. For this we use the explicit construction of $k_n X$ by the attaching map f_{n+1}^0 in Addendum 2.5.9. Recall that $Z_n X$ denotes the cycles in $C_n X$. We have the commutative diagram

(2.5.11)
$$\begin{tikzcd}
C_{n+1}P_{n-1}X \arrow[r, "f_{n+1}^0"] & \pi_n X^n \arrow[dr, "i_*"] & \\
Z_{n+1}P_{n-1}X \arrow[u, phantom, sloped, "\subset"] \arrow[r] \arrow[dr, "q"'] & \Gamma_n X \arrow[u, phantom, sloped, "\subset"] \arrow[r, "i_n X"] & \pi_n X \\
 & H_{n+1}P_{n-1}X \arrow[u, "\cong", "\theta"'] \arrow[ur, "\mu(k_n X)"'] &
\end{tikzcd}$$

Here q is the quotient map and $\mu(k_n X)$ is defined by the cocycle $i_* f_{n+1}^0$

(representing $k_n X$) as in (2.1.31)(1). This shows that the outside of the diagram commutes. On the other hand, the Hurewicz homorphism yields the composite

$$h': \pi_n X^n \to H_n X^n \subset C_n X^n = C_n P_{n-1} X$$

for which $h' f_{n+1}^0$ is the boundary in $C_* P_{n-1} X$. This shows that the inside of the diagram commutes. Hence we get the compatibility (b) which is equivalent to the equation

$$(i_n X)\Theta = \mu(k_n X). \tag{2.5.12}$$

For the intricate proof of the compatibility (c) we use the properties of the cellular boundary β described in (II.4.6) of Baues [CH]. For this we recall the following notation.

(2.5.13) Definition For a chain complex C we consider the commutative diagram

$$\begin{array}{ccccccc} \to & C_{n+1} & \to & C_n & \to & C_{n-1} & \to \\ \cdots & \downarrow & & \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_{n-1}} & \cdots \\ \to & 0 & \to & C_n/dC_{n+1} & \to & C_{n-1} & \to \end{array}$$

Here q_n is the quotient map and q_i is the identity for $i < n$. The bottom row is a chain complex $C^{(n)}$ which we call the *n-type* of C. Clearly $q: C \to C^{(n)}$ is a chain map which induces isomorphisms $q_*: H_i C \to H_i C^{(n)}$ for $i \leq n$. Moreover q is natural with respect to chain maps and chain homotopies.

The cellular map $p_{n-1}: X \to P_{n-1} X$ induces the chain map

$$\beta: C = C_* X \xrightarrow{C_* p_{n-1}} C_* P_{n-1} X \xrightarrow{q} (C_* P_{n-1} X)^{(n+1)} = P$$

where P is the $(n+1)$-type of $C_* P_{n-1} X$. Since p_{n-1} restricted to n-skeleta is the identity of X^n we see that also $\beta_i: C_i = P_i$ is the identity for $i \leq n$. Thus only $\beta_{n+1}: C_{n+1} \to P_{n+1}$ is relevant. The chain map β represents the *cellular boundary* invariant explained in more detail in (II.4.6) of Baues [CH]. In particular we prove in (II.4.12)(4) of Baues [CH].

(2.5.14) Lemma *For $i_* f_{n+1}^0$ in (2.5.11) there is a commutative diagram*

$$\begin{array}{ccc} C_{n+1} P_{n-1} X & \xrightarrow{i_* f_{n+1}^0} & \pi_n X \\ \downarrow{\scriptstyle q_{n+1}} & & \uparrow{\scriptstyle \Theta'}\,\cong \\ P_{n+1} & \xrightarrow{q} & \operatorname{cok} \beta_{n+1} \end{array}$$

where q_{n+1} and q are the quotient maps and where Θ' is an isomorphism.

Thus by Addendum 2.5.9 the composite $\Theta' q$ is a cocycle which represents $k_n X$; see also (II.4.14) in Baues [CH]. We now consider for the chain map β above the cofibre sequence of β in the cofibration category of chain complexes; see Baues [AH]. For this we choose first a factorization

$$\beta: C \rightarrowtail Z_\beta \xrightarrow{\sim} P$$

of β where $C \rightarrowtail Z_\beta$ is a cofibration so that Z_β and the quotient $D = Z_\beta / C$ are free. The short exact sequence $C \rightarrowtail Z_\beta \twoheadrightarrow D$ induces the long exact homology sequence in the top row of the following commutative diagram

(2.5.15)

$$\begin{tikzcd}
H_{n+1}C \arrow[r, "\beta_*"] \arrow[d, equal] & H_{n+1}P \arrow[r] \arrow[d, "\cong"', "\Theta"] & H_{n+1}D \arrow[r, "\partial"] \arrow[d, "\cong"', "\Theta'"] & H_nC \arrow[r, "\beta_*"] \arrow[d, equal] & H_nP \arrow[r] \arrow[d, "\Theta"] & 0 \\
H_{n+1}X \arrow[r, "b_{n+1}"'] & \Gamma_n X \arrow[r, "i_n"'] & \pi_n X \arrow[r, "h_n"'] & H_n X \arrow[r, two heads, "b_n"'] & \Gamma''_{n-1} X &
\end{tikzcd}$$

The bottom row is given by Whitehead's exact sequence and Θ' is induced by Θ' in Lemma 2.5.14. The isomorphisms Θ are described in Definition 2.5.7(4). Compare II.4.11 in Baues [CH] where the diagram is also obtained for the case that X is not simply connected.

We are going to apply Theorem 2.1.33 for the top row of (2.5.15); this yields the proof for the compatibility (c) of Theorem 2.5.10. For the quotient map $r: Z_\beta \to D$ we have the mapping cone C_r as defined in (2.1.32). Moreover by (II.8.24)(*) in Baues [AH] the diagram

$$\begin{tikzcd}
D \arrow[r, tail, "i"] \arrow[d, "-1"'] & C_r \arrow[r, two heads, "\pi"] \arrow[d, "\sim"] & sZ_\beta \arrow[d, "\sim"] \\
D \arrow[r, "\partial"] & sC \arrow[r, "s\beta"] & sP
\end{tikzcd}$$

commutes. Here the top row is defined as in (2.1.32)(1) and the bottom row is part of the cofibre sequence for β. Thus we obtain the commutative diagram

(2.5.16)
$$\begin{tikzcd}
H_{n+1}Z_\beta \arrow[r, "H_{n+1}r"] & H_{n+1}D \arrow[r, "i_*"] \arrow[d, "-1"', "\cong"] & H_{n+1}C_r \arrow[r, two heads, "\pi_*"] \arrow[d] & H_nZ_\beta \arrow[d] \\
& H_{n+1}D \arrow[r, "\partial"] & H_nC \arrow[r, two heads, "\beta_*"] & H_nP
\end{tikzcd}$$

Here the top row corresponds to (2.1.32)(2) and the bottom row is given by the top row of (2.5.15).

(2.5.17) Lemma *There exists a unitary class u in $H^{n+1}(D, H_{n+1}D)$ such that the composite*

$$H^{n+1}(D, H_{n+1}D) \xrightarrow{r^*} H^{n+1}(Z_\beta, H_{n+1}D) = H^{n+1}(P_{n-1}X, H_{n+1}D)$$

satisfies $\Theta'_*(r^*u) = k_nX \in H^{n+1}(P_{n-1}X, \pi_nX)$.

For the proof of the lemma we observe that we can choose Z_β such that $(Z_\beta)^{(n+1)} = P^{(n+1)}$ and $D_{n+1}^{(n+1)} = \operatorname{cok}\beta_{n+1}$, so that u is represented by $D_{n+1} \twoheadrightarrow D_{n+1}^{(n+1)} = \operatorname{cok}\beta_{n+1} \cong H_{n+1}D$. Finally we are ready to apply Theorem 2.1.33 which shows

$$(2.5.18) \qquad -(r^*u)_\dagger = \{H_{n+1}C_r\} \in \operatorname{Ext}(H_nZ_\beta, \operatorname{cok} H_{n+1}r).$$

Hence we get, by Lemma 2.5.17, Definition 2.5.13, and (2.5.15) the equivalent equation

$$(2.5.19) \qquad (k_nX)_\dagger = \{H_nX\} \in \operatorname{Ext}(\Gamma''_{n-1}X, \operatorname{cok} i_nX).$$

Hence the proof of Theorem 2.5.10 is complete. □

We derive from Theorem 2.5.10 on Postnikov invariants the following criterion for products of Eilenberg–Mac Lane spaces.

(2.5.20) Proposition *A simply connected space X is homotopy equivalent to a product of Eilenberg–Mac Lane spaces if and only if the Hurewicz homomorphism*

$$h: \pi_nX \to H_nX$$

is split injective for all n.

The proposition has a nice dual; see Proposition 2.6.15 below.

Proof Since h is injective we see that $i_nX: \Gamma_nX \to \pi_nX$ is trivial, $i_nX = 0$, for all n. Hence by Theorem 2.5.10(b), (c) we see that $k_nX = \Delta\{H_nX\}$ with $\{H_nX\} \in \operatorname{Ext}(\Gamma_{n-1}X, \pi_nX)$ given by the extension

$$\pi_nX \overset{h}{\rightarrowtail} H_nX \twoheadrightarrow \Gamma_{n-1}X.$$

Since h is split injective we get $\{H_nX\} = 0$ and hence $k_nX = 0$ for all n. Hence the Proposition is a consequence of Theorem 2.5.10(d) □

Remark There is a simple direct proof of Proposition 2.5.20. Using a retraction $r_n: H_n X \to \pi_n X$ of the Hurewicz homomorphism we can choose an element

$$\begin{cases} r_n' \in H^n(X, \pi_n X) = [X, K(\pi_n X, n)] \quad \text{with} \\ \mu(r_n') = r_n. \end{cases}$$

The collection of the elements r_n' yields a map from X into a product of Eilenberg–Mac Lane spaces which, by the Whitehead theorem, can be seen to be a homotopy equivalence.

(2.5.21) Example There exists a space Y for which the Hurewicz homomorphism $h_n Y$ is injective for all n but not split injective. In fact, such a space is the $(n+2)$-type $Y = P_{n+2}X(2^s\eta)$ where $s \geq 2$ is a power of 2. Here $X(2^s\eta)$ is a space in the list in Definition 12.5.3 with homotopy groups $\pi_n Y = \mathbb{Z}$, $\pi_{n+1}Y = 0 = H_{n+1}Y$, and $\pi_{n+2}Y = \mathbb{Z}/s$. The Hurewicz homomorphism satisfies $h_{n+2}Y: \mathbb{Z}/s \rightarrowtail \mathbb{Z}/2s$. This example shows that the condition 'split injective' in Proposition 2.5.20 cannot be replaced by the weaker condition 'injective'.

2.6 Boundary invariants of a homotopy type

In this section we introduce fundamental new homotopy invariants of a simply connected CW-space which we call boundary invariants. There are two possible ways of defining these invariants: on the one hand, we obtain them by use of the boundary operator in the Γ-sequence with coefficients; on the other hand, they can be defined via the Postnikov tower and pseudo-homology groups.

We first consider the boundary operator in the Γ-sequence with coefficients. Let A be an abelian group and let X be a simply connected CW-space. Using the boundary operator b^A in (2.3.2) one gets the following commutative diagram with short exact rows.

$$(2.6.1)\quad \begin{array}{ccccc} \mathrm{Ext}(A, H_{n+1}X) & \overset{\Delta}{\rightarrowtail} & H_n(A, X) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A, H_n X) \\ \big\downarrow{\scriptstyle (b_{n+1}X)_*} & & \big\downarrow{\scriptstyle b^A} & & \big\downarrow{\scriptstyle (b_n X)_*} \\ \mathrm{Ext}(A, \Gamma_n X) & \overset{\Delta}{\rightarrowtail} & \Gamma_{n-1}(A, X) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A, \Gamma_{n-1}X) \\ \big\downarrow{\scriptstyle i_*} & \text{push} & \big\downarrow{\scriptstyle i_*} & & \| \\ \mathrm{Ext}(A, \mathrm{cok}\, b_{n+1}X) & \overset{\Delta}{\rightarrowtail} & \dfrac{\Gamma_{n-1}(A, X)}{\Delta \mathrm{im}(b_{n+1}X)_*} & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A, \Gamma_{n-1}X) \end{array}$$

The left-hand column is exact. Moreover $\mathrm{im}(b_{n+1}X)_*$ is the image of the homomorphism $(b_{n+1}X)_*$ in the diagram and $i_\#$ is the quotient map; equivalently $i_\#$ is the projection for the cokernel of $\Delta(b_{n+1}X)_*$. On the other hand $i: \Gamma_n X \to \operatorname{cok} b_{n+1}X$ is the projection for the cokernel of $b_{n+1}X: H_{n+1}X \to \Gamma_n X$. Hence $i_\#$ and i_* in the diagram are surjective and one readily checks that the subdiagram 'push' is a push-out diagram of abelian groups. Using Definition 2.2.9 one gets the binatural inclusions

$$\frac{\Gamma''_{n-1}(A,X)}{\Delta\mathrm{im}(b_{n+1}X)_*} \subset \frac{\Gamma'_{n-1}(A,X)}{\Delta\mathrm{im}(b_{n+1}X)_*} \subset \frac{\Gamma_{n-1}(A,X)}{\Delta\mathrm{im}(b_{n+1}X)_*}. \tag{2.6.2}$$

Here Γ''_{n-1} and Γ'_{n-1} are actually functorial in $A \in \mathbf{Ab}$ while Γ_{n-1} is not. Let $\mathbf{spaces}_2$ be the homotopy category of simply connected CW-spaces. Then the left-hand group in (2.6.2), which we denote by

$$\mathfrak{Q}_{n-1}(A,X) = \frac{\Gamma''_{n-1}(A,X)}{\Delta\mathrm{im}(b_{n+1}X)_*}, \tag{2.6.3}$$

yields a bifunctor

$$\mathfrak{Q}_{n-1}: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{spaces}_2 \to \mathbf{Ab}.$$

Moreover we have a binatural short exact sequence

$$\mathrm{Ext}(A, \operatorname{cok} b_{n+1}X) \overset{\Delta}{\rightarrowtail} \mathfrak{Q}_{n-1}(A,X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A, \Gamma''_{n-1}X)$$

where $\Gamma''_{n-1}X = \mathrm{image}(b_n X: H_n X \to \Gamma_{n-1}X)$.

(2.6.4) Definition We define the *boundary invariant* $\beta_n X$, $n \geq 3$, of a simply connected CW-space X as follows. Consider diagram (2.6.1) where we set $A = H_n X$. Then we get

$$\beta_n X \in \mathfrak{Q}_{n-1}(H_n X, X) \subset \frac{\Gamma_{n-1}(H_n X, X)}{\Delta\mathrm{im}(b_{n+1}X)_*}$$

by

$$\beta_n X = i_\# b^A \mu^{-1}(1_{H_n X}).$$

Here $1_{H_n X} \in \mathrm{Hom}(A, H_n X)$ is the identity of $H_n X$. Since $\mathrm{image}(b_{n+1}X)_* = \mathrm{kernel}(i_*)$ we see that the element $\beta_n X = i_\# b^A(a)$ does not depend on the choice of $a \in H_n(A,X)$ with $\mu(a) = 1_{H_n X}$. The commutativity of the diagram implies the equation

$$(\beta_n X)_* = \mu(\beta_n X) = b_n X.$$

This shows that $\beta_n X$ is actually an element of the subgroup $\mathfrak{Q}_{n-1}(H_n X, X)$; compare the definition of Γ''_{n-1} in Definition 2.2.9. We have chosen the name 'boundary invariant' for $\beta_n X$ since, on the one hand, the equation $\mu(\beta_n X) = b_n X$ shows that $\beta_n X$ is a refinement of the secondary boundary operator $b_n X$ of J.H.C. Whitehead. On the other hand the boundary operator b^A in the Γ-sequence with coefficients is the crucial ingredient in the definition of $\beta_n X$.

(2.6.5) Addendum We can define the boundary invariant $\beta_n X$ by use of the element $\{\beta\} \in \Gamma_{n-1}(H_n X, X)$ with $\mu\{\beta\} = b_n X$ in Definition 2.3.5(16), namely

$$\beta_n X = i_{\#}\{\beta\}$$

where $i_{\#}$ is the quotient map in (2.6.1). This follows readily from the definition of the boundary operator b^A in terms of $\{\beta\}$ in Definition 2.3.5(20). The formula $\beta_n X = i_{\#}\{\beta\}$ shows the direct connection of $\beta_n X$ with the attaching maps in the space X; this connection will be heavily used in proofs below.

Given an element $\beta \in \mathfrak{Q}_{n-1}(A, X)$ we obtain elements

$$(2.6.6) \qquad \begin{cases} \beta_* = \mu(\beta) \in \operatorname{Hom}(A, \Gamma''_{n-1} X) \\ \beta_\dagger = \Delta^{-1} j^*(\beta) \in \operatorname{Ext}(\ker(\beta_*), \operatorname{cok}(b_{n+1} X)). \end{cases}$$

Here Δ and μ are given by the binatural short exact sequence in (2.6.3). Moreover $j\colon \ker(\beta_*) \subset A$ is the inclusion and $j^*\colon \mathfrak{Q}_{n-1}(A, X) \to \mathfrak{Q}_{n-1}(\ker(\beta_*), X)$ is induced by the bifunctor $\mathfrak{Q}_{n-1}$. As in (2.1.31)(4) one readily checks that the elements in (2.6.6) are well defined. As an example, we obtain, for $\beta = \beta_n X$, the element

$$(\beta_n X)_* = \mu(\beta_n X) = b_n X$$

where $b_n X\colon H_n X \twoheadrightarrow \Gamma''_{n-1} X$ is the surjective homomorphism given by $b_n X\colon H_n X \to \Gamma_{n-1} X$. Moreover we get by (2.6.6) the element

$$(2.6.7) \qquad (\beta_n X)_\dagger \in \operatorname{Ext}(\ker(b_n X), \operatorname{cok}(b_{n+1} X))$$

which can be compared with the element

$$(2.6.8) \qquad \{\pi_n X\} \in \operatorname{Ext}(\ker(b_n X), \operatorname{cok}(b_{n+1} X)).$$

Here $\{\pi_n X\}$ is given by the extension

$$\operatorname{cok}(b_{n+1} X) \rightarrowtail \pi_n X \twoheadrightarrow \ker(b_n X)$$

obtained by the exact sequence

$$H_{n+1}X \xrightarrow{b_{n+1}X} \Gamma_n X \to \pi_n X \to H_n X \xrightarrow{b_n X} \Gamma_{n-1}X.$$

The next result shows that the elements $(\beta_n X)_\dagger$ and $\{\pi_n X\}$ actually coincide. In fact the next result is the precise analogue of Theorem 2.5.10 on Postnikov invariants. The similarity of Theorem 2.5.10 and the following Theorem 2.6.9 emphasizes the duality between k-invariants and boundary invariants.

(2.6.9) Theorem on boundary invariants *With each* 1*-connected CW-complex* X *there is canonically associated a sequence of elements* $(\beta_3, \beta_4, \ldots)$ *with*

$$\beta_n = \beta_n X \in \mathfrak{Q}_{n-1}(H_n X, X)$$

such that the following properties are satisfied:

(*a*) *Naturality: for a map* $F\colon X \to Y$ *we have*

$$F_*(\beta_n X) = (H_n F)^*(\beta_n Y) \in \mathfrak{Q}_{n-1}(H_n X, Y).$$

(*b*) *Compatibility with* $b_n X$:

$$(\beta_n X)_* = b_n X \in \mathrm{Hom}(H_n X, \Gamma''_{n-1} X).$$

(*c*) *Compatibility with* $\{\pi_n X\}$:

$$(\beta_n X)_\dagger = \{\pi_n X\} \in \mathrm{Ext}(\ker(b_n X), \mathrm{cok}(b_{n+1} X)).$$

(*d*) *Vanishing condition: all boundary invariants are trivial, that is* $\beta_n = 0$ *for* $n \geq 3$, *if and only if* X *has the homotopy type of a one-point union of Moore spaces* $M(H_n, n)$, $n \geq 2$, *with* $H_n = H_n(X)$.

Proof The naturality is an easy consequence of the naturality of the operators $i_\#$ and b_n and μ. In fact

$$\begin{aligned} F_* \beta_n X &= F_* i_\# b_n \mu^{-1}(1_{H_n X}) \\ &= i_\# b_n \mu^{-1}\big((H_n F)_* 1_{H_n X}\big) \\ &= i_\# b_n \mu^{-1}\big((H_n F)^* 1_{H_n Y}\big) \\ &= (H_n F)^* i_\# b_n \mu^{-1}(1_{H_n Y}) \\ &= (H_n F)^* \beta_n Y. \end{aligned}$$

Compatibility with b_nX was already obtained in Definition 2.6.4. Finally we prove compatibility with $\{\pi_nX\}$ by the definition of β_nX in Addendum 2.6.5 above. We use the notation in Definition 2.3.5. We have the commutative diagram

$$\begin{tikzcd}
B_n \arrow[r, tail, "d"] \arrow[d, equal] & Z_n \arrow[r, two heads, "p"] & H_n \\
B_n \arrow[r, tail, "\partial"] & \ker(b_np) \arrow[r, two heads] \arrow[u, hook, "j"'] \arrow[d, equal] & \ker b_n \arrow[u, hook, "i"'] \arrow[d, equal] \\
& L & K
\end{tikzcd}$$

with exact rows. The inclusion i yields a map $\bar{i}$ for which the following diagram homotopy commutes

$$\begin{tikzcd}
M(B_n,n) \arrow[drr, "\pi"] & & \\
M(K,n-1) \arrow[u, "q"] \arrow[r, "\bar{i}"] & M(H_n,n-1) \arrow[r, "\beta"] & X^{n-1} \\
M(L,n-1) \arrow[u, hook] \arrow[r, "i"] & M(Z_n,n-1) \arrow[u, hook] \arrow[r, "b"] & X^{n-2} \arrow[u, hook]
\end{tikzcd}$$

Here q is the quotient map for $M(K,n-1)=C_\partial$. Since by definition of b in Definition 2.3.5(6) we know $\bar{i}bj\simeq 0$ there is a map π for which the composition

$$\bar{\pi}: B_n \xrightarrow{\pi} \pi_nX^{n-1} \overset{p}{\twoheadrightarrow} \Gamma_n \to \operatorname{cok} b_{n+1}$$

represents

$$\{\bar{\pi}\} = \Delta^{-1}\bar{i}^*(\beta_{n+1}X) \in \Delta^{-1}i^*(\beta_{n+1}X). \tag{1}$$

We show that $\bar{\pi}$ also represents the extension class $\{\pi_n\}$. Let $\bar{X}$ be the mapping cone of the restriction $v|_{M(Z_{n+1},n)\vee M(\iota B_{n-1},n-1)}$, see Definition 2.3.5(9) and (7). Then β in Definition 2.3.5(15) yields the map

$$\beta: M(H_n,n-1) \to X^{n-1} \subset \bar{X}$$

with $C_\beta = X^{n+1}$. Therefore we have the following diagram of cofibre sequences

$$\begin{tikzcd}
M(K,n-1) \arrow[r, "q"] \arrow[d, "\bar{i}"] & M(B_n,n) \arrow[r, "\partial"] \arrow[d, "\pi"] & M(L,n) \arrow[r] \arrow[d] & M(K,n) \arrow[d, "\Sigma\bar{i}"] \\
M(H_n,n-1) \arrow[r, "\beta"'] & \bar{X} \arrow[r] & C_\beta \arrow[r, "q"'] & M(H_n,n)
\end{tikzcd}$$

We apply the functor π_n to this diagram and we get the commutative diagram

$$\begin{array}{ccccc} B_n & \rightarrowtail & L & \twoheadrightarrow & K \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\ \pi_n \bar{X} & \xrightarrow[j]{} & \pi_n & \xrightarrow[h]{} & H_n \end{array} \tag{2}$$

Here $\pi_n C_\beta = \pi_n X^{n+1} = \pi_n$ is the homotopy group of X and $h = \pi_n(q)$ is the Hurewicz map. Therefore we have the group

$$h(\pi_n) = \ker b_n = K.$$

Since the top row of (2) is a free presentation of this group the map $j\pi\colon B_n \to \ker h$ represents $\{\pi_n\}$, see Lemma 2.1.30. In fact, $j\pi$ is the same as $\bar{\pi}$ in (1). This completes the proof of (c). Finally the definition of the boundary invariants shows that all $\beta_n(X)$ are trivial if X is a one-point union of Moore spaces; here we use the definition in Addendum 2.6.5 and Definition 2.3.5. On the other hand, the classification theorem in Chapter 3 below yields the inverse; namely the vanishing of all $\beta_n X$ implies that X has the homotopy type of a one-point union of Moore spaces. This completes the proof of Theorem 2.6.9. □

In our second definition of the boundary invariants we use the $(n-1)$-type

(2.6.10) $$p_{n-1}^X\colon X \to P_{n-1}X$$

of the simply connected space X; see Definition 2.5.6. Using pseudo-homology groups with coefficients in A we obtain the following diagram which resembles diagram (2.6.1); in fact we show below that the following diagram is naturally embedded in diagram (2.6.1).

(2.6.11)

$$\begin{array}{ccccc} \operatorname{Ext}(A, H_{n+1}X) & \overset{\Delta}{\rightarrowtail} & H_n(A,X) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(A, H_n X) \\ \downarrow{\scriptstyle (H_{n+1}p_{n-1}^X)_*} & & \downarrow{\scriptstyle (p_{n-1}^X)_*} & & \downarrow{\scriptstyle (H_n p_{n-1}^X)_*} \\ \operatorname{Ext}(A, H_{n+1}P_{n-1}X) & \overset{\Delta}{\rightarrowtail} & H_n(A, P_{n-1}X) & \xrightarrow{\mu} & \operatorname{Hom}(A, H_n P_{n-1}X) \\ \downarrow{\scriptstyle i_*} & \text{push} & \downarrow{\scriptstyle i_*} & & \| \\ \operatorname{Ext}(A, \operatorname{cok} H_{n+1}p_{n-1}^X) & \overset{\Delta}{\rightarrowtail} & \dfrac{H_n(A, P_{n-1}X)}{\Delta \operatorname{im}(H_{n+1}p_{n-1}^X)_*} & \xrightarrow{\mu} & \operatorname{Hom}(A, H_n P_{n-1}X) \end{array}$$

The rows are short exact and the left-hand column is exact. Moreover $\mathrm{im}(H_{n+1}p_{n-1}^X)_*$ is the image of the homomorphism $(H_{n+1}p_{n-1}^X)_*$ in the diagram and $i_\#$ is the quotient map; equivalently $i_\#$ is the projection for the cokernel of the composition $\Delta(H_{n+1}p_{n-1}^X)_*$. On the other hand, $i: H_{n+1}P_{n-1}X \to \mathrm{cok}\, H_{n+1}p_{n-1}^X$ is the projection for the cokernel of $H_{n+1}p_{n-1}^X: H_{n+1}X \to H_{n+1}P_{n-1}X$. Hence i_* and $i_\#$ in the diagram are surjective and hence the subdiagram 'push' in (2.6.11) is a push-out diagram of abelian groups. The naturality of the $(n-1)$-type (2.6.10) shows that the group

$$\mathfrak{Q}'_{n-1}(A, X) = \frac{H_n(A, P_{n-1}X)}{\Delta \mathrm{im}(H_{n+1}p_{n-1}^X)_*} \tag{2.6.12}$$

yields a functor

$$\mathfrak{Q}'_{n-1}: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{spaces}_2 \to \mathbf{Ab}$$

together with a binatural short exact sequence given by the bottom row of (2.6.11).

(2.6.13) Definition We define the *(homological) boundary invariant* $\beta'_n X$, $n \geq 3$, of a simply connected CW-space X as follows. Consider diagram (2.6.11) where we set $A = H_n X$. Then we get

$$\beta'_n X \in \mathfrak{Q}'_{n-1}(A, X) = \frac{H_n(A, P_{n-1}X)}{\Delta \mathrm{im}(H_{n+1}p_{n-1}^X)_*}$$

$$\beta'_n X = i_\#(p_{n-1}^X)_* \mu^{-1}(1_{H_n X}).$$

Here $1_{H_n X} \in \mathrm{Hom}(A, H_n X)$ is the identity of $H_n X$. As in Definition 2.6.4 we see that $\beta'_n X$ is a well-defined element. The naturality of the diagram (2.6.11) immediately yields the naturality of the invariant $\beta'_n X$, that is, for $f: X \to Y$ in $\mathbf{spaces}_2$ we have $f_*(\beta'_n X) = f^*(\beta'_n Y)$. Hence $\beta'_n X$ is a well-defined invariant of the homotopy type of X. The next result shows that we can identify the homological boundary invariant $\beta'_n X$ with the boundary invariant $\beta_n X$ in Definition 2.6.4.

(2.6.14) Theorem *For a simply connected CW-space X and an abelian group A there is a binatural isomorphism*

$$\Theta: \mathfrak{Q}'_{n-1}(A, X) \cong \mathfrak{Q}_{n-1}(A, X)$$

compatible with Δ and μ. Moreover for $A = H_n X$ the isomorphism Θ carries the homological boundary invariant $\beta'_n X$ to the boundary invariant $\beta_n X$, that is $\Theta(\beta'_n X) = \beta_n X$.

Proof By Definition 2.5.7 there are natural isomorphisms

$$\Theta: H_n P_{n-1} X \cong \Gamma''_{n-1} X \tag{1}$$

$$\Theta: H_{n+1} P_{n-1} X \cong \Gamma_n X \tag{2}$$

such that the diagram

$$\begin{tikzcd} H_{n+1}X \arrow[r, "H_{n+1}p^X_{n-1}"] \arrow[d, equal] & H_{n+1}P_{n-1}X \arrow[d, equal, "\Theta"] \\ H_{n+1}X \arrow[r, "b_{n+1}X"] & \Gamma_n X \end{tikzcd} \tag{3}$$

commutes. Hence $(b_{n+1}X)_*$ in (2.6.1) corresponds to $(H_{n+1}p^X_{n-1})_*$ in (2.6.11). We now show that in addition there is a binatural isomorphism

$$\Theta: H_n(A, P_{n-1}X) \cong \Gamma''_{n-1}(A, X) \tag{4}$$

such that the following diagram with short exact rows commutes

$$\begin{tikzcd} \mathrm{Ext}(A, H_{n+1}P_{n-1}X) \arrow[r, tail, "\Delta"] \arrow[d, equal, "\Theta_*"] & H_n(A, P_{n-1}X) \arrow[r, two heads, "\mu"] \arrow[d, equal, "\Theta"] & \mathrm{Hom}(A, H_n P_{n-1}X) \arrow[d, equal, "\Theta_*"] \\ \mathrm{Ext}(A, \Gamma_n X) \arrow[r, tail, "\Delta"] & \Gamma''_{n-1}(A, X) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, \Gamma''_{n-1}X) \end{tikzcd} \tag{5}$$

Moreover Θ in (4) makes the diagram

$$\begin{tikzcd} H_n(A, X) \arrow[r, "(p^X_{n-1})_*"] \arrow[d, "b^A"] & H_n(A, P_{n-1}X) \arrow[d, equal, "\Theta"] \\ \Gamma_{n-1}(A, X) \arrow[r, phantom, "\supset"] & \Gamma''_{n-1}(A, X) \end{tikzcd} \tag{6}$$

commutative, where b^A is the boundary operator in (2.6.1). Hence (3), (5), and (6) show that Θ in (4) induces an isomorphism as in the Theorem for which $\Theta\beta'_n X = \beta_n X$, this is readily seen by comparing diagram (2.6.1) with diagram (2.6.11). We obtain Θ in (4) as follows.

Using the boundary operator b^A in (2.3.2) and Definition 2.3.5 for the space $Y = P_{n-1}X$ we obtain the commutative diagram with short exact rows:

$$\begin{tikzcd} \mathrm{Ext}(A, H_{n+1}Y) \arrow[r, tail] \arrow[d, "(b_{n+1}Y)_*", "\cong"'] & H_n(A, Y) \arrow[r, two heads] \arrow[d, "b^A"] \arrow[dr, phantom, "\text{pull}"] & \mathrm{Hom}(A, H_n Y) \arrow[d, "(b_n Y)_*"] \\ \mathrm{Ext}(A, \Gamma_n Y) \arrow[r, tail] & \Gamma_{n-1}(A, Y) \arrow[r, two heads] & \mathrm{Hom}(A, \Gamma_{n-2} Y) \\ \mathrm{Ext}(A, \Gamma_n X) \arrow[r, tail] \arrow[u, "(\Gamma_n p)_*", "\cong"'] & \Gamma_{n-1}(A, X) \arrow[r] \arrow[u, "p_*", "\cong"'] & \mathrm{Hom}(A, \Gamma_{n-1} X) \arrow[u, "(\Gamma_{n-1} p)_*", "\cong"'] \end{tikzcd}$$

Here $p = p^X_{n-1}: X \to Y$ is the $(n-1)$-type of X in (2.6.10). We know that p induces isomorphisms $\Gamma_n p$, p_* and $\Gamma_{n-1} p$ as shown in the diagram. Also b^A in the diagram is injective by (2.3.2) since $\pi_n(A, Y) = 0$. Moreover $b_{n+1}Y$ is an isomorphism and $b_n Y$ is injective by Definition 2.5.7(3). This shows that the top part of the diagram is a pull-back diagram. Hence we obtain the isomorphism Θ in (4) by the composite $(p_*)^{-1} b^A$. In fact, by (1) the injection $b_n Y$ above corresponds to the inclusion $\Gamma''_{n-1} X \subset \Gamma_{n-1} X$ since the diagram

$$\begin{tikzcd} H_n Y \arrow[r, tail, "b_n Y"] \arrow[d, equal, "\Theta"'] & \Gamma_{n-1} Y \arrow[d, equal, "\Gamma_{n-1} p"] \\ \Gamma''_{n-1} X \arrow[r, phantom, "\subset"] & \Gamma_{n-1} X \end{tikzcd}$$

commutes; compare the definition of Γ''_{n-1} in Definition 2.2.9. The definition of Θ by $(p_*)^{-1} b_A$ shows that diagram (5) and diagram (6) commute. □

Finally we derive from the theorem on boundary invariants the following criterion for one-point unions of Moore spaces.

(2.6.15) Proposition *A simply connected space X is homotopy equivalent to a one-point union of Moore spaces if and only if the Hurewicz homomorphism*

$$h: \pi_n X \to H_n X$$

is split surjective for all n.

This is the precise dual of the criterion for products of Eilenberg–Mac Lane spaces in Proposition 2.5.20.

Proof Since h is surjective we see that $b_n X: H_n X \to \Gamma_{n-1} X$ is trivial, $b_n X = 0$, for all n. Hence by Theorem 2.6.9(b), (c) we get $\beta_n X = \Delta\{\pi_n X\}$ with $\{\pi_n X\} \in \operatorname{Ext}(H_n X, \Gamma_n X)$ given by the extension

$$\Gamma_n X \rightarrowtail \pi_n X \overset{h}{\twoheadrightarrow} H_n X.$$

Since h is split surjective we get $\{\pi_n X\} = 0$ and hence $\beta_n X = 0$ for all n. Hence the proposition follows from Theorem 2.6.9(d). □

Remark There is a simple direct proof of Proposition 2.6.15. Using a splitting $s_n: H_n X \to \pi_n X$ of the Hurewicz homomorphism we can choose an element

$$\begin{cases} s'_n \in \pi_n(H_n X, X) = [M(H_n X, n), X] \\ \text{with} \quad \mu(s'_n) = s_n. \end{cases}$$

The collection of these elements s'_n yields a map from a one-point union of Moore spaces to X which, by the Whitehead theorem, can be seen to be a homotopy equivalence.

(2.6.16) Example There exists a space X for which the Hurewicz homomorphism $h_n X$ is surjective for all n but not split surjective. In fact, such a space is given by $X = \Sigma^{n-1}\mathbb{R}P_4$, $n \geq 4$, as we show in (8.1.11). Hence the condition 'split' in Proposition 2.6.15 is necessary.

2.7 Homotopy decomposition and homology decomposition

In this section we describe both the 'homotopy decomposition' and the 'homology decomposition' of a simply connected CW-space. These concepts are Eckmann–Hilton dual to each other. The homotopy decomposition is also called the Postnikov decomposition or Postnikov tower associated with a space X. The homology decomposition was obtained by Eckmann and Hilton, and Brown and Copeland.

The 'homotopy decomposition' describes a construction of spaces using Eilenberg–Mac Lane spaces $K(\pi, n)$ as building blocks. This is explained more precisely in the following definition and theorem.

(2.7.1) Definition A *homotopy decomposition* (1-connected) is a system of the form

$$(\pi_2, \pi_3, \ldots; k_3, k_4, \ldots; X_2, X_3, \ldots).$$

Here $\pi_2, \pi_3, \ldots$ is a sequence of abelian groups and $X_2, X_3, \ldots$ is a sequence of CW-spaces which fit into the following diagram ($n \geq 3$).

$$\begin{array}{ccccc}
 & & \vdots & & \\
K(\pi_n, n) & \to & X_n & \xrightarrow{k_{n+1}} & K(\pi_{n+1}, n+2) \\
 & & \downarrow & & \\
K(\pi_{n-1}, n-1) & \to & X_{n-1} & \xrightarrow{k_n} & K(\pi_n, n+1) \\
 & & \downarrow & & \\
 & & \vdots & & \\
K(\pi_3, 3) & \to & X_3 & \xrightarrow{k_4} & K(\pi_4, 5) \\
 & & \downarrow & & \\
K(\pi_2, 2) & = & X_2 & \xrightarrow{k_3} & K(\pi_3, 4)
\end{array}$$

Each vertical arrow is a fibration in **Top** such that, for $n \geq 3$, the sequences

$$K(\pi_n, n) \xrightarrow{i_n} X_n \xrightarrow{q_n} X_{n-1} \xrightarrow{k_n} K(\pi_n, n+1) \quad (2)$$

are fibre sequences. That is, q_n is a principal fibration with classifying map k_n and fibre $K(\pi_n, n) = \Omega K(\pi_n, n+1)$. The classifying map k_n represents the cohomology class

$$k_n \in H^{n+1}(X_{n-1}, \pi_n) \tag{3}$$

which, for $n \geq 3$, yields the sequence of elements $k_3, k_4, \ldots$ in (1). The fibre sequences (2) imply that $\pi_j X_n = 0$ for $j > n$ so that X_n is a simply connected n-type. Hence the map $k_n: X_{n-1} \to K(\pi_n, n+1)$ is π_**-trivial*, that is, the map k_n induces the trivial homomorphism on homotopy groups. Let $\varprojlim(X_n)$ be the *inverse limit* in **Top** of the sequence of fibrations $X_2 \twoheadleftarrow X_3 \twoheadleftarrow \cdots$.

(2.7.2) Theorem *For each simply connected CW-space X there exists a homotopy decomposition, together with a map $f: X \to \varprojlim(X_n)$, which induces isomorphisms of homotopy groups $f_*: \pi_n X \cong \pi_n$.*

Compare, for example, G.W. Whitehead [EH]. The theorem shows that the homotopy type of X is determined by the homotopy decomposition of X which essentially is unique. Moreover an important feature of the homotopy decomposition is its *naturality* with respect to maps $X \to X'$; see G.W. Whitehead [EH] or Baues [OT]. In particular the homotopy type of X_n in Theorem 2.7.2 is an invariant of the homotopy type of X.

Proof of Theorem 2.7.2 We replace inductively the maps $q_n: P_n X \to P_{n-1} X$ in Definition 2.5.6 by fibrations so that we get a sequence of fibrations as in Definition 2.7.1 together with commutative diagrams

$$\begin{tikzcd} P_n X \arrow[r, "q_n"] \arrow[d, "\simeq"'] & P_{n-1} X \arrow[d, "\simeq"] \\ X_n \arrow[r, two heads, "q_n"] & X_{n-1} \end{tikzcd}$$

in which the vertical arrows are homotopy equivalences. Moreover we can choose inductively maps $f_n: X \to X_n$ with $q_n f_n = f_{n-1}$ such that $X \to P_n X \simeq X_n$ given by p_n in Definition 2.5.6 is homotopic to f_n. Then (f_n) induces the weak equivalence f in the statement of the theorem. The naturality of the homotopy decomposition can be derived from the functorial properties of the Postnikov section $X \to P_n X$ in Definition 2.5.6 and the naturality of the Postnikov invariants in Theorem 2.5.10. □

Next we explain the concept of a homology decomposition which uses Moore spaces $M(H_n, n)$ as building blocks.

(2.7.3) Definition A *homology decomposition* (1-connected) is a system of the form

$$(H_2, H_3, \ldots; k'_3, k'_4, \ldots; X_2, X_3, \ldots). \tag{1}$$

Here $H_2, H_3, \ldots$ is a sequence of abelian groups and $X_2, X_3, \ldots$ is a sequence of CW-spaces which fit into the following diagram ($n \geq 3$).

$$\begin{tikzcd}
 & \vdots & \\
M(H_n, n) & X_n \arrow[l] & M(H_{n+1}, n) \arrow[l, "k'_{n+1}"'] \\
M(H_{n-1}, n-1) & X_{n-1} \arrow[l] \arrow[u, tail] & M(H_n, n-1) \arrow[l, "k'_n"'] \\
 & \vdots \arrow[u, tail] & \\
M(H_3, 3) & X_3 \arrow[l] & M(H_4, 3) \arrow[l, "k'_4"'] \\
M(H_2, 2) \arrow[r, equal] & X_2 \arrow[u, tail] & M(H_3, 2) \arrow[l, "k'_3"']
\end{tikzcd}$$

Each vertical arrow is a cofibration in **Top** such that, for $n \geq 3$, the sequences

$$M(H_n, n) \xleftarrow{q_n} X_n \xleftarrow{i_n} X_{n-1} \xleftarrow{k'_n} M(H_n, n-1) \tag{2}$$

are cofibre sequences. That is, i_n is a principal cofibration with attaching map k'_n and cofibre $M(H_n, n) = \Sigma M(H_n, n-1)$; equivalently X_n is the mapping cone of the map k'_n. This map represents the homotopy class

$$k'_n \in \pi_{n-1}(H_n, X_{n-1}) \tag{3}$$

which, for $n \geq 3$, yields the sequence of elements $k'_3, k'_4, \ldots$ in (1). We require that these elements are *H_*-trivial*, that is, the map k'_n induces the trivial homomorphism on homology groups. Let $\varinjlim(X_n)$ be the *direct limit* in **Top** of the sequence of cofibrations $X_2 \rightarrowtail X_3 \rightarrowtail \cdots$.

(2.7.4) Theorem *For each simply connected CW-space X there exists a homology decomposition, together with a map $f\colon \varinjlim(X_n) \to X$, which induces isomorphisms of homology groups $f_*: H_n \cong H_n X$.*

Since the direct limit in Theorem 2.7.4 is a CW-space the map f is actually a homotopy equivalence. Thus we may construct any 1-connected homotopy type with homology groups $H_2, H_3, \ldots$ by a process of successively attaching cones $CM(H_n, n-1)$ via homologically trivial maps. Conversely, any such construction produces a homotopy type with homology groups $H_2, H_3, \ldots$. The homotopy classes k'_n of the attaching maps in Definition 2.7.3(3) are '*k'-invariants*' of the homology decomposition. These are dual to the k-invariants in the Postnikov decomposition. In fact, the homology decomposition was introduced by Eckmann and Hilton as a *dual* of the Postnikov decomposition. The homology decomposition turned out to have a disadvantage, namely it fails to be natural. In particular, the homotopy type of the section X_n of a homology decomposition is not an invariant of the homotopy type of X. Therefore, also, the k' invariants do not have the desired property of naturality. For this reason we have introduced in Section 2.6 above new invariants which we call the *boundary invariants* of X. The boundary invariants can be thought of as being exactly the ingredient of the k'-invariants which is natural. Moreover, these boundary invariants determine the homotopy type in the same way as the Postnikov invariants. We shall prove this in Chapter 3. For the proof of Theorem 2.7.4 we use the principal reduction in Definition 2.3.5(4). This proof also shows how the attaching map k'_n above is related to the boundary invariant $\beta_n X$.

Proof of Theorem 2.7.4 Let X be a 1-connected CW-complex. We may assume that $X^1 = *$. We choose splittings t for the short exact sequences

$$Z_n \rightarrowtail C_n \overset{d}{\twoheadrightarrow} B_{n-1}$$

where $B_{n-1} = dC_n$, $C = C_* X$. Then we have $C_n \cong tB_{n-1} \oplus Z_n$. The attaching map f of n-cells in X yields maps f_B and f_Z for which the following diagram commutes

$$\begin{tikzcd}
M(Z_n, n-1) \arrow[r, phantom, "\subset"] \arrow[dr, "f_Z"'] & M(C_n, n-1) \arrow[d, "f"] & M(tB_{n-1}, n-1) \arrow[l, phantom, "\supset"] \arrow[dl, "f_B"] \\
& X^{n-1} &
\end{tikzcd}$$

Let X_{n-1} be the mapping cone of f_B and let $i: X^{n-1} \subset X_{n-1}$ be the inclusion. Assume now X_{n-1} admits a homology decomposition as in the theorem. Then X_n admits one also since X_n is homotopy equivalent to the mapping cone of a map

$$k'_n = i\beta : M(H_n, n-1) \to X_{n-1}$$

which is homologically trivial. We obtain β as in Definition 2.3.5(15) by the map $v: V \to X^{n-1}$ with mapping cone $C_v \simeq X^{n+1}$. This completes the proof of Theorem 2.7.4. □

Remark The map β in the proof above is also used for the construction of the boundary invariant $\beta_n X$ in addendum (2.6.8). On the other hand, we obtain $\beta_n X$ directly by the homology decomposition as follows. Let $X = \varinjlim X_n$ be given by a homology decomposition as in Definition 2.7.3. Then X can be chosen to be a CW-complex for which the skeleta X^n satisfy

$$X^{n-1} \overset{i}{\subset} X_{n-1} \subset X^n.$$

Moreover k'_n admits a factorization β for which the following diagram homotopy commutes.

$$
(2.7.5)\qquad
\begin{array}{ccc}
 & \overset{k'_n}{\nearrow} & X_{n-1} \\
 & & \cup \\
M(H_n, n-1) & \xrightarrow{\beta} & X^{n-1} \\
\cup & & \cup \\
M(Z_n, n-1) & \longrightarrow & X^{n-2}
\end{array}
$$

Hence β represents an element $\beta_n X$ in $\mathfrak{L}_{n-1}(H_n, X)$ by definition of $\Gamma_{n-1}(A, X)$ in Definition 2.2.3; see Definition 2.2.9, Lemma 2.3.4 and the definition of $\mathfrak{L}_{n-1}(A, X)$ in (2.6.2). Hence diagram (2.7.5) describes exactly how to deduce from k'_n the boundary invariant $\beta_n X$.

We would like to emphasize that the classical concepts of homotopy decomposition and homology decomposition above do not have a strong impact on the classification of homotopy types. They only provide instructions on how to build a 1-connected CW-space with prescribed homotopy groups and homology groups respectively. The main problem is to decide whether two such constructions are actually homotopy equivalent.

A further deep problem is connected with the relationship between the homotopy decomposition and the homology decomposition of a given space X. In fact, the homotopy decomposition of a Moore space $M(A, n)$ (for example a sphere) involves the computation of all homotopy groups of the Moore space. Conversely the homology decomposition of an Eilenberg–Mac Lane space $K(A, n)$ requires the computation of all homology groups of $K(A, n)$ as achieved by the work of Eilenberg and Mac Lane, and Cartan. A basic result concerning the connection between the homotopy decomposition and the homology decomposition is given by the compatibility properties of Postnikov invariants and boundary invariants respectively; see Theorem 2.6.9(b, c) and Theorem 2.5.10(b, c). Rationally, that is for 1-connected rational spaces X, the connection between the homotopy decomposition and the homology decomposition of X is completely understood; see Baues and Lemaire [MM].

2.8 Unitary invariants of a homotopy type

We here describe the 'unitary invariants' of a homotopy type which were introduced in (V.1.7) of G.W. Whitehead [RA]. The definition of these invariants is similar to the definition of the boundary invariants β'_n in Definition 2.6.13. We show, however, that unitary invariants are not suitable for the classification of homotopy types; in fact there are spaces which can be distinguished by boundary invariants but not by unitary invariants.

In the definition of unitary invariants we use again the $(n-1)$-type

(2.8.1) $$p_{n-1}^X : X \to P_{n-1}X$$

of the simply connected space X. Using the cohomology groups with coefficients in an abelian group A we obtain the following diagram which resembles (2.6.11).

(2.8.2)

$$\begin{tikzcd}
\mathrm{Ext}(H_nP_{n-1}X, A) \arrow[r, tail, "\Delta"] \arrow[d, "(H_np_{n-1}^X)^*"] & H^{n+1}(P_{n-1}X, A) \arrow[r, two heads, "\mu"] \arrow[d, "(p_{n-1}^X)^*"] & \mathrm{Hom}(H_{n+1}P_{n-1}X, A) \arrow[d, "(H_{n+1}p_{n-1}^X)^*"] \\
\mathrm{Ext}(H_nX, A) \arrow[r, tail, "\Delta"] \arrow[d, "j^*"'] \arrow[dr, phantom, "\text{push}"] & H^{n+1}(X, A) \arrow[r, two heads, "\mu"] \arrow[d, "j^{\#}"] & \mathrm{Hom}(H_{n+1}X, A) \arrow[d, equal] \\
\mathrm{Ext}(\ker b_nX, A) \arrow[r, tail, "\Delta"] & \dfrac{H^{n+1}(X, A)}{\Delta\,\mathrm{im}(H_np_{n-1}^X)^*} \arrow[r, two heads, "\mu"] & \mathrm{Hom}(H_{n+1}X, A)
\end{tikzcd}$$

The rows are given by the short exact coefficient sequences. For the inclusion $j: \ker(b_nX) \subset H_nX$ the left-hand column is exact since we can identify

$$\begin{tikzcd}
H_nX \arrow[r, "H_np_{n-1}^X"] \arrow[d, equal] & H_nP_{n-1}X \arrow[d, equal, "\theta"] \\
H_nX \arrow[r, two heads, "b_nX"] & \Gamma''_{n-1}X \subset \Gamma_{n-1}X
\end{tikzcd} \tag{1}$$

Moreover we can identify

$$\begin{tikzcd}
H_{n+1}X \arrow[r, "H_{n+1}p_{n-1}^X"] \arrow[d, equal] & H_{n+1}P_{n-1}X \arrow[d, equal, "\theta"] \\
H_{n+1}X \arrow[r, "b_{n+1}X"] & \Gamma_nX
\end{tikzcd} \tag{2}$$

Compare the proof of Theorem 2.6.14. Let

$$\mathrm{im}(H_np_{n-1}^X)^* = (H_np_{n-1}^X)^*\mathrm{Ext}(H_nP_{n-1}X, A)$$

be the image of the homomorphism $(H_np_{n-1}^X)^*$ in diagram (2.8.2) and let $j^{\#}$ be the quotient map; equivalently $j^{\#}$ is the projection for the cokernel of the composite $\Delta(H_n p_{n-1}^X)$. Hence j^* and $j^\#$ are surjective and one readily checks that the subdiagram 'push' in (2.8.2) is a push-out diagram of abelian groups. The naturality of the $(n-1)$-type (2.8.1) shows that the group

(2.8.3) $$\mathfrak{A}^{n+1}(X,A) = \frac{H^{n+1}(X,A)}{\Delta \operatorname{im}(H_n p_{n-1}^X)^*}$$

yields a functor

$$\mathfrak{A}^{n+1}: \mathbf{spaces}_2^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab},$$

together with a binatural short exact sequence given by the bottom row of (2.8.2).

(2.8.4) Definition We define the *unitary invariants* $u^{n+1}(X)$, $n \geq 2$, of a simply connected CW-space X as follows. Consider diagram (2.8.2) where we set $A = H_{n+1}P_{n-1}X \overset{\theta}{=} \Gamma_n X$. Then we get

$$u^{n+1}(X) \in \mathfrak{A}^{n+1}(X, \Gamma_n X) = \frac{H^{n+1}(X, \Gamma_n X)}{\Delta \operatorname{im}(H_n p_{n-1}^X)^*}$$

$$u^{n+1}(X) = j^\#(p_{n-1}^X)^* \mu^{-1}(1_{\Gamma_n X}).$$

Here $1_{\Gamma_n X} \in \operatorname{Hom}(\Gamma_n X, A) = \operatorname{Hom}(H_{n+1}P_{n-1}X, A)$ is the identity of $\Gamma_n X$, hence an element $u \in \mu^{-1}(1_{\Gamma_n X})$ is a unitary class (see (1.2.32)), which via $j^\#(p_{n-1}^X)^*$ yields the invariant $u^{n+1}(X)$. One readily checks that $u^{n+1}(X)$ is well defined. Given an element $u \in \mathfrak{U}^{n+1}(X, A)$ we obtain elements

(2.8.5) $$\begin{cases} u_* = \mu(u) \in \operatorname{Hom}(H_{n+1}X, A) \\ u_\dagger = \Delta^{-1} q_*(u) \in \operatorname{Ext}(\ker b_n X, \operatorname{cok}(u_*)). \end{cases}$$

Here Δ and μ are defined by the binatural short exact sequence in (2.8.2). Moreover $q: A \to \operatorname{cok}(u_*)$ is the quotient map and

$$q_*: \mathfrak{A}^{n+1}(X, A) \to \mathfrak{A}^{n+1}(X, \operatorname{cok}(u_*))$$

is induced by the bifunctor $\mathfrak{A}^{n+1}$. As in (2.1.31)(4) one readily checks that the elements in (2.8.5) are well defined. As an example we obtain for $u = u^{n+1}X$ the element

$$(u^{n+1}X)_* = \mu(u^{n+1}X) = b_{n+1}X$$

and the element

$$(u^{n+1}X)_\dagger \in \operatorname{Ext}(\ker b_n X, \operatorname{cok} b_{n+1} X)$$

which can be compared with $\{\pi_n X\}$ in (2.6.8).

(2.8.6) Theorem on unitary invariants *With each* 1*-connected CW-complex* X *there is canonically associated a sequence of elements* $(u^4, u^5, \dots)$ *with*

$$u^{n+1} = u^{n+1}X \in \mathfrak{A}^{n+1}(X, \Gamma_n X), \quad n \geq 3,$$

such that the following properties are satisfied:

(*a*) *Naturality: for a map* $F\colon X \to Y$ *we have*

$$F^*(u^{n+1}Y) = (\Gamma_n F)_*(u^{n+1}X) \in \mathfrak{A}^{n+1}(X, \Gamma_n Y).$$

(*b*) *Compatibility with* $b_{n-1}X$:

$$(u^{n+1}X)_* = b_{n+1}X \in \mathrm{Hom}(H_{n+1}X, \Gamma_n X).$$

(*c*) *Compatibility with* $\{\pi_n X\}$:

$$-(u^{n+1}X)_\dagger = \{\pi_n X\} \in \mathrm{Ext}(\ker(b_n X), \mathrm{cok}(b_{n+1}X)).$$

(*d*) *Vanishing condition: all unitary invariants are trivial, that is* $u^{n+1} = 0$ *for* $n \geq 2$, *if and only if* X *has the homotopy type of a one-point union of Moore spaces* $M(H_n, n)$, $n \geq 2$, $H_n = H_n(X)$.

Proof Propositions (a) and (b) are clear by construction. Moreover (c) is a consequence of Theorem 2.1.33, compare (V.1.7) in G.W. Whitehead [RA]. Now assume $u^{n+1}X = 0$ for $n \geq 3$. Then $b_{n+1}X = 0$ for all n and also $\{\pi_n X\} = 0$. Hence $h\colon \pi_n X \to H_n X$ is split surjective and therefore we obtain (d) by Proposition 2.6.15. □

The theorem shows that unitary invariants have almost the same properties as boundary invariants in Theorem 2.6.9. The unitary invariants, however, are not suitable for classification as we show by the following example.

(2.8.7) Example Let X and Y be $(n-1)$-connected $(n+3)$-dimensional spaces defined as follows. The space X is the mapping cone of the map

$$i_1 \eta_n^2 q + i_2 \eta\colon M(\mathbb{Z}/2, n+1) \to S^n \vee M(\mathbb{Z}/r, n).$$

Here $q\colon M(\mathbb{Z}/2, n+1) \to S^{n+2}$ is the pinch map and $\eta_n^2\colon S^{n+2} \to S^n$ is the double Hopf map. Moreover i_1, i_2 are the inclusions and

$$\eta = \eta_r^2\colon M(\mathbb{Z}/2, n+1) \to M(\mathbb{Z}/r, n)$$

is a map which is non-trivial on the $(n+1)$-skeleton. Let Y be the mapping cone of η. The homology of X and $S^n \vee Y$ coincide and we have the injective maps

$$b_{n+2}(X) = b_{n+2}(S^n \vee Y): H_{n+2}X = \mathbb{Z}/2 \rightarrowtail \Gamma_{n-1}X = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

with $H_n X = \mathbb{Z} \oplus \mathbb{Z}/r$ and $\Gamma_{n-1}X = H_n(X) \otimes \mathbb{Z}/2$. Since $H_{n+3}X = 0$ the injectivity implies that $u^{n+3}(X) = u^{n+3}(S^n \vee Y) = 0$. Hence the unitary invariants do not suffice to distinguish between X and $S^n \vee Y$. In Chapter 8 we show via boundary invariants that X and $S^n \vee Y$ are not homotopy equivalent. In fact, we have $X = X(\varepsilon^2\eta_r)$ and $Y = X({}^2\eta_r)$ in the notation of Chapter 12, and X is not decomposable.

(2.8.8) Remark If X is $(n-1)$-connected, $n \geq 2$, then

$$u^{n+2}X \in H^{n+2}(X, \Gamma_n^1 H_n X)$$

is the *Pontrjagin–Steenrod element*; compare (V.1.9) in G.W. Whitehead [RA] and (I.6.8) in Baues [CH]. Thus formula (5.3.5) below can be derived from Theorem 2.8.6(c).

(2.8.9) Remark A crucial property of the bifunctors $\mathfrak{P}_{n-1}$ in Definition 2.5.7 and $\mathfrak{Q}_{n-1}$ in (2.6.3) is the fact that the $(n-1)$-type $p_{n-1}^X: X \to P_{n-1}X$ induces isomorphisms

$$(p_{n-1}^X)^*: \mathfrak{P}_{n-1}(P_{n-1}X, A) \cong \mathfrak{P}_{n-1}(X, A),$$

$$(p_{n-1}^X)_*: \mathfrak{Q}_{n-1}(A, X) \cong \mathfrak{Q}_{n-1}(P_{n-1}X, A).$$

These isomorphisms allow the definition of the detecting functor in the classification theorem (3.4.4). For the functor $\mathfrak{A}^{n+1}$ we do not have such an isomorphism.

3

ON THE CLASSIFICATION OF HOMOTOPY TYPES

In this chapter we describe fundamental new results on the classification of homotopy types. On the one hand, we get a classification by Postnikov invariants (k-invariants); on the other hand, we obtain a classification by boundary invariants. The general properties of these invariants lead us to introduce algebraic concepts which we call 'kype functors' E and 'bype functors' F, respectively. Here kype is a new word derived from the words k-invariant and type and similarly bype is derived from boundary invariant and type. A kype functor E and a bype functor F determine categories which we denote by

$$\mathbf{Kypes}(E), \quad \mathbf{kypes}(E)$$

and

$$\mathbf{Bypes}(F), \quad \mathbf{bypes}(F).$$

Our classification theorem shows that the objects of such categories are models of homotopy types. Hence a classification of homotopy types can be achieved by the computation of kype functors and bype functors, respectively. In later chapters we describe various examples of such computations which lead to optimal algebraic descriptions of certain classes of homotopy types. As a simple example we obtain the old results of J.H.C. Whitehead on the classification of 1-connected 4-dimensional homotopy types, see Section 3.5. The classification theorem of this chapter is the core of the book. We shall describe many applications and explicit examples of this theorem.

3.1 kype functors

The properties of k-invariants of a simply connected CW-space lead to the notion of 'kype'. Here kype is an amalgamation of the words k-invariant and type. In Section 3.4 we describe a classification theorem which shows that kypes are fundamental models of homotopy types.

(3.1.1) Definition Let $\mathbf{C}$ be a category. A *kype functor* on $\mathbf{C}$ is a functor

$$E\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab} \tag{1}$$

together with a binatural short exact sequence ($X \in \mathbf{C}$, $\pi \in \mathbf{Ab}$)

$$\mathrm{Ext}(E_0 X, \pi) \overset{\Delta}{\rightarrowtail} E(X, \pi) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(E_1 X, \pi) \tag{2}$$

where E_0, E_1: $\mathbf{C} \to \mathbf{Ab}$ are functors. We call (E_0, E_1, Δ, μ) the 'kype structure' of the functor E. We say that that the kype functor E is *split* if the exact sequence (2) admits a binatural splitting in $\mathbf{Ab}$. In this case E is completely determined by the pair of functors (E_0, E_1) since we can identify

$$E(X, \pi) = \mathrm{Ext}(E_0 X, \pi) \oplus \mathrm{Hom}(E_1 X, \pi) \tag{3}$$

by the splitting of (2). Moreover we say that E is *semitrivial* if $E_0 = 0$ is the trivial functor. Clearly in this case E is determined by E_1. All kype functors E are *semisplit* in the following sense: for each object $X \in \mathbf{C}$ there is a splitting homomorphism

$$s_X : \mathrm{Hom}(E_1 X, \pi) \to E(X, \pi) \tag{4}$$

with $\mu s_X = 1$ which is natural in π; that is for φ: $\pi \to \pi'$ in $\mathbf{Ab}$ we have $\varphi_* s_X = s_X \varphi_*$. We obtain s_X since the functor $\mathbf{Ab} \to \mathbf{Ab}$ carrying π to $\mathrm{Hom}(E_1 X, \pi)$ is projective in the functor category of functors $\mathbf{Ab} \to \mathbf{Ab}$. In Section 3.5 we describe examples of semitrivial kype functors and in Section 3.6 we deal with split kype functors.

Using a kype functor E we introduce the following category of E-kypes which is a kind of extended Grothendieck construction, see Remark 3.1.3.

(3.1.2) Definition Let E be a kype functor on the category $\mathbf{C}$. An E-*kype*

$$\bar{X} = (X, \pi, k, H, b)$$

is a tuple consisting of an object X in $\mathbf{C}$, abelian groups π and H, and elements

$$k \in E(X, \pi),$$

$$b \in \mathrm{Hom}(H, E_1 X)$$

such that the sequence

$$H \xrightarrow{b} E_1(X) \xrightarrow{\mu(k)} \pi \tag{1}$$

is exact. A morphism

$$(f, \varphi, \varphi_H) \colon (X, \pi, k, H, b) \to (X', \pi', k', H', b')$$

between such E-kypes is given by a morphism f: $X \to X'$ in $\mathbf{C}$ and by homomorphisms $\varphi: \pi \to \pi'$, $\varphi_H: H \to H'$ such that the following properties (2) and (3), are satisfied:

$$f^*(k') = \varphi_*(k) \in E(X, \pi') \tag{2}$$

Here the induced homomorphisms

$$E(X', \pi') \xrightarrow{f^*} E(X, \pi') \xleftarrow{\varphi_*} E(X, \pi)$$

are given by the bifunctor E. Moreover the diagram

$$\begin{array}{ccc} H & \xrightarrow{b} & E_1(X) \\ \downarrow{\scriptstyle \varphi_H} & & \downarrow{\scriptstyle f_* = E_1(f)} \\ H' & \xrightarrow{b'} & E_1(X') \end{array} \tag{3}$$

commutes. E-kypes and such morphisms form the 'category of E-kypes'. If the kype functor E is clear from the context we call an E-kype simply a *kype*. We say that an E-kype (X, π, k, H, b) is *injective* if b is an injective homomorphism. Let

$$\mathbf{kypes}(\mathbf{C}, E) \tag{4}$$

be the full subcategory of injective E-kypes. Moreover an E-kype (X, π, k, H, b) is *free* if H is a free abelian group. Let

$$\mathbf{Kypes}(\mathbf{C}, E) \tag{5}$$

be the full subcategory of such free E-kypes. We have the forgetful functor

$$\phi: \mathbf{Kypes}(\mathbf{C}, E) \to \mathbf{kypes}(\mathbf{C}, E) \tag{6}$$

which carries the free E-kype (X, π, k, H, b) to the injective E-kype (X, π, k, H', b') where H' is the image of b and where b' is the inclusion of this image.

Lemma *The functor ϕ is full and representative.*

Proof It is clear that each E-kype has a ϕ-realization by choosing a surjection $H \twoheadrightarrow \ker \mu(k)$ where H is free abelian. Hence ϕ is representative. Moreover ϕ is full since, by Definition 3.1.2 (2) the homomorphism f_* carries $\ker \mu(k)$ to $\ker \mu(k')$ and hence we can choose a homomorphism φ_H for which Definition 3.1.2 (3) commutes since H is free abelian. □

(3.1.3) Remark Any bifunctor

$$E: \mathbf{C}^{\mathrm{op}} \times \mathbf{K} \to \mathbf{L} \tag{1}$$

yields a category $\mathbf{Gro}(E)$ which is called the *Grothendieck construction* on E. Objects are triples (X, π, k) where $X \in Ob(\mathbf{C})$, $\pi \in Ob(\mathbf{K})$, and

$$k \in E(X, \pi). \tag{2}$$

Morphisms $(f, \varphi)\colon (X, \pi, k) \to (X', \pi', k')$ are given by morphisms $f\colon X \to X'$ in $\mathbf{C}$ and $\varphi\colon \pi \to \pi'$ in $\mathbf{K}$ such that the equation

$$f^*(k') = \varphi_*(k) \in E(X, \pi') \tag{3}$$

is satisfied; see Definition 3.1.2(2) above. Hence the category of E-kypes above is a kind of enriched Grothendieck construction. Moreover we have, for a kype functor E, the forgetful functor

$$\psi\colon \mathbf{kypes}(\mathbf{C}, E) \xrightarrow{\sim} \mathbf{Gro}(E)$$

which carries the injective E-kype (X, π, k, H, b) to (X, π, k). This functor ψ is easily seen to be an equivalence of categories. This way we identify an injective E-kype with an object in the Grothendieck construction of E.

(3.1.4) Remark Let E be a kype functor on $\mathbf{C}$ and let $\alpha\colon \mathbf{K} \to \mathbf{C}$ be a functor. Then we obtain a *kype functor* α^*E on $\mathbf{K}$ as follows. We define α^*E by the composite

$$\alpha^*E\colon \mathbf{K}^{\mathrm{op}} \times \mathbf{Ab} \xrightarrow{\alpha \times 1} \mathbf{C}^{\mathrm{op}} \times \mathbf{Ab} \xrightarrow{E} \mathbf{Ab} \tag{1}$$

and we obtain the kype structure of α^*E by $(E_0\alpha, E_1\alpha, \Delta, \mu)$ where (E_0, E_1, Δ, μ) is the kype structure of E. Moreover α induces functors

$$\alpha\colon \mathbf{kypes}(\mathbf{K}, \alpha^*E) \to \mathbf{kypes}(\mathbf{C}, E) \tag{2}$$

$$\alpha\colon \mathbf{Kypes}(\mathbf{K}, \alpha^*E) \to \mathbf{Kypes}(\mathbf{C}, E) \tag{3}$$

which carry (Y, π, k, H, b) to $(\alpha Y, \pi, k, H, b)$. The induced functors α in (2) and (3) are equivalences of categories if α is an equivalence of categories.

We now use the kype structure of the functor E in an essential way. Let $\bar{X} = (X, \pi, k, H, b)$ be an E-kype as above. Then we obtain elements

$$\text{(3.1.5)} \qquad \begin{aligned} k_* &= \mu(k) \in \mathrm{Hom}(E_1X, \pi) \\ k_\dagger &= \Delta^{-1}q_*(k) \in \mathrm{Ext}(E_0X, \mathrm{cok}(k_*)). \end{aligned}$$

as follows. The homomorphism $k_* = \mu(k)$ is given by k in $\bar{X}$ and by the natural transformation μ in Definition 3.1.1. For the quotient map

$$q\colon \pi \twoheadrightarrow \pi' = \mathrm{cok}(k_*) \tag{1}$$

(to the cokernel of k_*) we get via naturality a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}(E_0X,\pi) \arrow[r, tail, "\Delta"] \arrow[d, "q_*"] & E(X,\pi) \arrow[r, two heads, "\mu"] \arrow[d, "q_*"] & \mathrm{Hom}(E_1X,\pi) \arrow[d, "q_*"] \\
\mathrm{Ext}(E_0X,\pi') \arrow[r, tail, "\Delta"] & E(X,\pi') \arrow[r, "\mu"] & \mathrm{Hom}(E_1X,\pi')
\end{tikzcd} \tag{2}$$

Here $q_*\mu(k)=0$ implies that the element $k_\dagger=\Delta^{-1}q_*(k)$ in (3.1.5) is well defined. Let

$$\mathrm{cok}(k_*) \overset{i}{\rightarrowtail} H(k_\dagger) \overset{b_0}{\twoheadrightarrow} E_0X \tag{3}$$

be an extension of abelian groups which represents the element $k_\dagger$. Hence we obtain by $\bar{X}$ the exact sequence

$$H \xrightarrow{b} E_1X \xrightarrow{k_*} \pi \xrightarrow{i} H(k_\dagger) \xrightarrow{b_0} E_0X \to 0 \tag{4}$$

which we call the *Γ-sequence* of the E-kype $\bar{X}$. This sequence is in the following sense natural with respect to morphisms between E-kypes. We say that a morphism $f\colon X \to X'$ together with a commutative diagram in **Ab**

$$\begin{tikzcd}
H \arrow[r] \arrow[d] & E_1X \arrow[r] \arrow[d, "f_*"] & \pi \arrow[r] \arrow[d] & H(k_\dagger) \arrow[r] & E_0X \arrow[r] \arrow[d, "f_*"] & 0 \\
H' \arrow[r] & E_1X' \arrow[r] & \pi' \arrow[r] & H(k'_\dagger) \arrow[r] & E_0X' \arrow[r] & 0
\end{tikzcd} \tag{5}$$

is a *weak morphism* between Γ-sequences if there is a homomorphism $H(k_\dagger)\to H(k'_\dagger)$ which extends the diagram commutatively. Similarly we define a *weak isomorphism*. Each morphism (f,φ,φ_H) between E-kypes clearly induces a weak morphism between Γ-sequences for which the vertical arrows in (5) are φ_H, $E_1(f)$, φ, and $E_0(f)$, respectively.

3.2 bype functors

The properties of boundary invariants of a simply connected CW-space give rise to the notion of 'bype'; here bype is the amalgamation of the words boundary invariant and type. A bype is the true Eckmann–Hilton dual of a kype discussed in Section 3.1. To stress the duality between bypes and kypes this section is organized in the same manner as Section 3.1. Indeed the strength of this duality is amazing and new; see also Section 3.3. In Section 3.4 we describe a classification theorem which shows that bypes are fundamental new models of homotopy types where bypes are related to homology groups similarly to the way kypes are related to homotopy groups.

(3.2.1) Definition Let $\mathbf{C}$ be a category. A *bype functor* F on $\mathbf{C}$ is a functor

$$F: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab} \tag{1}$$

together with a binatural short exact sequence ($X \in \mathbf{C}$, $H \in \mathbf{Ab}$)

$$\mathrm{Ext}(H, F_1 X) \overset{\Delta}{\rightarrowtail} F(H, X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H, F_0 X) \tag{2}$$

where F_0, F_1: $\mathbf{C} \to \mathbf{Ab}$ are functors. We call (F_0, F_1, Δ, μ) the 'bype structure' of the functor F. We say that the bype functor F is *split* if the exact sequence (2) admits a binatural splitting in $\mathbf{Ab}$. In this case F is completely determined by the pair of functors (F_0, F_1) since we can identify

$$F(H, X) = \mathrm{Ext}(H, F_1 X) \oplus \mathrm{Hom}(H, F_0 X) \tag{3}$$

by the splitting of (2). Moreover we say that F is *semitrivial* if $F_0 = 0$ is the trivial functor. Clearly in this case F is determined by F_1. Each bype functor F is *semisplit* in the following sense: for each object $X \in \mathbf{C}$ there is a splitting homomorphism

$$s_X: \mathrm{Hom}(H, F_0 X) \to F(H, X) \tag{4}$$

with $\mu s_X = 1$ which is natural in $H \in \mathbf{Ab}$. For this we use a similar argument to that in Definition 3.1.1 (4).

We study bype functors in more detail in Section 3.3 where we show that they are actually 'equivalent' to kype functors by a duality isomorphism. We consider an example of a semitrivial bype functor in Section 5 and in Section 6 we describe an example of a split bype functor. Up to a few changes a bype functor is the exact analogue of a kype functor in Definition 3.1.1. We now introduce categories of bypes and Bypes respectively which are the analogues of the corresponding categories of kypes and Kypes in Section 3.1. In Sections 3.5 and 3.6 we describe such categories in case the bype (resp. kype) functors are semitrivial or split. It is interesting to have these simple cases in mind when reading the following definitions.

(3.2.2) Definition Let F be a bype functor on the category $\mathbf{C}$. An *F-bype*

$$\bar{X} = (X, H_0, H_1, b, \beta)$$

is a tuple consisting of an object X in $\mathbf{C}$, abelian groups H_0, H_1, and elements

$$b \in \mathrm{Hom}(H_1, F_1 X),$$

$$\beta \in F(H_0, X, b)$$

with the following properties. Using the homomorphism b and the bype structure of F we define the abelian group $F(H_0, X, b)$ by the push-out diagram

$$\begin{array}{ccccc}
\mathrm{Ext}(H_0, F_1X) & \overset{\Delta}{\rightarrowtail} & F(H_0, X) & \xrightarrow{\mu} & \mathrm{Hom}(H_0, F_0X) \\
\downarrow{\scriptstyle i_*} & \text{push} & \downarrow & & \| \\
\mathrm{Ext}(H_0, F_1X/K) & \overset{\Delta}{\rightarrowtail} & F(H_0, X, b) & \xrightarrow{\mu} & \mathrm{Hom}(H_0, F_0X)
\end{array}$$

Here $i\colon F_1X \twoheadrightarrow \mathrm{cok}(b) = F_1X/K$ is the quotient map for the cokernel of b with $K = \mathrm{image}(b)$. The element $\beta \in F(H_0, X, b)$ has the property that the sequence

$$H_0 \xrightarrow{\mu(\beta)} F_0X \to 0 \tag{1}$$

is exact, that is $\mu(\beta)$ is surjective. A morphism

$$(f, \varphi_0, \varphi_1)\colon (X, H_0, H_1, b, \beta) \to (X', H_0', H_1', b', \beta')$$

between such F-bypes is given by a morphism $f\colon X \to X'$ in $\mathbf{C}$ and by homomorphisms $\varphi_0\colon H_0 \to H_0'$, $\varphi_1\colon H_1 \to H_1'$ such that the following properties (2) and (3), are satisfied. The diagram

$$\begin{array}{ccc}
H_1 & \xrightarrow{b} & F_1(X) \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle f_* = F_1(f)} \\
H_1' & \xrightarrow{b'} & F_1(X')
\end{array} \tag{2}$$

is commutative. Hence f_* induces a homomorphism $f_*\colon \mathrm{cok}(b) \to \mathrm{cok}(b')$ between cokernels so that the bype functor F yields induced homomorphisms

$$F(H_0, X, b) \xrightarrow{f_*} F(H_0, X', b') \xleftarrow{\varphi_0^*} F(H_0', X', b')$$

with

$$f_*(\beta) = \varphi_0^*(\beta') \in F(H_0, X', b'). \tag{3}$$

F-bypes and such morphisms form the 'category of F-bypes'. If the bype functor F is clear from the context we call an F-bype also simply a *bype*.

We say that an F-bype (X, H_0, H_1, b, β) is *injective* if b is an injective homomorphism. Let

$$\mathbf{bypes}(\mathbf{C}, F) \tag{4}$$

be the full subcategory of injective F-bypes. Moreover an F-bype (X, H_0, H_1, b, β) is *free* if H_1 is a free abelian group. Let

$$\mathbf{Bypes}(\mathbf{C}, F) \tag{5}$$

be the full subcategory of free F-bypes. We have the forgetful functor

$$\phi\colon \mathbf{Bypes}(\mathbf{C}, F) \to \mathbf{bypes}(\mathbf{C}, F) \tag{6}$$

which carries the free F-bype (X, H_0, H_1, b, β) to the injective F-bype $(X, H_0, H_1', b', \beta)$ where H_1' is the image of b and where b' is the inclusion of this image. Hence cokernels $\operatorname{cok}(b) = \operatorname{cok}(b')$ coincide so that $\beta \in F(H_0, X, b) = F(H_0, X, b')$. As in Definition 3.1.2 we see

(3.2.3) Lemma *The functor ϕ is full and representative.*

(3.2.4) Remark Let F be a bype functor on $\mathbf{C}$ and let $\alpha\colon \mathbf{K} \to \mathbf{C}$ be a functor. Then we obtain similarly as in Remark 3.1.4 the bype functor α^*F on $\mathbf{K}$ and the induced functors

$$\alpha\colon \mathbf{bypes}(\mathbf{K}, \alpha^*F) \to \mathbf{bypes}(\mathbf{C}, E)$$

$$\alpha\colon \mathbf{Bypes}(\mathbf{K}, \alpha^*F) \to \mathbf{Bypes}(\mathbf{C}, F).$$

The induced functors are equivalences of categories if α is.

Now let $\bar{X} = (X, H_0, H_1, b, \beta)$ be F-bype as above. Then we obtain elements

$$\begin{aligned} \beta_* &= \mu(\beta) \in \operatorname{Hom}(H_0, F_0X) \\ \beta_\dagger &= \Delta^{-1} i^*\beta \in \operatorname{Ext}(\ker \beta_*, \operatorname{cok} b) \end{aligned} \tag{3.2.5}$$

as follows. The homomorphism β_* is given by β in $\bar{X}$ and by the natural transformation μ in the bottom row of Definition 3.2.2 (*). By the assumption in Definition 3.2.2 (1) the homomorphism β_* is surjective. Now let

$$i\colon \ker(\beta_*) = H_0' \rightarrowtail H_0 \tag{1}$$

be the inclusion of the kernel of β_*. Then i yields by naturality a commutative diagram

$$\begin{tikzcd} \operatorname{Ext}(H_0, \operatorname{cok} b) \arrow[r, tail, "\Delta"] \arrow[d, "i^*"] & F(H_0, X, b) \arrow[r, two heads, "\mu"] \arrow[d, "i^*"] & \operatorname{Hom}(H_0, F_0X) \arrow[d, "i^*"] \\ \operatorname{Ext}(H_0', \operatorname{cok} b) \arrow[r, tail, "\Delta"] & F(H_0', X, b) \arrow[r, two heads, "\mu"] & \operatorname{Hom}(H_0', F_0X) \end{tikzcd} \tag{2}$$

Here $i^*\mu(\beta)=0$ implies that the element $\beta_\dagger = \Delta^{-1}i^*(\beta)$ in (3.2.5) is well defined. Let

$$\operatorname{cok} b \rightarrowtail \pi(\beta_\dagger) \twoheadrightarrow \ker(\beta_*) \tag{3}$$

be an extension of abelian groups which represents the element $\beta_\dagger$. Hence we obtain by $\bar{X}$ the exact sequence

$$H_1 \xrightarrow{b} F_1X \xrightarrow{i} \pi(\beta_\dagger) \xrightarrow{h} H_0 \xrightarrow{\beta_*} F_0X \to 0 \tag{4}$$

which we call the *Γ-sequence* of $\bar{X}$. Here i is the composite of the quotient map and the inclusion in (3), and h is given by the projection in (3). The sequence is natural with respect to morphisms between F-bypes in the following sense. We say that a morphism $f\colon X \to X'$ in **C**, together with a commutative diagram in **Ab**,

$$\begin{array}{ccccccccc} H_1 & \to & F_1X & \to & \pi(\beta_\dagger) & \to & H_0 & \to & F_0X & \to 0 \\ \downarrow & & \downarrow{\scriptstyle f_*} & & & & \downarrow & & \downarrow{\scriptstyle f_*} & \\ H_1' & \to & F_1X' & \to & \pi(\beta_\dagger') & \to & H_0' & \to & F_0X' & \to 0 \end{array} \tag{5}$$

is a *weak morphism* between Γ-sequences if there is a homomorphism $\pi(\beta_\dagger) \to \pi(\beta_\dagger')$ which extends the diagram commutatively. Similarly we define weak isomorphisms. A morphism $(f, \varphi_0, \varphi_1)$ between F-bypes as in Definition 3.2.2 clearly induces a weak morphism between Γ-sequences for which the vertical arrows in (5) are φ_1, $F_1 f$, φ_0, and $F_0 f$, respectively.

3.3 Duality of bype and kype

Kype functors (E, E_0, E_1) form an abelian group $\operatorname{kext}_{\mathbf{C}}(E_1, E_0)$ and similarly bype functors (F, F_0, F_1) form an abelian group $\operatorname{bext}_{\mathbf{C}}(F_0, F_1)$. We show for $F_0 = E_0$ and $F_1 = E_1$ that there is a duality isomorphism

$$D\colon \operatorname{kext}_{\mathbf{C}}(E_1, E_0) \cong \operatorname{bext}_{\mathbf{C}}(E_0, E_1).$$

Hence a kype functor E determines up to equivalence a bype functor $F = D(E)$ and vice versa. We say in this case that E and F are dual to each other, see (3.3.6). Given $n \in \mathbb{Z}$ and a functor

$$K_*\colon \mathbf{C} \to \mathbf{Chain}_{\mathbb{Z}}/\simeq$$

we obtain induced kype and bype functors E and F, respectively, with

$$E(X, A) = H^{n+1}(K_*X, A)$$

$$F(A, X) = H_n(A, K_*X)$$

for $A \in \mathbf{Ab}$ and $X \in \mathbf{C}$. These turn out to be dual to each other; in particular E is split if and only if F is split, see Theorem 3.3.9. Hence the computation of the functor F also yields a computation of the dual functor E. This fact is of crucial importance in our main result on the classification of homotopy types in the next section.

Let $\mathbf{C}$ be a category. (We assume also that $\mathbf{C}$ is a small category so that the cohomology of $\mathbf{C}$ is an abelian group. There is, however, a canonical way to extend the following results to the case when $\mathbf{C}$ is not small.) Let

$$E_0, E_1 \colon \mathbf{C} \to \mathbf{Ab} \tag{3.3.1}$$

be two functors. We now consider kype functors E and bype functors F which are both associated with the functors $E_0 = F_0$ and $E_1 = F_1$.

(3.3.2) Definition A *kype extension* E of E_1 by E_0 is a semisplit kype functor E with structure (E_0, E_1, Δ, μ), that is, E is embedded in the binatural short exact sequence

$$\mathrm{Ext}(E_0X, \pi) \overset{\Delta}{\rightarrowtail} E(X, \pi) \xrightarrow{\mu} \mathrm{Hom}(E_1X, \pi)$$

with $X \in \mathbf{C}$, $\pi \in \mathbf{Ab}$. Two such extensions E, E' are *equivalent* if there is a binatural isomorphism τ: $E(X, \pi) \cong E'(X, \pi)$ with $\mu\tau = \mu$ and $\tau\Delta = \Delta$. Let

$$\mathrm{kext}_{\mathbf{C}}(E_1, E_0)$$

be the set of all equivalence classes of such kype extensions. Dually we define:

(3.3.3) Definition A *bype extension* E of E_0 by E_1 is a bype functor E with structure (E_0, E_1, Δ, μ) that is, E is embedded in the binatural short exact sequence

$$\mathrm{Ext}(H, E_1X) \overset{\Delta}{\rightarrowtail} E(H, X) \xrightarrow{\mu} \mathrm{Hom}(H, E_0X)$$

with $X \in \mathbf{C}$, $H \in \mathbf{Ab}$. Two such extensions E, E' are *equivalent* if there is a binatural isomorphism τ: $E(X, \pi) \cong E'(X, \pi)$ with $\mu\tau = \mu$ and $\tau\Delta = \Delta$. Let

$$\mathrm{bext}_{\mathbf{C}}(E_0, E_1)$$

be the set of such bype extensions.

We introduce for E_0, E_1 in (3.3.1) the bifunctor

$$\mathrm{Ext}(E_0, E_1) \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab} \tag{3.3.4}$$

which carries the pair (X, Y) of objects in $\mathbf{C}$ to the abelian group

$\mathrm{Ext}(E_0X, E_1Y)$. Hence $\mathrm{Ext}(E_0, E_1)$ is a $\mathbf{C}$-bimodule which we use as coefficients in the cohomology groups $H^*(\mathbf{C}, \mathrm{Ext}(E_0, E_1))$; compare Definition 1.1.15.

(3.3.5) Theorem *There are canonical bijections*

$$\mathrm{kext}_{\mathbf{C}}(E_1, E_0) \approx H^1(\mathbf{C}, \mathrm{Ext}(E_0, E_1)) \approx \mathrm{bext}_{\mathbf{C}}(E_0, E_1)$$

which carry the split extension to the zero-element in H^1 $(\mathbf{C}, \mathrm{Ext}(E_0, E_1))$.

The bijections in Theorem 3.3.5 yield an abelian group structure for the sets $\mathrm{kext}_{\mathbf{C}}(E_1, E_0)$ and $\mathrm{bext}_{\mathbf{C}}(E_0, E_1)$ such that we obtain an isomorphism

$$D\colon \mathrm{kext}_{\mathbf{C}}(E_1, E_0) \cong \mathrm{bext}_{\mathbf{C}}(E_0, E_1) \tag{3.3.6}$$

which we call the *duality* isomorphism. We say that a semisplit kype functor E is *dual* to the semisplit bype functor F if $E_0 = F_0$, $E_1 = F_1$, and $D\{E\} = \{F\}$. Here $\{E\}$ and $\{F\}$ denote the corresponding equivalence classes. Indeed using Theorem 3.3.5, each E yields up to equivalence a dual F and vice versa, showing that kype functors and bype functors are actually equivalent to each other. We point out that the bijections in Theorem 3.3.5 are reminiscent of Corollary 3.11 in Jibladze and Pirashvili [CA].

The elements in the first cohomology $H^1(\mathbf{C}, D)$ of the category $\mathbf{C}$ are represented by derivations as follows.

(3.3.7) Definition Let $\mathbf{C}$ be a (small) category and let D be a natural system on $\mathbf{C}$, for example $D = \mathrm{Ext}(E_0, E_1)$ that is $D(f) = \mathrm{Ext}(E_0X, E_1Y)$ for $f\colon X \to Y \in \mathbf{C}$. A *derivation*

$$d\colon \mathbf{C} \to D \tag{1}$$

is a function which associates with each $f\colon X \to Y \in \mathbf{C}$ an element $d(f) \in D(f)$ such that for composites gf in $\mathbf{C}$ we have the *derivation formula*

$$d(gf) = g_* d(f) + f^* d(g) \tag{2}$$

where g_* and f^* are defined by D. Suppose that there exists a function $a \in F^0(\mathbf{C}, D)$, which carries an object $X \in \mathbf{C}$ to an element $a(X) \in D(1_X)$, such that

$$d(f) = f_* a(X) - f^* a(Y). \tag{3}$$

Then d is called an *inner derivation* induced by a; we also write $d = \partial_a$ in this case. Let $\mathrm{Der}(\mathbf{C}, D)$ and $\mathrm{Ider}(\mathbf{C}, D)$ be the abelian groups of derivations and inner derivations respectively. Then we obtain the canonical isomorphism

$$H^1(\mathbf{C}, D) = \mathrm{Der}(\mathbf{C}, D)/\mathrm{Ider}(\mathbf{C}, D) \tag{4}$$

where the right-hand side is the quotient group, compare (IV.7.6) in Baues [AH].

Proof of Theorem 3.3.5 We define functions

$$\mathrm{kext}_{\mathbf{C}}(E_1, E_0) \underset{\tau_k'}{\overset{\tau_k}{\rightleftarrows}} H^1(\mathbf{C}, \mathrm{Ext}(E_0, E_1)) \tag{1}$$

as follows. Let E be a kype extension of E_1 by E_0 and choose splittings s_X as in Definition 3.1.1 (4). For $f\colon X \to Y \in \mathbf{C}$ we consider the diagram

$$\begin{tikzcd}
E(X, E_1X) \arrow[r, "(E_1f)_*"] & E(X, E_1Y) & E(Y, E_1Y) \arrow[l, "f^*"'] \\
\mathrm{Hom}(E_1X, E_1X) \arrow[u, "s_X"] \arrow[r, "(E_1f)_*"'] & \mathrm{Hom}(E_1X, E_1Y) \arrow[u, "s_X"] & \mathrm{Hom}(E_1Y, E_1Y) \arrow[u, "s_Y"] \arrow[l, "(E_1f)^*"]
\end{tikzcd} \tag{2}$$

where the left-hand square commutes. We now get a derivation $d_s\colon \mathbf{C} \to \mathrm{Ext}(E_0, \mathrm{E}_1)$ by

$$d_s(f) = \Delta^{-1}\big((E_1f)_* s_X(1_{E_1X}) - f^* s_Y(1_{E_1Y})\big). \tag{3}$$

For a different choice s'_X of splitting functions we obtain

$$a \in F^0(\mathbf{C}, \mathrm{Ext}(E_0, E_1))$$

by

$$a(X) = \Delta^{-1}\big(s_X(1_{E_1X}) - s'_X(1_{E_1X})\big) \tag{4}$$

and we immediately see that $d_s - d_{s'} = \partial_a$ is the inner derivation induced by α. Moreover, if $\tau\colon E \to E'$ is an equivalence we define splitting functions s'_X for E' by $s'_X = \tau s_X$ so that in this case $d_s = d_{s'}$. This shows that the function τ_k in (1) is well defined by

$$\tau_k\{E\} = \{d_s\}. \tag{5}$$

Next we construct the function τ'_k in (1). For derivation $d\colon \mathbf{C} \to \mathrm{Ext}(E_0, E_1)$ we define a kype functor E_d with structure (E_0, E_1, Δ, μ) as follows. For $X \in \mathbf{C}$, $\pi \in \mathbf{Ab}$ let

$$E_d(X, \pi) = \mathrm{Ext}(E_0X, \pi) \oplus \mathrm{Hom}(E_1X, \pi) \tag{6}$$

with Δ = inclusion and μ = projection. For $g\colon \pi \to \pi' \in \mathbf{Ab}$ we define

$$g_* = E_d(X, g) = \mathrm{Ext}(E_0X, g) \oplus \mathrm{Hom}(E_1X, g).$$

Moreover for $f: X \to Y \in \mathbf{C}$ we define

$$f^* = E_d(f, \pi): E_d(Y, \pi) \to E_d(X, \pi)$$

by the conditions

$$\Delta \operatorname{Ext}(E_0 f, \pi) = f^* \Delta$$

$$\mu f^* = \operatorname{Hom}(E_1 f, \pi)$$

and by the following condition. Let s_X: $\operatorname{Hom}(E_1 X, \pi) \to E_d(X, \pi)$ be the inclusion of the second summand in (6). Then we set, for $\varphi \in \operatorname{Hom}(E_1 Y, \pi)$,

$$f^* s_Y(\varphi) = s_X(\varphi E_1(f)) - \Delta(\varphi_* d(f)). \tag{7}$$

The derivation formula for d shows that E_d is a well-defined kype functor with structure (E_0, E_1, Δ, μ). For $a \in F^0(\mathbf{C}, \operatorname{Ext}(E_0, E_1))$ we construct below an equivalence of kype extensions

$$\phi_a: E_d \cong E_{d + \partial_a}. \tag{8}$$

This shows that the function τ_k' in (1) given by

$$\tau_k'\{d\} = \{E_d\} \tag{9}$$

is well defined. One readily checks that τ_k' is the inverse of τ_k. The equivalence ϕ_a is determined by the conditions $\mu \phi_a = \mu$, $\Delta \phi_a = \Delta$, and by

$$\phi_a s_X(\varphi) = s_X(\varphi) - \Delta(\varphi_* a(X)) \tag{10}$$

for $\varphi \in \operatorname{Hom}(E_1 X, \pi)$. We clearly have $g_* \phi_a = \phi_a g_*$. Moreover we get $f^\# \phi_a = \phi_a f^*$, with $f^\#$ induced by $E_{d+\partial_a}$, by the following equations where $\varphi \in \operatorname{Hom}(E_1 Y, \pi)$.

$$\begin{aligned}
f^\# \phi_a s_Y(\varphi) &= f^\#(s_Y(\varphi) - \Delta \varphi_* a(Y)) \\
&= f^\# s_Y(\varphi) - \Delta f^* \varphi_* a(Y) \\
&= s_X(\varphi E_1(f)) - \Delta(\varphi_* (d + \partial_a)(f)) - \Delta f^* \varphi_* a(Y) \\
&= s_X(\varphi E_1(f)) + \Delta(\varphi_* [-d(f) - \partial_a(f) - f^* a(Y)]) \\
&= s_X(\varphi E_1(f)) + \Delta \varphi_* [-d(f) - f_* a(X)] \\
\phi_a f^* s_Y(\varphi) &= \phi_a(s_X(\varphi E_1(f)) - \Delta(\varphi_* d(f))) \\
&= \phi_a s_X(\varphi E_1(f)) - \Delta \varphi_* d(f) \\
&= s_X(\varphi E_1(f)) - \Delta(\varphi_* (E_1 f)_* a(X)) - \Delta \varphi_* d(f).
\end{aligned}$$

This completes the proof that τ_k in (1) is a bijection. In a completely analogous fashion we obtain the bijection

$$\tau_b\colon \mathrm{bext}_{\mathbf{C}}(E_0, E_1) \approx H^1(\mathbf{C}, \mathrm{Ext}(E_0, E_1)). \tag{11}$$

For this let F be a bype extension of E_0 by E_1 and choose a splitting s_X as in Definition 3.2.1 (4). For $f\colon X \to Y \in \mathbf{C}$ consider the diagram

$$\begin{tikzcd}
F(E_0X, X) \arrow[r, "f_*"] & F(E_0X, Y) & F(E_0Y, Y) \arrow[l, "(E_0f)^*"'] \\
\mathrm{Hom}(E_0X, E_0X) \arrow[u, "s_X"] \arrow[r, "(E_0f)_*"'] & \mathrm{Hom}(E_0X, E_0Y) \arrow[u, "s_Y"] & \mathrm{Hom}(E_0Y, E_0Y) \arrow[u, "s_Y"] \arrow[l, "(E_0f)^*"]
\end{tikzcd}$$

where the right-hand square commutes. We obtain a derivation $d_s'\colon \mathbf{C} \to \mathrm{Ext}(E_0, E_1)$ by

$$d_s'(f) = \Delta^{-1}\big(-f_* s_X(1_{E_0X}) + (E_0f)^* s_Y(1_{E_0Y})\big). \tag{12}$$

The bijection τ_b is now defined by

$$\tau_b\{F\} = \{d_s'\}. \tag{13}$$

We leave it to the reader to show that τ_b is well defined and a bijection. □

The following definition yields many examples of kype functors and bype functors respectively.

(3.3.8) Definition Recall that $\mathbf{Chain}_{\mathbb{Z}}$ denotes the category of chain complexes (C_*, d) of abelian groups. We say that (C_*, d) is a *free* chain complex if all chain groups C_n, $n \in \mathbb{Z}$, are free abelian. Now let $\mathbf{C}$ be a category and let

$$K_*\colon \mathbf{C} \to \mathbf{Chain}_{\mathbb{Z}}/\simeq$$

be a functor which carries each object $X \in \mathbf{C}$ to a free chain complex $K_*(X)$. Using cohomology and pseudo-homology we define, for $X \in \mathbf{C}$, $A \in \mathbf{Ab}$,

$$E(X, A) = H^{n+1}(K_*X, A),$$

$$F(A, X) = H_n(A, K_*X).$$

Then E is a kype functor and F is a bype functor with the structure given by the universal coefficient sequences

$$\mathrm{Ext}(H_nK_*X, A) \overset{\Delta}{\rightarrowtail} H^{n+1}(K_*X, A) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H_{n+1}K_*X, A)$$

$$\mathrm{Ext}(A, H_{n+1}K_*X) \overset{\Delta}{\rightarrowtail} H_n(A, K_*X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A, H_nK_*X).$$

Here the functors $E_0 = F_0$ and $E_1 = F_1$ are given by the homology groups

$$E_0(X) = H_n(K_* X), E_1(X) = H_{n+1}(K_* X).$$

(3.3.9) Theorem *The kype functor E with $E(X, A) = H^{n+1}(K_* X, A)$ and the bype functor F with $F(A, X) = H_n(A, K_* X)$ are dual to each other in the sense of* (3.3.6). *In particular, E is split if and only if F is split.*

Proof For the chain complex $K_* = K_* X$ let $Z_n = \text{kernel}(d\colon K_n \to K_{n-1})$ and $B_n = \text{image}(d\colon K_{n+1} \to K_n)$ so that the homology $H_n = H_n K_* X$ has a short free resolution

$$B_n \overset{i}{\rightarrowtail} Z_n \overset{q}{\twoheadrightarrow} H_n. \tag{1}$$

In fact, since we assume $K_* X$ to be free we see that also B_n and Z_n are free abelian groups. Moreover, the short exact sequence

$$H_{n+1} \overset{j}{\rightarrowtail} K_{n+1}/B_{n+1} \overset{d}{\twoheadrightarrow} B_n \tag{2}$$

admits a splitting s_X of d and a retraction r_X of j since B_n is free abelian. Putting (1) and (2) together we obtain the exact sequence

$$0 \to H_{n+1} \to K_{n+1}/B_{n+1} \xrightarrow{d_X} Z_n \to H_n \to 0. \tag{3}$$

Let A be an abelian group with short free resolution

$$A_1 \overset{d_A}{\rightarrowtail} A_0 \twoheadrightarrow A$$

and let $s^n d_A$ be the corresponding chain complex concentrated in degrees n and $n+1$. Moreover let $s^n A$ be in the chain complex concentrated in degree n with $(s^n A)_n = A$. Then we obtain

$$E(X, A) = H^{n+1}(K_* X, A) = [K_* X, s^{n+1} A] = [d_X, sA]$$

with

$$[d_X, sA,] = \text{Hom}(K_{n+1}/B_{n+1}, A)/d_X^* \text{Hom}(Z_n, A).$$

On the other hand we get

$$F(A, X) = H_n(A, K_* X) = [s^n d_A, K_* X] = [d_A, d_X].$$

Here d_X is the chain complex concentrated in degrees 0, 1 given by d_X in (3). Moreover $[-,-]$ denotes the group of homotopy classes of chain maps. We now define a splitting function

$$s_X^E\colon \text{Hom}(H_{n+1}, A) \to E(X, A), \quad \text{resp.} \quad s_X^F\colon \text{Hom}(A, H_n) \to F(A, X)$$

by use of r_X, resp. s_X, above. We set, for $\varphi\colon H_{n+1}\to A$,

$$s_X^E(\varphi)=\{\varphi r_X\}\in[d_X,s_A]. \tag{4}$$

Since i in (1) is a short free resolution of H_n we obtain the isomorphism

$$\mathrm{Hom}(A,H_n)=[d_A,i]$$

which carries the homomorphism φ to the chain map (φ_0,φ_1). Then s_X^F is given by

$$s_X^F(\varphi)=\{(s_X\varphi_0,\varphi_1)\}\in[d_A,d_X]. \tag{5}$$

One readily checks that s_X^F and s_X^F are both natural in A.

We now consider the derivation $d\colon \mathbf{C}\to \mathrm{Ext}(E_0,E_1)$ given by s_X^E, resp. s_X^F; compare Theorem 3.3.5 (3). Let $f\colon X\to Y\in\mathbf{C}$ and let

$$f_*: K_*X=K_*\to K_*Y=K_*'$$

be a chain map representing $K_*(f)$. We obtain by (1) the equation

$$\mathrm{Ext}(H_n,H'_{n+1})=\mathrm{Hom}(B_n,H'_{n+1})/i^*\,\mathrm{Hom}(Z_n,H'_{n+1}). \tag{6}$$

Moreover we obtain the following diagrams

$$\begin{tikzcd}
K_{n+1}/B_{n+1} \arrow[r, "r_X"] \arrow[d, "f_{n+1}"] & H_{n+1}=H_{n+1}K_*X \arrow[d, "(f_{n+1})_*"] \\
K'_{n+1}/B'_{n+1} \arrow[r, "r_Y"'] & H'_{n+1}=H_{n+1}K_*Y
\end{tikzcd}$$

$$\begin{tikzcd}
B_n \arrow[r, "s_X"] \arrow[d, "f_n"] & K_{n+1}/B_{n+1} \arrow[d, "f_{n+1}"] \\
B'_n \arrow[r, "s_Y"'] & K'_{n+1}/B'_{n+1}
\end{tikzcd}$$

These diagrams need not commute. They define, however, factorizations ∂_f, resp. δ_f, given by

$$(E_1f)r_X-r_Yf_{n+1}=\partial_f d\colon K_{n+1}/B_{n+1}\xrightarrow{d}B_n\xrightarrow{\partial_f}H'_{n+1} \tag{7}$$

$$-f_{n+1}s_X+s_Yf_n=j\delta_f\colon B_n\xrightarrow{\delta_f}H'_{n+1}\xrightarrow{j}K'_{n+1}/B'_{n+1}. \tag{8}$$

One readily checks that the derivation d_s defined in Theorem 3.3.5 (3) by $s=s_X^E$ satisfies

$$d_s(f)=\{\partial_f\}. \tag{9}$$

Similarly the derivation d'_s defined in Theorem 3.3.5 (12) by $s = s_X^F$ satisfies

$$d'_s(f) = \{\delta_f\}. \tag{10}$$

The right-hand sides of (9) and (10) denote the cosets in (6) given by (7) and (8) respectively. We claim that r_X can be chosen via s_X such that $\partial_f = \delta_f$. This implies that E and F are dual and the proof of Theorem 3.3.9 is complete. In fact, we can define r_X above by the formula

$$r_X(z) = j^{-1}(z - s_X d(z)) \tag{11}$$

where we use j and d in (2), $z \in K_{n+1}/B_{n+1}$. Now (11) implies $\partial_f = \delta_f$ since we get the following equations.

$$\begin{aligned} j\,\partial_f d(z) &= j((E_1 f)r_X(z) - r_Y f_{n+1}(z)) \\ &= jE_1(f)j^{-1}(z - s_X d(z)) - jj^{-1}(f_{n+1}(z) - s_Y\, df_{n+1}(z)) \\ &= f_{n+1}(z) - f_{n+1}s_X d(z) - f_{n+1}(z) + s_Y f_n\, d(z) \\ &= j\,\delta_f d(z). \end{aligned}$$

□

(3.3.10) Remark If E and F are dual to each other there should be a connection between the corresponding categories of E-kypes and F-bypes, respectively, which we do not know in general. The detecting functors Λ, Λ' in the classification theorem of the next section, for example, yield such a connection. Also if E and F are both split we describe in Section 3.6 the relation between E-kypes and F-bypes.

3.4 The classification theorem

We describe in this section fundamental results on the classification of homotopy types by using kype functors and bype functors as defined in Sections 3.1 and 3.2. These results are the main motivation to study such functors. Recall that $\mathbf{types}_m^r$ denotes the full homotopy category of $(m-1)$-connected $(m+r)$-types and that $\mathbf{spaces}_m^r$ denotes the full homotopy category of $(m-1)$-connected CW-spaces X with $\dim X \le m + r$. For any full subcategory $\mathbf{C} \subset \mathbf{types}_m^{r-1}$ let

$$\text{(3.4.1)} \qquad \begin{aligned} \mathbf{types}_m^r(\mathbf{C}) &\subset \mathbf{types}_m^r, \\ \mathbf{spaces}_m^{r+1}(\mathbf{C}) &\subset \mathbf{spaces}_m^{r+1} \end{aligned}$$

be the full subcategories of objects X for which the $(n-1)$-type $P_{n-1}X$ is an object in $\mathbf{C}$, $n = m + r$. If $\mathbf{C}$ is the whole category $\mathbf{types}_m^{r-1}$ then the inclusions in (3.4.1) are equations; in this case we can omit $\mathbf{C}$ in the notation. We always assume that $m \geq 2$. We now introduce a kype functor E and a bype functor F on $\mathbf{C}$ with the property $E_0 = F_0$ and $E_1 = F_1$. We obtain the kype functor

$$E: \mathbf{C}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab} \tag{3.4.2}$$

by the cohomology group $(n = m + r)$

$$E(X, \pi) = H^{n+1}(X, \pi) \tag{1}$$

for $X \in \mathbf{C}$, $\pi \in \mathbf{Ab}$. The kype structure of E is given by the universal coefficient sequence

$$\mathrm{Ext}(H_n X, \pi) \overset{\Delta}{\rightarrowtail} H^{n+1}(X, \pi) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H_{n+1} X, \pi) \tag{2}$$

where $E_0 X = H_n X$ and $E_1 X = H_{n+1} X$ are the homology groups. On the other hand, we define the bype functor

$$F: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C} \tag{3.4.3}$$

in two ways, by the functor Γ''_{n-1} or by the pseudo-homology,

$$F(H, X) = \Gamma''_{n-1}(H, X) \tag{1}$$

$$F(H, X) = H_n(H, X) \tag{2}$$

with $H \in \mathbf{Ab}$ and $X \in \mathbf{C}$. Recall that $\Gamma''_{n-1}(X) = \ker\ (i_{n-1} X) = \mathrm{image}(b_n X) \subset \Gamma_{n-1} X$ leads to the definition of $\Gamma''_{n-1}(H, X)$ by the pull-back diagram, see Definition 2.2.9,

$$\begin{array}{ccccc} \mathrm{Ext}(H, \Gamma_n X) & \overset{\Delta}{\rightarrowtail} & \Gamma_{n-1}(H, X) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(H, \Gamma_{n-1} X) \\ \| & & \uparrow & & \uparrow \\ \mathrm{Ext}(H, \Gamma_n X) & \rightarrowtail & \Gamma''_{n-1}(H, X) & \twoheadrightarrow & \mathrm{Hom}(H, \Gamma''_{n-1} X) \end{array} \tag{3}$$

Moreover we have by Theorem 2.6.14 (5) the commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}(H, H_{n+1} X) & \overset{\Delta}{\rightarrowtail} & H_n(H, X) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(H, H_n X) \\ \|\,\theta_* & & \|\,\theta & & \|\,\theta_* \\ \mathrm{Ext}(H, \Gamma_n X) & \overset{\Delta}{\rightarrowtail} & \Gamma''_{n-1}(H, X) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(H, \Gamma''_{n-1} X) \end{array} \tag{4}$$

Here Θ is a natural isomorphism. The short exact sequences of this diagram describe the bype structure of the bype functor F with

$$F_0 X = \Gamma''_{n-1} X = H_n X = E_0 X, \tag{5}$$

$$F_1 X = \Gamma_n X = H_{n+1} X = E_1 X. \tag{6}$$

Since $X \in \mathbf{C}$ is a 1-connected $(n-1)$-type the isomorphisms in (5) and (6) are given by the secondary boundary

$$\begin{aligned} b_{n+1}X&: H_{n+1}X \cong \Gamma_n X, \\ b_n X&: H_n X \cong \Gamma''_{n-1} X \subset \Gamma_{n-1} X \end{aligned} \tag{7}$$

which we use as identification.

Remark The kype functor E in (3.4.2) and the bype functor F in (3.4.3) are examples of functors as defined in Definition 3.3.8. In fact, for $\mathbf{C} \subset \mathbf{types}_m^{r-1}$ we have the functor

$$K_* : \mathbf{C} \to \mathbf{Chain}_{\mathbb{Z}}/\simeq$$

which carries an object X in $\mathbf{C}$ to its singular chain complex. Then clearly

$$E(X, \pi) = H^{n+1}(K_* X, \pi) = H^{n+1}(X, \pi)$$

$$F(H, X) = H_n(H, K_* X) = H_n(H, X).$$

Hence Theorem 3.3.9 implies that E and F are dual to each other, in particular E is split if and only if F is split. In fact, the duality can be used for the computation of the functor E as follows. First compute $\Gamma_{n-1}(H, X)$ in (3.4.3) (3) and then form the pull-back $\Gamma''_{n-1}(H, X)$ which, by (3.4.3) (2), yields $F(H, X)$ together with its functorial properties. Next use duality to derive from F the functor E. We shall apply this method in various examples.

We are now ready to state our main general result on the classification of homotopy types. For this recall that a *detecting functor* λ: $\mathbf{A} \to \mathbf{B}$ reflects isomorphisms, is full, and representative. In particular λ induces a 1–1 correspondence between isomorphism classes of objects in $\mathbf{A}$ and isomorphism classes of objects in $\mathbf{B}$. A 'good classification theorem' in algebraic topology can often be stated by saying that a certain functor λ: $\mathbf{A} \to \mathbf{B}$ is a detecting functor. Here $\mathbf{B}$ is supposed to be a category which is appropriate for computation and $\mathbf{A}$ is a more intricate topological category. The next result is crucial for this book; it was announced in Baues [HT].

(3.4.4) Classification theorem *Let $m \geq 2$ and let* $\mathbf{C}$ *be a full subcategory of* $\mathbf{types}_m^{r-1}$ *and let E and F be defined as in* (3.4.2) *and* (3.4.3). *Then there are detecting functors*

$$\Lambda: \mathbf{spaces}_m^{r+1}(\mathbf{C}) \to \mathbf{Kypes}(\mathbf{C}, E)$$

$$\Lambda': \mathbf{spaces}_m^{r+1}(\mathbf{C}) \to \mathbf{Bypes}(\mathbf{C}, F).$$

Moreover the Γ-sequences of both $\Lambda(X)$ and $\Lambda'(X)$ with $X \in \mathbf{spaces}_m^{r+1}(\mathbf{C})$ are naturally weakly isomorphic to the part

$$H_{n+1}X \to \Gamma_n X \to \pi_n X \to H_n X \to \ker(i_{n-1}X) \to 0$$

of Whitehead's exact sequence, $n = m + r$. In addition one has detecting functors

$$\lambda: \mathbf{types}_m^r(\mathbf{C}) \to \mathbf{kypes}(\mathbf{C}, E) = \mathbf{Gro}(E),$$

$$\lambda': \mathbf{types}_m^r(\mathbf{C}) \to \mathbf{bypes}(\mathbf{C}, F).$$

We give an explicit definition of the detecting functors as follows.

(3.4.5) Definition Recall that $m \geq 2$ and $n = m + r$. For the definition of Λ and λ we use the Postnikov invariants $k_n X$ and for the definition of Λ' and λ' we use the boundary invariants $\beta_n X$. Let

$$X \in \mathbf{spaces}_m^{r+1}(\mathbf{C}). \tag{1}$$

Then the $(n-1)$-type $P_{n-1}X$ of X is an object in $\mathbf{C}$ and we set

$$\Lambda(X) = (P_{n-1}X, \pi_n X, k_n X, H_{n+1}X, b_{n+1}X)$$

$$\Lambda'(X) = (P_{n-1}X, H_n X, H_{n+1}X, b_{n+1}X, \beta_n X).$$

Next let

$$X \in \mathbf{types}_m^r(\mathbf{C}). \tag{2}$$

Then we again have $P_{n-1}X \in \mathbf{C}$ and we set

$$\lambda(X) = (P_{n-1}X, \pi_n X, k_n X), \quad \text{see}(1.3)$$

$$\lambda'(X) = (P_{n-1}X, H_n X, H_{n+1}X, b_{n+1}X, \beta_n X).$$

Since X is a 1-connected n-type we see that

$$b_{n+1}X: H_{n+1}X \cong \Gamma_n'' X \subset \Gamma_n X = F_1(X)$$

is injective.

On the proof of Theorem 3.4.4 Only the detecting functor

$$\lambda\colon \mathbf{types}'_m(\mathbf{C}) \to \mathbf{Gro}(E)$$

is a classical result, due to Postnikov. The other detecting functors $\Lambda, \Lambda', \lambda'$ have not appeared in the literature. We prove that λ, λ', and Λ are detecting functors in Section 3.7. The proof that Λ' is a detecting functor is highly sophisticated; it involves most of the theory in the chapter of CW-towers, see Section 4.7. □

The functors of the theorem are part of the following diagram which commutes up to canonical natural isomorphisms.

$$\begin{tikzcd}
\mathbf{Kypes}(\mathbf{C}, E) \arrow[d, "\phi"] & \mathbf{spaces}^{r+1}_m(\mathbf{C}) \arrow[l, "\Lambda"'] \arrow[r, "\Lambda'"] \arrow[d, "P_{m+r}"] & \mathbf{Bypes}(\mathbf{C}, F) \arrow[d, "\phi"] \\
\mathbf{kypes}(\mathbf{C}, E) & \mathbf{types}'_m(\mathbf{C}) \arrow[l, "\lambda"'] \arrow[r, "\lambda'"] & \mathbf{bypes}(\mathbf{C}, F)
\end{tikzcd} \tag{3.4.6}$$

Here P_{m+r} is the Postnikov functor. All functors in the diagram are full and representative. But ϕ and P_{m+r} are not detecting functors since they do not reflect isomorphisms. We deduce from Theorem 3.4.4 the next result on the realizability of Γ-sequences. Recall that an *n-equivalence* $X \to Y$ is a map which induces isomorphisms $\pi_i X \cong \pi_i Y$ for $i \le n$.

(3.4.7) Theorem on the realizability of the Hurewicz homomorphism *Let Y be a simply connected $(n-1)$-type and let*

$$H_1 \xrightarrow{b_1} \Gamma_n Y \to \pi \to H_0 \xrightarrow{b_0} \ker(i_{n-1}Y) \to 0$$

be an exact sequence of abelian groups where H_1 is free abelian and where $\Gamma_n Y$ and $i_{n-1}Y\colon \Gamma_{n-1}Y \to \pi_{n-1}Y$ are given by Y. Then there exists an $(n+1)$-dimensional CW-complex X, an $(n-1)$-equivalence $p\colon X \to Y$, and a commutative diagram

$$\begin{tikzcd}
H_{n+1}X \arrow[r] \arrow[d, "\cong"', "\varphi_1"] & \Gamma_n X \arrow[r] \arrow[d, "\cong"', "p_*"] & \pi_n X \arrow[r] \arrow[d, "\cong"', "\psi"] & H_n X \arrow[r] \arrow[d, "\cong"', "\varphi_0"] & \ker(i_{n-1}X) \arrow[r] \arrow[d, "\cong"', "p_*"] & 0 \\
H_1 \arrow[r] & \Gamma_n Y \arrow[r] & \pi \arrow[r] & H_0 \arrow[r] & \ker(i_{n-1}Y) \arrow[r] & 0
\end{tikzcd}$$

in which all vertical arrows are isomorphism.

We can prove Theorem 3.4.7 either by the detecting factor Λ or by the detecting functor Λ' in Theorem 3.4.4. In the following proof we use Λ'.

Proof Let the bype functor F be defined as in Theorem 3.4.4 with $\mathbf{C} = \mathbf{types}_2^r$, $r+2 = n-1$. Then we find an F-bype

$$\bar{Y} = (Y, H_0, H_1, b, \beta).$$

Here H_0 and H_1 are given by the exact sequence and K is the kernel of $\Gamma_n Y \to \pi$ and b is given by b_1. Let $\{\pi\}$ be the element

$$\{\pi\} \in \mathrm{Ext}(\ker(b_0), \Gamma_n Y/K)$$

determined by the extension $\Gamma_n Y/K \rightarrowtail \pi \to \ker(b_0)$ which we deduce from the exact sequence in the theorem. Then there exists an element β with

$$\beta_* = b_0 \quad \text{and} \quad \beta_\dagger = \{\pi\}.$$

This follows since μ in (3.2.5) is surjective and since for the inclusion i: $\ker(b_0) \subset H_0$ also the induced map i^*: $\mathrm{Ext}(H_0, F_1X/K) \to \mathrm{Ext}(\ker(b_0), F_1X/K)$ in (3.2.5) (2) is surjective. Hence the Γ-sequence of $\bar{Y}$ is weakly isomorphic to the exact sequence (3.4.7). Now let X be a Λ'-realization of $\bar{Y}$. Thus we obtain the proposition by the natural isomorphism of Γ-sequences in Theorem 3.4.4. □

(3.4.8) Remark Theorem 3.4.7 shows that each exact sequence

$$H_1 \to \Gamma_n Y \to \pi \to H_0 \to \ker i_{n-1} Y \to 0$$

is realizable. We do not know however what morphisms between such sequences are realizable. More precisely let X, X' be 1-connected $(n+1)$-dimensional CW-complexes with $(n-1)$-types $Y = P_{n-1}X$ and $Y' = P_{n-1}X'$ respectively. Then we consider the commutative diagram in $\mathbf{Ab}$

$$\begin{tikzcd}
H_1 \arrow[r] \arrow[d, "\varphi_1"] & \Gamma_n Y \arrow[r] \arrow[d, "f_*"] & \pi \arrow[r] \arrow[d, "\varphi_\pi"] & H_0 \arrow[r] \arrow[d, "\varphi_0"] & \ker i_{n-1} Y \arrow[r] \arrow[d, "f_*"] & 0 \\
H_1' \arrow[r] & \Gamma_n Y' \arrow[r] & \pi' \arrow[r] & H_0' \arrow[r] & \ker i_{n-1} Y' \arrow[r] & 0
\end{tikzcd} \tag{1}$$

where $f \in [Y, Y']$ and where the top row and the bottom row are Whitehead's exact sequence for X and X' respectively. The detecting functor Λ shows that $(f, \varphi_\pi, \varphi_1)$ is realizable by a map $\bar{f}$: $X \to X'$ if and only if

$$f^*(k_n X) = (\varphi_\pi)_*(k_n X'). \tag{2}$$

On the other hand, the detecting functor Λ' in Theorem 3.4.4 shows that $(f, \varphi_0, \varphi_1)$ is realizable by a map $X \to X'$ if and only if

$$f_*(\beta_n X) = \varphi_0^*(\beta_n X'). \tag{3}$$

What is the condition that $(f, \varphi_0, \varphi_1, \varphi_\pi)$ is realizable by a map $X \to X'$? Clearly (2) and (3) must be satisfied but is this a sufficient condition? Moreover what pairs of invariants (β_n, k_n) are realizable? Hence we are searching for an unknown category $\mathbf{U}$ for which the diagram of detecting functors

$$\begin{tikzcd} & \mathbf{spaces}_m^{r+1}(\mathbf{C}) \arrow[dl, "\Lambda"'] \arrow[d] \arrow[dr, "\Lambda'"] & \\ \mathbf{Kypes}(\mathbf{C}, E) & \mathbf{U} \arrow[l] \arrow[r] & \mathbf{Bypes}(\mathbf{C}, E) \end{tikzcd} \tag{4}$$

commutes and for which $\mathbf{U}$ is given by an algebraic structure like E, F on $\mathbf{C}$. This is the *unification problem*. Since Λ and Λ' are detecting functors we see that they induce a 1–1 correspondence between isomorphism classes of objects in $\mathbf{Kypes}(\mathbf{C}, E)$ and $\mathbf{Bypes}(\mathbf{C}, F)$ respectively. In this sense the k-invariant $k_n(X)$ determines the boundary invariant $\beta_n(X)$ and vice versa, but it is unclear how this connection between $k_n(X)$ and $\beta_n(X)$ could be described algebraically. Below we show that $k_n X$ is 'orthogonal' to $\beta_n(X)$. Moreover, in the case that E and F are split, we describe a possible candidate for the category $\mathbf{U}$ in Definition 3.6.1 (6), namely $\mathbf{U} = \mathbf{S}(E_0, E_1)$.

(3.4.9) Definition Let X be a space and let $C_* X$ be the singular chain complex of X. Then we know that the cohomology and pseudo-homology can be described by sets of homotopy classes of chain maps

$$H^{n+1}(X, A) = [C_* X, C_* M(A, n+1)]$$

$$H_n(B, X) = [C_* M(B, n), C_* X].$$

Hence we get by composition of chain maps a pairing

$$H_n(B, X) \otimes H^{n+1}(X, A) \to \mathrm{Ext}(B, A)$$

which carries $\beta \otimes k$ to $\langle \beta, k \rangle = k \circ \beta$ in

$$[C_* M(B, n), C_* M(A, n+1)] = \mathrm{Ext}(B, A).$$

(3.4.10) Proposition *Let X be a simply connected CW-space with $(n-1)$-type $p_{n-1}\colon X \to Y = P_{n-1} X$ and let*

$$k_n X \in H^{n+1}(Y, \pi_n X), \quad \text{resp.} \quad \beta_n X \in H_n(H_n X, Y)/\Delta\, \mathrm{image}\,(p_{n-1})_*,$$

be the k-invariant, resp. boundary invariants, of X. Then $k_n X$ and $\beta_n X$ are orthogonal with respect to the pairing $\langle , \rangle$ in Definition 3.4.9, that is

$$\langle \beta, k_n X \rangle = 0$$

for all $\beta \in H_n(H_n X, Y)$ representing $\beta_n X$.

Proof Let $H = H_n X$ and $\pi = \pi_n X$ and consider the composite

$$k_n p_{n-1}: X \to Y \to K(\pi, n+1)$$

where k_n is given by $k_n(X)$. The Postnikov tower shows that $k_n p_{n-1} \simeq 0$ is null homotopic. We now obtain the following commutative diagram

$$\begin{array}{ccccc} H_n(H,X) & \xrightarrow{b_n} & \Gamma''_{n-1}(H,X) & \longrightarrow & \mathrm{Ext}(H,\pi) \\ \| & & \| & & \| \\ H_n(H,X) & \xrightarrow{(p_{n-1})_*} & H_n(H,Y) & \xrightarrow{(k_n)_*} & H_n(H,K(\pi,n+1)) \end{array}$$

Since $\beta \in \beta_n X$ is in the image of b_n and $(p_{n-1})_*$ respectively, we see that

$$(k_n)_* \beta = 0$$

since $k_n p_{n-1} \simeq 0$. Here $(k_n)_* \beta$ represents $\langle \beta, k_n X \rangle$. □

3.5 The semitrivial case of the classification theorem and Whitehead's classification

In this section we describe the homotopy classification of $(m-1)$-connected $(m+2)$-dimensional CW-spaces by simple algebraic invariants. This corresponds to well-known results of J.H.C. Whitehead [CE], [SC], [HT]. We obtain the classification by applying the classification theorem 3.4.4 to the simple case $r = 1$; in fact, for $r = 1$ only semitrivial kype functors and semitrivial bype functors are relevant. Recall that for an Eilenberg–Mac Lane space $K(A, m)$, $m \geq 2$, one has natural isomorphisms

$$(3.5.1) \qquad \begin{cases} H_{m+2}K(A,m) = \Gamma_{m+1}K(A,m) = \Gamma^1_m(A) \\ H_{m+1}K(A,m) = \Gamma_m K(A,m) = 0. \end{cases}$$

Here Γ^1_m: **Ab** $\to$ **Ab** is the algebraic functor with $\Gamma^1_2(A) = \Gamma(A)$ (given by Whitehead's quadratic functor Γ) and with $\Gamma^1_m(A) = A \otimes \mathbb{Z}/2$ for $m \geq 3$. For an $(m-1)$-connected space X we have a natural isomorphism (see Theorem 2.1.22)

$$(3.5.2) \qquad \Gamma_{m+1}X \cong \Gamma^1_m(H_m X)$$

so that Whiteheads sequence yields the exact sequence

$$(3.5.3) \qquad H_{m+2}X \xrightarrow{b} \Gamma^1_m(H_m X) \xrightarrow{i} \pi_{m+1}X \xrightarrow{h} H_{m+1}X \longrightarrow 0$$

which is natural for maps between $(m-1)$-connected spaces. Here $H_{m+2}X$

is free abelian if $\dim X \le m+2$. We now introduce the following categories with objects being exact sequences as in (3.5.3).

(3.5.4) Definition Let $\mathbf{C}$ be a category and let $G\colon \mathbf{C} \to \mathbf{Ab}$ be a functor. A *G-sequence* is an object H in $\mathbf{C}$ together with an exact sequence of abelian groups

$$H_1 \to G(H) \to \pi \to H_0 \to 0. \tag{1}$$

A *morphism* between G-sequences is a morphism $\varphi\colon H \to H'$ in $\mathbf{C}$ together with a commutative diagram

$$\begin{array}{ccccccccc} H_1 & \longrightarrow & G(H) & \longrightarrow & \pi & \longrightarrow & H_0 & \longrightarrow & 0 \\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi_0} & & \\ H_1' & \longrightarrow & G(H') & \longrightarrow & \pi' & \longrightarrow & H_0' & \longrightarrow & 0 \end{array} \tag{2}$$

A *weak morphism* between G-sequences is a triple $(\varphi_1, \varphi_0, \varphi)$ for which there exists ψ such that the diagram commutes. The G-sequence (1) is *free* if H_1 is free abelian and is *injective* if $H_1 \to G(H)$ is injective. Let

$$\mathbf{K}(G), \quad \text{resp.} \quad \mathbf{k}(G) \tag{3}$$

be the categories consisting of free, resp. injective G-sequences and morphisms as above. Clearly $\mathbf{k}(G) = \mathbf{Gro}(\mathrm{Hom}(G,-))$ is the Grothendieck construction of the bifunctor

$$\mathrm{Hom}(G,-)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

which carries (H, π) to the abelian group of homomorphisms $\mathrm{Hom}(G(H), \pi)$. Moreover let

$$\mathbf{B}(G), \quad \text{resp.} \quad \mathbf{b}(G) \tag{4}$$

be the categories consisting of free, resp. injective G-sequences and weak morphisms. The proof of the next lemma is left as an exercise.

(3.5.5) Lemma *Let E be a semitrivial kype functor with $E_0 = 0$ and $E_1 = G$. Then one has canonical equivalence of categories*

$$\mathbf{Kypes}(\mathbf{C}, E) = \mathbf{K}(G),$$

$$\mathbf{kypes}(\mathbf{C}, E) = \mathbf{k}(G).$$

Let F be a semitrivial bype functor with $F_0 = 0$ and $F_1 = G$. Then one has canonical equivalences of categories

$$\mathbf{Bypes}(\mathbf{C}, F) = \mathbf{B}(G),$$

$$\mathbf{bypes}(\mathbf{C}, F) = \mathbf{b}(G).$$

In the next result we consider the case when $G = \Gamma_m^1$: **Ab** $\to$ **Ab** is the functor given by (3.5.1). Recall that $\mathbf{spaces}_m^2$ is the full homotopy category of $(m-1)$-connected $(m+2)$-dimensional CW-spaces.

(3.5.6) Theorem of J.H.C. Whitehead *For $m \geq 2$ one has detecting functors*

$$\Lambda\colon \mathbf{spaces}_m^2 \to \mathbf{K}(\Gamma_m^1)$$

$$\Lambda'\colon \mathbf{spaces}_m^2 \to \mathbf{B}(\Gamma_m^1)$$

These functors carry a space X to the exact sequence in (3.5.3).

Proof Consider Theorem 3.4.4 for the special case $r = 1$ and $\mathbf{C} = \mathbf{types}_m^0$. Then one has an equivalence of categories $\mathbf{C} = \mathbf{Ab}$ and by (3.4.2) and (3.5.1) the kype functor E on **Ab** is given by

$$E(A, \pi) = \mathrm{Hom}(\Gamma_m^1(A), \pi) \tag{1}$$

with $E_0 = 0$ and $E_1 = \Gamma_m^1$. Hence one has, by Lemma 3.5.5, an equivalence of categories

$$\mathbf{Kypes}(\mathbf{C}, E) = \mathbf{K}(\Gamma_m^1) \tag{2}$$

and the detecting functor Λ in Definition 3.4.5 corresponds to the functor Λ in Lemma 3.5.5.

We now consider the detecting functor Λ' in Theorem 3.4.4 for the special case $r = 1$ with $\mathbf{C}$ as above. Then we have by (3.5.1) and (3.4.3) the bype functor F on **Ab** given by

$$F(H, A) = \mathrm{Ext}(H, \Gamma_m^1(A)) \tag{3}$$

with $F_0 = 0$ and $F_1 = \Gamma_m^1$. Moreover we have by Lemma 3.5.5 an equivalence of categories

$$\mathbf{Bypes}(\mathbf{C}, F) = \mathbf{B}(\Gamma_m^1). \tag{4}$$

Thus the detecting functor Λ' in Definition 3.4.5 yields the functor Λ' in Lemma 3.5.5. □

(3.5.7) Remark One has a forgetful functor

$$\phi\colon \mathbf{K}(G) \to \mathbf{B}(G)$$

such that for $G = \Gamma_m^1$ the functors in Lemma 3.5.5 satisfy $\phi\Lambda = \Lambda'$. Since ϕ and Λ are both detecting functors this also shows that Λ' is a detecting functor.

Recall that $\mathbf{types}_m^1$ is the full homotopy category of $(m-1)$-connected $(m+1)$-types.

(3.5.8) Theorem *For $m \geq 2$ one has detecting functors*

$$\lambda\colon \mathbf{types}_m^1 \to \mathbf{k}(\Gamma_m^1) = \mathbf{Gro}\big(\mathrm{Hom}(\Gamma_m^1, -)\big)$$

$$\lambda'\colon \mathbf{types}_m^1 \to \mathbf{b}(\Gamma_m^1).$$

These functors carry $X \in \mathbf{types}_m^1$ to the exact sequence (3.5.3).

Proof For the kype functor E in Theorem 3.5.6 (1) we have by Lemma 3.5.5

$$\mathbf{kypes}(\mathbf{C}, E) = \mathbf{k}(\Gamma_m^1)$$

and for the bype functor F in Theorem 3.5.6 (3) we have an equivalence of categories

$$\mathbf{bypes}(\mathbf{C}, F) = \mathbf{b}(\Gamma_m^1)$$

Hence Theorem 3.5.9 is a consequence of the classification result Theorem 3.4.4. □

3.6 The split case of the classification theorem

We here discuss the classification theorem 3.4.4 in case the bype functor F and the kype functor E are split with $E_0 = F_0$, $E_1 = F_1$. In this case E-kypes and F-bypes can both be described by chain complexes

$$H_1 \to E_1(X) \to R \to E_0(X) \to 0$$

which are exact at $E_1(X)$ and $E_0(X)$. This simplifies the description of the corresponding categories of E-kypes and F-bypes considerably. Hence for split functors E, F we get a new kind of classification theorem derived from Theorem 3.4.4. As an illustration we consider the homotopy classification of $(m-1)$-connected $(m+3)$-dimensional polyhedra X with trivial homotopy groups $\pi_{m+1}X = 0$, $m \geq 4$. In this case the bype and kype functors are split and are of a particularly easy form. Various other examples of split bype and kype functors are described in Chapter 6.

(3.6.1) Definition Let $\mathbf{C}$ be a category and let $E_0, E_1\colon \mathbf{C} \to \mathbf{Ab}$ be two functors. An *(E_0, E_1)-sequence* is an object X in $\mathbf{C}$ together with a chain complex

$$H_1 \xrightarrow{b} E_1(X) \xrightarrow{\partial} R \xrightarrow{\delta} E_0(X) \longrightarrow 0 \tag{1}$$

of abelian groups which is exact in $E_1(X)$ and $E_0(X)$. A *morphism* between such sequences is given by a morphism $f\colon X \to X'$ in **C** and by a commutative diagram in **Ab**

$$\begin{array}{ccccccccc} H_1 & \xrightarrow{b} & E_1(X) & \xrightarrow{\partial} & R & \xrightarrow{\delta} & E_0(X) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle f_*} & & \\ H_1' & \xrightarrow{b'} & E_1(X') & \xrightarrow{\partial'} & R' & \xrightarrow{\delta'} & E_0(X') & \longrightarrow & 0 \end{array} \tag{2}$$

The (E_0, E_1)-sequences and such morphisms form a well-defined category. We say that the sequence (1) is *free* if H_1 is free abelian and we say that (1) is *injective* if b is injective. Let

$$\mathbf{S}(E_0, E_1), \quad \text{resp.} \quad \mathbf{s}(E_0, E_1) \tag{3}$$

be the full subcategories of free, resp. injective (E_0, E_1)-sequences. We introduce two natural equivalence relations $\overset{k}{\sim}$ and $\overset{b}{\sim}$ as follows. Let (φ_1, f, r) and (φ_1', f', r') be morphisms as in (2). We set $(\varphi_1, f, r) \overset{k}{\sim} (\varphi_1', f', r')$ if $\varphi_1 = \varphi_1'$, $f = f'$ and if r and r' induce the same homomorphism.

$$r_* = r_*'\colon \text{kernel}(\delta) \to \text{kernel}(\delta'). \tag{4}$$

On the other hand, we set $(\varphi_1, f, r) \overset{b}{\sim} (\varphi_1' f', r')$ if $\varphi_1 = \varphi_1'$, $f = f'$ and if r and r' induce the same homomorphism

$$r_* = r_*'\colon \text{cokernel}(\partial) \to \text{cokernel}(\partial'). \tag{5}$$

One has the obvious quotient functors

$$\mathbf{S}(E_0, E_1)/\overset{b}{\sim} \;\leftarrow \mathbf{S}(E_0, E_1) \to \mathbf{S}(E_0, E_1)/\overset{k}{\sim} \tag{6}$$

$$\mathbf{s}(E_0, E_1)/\overset{b}{\sim} \;\leftarrow \mathbf{s}(E_0, E_1) \to \mathbf{s}(E_0, E_1)/\overset{k}{\sim} \tag{7}$$

all of which are easily seen to be detecting functors, in fact linear extensions of categories. We associate with an object (1) the following exact sequence

$$\begin{array}{ccccccccccc} H_1 & \xrightarrow{b} & E_1(X) & \xrightarrow{i} & \ker(\delta) & \xrightarrow{j} & \operatorname{cok}(\partial) & \xrightarrow{b_0} & E_0(X) & \longrightarrow & 0 \\ & & & & \| & & \| & & & & \\ & & & & \pi & & H_0 & & & & \end{array} \tag{8}$$

Here b_0 is induced by δ and i is induced by ∂. Moreover j is the composite

$$j\colon \ker(\delta) \subset R \twoheadrightarrow \operatorname{cok}(\partial)$$

of the inclusion and the quotient map. Morphisms between(E_0, E_1)-sequences clearly induce morphisms between the corresponding exact sequences in (8) which we call *Γ-sequences*.

(3.6.2) Lemma *Let E be a split kype functor given by E_0, E_1 and let F be a split bype functor given by F_0, F_1; see Definitions* 3.1.1 *and* 3.1.2. *Then there are canonical equivalences of categories*

$$\tau\colon \mathbf{S}(E_0,E_1)/\overset{k}{\sim} \;\to \mathbf{Kypes}(\mathbf{C},E)$$

$$\tau\colon \mathbf{s}(E_0,E_1)/\overset{k}{\sim} \;\to \mathbf{kypes}(\mathbf{C},E)$$

$$\tau'\colon \mathbf{S}(F_0,F_1)/\overset{b}{\sim} \;\to \mathbf{Bypes}(\mathbf{C},F)$$

$$\tau'\colon \mathbf{s}(F_0,F_1)/\overset{b}{\sim} \;\to \mathbf{bypes}(\mathbf{C},F).$$

Proof The equivalence τ carries the (E_0, E_1)-sequence

$$S = \{H_1 \xrightarrow{b} E_1(X) \xrightarrow{\partial} R \xrightarrow{\delta} E_0(X) \longrightarrow 0\}$$

to the kype

$$\tau(S) = (X,\pi,k,H_1,b) \quad \text{with} \quad \pi = \ker(\delta)$$

where $k \in E(X,\pi) = \mathrm{Ext}(E_0X,\pi) \times \mathrm{Hom}(E_1X,\pi)$ is given by $i\colon E_1(X) \to \pi$ in Definition 3.6.1 (8) and by the extension

$$0 \to \pi \to R \to E_0X \to 0$$

given by δ. Now one readily checks that τ in the statement of the lemma is an equivalence of categories.

Next let S' be the (F_0, F_1)-sequence

$$S' = \{H_1 \xrightarrow{b} F_1(X) \xrightarrow{\partial} R \xrightarrow{\delta} F_0(X) \longrightarrow 0\}.$$

Then τ' carries S' to the bype

$$\tau'(S') = (X,H_0,H_1,b,\beta) \quad \text{with} \quad H_0 = \mathrm{cok}(\partial).$$

Here $\beta \in F(H_0,X,b) = \mathrm{Ext}(H_0,\mathrm{cok}\, b) \times \mathrm{Hom}(H_0,F_0X)$ is given by $b_0\colon H_0 \to F_0X$ as in Definition 3.6.1 (8) and by the extension

$$0 \to \mathrm{cok}(b) \xrightarrow{\partial'} R \to H_0 \to 0$$

where ∂' is induced by ∂. Again one readily checks that τ' in the statement of the lemma is an equivalence of categories. □

We are now ready to formulate an addendum to the classification theorem 3.4.4 which deals with the case when the bype and kype functors are split.

(3.6.3) Classification theorem *Let $m \geq 2$ and let* $\mathbf{C}$ *be a full subcategory of* $\mathbf{types}_m^{r-1}$ *and let E and F be defined as in* (3.4.2) *and* (3.4.3). *Then the kype functor E on* $\mathbf{C}$ *is split if and only if the bype functor F on* $\mathbf{C}$ *is split. If E and F are split we obtain, with the homology functors* $(n = m + r)$

$$H_n, H_{n+1}: \mathbf{C} \to \mathbf{Ab},$$

the following detecting functors:

$$\Lambda: \mathbf{spaces}_m^{r+1}(\mathbf{C}) \to \mathbf{S}(H_n, H_{n+1})/\overset{k}{\sim}$$

$$\Lambda': \mathbf{spaces}_m^{r+1}(\mathbf{C}) \to \mathbf{S}(H_n, H_{n+1})/\overset{b}{\sim}$$

$$\lambda: \mathbf{types}_m^{r}(\mathbf{C}) \to \mathbf{s}(H_n, H_{n+1})/\overset{k}{\sim}$$

$$\lambda': \mathbf{types}_m^{r}(\mathbf{C}) \to \mathbf{s}(H_n, H_{n+1})/\overset{b}{\sim}\,.$$

Moreover the Γ-sequences of $\Lambda(X)$ or $\Lambda'(X)$ given by Definition 3.6.1 (8) *are natural weakly isomorphic to the part*

$$H_{n+1}X \to \Gamma_n X \to \pi_n X \to H_n X \to \ker(i_{n-1}X) \to 0$$

of Whitehead's exact sequences. Here we use $\Gamma_n X = H_{n+1}P_{n-1}X$ *and* $\ker(i_{n-1}X) = H_n P_{n-1}X$.

Proof We apply the remark below (3.4.3)(7). Hence the theorem is a consequence of Lemma 3.6.2 and the classification theorem 3.4.4. □

Remark We do not see that there is a functor

$$\mathbf{spaces}_m^{r+1}(\mathbf{C}) \xrightarrow{?} \mathbf{S}(E_0, E_1)$$

which induces the functors Λ and Λ'. Theorem 3.6.3, however, suggests that there might be such a functor in case E and F are split.

We now consider an example for the classification theorem 3.6.3. For any abelian group A we have the exact sequence

$$0 \to A * \mathbb{Z}/2 \to A \xrightarrow{2} A \to A \otimes \mathbb{Z}/2 \to 0 \tag{3.6.4}$$

where $A * \mathbb{Z}/2$ is the 2-torsion of A and where $A/2A = A \otimes \mathbb{Z}/2$ is the tensor product of A with $\mathbb{Z}/2$. Thus we obtain functors $*\mathbb{Z}/2, \otimes\mathbb{Z}/2$: $\mathbf{Ab} \to \mathbf{Ab}$ which carry A to $A * \mathbb{Z}/2$ and $A \otimes \mathbb{Z}/2$ respectively. The next result is an application of the classification theorem 3.6.3. Let

$$\mathbf{spaces}(m, m+2)_\pi \subset \mathbf{spaces}_m^3$$

be the full homotopy category of $(m-1)$-connected $(m+3)$-dimensional CW-spaces X with $\pi_{m+1}X = 0$.

(3.6.5) Theorem *Let $m \geq 4$. Then there are detecting functors*

$$\mathbf{spaces}(m, m+2)_\pi \xrightarrow{\Lambda} \mathbf{S}(\otimes \mathbb{Z}/2, * \mathbb{Z}/2)/\overset{k}{\sim}$$

$$\mathbf{spaces}(m, m+2)_\pi \xrightarrow{\Lambda'} \mathbf{S}(\otimes \mathbb{Z}/2, * \mathbb{Z}/2)/\overset{b}{\sim}\ .$$

Moreover the Γ-sequences of $\Lambda(X)$ and $\Lambda'(X)$ with $A = \pi_m X = H_m X$ are natural weakly isomorphic to the part

$$H_{m+3}X \to \Gamma_{m+2}X \to \pi_{m+2}X \to H_{m+2}X \to \Gamma_{m+1}X \to 0$$

of Whitehead's exact sequence.

Theorem 3.6.5 implies by Definition 3.6.1 (6) that there is a 1–1 correspondence between $(m-1)$-connected $(m+3)$-dimensional homotopy types X with $\pi_{m+1}X = 0$ and $m \geq 4$ and isomorphism classes of objects

$$H_1 \to A * \mathbb{Z}/2 \to R \to A \otimes \mathbb{Z}/2 \to 0$$

in $\mathbf{S}(\otimes\mathbb{Z}/2, *\mathbb{Z}/2)$; see Definition 3.6.1. This is indeed a simple description of such homotopy types.

Proof of Theorem 3.6.5 We consider the case $r = 2$ in the classification theorem 3.6.3 where we set

$$\mathbf{C} = \mathbf{types}_m^0 = \mathbf{Ab}$$

Eilenberg and Mac Lane [II] show that the kype functor E is split with

$$E_0 X = H_{m+2}K(A, m) = A \otimes \mathbb{Z}/2$$

$$E_1 X = H_{m+3}K(A, m) = A * \mathbb{Z}/2$$

for $X = K(A, m) \in \mathbf{C}$. Hence, by Theorem 3.3.9, also the bype functor F is split with $F_0 = E_0$ and $F_1 = E_1$. Below we shall prove that the bype functor F is split independently of Theorem 3.3.9. Now the application of Theorem 3.6.3 completes the proof. □

3.7 Proof of the classification theorem

We assume that $m \geq 2$, $n = m + r$, and

(3.7.1) $$\mathbf{C} = \mathbf{types}_m^{r-1}.$$

The kype functor E and the bype functor F on $\mathbf{C}$ are given by $E(X,\pi) = H^{n+1}(X,\pi)$ and $F(H,X) = \Gamma''_{n-1}(H,X)$ with $X \in \mathbf{C}, \pi, H \in \mathbf{Ab}$; see (3.4.2) and (3.4.3). We first show the classical result of Postnikov:

(3.7.2) Proposition *The functor*

$$\lambda\colon \mathbf{types}^r_m \to \mathbf{Gro}(E)$$

$$\lambda(X) = (P_{n-1}X, \pi_n X, k_n X)$$

is a detecting functor.

Proof The functor λ reflects isomorphisms by the Whitehead theorem. Moreover an object (Y,π,k) in $\mathbf{Gro}(E)$ with $k \in E(Y,\pi)$ is λ-realizable by choosing X with $\lambda(X) \cong (Y,\pi,k)$ as follows. Let $X = P_k$ be the fibre of a map $k\colon Y \to K(\pi, n+1)$ which represents the cohomology class k. Each morphism $(f,\varphi)\colon \lambda(X) \to \lambda(X')$ is λ-realizable since the diagram

$$\begin{array}{ccc} Y & \xrightarrow{k} & K(\pi, n+1) \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \varphi} \\ Y' & \xrightarrow{k'} & K(\pi', n+1) \end{array}$$

homotopy commutes by condition (3) of Remark 3.1.3. Hence there is an associated principal map

$$X \simeq P_k \to P_{k'} \simeq X'$$

which realizes (f,φ), see (V. §6) in Baues [AH]. □

Next we consider the functor Λ in Theorem 3.4.4.

(3.7.3) Proposition *The functor*

$$\Lambda\colon \mathbf{spaces}^{r+1}_m \to \mathbf{Kypes}(\mathbf{C}, E)$$

$$\Lambda(X) = (P_{n+1}X, \pi_n X, H_{n+1}X, b_{n+1}X)$$

is a detecting functor.

Here we use $(P_{n-1}X, \pi_n X, k_n X) \in \mathbf{Gro}(E)$ as in Proposition 3.7.2 and

$$b_{n+1}X\colon H_{n+1}X \to \Gamma_n X = H_{n+1}P_{n-1}X$$

is given by the secondary boundary operator and the isomorphism θ. By Theorem 2.5.10 (c) we see that b_{n+1} surjects to the kernel of $\mu(k_n X) = (k_n X)_* = i_n X$. Hence naturality of $k_n X$ shows that the functor Λ in Proposition 3.7.3 is well defined.

Proof of Proposition 3.7.3 It is clear by the Whitehead theorem that Λ reflects isomorphisms. We now show that Λ' is representative, that is, each kype $(Y, \pi, k, H, b) = \bar{Y}$ has a Λ-realization X with $\Lambda'(X) \cong \bar{Y}$. For the construction of X we first choose an n-type U with $\lambda(U) \cong (Y, \pi, k)$; we can do this by Proposition 3.7.2. Here we may assume that U is a CW-complex. We now construct X together with a map $X \to U$ which induces isomorphisms of homotopy groups π_i for $i \geq n$ so that $\lambda X = \lambda U$. For the cellular chains $C_* U$ and the skeleton U^n of U we obtain the following commutative diagram with exact columns

$$\begin{tikzcd}
H \arrow[r, two heads, "b"] \arrow[d, "s"] & \ker(k_*) \arrow[r, phantom, "\subset"] \arrow[d, tail] & \Gamma_n Y \arrow[r, "k_*"] \arrow[d, tail] & \pi \arrow[d, equal] \\
C_{n+1}U \arrow[r, two heads, "f"] & \ker(i_*) \arrow[r, phantom, "\subset"] \arrow[d, two heads] & \pi_n U^n \arrow[r, "i_*"] \arrow[d] & \pi_n U^{n+1} \\
 & K \arrow[r, phantom, "\subset"] & C_n U &
\end{tikzcd}$$

Here $\Gamma_n Y = \Gamma_n U$ is a subgroup of $\pi_n U^n$ and $\mu(k) = k_*$ is a restriction of i_* by Theorem 2.5.10 (b). This shows that the quotient $K = \ker(i_*)/\ker(k_*)$ injects into the free abelian group $C_n U$ and hence K is free abelian. Therefore we can choose a splitting t of the surjection $C_{n+1}U \twoheadrightarrow K$ given by the attaching map f of $(n+1)$-cells in U. We define the CW-complex $X = X^{n+1}$ by the n-skeleton $X^n = U^n$ and by the attaching map of $(n+1)$-cells $g\colon C_{n+1}X \to \pi_n X^n$. Here $C_{n+1}X$ is the free abelian group $H \oplus K$ and g is the composite

$$g\colon C_{n+1}X = H \oplus K \xrightarrow{(s,t)} C_{n+1}U \xrightarrow{f} \pi_n U^n \tag{2}$$

where $s\colon H \to C_{n+1}U$ is any homomorphism for which diagram (1) commutes. Since $g = f \circ (s, t)$ we obtain a map $X \to U$ which is the identity on the n-skeleton and which induces (s, t) on cellular chains C_{n+1}. Since by construction of g

$$\text{image}(g) = \ker(i_*) \tag{3}$$

we get $\pi_n X = \pi_n U = \pi$. Moreover the construction of g shows $H_{n+1}X = H$ and $b_{n+1}X = b$ so that X in fact is a Λ-realization of $\bar{Y}$ above.

It remains to show that the functor Λ is full. For this let X, Y be CW-complexes in $\mathbf{spaces}_m^{r+1}$ and let

$$(f, \varphi, \varphi_H)\colon \Lambda'(X) \to \Lambda'(Y) \tag{4}$$

be a morphism between kypes. Let $U = P_n X$ with $U^{n+1} = X^{n+1} = X$ and in the same way let $V = P_n Y$ with $V^{n+1} = Y^{n+1} = Y$. Since $(f, \varphi)\colon \lambda(X) \to \lambda(Y)$

is a map of **Gro**(E) we obtain by Proposition 3.7.2 a cellular map $g'\colon U \to V$ which realizes (f, φ). Hence the map g' restricted to the $(n+1)$-skeleton yields a map $g\colon X \to Y$ which realizes (f, φ) but which need not realize φ_H. We obtain the following diagram, where the left-hand side is defined by X and the right-hand side is defined by Y; compare diagram (1).

$$\begin{tikzcd}
H \arrow[rrr, "\varphi_H"] \arrow[d, equal] \arrow[drrr, phantom, "(*)"] & & & H' \arrow[d, equal] \\
H_{n+1}X \subset \arrow[d, "b"] & C_{n+1}X \arrow[r, "g_*"'] \arrow[d, "f_X"] & C_{n+1}Y \arrow[d, "f_Y"] & \supset H_{n+1}Y \arrow[d, "b'"] \\
\Gamma_n X \subset \arrow[rrr, bend right, "\Gamma_n g = \Gamma_n f"'] & \pi_n X^n \arrow[r, "g_*^n"] & \pi_n Y^n & \supset \Gamma_n Y
\end{tikzcd}$$

All subdiagrams commute except possibly subdiagram (*). We have however by Definition 3.1.2 (3) the equation

$$b'\varphi_H = (\Gamma_n g)b. \tag{6}$$

The maps f_X, f_Y are the attaching maps of $(n+1)$-cells in X and Y respectively. Now (6) and (5) imply that the difference $\varphi_H - g_*\colon H \to C_{n+1}Y$ given by diagram (*) satisfies

$$f_Y(\varphi_H - g_*) = 0. \tag{7}$$

Since H is free abelian and since the sequence

$$\begin{tikzcd}
\pi_{n+1}Y^{n+1} \arrow[r, "j"] & \pi_{n+1}(Y^{n+1}, Y^n) \arrow[r, "\partial"] \arrow[d, equal] & \pi_n Y^n \\
& C_{n+1}X \arrow[ur, "f_Y"'] &
\end{tikzcd}$$

is exact we can choose by (7) a homomorphism

$$\alpha\colon H \to \pi_{n+1}Y^{n+1} \quad \text{with} \quad j(\alpha) = \varphi_H - g_*. \tag{8}$$

Now H is a direct summand of $C_{n+1}X$ for which we choose a retraction r so that we get a map

$$h = g + \alpha r\colon X \to Y \tag{9}$$

by the action in (4.2.5). The map h is cellular and satisfies $h^n = g^n$ on the n-skeleton. We claim that h is in fact a Λ-realization of (f, φ, φ_H) above. Indeed since $h^n = g^n$ the map h realizes (f, φ) and we compute $H_{n+1}(h)$ by

$$C_{n+1}(h) = (g + \alpha r)_* = g_* + j\alpha r = g_* + (\varphi_H - g_*)r \tag{10}$$

so that the restriction of $C_{n+1}(h)$ to H yields φ_H. Hence h realizes also φ_H. Therefore the proof of Proposition 3.7.3 is complete. □

Finally we consider the functor λ' in the classification theorem 3.4.4.

(3.7.4) Proposition *The functor* λ': $\mathbf{types}_m^r \to \mathbf{bypes}(\mathbf{C}, F)$ *with*

$$\lambda'(X) = (P_{n-1}X, H_nX, H_{n+1}X, b_{n+1}X, \beta_nX)$$

is a detecting functor.

We here only show that Proposition 3.7.4 is a corollary of the corresponding result:

(3.7.5) Proposition *The functor*

$$\Lambda' : \mathbf{spaces}_m^{r+1} \to \mathbf{Bypes}(\mathbf{C}, F)$$

with

$$\Lambda'(X) = (P_{n-1}X, H_nX, H_{n+1}X, b_{n+1}X, \beta_nX)$$

is a detecting functor.

The highly sophisticated proof of Proposition 3.7.5 involves most of the theory of Chapters 2 and 4. In particular the new concept of towers of categories is crucial for this proof. The final proof of Proposition 3.7.5 is given in Section 4.7.

Proof of Proposition 3.7.4 We derive Proposition 3.7.4 from Proposition 3.7.5. Again the Whitehead theorem shows that the functor λ' reflects isomorphisms. In fact, recall that β_nX determines $\{\pi_nX\}$ in Theorem 2.6.9 (c). Hence the five lemma shows that an isomorphism between bypes yields an isomorphism on π_*X. Next we consider realizability of objects. For this we use the commutative diagram of functors in (3.4.6). In this diagram it is easy to see that each bype T has a ϕ-realization T'. Since Λ' is a detecting functor we find a Λ'-realization T'' of T'. Hence $P_{m+r}T''$ is a λ'-realization of T since diagram (3.4.6) commutes.

Similarly we see that the functor λ' is full: let T_1, T_2 be objects in $\mathbf{types}_m^r$ and let f: $\lambda'T_1 \to \lambda'T_2$ be a morphism between bypes. Let $T_1', T_2' \in \mathbf{spaces}_m^{r+1}$ be the $(m+r+1)$-skeleta of T_1 and T_2 respectively. Then clearly $PT_1' = T_1$ and $PT_2' = T_2$. Moreover we can choose a morphism f': $\lambda T_1' \to \lambda T_2'$ with $\phi f' = f$. For this we only choose a homomorphism f' such that the diagram

$$\begin{tikzcd}
H_{n+1}T_1' \arrow[r, two heads] \arrow[d, "f'"] & bH_{n+1}T_1 \arrow[d, "f_*"] \\
H_{n+1}T_2' \arrow[r, two heads] & bH_{n+1}T_2
\end{tikzcd}$$

commutes. This is possible since the horizontal arrows are surjective and since $H_{n+1}T_1'$ is free abelian. Now, since Λ' is a detecting functor, we find a Λ'-realization f'': $T_1 \to T_2$ of f' and hence Pf'' is a λ'-realization of f. This completes the proof that λ' is a detecting functor. □

4

THE CW-TOWER OF CATEGORIES

A CW-complex X is obtained inductively by constructing the skeleta X^n, $n \geq 0$. Since we here only consider homotopy types of simply connected CW-complexes we may assume that X is *reduced* in the sense that the 1-skeleton of X consists of a single point, $X^1 = *$. However, the skeletal filtration has the disadvantage that the homotopy type of X^n is not well defined by the homotopy type of X. For this reason J.H.C. Whitehead introduced the $(n-1)$-type of X which is also the $(n-1)$-type of X^n and which can be obtained from X^n by 'killing' homotopy groups $\pi_m X^n$, $m \geq n$. The $(n-1)$-type is the $(n-1)$-section of the Postnikov tower of X which we denote by $P_{n-1}(X)$. It is a classical result that the homotopy type of $P_{n-1}(X)$ is well defined by the homotopy type of X. This fact justifies the construction of $P_{n-1}(X)$, though it is to some extent absurd to replace a nice n-dimensional CW-complex X^n by a CW-complex $P_{n-1}(X)$ which in general is infinite dimensional, and the homology of which is hard to compute.

We here study the skeletal filtration of a CW-complex X rather than the Postnikov decomposition. We deduce from the skeletal filtration the object

$$r_{n+1}(X) = (C, f_{n+1}, X^n)$$

which is a triple consisting of an algebraic part C (which is the cellular chain complex of X) and of a topological part X^n (which is the n-skeleton of X). Moreover f_{n+1} is the homotopy class of the attaching map of $(n+1)$-cells of X given by a homomorphism $C_{n+1} \to \pi_n X^n$. We call such a triple a *homotopy system of order* $(n+1)$. The crucial point is that homotopy systems of order $(n+1)$ form a homotopy category $\mathbf{H}_{n+1}/\simeq$ and that the homotopy type of the object $r_{n+1}(X)$ in $\mathbf{H}_{n+1}/\simeq$ depends only on the homotopy type of X. Hence enriching the n-skeleton X^n by an algebraic part (C, f_{n+1}) yields a new invariant $r_{n+1}(X)$ of the homotopy type of X which has the same kind of naturality as the Postnikov section $P_{n-1}(X)$. We consider the sequence

$$r_3(X), r_4(X), \ldots, r_{n+1}(X), \ldots$$

of homotopy systems to be the true Eckmann–Hilton dual of the sequence

$$P_2(X), P_3(X), \ldots, P_{n-1}(X), \ldots$$

of Postnikov sections of the simply connected CW-space X. It is Postnikov's result that the homotopy type of $P_n(X)$ is determined by the pair

$$(P_{n-1}(X), k_n(X))$$

where $k_n(X)$ is the nth k-invariant of X. Our main result here shows that, on the other hand, the homotopy type of $r_{n+1}(X)$ is determined by the triple (see Theorem 6.6.4)

$$(r_n(X), \beta_n(X), b_{n+1}(X)).$$

Here $b_{n+1}(X)$ is the second boundary homomorphism in Whitehead's certain exact sequence and $\beta_n(X)$ is the boundary invariant introduced in Chapter 2. Since $\beta_{n+1}(X)$ determines $b_{n+1}(X)$ we see that the sequence of boundary invariants

$$\beta_3(X), \beta_4(X), \ldots$$

determines the homotopy type of X in a similar way as the sequence of k-invariants

$$k_3(X), k_4(X), \ldots.$$

Moreover each simply connected homotopy type can be built by the inductive construction of either sequence. Further dual properties of boundary invariants and k-invariants are described in Chapters 2 and 3. The categories $\mathbf{H}_{n+1}/\simeq$ of homotopy systems of order $(n+1)$ yield the *CW-tower* of categories. This is a sequence of functors ($n \geq 3$)

$$\mathbf{spaces}_2 \xrightarrow{r} \mathbf{H}_{n+1}/\simeq \xrightarrow{\lambda} \mathbf{H}_n/\simeq \to \cdots \xrightarrow{\lambda} \mathbf{H}_3/\simeq$$

which approximates the homotopy category $\mathbf{spaces}_2$ of simply connected CW-spaces. Each functor λ is embedded in an exact sequence

$$H^n\Gamma_n + \to \mathbf{H}_{n+1}/\simeq \xrightarrow{\lambda} \mathbf{H}_n/\simeq \xrightarrow{\mathcal{O}} H^{n+1}\Gamma_n.$$

Here $H^m\Gamma_n$ denotes an $\mathbf{H}_n/\simeq$-bimodule. In Section 4.1 we recall the useful language concerning such exact sequence. (The CW-tower of categories is studied for non-simply connected spaces in Baues [AH] and [CH]. We here deal only with the simply connected case. This simplifies these towers considerably; compare also the final chapter in [AH]).

The proofs of our main results are based on properties of the CW-tower of categories. In particular in Section 4.6 we relate the obstruction operator $\mathcal{O}$ in the CW-tower with the secondary boundary operator of Whitehead and the boundary invariants in Chapter 2. This leads in Section 4.7 to a proof of the classification theorem in Chapter 3. Moreover we prove a theorem on the action $H^n\Gamma_n+$ in the CW-tower which is useful for the classification of homotopy classes of maps; see Section 4.8.

4.1 Exact sequences for functors

The concept of exact sequences for groups is well known in algebraic topology. We can consider a group to be a category with a single object in which all morphisms are equivalences. Therefore there might be a more general notion of an exact sequence for categories and functors. Motivated by the CW-tower of categories we introduce in this section an exact sequence for a functor λ of the form

$$D+ \to \mathbf{A} \xrightarrow{\lambda} \mathbf{B} \xrightarrow{\mathscr{O}} H.$$

Here, however, D and H are not categories but natural systems of abelian groups on $\mathbf{B}$, for example $\mathbf{B}$-bimodules. Such natural systems serve as coefficients of cohomology groups $H^n(\mathbf{B}, D)$ of the (small) category $\mathbf{B}$. Special exact sequences are the linear extensions of $\mathbf{B}$ by D. Exact sequences for a functor λ and linear extensions arise frequently in algebraic topology and in many other fields of mathematics; see Baues [AH], [CH]. The examples here are mainly derived from the CW-tower of categories.

As usual let $\mathbf{Ab}$ be the category of abelian groups. For a category $\mathbf{B}$ a $\mathbf{B}$*-module* M is a functor $M: \mathbf{B} \to \mathbf{Ab}$ and a $\mathbf{B}$*-bimodule* D is a functor

$$D: \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Ab}.$$

Here $\mathbf{B}^{\mathrm{op}}$ is the opposite category and $\mathbf{B}^{\mathrm{op}} \times \mathbf{B}$ is the product category. For objects X, Y in $\mathbf{B}$ the set $D(X, Y)$ is an abelian group contravariant in X and covariant in Y. For example if $\mathbf{B} \subset \mathbf{Top}^*/\simeq$ is a subcategory of the homotopy category of pointed spaces we have the $\mathbf{B}$-bimodule D,

(4.1.1) $$D(X, Y) = H^n(X, \pi_m Y),$$

given by the nth cohomology of X with coefficients in the mth homotopy group of Y. This bimodule arises often in obstruction theory. The following notion of a 'natural system of abelian groups on $\mathbf{B}$' generalizes the notion of a $\mathbf{B}$-bimodule. For this recall that the *category of factorizations* in $\mathbf{B}$, denoted by $F\mathbf{B}$, is given as follows. Objects are morphisms f, g in $\mathbf{B}$ and morphisms $f \to g$ are pairs (a, b) for which

$$\begin{array}{ccc} Y & \xrightarrow{b} & Y' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ X & \xleftarrow{a} & X' \end{array}$$

commutes in $\mathbf{B}$. Thus bfa is a factorization of g. A *natural system* (of abelian groups) on $\mathbf{B}$ is a functor

(4.1.2) $$D: F\mathbf{B} \to \mathbf{Ab}$$

that is, an $F\mathbf{B}$-module. This functor carries the object f to $D_f = D(f)$ and carries (a, b) to $D(a, b) = a^* b_*$ with $a^* = D(a, 1)$ and $b_* = D(1, b)$. A $\mathbf{B}$-bimodule D is also a natural system by setting $D_f = D(X, Y)$ for $f: X \to Y$, that is, in this case D_f depends only on the source and the target of f. A functor $\lambda: \mathbf{A} \to \mathbf{B}$ induces the function

$$\lambda: \mathbf{A}(X,Y) \to \mathbf{B}(\lambda X, \lambda Y)$$

between morphism sets; here X and Y are objects in $\mathbf{A}$. For a morphism $f_0: X \to Y$ in $\mathbf{A}$ with $f = \lambda f_0$ we thus have the subset

(4.1.3) $$\lambda^{-1}(f) \subset \mathbf{A}(X,Y) \quad \text{with} \quad f_0 \in \lambda^{-1}(f).$$

Now recall the definition of a linear extension of categories in Definition 1.1.9:

(4.1.4) $$D + \rightarrowtail \mathbf{A} \overset{\lambda}{\twoheadrightarrow} \mathbf{B}.$$

The next definition of an exact sequence for a functor λ generalizes the notion of a linear extension in two ways. On the one hand, λ needs not to be full but its image can be described by an obstruction operator $\mathcal{O}$; on the other hand, the action of D_f on $\lambda^{-1}(f)$ need not be effective.

(1.4.5) Definition Let $\lambda: \mathbf{A} \to \mathbf{B}$ be a functor and let D and H be natural systems of abelian groups on $\mathbf{B}$. We call the sequence

$$D \xrightarrow{+} \mathbf{A} \xrightarrow{\lambda} \mathbf{B} \xrightarrow{\mathcal{O}} H$$

an *exact sequence* for λ if the following properties are satisfied.

(a) For each morphism $f_0: X \to Y$ in $\mathbf{A}$ the abelian group D_f, $f = \lambda f_0$, acts transitively on the set of morphisms $\lambda^{-1}(f) \subset \mathbf{A}(X,Y)$. Let $I_{f_0} = \{\alpha \in D_f, f_0 + \alpha = f_0\}$ be the isotopy group.

(b) The linear distributivity law (Definition 1.1.9) (c) is satisfied.

(c) For all objects X,Y in $\mathbf{A}$ and for all morphisms $f: \lambda X \to \lambda Y$ in $\mathbf{B}$ an *obstruction element* $\mathcal{O}_{X,Y}(f) \in H(f)$ is given such that $\mathcal{O}_{X,Y}(f) = 0$ if and only if there is a morphism $f_0: X \to Y$ with $\lambda f_0 = f$.

(d) $\mathcal{O}$ is a *derivation*, that is $\mathcal{O}_{X,Z}(gf) = g_* \mathcal{O}_{X,Y}(f) + f^* \mathcal{O}_{Y,Z}(g)$ for $f: \lambda X \to \lambda Y$, $g: \lambda Y \to \lambda Z$.

(e) For all objects X in $\mathbf{A}$ and for all $\alpha \in H(1_{\lambda X})$ there is an object Y in $\mathbf{A}$ with $\lambda Y = \lambda X$ and $\mathcal{O}_{X,Y}(1_{\lambda X}) = \alpha$; we write $X = Y + \alpha$ in this case.

A *tower of categories* is a diagram ($i \in \mathbb{Z}$)

(f)
$$\begin{tikzcd} & {} \arrow[d] & \\ D_i \arrow[r] & \mathbf{H}_i \arrow[r] \arrow[d, "\lambda"] & \Gamma_{i+1} \\ D_{i-1} \arrow[r] & \mathbf{H}_{i-1} \arrow[r] \arrow[d] & \Gamma_i \\ & {} & \end{tikzcd}$$

where $D_i \to \mathbf{H}_i \to \mathbf{H}_{i-1} \to \Gamma_i$ is an exact sequence.

We say that D *acts on* λ if Definition 4.1.5(a) above is satisfied. Moreover, D *acts linearly* on λ if (a) and (b) are satisfied. We say that D *acts effectively* if all isotropy groups in (a) are trivial. A linear extension as in (4.1.4) yields an exact sequence

$$D \xrightarrow{+} \mathbf{E} \to \mathbf{C} \xrightarrow{\mathscr{O}} 0$$

where 0 is a trivial natural system. On the other hand, each exact sequence as in Definition 4.1.5 yields a linear extension of categories

(4.1.6) $$D/I \overset{+}{\rightarrowtail} \mathbf{A} \twoheadrightarrow \lambda\mathbf{A}.$$

Here $\lambda\mathbf{A}$ is the image category of $\lambda: \mathbf{A} \to \mathbf{B}$. The natural system D/I on λA is given by $(D/I)(f) = D_f/I_{f_0}$, $f_0 \in \lambda^{-1}(f)$; see Definition 4.1.5(a).

Next, we consider the *groups of automorphisms* in an exact sequence. Let A be an object in $\mathbf{A}$. Then we obtain by the properties in Definition 4.1.5 the exact sequence

(4.1.7) $$D(1_{\lambda A}) \xrightarrow{1+} \mathrm{Aut}_{\mathbf{A}}(A) \xrightarrow{\lambda} \mathrm{Aut}_{\mathbf{B}}(\lambda A) \xrightarrow{\bar{\mathscr{O}}} H(1_{\lambda A}).$$

Here λ is the homomorphism of groups induced by λ and 1^+ is the homomorphism of groups given by $1^+(\alpha) = 1_{\lambda A} + \alpha$. Moreover, the function $\bar{\mathscr{O}}$ is defined by $\bar{\mathscr{O}}(f) = (f^{-1})_* \mathscr{O}_{A,A}(f)$. In fact, $\bar{\mathscr{O}}$ is a derivation of groups with $\bar{\mathscr{O}}(fg) = \bar{\mathscr{O}}(f)^g + \bar{\mathscr{O}}(f)$. Here we set $x^g = g^*(g^{-1})_*(x)$ for $x \in H(1_{\lambda A})$. Compare (IV.4.11) Baues [AH].

(4.1.8) Lemma *A functor λ in an exact sequence (Definition 4.1.5) reflects isomorphisms.*

Compare (IV.4.11) in Baues [AH]. The lemma implies that a weak linear extension is a detecting functor. Recall that $\mathrm{Real}_\lambda(B)$ for an object B in $\mathbf{B}$ denotes the class of realizations of B in $\mathbf{A}$; see (1.1.6).

(4.1.9) Lemma *Let λ be a functor in an exact sequence as in Definition 4.1.5 and assume $\mathrm{Real}_\lambda(B)$ is not empty. Then the group $H(1_B)$ acts transitively and effectively on $\mathrm{Real}_\lambda(B)$ by Definition 4.1.5(e). In particular $\mathrm{Real}_\lambda(B)$ is a set.*

Compare (IV.4.12) in Baues [AH].

4.2 Homotopy systems of order $(n + 1)$

Homotopy systems of order $(n + 1)$ are triples (C, f_{n+1}, X^n) consisting of a chain complex of free abelian groups C, an attaching map f_{n+1}, and an n-dimensional CW-complex X^n. Such homotopy systems (defined more precisely below) are motivated by the following properties of CW-complexes. Let X be a CW-complex with trivial 1-skeleton $X^1 = *$. Then the *cellular chain complex* $C = C_*(X)$ is given by

$$C_n = C_n(X) = H_n(X^n, X^{n-1})$$

with the boundary $d: C_n \to C_{n-1}$ defined by the triple (X^n, X^{n-1}, X^{n-2}). Since $X^1 = *$ we have the Hurewicz isomorphism h in the composition

(4.2.1)

$$f_{n+1}: C_{n+1} = H_{n+1}(X^{n+1}, X^n) \overset{h}{\cong} \pi_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial} \pi_n(X^n)$$

where ∂ is the boundary in the homotopy exact sequence of the pair (X^{n+1}, X^n). One readily checks that f_{n+1} satisfies $f_{n+1}d = 0$. The set Z_{n+1} of $(n + 1)$-cells in X is a basis of the free abelian group C_{n+1} of cellular $(n + 1)$-chains. Therefore f_{n+1} describes the homotopy class of a map

$$f_{n+1}: M(C_{n+1}, n) = \bigvee_{Z_{n+1}} S^n \to X^n \tag{1}$$

which is the *attaching map of $(n + 1)$-cells* in X; in fact one has a homotopy equivalence under X^n

$$c: X^{n+1} \simeq C_f \tag{2}$$

where the right-hand side is the mapping cone of $f = f_{n+1}$. By definition of f_{n+1} and d we see that the diagram

$$\begin{tikzcd} C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d"'] & \pi_n(X^n) \arrow[d, "j"] \\ C_n \arrow[r, "h", "\cong"'] & \pi_n(X^n, X^{n-1}) \end{tikzcd} \tag{3}$$

commutes, where h again is the Hurewicz isomorphism and where j is from the homotopy exact sequence of the pair (X^n, X^{n-1}).

Recall that **CW** is the category of CW-complexes with trivial 0-skeleton and of cellular maps. Let $\mathbf{CW}^n$ be the full subcategory consisting of n-dimensional CW-complexes. Moreover morphisms in the quotient category

$\mathbf{CW}/\overset{\circ}{\simeq}$ are 0-homotopy classes of cellular maps, where a 0-homotopy is a homotopy running through cellular maps.

(4.2.2) Definition Let $n \geq 2$. A (reduced) *homotopy system of order* $(n+1)$ is a triple (C, f_{n+1}, X^n) where X^n is an n-dimensional CW-complex with trivial 1-skeleton and where C is a chain complex of free abelian groups which coincide with $C_* X^n$ in degree $\leq n$. Moreover

$$f_{n+1}: C_{n+1} \to \pi_n X^n \tag{1}$$

is a homomorphism of abelian groups which satisfies the *cocycle condition*

$$f_{n+1} d = 0 \tag{2}$$

and for which diagram (4.2.1)(3) commutes. A *map* between homotopy systems of order $(n+1)$ is a pair (ξ, η),

$$(\xi, \eta): (C, f_{n+1}, X^n) \to (C', g_{n+1}, Y^n) \tag{3}$$

with the following properties. The map $\eta: X^n \to Y^n$ is a morphism in $\mathbf{CW}/\overset{\circ}{\simeq}$ and $\xi: C \to C'$ is a chain map which coincides with $C_* \eta$ in degree $\leq n$ and for which the following diagram commutes:

$$\begin{tikzcd} C_{n+1} \arrow[r, "\xi_{n+1}"] \arrow[d, "f_{n+1}"] & C'_{n+1} \arrow[d, "g_{n+1}"] \\ \pi_n X^n \arrow[r, "\eta_*"] & \pi_n Y^n \end{tikzcd} \tag{4}$$

Let $\mathbf{H}_{n+1}$ be the category of such (reduced) homotopy systems of order $(n+1)$. Clearly composition is defined by $(\xi, \eta)(\xi', \eta') = (\xi\xi', \eta\eta')$. By (4.2.1) we have the obvious functors

(4.2.3) $$\mathbf{CW}_2/\overset{\circ}{\simeq} \xrightarrow[r_{n+1}]{} \mathbf{H}_{n+1} \xrightarrow[\lambda]{} \mathbf{H}_n$$

with $\lambda r_{n+1} = r_n$. Here $\mathbf{CW}_2$ is the full subcategory in $\mathbf{CW}$ consisting of CW-complexes with trivial 1-skeleton. Clearly the functor r_{n+1} takes X to the triple

$$r_{n+1} X = (C_* X, f_{n+1}, X^n)$$

see (4.2.1), and the functor λ carries (C, f_{n+1}, X^n) to (C, f_n, X^{n-1}) where X^{n-1} is the $(n-1)$-skeleton of X^n and where f_n is the attaching map of n-cells in X^n. For the definition of the homotopy relation on the category $\mathbf{H}_{n+1}$ we need the coaction μ which is a map under X^{n-1}

(4.2.4) $$\mu: X^n \to X^n \vee M(C_n, n).$$

Here $M(C_n, n)$ is the Moore space of C_n in degree n. The coaction μ is obtained by the corresponding coaction on the mapping cone C_f with $f = f_n$, see (4.2.1)(2), that is $\mu = (c \vee 1)\mu_f c'$ where c' is a homotopy inverse of c under X^{n-1}. The coaction μ induces an action $+$ on the set of homotopy classes $[X^n, Y]$ in **Top**$^*/\simeq$, namely

$$(4.2.5) \qquad [X^n,Y] \times E(X^n,Y) \xrightarrow{+} [X^n,Y]$$

with $F + \alpha = \mu^*(F, \alpha)$. Here we set

$$E(X^n,Y) = [M(C_n,n),Y] = \operatorname{Hom}(C_n, \pi_n Y). \tag{1}$$

Since we assume that X^n has trivial 1-skeleton the action (4.2.5) induces an action

$$[X^n,Y] \times H^n(X^n, \pi_n Y) \xrightarrow{+} [X^n,Y] \tag{2}$$

with $F + \{\alpha\} = F + \alpha$. The isotropy groups of this action are considered in Section 4.8 below.

(4.2.6) Definition Let

$$(\xi,\eta),(\xi',\eta'):(C,f_{n+1},X^n) \to (C',g_{n+1},Y^n)$$

be maps in $\mathbf{H}_{n+1}$. We set $(\xi,\eta) \simeq (\xi',\eta')$ if there exist homomorphisms α_{j+1}: $C_j \to C'_{j+1}$, $j \geq n$, such that:

(a) $\{\eta\} + g_{n+1}\alpha_{n+1} = \{\eta'\}$; and

(b) $\xi'_k - \xi_k = \alpha_k d + d\alpha_{k+1}$, $k \geq n+1$.

The action $+$ in (a) is defined by (4.2.5) above; $\{\eta\}$ denotes the homotopy class of η in $[X^n, Y^n]$, that is in **Top**$^*/\simeq$. We call $\alpha:(\xi,\eta) \simeq (\xi',\eta')$ a *homotopy* in $\mathbf{H}_{n+1}$.

One can check that the homotopy relation is a natural equivalence relation on $\mathbf{H}_{n+1}$ and that the functors in (4.2.3) induce functors

$$(4.2.7) \qquad \mathbf{CW}_2/\simeq \xrightarrow[r_{n+1}]{} \mathbf{H}_{n+1}/\simeq \xrightarrow[\lambda]{} \mathbf{H}_n/\simeq.$$

(We refer the reader to Baues [AH] where we actually study homotopy systems in any cofibration category. Reduced homotopy systems as in Definition 4.2.2 are considered in the final chapter of Baues [AH].) Let **Chain**$_2$ be the category of chain complexes C of free abelian groups with $C_i = 0$ for $i \leq 1$. We observe that the forgetful functor

$$(4.2.8) \qquad \tilde{C}_*:\mathbf{H}_3 \xrightarrow{\approx} \mathbf{Chain}_2, \quad (C,f_3,X^2) \mapsto C/C_0$$

is actually an isomorphism of categories and that $H_3/\simeq = \mathbf{Chain}_2/\simeq$ is given by the usual homotopy relation for chain maps. This way we can identify r_3 with the classical cellular chain functor (reduced)

(4.2.9) $$\tilde{C}_* = r_3 : \mathbf{CW}_2/\overset{\circ}{\simeq} \to \mathbf{Chain}_2 = \mathbf{H}_3.$$

Hence the functors r_{n+1} and λ in (4.2.7) lead to a sequence of functors which decompose the chain functor. We study the properties of this sequence in the next section.

4.3 The CW-tower of categories

The categories of homotopy systems introduced in Section 4.2 above form a sequence of categories and functors ($n \geq 3$)

$$C_* : \mathbf{CW}_2/\simeq \xrightarrow{r} H_{n+1}/\simeq \xrightarrow{\lambda} H_n/\simeq \to \cdots \xrightarrow{\lambda} \mathbf{H}_3/\simeq = \mathbf{Chain}_2/\simeq$$

such that the composite is the cellular chain functor C_*. We now show that each functor λ is embedded in an exact sequence as discussed in Section 4.1. We call the collection of these exact sequences the CW-tower of categories. We first observe that the Postnikov functor P_n which carries X to its n-type $P_n X$ admits a factorization

(4.3.1) $$P_n : \mathbf{CW}_2/\simeq \xrightarrow[r_{n+1}]{} \mathbf{H}_{n+1}/\simeq \xrightarrow[P_n]{} n\textbf{-types}.$$

This is clear since $P_n X = P_n X^{n+1}$ is given by the $(n+1)$-skeleton of X and $r_{n+1}X$ determines the homotopy type of X^{n+1} under X^n by (4.2.1)(2).

Next we consider Whitehead's functor Γ_n which carries a CW-complex X to the group $\Gamma_n X = \mathrm{image}(\pi_n X^{n-1} \to \pi_n X^n)$. This functor admits a factorization through r_n. In fact, there is a functor

(4.3.2) $$\Gamma_n : \mathbf{H}_n/\simeq \to \mathbf{Ab}$$

with $\Gamma_n r_n X = \Gamma_n X$. We define $\Gamma_n Y$ for an object $Y = (C', g_n, Y^{n-1})$ in $\mathbf{H}_n$ as follows. Let Y^n be a CW-complex such that g_n is the attaching map of n-cells in Y^n. Then Y^n is well defined by (g_n, Y^{n-1}) up to homotopy equivalence under Y^{n-1}. Hence $\Gamma_n Y = \Gamma_n Y^n$ is well defined. A map (ξ, η): $X \to Y$ in $\mathbf{H}_n$ admits an extension $\bar{\eta} : X^n \to Y^n$ of η so that also the induced map $\Gamma_n(\xi, \eta) = \Gamma_n \bar{\eta}$ is well defined. It is clear that Γ_n in (4.3.2) factors through the functor

$$P_{n-1} : \mathbf{H}_n/\simeq \to (n-1)\textbf{-types}$$

given by (4.3.1), that is $\Gamma_n Y = \Gamma_n P_{n-1} Y$ for an object Y in $\mathbf{H}_n$. We use the functor Γ_n in (4.3.2) for the definition of the bimodule

$$H^m\Gamma_n : (\mathbf{H}_n/\simeq)^{\mathrm{op}} \times \mathbf{H}_n/\simeq \;\to \mathbf{Ab} \tag{4.3.3}$$

which carries a pair (X, Y) of objects in $\mathbf{H}_n$ to

$$H^m\Gamma_n(X, Y) = H^m(X, \Gamma_n Y).$$

Here we set $H^m(X, -) = H^m(C, -)$ for the object $X = (C, f_n, X^{n-1})$ in $\mathbf{H}_n$. We are now ready to state the following theorem which establishes the *CW-tower of categories*.

(4.3.4) Theorem *The functors* λ *in* (4.2.3) *and* (4.2.7) *are part of the following commutative diagram in which the rows are exact sequences in the sense of Section* 4.1, $n \geq 3$.

$$\begin{tikzcd}
H^n\Gamma_n + \arrow[r] \arrow[d, "1"] & \mathbf{H}_{n+1} \arrow[r, "\lambda"] \arrow[d, "q_{n+1}"] & \mathbf{H}_n \arrow[r, "\mathcal{O}"] \arrow[d, "q_n"] & H^{n+1}\Gamma_n \arrow[d, "1"] \\
H^n\Gamma_n + \arrow[r] & \mathbf{H}_{n+1}/\simeq \arrow[r, "\lambda"] & \mathbf{H}_n/\simeq \arrow[r, "\mathcal{O}"] & H^{n+1}\Gamma_m
\end{tikzcd}$$

Here q_n is the quotient functor and 1 denotes the identity. The functor q_{n+1} is equivariant with respect to the action of $H^n\Gamma_n$. We describe the action and the obstruction operator explicitly below. (The theorem is proved in a more general form in VI.5.11 of Baues [AH], see also II.3.3 in Baues [CH].)

With the notation in Definition 4.1.5(f) the exact sequences in Theorem 4.3.4 yield towers of categories which approximate $\mathbf{CW}_2/\overset{\circ}{\simeq}$ and $\mathbf{CW}_2/\simeq$ respectively. In particular we obtain the tower of homotopy categories

$$\begin{tikzcd}
 & \mathbf{CW}_2/\simeq \arrow[d] & \\
 & \vdots & \\
H^n\Gamma_n \arrow[r, "+"] & \mathbf{H}_{n+1}/\simeq \arrow[d, "\lambda"] & \\
 & \mathbf{H}_n/\simeq \arrow[r, "\mathcal{O}"] \arrow[d] & H^{n+1}\Gamma_n \\
 & \vdots & \\
H^3\Gamma_3 \arrow[r, "+"] & \mathbf{H}_4/\simeq \arrow[d, "\lambda"] & \\
 & \mathbf{H}_3/\simeq \arrow[r, "\mathcal{O}"] & H^4\Gamma_3
\end{tikzcd}$$

which somehow resembles the Postnikov tower of a space since we have obstructions and actions as we are used to in Postnikov towers. We now describe the obstruction operator. Let X, Y be objects in $\mathbf{H}_{n+1}$ and let $(\xi, \eta): \lambda X \to \lambda Y$ be a map in $\mathbf{H}_n$. Then an element

(4.3.5) $$\mathscr{O}_{X,Y}(\xi, \eta) \in H^{n+1}(X, \Gamma_n Y)$$

is defined such that $\mathscr{O}_{X,Y}(\xi, \eta) = 0$ if and only if there exists a map $(\xi, \overline{\eta})$: $X \to Y$ in $\mathbf{H}_{n+1}$ with $\lambda(\xi, \overline{\eta}) = (\xi, \eta)$. We define the *obstruction* $\mathscr{O}_{X,Y}(\xi, \eta)$ in (4.3.5) as follows. Since (ξ, η) is a map in $\mathbf{H}_n$ we can choose a map $F: X^n \to Y^n$ in $\mathbf{CW}/\overset{\circ}{\simeq}$ which extends η and for which $C_* F$ coincides with ξ in degree $\leq n$. The diagram

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{\xi_{n+1}} & C'_{n+1} \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle g_{n+1}} \\ \pi_n X^n & \xrightarrow[F_*]{} & \pi_n Y^n \end{array} \tag{1}$$

need not be commutative. The difference

$$\mathscr{O}(F) = -g_{n+1}\xi_{n+1} + F_* f_{n+1} \tag{2}$$

maps C_{n+1} to the subgroup $\Gamma_n Y \subset \pi_n Y^n$ and this difference is a cocycle in $\mathrm{Hom}(C_{n+1}, \Gamma_n Y)$. The obstruction

$$\mathscr{O}_{X,Y}(\xi, \eta) = \{\mathscr{O}(F)\} \in H^{n+1}(X, \Gamma_n Y) \tag{3}$$

is the cohomology class represented by the cocycle $\mathscr{O}(F)$. This class does not depend on the choice of F in (1). Moreover, $\mathscr{O}_{X,Y}(\xi, \eta)$ depends only on the homotopy class of (ξ, η) in $\mathbf{H}_n/\simeq$. In addition the properties in Definition 4.1.5 (c), (d), (e) are satisfied.

Next we consider the action $+$ in Theorem 4.3.4. Let X, Y be CW-complexes with trivial 1-skeleton or objects in $\mathbf{H}_m$. For $n \leq m$ we denote by $[X, Y]^n$ the set of all morphisms $X_0 \to Y_0$ in $\mathbf{H}_n/\simeq$ where X_0 and Y_0 are the images of X and Y respectively in the category $\mathbf{H}_n$. Here we use the functors in the CW-tower. The functor λ yields the function

(4.3.6) $$\lambda: [X, Y]^{n+1} \to [X, Y]^n.$$

Now let X and Y be objects in $\mathbf{H}_{n+1}$. Then the action $+$ in the bottom row of the commutative diagram of Theorem 4.3.4 is of the form

(4.3.7) $$[X, Y]^{n+1} \times H^n(X, \Gamma_n Y) \xrightarrow{+} [X, Y]^{n+1}$$

and satisfies $\lambda f = \lambda g$ if and only if $g = f + \alpha$ for an appropriate α. We describe the action as follows. Let $(\xi, \eta): X \to Y$ be a map in $\mathbf{H}_{n+1}$ and let $\{\alpha\} \in H^n(X, \Gamma_n Y)$ be the class represented by the cocycle

$$\alpha \in \operatorname{Hom}(C_n, \Gamma_n Y). \tag{1}$$

Then we obtain by $i: \Gamma_n Y \subset \pi_n Y^n$ the composite $i\alpha$ such that $\eta + i\alpha$ with

$$\eta + i\alpha: X^n \xrightarrow{\mu} X^n \vee M(C_n, n) \xrightarrow{(\eta, i\alpha)} Y^n \tag{2}$$

is a map in $\mathbf{CW}/\overset{\circ}{\simeq}$ defined by μ in (4.2.4). We now set

$$\{(\xi, \eta)\} + \{\alpha\} = \{(\xi, \eta + i\alpha)\} \tag{3}$$

where $\{(\xi, \eta)\} \in [X, Y]^{n+1}$ is the homotopy class of (ξ, η) in $\mathbf{H}_{n+1}$. In Baues [AH] we check that (3) yields a well-defined action in (4.3.7). Moreover we get the action in the top row of the commutative diagram of Theorem 4.3.4 by

$$(\xi, \eta) + \{\alpha\} = (\xi, \eta + i\alpha). \tag{4}$$

Here in fact $(\xi, \eta + i\alpha)$ depends only on the cohomology class $\{\alpha\}$. The actions in (3) and (4) satisfy the properties of Definition 4.1.5(a), (b). In Section 4.8 we study the isotropy groups of the action $+$ in (3).

For CW-complexes X, Y with trivial 1-skeleton the CW-tower yields the following diagram of exact sequences of sets

(4.3.8)
$$\begin{array}{ccccc}
 & & [X,Y] & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^n(X,\Gamma_n Y) & \xrightarrow{+} & [X,Y]^{n+1} & & \\
 & & \downarrow{\scriptstyle\lambda} & & \\
 & & [X,Y]^n & \xrightarrow{\mathcal{O}} & H^{n+1}(X,\Gamma_n Y) \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^3(X,\Gamma_3 Y) & \xrightarrow{+} & [X,Y]^4 & & \\
 & & \downarrow{\scriptstyle\lambda} & & \\
 & & [X,Y]^3 & \xrightarrow{\mathcal{O}} & H^4(X,\Gamma_3 Y)
\end{array}$$

Here $[X, Y]^3$ is the set of homotopy classes of chain maps $\tilde{C}_* X \to \tilde{C}_* Y$. Exactness means that

(4.3.9) $$\operatorname{kernel}(\mathcal{O}) = \operatorname{image}(\lambda)$$

and $\lambda(f)=\lambda(g)$ if and only if there is an α with $g=f+\alpha$. Moreover, for an N-dimensional CW-complex $X=X^N$ the map

$$r_n\colon [X,Y]\to[X,Y]^n$$

is bijective for $n=N+1$ and is surjective for $n=N$. This follows from the definition of $\mathbf{H}_n/\simeq$.

Next we derive from the CW-tower a structure theorem for the group of homotopy equivalences. For a CW-complex X in $\mathbf{CW}_2$ let $\mathrm{Aut}(X)=\mathfrak{E}(X)\subset[X,X]$ be the *group of homotopy equivalences* of X. Moreover, let $E_n(X)\subset[X,X]^n$, $n\geq 3$, be the group of equivalences of r_nX in $\mathbf{H}_n/\simeq$. Then the CW-tower yields the following tower of groups where the arrows $\bar{\mathcal{O}}$ denote derivations and where all the other arrows are homomorphisms between groups.

$$\begin{tikzcd}
 & \mathrm{Aut}(X) \arrow[d] & \\
 & \vdots & \\
H^n(X,\Gamma_nX) \arrow[r, "1^+"] & E_{n+1}(X) \arrow[d, "\lambda"] & \\
 & E_n(X) \arrow[r, "\bar{\mathcal{O}}"] \arrow[d] & H^{n+1}(X,\Gamma_nX) \\
 & \vdots & \\
H^3(X,\Gamma_3X) \arrow[r, "1^+"] & E_4(X) \arrow[d, "\lambda"] & \\
 & E_3(X) \arrow[r, "\bar{\mathcal{O}}"] & H^4(X,\Gamma_3X)
\end{tikzcd} \tag{4.3.10}$$

Here we define the derivation $\bar{\mathcal{O}}$ by the obstruction $\mathcal{O}$ as in (4.1.7) and we set $1^+(\alpha)=1+\alpha$ where 1 is the identity and where $1+\alpha$ is given by (4.3.7). We have exactness

$$\mathrm{image}(1^+)=\mathrm{kernel}(\lambda),\quad \mathrm{image}(\lambda)=\mathrm{kernel}(\bar{\mathcal{O}}).$$

Moreover, as in (4.3.9) we see that for $X=X^N$ the homomorphism

$$r_n\colon \mathrm{Aut}(X)\to E_n(X) \tag{4.3.11}$$

is an isomorphism for $n=N+1$ and is an epimorphism for $n=N$. Finally we derive from Lemma 4.1.8 the following Whitehead theorem for homotopy systems.

(4.3.12) Lemma *A map* $(\xi, \eta): X \to Y$ *in* $\mathbf{H}_n$ *is a homotopy equivalence in* $\mathbf{H}_n/\simeq$ *if and only if* $\xi_*: H_*X \to H_*Y$ *is an isomorphism. Here we set* $H_nX = H_nC$ *for* $X = (C, f_n, X^{n-1})$.

4.4 Boundary invariants for homotopy systems

We have seen that Whitehead's groups Γ_nX are also defined for homotopy systems X in $\mathbf{H}_n$. In the same way we obtain the Γ-groups with coefficients in A

$$\Gamma''_{n-1}(A, X) \subset \Gamma_{n-1}(A, X)$$

for such homotopy systems, see Section 2.2. Here Γ''_{n-1} is a bifunctor

$$\Gamma''_{n-1}: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{H}_n \to \mathbf{Ab} \tag{4.4.1}$$

which fits into the binatural exact sequence

$$\operatorname{Ext}(A, \Gamma_nX) \overset{\Delta}{\rightarrowtail} \Gamma''_{n-1}(A, X) \overset{\mu}{\twoheadrightarrow} \operatorname{Hom}(A, \Gamma''_{n-1}X).$$

Recall that Γ''_{n-1} is the kernel of $i_{n-1}X: \Gamma_{n-1}X \to \pi_{n-1}X$, see Definition 2.2.9. Here we define $\Gamma_{n-1}X$ for a homotopy system $X = (C, f_n, X^{n-1})$ by $\Gamma_{n-1}X = \Gamma_{n-1}X^{n-1}$. We define the homology of X by the homology of C, that is $H_nX = H_nC$. Let $b_{n+1}: H_{n+1}X \to \Gamma_nX$ be a homomorphism and let i be the quotient map in the exact sequence

$$H_{n+1}X \xrightarrow{b_{n+1}} \Gamma_nX \xrightarrow{i} i\Gamma_nX \to 0.$$

Then we define $\Omega_{n-1}(A, X)$ by the push-put diagram (compare (2.6.6))

$$\tag{4.4.2}
\begin{array}{ccccc}
\operatorname{Ext}(A, \Gamma_nX) & \overset{\Delta}{\rightarrowtail} & \Gamma''_{n-1}(A, X) & \xrightarrow{\mu} & \operatorname{Hom}(A, \Gamma''_{n-1}X) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i} & & \| \\
\operatorname{Ext}(A, i\Gamma_nX) & \overset{\Delta}{\rightarrowtail} & \Omega_{n-1}(A, X) & \xrightarrow{\mu} & \operatorname{Hom}(A, \Gamma''_{n-1}X)
\end{array}$$

Hence $\Omega_{n-1}(A, X)$ is functorial in A and functorial in X for those maps $F: X \to Y$ in $\mathbf{H}_n$ for which the diagram

$$\begin{array}{ccc}
H_{n+1}X & \xrightarrow{F_*} & H_{n+1}Y \\
\downarrow{\scriptstyle b_{n+1}} & & \downarrow{\scriptstyle b_{n+1}} \\
\Gamma_nX & \xrightarrow{F_*} & \Gamma_nY
\end{array}$$

commutes. We call such maps b_{n+1}*-proper*. Now let $X = (C, f_{n+1}, X^n)$ be a homotopy system of degree $(n+1)$ in $\mathbf{H}_{n+1}$ and let $\lambda X=(C,f_n,X^{n-1})$ be the corresponding homotopy system of degree n in $\mathbf{H}_n$ given by the functor $\lambda:\mathbf{H}_{n+1}\to\mathbf{H}_n$; see (4.2.3). We associate with X *boundary invariants*

$$(4.4.3)\qquad \begin{cases} b_{n+1}=b_{n+1}X\in\operatorname{Hom}(H_{n+1}X,\Gamma_n(\lambda X)),\\ \beta_n=\beta_nX\in\mathfrak{Q}_{n-1}(H_nX,\lambda X)\quad\text{with}\\ \mu\beta_nX=b_nX:H_nX\to\Gamma''_{n-1}\lambda X\subset\Gamma_{n-1}\lambda X.\end{cases}$$

The abelian group $\operatorname{Hom}(H_{n+1}X,\Gamma_n\lambda X)$ is determined by λX and the abelian group $\mathfrak{Q}_{n-1}(H_nX,\lambda X)$ is determined by the pair $(\lambda X,b_{n+1}X)$ as in (4.4.2). We define the secondary boundary homomorphism $b_{n+1}X$ by the following commutative diagram where Z_{n+1} is the group of $(n+1)$-cycles in C.

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{f_{n+1}} & \pi_nX^n\\ \cup & & \cup\\ Z_{n+1} & \dashrightarrow & \Gamma_n(\lambda X)\\ & {}_{q}\searrow \quad \nearrow_{b_{n+1}} & \\ & H_{n+1}X & \end{array}\qquad (1)$$

Here q is the quotient map for the homology $H_{n+1}X=H_{n+1}C$. We observe that b_{n+1} is well defined by the cocycle condition of Definition (4.2.1)(3), since the kernel of j in (4.2.1)(3) is $\Gamma_n(\lambda X)$.

We can choose a CW-complex X^{n+1} with n-skeleton X^n and attaching maps f_{n+1}, that is, X^{n+1} is the mapping cone of $f_{n+1}:M(C_{n+1},n)\to X^n$, see (4.2.1). Then $b_{n+1}q$ above coincides with Whitehead's secondary boundary $b_{n+1}X^{n+1}$ of X^{n+1}; compare (2.1.17). Using the CW-complex X^{n+1} we define the boundary invariant β_nX in (4.4.2) by the corresponding boundary invariant β_nX^{n+1} in (2.6.7), that is

$$\beta_nX=\beta_nX^{n+1}\in\mathfrak{Q}_{n-1}(H_nX,\lambda X).\qquad (2)$$

Here the right-hand group coincides with $\mathfrak{Q}_{n-1}(H_nX^{n+1},X^{n+1})$ used in (2.6.7) for the definition of β_nX^{n+1}. By naturality of boundary invariants we get:

(4.4.4) Proposition *Let $\bar{F}:X\mapsto Y$ be a map in $\mathbf{H}_{n+1}/\simeq$ and let $F:\lambda X\to Y$ be the induced map in $\mathbf{H}_n/\simeq$. Then we have the equations*

(a) $(\Gamma_nF)_*b_{n+1}X=(H_{n+1}F)^*b_{n+1}Y$;

(b) $F_*\beta_nX=(H_nF)^*\beta_nY$.

Here (a) shows that F is b-proper so that F_* in (b) is well defined; see (4.4.2).

Proof For $\bar{F} = (\xi, \eta)$ there is a map $F^{n+1}: X^{n+1} \to Y^{n+1}$ which extends η and for which $C_* F^{n+1}$ coincides with ξ in degree $\leq n+1$. Thus the naturality of boundary invariants with respect to F^{n+1} yields the result, see Theorem 2.6.9. □

We now study the realizability of boundary invariants.

(4.4.5) Theorem *Let X be an object in $\mathbf{H}_n$ with secondary boundary $b_n X$: $H_n X \to \Gamma''_{n-1} X \subset \Gamma_{n-1} X$. Then for each element*

$$b_{n+1} \in \mathrm{Hom}(H_{n+1} X, \Gamma_n X)$$

and for each element

$$\beta_n \in \mathfrak{Q}_{n-1}(H_n X, X) \quad \textit{with} \quad \mu\beta_n = b_n X$$

there is an object $\bar{X}$ in $\mathbf{H}_{n+1}$ with $\lambda\bar{X} = X$ and $b_{n+1}\bar{X} = b_{n+1}$ and $\beta_n \bar{X} = \beta_n$.

Proof Let $X = (C, f_n, X^{n-1})$. For b_{n+1} and β_n we find a map $v: V \to X^{n-1}$ as in the construction of the boundary operator in Addendam 2.6.5 and Definition 2.3.5(16). We choose v compatible with b_{n+1} and β_n in the statement of the theorem, see Definition 2.3.5(16). Then the mapping cone of v yields the CW-complex $X^{n+1} = C_v$ with $b_{n+1} X^{n+1} = b_{n+1} q$ and $\beta_n X^{n+1} = \beta_n$. Now let f_{n+1} be the attaching map of $(n+1)$-cells in X^{n+1}. Then $\bar{X} = (C, f_{n+1}, X^n)$ satisfies the proposition where X^n is the n-skeleton of X^{n+1}. By definition of v the skeleton X^n is also obtained by the attaching map f_n in X so that $\lambda\bar{X} = X$. □

4.5 Three formulas for the obstruction operator

We show that the boundary invariants in Section 4.4 can be used to compute the obstruction operator in the CW-tower. Let X and Y be objects in $\mathbf{H}_{n+1}$ and let $F: \lambda X \to \lambda Y$ be a map in $\mathbf{H}_n$. Then the obstruction element

(4.5.1) $$\mathcal{O}_{X,Y}(F) \in H^{n+1}(X, \Gamma_n \lambda Y)$$

is defined with the property that $\mathcal{O}_{X,Y}(F) = 0$ if and only if there is a map $F_0: X \to Y$ with $\lambda F_0 = F$; compare (4.3.5). The cohomology group in (4.5.1) is embedded in the universal coefficient sequence

(4.5.2)

$$\mathrm{Ext}(H_n X, \Gamma_n \lambda Y) \overset{\Delta}{\rightarrowtail} H^{n+1}(X, \Gamma_n \lambda Y) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H_{n+1} X, \Gamma_n \lambda Y).$$

We use the operators Δ and μ in this short exact sequence in the next theorem in which two formulas describe the relation between the obstruction element (4.5.1) and the boundary invariants (4.4.3).

(4.5.3) Theorem *Let X and Y be objects in $\mathbf{H}_{n+1}$ and let $F\colon \lambda X \to \lambda Y$ be a map in $\mathbf{H}_n/\simeq$. Then the element $\mu\mathcal{O}_{X,Y}(F)$ is the difference of homomorphisms in the diagram*

$$(a)\qquad \begin{array}{ccc} H_{n+1}X & \xrightarrow{H_{n+1}F} & H_{n+1}Y \\ {\scriptstyle b_{n+1}X}\downarrow & & \downarrow{\scriptstyle b_{n+1}Y} \\ \Gamma_n\lambda X & \xrightarrow{\Gamma_n F} & \Gamma_n\lambda Y \end{array}$$

that is

$$\mu\mathcal{O}_{X,Y}(F) = (\Gamma_n F)(b_{n+1}X) - (b_{n+1}Y)(H_{n+1}F).$$

If diagram (a) commutes then the following equation holds in $\operatorname{Ext}(H_n X, i\Gamma_n\lambda Y)$ *where*

$$H_{n+1}Y \xrightarrow{b_{n+1}Y} \Gamma_n\lambda Y \xrightarrow{i} i\Gamma_n\lambda Y \to 0$$

is exact:

$$(b)\qquad -i_*\Delta^{-1}\,\mathcal{O}_{X,Y}(F) = \Delta^{-1}(F_*\,\beta_n X - (H_n F)^*\beta_n Y).$$

Here the left-hand side is given by Δ in (4.5.2) *and is well defined since we assume that diagram (a) commutes. The right-hand side is obtained by Δ in* (4.4.2) *and is well defined since F is b_n-proper by Proposition* 4.4.4(a).

Proof Let $X = (C, f_{n+1}, X^n)$ and let $Y = (C', g_{n+1}, Y^n)$ and let

$$\eta'\colon X^n \to Y^n$$

be a map associated with $F = (\xi, \eta)\colon \lambda X \to \lambda Y$, that is, η' extends η and $C_*(\eta')$ coincides with ξ in degree $\leq n$. Then the cohomology class $\mathcal{O}_{X,Y}(F)$ is represented by the cocycle

$$\mathcal{O}(F) = -g_{n+1}\xi_{n+1} + \eta'_* f_{n+1} \tag{1}$$

in $\operatorname{Hom}(C_{n+1}, \pi_n Y^n)$. This cocycle actually maps to the subgroup $\Gamma_n Y \subset \pi_n Y^n$. Moreover $\mu\mathcal{O}_{X,Y}(F)$ is represented by the restriction

$$\mu\mathcal{O}_{X,Y}(F)q = \mathcal{O}(F)\,|\,Z_{n+1} \tag{2}$$

where $q\colon Z_{n+1} \to H_{n+1}X$ is the quotient map. Now (a) is an easy consequence of the following diagram in which all subdiagrams except the one in the middle commute.

$$
\begin{tikzcd}
Z_{n+1} \arrow[rrr, "\xi^Z_{n+1}"] \arrow[dr] \arrow[ddd, "(b_{n+1}X)q"'] & & & Z'_{n+1} \arrow[dl] \arrow[ddd, "(b_{n+1}Y)q"] \\
& C_{n+1} \arrow[r, "\xi_{n+1}"'] \arrow[d, "f_{n+1}"] & C'_{n+1} \arrow[d, "g_{n+1}"] & \\
& \pi_n X^n \arrow[r, "\eta'_*"'] & \pi_n Y^n & \\
\Gamma_n X \arrow[ur] \arrow[rrr, "\Gamma_n(F)"'] & & & \Gamma_n Y \arrow[ul]
\end{tikzcd}
\tag{3}
$$

Now assume that the diagram in (a) commutes. This implies that the exterior square of (3) commutes. Hence we get via the exact sequence

$$
0 \to Z_{n+1} \to C_{n+1} \to B_n \to 0 \tag{4}
$$

the diagram

$$
\begin{tikzcd}
B_n \arrow[r, "\xi^B_n"'] \arrow[d, "f'_{n+1}"'] & B'_n \arrow[d, "g'_{n+1}"] \\
\pi_n X^n / f_{n+1} Z_{n+1} \arrow[r, "\eta'_*"'] & \pi_n Y^n / g_{n+1} Z'_{n+1}
\end{tikzcd}
\tag{5}
$$

as a quotient of diagram (3). Here f'_{n+1} and g'_{n+1} are induced by f_{n+1} and g_{n+1} respectively since we use (4). The difference

$$
\Delta = -g'_{n+1}\xi^B_n + \eta'_* f'_{n+1} \tag{6}
$$

maps to the subgroup

$$
i\Gamma_n \lambda Y = \operatorname{cok} b_{n+1} Y \subset \pi_n Y^n / g_{n+1} Z'_{n+1} \tag{7}
$$

Moreover, Δ represents the element

$$
\{\Delta\} = i_* \Delta^{-1} \mathscr{O}_{X,Y}(F) \in \operatorname{Ext}(H_n X, i\Gamma_n \lambda Y). \tag{8}
$$

Now let X' and Y' be the mapping cones of

$$
f_{n+1} \mid Z_{n+1} : M(Z_{n+1}, n) \to X^n
$$

and

$$
g_{n+1} \mid Z'_{n+1} : M(Z'_{n+1}, n) \to Y^n
$$

respectively. Since the exterior square of diagram (3) commutes we can find an extension

$$
\eta'' : X' \to Y' \tag{9}
$$

of η' such that $C_{n+1}(\eta'') = \xi^Z_{n+1}$. Moreover we have isomorphisms

$$\begin{cases} \pi_n X' = \pi_n X^n / f_{n+1} Z_{n+1} & \text{and} \\ \pi_n Y' = \pi_n Y^n / g_{n+1} Z'_{n+1}. \end{cases} \tag{10}$$

Using (5) and (10) we have the following diagram in which φ is a map between Moore spaces which induces $H_n F$ in homology

$$\begin{tikzcd}
M(H_n, n-1) \arrow[rrr, "\varphi"] \arrow[dr, "-q"] \arrow[ddd, "\beta"'] & & & M(H'_n, n-1) \arrow[dl, "-q"'] \arrow[ddd, "\beta'"] \\
& M(B_n, n) \arrow[r, "\xi^B_n"'] \arrow[d, "f'_{n+1}"] & M(B'_n, n) \arrow[d, "g'_{n+1}"] & \\
& X' \arrow[r, "\eta''"'] & Y'' & \\
X^{n-1} \arrow[ur] \arrow[rrr, "\eta"'] & & & Y^{n-1} \arrow[ul]
\end{tikzcd} \tag{11}$$

Here $q: M(H_n, n-1) \to M(B_n, n)$ is the pinch map since we obtain $M(H_n, n-1)$ by the presentation

$$0 \to B_n \to Z_n \to H_n \to 0$$

where B_n and Z_n are free abelian groups. Moreover β (resp. β') is chosen for X (resp. Y) as in Definition 2.3.5(15). By (2.6.8) the map β (resp. β') represents the boundary invariant $\beta_n X$ (resp. $\beta_n Y$). All small subdiagrams of (11) except the one in the middle homotopy commute. This shows that $-\Delta$, with Δ in (b), also represents the right-hand side of the equation in (b) and hence the proof of this formula is complete by (8). □

Let $X = (C, f_{n+1}, X^n)$ and $Y = (C', g_{n+1}, Y^n)$ again be objects in $\mathbf{H}_{n+1}$ and let $F = (\xi, \eta): \lambda X \to \lambda Y$ be a map in $\mathbf{H}_n$. Thus $\xi: C \to C'$ is a chain map and $\eta: X^{n-1} \to Y^{n-1}$ is given by a cellular map for which $C_* \eta$ coincides with ξ in degree $\leq n-1$. Let

$$\text{(4.5.4)} \qquad \{q\delta\} \in \operatorname{Ext}(H_n X, H_{n+1} Y)$$

be represented by a homomorphism δ which is part of the following composition

$$C_{n+1} \overset{d}{\twoheadrightarrow} B_n \xrightarrow{\delta} Z'_{n+1} \overset{i}{\rightarrowtail} C'_{n+1}.$$

Here d is the boundary in C and i is the inclusion of cycles in C'. Moreover $q: Z'_{n+1} \twoheadrightarrow H_{n+1} Y$ is the quotient map so that $q\delta$ represents an element in the Ext-group (4.5.4). Using δ we define the following *δ-deformation* $\xi + \delta$ of the chain map ξ, namely let $\xi + \delta$ be the chain map

(4.5.5)

$$\xi + \delta : C \to C' \quad \text{with} \quad \begin{cases} (\xi + \delta)_{n+1} = \xi_{n+1} + i\delta d & \text{and} \\ (\xi + \delta)_i = \xi_i & \text{otherwise.} \end{cases}$$

One readily checks that $(\xi + \delta, \eta): \lambda X \to \lambda Y$ is again a well defined map in $\mathbf{H}_n$ for all δ. Our third formula describes the obstruction for this map. Recall that $F = (\xi, \eta)$ is a b_{n+1}-proper map if the diagram in Theorem 4.5.3(a) commutes.

(4.5.6) Proposition *Let X and Y be objects in $\mathbf{H}_{n+1}$ and let $(\xi, \eta): \lambda X \to \lambda Y$ be a b_{n+1}-proper map in $\mathbf{H}_n$. Then also $(\xi + \delta, \eta)$ is a b_{n+1}-proper map and one has the formula*

$$\Delta^{-1}\mathscr{O}_{X,Y}(\xi + \delta, \eta) = \Delta^{-1}\mathscr{O}_{X,Y}(\xi, \eta) - (b_{n+1}Y)_*\{q\delta\}$$

in $\operatorname{Ext}(H_n X, \Gamma_n \lambda Y)$. *Here Δ is the operator in* (4.5.2).

Proof Clearly $(\xi + \delta, \eta)$ is b_{n+1}-proper since $H_{n+1}(\xi) = H_{n+1}(\xi + \delta)$. Now the formula is an easy consequence of the fact that the following diagram commutes

$$\begin{array}{ccc} C'_{n+1} & \xrightarrow{\quad g_{n+1} \quad} & \pi_n Y^n \\ \uparrow & & \uparrow \\ Z'_{n+1} \twoheadrightarrow H_{n+1} & \xrightarrow[b_{n+1}Y]{} & \Gamma_n Y \end{array}$$

□

Remark In Baues [CH] II.5.4 and (II.5.6) we have shown that the formulas of Theorem 4.5.3(a) and Proposition 4.5.6 have a generalization for the non-simply connected case. The formula of Theorem 4.5.3(b), however, is not available in the non-simply connected case. The formula in Baues [CH] II.5.4(2) corresponds via Theorem 2.6.9(c) to the formula in Theorem 4.5.3(b).

4.6 λ-Realizability

Using the formula in Section 4.5 we obtain a crucial result on the λ-realizability of maps in $\mathbf{H}_n$. This leads to a classification of homotopy types of objects in $\mathbf{H}_{n+1}$.

(4.6.1) Theorem *Let X and Y be objects in $\mathbf{H}_{n+1}$ and let $F = (\xi, \eta)\colon \lambda X \to \lambda Y$ be a map in $\mathbf{H}_n$. Moreover assume that F satisfies the equations*

(a) $(\Gamma_n F)_* b_{n+1} X = (H_{n+1} F)^* b_{n+1} Y$; *and*

(b) $F_* \beta_n X = (H_n F)^* \beta_n Y$.

Then there exists $\{q\delta\} \in \mathrm{Ext}(H_n X, H_{n+1} Y)$ such that the δ-deformation $F_\delta = (\xi + \delta, \eta)$ of F in (4.5.5) is λ-realizable by a map $\bar{F}\colon X \to Y$ with $\lambda \bar{F} = F_\delta$.

In Proposition 4.4.4 we have seen that (a) and (b) are always true if F is λ-realizable. Now Theorem 4.6.1 shows that these equations are the criterion for the λ-realizability up to δ-deformation.

Proof of Theorem 4.6.1 The exact sequence

$$H_{n+1} Y \xrightarrow[b_{n+1}Y]{} \Gamma_n \lambda Y \xrightarrow[i]{} i\Gamma_n \lambda Y \to 0 \tag{1}$$

induces the exact sequence of Ext groups

$$\mathrm{Ext}(H_n X, H_{n+1} Y) \xrightarrow[(b_{n+1}Y)_*]{} \mathrm{Ext}(H_n X, \Gamma_n \lambda Y) \xrightarrow[i_*]{} \mathrm{Ext}(H_n X, i\Gamma_n \lambda Y) \to 0. \tag{2}$$

Now (a) implies by Theorem 4.5.3(a) that $\mu \mathscr{O}_{X,Y}(F) = 0$ and (b) implies by Theorem 4.5.3(b) that for i_* in (2) the element

$$i_* \Delta^{-1} \mathscr{O}_{X,Y}(F) = 0 \tag{3}$$

is trivial. Hence by exactness of (2) there is an element $\{q\delta\} \in \mathrm{Ext}(H_n X, H_{n+1} Y)$ with

$$\Delta(b_{n+1} Y)_* \{q\delta\} = \mathscr{O}_{X,Y}(F). \tag{4}$$

Now Proposition 4.5.6 shows that the δ-deformation of (ξ, η) satisfies

$$\mathscr{O}_{X,Y}(\xi + \delta, \eta) = \mathscr{O}_{X,Y}(\xi, \eta) - \Delta(b_{n+1} Y)_* \{q\delta\}$$

and this element is trivial by (4). Hence the obstruction property in (4.3.5) shows that there is a map $\bar{F} = (\xi + \delta, \bar{\eta})\colon X \to Y$ in $\mathbf{H}_{n+1}$ with $\lambda \bar{F} = (\xi + \delta, \eta)$. □

We use the equations (a) and (b) in Theorem 4.6.1 for the definition of the following category.

(4.6.2) Definition Objects in the category $\mathbf{H}_n^b/\simeq$ are triplets (X, b_{n+1}, β_n) where X is an object in $\mathbf{H}_n$ and where b_{n+1} and β_n are elements

$$b_{n+1} \in \operatorname{Hom}(H_{n+1}X, \Gamma_n X), \tag{1}$$

$$\beta_n \in \mathfrak{Q}_{n-1}(H_n X, X) \quad \text{with} \quad \mu\beta_n = b_n X \tag{2}$$

The groups in (1) and (2) are defined in (4.3.2) and (4.4.2) above. A morphism $F:(X, b_{n+1}, \beta_n) \to (Y, b'_{n+1}, \beta'_n)$ between such triples is a map $F: X \to Y$ in $\mathbf{H}_n/\simeq$ for which the equations

$$(\Gamma_n F)_* b_{n+1} = (H_{n+1}F)^* b'_{n+1}$$

and

$$F_* \beta_n = (H_n F)^* \beta'_n \tag{3}$$

are satisfied. By Proposition 4.4.4 we have the functor

$$\lambda^b : \mathbf{H}_{n+1}/\simeq \to \mathbf{H}_n^b/\simeq \tag{4.6.3}$$

which carries X in $\mathbf{H}_{n+1}$ to the triple $\lambda^b(X) = (\lambda X, b_{n+1}X, \beta_n X)$ given by the boundary invariants of X in (4.4.3). This functor is not full but satisfies by Theorem 4.6.1 the realizability condition for maps up to δ-deformation. Moreover by Theorem 4.4.5 this functor satisfies the realizability condition for objects. This yields the following classification of equivalence classes of objects in the category $\mathbf{H}_{n+1}/\simeq$.

(4.6.3) Theorem *The homotopy type of an object X in $\mathbf{H}_{n+1}/\simeq$ is completely determined by the triple $(\lambda X, b_{n+1}X, \beta_n X)$. In fact, the functor λ^b above induces a 1–1 correspondence between equivalence classes of objects in $\mathbf{H}_{n+1}/\simeq$ and equivalence classes of objects in $\mathbf{H}_n^b/\simeq$.*

Proof Surjectivity of the correspondence follows from Theorem 4.4.5. We now check injectivity. Let X and Y be objects in $\mathbf{H}_{n+1}$ and let

$$F = (\xi, \eta) : (\lambda X, b_{n+1}X, \beta_n X) \to (\lambda Y, b_{n+1}Y, \beta_n Y)$$

be an equivalence in $\mathbf{H}_n^b/\simeq$, that is $F: \lambda X \to \lambda Y$ is a homotopy equivalence in $\mathbf{H}_n/\simeq$ which satisfies the equations (a) and (b) in Theorem 4.6.1. Then Theorem 4.6.1 shows that there is an element δ and a map $\bar{F} = (\xi + \delta, \bar{\eta})$: $X \to Y$ with $\lambda\bar{F} = (\xi + \delta, \eta)$. Here in fact $\bar{F}$ is a homotopy equivalence in $\mathbf{H}_{n+1}$ since $\bar{F}$ induces an isomorphism in homology; see Lemma 4.3.12. In fact we have

$$H_*\bar{F} = H_*(\xi + \delta, \bar{\eta}) = H_*(\xi + \delta) = H_*(\xi) = H_*(\xi, \eta) = H_* F$$

where $H_* F$ is an isomorphism since F is a homotopy equivalence. □

4.7 Proof of the boundary classification theorem

In this section we complete the proof of the classification theorem 3.4.4. It remains to prove that Λ' in Proposition 3.7.5 is a detecting functor. We again assume that $m \geq 2$ and that

$$\mathbf{C} = \mathbf{types}_m^{r-1} \tag{4.7.1}$$

with $n = m + r$. The bype functor F on $\mathbf{C}$ is given by $F(H, X) = \Gamma''_{n-1}(H, X)$. We consider the functor

$$\begin{aligned} &\Lambda' : \mathbf{spaces}_m^{r+1} \to \mathbf{Bypes}(\mathbf{C}, F) \quad \text{with} \\ &\Lambda'(X) = (P_{n-1}X, H_nX, H_{n+1}X, b_{n+1}X, \beta_nX) \end{aligned} \tag{4.7.2}$$

(4.7.3) Theorem *The functor Λ' is a detecting functor.*

This is the reformulation of Proposition 3.7.5. It is enough to consider the case $m = 2$. Then we get the equivalence of categories

$$(\mathbf{CW}_2)^{n+1}/\simeq \; \to \mathbf{spaces}_2^{r+1}.$$

Here $(\mathbf{CW}_2)^{n+1}$ is the category of CW-complexes X with $X^1 = *$ and $\dim X \leq n + 1$ and of cellular maps. We obtain a proof of Theorem 4.7.3 by use of the following commutative diagram of functors.

$$\begin{array}{ccc} (\mathbf{CW}_2)^{n+1}/\simeq & \xrightarrow{\Lambda'} & \mathbf{Bypes}(\mathbf{C}, F) \\ \downarrow r & & \uparrow \lambda'' \\ (\mathbf{H}_{n+1}/\simeq)^{n+1} & \xrightarrow{\lambda^b} & (\mathbf{H}_n^b/\simeq) \end{array} \tag{4.7.4}$$

Here $(\mathbf{H}_{n+1}/\simeq)^{n+1}$ denotes the full subcategory of $\mathbf{H}_{n+1}/\simeq$ consisting of objects $X = (C, f_{n+1}, X^n)$ with $\dim(X) \leq n + 1$ or equivalently with $C_i = 0$ for $i > n + 1$. Similarly the category $(H_n^b/\simeq)^{n+1}$ consists of $(n + 1)$-dimensional objects in $\mathbf{H}_n^b$. The functor λ^b in the bottom row of (4.7.4) is a restriction of the corresponding functor λ^b in (4.6.3). Moreover the functor r in (4.7.4) is the restriction of the functor r_{n+1} in (4.2.7). The CW-tower shows that r is a detecting functor. Finally we obtain the functor λ'' in (4.7.4) as follows. For this we use the functor

$$P_{n-1} : \mathbf{H}_n/\simeq \; \to (n-1)\text{-}\mathbf{types} \tag{4.7.5}$$

defined as in (4.3.1). Then λ'' carries an object (X, b_{n+1}, β_n) in $\mathbf{H}_n^b/\simeq$ to the object

$$\lambda''(X, b_{n+1}, \beta_n) = (P_{n-1}X, H_nX, \beta_n, H_{n+1}X, b_{n+1}). \tag{4.7.6}$$

The definition of Λ' in the classification theorem and the definition of λ^b in (4.6.3) show that diagram (4.7.4) commutes. For this we use identifications

$$\Gamma_i X \cong \Gamma_i P_{n-1} X \quad \text{for} \quad i \leq n \tag{1}$$

$$\Gamma''_{n-1}(A, X) \cong \Gamma''_{n-1}(A, P_{n-1} X) \tag{2}$$

which are available for objects X in $\mathbf{H}_n$ in the same way as for spaces X. Now it is obvious by (4.7.5) and (4.7.6) how to define λ'' on maps, namely

$$\lambda''(F) = (P_{n-1} F, H_n F, H_{n+1} F). \tag{3}$$

Using the naturality of the identifications (1) and (2) we see that λ'' as a well-defined functor. Below we show

(4.7.7) Theorem *The functor λ'' is a detecting functor.*

Using the proposition and the results in Section 4.6 we can show that also Λ' in (4.7.4) is a detecting functor as follows.

Proof of Theorem 4.7.3 Let X_0 be an object in $\mathbf{Bypes}(\mathbf{C}, F)$. We first find a λ''-realization X_1 by Theorem 4.7.7, then we find a λ^b-realization X_2 of X_1 by Theorem 4.4.5, and then we get an r-realization X_3 of X_2 since r is a detecting functor. Hence X_3 is a Λ'-realization of X_0 by the commutativity of (4.7.4). The Whitehead theorem shows that Λ' reflects isomorphisms hence it remains to show that Λ' is a full functor. For this let X_3, Y_3 be objects in $(\mathbf{CW}_2)^{n+1}/\simeq$ and let $X_2 = r(X_3)$, $X_1 = \lambda^b(X_2)$, $X_0 = \lambda''(X_1) = \Lambda'(X_3)$, and let Y_2, Y_1, and Y_0 be given accordingly by Y_3. Then any map $F_0\colon X_0 \to Y_0$ admits a λ''-realization $F_1 = (\xi, \eta)\colon X_1 \to Y_1$. By Theorem 4.6.1 there exists δ such that $F_1' = (\xi + \delta, \eta)$ admits a λ^b-realization $F_1\colon X_2 \to Y_2$ which in turn admits an r-realization $F_3\colon X_3 \to Y_3$. Since $\xi + \delta$ induces the same homology homomorphism as ξ we see that

$$\lambda''(\xi + \delta, \eta) = \lambda''(\xi, \eta).$$

Hence we get $\Lambda' F_3 = F_0$ and thus Λ' is full. □

Proof of Theorem 4.7.7 We first consider the λ''-realizability of objects. For this let Y be a 1-connected $(n-1)$-type, and let H_n, H_{n+1} be abelian groups with H_{n+1} free abelian. Moreover let b_n be a homomorphism for which the sequence

$$H_n \xrightarrow{b_n} \Gamma_{n-1} Y \xrightarrow{i_{n-1} Y} \pi_n Y \tag{1}$$

is exact; compare Definition 3.2.2(1). We construct an object $X = (C, f_n, X^{n-1})$ in $(\mathbf{H}_n/\simeq)^{n+1}$ which realizes the tuple (Y, H_n, H_{n+1}, b_n) that is, there are isomorphisms

$$\begin{cases} H_{n+1} = H_{n+1}C & (2) \\ H_n = H_nC & (3) \\ Y = P_{n+1}X & (4) \\ b_n = b_nX : H_nX \to \Gamma_{n-1}X = \Gamma_{n-1}Y. & (5) \end{cases}$$

We construct X as follows. Let Z_n be a free abelian group and let $q: Z_n \twoheadrightarrow H_n$ be a surjection with kernel B_n. Then we define by the cellular chain complex C_*Y of Y with cycles Z_*Y and boundaries B_*Y the chain complex C as follows:

$$\begin{cases} C_{n+1} = H_{n+1} \oplus B_n & (6) \\ C_n = Z_n \oplus B_{n-1}Y & (7) \\ C_j = C_jY \quad \text{for} \quad j \le n-1 & (8) \end{cases}$$

The boundary $d: C_{n+1} \to C_n$ is trivial on H_{n+1} and is the inclusion $B_n \subset Z_n$ if restricted to B_n; moreover $d: C_n \to C_{n-1}$ is trivial on Z_n and is the inclusion on $B_{n-1}Y$. Then clearly (2) and (3) are satisfied. Next we define X^{n-1} by the $(n-1)$-skeleton of Y, that is $X^{n-1} = Y^{n-1}$.

For the construction of

$$f_n : C_n = Z_n \oplus B_{n-1}Y \to \pi_{n-1}Y^{n-1} \tag{9}$$

we make the following choices. Let d^Y be the boundary in C_*Y. We choose a splitting t,

$$C_nY \underset{t}{\overset{d^Y}{\rightleftarrows}} B_{n-1}Y \tag{10}$$

with $dt = 1$. Moreover we choose a homomorphism b' for which the following diagram commutes

$$\begin{tikzcd} B_n \arrow[d] & \subset & Z_n \arrow[r, two heads, "a"] \arrow[d, "b'"] & H_b \arrow[r, "b_n"] \arrow[d, "b"] & \Gamma_{n-1}Y \\ B_n(Y) & \subset & Z_n(Y) \arrow[r, two heads, "q'"'] & H_nY \arrow[ur, "b_nY"'] & \end{tikzcd} \tag{11}$$

Here b' exists since b_nY is injective and since the image of b_nY is the kernel of $i_{n-1}Y$ in (1) which in turn is the image of b_n by exactness in (1). Hence b in the diagram is well defined and surjective. Therefore the lift b' exists since Z_n is free abelian. In fact we can choose Z_n in such a way that b' is surjective. For this choose a surjection $Z_n \twoheadrightarrow A$ where A is the pull back of q'

and b, where q' is the quotient map in (11). Using b' and t we get the surjective homomorphism

$$\varphi: C_n = Z_n \oplus B_{n-1}(Y) \twoheadrightarrow C_n(Y) \tag{12}$$

for which $\varphi \mid Z_n$ is the composite

$$Z_n \xrightarrow{b'} Z_n(Y) \subset C_n(Y)$$

for which $\varphi \mid B_{n-1}(Y) = t$.

Now let $(C_* Y, g_n, Y^{n-1})$ be the homotopy system of order n given by Y with

$$g_n : C_n(Y) \to \pi_{n-1} Y^{n-1}.$$

We define f_n in (9) by the composite

$$f_n = g_n \varphi. \tag{13}$$

Then $X = (C, f_n, Y^{n-1})$ is a well-defined homotopy system of order n. In fact, f_n satisfies the cocycle condition since g_n does and since $b'(B_n) \subset B_n(Y)$. Moreover $h\alpha = jf_n$ in (4.2.1)(3) is satisfied since we know that $hd^Y = jg_n$ is satisfied.

We now observe that $\pi_{n-1} X = \pi_{n-1} Y$ since φ above is surjective. Hence we get the homotopy equivalence (4). Moreover we get (5) by the definition of $b_n X$ in (4.4.3)(1) and by the commutative diagram (11). This completes the proof that $X = (C, f_n, X^{n-1})$ is a realization of (Y, H_n, H_{n+1}, b_n). This, however, also implies that (X, b_{n+1}, β_n) is a λ''-realization of the object $(Y, H_n, \beta_n, H_{n+1}, b_{n+1})$. For this we use the natural identifications in (4.7.6)(1), (2).

By the homological Whitehead theorem it is clear that λ'' reflects isomorphisms. Therefore it remains to show that λ'' is full. For this let $X^b = (X, b_{n+1}, \beta_n)$ and $Y^b = (Y, b'_{n+1}, \beta'_n)$ be objects in $(\mathbf{H}_n^b/\simeq)^{n+1}$ with $X = (C, f_n, X^{n-1})$ and $Y = (C', g_n, Y^{n-1})$ and let $(F; \varphi_n, \varphi_{n+1}): X^b \to Y^b$ be a map in $\mathbf{Bypes}(\mathbf{C}, F)$, that is

$$\varphi_n : H_n \to H'_n,\ \varphi_{n+1} : H_{n+1} \to H'_{n+1} \tag{14}$$

are homomorphisms (with $H_* = H_* X$ and $H'_* = H_* Y$) and

$$F: P_{n-1} X \to P_{n-1} Y \tag{15}$$

is a map which we can assume to be cellular. Recall that the n-skeleton of $P_{n+1} X$ coincides with X^n where X^n is the mapping cone of f_n; in the same way the n-skeleton of $P_{n-1} Y$ coincides with Y^n where Y^n is the mapping cone of g_n. Thus the cellular map F in (15) yields a restriction $(F^n, F^{n-1}): (X^n, X^{n-1}) \to (Y^n, Y^{n-1})$. We now show that there exists a λ'-realization

$$(\xi, F^{n-1}): X \to Y \tag{16}$$

of $(F, \varphi_n, \varphi_{n+1})$. For this we choose splittings of $C_k \twoheadrightarrow B_{k-1}$ and $C'_k \twoheadrightarrow B'_{k-1}$. Hence we get

$$C_k = Z_k \oplus B_{k-1}, \quad C'_k = Z'_k \oplus B_{k-1} \tag{17}$$

with $Z_{n+1} = H_{n+1}$, $Z'_{n+1} = H'_{n+1}$ since C and C' are $(n+1)$-dimensional. We consider the following diagram

$$\begin{tikzcd}
C_n \arrow[d, "\xi'_n"'] \arrow[rrrr, bend left, "f_n"] & \supset Z_n \arrow[r, two heads, "q"] \arrow[d, "\xi'_n"] \arrow[r, phantom, shift right=6ex, "(*)"] & H_n \arrow[r, "b_n"] \arrow[d, "\varphi_n"] & \Gamma_{n-1}X \arrow[d, "F_*"] & \subset \pi_{n-1}X^{n-1} \arrow[d, "F_*^{n-1}"] \\
C'_n \arrow[rrrr, bend right, "g_n"'] & \supset Z'_n \arrow[r, two heads, "q'"'] & H'_n \arrow[r, "b'_n"'] & \Gamma_{n-1}Y & \subset \pi_{n-1}Y^{n-1}
\end{tikzcd}$$

Here $\xi'_n = C_n(F^n)$ is induced by F^n. This implies that the exterior square of the diagram commutes. Therefore ξ'_n admits the restriction $\xi'_n: Z_n \to Z'_n$, also denoted by ξ'_n. Since $(F, \varphi_n, \varphi_{n+1})$ is a map between bypes we see that for $b_n = \mu\beta_n$ and $b'_n = \mu\beta'_n$ the equation

$$F_* b_n = b'_n \varphi_n \tag{18}$$

is satisfied. In fact, (18) is a consequence of the equation $F_*(\beta_n) = \varphi_n^*(\beta'_n)$; compare Definition 3.2.2(3). Hence all subdiagrams of the diagram above commute except possibly diagram $(*)$. We have however $F_* b_n q = b'_n q' \xi'_n$ so that the difference

$$\Delta = \varphi_n q - q'\xi'_n: Z_n \to \text{kernel } b'_n \subset H'_n \tag{19}$$

maps to the kernel of b'_n. Since $b'_n q' = b_n Y^n$ we see that we have surjections

$$q'q'': \pi_n Y^n \xrightarrow{q''} \text{kernel } b_n Y^n \xrightarrow{q'} \text{kernel } b'_n$$

and hence we can choose a homomorphism

$$\beta: Z_n \to \pi_n Y^n \quad \text{with} \quad q'q''\beta = \Delta. \tag{20}$$

Let $p: C_n \to Z_n$ be the projection. Then we obtain by the action in (4.2.5) a cellular map $F^n + \beta p: X^n \to Y^n$ which extends F^{n-1}. We can therefore replace ξ'_n in the diagram by

$$\xi_n = C^n(F^n + \beta p) \tag{21}$$

with the effect that then all subdiagrams of the diagram are commutative. In particular we get for subdiagram $(*)$

$$\begin{aligned}\varphi_n q - q'\xi_n &= \varphi_n q - q'C^n(F^n + \beta p) \mid Z_n\\ &= \varphi_n q - q'(\xi_n' + q''\beta)\\ &= \varphi_n q - q'\xi_n' - q'q''\beta = \Delta - \Delta = 0.\end{aligned}$$

Hence the restriction $\xi_n^B \colon B_n \to B_n'$ of ξ_n is well defined and we can set

$$\xi_{n+1} = \varphi_{n+1} \oplus \xi_n^B \tag{22}$$

by use of (17). Moreover ξ coincides with $C_* F^{n-1}$ in degree $\leq n-1$ so that a chain map $\xi \colon C \to C'$ is well defined. Moreover it is clear that (ξ, F^{n-1}) is a λ''-realization of $(F, \varphi_n, \varphi_{n+1})$ since we have the natural isomorphisms (4.7.6)(1), (2). □

The proof of Theorem 4.7.7 above completes the proof of the classification theorem 3.4.4; compare Theorem 4.7.3.

4.8 The computation of isotropy groups in the CW-tower

We consider two different actions of abelian groups on certain sets of homotopy classes and we describe a general method for the computation of isotropy groups of these actions. For an n-dimensional simply connected CW-complex X^n and a space Y we have the action

(4.8.1) $$[X^n, Y] \times H^n(X^n, \pi_n Y) \xrightarrow{+} [X^n, Y]$$

defined in (4.2.5)(2). Here $[X^n, Y]$ is the set of homotopy classes in **Top**$^*/\simeq$. The cofibre sequence for $j \colon X^{n-1} \subset X^n$ shows that $j^* \colon [X^n, Y] \to [X^{n-1}, Y]$ satisfies for elements $\eta, \eta' \in [X^n, Y]$

$$j^*(\eta) = j^*(\eta') \Leftrightarrow \exists \alpha \in H^n(X^n, \pi_n Y) \quad \text{with} \quad \eta + \alpha = \eta'. \tag{1}$$

Thus the action (4.8.1) is useful for the inductive computation of the sets $[X^n, Y]$ with $n \geq 1$. Let $I(\eta) \subset H^n(X^n, \pi_n Y)$ be the isotropy group of the action in η, that is

$$I(\eta) = \{\alpha \in H^n(X^n, \pi_n Y); \eta + \alpha = \eta\}. \tag{2}$$

Hence by (1) the orbit of η is given by

$$\eta \in (j^*)^{-1}(\eta_0) \approx H^n(X^n, \pi_n Y)/I(\eta) \tag{3}$$

where $\eta_0 = j^*\eta$ and where the right-hand side is the quotient group. The bijection carries $\{\alpha\}$ to $\eta + \alpha$. The group $I(\eta)$ depends actually only on η_0. For the computation of $I(\eta)$ in (2) one needs a spectral sequence; compare Baues [AH] VI.5.9 (4).

On the other hand, let now X and Y be homotopy systems of degree $n+1$ and let $[X,Y]^{n+1}$ be the set of homotopy classes of maps $X \to Y$ in $\mathbf{H}_{n+1}/\simeq$. Then we have by (4.3.7) the action

$$(4.8.2) \qquad [X,Y]^{n+1} \times H^n(X, \Gamma_n Y) \xrightarrow{+} [X,Y]^{n+1}.$$

The functor $\lambda\colon H_{n+1}/\simeq \to H_n/\simeq$ yields the function $\lambda\colon [X,Y]^{n+1} \to [X,Y]^n$ which satisfies for elements $F, F' \in [X,Y]^{n+1}$

$$\lambda(F) = \lambda(F') \Leftrightarrow \exists \alpha \in H^n(X, \Gamma_n Y) \quad \text{with} \quad F + \alpha = F'. \tag{1}$$

Thus (4.8.2) is useful for the inductive computation of the sets $[X,Y]^n, n \geq 1$; compare for this the tower of homotopy sets in (4.3.8). Let $I(\xi, \eta) \subset H^n(X, \Gamma_n Y)$ be the isotropy group of the action in $\{(\xi, \eta)\} = F$ where $\xi\colon C \to C'$ is a chain map and where $\eta\colon X^n \to Y^n$ is a cellular map with $X = (C, f_{n+1}, X^n)$ and $Y = (C', g_{n+1}, Y^n)$. We have

$$I(\xi, \eta) = \{\alpha \in H^n(X, \Gamma_n Y); F + \alpha = F\}. \tag{2}$$

Hence the orbit of $F = \{(\xi, \eta)\}$ is given by

$$F \in \lambda^{-1}(F_0) \approx H^n(X, \Gamma_n Y)/I(\xi, \eta) \tag{3}$$

where $F_0 = \lambda F$. The bijection carries the coset $\{\alpha\}$ to $F + \alpha$. The group $I(\xi, \eta)$ depends only on F_0. For the computation of $I(\xi, \eta)$ we have the following result. Let

$$j\colon H^n(X, \Gamma_n Y) \subset H^n(X^n, \Gamma_n Y) \to H_n(X^n, \pi_n Y^n) \tag{4}$$

be induced by the inclusions $C_* X^n \subset C$ and $\Gamma_n Y \subset \pi_n Y^n$ respectively. Moreover let

$$H_{n+1} Y \xrightarrow{b_{n+1} Y} \Gamma_n Y \xrightarrow{i} i\Gamma_n Y \to 0 \tag{5}$$

be an exact sequence defined by the secondary boundary homomorphism b_{n+1} of Y and let

$$0 \to i\Gamma_n Y \rightarrowtail \pi_n Y \to \ker b_n Y \to 0 \tag{6}$$

be the short exact sequence given by Whitehead's exact sequence. Compare Theorem 2.6.9(c) where we show that the extension element $\{\pi_n Y\}$ given by (6) satisfies

$$(\beta_n Y)_\dagger = \{\pi_n Y\} \in \operatorname{Ext}(\ker(b_n Y), i\Gamma_n Y) \tag{7}$$

so that the extension (6) is determined by the boundary invariant $\beta_n Y$. Let

$$\tau\colon \operatorname{Hom}(H_{n-1} X, \ker(b_n Y)) \to \operatorname{Ext}(H_{n-1} X, i\Gamma_n Y) \tag{8}$$

be the boundary homomorphism associated to the extension (6), that is

$$\tau(\alpha) = \alpha^* \{\pi_n Y\} \tag{9}$$

for $\alpha: H_{n-1}X \to \ker(b_nY)$. As usual the sum of subgroups U_1, U_2 in an abelian group U is given by $U_1 + U_2 = \{x+y; x \in U_1, y \in U_2\}$.

(4.8.3) Theorem *The isotropy group $I(\xi, \eta)$ is the following sum of three subgroups in $H^n(X, \Gamma_n Y)$:*

$$I(\xi, \eta) = j^{-1}I(\eta) + (b_{n+1}Y)_* H^n(X, H_{n+1}Y) + \Delta(i_*^{-1} \operatorname{image} \tau).$$

Here $I(\eta) \subset H^n(X^n, \pi_n Y^n)$ is the isotropy group of $\eta \in [X^n, Y^n]$ in (4.8.1)(2) and j is the homomorphism in (4) above. The homomorphism i_* and Δ,

$$\operatorname{Ext}(H_{n-1}X, i\Gamma_n Y) \xleftarrow{i_*} \operatorname{Ext}(H_{n-1}X, \Gamma_n Y) \overset{\Delta}{\rightarrowtail} H^n(X, \Gamma_n Y)$$

are given by the surjection i in (5) and by the universal coefficient theorem respectively. This shows that for the boundary τ in (9) the subgroup $\Delta(i_*^{-1} \operatorname{image} \tau)$ is well defined. Moreover $(b_{n+1}Y)_* H^n(X, H_{n-1}Y)$ is the image of the coefficient homomorphism $(b_{n+1}Y)_*: H^n(X, H_{n+1}Y) \to H^n(X, \Gamma_n Y)$ induced by $b_{n+1}Y$ in (5) above.

Proof of Theorem 4.8.3 The result in Baues [AH] VI.5.16 shows that

$$I(\xi, \eta) = j^{-1}(I(\eta) + A_n(X, Y))$$

where $I(\eta) \subset \operatorname{image} j$ so that

$$I(\xi, \eta) = j^{-1}I(\eta) + j^{-1}A_n(X, Y).$$

We now show

$$j^{-1}A_n(X, Y) = (b_{n+1}Y)_* H^n(X, H_{n+1}Y) + \Delta(i_*^{-1} \operatorname{image} \tau). \tag{1}$$

This implies the proposition of the theorem. For the proof of (1) we first recall the definition of the subgroup $A_n(X, Y) \subset H^n(X^n, \pi_n Y^n)$. Let $g_{n+1}: C_{n+1} \to \pi_n Y^n$ be the attaching map in Y. Then $A_n(X, Y)$ is the set of all cohomology classes $\{g_{n+1}\alpha_{n+1}\}$ for all which there exist

$$\begin{aligned} &\alpha_{j+1}: C_j \to C'_{j+1}, \quad j \geq n \quad \text{with} \\ &\alpha_k d + d'\alpha_{k+1} = 0 \quad \text{for} \quad k \geq n+1. \end{aligned} \tag{2}$$

Compare Baues [AH] VI.5.16(7). Consider the diagram

$$\begin{tikzcd} C_{n+1} \arrow[r, "d"] \arrow[d, "\alpha_{n+3}"] & C_{n+1} \arrow[r, "d"] \arrow[d, "\alpha_{n+2}"] & C_n \arrow[d, "\alpha_{n+1}"] & \\ C'_{n+3} \arrow[r, "d'"'] & C'_{n+2} \arrow[r, "d'"'] & C'_{n+1} \arrow[r, "g_{n+1}"'] & \pi_n Y^n \end{tikzcd} \tag{3}$$

Let $t: B_n \to C_{n+1}$ be a splitting of d with $B_n = dC_{n+1}$. A sequence of maps $(\alpha_j, j \geq n)$ satisfying (2) exists if and only if there exists a commutative diagram

$$\begin{tikzcd} tB_n \arrow[r, tail, "d"] \arrow[d, "\bar{\alpha}"] & C_n \arrow[d, "\alpha_{n+1}"] \\ C'_{n+2} \arrow[r, "d'"'] & C'_{n+1} \end{tikzcd} \tag{4}$$

or equivalently if and only if $\alpha_{n+1}(B_n) \subset B'_{n+1}$ with $B'_{n+1} = d'C_{n+2}$. Clearly this condition is necessary by (2) where we set $k = n+1$. On the other hand, this condition is sufficient since $C_{n+1} = Z_{n+1} \oplus tB_n$ and since we can define α_j, $j \geq n+2$, by

$$\begin{gathered} \alpha_{n+2} \mid tB_n = -\bar{\alpha} \\ \alpha_{n+2} \mid Z_{n+1} = 0 \\ \alpha_j = 0 \quad \text{for} \quad j \geq n+3. \end{gathered} \tag{5}$$

Hence (4) shows

$$A_n(X,Y) = \{\{g_{n+1}\alpha_{n+1}\};\ \alpha_{n+1} \in \operatorname{Hom}(C_n, C'_{n+1}) \quad \text{and} \quad \alpha_{n+1}B_n \subset B'_{n+1}\}. \tag{6}$$

The attaching map g_{n+1} is embedded in the following commutative diagram

$$\begin{tikzcd} Z'_{n+1} \arrow[r, "b'_{n+1}p_{n+1}"] \arrow[d, tail] & \Gamma_n Y \arrow[r, two heads, "i"] \arrow[d, tail, "i_\Gamma"] & i\Gamma_n Y \arrow[d, tail] \\ C'_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d'"] & \pi_n Y^n \arrow[r, two heads, "q_Y"] \arrow[d, "d''"] & \pi_n Y \arrow[d] \\ B'_n \arrow[r, tail, "i_B"] \arrow[d, equal] & \ker(b_n Y^n) \arrow[r, two heads, "p"] \arrow[d, phantom, "\cap"] & \ker(b_n Y) \arrow[d, phantom, "\cap"] \\ B'_n \arrow[r, tail] & Z'_n \arrow[r, "p_n"'] & H_n Y \end{tikzcd} \tag{7}$$

The vertical arrows are the obvious inclusions and surjections respectively. The row i_B, p is short exact and hence a free resolution of the group $\ker(b_n Y)$. The map q_Y is induced by the inclusion $Y^n \subset Y^{n+1}$ where Y^{n+1} is the mapping cone of g_{n+1} and $\pi_n Y = \pi_n Y^{n+1}$. All rows of the diagram are exact. Moreover all columns contain short exact sequences in the upper part. We now choose splittings t_1 and t_0,

$$\pi_n Y^n \underset{t_1}{\overset{d''}{\rightleftarrows}} \ker(b_n Y^n)$$

$$C'_{n+1} \underset{t_0}{\overset{d'}{\rightleftarrows}} B'_n$$

of the surjective homomorphisms d'' and d' respectively in diagram (7) above. We define

$$\bar{\beta} = i_\Gamma^{-1}(t_1 i_B - g_{n+1} t_0) \in \mathrm{Hom}(B'_n, \Gamma_n Y) \tag{8}$$

Hence the following diagram commutes.

$$\begin{tikzcd} B'_n \arrow[r, "i\bar{\beta}"] \arrow[d, tail] & i\Gamma_n Y \arrow[d, tail] \\ \ker(b_n Y^n) \arrow[r, "q_Y t_1"] \arrow[d] & \pi_n Y \arrow[d] \\ \ker(b_n Y) \arrow[r, equal] & \ker(b_n Y) \end{tikzcd} \tag{9}$$

This is clear since $q_Y g_{n+1} = 0$. By (9) we see that $i\bar{\beta}$ represents the extension $\{\pi_n Y\} \in \mathrm{Ext}(\ker(b_n Y), i\Gamma_n Y)$.

We now consider $j^{-1}A_n(X,Y)$ where

$$j: H^n(X, \Gamma_n Y) \rightarrowtail H^n(X^n, \pi_n Y^n) \tag{10}$$

is actually an inclusion defined by $C_* X^n \subset C$ and by $i_\Gamma: \Gamma_n Y \subset \pi_n Y^n$; see (4.8.2)(4). A map α_{n+1} as in (6) induces a map $\bar{\alpha}_{n+1}$ for which the following diagram commutes

$$\begin{tikzcd} C_n/B_n \arrow[d, "\bar{\alpha}_{n+1}"] & C_n \arrow[l, "p"'] \arrow[d, "g_{n+1}\alpha_{n+1}"] \\ C'_{n+1}/B'_{n+1} \arrow[r, "\bar{g}_{n+1}"] \arrow[d, "t_0", "\cong"'] & \pi_n Y^n \arrow[d, "t_1", "\cong"'] \\ H_{n+1}(Y) \oplus B'_n \arrow[r, "M"] & \Gamma_n(Y) \oplus \ker(b_n Y^n) \end{tikzcd} \tag{11}$$

Here p is the quotient map and $\bar{g}_{n+1}$ is induced by g_{n+1}. The isomorphisms t_0, t_1 are induced by the corresponding splittings above. Moreover (7) and (8) show that the matrix M in (11) is given by

$$M = \begin{pmatrix} b_{n+1} Y & -\bar{\beta} \\ 0 & i_B \end{pmatrix}. \tag{12}$$

The isomorphism t_1 in (11) yields the following diagram with j as in (10) and where we set $K = \ker(b_n Y^n)$.

$$\begin{tikzcd}
 & H_n(X^n, \pi_n Y^n) & \\
H^n(X, K) & H_n(X, \Gamma_n Y \oplus K) \arrow[l] \arrow[u, "\bar{j}"] & H^n(X, \Gamma_n Y) \arrow[l, tail] \arrow[ul, "j"'] \\
 & H^n(X, H_{n-1} Y \oplus B'_n) \arrow[ul, "(0, i_B)_*"] \arrow[u, "M_*"] & \ker(0, i_B)_* \arrow[l, tail] \arrow[u, "(b_{n+1}Y, -\bar{\beta})_*"']
\end{tikzcd} \tag{13}$$

The definition of M in (11) together with (6) show

$$A_n(X, Y) = \mathrm{image}(\bar{j} M_*). \tag{14}$$

Hence we get by a diagram chase in (13) the formula

$$j^{-1} A_n(X, Y) = \left(b_{n+1}Y, -\bar{\beta}\right)_* \ker(0, i_B)_* . \tag{15}$$

Since i_B is injective one gets

$$\ker(0, i_B)_* = H^n(X, H_{n+1} Y) \oplus \Delta \ker(i_\#) \tag{16}$$

where $i_\# = \mathrm{Ext}(H_{n-1}X, i_B)$. Moreover (9) and the six-term exact sequence for the boundary τ in (4.8.2)(8) yield the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(H_{n-1}X, \ker b_n Y) \arrow[d] \arrow[r, equal] & \mathrm{Hom}(H_{n-1}X, \ker b_n Y) \arrow[d, "\tau"] \\
\mathrm{Ext}(H_{n-1}X, B'_n) \arrow[r, "(i\bar{\beta})_*"] \arrow[d, "i_\#"'] \arrow[dr, "\bar{\beta}_*"] & \mathrm{Ext}(H_{n-1}X, i\Gamma_n Y) \\
\mathrm{Ext}(H_{n-1}X, K) & \mathrm{Ext}(H_{n-1}X, \Gamma_n Y) \arrow[u, "i_*"']
\end{tikzcd} \tag{17}$$

The left column is exact. A diagram chase in (17) shows

$$\bar{\beta}_* \ker i_\# = i_*^{-1}(\mathrm{image}\, \tau) \tag{18}$$

Thus (15), (16) and (18) yield formula (1) and the proof of Theorem 4.8.3 is complete. □

5

SPANIER–WHITEHEAD DUALITY AND THE STABLE CW-TOWER

We consider some aspects of combinatorial homotopy theory which arise in the stable range. In particular, we use Spanier–Whitehead duality which carries homotopy groups to cohomotopy groups. Whitehead's certain exact sequence for homotopy groups corresponds in this way to a dual sequence for cohomotopy groups. The secondary boundary in the dual sequence is related to cohomology operations which appear in the Atiyah–Hirzebruch spectral sequence. Moreover we describe the Spanier–Whitehead dual of the stable CW-tower.

5.1 Cohomotopy groups

We introduce cohomotopy groups and we describe the dual of Whitehead's certain exact sequence which embeds cohomotopy groups and cohomology groups in a long exact sequence. To stress the duality we develop the theory along parallel lines which are dual to each other.

Let X be a CW-complex with base point. *Homotopy groups* and *cohomotopy groups* are defined by the set of homotopy classes in **Top***$/\simeq$

(5.1.1)
$$\pi_n X = [S^n, X],$$
$$\pi^n X = [X, S^n].$$

Here $\pi^n X$ in general is not a group, but in the *stable range*, $\dim X \leq 2n-1$, we have the suspension isomorphism Σ: $[X, S^n] = [\Sigma X, S^{n+1}]$ which yields an abelian group structure for $\pi^n X$. The dual of the skeleton X^n is the *coskeleton* X_n which is the quotient space

(5.1.2)
$$X_n = X/X^{n-1}.$$

Hence the coskeleton of the m-skeleton, $m \geq n$, is $X_n^m = X^m/X^{n-1}$ which is also the m-skeleton of X_n with $X_n^m \subset X_n$. We point out that X_n^n is a one-point union of n-spheres or equivalently a Moore space of a free abelian group

(5.1.3)
$$X_n^n = \bigvee_{Z_n} S^n = M(C_n X, n).$$

Here Z_n is the set of n-cells in X and $C_n X$ is given by the cellular chain complex $C_* X$ of X. Let $C^* X$ be the *cellular cochain complex* of X, that is

$$\text{(5.1.4)}\qquad \begin{aligned} C_n X &= H_n(X^n, X^{n-1}) = H_n(X_n^n) \\ C^n X &= \mathrm{Hom}(C_n X, \mathbb{Z}). \end{aligned}$$

The coboundary $d\colon C^n X \to C^{n+1} X$ is induced by the boundary $d\colon C_{n+1} X \to C_n X$. We have the attaching map of n-cells

$$\text{(5.1.5)}\qquad f_n\colon \bigvee_{Z_n} S^{n-1} = M(C_n X, n-1) \to X^{n-1}.$$

Dually we have the *coattaching map*

$$f^n\colon X_{n+1} \to \bigvee_{Z_n} S^{n+1} = M(C_n X, n+1).$$

This map is obtained by the cofibre sequence

$$X_n^n \rightarrowtail X_n \to X_{n+1} \xrightarrow{f^n} \Sigma X_n^n \to \cdots$$

given by the inclusion $X_n^n \subset X_n$ of the n-skeleton. We clearly have $\pi_{n-1} M(C_n X, n-1) = C_n X$ and $\pi^{n+1} M(C_n X, n+1) = C^n X$. Hence by applying the functors π_{n-1} and π^{n+1} to f_n and f^n respectively we get the induced homomorphisms

$$\text{(5.1.6)}\qquad f_n\colon C_n X \to \pi_{n-1} X^{n-1}, \quad \text{resp.} \quad f^n\colon C^n X \to \pi^{n+1} X^{n+1}$$

which actually determine the homotopy classes of the corresponding maps in (5.1.5). Therefore we denote the homomorphisms (5.1.6) and the corresponding maps in (5.1.5) by the same symbol. We write

$$X = X_N^M \quad (X \text{ is an } A_N^{M-N}\text{-polyhedron})$$

if X is an M-dimensional CW-complex with trivial $(N-1)$-skeleton $X^{n-1} = *$. If $M < 2N - 1$ we obtain the following two exact sequences of abelian groups which are 'dual' to each other.

(5.1.7)

$$0 \to \Gamma_M X \xrightarrow{i} \pi_M X \xrightarrow{h} H_M X \xrightarrow{b} \Gamma_{M-1} X \to \cdots \to H_{N+1} X \to 0 \to \pi_N X = H_N X,$$

$$0 \to \Gamma^N X \xrightarrow[i]{} \pi^N X \xrightarrow[h]{} H^N X \xrightarrow[b]{} \Gamma^{N+1} X \to \cdots \to H^{M+1} X \to 0 \to \pi^M X = H^M X.$$

The top row is the exact sequence of J.H.C. Whitehead. The bottom row is defined as follows: Let $M \leq n \leq N$. The inclusion $X^{n-1} \to X^n$ and the projection $X_n \to X_{n+1}$ yield the *Γ-groups*

$$\text{(5.1.8)} \qquad \begin{aligned} \Gamma_n X &= \operatorname{Im}(\pi_n X^{n-1} \to \pi_n X^n), \\ \Gamma^n X &= \operatorname{Im}(\pi^n X_{n+1} \to \pi^n X_n). \end{aligned}$$

Now the inclusion $X^n \to X$ and the projection $X \to X_n$ induce the maps

$$i = i_n : \Gamma_n X \subset \pi_n X^n \to \pi_n X \quad \text{and} \quad i = i^n : \Gamma^n X \subset \pi^n X_n \to \pi^n X \tag{1}$$

respectively. Moreover, we have the *Hurewicz homorphism*

$$\begin{aligned} h &= h_n : \pi_n X \to H_n X = H_n(X, \mathbb{Z}), \\ h &= h^n : \pi^n X \to H^n X = H^n(X, \mathbb{Z}) \end{aligned} \tag{2}$$

by $h_n(\alpha) = \alpha_*\{e_n\}$, $h^n(\beta) = \beta^*\{e^n\}$. Here $\{e_n\} \in H_n(S^n, \mathbb{Z})$ and $\{e_n\} \in H^n(S^n, \mathbb{Z})$ are generators which are dual to each other. Next we obtain the *secondary boundaries*

$$\begin{aligned} b &= b_n : H_n X \to \Gamma_{n-1} X, \\ b &= b^n : H^n X \to \Gamma^{n+1} X \end{aligned} \tag{3}$$

by the maps in (5.1.6) as follows. Let $Z_n X$ and $Z^n X$ be the group of n-cycles and n-cocyles respectively. Then we have commutative diagrams

$$\begin{tikzcd} Z_n X \subset C_n X \arrow[r, "f_n"] \arrow[d] & \pi_{n-1} X^{n-1} \\ H_n X \arrow[r, dashed, "b_n"] & \Gamma_{n-1} X, \arrow[u, tail] \end{tikzcd}$$

$$\begin{tikzcd} Z^n X \subset C^n X \arrow[r, "f^n"] \arrow[d] & \pi^{n+1} X^{n+1} \\ H^n X \arrow[r, dashed, "b_n"] & \Gamma^{n+1} X. \arrow[u, tail] \end{tikzcd}$$

(5.1.9) Proposition *For $X = X_N^M$ and $M \leq 2N - 1$ the sequences in* (5.1.7) *are exact.*

Proof The sequences of J.H.C. Whitehead is extracted from the homotopy exact couple of X. In the same way we extract the bottom row of (5.1.7) from the cohomotopy exact couple which is available in the stable range. □

5.2 Spanier–Whitehead duality

We recall some facts about Spanier–Whitehead duality which carries homotopy groups in the stable range to cohomotopy groups. Moreover Spanier–Whitehead duality carries Whitehead's exact sequence to the dual sequence for cohomotopy groups described in Section 5.1. Let $\mathbf{A}_N^{M-N}$ be the full homotopy category of all finite A_N^{M-N}-polyhedra or equivalently of all finite CW-complexes $X = X_N^M$ with $\dim X \leq M$ and trivial $(N-1)$-skeleton $X^{N-1} = *$. This is a full subcategory of $\mathbf{Top}^*/\simeq$. In the stable range $M < 2N - 1$ *Spanier–Whitehead duality* is a contravariant isomorphism of additive categories

$$(5.2.1) \qquad D: \mathbf{A}_N^{M-N} \xrightarrow{\cong} \mathbf{A}_N^{M-N}.$$

This isomorphism carries X to $DX = X^*$ and carries the homotopy class of $f: X \to Y$ to the homotopy class $Df = f^*: Y^* \to X^*$. The isomorphism D satisfies $DD =$ identity, that is

$$X^{**} = X, \quad f^{**} = f.$$

The functor D is determined by $(N+M)$-duality maps as follows. Recall that for pointed CW-complexes X, Y we have the *smash product* $X \wedge Y = X \times Y / X \vee Y$ which satisfies $S^n \wedge S^m = S^{n+m}$ and $S^1 \wedge X = \Sigma X$.

(5.2.2) Definition Let $X = X_N^M, M < 2N - 1$. An $(N+M)$*-duality map* is a CW-complex $X^* = (X^*)_N^M$ in $\mathbf{A}_N^{M-N}$ together with a map

$$D_X: X^* \wedge X \to S^{N+M} \tag{1}$$

such that the following compositions are isomorphisms for $N \leq q \leq M$:

$$\begin{tikzcd} \pi_{N+q}(X^*) \arrow[r, "\wedge X"] \arrow[d, dashed, "\cong"'] & {[S^{N+q} \wedge X, X^* \wedge X]} \arrow[d, "(D_X)_*"] \\ \pi^{M-q}(X) \arrow[r, phantom, "\overset{\Sigma^{N+q}}{=}"] & {[S^{N+q} \wedge X, S^{N+M}]} \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} \pi_{N+q}(X) \arrow[r, "X^* \wedge"] \arrow[d, dashed, "\cong"'] & {[X^* \wedge S^{N+q}, X^* \wedge X]} \arrow[d, "(D_X)_*"] \\ \pi^{M-q}(X^*) \arrow[r, phantom, "\overset{\Sigma^{N+q}}{=}"] & {[X^* \wedge S^{N+q}, S^{N+M}]} \end{tikzcd} \tag{3}$$

Spanier and Whitehead have shown that for each X in $\mathbf{A}_N^{M-N}$ there exists an $(N+M)$-duality map; compare Spanier and Whitehead [DH] and Chapter 8, Exercises F in Spanier [AT].

Remark Geometrically one obtains an $(N+M)$-duality map D_X in Definition 5.2.2 as follows. Let X be embedded in $S^{n+1}, n = 2M$, and let Y be a finite CW-complex which is a strong deformation retract of the complement $S^{n+1}-X$ with $Y\subset S^{n+1}-X$. Then pick a point $\alpha\in S^{n+1}-X\cup Y$ and consider the inclusion

$$X\cup Y\subset S^{n+1}-\{\alpha\}\approx \mathbb{R}^{n+1}. \tag{1}$$

Since $X\cap Y=\varnothing$ is empty we have

$$X\times Y\subset \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}-\Delta \tag{2}$$

where $\Delta=\{(x,x)\mid x\in\mathbb{R}^{n+1}\}$ is the diagonal of $\mathbb{R}^{n+1}$. We obtain a deformation retraction r with $ri=1$,

$$S^n\xrightarrow{\ i\ }\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}-\Delta\xrightarrow{\ r\ }S^n \tag{3}$$

given by $i(x)=(x,0)$ for $x\in S^n=\{x\in\mathbb{R}^{n+1},\|x\|=1\}$ and $r(x,y)=(x-y)/\|x-y\|$. The composition of r and (2) yields the map

$$X\times Y\to S^n \tag{4}$$

which is null-homotopic on $X\vee Y$. Hence this map induces a map

$$D\colon X\wedge Y\simeq C_j\to S^n \tag{5}$$

where C_j is the mapping cone of the inclusion $j\colon X\vee Y\subset X\times Y$. The map D is then an n-duality map in the sense of Definition 5.2.2. There is a CW-complex X^* in $\mathbf{A}_N^{M-N}$ together with a homotopy equivalence

$$\Sigma^{M-N}X^*\simeq Y. \tag{6}$$

Moreover there is a map

$$D_X\colon X\wedge X^*\to S^{M+N} \tag{7}$$

for which the $(M-N)$-fold suspension $\Sigma^{M-N}D_X$ is homotopic to

$$\Sigma^{M-N}(X\wedge X^*)\simeq X\wedge Y\xrightarrow{\ D\ }S^n=S^{2M}.$$

The map D_X is the duality map in Definition 5.2.2; compare Cohen [SH]. We are now ready for the definition of the duality functor D in (5.2.1).

(5.2.3) Definition of D Choose for $X=\mathbf{A}_N^{M-N}$ an $(N+M)$-duality map and set $DX=X^*$. For X^* we can choose $DX^*=X$ and for a sphere S^{N+q} we choose $DS^{N+q}=(S^{N+q})^*=S^{M-q}$. Moreover, D is defined on maps by the following commutative diagram

$$\begin{tikzcd}
& {[X,Y]} \arrow[rr, dashed, "D"] \arrow[dl, "Y^*\wedge"'] \arrow[ddr, "\cong"] & & {[Y^*,X^*]} \arrow[dr, "\wedge X"] \arrow[ddl, "\cong"'] & \\
{[Y^*\wedge X, Y^*\wedge Y]} \arrow[drr, "(D_Y)_*"'] & & & & {[Y^*\wedge X, X^*\wedge X]} \arrow[dll, "(D_X)_*"] \\
& & {[Y^*\wedge X, S^{M+N}]} & &
\end{tikzcd}$$

in which the compositions $(D_X)_*(\wedge X)$ and $(D_Y)_*(Y^* \wedge)$ are isomorphisms.

The inductive construction of X^* in Chapter 8, Exercises F, in Spanier [AT], shows that the functor D can be chosen such that the following properties are satisfied. In the following let $0 \le q \le M-N$. For $X = X_N^M$ the cells e_{N+q} form a basis of $C_{N+q}X$. Let e^{N+q} be given by the dual basis in $C^{N+q}X = \mathrm{Hom}(C_{N+q}, \mathbb{Z})$. Now each cell e_{N+q} in X is in 1–1 correspondence to a cell e^*_{M-q} in X^*. This correspondence yields the identification

(5.2.4) $$C^{N+q}X = C_{M-q}X^*, e^{N+q} \mapsto e^*_{M-q}.$$

We call e^*_{M-q} the *dual cell* of e_{N+q}. Moreover, for skeleta and coskeleta the following equation holds:

(5.2.5) $$(X^{N+q})^* = (X^*)_{M-q};$$

compare the notion in (5.1.2). The attaching and coattaching maps yield the commutative diagram

(5.2.6) $$\begin{tikzcd} M(C_{N+q}, N+q-1)^* \arrow[d, equal] & (X^{N+q-1})^* \arrow[l, "f^*_{N+q}"'] \arrow[d, equal] \\ M(C_{M-q}X^*, M-q+1) & (X^*)_{M-q+1} \arrow[l, "f_{M-q}"] \end{tikzcd}$$

Here $M(A,k)$ denotes the Moore space of A in degree k; we use the fact that for a finitely generated free abelian group A we have $M(A, N+q)^* = M(A^*, M-q)$, $A^* = \mathrm{Hom}(A, \mathbb{Z})$. By Definition 5.2.3 and (5.2.4) the dual of a map $\alpha: S^{N+q} \to S^{N+q'}$ between spheres is $\alpha^* = \Sigma^k\alpha: S^{M-q'} \to S^{M-q}$ with $k = M-N-q'-q$. Hence by (5.2.6) the dual of the mapping cone $C_\alpha = S^{N+q'} \cup_\alpha e^{N+q+1}$ is the mapping cone of $\alpha^* = \Sigma^k\alpha$ or equivalently

(5.2.7) $$(C_\alpha)^* = \Sigma^k C_\alpha.$$

If $\alpha: S^{N+q} \to S^{N+q}$ is a map of degree n we obtain the Moore space $C_\alpha = M(\mathbb{Z}/n, N+q)$ of the finite cyclic group $\mathbb{Z}/n$. Hence the dual of this Moore space is, by (5.2.7), again a Moore space

(5.2.8) $$M(\mathbb{Z}/n, N+q)^* = M(\mathbb{Z}/n, M-q-1).$$

From (5.2.4) and (5.2.6) we derive the commutative diagram

(5.2.9) $$\begin{tikzcd} C^{N+q}X \arrow[d, equal] & C^{N+q+1}X \arrow[l, "d^*"'] \arrow[d, equal] \\ C_{M-q}X^* & C_{M-q+1}X^* \arrow[l, "d"] \end{tikzcd}$$

Here d is the boundary in the cellular chain complexes of X and X^* respectively. This yields for any coefficient group the isomorphism

(5.2.10) $$H_{M-q}(X^*,G) = H^{N+q}(X,G)$$

which is natural in X and G. For example if $X = M(A, N+q)$ is the Moore space of a finitely generated abelian group A then we have $H^{N+q}(X) = \mathrm{Hom}(A,\mathbb{Z})$ and $H^{N+q+1}(X) = \mathrm{Ext}(A,\mathbb{Z})$ so that the homology groups of the dual $X^* = M(A, N+q)^*$ are by (5.2.10)

$$H_i M(A,N+q)^* = \begin{cases} \mathrm{Hom}(A,\mathbb{Z}), & i = M-q \\ \mathrm{Ext}(A,\mathbb{Z}), & i = M-q-1 \\ 0 & \text{otherwise} \end{cases}$$

This implies that we have a homotopy equivalence

(5.2.11)
$$M(A,N+q)^* \simeq M(\mathrm{Hom}(A,\mathbb{Z}), M-q) \vee M(\mathrm{Ext}(A,\mathbb{Z}), M-q-1)$$

where the right-hand side is a one-point union of Moore spaces.

For homotopy and cohomotopy groups we have by Definition 5.2.3 the isomorphism D:

(5.2.12) $$\pi_{M-q}(X^*) = \pi^{N+q}(X)$$

which we use an identification. Similarly we get for the Γ-groups the isomorphism

(5.2.13) $$\Gamma_{M-q}(X^*) = \Gamma^{N+q}(X).$$

Now the exact sequences in (5.1.7) have the following property.

(5.2.14) Proposition *For $X = X_N^M$ in $\mathbf{A}_N^{M-N}$ with $M < 2N-1$ there is the natural commutative diagram of exact sequences*:

$$\begin{array}{ccccccccccccccc}
0 \to & \Gamma_M X^* & \to & \pi_M X^* & \to & H_M X^* & \to & \cdots & \to & H_{N+1} X^* & \to 0 \to & \pi_N X^* & = & H_N X^* \\
 & \| & & \| & & \| & & & & \| & & \| & & \\
0 \to & \Gamma^N X & \to & \pi^N X & \to & H^N X & \to & \cdots & \to & H^{M-1} X & \to 0 \to & \pi^N X & = & H^N X
\end{array}$$

This shows that each result on the operators in the exact sequence of J.H.C. Whitehead yields a dual result on the dual operators. Similarly, it is well known that homology and cohomology operations behave well with respect to the isomorphism in (5.2.10); see 27.23 in Gray [GT] and Maunder [CO]. For example the *Steenrod squares* have the property that for $X = X_N^M$ $(M < 2N - 1)$ the following diagram commutes

$$
\text{(5.2.15)} \qquad
\begin{array}{ccc}
H^{N+q}(X,\mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{N+q+i}(X,\mathbb{Z}/2) \\
\| & & \| \\
H_{M-q}(X^*,\mathbb{Z}/2) & \xrightarrow[Sq_i]{} & H_{M-q-i}(X^*,\mathbb{Z}/2)
\end{array}
$$

Since $H^*(X,\mathbb{Z}/2)$ and $H_*(X,\mathbb{Z}/2)$ are dual vector spaces over $\mathbb{Z}/2$ we can consider the $\mathbb{Z}/2$-dual operation $(Sq_i)^* = \mathrm{Hom}(Sq_i, \mathbb{Z}/2)$. This operation does not coincide with Sq^i. The connection is defined by the automorphism χ of the Steenrod algebra: $\chi(Sq^i) = (Sq_i)^*$; see for example Gray [HT]; however, $(Sq_2)^* = Sq^2$. We denote by $Sq^i_{\mathbb{Z}}, Sq^{\mathbb{Z}}_i$ the integral operations which are the composites

$$
\text{(5.2.16)} \qquad
\begin{aligned}
&Sq^i_{\mathbb{Z}}\colon H^n(X,\mathbb{Z}) \xrightarrow{p_*} H^n(X,\mathbb{Z}/2) \xrightarrow{Sq^i} H^{n+i}(X,\mathbb{Z}/2),\\
&Sq^{\mathbb{Z}}_i\colon H_n(X,\mathbb{Z}) \xrightarrow[p^*]{} H_n(X,\mathbb{Z}/2) \xrightarrow[Sq_i]{} H_{n-i}(X,\mathbb{Z}/2)
\end{aligned}
$$

where $p\colon \mathbb{Z} \to \mathbb{Z}/2$ is the quotient map. The *Adem relation* $Sq^2 Sq^2_{\mathbb{Z}} = 0$ yields the secondary operations ϕ^* and ϕ_*; see Mosher and Tangora [CO],

$$
\text{(5.2.17)} \qquad
\begin{aligned}
&H^n(X,\mathbb{Z}) \supset \ker Sq^2_{\mathbb{Z}} \xrightarrow{\phi^*} H^{N+3}(X,\mathbb{Z}/2)/Sq^2 H^{n+1}(X,\mathbb{Z}/2),\\
&H_n(X,\mathbb{Z}) \supset \ker Sq^{\mathbb{Z}}_2 \xrightarrow[\phi_*]{} H_{n-3}(X,\mathbb{Z}/2)/Sq^2 H_{n-1}(X,\mathbb{Z}/2).
\end{aligned}
$$

Again these operations satisfy Spanier–Whitehead duality, that is for $X = X_N^M$, $M < 2N - 1$, we have the commutative diagram

$$
\begin{array}{ccccc}
H^{N+q}(X,\mathbb{Z}) & \supset & \ker Sq^2_{\mathbb{Z}} & \xrightarrow{\phi^*} & H^{N+q+3}(X,\mathbb{Z}/2)/Sq^2 H^{N+q+1}(X,\mathbb{Z}/2) \\
\| & & \| & & \| \\
H_{M-q}(X^*,\mathbb{Z}) & \supset & \ker Sq^{\mathbb{Z}}_2 & \xrightarrow[\phi_*]{} & H_{M-q-3}(X^*,\mathbb{Z}/2)/Sq_2 H_{M-q-1}(X^*,\mathbb{Z}/2)
\end{array}
$$

compare Maunder [CO].

5.3 Cohomology operations and homotopy groups

There are some old results which relate the operators in the stable Γ-sequence of J.H.C. Whitehead with homology operations. Using Spanier–Whitehead duality one has the dual results for cohomotopy groups and cohomology operations.

Let Y be an $(n-1)$-connected CW-space. Then $\pi_n Y, \pi_{n+1}Y, \pi_{n+2}Y, \ldots$ are called the first, second, third,... non-vanishing groups of Y respectively. Dually we have for an n-dimensional CW-space X the first, second, third,... non-vanishing cohomotopy groups given by $\pi^n X, \pi^{n-1}X, \pi^{n-2}X, \ldots$ respectively. It is a classical problem to compute for small values of $k = 1, 2, 3, \ldots$ the kth non-vanishing homotopy groups and cohomotopy groups in terms of homology and cohomology. We here consider this problem in the stable range. Since we apply Spanier–Whitehead duality we assume that all CW-complexes considered are finite. Recall that we write $Y = Y_N$ if Y is a CW-complex with trivial $(N-1)$-skeleton and that we write $X = X^M$ is X is a CW-complex of dimension $\leq M$. The first non-vanishing homotopy group was computed by Hurewicz and the first non-vanishing cohomotopy group was considered by Hopf. Their results yield isomorphisms

$$
(5.3.1) \qquad \begin{aligned} \pi_N Y &= H_N(Y, \mathbb{Z}) \quad \text{for} \quad Y = Y_N, N \geq 2, \\ \pi^M X &= H^M(X, \mathbb{Z}) \quad \text{for} \quad X = X^M, M \geq 1. \end{aligned}
$$

These isomorphisms are also consequences of the exact sequences in (5.1.7). For the *second non-vanishing group* we have by (5.1.7) the following exact sequences ($Y = Y_N$, $X = X^M$)

$$
(5.3.2) \qquad \begin{aligned} & H_{N+2}Y \xrightarrow{b_{N+2}} \Gamma_{N+1}Y \xrightarrow{i} \pi_{N+1}Y \twoheadrightarrow H_{N+1}Y, \\ & H^{M-2}X \xrightarrow{b^{M-2}} \Gamma^{M-1}X \to \pi^{M-1}X \twoheadrightarrow H^{M-1}X. \end{aligned}
$$

(Here it is enough to consider $Y = Y_N^{N+2}$ and $X = X^M_{M-2}$, $M = N+2$, so that X and Y are A_N^2-polyhedra.) In the stable range there is the isomorphism

$$
(5.3.3) \qquad \eta^*\colon H_N(Y, \mathbb{Z}/2) = \pi_N Y \otimes \mathbb{Z}/2 \xrightarrow{\cong} \Gamma_{N+1}, N > 2,
$$

where η is the Hopf element. Since η is detected by Sq^2 it is easy to see that the secondary operator in (5.3.2) is determined by the commutative diagram

$$
\text{(a)} \qquad \begin{array}{ccc} H_{N+2}Y & \xrightarrow{b_{N+2}} & \Gamma_{N+1}Y \\ & \searrow^{Sq^2_2} & \| \, \eta^* \\ & & H_n(Y, \mathbb{Z}/2) \end{array} \qquad (N > 2)
$$

By Spanier–Whitehead duality (see (5.2.15)) the secondary coboundary operator is given by the commutative diagram

$$
\text{(b)} \qquad \begin{array}{ccc} H^{M-2}X & \xrightarrow{b^{M-2}} & \Gamma^{M-1}X \\ & \searrow^{Sq^2_2} & \| \, \eta_* \\ & & H^M(X, \mathbb{Z}/2) \end{array} \qquad (M > 3)
$$

Hence we obtain by (5.3.2) the classical results (5.3.4) and (5.3.4)′ below on the second non-vanishing groups. These results are due to J.H.C. Whitehead [CE] and Steenrod [PC] respectively and the results are dual to each other by Spanier–Whitehead duality.

For $Y = Y_N$, $N > 2$, there is the natural short exact sequence (see also Theorem 15 in 8.5 of Spanier [AT])

(5.3.4)
$$\frac{H_N(Y,\mathbb{Z}/2)}{Sq_2^{\mathbb{Z}} H_{N+2}Y} \rightarrowtail \pi_{N+1}Y \twoheadrightarrow H_{N+1}Y.$$

For $X = X^M$, $M > 3$, we have the natural exact sequence

(5.3.4)′
$$\frac{H^M(X,\mathbb{Z}/2)}{Sq_{\mathbb{Z}}^2 H^{M-2}Z} \rightarrowtail \pi^{M-1}X \twoheadrightarrow H^{M-1}X.$$

It remains to solve the extension problem for these sequences. Recall that for any coefficient group G the map $q: G \to G \otimes \mathbb{Z}/2$ yields the squaring operation:

$$Sq^2 = Sq_G^2 : H^N(Y,G) \xrightarrow{q_*} H^N(Y, G \otimes \mathbb{Z}/2) \xrightarrow{Sq^2 \otimes G} H^{N+2}(Y, G \otimes \mathbb{Z}/2).$$

Let $H_i = H_i(Y,\mathbb{Z})$ and let $H_i(2) = H_i(Y,\mathbb{Z}/2)$. For the fundamental class $i = \mathrm{id} \in H^N(Y, \pi_N Y) = \mathrm{Hom}(H_N, H_N)$ we have the element

$$Sq^2(i) \in H^{N+2}(Y, H_N(2))$$

where $H_N(2) = H_N \otimes \mathbb{Z}/2 = \pi_N Y \otimes \mathbb{Z}/2$ since Y is $(n-1)$-connected. For the projection $p: H_N(2) \to \operatorname{cok} Sq_2^{\mathbb{Z}}$ the universal coefficient theorem yields the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}(H_{N+1}, H_N(2)) & \rightarrowtail & H^{N+2}(Y, H_N(2)) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(H_{N+2}, H_N(2)) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\mathrm{Ext}(H_{N+1}, \operatorname{cok} Sq_2^{\mathbb{Z}}) & \overset{\Delta}{\rightarrowtail} & H^{N+2}(Y, \operatorname{cok} Sq_2^{\mathbb{Z}}) & \to & \mathrm{Hom}(H_{N+2}, \operatorname{cok} Sq_2^{\mathbb{Z}})
\end{array}$$

where $\mu Sq^2(i) = Sq_2^{\mathbb{Z}}$. Thus $p_* \mu Sq^2(i) = 0$. This shows that the element $\Delta^{-1} p_* Sq^2(i)$ is well defined. Now the extension class of (5.3.4) is given by the formula (see Remark 2.8.8)

(5.3.5)
$$\{\pi_{N+1}Y\} = \Delta^{-1} p_* Sq^2(i) \in \mathrm{Ext}(H_{N+1}, \operatorname{cok} Sq_2^{\mathbb{Z}}).$$

If we put $Y = DX$ we get by Spanier–Whitehead duality the extension class of (5.3.4)′ as follows. We have for any coefficient group G the squaring operation

$$Sq_2 = Sq_2^G : H_M(X,G) \xrightarrow{q_*} H_M(X, G \otimes \mathbb{Z}/2) \xrightarrow{Sq_2 \otimes G} H_{M-2}(X, G \otimes \mathbb{Z}/2).$$

Let $H^i = H^i(X,\mathbb{Z})$ and let $H^i(2) = H^i(X,\mathbb{Z}/2)$. For the fundamental class $i = \mathrm{id} \in H_M(X, \pi^M X) = H_M(X, H^M) = \mathrm{Hom}(H^M, \mathbb{Z}) \otimes H^M = \mathrm{Hom}(H^M, H^M)$ we have the element

$$Sq_2(i) \in H_{M-2}(X, H^M(2))$$

where $H^M(2) = H^M \otimes \mathbb{Z}/2 = \pi^M \otimes \mathbb{Z}/2$ since $\dim X \le M$. Now the universal coefficient theorem yields for the projection $p: H^M(2) \to \mathrm{cok}\, Sq_{\mathbb{Z}}^2$ the commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}(H^{M-1}, H^M(2)) & \rightarrowtail & H_{M-2}(X, H^M(2)) & \xrightarrow{\mu} & \mathrm{Hom}(H^{M-2}, H^M(2)) \\ \downarrow & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ \mathrm{Ext}(H^{M-1}, \mathrm{cok}\, Sq_{\mathbb{Z}}^2) & \overset{\Delta}{\rightarrowtail} & H_{M-2}(X, \mathrm{cok}\, Sq_{\mathbb{Z}}^2) & \longrightarrow & \mathrm{Hom}(H^{M-2}, \mathrm{cok}\, Sq_{\mathbb{Z}}^2) \end{array}$$

Here we use the fact that C_*X is the dual of C^*X, that is $C_*X = \mathrm{Hom}(C^*X, \mathbb{Z})$, since X is a finite CW-complex. Now we have $\mu Sq_2(i) = Sq_{\mathbb{Z}}^2$ so that $p_* \mu Sq_2(i) = 0$. Therefore the element $\Delta^{-1} p_* Sq_2(i)$ is well defined. Dually to (5.3.4) we get the extension class of (5.3.4)′ by the formula

(5.3.5)′ $\qquad \{\pi^{M-1} X\} = \Delta^{-1} p_* Sq_2(i) \in \mathrm{Ext}(H^{M-1}, \mathrm{cok}\, Sq_{\mathbb{Z}}^2).$

Remark The equation in (5.3.5) is due to J.H.C. Whitehead, see §18 in J.H.C. Whitehead [CE]. G.W. Whitehead reformulated this result in V.1.9 of G.W. Whitehead [RA], see also page 570 in G.W. Whitehead's book. Also Chow [SO] considers the extension problem (5.3.5). Again Shen [NH] considers the extension element (5.3.5)′ which, as we have seen, is dual to (5.3.5). A proof of (5.3.5) is also given for the unstable case in Baues [CH].

Next we describe the *third non-vanishing group*. This group was considered by Hilton [GC] who computed the homotopy group π_{N+2} of an A_N^2-polyhedron, $N > 2$. On the other hand, Shen [NB] computed the third non-vanishing cohomotopy group. We obtain these results as follows. By the exact sequences (5.1.7) and (5.3.3)(a),(b) we get the exact sequences $(Y = Y_N, X = X^M)$:

$$(5.3.6) \qquad \begin{array}{l} H_{N+3} Y \xrightarrow{b_{N+3}} \Gamma_{N+2} Y \to \pi_{N+2} Y \overset{h}{\twoheadrightarrow} \ker Sq_2^{\mathbb{Z}} \ (\subset H_{N+2} Y), \\ H^{M-3} X \xrightarrow{b^{M-3}} \Gamma^{M-2} X \to \pi^{M-2} X \twoheadrightarrow \ker Sq_{\mathbb{Z}}^2 \ (\subset H^{M-2} X), \end{array}$$

where $N > 2, M > 3$. Here the groups $\Gamma_{N+2} Y$ and $\Gamma^{M-2} X$ depend actually only on the A_N^2-polyhedra X_{M-2}^M where we set $N = M - 2$. These groups can be computed by the following result.

(5.3.7) Theorem *The groups $\Gamma_{N+2}Y$ and $\Gamma^{M-2}X$ with $Y = Y_N$, $X = X^M$ and $N = M - 2 > 3$ are embedded in the natural diagrams (a) and (b) below in which the column and each row is a short exact sequence*

$$\begin{tikzcd}
H_{N+1}Y \otimes \mathbb{Z}/2 \arrow[r, tail, "\mu"] & H_{N+1}(Y,\mathbb{Z}/2) \arrow[r, two heads, shift left, "\Delta"] & H_N Y * \mathbb{Z}/2 \arrow[l, dashed, shift left] \\
\pi_{N+1}Y \otimes \mathbb{Z}/2 \arrow[u, two heads, "h"] \arrow[r, tail, "\eta^*"] & \Gamma_{N+2}Y \arrow[u, two heads] \arrow[r, two heads] & H_N Y * \mathbb{Z}/2 \arrow[u, equal] \\
\dfrac{H_N(Y,\mathbb{Z}/2)}{Sq_2 H_{N+2}(Y,\mathbb{Z}/2)} \arrow[u, tail] & &
\end{tikzcd} \tag{a}$$

$$\begin{tikzcd}
H^{M-1}X \otimes \mathbb{Z}/2 \arrow[r, tail] & H^{M-1}(X,\mathbb{Z}/2) \arrow[r, two heads, shift left] & H^M X * \mathbb{Z}/2 \arrow[l, dashed, shift left] \\
\pi^{M-1}X \otimes \mathbb{Z}/2 \arrow[u, two heads] \arrow[r, tail, "\eta_*"] & \Gamma^{M-2}X \arrow[u, two heads] \arrow[r, two heads] & H^M X * \mathbb{Z}/2 \arrow[u, equal] \\
\dfrac{H^M(X,\mathbb{Z}/2)}{Sq^2 H^{M-2}(X,\mathbb{Z}/2)} \arrow[u, tail] & &
\end{tikzcd} \tag{b}$$

The columns are induced by (5.3.4) and (5.3.4)′ respectively and the top rows are given by the universal coefficient theorem; in (b) we again use the assumption that X is a finite CW-complex. The maps η^* and η_* are induced by the Hopf maps. Clearly (b) is the Spanier–Whitehead dual of (a). J.H.C. Whitehead considers diagram (a) in 1.12 of [GD] and 1.18 of [NT]. The extension problems for the groups $\Gamma_{N+2}Y$ and $\Gamma^{M-2}X$ in (a) and (b) respectively are solved. We shall describe the extension by use of the computation of the homotopy group $\pi_{N+2}M(A,n)$ of a Moore space.

For the computation of $\pi_{N+2}Y$ and $\pi^{M-2}X$ one has also to consider the secondary boundary operators b_{N+2} and b^{M-3}. These operators are studied in the next result.

(5.3.8) Theorem *Let $Y = Y_N$ and let $X = X^M$ for $N = M - 2 > 3$. Then the secondary boundary operators b_{N+3} and b^{M-3} are embedded in the commutative diagrams (a) and (b) respectively where ϕ^* and ϕ_* are the secondary operations of Adem in (5.2.17).*

(a)
$$\begin{tikzcd}
\ker Sq_2^{\mathbb{Z}} \arrow[d, "\phi_*"] & \subset & H_{N+3}Y \arrow[d, "b_{n+3}"] \arrow[dr, "Sq_2^{\mathbb{Z}}"] & \\
\dfrac{H_N(Y,\mathbb{Z}/2)}{Sq_2 H_{N+2}(Y,\mathbb{Z}/2)} \arrow[rr, tail, "\eta^* i"] & & \Gamma_{N+2}Y \arrow[r, two heads] & H_{N+1}(Y,\mathbb{Z}/2)
\end{tikzcd}$$

(b)
$$\begin{tikzcd}
\ker Sq^2_{\mathbb{Z}} \arrow[d, "\phi^*"] & \subset & H^{M-3}X \arrow[d, "b^{M-3}"] \arrow[dr, "Sq^2_{\mathbb{Z}}"] & \\
\dfrac{H^M(X,\mathbb{Z}/2)}{Sq^2 H^{M-2}(X,\mathbb{Z}/2)} \arrow[rr, tail, "\eta^* i"] & & \Gamma^{M-2}X \arrow[r, two heads] & H^{M-1}(X,\mathbb{Z}/2)
\end{tikzcd}$$

The exact rows of these diagrams are obtained by Theorem 5.3.7. Again (b) is Spanier–Whitehead dual to (a); in fact we can assume $Y = Y_N^{N+3}$ and $X = X_{M-3}^M$, $N = M - 3 \geq 4$. Then we are in the stable range so that we can apply duality. The result remains true for $N = 4$ by the suspension isomorphism. We see that ϕ^* in (b) is actually the Adem operation since both ϕ^* and b^{M-3} detect the double Hopf map; see also Section 8.5.

(5.3.9) Corollary *Let $Y = Y_N$ and let $X = X^M$ and let $N = M - 2 > 3$. Then the Hurewicz homomorphism maps the fourth non-vanishing group surjectively to the kernel of the Adem operation:*

$$\pi_{N+3}Y \twoheadrightarrow \ker \phi_* \subset H_{N+3}Y,$$

$$\pi^{M-3}X \twoheadrightarrow \ker \phi^* \subset H^{M-3}X.$$

We can proceed in a similar way in the discussion of the groups and operators of the Γ-sequences (5.1.7). The secondary boundaries b^{M-4} and b_{N+4}, however, involve six different cohomology operations in the stable range. This is seen by the following spectral sequence which corresponds to the Atiyah–Hirzebruch spectral sequence for the stable cohomotopy groups; see Hilton [GC].

(5.3.10) Remark Let M be large and assume X is a CW-complex of dimension M, $X = X^M$. For $q < (M-1)/2$ we have the suspension isomorphism

$$\pi^{M-q}(X) = [X, S^{M-q}] \cong [\Sigma^q X, S^M] \tag{1}$$

which allows us to replace the cohomotopy group $\pi^{M-q}(X)$ by the group $[\Sigma^q X, S^M]$ The latter group can be computed for $q \geq 0$ by a spectral sequence

$$\{E_r^{s,t}, d_r\colon E_r^{s,t} \to E_r^{s+r,t+r+1}, r \geq 1\}. \tag{2}$$

The E_2-term is given by the cohomology groups of X

$$E_2^{s,t} = H^s(X, \pi_t(S^M)) \tag{3}$$

with coefficients in homotopy groups of spheres. Let $K^{s,q} \subset [\Sigma^q X, S^M]$ be the kernel of the restriction map

$$[\Sigma^q X, S^M] \to [\Sigma^q X^s, S^M]$$

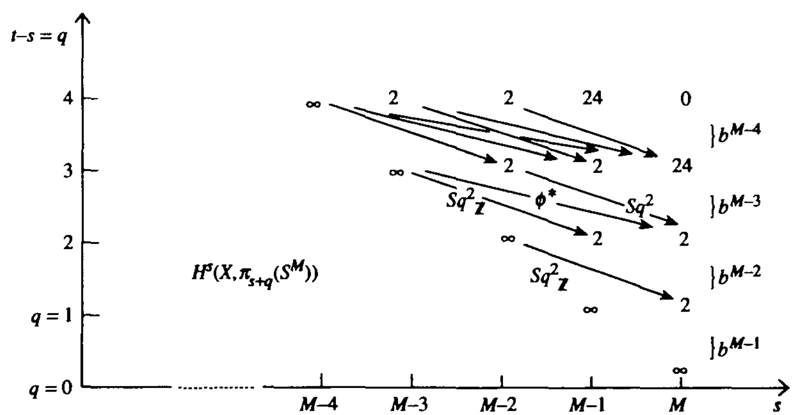
given by the inclusion $X^s \subset X$ of the s-skeleton. Then we have the filtration

$$\cdots \subset K_{s,q} \subset \cdots \subset K_{0,q} = [\Sigma^q X, S^M].$$

The associated graded group of this filtration is the ∞-term of the spectral sequence, that is

$$E_\infty^{s,q+s} = K_{s-1,q}/K_{s,q}. \tag{4}$$

This spectral sequence is a special case of the *homotopy spectral sequence*, see (III. §2) case (A) and (III.5.10) in Baues [AH]. We picture the E^2-term $E_2^{s,q+s}$ in the following diagram; by (4) the qth row in the diagram contributes to the group (1).



The coefficients in low degrees are

i	0	1	2	3	4
$\pi_{M+i}(S^M)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0

Therefore the E_2-term yields the diagram above in which we describe the coefficients by $\infty, 2, 4, 24, 0$. The diagram indicates the possible differentials. The differentials in the row b^{M-q} are those (higher-order) cohomology operations which are involved in the secondary boundary operator b^{M-q} $(q = 0, 1, 2, \ldots)$. For example the row b^{M-2} in the diagram corresponds exactly to the result in (5.3.5)(b) and the row b^{M-3} in the diagram corresponds to the result in Theorem (5.3.8)(b). Therefore the spectral sequence above is the precise extension of the classical results described in (5.3.2) and Theorem (5.3.8). The six differentials in the row $q = 4$, however, do not correspond to classical cohomology operations in the way that those in the rows $q = 2$ and $q = 3$ do.

Spanier–Whitehead duality yields a dual spectral sequence consisting of homotopy groups and homology operations which can be used for the computation of stable homotopy groups of a finite CW-complex; this extends the duality discussed in (5.3.2) and Theorem (5.3.8).

5.4 The stable CW-tower and its dual

Let $N < M < 2N - 1$. The *stable CW-tower* is the restriction of the CW-tower of categories to the subcategory of CW-complexes $X = X_N^M$ with $\dim X \leq M$ and trivial $(N-1)$-skeleton $X^{N-1} = *$. The stable CW-tower is defined by homotopy systems which consist of a chain complex C and a CW-complex X^n which realizes C in degree $\leq n$. We now consider dually *cohomotopy systems* which consist of a cochain complex D and a CW-complex Y_n which realizes D in degree $\geq n$. Cohomotopy systems yield a tower of categories which is Spanier–Whitehead dual to the stable CW-tower.

We fix N and M as above. Let $\mathbf{A}_N^{M-N}$ be the full homotopy category of finite CW-complexes of the form $X = X_N^M$; see Section 5.2

(5.4.1) Definition We define an (N, M)*-homotopy system* of order $(q+1)$ and an (N, M)*-cohomotopy system* of order $(q+1)$ to be triples

$$(C, f, X_N^{N+q}), \quad \text{resp.} \quad \left(D, g, Y_{M-q}^M\right)$$

with the following property (a), resp. (a') where (a') is the dual of (a). As usual let $\tilde{C}_*$ and $\tilde{C}^*$ be the reduced cellular chain complex and cochain complex respectively.

(a) $C = (C, d)$ is a chain complex of finitely generated free abelian groups with $C_i = 0$ for $i < N$ and $i > M$ and C coincides with $\tilde{C}_* X_N^{N+q}$ in degree $\leq N + q$. Moreover

$$f \colon C_{N+q+1} \to \pi_{N+q} X^{N+q}$$

is a homorphism such that $fd = 0$ and such that

$$d\colon C_{N+q+1} \xrightarrow{f} \pi_{N+q} X^{N+q} \xrightarrow{h} H_{N+q} X^{N+q} \subset C_{N+q}$$

is the differential of C.

(a′) $D = (D, d)$ is a cochain complex of finitely generated free abelian groups with $D_i = 0$ for $i < N$ and $i > M$ and D coincides with $\tilde{C}^* Y^M_{M-q}$ in degree $\geq M - q$. Moreover,

$$g\colon D^{M-q-1} \to \pi^{M-q} Y^M_{M-q}$$

is a homomorphism such that $gd = 0$ such that

$$d\colon D^{M-q-1} \xrightarrow{g} \pi^{M-q} Y^M_{M-q} \xrightarrow{h} H^{M-q} Y^M_{M-q} \subset D^{M-q}$$

is the differential of D.

We obtain maps

(b) $$(\xi, \eta)\colon (C, f, X^{N+q}_N) \to (C', F', U^{N+q}_N)$$

(b′) $$(\xi, \eta)\colon \left(D', g', V^M_{M-q}\right) \to \left(D, g, Y^K_{M-q}\right)$$

between such homotopy systems, resp. such cohomotopy systems, as follows.

In (b) the map $\xi\colon C \to C'$ is a chain map and $\eta\colon X^{N+q}_N \to U^{N+q}_N$ is a map in $\mathbf{CW}/\overset{0}{\simeq}$ such that ξ coincides with $\tilde{C}_* \eta$ in degree $\leq N + q$ and such that $f'\xi = \eta_* f$ on C_{N+q+1}.

In (b′) the map $\xi\colon D \to D'$ is a cochain map and $\eta\colon V^M_{M-q} \to Y^M_{M-q}$ is a map in $\mathbf{CW}/\overset{0}{\simeq}$ such that ξ coincides with $\tilde{C}^* \eta$ in degree $\geq M - q$ and such that $g'\xi = \eta^* g$ on D^{M-q-1}. Let $\mathbf{H}_{(q+1)}$ and $\mathbf{H}^{(q+1)}$ be the category of such homotopy systems and cohomotopy systems respectively.

Remark An (N, M)-homotopy system of order $(q + 1)$ is a special homotopy system of order $N + q + 1$ in the sense of Chapter 4. We recall above the definition of homotopy system so that the duality between homotopy systems and cohomotopy systems becomes evident.

(5.4.2) Definition We define the *homotopy relation* $\simeq$ on $H_{(q)}$. For maps as in (b) we set $(\xi, \eta) \simeq (\bar{\xi}, \bar{\eta})$ if there exist homomorphisms $\alpha_{j+1}\colon C_j \to C'_{j+1}$, $j \geq N + q$, such that in the abelian group $[X^{N+q}_N, U^{N+q}_N]$ we have the equation

$$\{\bar{\eta}\} - \{\eta\} = p^*(f'\alpha_{N+q+1})$$

where $p\colon X_N^{N+q} \to X_{N+q}^{N+q}$ is the quotient map and where p^* is defined on

$$\mathrm{Hom}\big(C_{N+q}, \pi_{N+q} U_N^{N+q}\big) = \big[X_{N+q}^{N+q}, U_N^{N+q}\big];$$

compare the convention in (5.1.2). Moreover:

$$\bar{\xi}_k - \xi_k = \alpha_k d_k + d_{k+1}\alpha_{k+1},\ k > N+q.$$

Dually we define the *homotopy relation* $\simeq$ on $H^{(q)}$. For maps as in (b′) we set $(\xi, \eta) \simeq (\bar{\xi}, \bar{\eta})$ if there exist homomorphisms

$$\alpha^{j-1}\colon D^j \to (D')^{j-1},\ j \le M-q,$$

such that in the abelian group $[V_{M-q}^M, Y_{M-q}^M]$ we have the equation

$$\{\bar{\eta}\} - \{\eta\} = i_*(g(\alpha^{M-q-1})).$$

Here $i\colon X_{M-q}^{M-q} \to X_{M-q}^M$ is the inclusion and i_* is defined on

$$\mathrm{Hom}\big(D^{M-q}, \pi^{M-q}\big(V_{M-q}^M\big)\big) = \big[V_{M-q}^M, X_{M-q}^{M-q}\big].$$

Moreover,

$$\bar{\xi}^k - \xi^k = \alpha^k d^k + d^{k-1}\alpha^{k-1},\ k < M-q.$$

The categories $\mathbf{H}_{(q)}$, $\mathbf{H}_{(q)}/\simeq$, $\mathbf{H}^{(q)}$, and $\mathbf{H}^{(q)}/\simeq$ are additive categories by the addition law on morphisms: $(\xi, \eta) + (\xi', \eta') = (\xi + \xi', \eta + \eta')$. Moreover we have functors

$$\text{(5.4.3)}\qquad \begin{aligned} &\lambda\colon \mathbf{H}_{(q+1)} \to \mathbf{H}_{(q)},\ (C, f, X_N^{N+q}) \mapsto (C, f', X_N^{N+q-1}) \\ &\lambda\colon \mathbf{H}^{(q+1)} \to \mathbf{H}^{(q)},\ \big(D, g, Y_{M-q}^M\big) \mapsto \big(D, g', Y_{M-q+1}^M\big). \end{aligned}$$

Here f' is the attaching map of $(N+q)$-cells in X_N^{N+q} and g' is the coattaching map of $(M-q)$-cells in Y_{M-q}^M; see (5.1.5). These functors induce functors between the corresponding homotopy categories. We have obvious isomorphisms of categories

$$\text{(5.4.4)}\qquad \begin{aligned} &\mathbf{H}_{(1)} = \mathbf{FChain}_N^{M-N} \quad \text{(covariant)}, \\ &\mathbf{H}^{(1)} = \mathbf{FCochain}_N^{M-N} \quad \text{(contravariant)}. \end{aligned}$$

Here $\mathbf{FChain}_N^{M-N}$ is the category of finitely generated free chain complexes C with $C_i = 0$ for $i < N$ and $i > M$. Similarly $\mathbf{FCochain}_N^{M-N}$ is the category of finitely generated free cochain complexes D with $D^i = 0$ for $i < N$ and $i > M$.

The next theorem describes the *stable CW-tower* and its *dual*. The stable CW-tower is just the restriction of the CW-tower in Section 4.3 to the stable subcategory $\mathbf{A}_N^{M-N}$; the new feature is the *dual of the stable CW-tower* obtained in this theorem.

(5.4.5) Theorem *Let $N < M < 2N-1$. The categories $\mathbf{H}_{(q)}$ form a tower of categories. Dually the categories $\mathbf{H}^{(q)}$ form a tower of categories. Both towers approximated the homotopy category $\mathbf{A}_N^{M-N}$. Moreover Spanier–Whitehead duality yields a contravariant isomorphism between these towers of categories as indicated in the following diagram.*

$$
\begin{tikzcd}
 & \mathbf{A}_N^{M-N} \arrow[rr, "D"] \arrow[d, "\cong"] & & \mathbf{A}_N^{M-N} \arrow[d, "\cong"] & \\
 & \mathbf{H}_{(M-N+1)} \arrow[d] & & \mathbf{H}^{(M-N+1)} \arrow[d] & \\
 & \vdots & & \vdots & \\
H^{N+q}\Gamma_{N+q} \arrow[rr, dashed, "T"] \arrow[dr, "+"] & \vdots & H_{M-q}\Gamma^{M-q} \arrow[dr, "+"] & \vdots & \\
 & \mathbf{H}_{(q+1)} \arrow[rr] \arrow[d] & & \mathbf{H}^{(q+1)} \arrow[d] & \\
 & \mathbf{H}_{(q)} \arrow[rr] \arrow[d] \arrow[dr, "\mathcal{O}"] & & \mathbf{H}^{(q)} \arrow[d] \arrow[dr, "\mathcal{O}"] & \\
 & \vdots & H^{N+q+1}\Gamma_{N+q} \arrow[rr, dashed, "T"] & \vdots & H_{N-q-1}\Gamma^{M-q} \\
 & \mathbf{H}_{(1)} \arrow[d, equal] & & \mathbf{H}^{(1)} \arrow[d, equal, "*"] & \\
 & \mathbf{FChain}_N^{M-N}/\simeq \arrow[rr] & & \mathbf{FCochain}_N^{M-N}/\simeq &
\end{tikzcd}
$$

The Spanier–Whitehead duality isomorphism D on $\mathbf{A}_N^{M-N}$ is defined in (5.2.1). We now define D on $\mathbf{H}_{(q+1)}$ by

$$D(C, f, X_N^{N+q}) = \left(\overline{D}C, Df, DX_N^{N+q}\right). \tag{5.4.6}$$

Here the cochain complex $\overline{D}C$ is given by $(\overline{D}C)^{M-q} = C_{N+q}$. Thus $\overline{D}$ is a covariant functor by $(\overline{D}\xi)^{M-q} = \xi_{N+q}$. We define D on maps (ξ, η) by

$D(\xi,\eta)=(\bar{D}\xi, D\eta)$. The properties of Spanier–Whitehead duality in Section 5.2 show that D is a well-defined contravariant isomorphism

$$D\colon \mathbf{H}_{(q+1)} \cong \mathbf{H}^{(q+1)}$$

of categories. We obtain the obstruction operator $\mathcal{O}$ on $\mathbf{H}^{(q)}$ and the action $+$ on $\mathbf{H}^{(q+1)}$ in the same but dual way as in Section 4.3. For this we use the $\mathbf{H}^{(q)}$-bimodules $H_{m-\varepsilon}\Gamma^m$ with $m=M-q$, $\varepsilon\geq 0$, which are defined by the homology groups

$$(H_{m-\varepsilon}\Gamma^m)(V,Y)=H_{m-\varepsilon}(D^*,\Gamma^m V) \tag{5.4.7}$$

where $Y=(D,g,Y_m^M)$ and $V=(D',g',V_m^M)$ are objects as in Definition 5.4.1(b). The chain complex $D^*=\operatorname{Hom}(D,\mathbb{Z})$ is the dual of the cochain complex D and $\Gamma^m(V)=\Gamma^m(\bar{V})$ is defined as in (5.1.8) by the space $\bar{V}$ which realizes the coattaching map g. Hence (5.4.7) is contravariant in V and covariant in Y. The natural isomorphism T in the diagram of Theorem 5.4.5 is given by $(X,Y\in \mathbf{H}_{(q)})$

$$H_{M-q-\varepsilon}(DX,\Gamma^{M-q}DY)\overset{T}{\cong} H^{N+q+\varepsilon}(X,\Gamma_{N+q}Y) \tag{5.4.8}$$

where we use (5.2.10) and (5.2.13).

Using Theorem 5.4.5 each result on the stable CW-tower corresponds to a dual result for the dual tower. In particular the results on boundary invariants in Chapter 4 have interesting dual formulations.

6

EILENBERG–MAC LANE AND MOORE FUNCTORS

A EILENBERG–MAC LANE FUNCTORS

We consider three types of Eilenberg–Mac Lane functors given by homology, cohomology, and pseudo-homology of Eilenberg–Mac Lane spaces. While the homology and cohomology of Eilenberg–Mac Lane spaces is extensively studied (see Eilenberg and Mac Lane [I], [II], Cartan [HC], and Decker [IH]), the pseudo-homology of Eilenberg–Mac Lane spaces is not treated in the literature. In our theory of boundary invariants, however, the pseudo-homology arises naturally in the same way as the cohomology in the theory of k-invariants. For this reason we describe the cohomology and pseudo-homology of Eilenberg–Mac Lane spaces along parallel lines. We classify $(m-1)$-connected $(n+1)$-dimensional homotopy types X with

$$\pi_i(X) = 0 \quad \text{for} \quad m < i < n.$$

Such homotopy types X have an $(n-1)$-type $P_{n-1}(X)$ which is an Eilenberg–Mac Lane space. Therefore our classification theorem (Section 3.4) yields explicit algebraic models of such homotopy types in terms of the Eilenberg–Mac Lane functors.

6.1 Homology of Eilenberg–Mac Lane spaces

For $m \geq 2$ the homotopy category $\mathbf{types}_m^0$ of $(m-1)$-connected m-types is equivalent to the category $\mathbf{Ab}$ of abelian groups. In fact, each $(m-1)$-connected m-type X is an Eilenberg–Mac Lane space and one has the equivalence of categories

(6.1.1) $$\mathbf{Ab} \xrightarrow{\sim} \mathbf{types}_m^0$$

which carries the abelian group A to the space $K(A, m)$. The inverse of this equivalence carries X to the abelian group $\pi_m(X)$. Using this equivalence of categories we identify homomorphisms $A \to B$ in $\mathbf{Ab}$ with homotopy classes of maps $K(A, m) \to K(B, m)$, that is

$$\mathrm{Hom}(A, B) = [K(A, m), K(B, m)]. \tag{1}$$

In particular, the homomorphism μ: $A \times A \to A$ given by addition in A (with $\mu(x,y) = x + y$ for $x, y \in A$) yields up to homotopy a map

$$\mu\colon K(A,m) \times K(A,m) \simeq K(A \times A, m) \to K(A,m) \tag{2}$$

which gives $K(A,m)$ the structure of a homotopy commutative H-space. The multiplication μ can also be obtained by loop addition since we have a canonical homotopy equivalence

$$K(A,m) \simeq \Omega K(A, m+1) \tag{3}$$

where ΩX denotes the loop space of X. The loop space functor

$$\Omega : \mathbf{types}^0_{m+1} \xrightarrow{\sim} \mathbf{types}^0_m \tag{4}$$

is an equivalence of categories compatible with the equivalence (6.1.1).

Remark We point out that the equivalence of categories in (6.1.1) is actually induced by a functorial construction of the space $K(A,m)$ as follows. For a topological monoid M let $B(M)$ be the classifying space of M; the space $B(M)$ is obtained as the realization of a simplicial space as for example in Baues [GL]. If M is abelian then $B(M)$ turns out to be again an abelian topological monoid in a canonical way so that in this case iteration is possible. Hence one obtains a functorial construction of the Eilenberg–Mac Lane space by the m-fold iterated classifying space $K(A,m) = BB \cdots B(A)$. Here the abelian group A is an abelian topological monoid with the discrete topology. Compare also Segal [CC].

(6.1.2) Definition The (classical) *Eilenberg–Mac Lane functor* $H_n(-,m)$: $\mathbf{Ab} \to \mathbf{Ab}$ is the composite

$$\mathbf{Ab} \xrightarrow{\sim} \mathbf{types}^0_m \xrightarrow{H_n} \mathbf{Ab}$$

of the equivalence (6.1.1) and the homology functor H_n of degree n. Hence the functor $H_n(-,m)$ carries the abelian group A to the homology group $H_n(A,m) = H_n(K(A,m))$ of the Eilenberg–Mac Lane space $K(A,m)$. The total homology

$$H(A,m) = H_*(K(A,m))$$

is a graded commutative ring via the multiplication

$$H(A,m) \otimes H(A,m) \xrightarrow{\times} H_*(K(A,m) \times K(A,m)) \xrightarrow{\mu_*} H(A,m).$$

Here we use the cross-product in homology and the multiplication μ in (6.1.1)(2).

(6.1.3) Definition We also define *Eilenberg–Mac Lane bifunctors*

$$H^n_{(m)}, H^{(m)}_n\colon \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

be the cohomology $H^n_{(m)}(A,B) = H^n(K(A,m),B)$ and the pseudo-homology $H^{(m)}_n(B,A) = H_n(B,K(A,m))$ respectively. Here B is the group of coefficients.

The universal coefficient sequences yield short exact sequences

$$\text{(6.1.4)}\quad \mathrm{Ext}(H_n(A,m),B) \overset{\Delta}{\rightarrowtail} H^{n+1}_{(m)}(A,B) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H_{n+1}(A,m),B)$$

$$\text{(6.1.5)}\quad \mathrm{Ext}(B,H_{n+1}(A,m)) \overset{\Delta}{\rightarrowtail} H^{(m)}_n(B,A) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(B,H_n(A,m))$$

which are natural in $A, B \in \mathbf{Ab}$.

(6.1.6) Proposition *The functor $H^{n+1}_{(m)}$ is, by (6.1.4), a kype functor on* **Ab** *and the functor $H^{(m)}_n$ is, by (6.1.5), a bype functor on* **Ab**. *Moreover $H^{n+1}_{(m)}$ is dual to $H^{(m)}_n$, in particular the kype functor $H^{n+1}_{(m)}$ is split if and only if the bype functor $H^{(m)}_n$ is split.*

Proof Let K_*: $\mathbf{Ab} \to \mathbf{Chain}_{\mathbb{Z}}/\simeq$ be the functor which carries an abelian group A to the singular chain complex $K_*(A) = C_*K(A,m)$. Then we have

$$H^{n+1}_{(m)}(A,B) = H^{n+1}(K_*(A),B)$$

$$H^{(m)}_n(B,A) = H_n(B,K_*(A)).$$

Hence the proposition is a consequence of Theorem 3.3.9. □

The next result is due to Decker [IH].

(6.1.7) Theorem *Let $k < m$ for m odd, and for m even let $k \le m$. Then the kype functor $H^{m+k}_{(m)}$ is split.*

Using Proposition 6.1.6 we get the corresponding result for the dual bype functor:

(6.1.8) Theorem *Let $k < m-1$ for m odd, and for m even let $k \le m-1$. Then the bype functor $H^{(m)}_{m+k}$ is split.*

6.2 Some functors for abelian groups

In order to give explicit descriptions of some Eilenberg–Mac Lane functors we have to introduce various basic functors and constructions on abelian groups, some of which are quite bizarre. We start with the classical torsion product.

(6.2.1) Definition The *torsion product* is a functor

$$\mathrm{Tor}\colon \mathbf{Ab}\times\mathbf{Ab}\to\mathbf{Ab}.$$

For abelian groups A and B let $\mathrm{Tor}(A,B)$ be the abelian group whose generators are the symbols $\tau_h(a,b)$ for all positive integers h and all pairs of elements $a\in A$, $b\in B$ such that $ha=0$ and $hb=0$. These generators are subject to the relations

$$\begin{aligned}
\tau_h(a,b_1+b_2)&=\tau_h(a,b_1)+\tau_h(a,b_2), && ha=0=hb_1=hb_2\\
\tau_h(a_1+a_2,b)&=\tau_h(a_1,b)+\tau_h(a_2,b), && ha_1=ha_2=0=hb\\
\tau_{hk}(a,b)&=\tau_h(ka,b), && hka=0=hb\\
\tau_{hk}(a,b)&=\tau_h(a,kb), && ha=0=hkb.
\end{aligned}$$

The torsion product, so defined by Eilenberg and Mac Lane [II], agrees with the usual definition derived from the tensor product of abelian groups. For this we choose a short exact sequence

$$0\to R\xrightarrow{d}F\to A\to 0 \tag{6.2.2}$$

where F and hence R are free abelian. We call $d=d_A$ a *short free resolution* of A. Then $A*B$ is the kernel of $d\otimes B$ in the exact sequence

$$0\to A*B\to R\otimes B\to F\otimes B\to A\otimes B\to 0.$$

This kernel is independent, up to a canonical isomorphism, of the choice of the short free resolution d_A. In particular we choose $F=\mathbb{Z}[A]$ to be the free abelian group with generators $[a]$, for all $a\in A$, and R to be the subgroup of F generated by all $[a_1]-[a_1+a_2]+[a_2]$, for $a_1,a_2\in A$. Then the correspondence $[a]\mapsto a$ induces an isomorphism $F/R\cong A$. Using this *standard resolution* of A we obtain

$$\begin{aligned}\mathrm{Tor}(A,B)&\cong A*B\subset R\otimes B\\ \tau_h(a,b)&\mapsto (h[a])\otimes b.\end{aligned} \tag{6.2.3}$$

To justify the definition of the isomorphism we observe that $h[a]\in R$ since $ha=0$ in A and that

$$(d\otimes B)(h[a])\otimes b=h[a]\otimes b=[a]\otimes hb=0$$

since $hb=0$. For a further discussion of the binatural isomorphism (6.2.3) compare 11.3 in Eilenberg and Mac Lane [II].

Next we consider functors which we derive from Whitehead's Γ-functor and the exterior square Λ^2 in Section 1.2. We have natural homomorphisms

$$\begin{aligned}\Gamma(A)&\xrightarrow{H}A\otimes A\xrightarrow{P}\Gamma(A)\\ \Lambda^2(A)&\xrightarrow{H}A\otimes A\xrightarrow{P}\Lambda^2(A).\end{aligned} \tag{6.2.4}$$

Here H in the top row is defined by $H\gamma(a) = a \otimes a$ and P is the Whitehead product defined by $P(a \otimes b) = [a, b] = \gamma(a+b) - \gamma(a) - \gamma(b)$ where γ: $A \to \Gamma(A)$ is the universal quadratic map. On the other hand, H in the bottom row is given by $H(a \wedge b) = a \otimes b - b \otimes a$ and we set $P(a \otimes b) = a \wedge b$. The homomorphisms H, P satisfy

$$PHP = 2P \quad \text{and} \quad HPH = 2H$$

so that (6.2.4) describes quadratic $\mathbb{Z}$-modules in the sense of Definition 6.13.5 below. Recall that $\mathbf{Chain}_{\mathbb{Z}}$ denotes the category of chain complexes $C_* = \{C_n, d_n\}_{n \in \mathbb{Z}}$. A cochain complex $C^* = \{C^n, \partial^n\}_{n \in \mathbb{Z}}$ is identified with the chain complex C_* given by $C_n = C^{-n}$, $d_n = \partial^{-n}$.

(6.2.5) Definition Whitehead's quadratic functor Γ and the exterior square Λ^2 induce *chain functors*

$$\Gamma_*, \Lambda^2_* : \mathbf{Ab} \to \mathbf{Chain}_{\mathbb{Z}}/\simeq \tag{1}$$

as follows. We choose for each abelian group A a short free resolution d denoted by $d = d_A$: $X_1 \to X_0$. Then we define for $F = \Gamma$ or $F = \Lambda^2$ the chain complex $F_*(d_A)$ by

$$\begin{array}{ccccc} F_2(d_A) & & F_1(d_A) & & F_0(d_A) \\ \| & & \| & & \| \\ X_1 \otimes X_1 & \xrightarrow{\delta_2} & F(X_1) \oplus X_1 \otimes X_0 & \xrightarrow{\delta_1} & F(X_0) \end{array} \tag{2}$$

$$\delta_1 = (F(d_A), P(d_A \otimes X_0)),$$

$$\delta_2 = (P, -X_1 \otimes d_A).$$

The chain functors $F_* = \Gamma_*$, Λ^2_* are examples of the quadratic chain functors in Definition 6.14.3 below. We consider d_A as a chain complex concentrated in only two degrees. The homology of d_A is the abelian group A. For a homomorphism φ: $A \to B$ we can choose a chain map

$$(\varphi_0, \varphi_1) \colon d_A \to d_B,$$

which induces φ in homology. This chain map induces

$$(\varphi_0, \varphi_1)_* : F_*(d_A) \to F_*(d_B)$$

given by $(\varphi_0, \varphi_1)_* = (F(\varphi_0), F(\varphi_1) \oplus \varphi_1 \otimes \varphi_0, \varphi_1 \otimes \varphi_1)$. The homotopy class of $(\varphi_0, \varphi_1)_*$ depends only on the homomorphism φ. Hence we obtain well-defined functors Γ_* and Λ^2_* in (1) which carry A to $\Gamma_*(d_A)$, resp. $\Lambda^2_*(d_A)$ and which carry a homomorphism φ to the homotopy class of $(\varphi_0, \varphi_1)_*$.

We now consider the homology of the chain functors. As usual we define for a chain complex $C_* = \{C_n, d_n\}$, resp. cochain complex $C^* = \{C^n, \partial^n\}$ the homology groups

$$H_n(C_*) = \text{kernel}(d_n)/\text{image}(d_{n+1})$$

$$H^n(C^*) = \text{kernel}(\partial^n)/\text{image}(\partial^{n-1}).$$

We leave it to the reader to show

(6.2.6) Proposition *One has $H_2\Gamma_*(d_A) = 0$ and $H_2\Lambda^2_*(d_A) = 0$. Moreover one has natural isomorphisms $\Gamma(A) = H_0\Gamma_*(d_A)$ and $\Lambda^2(A) = H_0\Lambda^2_*(d_A)$.*

The remaining homology is used in the following definition.

(6.2.7) Definition Using the chain functor in Definition 6.2.5 we define the *torsion functors*

$$\Gamma T, \Lambda^2 T\colon \mathbf{Ab} \to \mathbf{Ab}$$

by $\Gamma T(A) = H_1\Gamma_*(d_A)$ and $\Lambda^2 T(A) = H_1\Lambda^2_*(d_A)$. These are quadratic functors with cross-effect $\Gamma T(A \mid B) = A * B = \Lambda^2 T(A \mid B)$ given by the torsion product in Definition 6.2.1. Compare Section 6.13 below.

(6.2.8) Remark Using the notation in Section 6.14 we have $\Gamma T(A) = A *' \mathbb{Z}^\Gamma$ and $\Lambda^2 T(A) = A *' \mathbb{Z}^\Lambda$ so that the cross-effect in Definition 6.2.7 are obtained by 7.7 in Baues [QF].

The torsion functors above yield an interpretation of the bizarre functors Ω and R introduced by Eilenberg and Mac Lane [II], §13 and §22.

(6.2.9) Theorem *One has a natural isomorphism $\Gamma T(A) \cong R(A)$.*

Proof We here recall the definition of the functors R and we define the isomorphism in terms of generators. Let $R(A)$ denote the quotient group

$$R(A) = \text{Tor}(A, A) \oplus \Gamma({}_2A)/L_5(A) \tag{1}$$

with ${}_2A = \text{kernel}(2\colon A \to A) = \mathbb{Z}/2 * A$. Here $L_5(A)$ is the subgroup generated by the relations ($h \in \mathbb{Z}, a, s, t \in A$)

$$\tau_h(a, a) = 0 \qquad ha = 0 \tag{2}$$

$$[s, t] = \tau_2(s, t) \qquad 2s = 2t = 0 \tag{3}$$

where $[s, t] = \gamma(s + t) - \gamma(s) - \gamma(t) \in \Gamma({}_2A)$ is the Whitehead product. We construct special cycles (4) and (5) below in $\Gamma_1(d_A)$. Let $a, b, x \in A$ and let d_A be the standard resolution so that $X_0 = \mathbb{Z}[A]$ is freely generated by elements $[a]$, $a \in A$. We then define for $ha = hb = 0$ the element

$$\Theta_h(a,b) = (h[a]) \otimes [b] - (h[b]) \otimes [a] \in \Gamma_1(d_A) \tag{4}$$

where $h[a]$, $h[b] \in X_1 \subset X_0$. The element (4) is obviously a cycle, that is $\delta_1 \Theta_h(a,b) = 0$. Moreover we obtain for $2x = 0$ the element

$$\Theta_2(x) = \gamma(2[x]) - (2[x]) \otimes [x] \in \Gamma_1(d_A) \tag{5}$$

which is a cycle since $\gamma(2[x]) = 4\gamma([x]) = 2[[x],[x]]$ in $\Gamma(X_0)$. Using the elements (4) and (5) we define the isomorphism

$$\Theta \colon R(A) \cong \Gamma T(A) \tag{6}$$

by $\Theta\{\tau_h(a,b)\} = \{\Theta_h(a,b)\}$ and $\Theta\{\gamma(x)\} = \{\Theta_2(x)\}$, see (1). We study the Γ-torsion $\Gamma T(A)$ in more detail in Section 11.2. □

(6.2.10) Theorem *One has a natural isomorphism $\Lambda^2 T(A) \cong \Omega(A)$.*

Proof We recall the definition of $\Omega(A)$; see Eilenberg and Mac Lane [II], §13. We define the group $\Omega(A)$ to be the abelian group generated by the symbols $w_h(x)$, for positive integers h and elements $x \in A$ with $hx = 0$, subject to the relations

$$w_{hk}(x) = k w_h(x) \qquad hx = 0 \tag{1}$$

$$k w_{hk}(x) = w_k(kx) \qquad hkx = 0 \tag{2}$$

$$w_h(kx \mid y) = w_{hk}(x \mid y) \qquad hkx = hy = 0 \tag{3}$$

$$w_h(x|y|z) = 0 \qquad hx = hy = hz = 0. \tag{4}$$

Here we use for a function $f \colon A \to B$ the notation

$$f(x \mid y) = f(x+y) - f(x) - f(y), \tag{5}$$

$$f(x|y|z) = f(x+y+z) - f(x+y) - f(x+z) - f(y+z) + f(x) + f(y) + f(z). \tag{6}$$

Hence $f(x \mid y)$ is bilinear in x, y if and only if $f(x|y|z) = 0$. Now let $d_A \colon X_1 \to X_0$ be the standard resolution of A so that $X_0 = \mathbb{Z}[A]$ is freely generated by elements $[a]$, $a \in A$. We then define for $x \in A$ with $hx = 0$ the element

$$\Theta_h(x) = (h[x]) \otimes [x] \in X_1 \otimes X_0 \subset \Lambda_1^2(d_A) \tag{7}$$

where $h[x] \in X_1 \subset X_0$. One readily checks that $\Theta_h(x)$ is a cycle, that is $\delta_1 \Theta_h(x) = 0$. Using the element (7) we define the isomorphism

$$\Theta\colon \Omega(A) \cong \Lambda^2 T(A) \tag{8}$$

by $\Theta(w_h(x)) = \{\Theta_h(x)\}$. For a cyclic group $\mathbb{Z}/m\mathbb{Z}$, $m > 1$, generated by x we obtain the cyclic group

$$\Omega(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \tag{9}$$

generated by $w_m(x)$. □

We now use the chain functors for the definition of torsion bifunctors. For this recall that $[C_*, K_*]$ denotes the set of homotopy classes of chain maps $C_* \to K_*$. For an abelian group B let

$$d_B\colon X_1 \to X_0, \qquad sd_B\colon Y_2 \to Y_1$$

be short free resolutions of B considered as chain complexes.

(6.2.11) Definition We introduce the *torsion bifunctors*

$$\Gamma T_\#, \Gamma T^\#, \Lambda^2 T_\#, \Lambda^2 T^\#\colon \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}.$$

Here we use for $F = \Gamma$ or $F = \Lambda^2$ the pseudo-homology and cohomology of $F_* d_A$ with coefficients in B which defines:

$$FT_\#(B, A) = [d_B, F_* d_A] = H_0(B, F_* d_A)$$

$$FT^\#(A, B) = [F_* d_A, sd_B] = H^1(F_* d_A, B).$$

Using the universal coefficient sequences one has the following natural short exact sequences.

$$\begin{aligned}
&\mathrm{Ext}(B, \Gamma T(A)) \rightarrowtail \Gamma T_\#(B, A) \twoheadrightarrow \mathrm{Hom}(B, \Gamma(A))\\
&\mathrm{Ext}(\Gamma(A), B) \rightarrowtail \Gamma T^\#(A, B) \twoheadrightarrow \mathrm{Hom}(\Gamma T(A), B)\\
&\mathrm{Ext}(B, \Lambda^2 T(A)) \rightarrowtail \Lambda^2 T_\#(B, A) \twoheadrightarrow \mathrm{Hom}(B, \Lambda^2(A))\\
&\mathrm{Ext}(\Lambda^2(A), B) \rightarrowtail \Lambda^2 T^\#(A, B) \twoheadrightarrow \mathrm{Hom}(\Lambda^2 T(A), B).
\end{aligned}$$

Hence $\Gamma T_\#$ is a bype functor dual to the kype functor $\Gamma T^\#$ and $\Lambda^2 T_\#$ is a bype functor dual to the kype functor $\Lambda^2 T^\#$. These functors are linear in B and quadratic in A; see Section 6.13 below.

Next we describe a further pair of bifunctors $L^\#$, $L_\#$ which are dual to each other.

(6.2.12) Definition As in Eilenberg and Mac Lane [II] §27 one obtains a bifunctor

$$L^{\#}: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

as follows. Let $L^{\#}(A, B)$ be the abelian group consisting of all pairs (a, b) where b: $\Lambda^2(A) \to B$ is a homomorphism and where a: $A \to B \otimes \mathbb{Z}/2$ is a function satisfying the condition

$$a(x+y) - a(x) - a(y) = b(x \wedge y) \otimes 1$$

for $x, y \in A$. One has the natural short exact sequence

$$\mathrm{Ext}(A \otimes \mathbb{Z}/2, B) \overset{\Delta}{\rightarrowtail} L^{\#}(A, B) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(\Lambda^2(A), B)$$

with $\mu(a, b) = b$ and $\Delta(a) = (\Theta a, 0)$ where

$$\Theta: \mathrm{Ext}(A \otimes \mathbb{Z}/2, B) = \mathrm{Hom}(A \otimes \mathbb{Z}/2, B \otimes \mathbb{Z}/2) = \mathrm{Hom}(A, B \otimes \mathbb{Z}/2).$$

Hence $L^{\#}$ is a kype functor. There is a different definition of $L^{\#}$ by use of the 'quadratic Hom functor' in Definition 6.13.14 below. For this we use the 'quadratic $\mathbb{Z}$-module'

$$L'(B) = (\mathbb{Z}/2 \otimes B \xrightarrow{0} B \xrightarrow{q} \mathbb{Z}/2 \otimes B)$$

where q is the quotient map. Then we have a canonical binatural isomorphism

$$L^{\#}(A, B) = \mathrm{Hom}_{\mathbb{Z}}(A, L'(B)).$$

(6.2.13) (A) Definition We introduce the bifunctor

$$L_{\#}: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

as follows. If A and B are finitely generated let $L_{\#}(A, B)$ be the abelian group generated by elements (b, α) and (β) with $b \in B$, $\alpha \in \mathrm{Hom}(A, \mathbb{Z}/2)$, and $\beta \in \mathrm{Ext}(A, \Lambda^2(B))$. The relations are

$$(\beta + \beta') = (\beta) + (\beta')$$
$$(b, \alpha + \alpha') = (b, \alpha) + (b, \alpha')$$
$$(b + b', \alpha) = (b, \alpha) + (b', \alpha) + (b \wedge b')_*(\partial\alpha).$$

Here ∂: $\mathrm{Hom}(A, \mathbb{Z}/2) \to \mathrm{Ext}(A/\mathbb{Z})$ is the natural connecting homomorphism induced by the short exact sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$. Moreover $(b \wedge b')_*$: $\mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(A, \Lambda^2 B)$ is induced by the homomorphism $\mathbb{Z} \to \Lambda^2 B$ which carries 1 to $b \wedge b'$. It is obvious how to define induced maps for the bifunctor $L_\#$. One has the natural short exact sequence

$$\mathrm{Ext}(A, \Lambda^2 B) \overset{\Delta}{\rightarrowtail} L_\#(A, B) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A, B \otimes \mathbb{Z}/2)$$

with $\Delta(\beta) = \beta$ and $\mu(\beta) = 0$, $\mu(b, \alpha) = b_*(\alpha)$ where b_* is induced by the homomorphism $Z/2 \to B \otimes \mathbb{Z}/2$ which carries the generator 1 to $b \otimes 1$. The exact sequence shows that $L_\#$ is a bype functor on **Ab**. The definition of $L_\#$ can also be obtained by the quadratic tensor product in Definition 6.13.13. In fact, we have the binatural isomorphism

$$L_\#(A, B) = B \otimes_Z L(A)$$

where

$$L(A) = (\mathrm{Hom}(A, \mathbb{Z}/2) \xrightarrow{\partial} \mathrm{Ext}(A, \mathbb{Z}) \xrightarrow{0} \mathrm{Hom}(A, \mathbb{Z}/2)).$$

Here ∂ is the connecting homomorphism induced by the exact sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$.

(6.2.13) (B) Definition For arbitrary abelian groups A and B we have to use the following more intricate definition of the bifunctor $L_\#$ above. Let $\mathbb{Z}[B]$ be the free abelian group generated by the set B. We have the following commutative diagram with short exact rows

$$\begin{tikzcd}
K_2(B) \arrow[r, tail, "i"] \arrow[d, "\gamma_B"] & \mathbb{Z}[B] \otimes \mathbb{Z}/2 \arrow[r, two heads, "p_2"] \arrow[d, "\gamma'"] & B \otimes \mathbb{Z}/2 \arrow[d, equal] \\
\Lambda^2(B) \otimes \mathbb{Z}/2 \arrow[r, tail, "{[1,1]}"] & \Gamma(B) \otimes \mathbb{Z}/2 \arrow[r, two heads, "\sigma"] & B \otimes \mathbb{Z}/2
\end{tikzcd}$$

compare Section 1.2. Here $K_2(B)$ is the kernel of the canonical projection p_2 with $p_2([b] \otimes 1) = b \otimes 1$ and γ' is the homomorphism defined by $\gamma'([b] \otimes 1) = (\gamma b) \otimes 1$ for $b \in B$. Using the natural transformation γ_B we obtain the following push-out diagram

$$\begin{tikzcd}
\mathrm{Hom}(A, K_2(B)) \arrow[r, tail] \arrow[d, "\gamma"] \arrow[dr, phantom, "\text{push}"] & \mathrm{Hom}(A, \mathbb{Z}[B] \otimes \mathbb{Z}/2) \arrow[d] \arrow[dr, "(p_2)_*"] & \\
\mathrm{Ext}(A, \Lambda^2 B) \arrow[r, tail, "\Delta"] & L_\#(A, B) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, B \otimes \mathbb{Z}/2)
\end{tikzcd}$$

where $\gamma(\alpha) = \alpha^* \gamma_B$ with $\gamma_B \in \mathrm{Hom}(K_B, \Lambda^2(B) \otimes \mathbb{Z}/2) = \mathrm{Ext}(K_B, \Lambda^2(B))$ as above. This completes the definition of the bifunctor $L_\#$ in Definition 6.2.13(A). One can check that Definition 6.2.13(A) coincides with the one here since we have the isomorphism of quadratic $\mathbb{Z}$-modules

$$L(\mathbb{Z}/2) = (\mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2) = \mathbb{Z}^\Gamma \otimes \mathbb{Z}/2$$

so that $B \otimes L(\mathbb{Z}/2) = \Gamma(B) \otimes \mathbb{Z}/2$.

(6.2.14) Theorem *The functors $L_{\#}$ and $L^{\#}$ above are dual to each other.*

Proof Our proof is indirect. In fact, Eilenberg and Mac Lane [II] show

$$H^6_{(3)}(A,B) = L^{\#}(A,B) \oplus \mathrm{Hom}(A * \mathbb{Z}/2, B).$$

On the other hand, we show below

$$H_5^{(3)}(B,A) = L_{\#}(B,A) \oplus \mathrm{Ext}(B, A * \mathbb{Z}/2).$$

This implies that $L^{\#}$ and $L_{\#}$ are dual to each other since we know that $H^6_{(3)}$ and $H_5^{(3)}$ are dual to each other; see Proposition 6.1.6. □

6.3 Examples of Eilenberg–Mac Lane functors

Many cases of Eilenberg–Mac Lane functors are computed explicitly in the literature. Compare Eilenberg and Mac Lane [II], Cartan [HC], Serre [CM], and Decker [IH] for the computation of the groups

$$H_n(A,m) = H_n(K(A,m)),$$

$$H^n_{(m)}(A,B) = H^n(K(A,m),B).$$

The pseudo-homology

$$H_n^{(m)}(B,A) = H_n(B,K(A,m))$$

is not treated in the literature. By the work of Cartan one has a small model of the chain complex $C_* K(A,m)$ which can be used for the computation of $H_n^{(m)}(B,A)$ as a functor in A and B. On the other hand, we can use the duality of Proposition 6.1.6 which shows that the functors $H^{n+1}_{(m)}$ and $H_n^{(m)}$ determine each other; see Section 3.3. The duality, however, does not give us an appropriate formula for $H_n^{(m)}$ if we know such a formula for $H^{n+1}_{(m)}$ or vice versa.

We now describe, for small values of m and $r = n - m$, some of the Eilenberg–Mac Lane functors above. For the classical Eilenberg–Mac Lane functor $H_{m+r}(-,m)$ one has the following list of natural isomorphisms, $A \in \mathbf{Ab}$, $m \geq 2$.

(6.3.1) $$H_m(A,m) = A \quad \text{and} \quad H_{m+1}(A,m) = 0$$

(6.3.2) $$H_{m+2}(A,m) = \begin{cases} \Gamma(A) & m = 2 \\ A \otimes \mathbb{Z}/2 & m \geq 3. \end{cases}$$

Here Γ is Whitehead's functor (see Section 1.2).

(6.3.3) $$H_{m+3}(A,m) = \begin{cases} \Gamma T(A) & m = 2 \\ \Lambda^2(A) \oplus A * \mathbb{Z}/2 & m = 3 \\ A * \mathbb{Z}/2 & m \geq 4. \end{cases}$$

Here $A * B$ is the torsion product of abelian groups and $\Lambda^2(A)$ is the exterior square. Moreover $\Gamma T(A)$ is the Γ-torsion of A (see Section 6.2); this is $R(A)$ in the notation of Eilenberg and Mac Lane [II]; see Theorem 6.2.9.

$$(6.3.4)\qquad H_{m+4}(A,m)=\begin{cases}\Gamma_6(A) & m=2\\ \Lambda^2T(A)\oplus A\otimes\mathbb{Z}/3 & m=3\\ \Gamma(A)\oplus A\otimes\mathbb{Z}/3 & m=4\\ A\otimes(\mathbb{Z}/2\oplus\mathbb{Z}/3) & m\geq 5.\end{cases}$$

Here $\Lambda^2T(A)$ is a Λ^2-torsion of A (see Section 6.2); this is $\Omega(A)$ in the notation of Eilenberg and Mac Lane [II]; see Theorem 6.2.10. Moreover $\Gamma_6(A)$ is part of the free algebra with divided powers $\Gamma_* A$ generated by A; see Eilenberg and Mac Lane [II].

$$(6.3.5)\qquad H_{m+5}(A,m)=\begin{cases}\text{see Decker [IH]} & m=2\\ A\otimes A\otimes\mathbb{Z}/2\oplus A\otimes Z/3 & m=3\\ \Gamma T(A)\oplus A*\mathbb{Z}/3 & m=4\\ \Lambda^2(A)\oplus A*(\mathbb{Z}/2\oplus\mathbb{Z}/3) & m=5\\ A*(\mathbb{Z}/2\oplus\mathbb{Z}/3) & m\geq 6.\end{cases}$$

The results of (6.3.1)–(6.3.5) were essentially obtained by Eilenberg and Mac Lane [II]. One can find further explicit functorial descriptions of $H_{m+r}(A,m)$ in Decker [IH], in particular, in the metastable range $r<2m$. Next we consider for $r\leq 4$ the bifunctors

$$(6.3.6)\qquad H_{(m)}^{m+r+1},\ H_{m+r}^{(m)}\colon \mathbf{Ab}^{\mathrm{op}}\times\mathbf{Ab}\to\mathbf{Ab}$$

which are dual to each other with kype and bype structure given by (6.1.5) and (6.1.4) respectively. Hence we have in the *split case*, see for example Theorems 6.1.7 and 6.1.8, the natural isomorphisms

(6.3.7)

$$\begin{cases}H_{(m)}^{m+r+1}(A,B)=\mathrm{Ext}(H_{m+r}(A,m),B)\oplus\mathrm{Hom}(H_{m+r+1}(A,m),B)\\ H_{m+r}^{(m)}(B,A)=\mathrm{Ext}(B,H_{m+r+1}(A,m))\oplus\mathrm{Hom}(B,H_{m+r}(A,m))\end{cases}$$

where the right-hand side is determined by the functors in (6.3.1)–(6.3.5). We now describe formulas for the functors in (6.3.6) in the following lists, where split means that one has to apply formula (6.3.7); moreover '$\oplus$ split' means that one has to use formulas as in (6.3.7) for the remaining terms described in the same line; compare the example on $H^{10}_{(5)}$ and $H^{(5)}_9$ in (6.3.12) below.

(6.3.8)

	$H^{m+2}_{(m)}(A,B)$	$H^{(m)}_{m+1}(B,A)$	$H_{m+1}(A,m)$	$H_{m+2}(A,m)$
$m=2$	$\mathrm{Hom}(\Gamma A,B)$	$\mathrm{Ext}(B,\Gamma A)$	0	$\Gamma(A)$
$m\geq 3$	$\mathrm{Hom}(A\otimes\mathbb{Z}/2,B)$	$\mathrm{Ext}(B,A\otimes\mathbb{Z}/2)$	0	$A\otimes\mathbb{Z}/2$

(6.3.9)

	$H^{m+3}_{(m)}(A,B)$	$H^{(m)}_{m+2}(B,A)$	$H_{m+2}(A,m)$	$H_{m+3}(A,m)$
$m=2$	$\Gamma T^{\#}(A,B)$	$\Gamma T_{\#}(B,A)$	$\Gamma(A)$	$\Gamma T(A)$
$m=3$	$L^{\#}(A,B)\oplus$ $\mathrm{Hom}(A*\mathbb{Z}/2,B)$	$L_{\#}(B,A)\oplus$ $\mathrm{Ext}(B,A*\mathbb{Z}/2)$	$A\otimes\mathbb{Z}/2$	$\Lambda^2(A)\oplus$ $A*\mathbb{Z}/2$
$m\geq 4$	split	split	$A\otimes\mathbb{Z}/2$	$A*\mathbb{Z}/2$

For the definition of the torsion bifunctors $\Gamma T^{\#}$ and $\Gamma T_{\#}$, see Definition 6.2.11 and for the definition of $L^{\#}$, $L_{\#}$ see Definition 6.2.13 and Theorem 6.2.14.

(6.3.10)

	$H^{m+4}_{(m)}(A,B)$	$H^{(m)}_{m+3}(B,A)$	$H_{m+3}(A,m)$	$H_{m+4}(A,m)$
$m=2$			$\Gamma T(A)$	$\Gamma_6(A)$
$m=3$	$\Lambda^2 T^{\#}(A,B)$ $\oplus$split	$\Lambda^2 T_{\#}(B,A)$ $\oplus$split	$\Lambda^2(A)$ $\oplus A*\mathbb{Z}/2$	$\Lambda^2 T(A)$ $\oplus A\otimes\mathbb{Z}/3$
$m=4$	split	split	$A*\mathbb{Z}/2$	$\Gamma(A)\oplus A\otimes\mathbb{Z}/3$
$m\geq 15$	split	split	$A*\mathbb{Z}/2$	$A\otimes\mathbb{Z}/6$

The split cases are consequences of Theorems 6.1.7 and 6.1.8.

(6.3.11)

	$H^{m+5}_{(m)}(A,B)$	$H^{(m)}_{m+4}(B,A)$	$H_{m+4}(A,m)$	$H_{m+5}(A,m)$
$m=2$			$\Gamma_6(A)$	see Decker [IH]
$m=3$			$\Lambda^2\mathrm{T}(A)$ $\oplus A\otimes\mathbb{Z}/3$	$A\otimes A\otimes\mathbb{Z}/2$ $\oplus A*\mathbb{Z}/3$
$m=4$	$\Gamma T^{\#}(A,B)$ $\oplus$split	$\Gamma T_{\#}(B,A)$ $\oplus$split	$\Gamma(A)$ $\oplus A\otimes\mathbb{Z}/3$	$\Gamma T(A)$ $\oplus A*\mathbb{Z}/3$
$m=5$	$L^{\#}(A,B)$ $\oplus$split	$L_{\#}(B,A)$ $\oplus$split	$A\otimes\mathbb{Z}/2$ $\oplus A\otimes\mathbb{Z}/3$	$\Lambda^2(A)$ $\oplus A*\mathbb{Z}/6$
$m\geq 6$	split	split	$A\otimes\mathbb{Z}/6$	$A*\mathbb{Z}/6$

The split case $m \geq 6$ is again a consequence of Theorem 6.1.7 and 6.1.8. For example the case $m = 5$ in the list means that one has natural isomorphisms of kype functors and bype functors respectively:

(6.3.12)

$$H^{10}_{(5)}(A,B) = L^{*}(A,B) \oplus \mathrm{Ext}(A \otimes \mathbb{Z}/3, B) \oplus \mathrm{Hom}(A * \mathbb{Z}/6, B),$$
$$H^{(5)}_{9}(B,A) = L_{\#}(B,A) \oplus \mathrm{Ext}(B, A * \mathbb{Z}/6) \oplus \mathrm{Hom}(B, A \otimes \mathbb{Z}/3).$$

For the definition of L^{*} and $L_{\#}$ see Definitions 6.2.12 and 6.2.13.

Proof for the lists (6.3.8)–(6.3.11) One finds all computations of the cohomology H^{m+r+1} for $r \leq 4$ in §27 of Eilenberg and Mac Lane [II]. The case $H^{5}_{(2)}$ is not treated by Eilenberg and Mac Lane. This case is obtained as follows. We shall show in Theorem 9.4.2 that $H^{(2)}_{4} = \Gamma T_{\#}$. Since ΓT^{*} is dual to $\Gamma T_{\#}$ and since $H^{5}_{(2)}$ is dual to $H^{(2)}_{4}$ this also implies $H^{5}_{(2)} = \Gamma T^{*}$. Hence we use here duality in a crucial way. We also use duality for the computation of $H^{(m)}_{m+r}$. For example $H^{10}_{(5)}$ in (6.3.11), $m = 5$, is known by Eilenberg and Mac Lane [II]. This yields $H^{(5)}_{9}$ since we have the duality of L^{*} and $L_{\#}$ in Theorem 6.2.14. The cases $H^{7}_{(3)}$, $H^{(3)}_{6}$, $H^{9}_{(4)}$, and $H^{(4)}_{8}$ are treated in the Diplomarbeit of J. Wendelken [KEM]. □

6.4 On $(m-1)$-connected $(n+1)$-dimensional homotopy types with $\pi_i X = 0$ for $m < i < n$

We consider homotopy types of $(n+1)$-dimensional spaces X for which the $(n-1)$-type $P_{n-1}(X)$ is an Eilenberg–Mac Lane space $K(A,m)$, $m \geq 2$. For such homotopy types the classification theorem in Chapter 3 can be applied effectively since the category **C** in this case is just the category of $(m-1)$-connected m-types which is equivalent to the category of abelian groups. Moreover the bype and kype functors in question are the Eilenberg–Mac Lane bifunctors discussed in Section 6.1; explicit examples of Eilenberg–Mac Lane functors are described in Section 6.3 above.

Let **spaces**$(m,n)_{\pi}$ be the full homotopy category of $(m-1)$-connected $(n+1)$-dimensional CW-spaces X with $\pi_i(X) = 0$ for $m < i < n$. Moreover let **types**$(m,n)_{\pi}$ be the full homotopy category of $(m-1)$-connected n-types Y with $\pi_i(Y) = 0$ for $m < i < n$. Hence Y has at most two non-trivial homotopy groups $\pi_m Y$ and $\pi_n Y$. Recall that we have the Eilenberg–Mac Lane functors $H^{n+1}_{(m)}$ and $H^{(m)}_{n}$ in Sections 6.1 and 6.3 which are kype functors and bype functors respectively. The next result is an immediate application of the classification theorem 3.4.4; it gives us explicit algebraic models of homotopy types.

(6.4.1) Classification theorem *Let $2 \le m < n$. Then one has detecting functors*

$$\Lambda: \mathbf{spaces}(m,n)_\pi \to \mathbf{Kypes}\big(\mathbf{Ab}, H^{n+1}_{(m)}\big)$$

$$\Lambda': \mathbf{spaces}(m,n)_\pi \to \mathbf{Bypes}(\mathbf{Ab}, H^{(m)}_n)$$

$$\lambda: \mathbf{types}(m,n)_\pi \to \mathbf{kypes}\big(\mathbf{Ab}, H^{n+1}_{(m)}\big) = \mathbf{Gro}\big(H^{n+1}_{(m)}\big)$$

$$\lambda': \mathbf{types}(m,n)_\pi \to \mathbf{bypes}(\mathbf{Ab}, H^{(m)}_n).$$

Here the categories on the right-hand side are the purely algebraic categories given by the kype functor $H^{n+1}_{(m)}$ and the bype functor $H^{(m)}_n$; see Sections 3.1 and 3.2. The detecting functor λ in the classification theorem is well known in the literature. The more sophisticated detecting functors Λ, Λ', and λ', however, yield new results on the algebraic classification of homotopy types.

Proof of Theorem 6.4.1 Consider Theorem 3.4.4 where we take $m + r = n$ and

$$\mathbf{C} = \mathbf{types}^0_m \xleftarrow{\sim} \mathbf{Ab}. \qquad (*)$$

Then we have

$$\mathbf{spaces}^{r+1}_m(\mathbf{C}) = \mathbf{spaces}(m,n)_\pi$$

$$\mathbf{types}^r_m(\mathbf{C}) = \mathbf{types}(m,n)_\pi$$

and E, resp. F, in Theorem 3.4.4 coincide with $H^{n+1}_{(m)}$, resp. $H^{(m)}_n$, by use of the equivalence $(*)$. This immediately yields the proposition of Theorem 6.4.1. □

Recall that the $H^{n+1}_{(m)}$-kypes used in Theorem 6.4.1 are tuples

$$\bar{X} = (A, \pi, k, H, b)$$

where A, π, H are abelian groups and

$$(6.4.2) \qquad \begin{cases} k \in H^{n+1}_{(m)}(A, \pi) \\ b \in \mathrm{Hom}(H, H_{n+1}(A, m)) \end{cases}$$

such that the sequence $H \xrightarrow{b} H_{n+1}(A,m) \xrightarrow{\mu(k)} \pi$ is exact. The kype $\bar{X}$ is free if H is free abelian; then $\bar{X}$ is an object in $\mathbf{Kypes}(\mathbf{Ab}, H^{n+1}_{(m)})$ and thus $\bar{X}$ determines via Λ in Theorem 6.4.1 a unique homotopy type X in

spaces$(m,n)_\pi$ with $\Lambda(X)\cong\bar{X}$. In (3.1.5) we associate with $\bar{X}$ the exact Γ-sequence which is the top row in the commutative diagram, $A=\pi_m X$,

(6.4.3)

$$\begin{array}{ccccccccc}
H & \xrightarrow{b} & H_{n+1}(A,m) & \xrightarrow{\mu(k)} & \pi & \to H(k_\dagger) & \to H_n(A,m) & \to 0 \\
\| & & \| & & \| & \downarrow\cong & \| & \\
H_{n+1}X & \longrightarrow & \Gamma_n X & \longrightarrow & \pi_n X & \to H_n X & \to \Gamma_{n-1}X & \to 0
\end{array}$$

which is a natural 'weak' isomorphism of exact sequences. The bottom row is part of Whitehead's certain exact sequence for X; compare Theorem 3.4.4. In particular we get:

(6.4.4) Theorem *Let X be an $(m-1)$-connected space, $m\geq 2$, with $\pi_i X=0$ for $m<i<n$. Then the homology $H_n X$ is determined as an abelian group by the k-invariant $k\in H^{n+1}_{(m)}(\pi_m X,\pi_n X)$ of X since we have the isomorphism $H_n X\cong H(k_\dagger)$ in* (6.4.3).

Remark The main theorem of Eilenberg and Mac Lane [RH] shows that the cohomology $H^n(X,G)$ of a space X as in (6.4.4) is determined up to isomorphism by the k-invariant k of X. Their description of $H^n(X,G)$ relies on a choice of a cocycle representing k. The direct computation of $H_n(X)=H(k_\dagger)$ in terms of k in Theorem 6.4.4, however, was not achieved and seems to be new; compare the remark following the theorem of Postnikov invariants in Theorem 2.5.10. Using Theorem 6.4.4 we obtain $H^n(X,G)$ as an abelian group by

$$H^n(X,G)\cong \mathrm{Ext}(H_{n-1}(\pi_n X,m),G)\oplus \mathrm{Hom}\big(H(k_\dagger),G\big)$$

where we use the universal coefficient formula.

On the other hand, recall that the $H_n^{(m)}$*-bypes* in Theorem 6.4.1 are tuples

(6.4.5) $$\bar{X}=(A,H_0,H_1,b,\beta).$$

Here A,H_0,H_1 are abelian groups and

$$\begin{cases} b\in \mathrm{Hom}(H_1,H_{n+1}(A,m)), \\ \beta\in H_n^{(m)}(H_0,A,b)=\mathrm{cok}(\Delta\,\mathrm{Ext}(H_0,b)), \end{cases}$$

$$\Delta\,\mathrm{Ext}(H_0,b)\colon \mathrm{Ext}(H_0,H_1)\to \mathrm{Ext}(H_0,H_{n+1}(A,m))\to H_n^{(m)}(H_0,A),$$

such that $\mu(\beta)\colon H_0\to H_n(A,m)$ is surjective. The bype $\bar{X}$ is free if H_1 is free abelian; then $\bar{X}$ is an object in **Bypes(Ab**, $H_n^{(m)}$) and thus $\bar{X}$ determines via Λ' in Theorem 6.4.1 a unique homotopy type in X in **spaces**$(m,n)_\pi$ with $\Lambda'(X)\cong\bar{X}$. In (3.2.5) we associate with $\bar{X}$ the exact Γ-sequence which is the top row in the commutative diagram, $A = H_m X$,

(6.4.6)

$$\begin{array}{ccccccccc}
H_1 & \xrightarrow{b} & H_{n+1}(A,m) & \to & \pi(\beta_\dagger) & \to & H_0 & \xrightarrow{\mu(\beta)} & H_n(A,m) \to 0 \\
\| & & \| & & \downarrow\cong & & \| & & \| \\
H_{n+1}X & \xrightarrow{b_{n+1}X} & \Gamma_n X & \longrightarrow & \pi_n(X) & \to & H_n X & \xrightarrow{b_n X} & \Gamma_{n-1}X \to 0.
\end{array}$$

This is a natural 'weak' isomorphism of exact sequences; compare Theorem 3.4.4. Dually to Theorem 6.4.4 we thus get:

(6.4.7) Theorem *Let X be an $(m-1)$-connected space, $m \geq 2$, with $\pi_i(X)=0$ for $m<i<n$. Then the homotopy group $\pi_n X$ is determined as an abelian group by the boundary invariant $\beta \in H_n^{(m)}(H_n X, H_m X, b_{n+1}X)$ of X since we have the isomorphism $\pi_n X \cong \pi\{\beta_\dagger\}$ in (6.4.6).*

6.5 Split Eilenberg–Mac Lane functors

We describe Eilenberg–Mac Lane sequences. Such sequences are algebraic models of $(m-1)$-connected $(n+1)$-dimensional homotopy types X with $\pi_i(X)=0$ for $m<i<n$ in the case when the Eilenberg–Mac Lane functors $H_{(m)}^{n+1}$ and $H_n^{(m)}$ are split.

(6.5.1) Definition Let $n>m\geq 2$. An Eilenberg–Mac Lane *(m,n)-sequence* S is an abelian group A together with a chain complex of abelian groups

$$S=\{H_1 \xrightarrow{b} H_{n+1}(A,m) \xrightarrow{\partial} T \xrightarrow{\delta} H_n(A,m) \to 0\}$$

which is exact in $H_{n+1}(A,m)$ and $H_n(A,m)$. Here we use the classical Eilenberg–Mac Lane functors $H_n(-,m)$ in Definition 6.1.2. A proper morphism between (m,n)-sequences is a homomorphism $f\colon A \to A'$ together with a commutative diagram

$$\begin{array}{ccccccc}
H_1 & \to & H_{n+1}(A,m) & \to & R & \to & H_n(A,m) \to 0 \\
\downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle f_*} \\
H_1' & \to & H_{n+1}(A',m) & \to & R' & \to & H_n(A',m) \to 0.
\end{array}$$

We call an (m,n)-sequence S *free* if H_1 is free and *injective* if b is injective.

Let **S**(m,n), resp. **s**(m,n) be the categories consisting of free, resp. injective, (m,n)-sequences and proper morphisms. These categories coincide with the categories **S**(E_0, E_1), resp. **s**(E_0, E_1) in Definition 3.6.1 where we set $E_0 = H_n(-,m)$ and $E_1 = H_{n+1}(-,m)$. As in Definition 3.6.1 we have the natural equivalence relations $\overset{k}{\sim}$ and $\overset{b}{\sim}$ on these categories.

(6.5.2) Classification theorem *Let $2 \le m < n = m + r$ and assume $H_{(m)}^{n+1}$ or equivalently $H_n^{(m)}$ are split; this is the case for $r < m$ or for m even and $r \le m$; see Theorems* 6.1.7 *and* 6.1.8. *Then one has detecting functors*

$$\Lambda: \mathbf{spaces}(m,n)_\pi \to \mathbf{S}(m,n)/\overset{k}{\sim}$$

$$\Lambda': \mathbf{spaces}(m,n)_\pi \to \mathbf{S}(m,n)/\overset{b}{\sim}$$

$$\lambda: \mathbf{types}(m,n)_\pi \to \mathbf{s}(m,n)/\overset{k}{\sim}$$

$$\lambda': \mathbf{types}(m,n)_\pi \to \mathbf{s}(m,n)/\overset{b}{\sim} .$$

Proof We apply the classification theorem 3.6.3; compare the proof of Theorem 6.4.1 above. □

As a special case of Theorem 6.5.2 one obtains for $m \ge 4$, $n = m + 2$, the example discussed in Theorem 3.6.5. The theorem shows that (m,n)-sequences are algebraic models of homotopy types in case $H_n^{(m)}$ or $H_{(m)}^{n+1}$ are split. In fact, proper isomorphism classes of free (m,n)-sequences in **S**(m,n) are in 1–1 correspondence (via Λ or Λ') with homotopy types in **spaces**$(m,n)_\pi$. On the other hand, proper isomorphism classes of injective (m,n)-sequences in **s**(m,n) are in 1–1 correspondence (via λ or λ') with homotopy types in **types**$(m,n)_\pi$. Here we use the detecting functors in Definition 3.6.1(6), (7).

Let X be the unique homotopy type in **spaces**$(m,n)_\pi$ corresponding via Λ (or Λ') to a free (m,n)-sequence S as in Definition 6.5.1, so that $\Lambda(X) \cong S$ (or $\Lambda'(X) \cong S$). In Definition 3.6.1(8) we associate with S the exact Γ-sequence which is the top row in the commutative diagram $(A = H_m X)$

(6.5.3)

$$\begin{array}{ccccccccccc}
H_1 & \to & H_{n+1}(A,m) & \to & \ker(\delta) & \to & \operatorname{cok}(\partial) & \to & H_n(A,m) & \to & 0 \\
\| & & \| & & \downarrow\cong & & \downarrow\cong & & \| & & \\
H_{n+1}X & \to & \Gamma_n X & \to & \pi_n X & \to & H_n X & \to & \Gamma_{n-1} X & \to & 0.
\end{array}$$

This is a natural 'weak' isomorphism of exact sequences. The same diagram is available in case S is an injective (m,n)-sequence and X is a homotopy type in $\mathbf{types}(m,n)_\pi$.

(6.5.4) Example For $m=4$ and $n=7$ we know that $H_7^{(4)}$ and $H_{(4)}^8$ are split and that

$$H_7(A,4)=A*\mathbb{Z}/2$$

$$H_8(A,4)=\Gamma(A)\oplus A\otimes\mathbb{Z}/3;$$

see (6.3.3) and (6.3.4). Hence 3-connected 8-dimensional homotopy types X with $\pi_5X=\pi_6X=0$ are in 1–1 correspondence with proper isomorphism classes of chain complexes in **Ab**

$$S=\{H_1\to\Gamma(A)\oplus A\otimes\mathbb{Z}/3\xrightarrow{\partial}R\xrightarrow{\delta}A*\mathbb{Z}/2\to 0\}$$

which are exact in $A*\mathbb{Z}/2$ and $\Gamma(A)\oplus A\otimes\mathbb{Z}/3$ and for which H_1 is free abelian. If X is the homotopy type corresponding to S then one has the commutative diagram of exact sequences

$$\begin{array}{ccccccccccc}
H_1 & \to & \Gamma(A)\oplus A\otimes\mathbb{Z}/3 & \to & \ker(\delta) & \to & \operatorname{cok}(\partial) & \to & A*\mathbb{Z}/2 & \to & 0\\
\| & & \| & & \downarrow\cong & & \downarrow\cong & & \| & & \\
H_8X & \longrightarrow & \Gamma_7X & \longrightarrow & \pi_7X & \to & H_7X & \to & \Gamma_6X & \to & 0
\end{array}$$

which is a natural 'weak' isomorphism of exact sequences. For example, for $A=\mathbb{Z}/2$ with $\Gamma(\mathbb{Z}/2)=\mathbb{Z}/4$ the $(4,7)$-sequence

$$S=\{\mathbb{Z}\xrightarrow{2}\mathbb{Z}/4\xrightarrow{8}\mathbb{Z}/16\xrightarrow{1}\mathbb{Z}/2\to 0\}$$

corresponds to a 3-connected 8-dimensional homotopy type X with

$$\begin{array}{ccccccccccc}
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4 & \xrightarrow{4} & \mathbb{Z}/8 & \xrightarrow{2} & \mathbb{Z}/8 & \xrightarrow{1} & \mathbb{Z}/2 & \to & 0\\
\| & & \| & & \| & & \| & & \| & & \\
H_8X & \longrightarrow & \Gamma_7X & \longrightarrow & \pi_7X & \longrightarrow & H_7X & \longrightarrow & \Gamma_6X & \to & 0
\end{array}$$

and $\pi_6X=\pi_5X=0$ and $\pi_4X=H_4X=\mathbb{Z}/2$ and $H_6X=\Gamma_5X=\mathbb{Z}/2$, $H_5X=\Gamma_4X=0$.

6.6 A transformation from homotopy groups of Moore spaces to homology groups of Eilenberg–Mac Lane spaces

There is a connection between the homotopy groups of Moore spaces and the homology groups of Eilenberg–Mac Lane spaces. For example, it was observed by Eilenberg and Mac Lane, and J.H.C. Whitehead that

$$\pi_3 M(A,2) = \Gamma A = H_4 K(A,2).$$

We here describe further results of this kind. We have the canonical map

$$k\colon M(A,n) \to K(A,n) \quad (n \geq 2)$$

which induces the identity on $\pi_n = A$. This map induces a natural transformation Q which carries the homotopy groups of a Moore space to the homology groups of an Eilenberg–Mac Lane space ($N > n$):

(6.6.1)
$$\begin{tikzcd} \pi_N M(A,n) \arrow[r, "Q = b^{-1}(\Gamma_N k) i^{-1}"'] & H_{N+1} K(A,n) \arrow[d, "b", "\cong"'] \\ \Gamma_N M(A,n) \arrow[u, "i"', "\cong"] \arrow[r, "\Gamma_N(k)"'] & \Gamma_N K(A,n). \end{tikzcd}$$

Here i and b are operators of the exact sequence of J.H.C. Whitehead which are isomorphisms for $N > n$. We call an element $\alpha \in \pi_N M(A,n)$ *strictly decomposable* if there is a homotopy commutative diagram

(6.6.2)
$$\begin{tikzcd} S^N \arrow[rr, "\alpha"] \arrow[dr] & & M(A,n) \\ & X_{n+1}^{N-1} \arrow[ur] & \end{tikzcd}$$

where X_{n+1}^{N-1} is a CW-complex which is n-connected and $(N-1)$-dimensional.

(6.6.3) Lemma *If $\alpha \in \pi_N M(A,n)$ is strictly decomposable then $Q\alpha = 0$.*

Proof If α admits a factorization as above the composition

$$X_{n+1}^{N-1} \to M(A,n) \to K(A,n)^{N-1} \subset K(A,n)^N$$

is null-homotopic since $[X_{n+1}^{N-1}, K(A,n) = 0]$. This implies the lemma. □

(6.6.4) Lemma *Let $\alpha \in \pi_N M(A,n)$ with $N > n$. Then $Q[\alpha, \beta] = 0$ for any Whitehead product $[\alpha, \beta]$.*

Proof Let $\beta \in \pi_M M(A,n)$, $M \geq n$. Then the composite

$$[\alpha, \beta]\colon S^{N+M-1} \to S^N \vee S^M \to M(A,n) \to K(A,n)^{N+M-1}$$

is trivial since $n < N < N + M - 1$, $(n \geq 2)$. □

For $\alpha, \beta \in A = \pi_n M(A, n)$, however, the element $Q[\alpha, \beta]$ is not necessarily trivial. This follows from Theorem 6.6.6 below.

(6.6.5) Theorem *For $N > 2n - 1$ we have a commutative diagram:*

$$\begin{tikzcd}
 & \pi_N(S^n) \times A \arrow[r, "\otimes"] \arrow[d, "c_1"] & \pi_N(S^n) \otimes A \arrow[d, "c_2"] \\
 & \pi_N M(A, N) \arrow[r, "Q"] \arrow[d, "j"] \arrow[dl, "j"'] & H_{N+1}K(A, n) \arrow[d, "p"] \\
\pi_N(M(A,n), M(A,n)^n) \supset & j\pi_N M(A, n) \arrow[r, "Q'"] & \operatorname{cok} c_2 .
\end{tikzcd}$$

Here j is defined by the n-skeleton $M(A, n)^n$ of $M(A, n)$. Moreover, for $\alpha \in A = \pi_n M(A, n)$ and $\xi \in \pi_N S^n$ we define the function c_1 by the composite $c_1(\xi, \alpha) = \alpha \circ \xi$. We show that Qc_1 is bilinear, therefore we obtain the factorization c_2. Moreover, we show that for the projection p above the composition pQ factors through j.

Proof of Theorem 6.6.5 Qc_1 is bilinear. This follows from the left distributivity law and from Lemmas 6.6.4 and 6.6.3. Clearly by definition of j we have $jc_1 = 0$. Moreover, by the Hilton–Milnor formula for $\pi_n M(A, n)^n$ we see that the kernel of j is spanned by the image of c_1 and by composites $W\xi$ where W is a Whitehead product. Now we obtain the result by Lemmas 6.6.3 and 6.6.4 since $N > 2n - 1$. □

For $N = n + 2$ the kernel of Q is actually generated by elements as in Lemma 6.6.4 and Theorem 6.6.5. The image of Q is a subfunctor which carries A to $Q\pi_N M(A, n) \subset H_{N+1}K(A, n)$. While the homology groups $H_j K(A, n)$ were extensively studied, the groups $Q\pi_N M(A, n)$ seem to be unknown. These groups are non-trivial since we get:

(6.6.6) Theorem *For $n \geq 2$ we have the isomorphism*

$$Q\colon \pi_{n+1} M(A, n) \cong H_{n+2} K(A, n) = \begin{cases} \Gamma A & \text{for } n = 2 \\ A \otimes \mathbb{Z}_2 & \text{for } n \geq 3 \end{cases}$$

and the surjection

$$Q\colon \pi_{n+2} M(A, n) \twoheadrightarrow H_{n+3} K(A, n) = \begin{cases} \Gamma T(A) & \text{for } n = 2 \\ A * \mathbb{Z}_2 \oplus \Lambda^2(A) & \text{for } n = 3 \\ A * \mathbb{Z}_2 & \text{for } n \geq 4. \end{cases}$$

Proof The isomorphism is obtained since, by a result of J.H.C. Whitehead, $\Gamma_{n+1}X = \Gamma_n^1(H_n X)$ for any $(n-1)$-connected space X. Below in Chapters 7, 8, and 11 we shall compute $\Gamma_{n+2}X$ for any $(n-1)$-connected space X. This yields the surjection, $n \geq 2$. □

The operator Q is available for Γ-groups with coefficients as follows $(n < N)$:

(6.6.7)
$$\begin{tikzcd} \pi_N(B, M(A,n)) \arrow[r, "Q = b^{-1}k_* i^{-1}"'] & H_{N+1}(B, K(A,N)) \arrow[d, "\cong"', "b"] \\ \Gamma_N(B, M(A,n)) \arrow[u, "i", "\cong"'] \arrow[r, "k_*"'] & \Gamma_N(B, K(A,n)). \end{tikzcd}$$

Here the isomorphisms i and b are given by the exact sequence in Section 2.3. The pseudo-homology is the group of homotopy classes of chain maps

(i)
$$H_N(B, K(A,n)) = [C_* M(B,N), C_* K(A,n)].$$

This is a bifunctor on pairs (B, A) of abelian groups. The operator Q is embedded in the following commutative diagram, the rows of which are short exact sequences:

(ii)

$$\begin{tikzcd} \mathrm{Ext}(B, \pi_{N+1}M(A,n)) \arrow[r, tail, "\Delta"] \arrow[d, "Q_*"] & \pi_N(B, M(A,n)) \arrow[r, two heads, "\mu"] \arrow[d, "Q"] & \mathrm{Hom}(B, \pi_N M(A,n)) \arrow[d, "Q_*"] \\ \mathrm{Ext}(B, H_{N+2}K(A,n)) \arrow[r, tail, "\Delta"] & H_{N+1}(B, K(A,n)) \arrow[r, two heads, shift left] & \mathrm{Hom}(B, H_{N+1}K(A,n)). \arrow[l, dashed, shift left] \end{tikzcd}$$

Here Q_* is induced by Q in (6.6.2). This diagram is easily obtained from the definition in (6.6.7), compare Section 2.3. For $N = n+1$ the right column of (ii) is an isomorphism and the left column is surjective by Theorem 6.6.6. Therefore we get the

(6.6.8) Corollary *For $N = n+1$ diagram* (ii) *is a push-out diagram and*

$$Q: \pi_{n+1}(B, M(A,n)) \twoheadrightarrow H_{n+2}(B; K(A,n))$$

is surjective, $n \geq 2$.

The fundamental importance of the subfunctors

$$Q\pi_N M(A,n) \subset H_{N+1}K(A,n),$$

$$Q\pi_N(B; M(A,n)) \subset H_{N+1}(B; K(A,n))$$

is described by the following fact. Let X be any $(n-1)$-connected CW-complex with $\pi_n X = H_n X = A$. Then we have the homotopy commutative diagram

$$\begin{tikzcd} M(A,n) \arrow[rr, "k"] \arrow[dr, "i"'] & & K(A,n) \\ & X \arrow[ur, "k_X"'] & \end{tikzcd} \tag{6.6.9}$$

where k_X is the fundamental class of X and where i induces the identity on H_n. The map k is the one in (6.6.1). The map k_X induces homomorphisms

$$Q\colon \Gamma_N X \to \Gamma_N K(A,n) \cong H_{N+1}K(A,n)$$

$$Q\colon \Gamma_N(B; X) \to \Gamma_N(B; K(A,n)) \cong H_{N+1}(B, K(A,n))$$

and (6.6.9) immediately implies:

(6.6.10) Proposition *For all $(n-1)$-connected CW-complexes X with $\pi_n X = H_n X = A$, $n \geq 2$, we have inclusions*

$$Q\pi_N M(A,n) \underset{i}{\subset} Q\Gamma_N X \underset{j}{\subset} H_{N+1}K(A,n),$$

$$Q\pi_N(B; M(A,n)) \underset{i}{\subset} Q\Gamma_N(B; X) \underset{j}{\subset} H_{N+1}(B; K(A,n)).$$

Here i is the identity if $X = M(A,n)$ and j is the identity if $X = K(A,n)$.

This fact clearly shows that a computation of $\Gamma_N X$ involves the computation of $Q\pi_N M(A,n)$. For example we derive from (6.6.7) and Corollary 6.6.8.

(6.6.11) Theorem *For all $(n-1)$-connected CW-complexes X with $\pi_n X = H_n X = A$, $n \geq 2$, we have surjective maps*

$$Q\colon \Gamma_{n+2} X \twoheadrightarrow H_{n+3}K(A,n),$$

$$Q\colon \Gamma_{n+1}(B; X) \twoheadrightarrow H_{n+2}(B; K(A,n)).$$

In fact, we will compute the groups $\Gamma_{n+2}X$ and $\Gamma_{n+1}(B; X)$ for any $(n-1)$-connected space X, $n \geq 2$.

B MOORE FUNCTORS

Moore functors are dual to Eilenberg–Mac Lane functors. We used Eilenberg–Mac Lane functors for the classification of $(m-1)$-connected $(n+1)$-dimensional homotopy types X which have trivial homotopy groups

$$\pi_i(X) = 0 \quad \text{for} \quad m < i < n. \tag{$*$}$$

Now we use Moore functors for the classification of $(m-1)$-connected $(n+1)$-dimensional homotopy types X which have trivial homology groups

$$H_i(X) = 0 \quad \text{for} \quad m < i < n. \tag{$**$}$$

We assume that $m \geq 2$ and $n = m + r \geq 4$. For $r = 2$ such spaces are part of the classification in Chapters 8, 9, and 12. In both cases $(*)$ and $(**)$ we use the classification theorem (Section 3.4). If we want to apply this theorem we have to choose a full subcategory $\mathbf{C} \subset \mathbf{types}_m^{r-1}$. In case $(*)$ the category $\mathbf{C}$ is the category of Eilenberg–Mac Lane spaces $K(A,m)$ which is equivalent to the category of abelian groups. In case $(**)$ the category $\mathbf{C}$ is the full homotopy category of '(m,n)-Moore types' which can be described in terms of the homotopy category $\mathbf{M}^m$ of Moore spaces $M(A,m)$. There are algebraic categories equivalent to the category $\mathbf{M}^m$, for example for $m \geq 3$ we have the equivalence $\mathbf{G} \cong \mathbf{M}^m$ in Theorem 1.6.7. The classification theorem Section 3.4 describes bype and kype functors on $\mathbf{C}$ which in case $(*)$ are the Eilenberg–Mac Lane functors and which in case $(**)$ are functors which we call 'Moore functors'. They are given by homotopy groups of Moore spaces. In the stable and metastable range we describe various algebraic properties of such Moore functors. For the metastable range we need the basic theory of quadratic functors which we describe in Sections 6.13 and 6.14.

6.7 Moore types and Moore functors

Let $n = m + r$ with $m, r \geq 2$.

(6.7.1) Definition An $(m-1)$-connected $(n-1)$-type X is an *(m,n)-Moore type* if the homology groups of X satisfy $H_i(X) = 0$ for $m < i < n$. Let $\mathbf{Moore}(m,n)$ be the full homotopy category of (m,n)-Moore types in $\mathbf{Top}^*/\simeq$.

In this section we describe an algebraic category equivalent to the category of (m,n)-Moore types and we study kype and bype functors on the category $\mathbf{Moore}(m,n)$ which we call Moore functors. Recall that $\mathbf{M}^m$ denotes the full homotopy category consisting of Moore spaces $M(A,m)$ of degree m. Algebraic models of this category are studied in Chapters 1 and 10. We now use homotopy groups $\pi_{n-1}M(A,m)$ to define an 'enriched' category of Moore spaces which is equivalent to $\mathbf{Moore}(m,n)$.

Definition Let $\mathbf{M}(m,n)$ be the following category. An object is a pair $(M(A,m),i)$, also denoted by (A,i), where

$$i\colon \pi_{n-1}M(A,m) \twoheadrightarrow \pi \tag{1}$$

is a surjective homomorphism. For $r \geq 3$ a morphism

$$(\overline{\varphi},\psi)\colon (M(A,m),i) \to (M(A',m),i) \tag{2}$$

is given by a homotopy class

$$\overline{\varphi} \in [M(A,m),M(A',m)] \tag{3}$$

(which is a morphism in $\mathbf{M}^m$) and by a homomorphism $\psi\colon \pi \to \pi'$ such that the diagram

$$\begin{tikzcd} \pi_{n-1}M(A,m) \arrow[r, "\overline{\varphi}_*"] \arrow[d, "i"] & \pi_{n-1}M(A',m) \arrow[d, "i'"] \\ \pi \arrow[r, "\psi"] & \pi' \end{tikzcd} \tag{4}$$

commutes. For $r = 2$ a morphism is an equivalence class $\{\overline{\varphi},\psi\}$ of a pair $(\overline{\varphi},\psi)$ as above. Here we need the action $+$ of $\mathrm{Ext}(A,\pi_{m+1}M(A',m))$ on the set (3); see Section 1.3. Since for $r = 2$ we have $m+1 = n-1$ we thus get an action of $\mathrm{Ext}(A,\ker i')$ on the set (3). Now we set $(\overline{\varphi},\psi) \sim (\overline{\varphi}_0,\psi_0)$ if $\psi = \psi_0$ and if there is $\alpha \in \mathrm{Ext}(A,\ker i')$ with $\overline{\varphi} = \overline{\varphi}_0 + \alpha$. This equivalence relation defines the class $\{\overline{\varphi},\psi\}$ which is a morphism in $\mathbf{M}(m,m+1)$.

(6.7.3) Proposition *For $n = m + r$ with $m,r \geq 2$ there is an equivalence of categories*

$$\Theta\colon \mathbf{Moore}(m,n) \xrightarrow{\sim} \mathbf{M}(m,n).$$

This result is a consequence of proposition (III.8.8) of Baues [CH]. We obtain the equivalence in Proposition 6.7.3 as follows. For each Moore type X in $\mathbf{Moore}(m,n)$ choose a map

$$i\colon M(A,m) \to X \tag{1}$$

which induces the identity $A = H_m X$ in homology. Then the functor Θ in Proposition 6.7.3 carries X to the pair

$$(M(A,m), i_*\colon \pi_{n-1}M(A,m) \twoheadrightarrow \pi_{n-1}X) \tag{2}$$

where i_* is surjective since $H_{n-1}X = 0$. Since $r \geq 2$ we can assume that i is the $(n-1)$-skeleton of X. Then Θ carries a map $f\colon X \to X'$ to $(\bar{\varphi}, \pi_{n-1}f)$ where $\bar{\varphi}$ is the restriction of f to the $(n-1)$-skeleton.

We now introduce functors on the category in Proposition 6.7.3 which we call 'Moore functors'.

(6.7.4) Definition The *Moore functors*

$$M_0, M_1\colon \mathbf{M}(m,n) \to \mathbf{Ab}$$

are defined as follows. For an object $(A,i) = (M(A,m),i)$ let

$$M_0(A,i) = \text{kernel}(i\colon \pi_{n-1}M(A,m) \twoheadrightarrow \pi)$$

and

$$M_1(A,i) = \text{cokernel}(\eta\colon M_0(A,i) \rightarrowtail \pi_n M(A,m)).$$

Here η is the restriction of the map $\eta_n^*\colon \pi_{n-1}M(A,m) \to \pi_n M(A,m)$ induced by the Hopf map η_n. The function η_n^* is a homomorphism since $n = m + r \geq 4$. It is clear how to define M_0 and M_1 on morphisms.

We can describe the Moore functors also by use of homology groups. For this let $X(A,i)$ be the (m,n)-Moore type corresponding to $(M(A,m),i)$ via the equivalence Θ in Proposition 6.7.3.

(6.7.5) Proposition *There are isomorphisms*

$$M_0(A,i) \cong H_n X(A,i)$$

$$M_1(A,i) \cong H_{n+1}X(A,i) \cong \Gamma_n X(A,i)$$

which are natural in $(A,i) = (M(A,m),i) \in \mathbf{M}(m,n)$.

Proof Since $M(A,m)$ is the $(n-1)$-skeleton of $X(A,i)$ we have

$$\Gamma_{n-1}X(A,i) = \pi_{n-1}M(A,m).$$

Moreover the operator i_{n-1} in Whitehead's exact sequence coincides with i, that is

$$i_{n-1}\colon \Gamma_{n-1}X(A,i) = \pi_{n-1}M(A,m) \xrightarrow{i} \pi = \pi_{n-1}X(A,i).$$

This implies $H_n X(A,i) = \ker(i) = M_0(A,i)$ since $\pi_n X(A,i) = 0$. In a similar way we get $H_{n+1}X(A,i) = \Gamma_n X(A,i) \cong M_1(A,i)$. □

Next we define bifunctors

$$(6.7.6)\qquad \begin{aligned} M^{\#}&: \mathbf{M}(m,n)^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab} \\ M_{\#}&: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{M}(m,n) \to \mathbf{Ab}, \end{aligned}$$

which we also call *Moore functors*. We define these functors by cohomology and pseudo-homology groups with coefficients:

$$M^{\#}(A,i;B) = H^{n+1}(X(A,i),B)$$

$$M_{\#}(B;A,i) = H_n(B,X(A,i)).$$

Here again we use the equivalence in Proposition 6.7.3. Hence $M^{\#}$ and $M_{\#}$ are kype and bype functors respectively with structure (M_0, M_1, Δ, μ); see Proposition 6.7.5 and Theorem 3.3.9. Moreover $M_{\#}$ and $M^{\#}$ are dual to each other. We can describe the bype functor $M_{\#}$ together with its bype structure (M_0, M_1, Δ, μ) by the following push-out–pull-back diagram in which the rows are short exact.

(6.7.7)

$$\begin{tikzcd}
\mathrm{Ext}(B,\pi_n M(A,m)) \arrow[r, tail] \arrow[d, "q_*"] \arrow[dr, phantom, "\text{push}"] & \pi_{n-1}(B,M(A,m)) \arrow[r, two heads, "\mu"] \arrow[d] & \mathrm{Hom}(B,\pi_{n-1}M(A,m)) \arrow[d, equal] \\
\mathrm{Ext}(B,M_1(A,i)) \arrow[r, tail] \arrow[d, equal] & \Gamma_{n-1}(B,X(A,i)) \arrow[r, two heads] \arrow[dr, phantom, "\text{pull}"] & \mathrm{Hom}(B,\pi_{n-1}M(A,m)) \\
\mathrm{Ext}(B,M_1(A,i)) \arrow[r, tail, "\Delta"] & M_{\#}(B;A,i) \arrow[u] \arrow[r, two heads, "\mu"] & \mathrm{Hom}(B,M_0(A,i)). \arrow[u, "j_*"]
\end{tikzcd}$$

Here q is the projection and j is the inclusion; see Definition 6.7.4. One obtains this diagram similarly as in the proof of Proposition 6.7.5 by the exact sequence in Section 2.3. The diagram shows that $M_{\#}$ can be computed by the homotopy groups of a Moore space with coefficients,

$$\pi_{n-1}(B,M(A,n)) = [M(B,n-1),M(A,m)]. \tag{1}$$

The top row in the diagram is the universal coefficient sequence. Though (1) is not a functor in B one can check that the push–pull group $M_{\#}(B;A,i)$ in (6.7.7) is a functor in B, so that diagram (6.7.7) can be used to define the bifunctor $M_{\#}$ in Proposition 6.7.6 in an alternative way. Diagram (6.7.7) yields a method of computation for $M_{\#}$. Using duality the functor $M_{\#}$ then determines the cohomology functor $M^{\#}$ which in general by the definition in (6.7.6) cannot be easily computed.

As we mentioned already the category $\mathbf{Moore}(m,n) \equiv \mathbf{M}(m,n)$ can be considered to be an algebraic category. For example we can use the equivalence $\mathbf{M}^m = \mathbf{G}$ for $m \geq 3$, see Theorem 1.6.7. Then one has to compute the functor π_{n-1}: $\mathbf{M}^m = \mathbf{G} \to \mathbf{Ab}$ in terms of $\mathbf{G}$; this leads to an explicit description of $\mathbf{M}(m,n)$. We shall describe explicit examples below. For $m \geq 2$ we compute the Moore functors on $\mathbf{M}(m, m+2)$ in Chapters 8, 9, and 11.

6.8 On $(m-1)$-connected $(n+1)$-dimensional homotopy types X with $H_i X = 0$ for $m < i < n$

Let $m, r \geq 2$ and $n = m + r$. The homology decomposition (Section 2.7) shows:

(6.8.1) Lemma *Let X be an $(m-1)$-connected $(n+1)$-dimensional CW-space with $H_i(X) = 0$ for $m < i < n$. Then there exists a map*

$$f\colon M(H_{n+1}X, n) \vee M(H_n X, n-1) \to M(H_m X, m)$$

such that X is homotopy equivalent to the mapping cone of f.

We get the following application of the classification theorem (Section 3.4) for homotopy types as in the lemma. Let $\mathbf{spaces}(m,n)_H$ be the full homotopy category of $(m-1)$-connected $(n+1)$-dimensional CW-spaces X with $H_i X = 0$ for $m < i < n$. Moreover let $\mathbf{types}(m,n)_H$ be the full homotopy category of $(m-1)$-connected n-types Y with $H_i Y = 0$ for $m < i < n$.

(6.8.2) Classification theorem *Let $n = m + r$ with $m, r \geq 2$. Using the Moore functors $M_\#$ and $M^\#$ on $\mathbf{M}(m,n)$ in Section 6.7 one has detecting functors:*

$$\Lambda\colon \mathbf{spaces}(m,n)_H \to \mathbf{Kypes}(\mathbf{M}(m,n), M^\#)$$

$$\Lambda'\colon \mathbf{spaces}(m,n)_H \to \mathbf{Bypes}(\mathbf{M}(m,n), M_\#)$$

$$\lambda\colon \mathbf{types}(m,n)_H \to \mathbf{kypes}(\mathbf{M}(m,n), M^\#)$$

$$\lambda'\colon \mathbf{types}(m,n)_H \to \mathbf{bypes}(\mathbf{M}(m,n), M_\#).$$

Here the categories on the right-hand side are purely algebraic in case one is able to compute the Moore functors $M_\#$ and $M^\#$ respectively.

(6.8.3) Addendum For a space X in $\mathbf{spaces}(m,n)_H$ the algebraic Γ-sequences associated with $\Lambda(X)$ and $\Lambda'(X)$ respectively are weakly isomorphic to the following part of Whitehead's exact sequence

$$\begin{tikzcd}
H_{n+1}X \arrow[r] & \Gamma_n X \arrow[r] \arrow[d, equal] & \pi_n X \arrow[r] & H_n X \arrow[r] & \Gamma_{n-1} X \arrow[r, "i_{n-1}"] \arrow[d, equal] & \pi_{n-1} X \arrow[r] \arrow[d, equal] & 0 \\
& M_1(A,i) & & & \pi_{n-1} M(A,m) \arrow[r, "i"] & \pi &
\end{tikzcd}$$

where $A = H_m X$. The map i_{n-1} corresponds to the homomorphism i given by the $(n-1)$-type $P_{n-1}(X) = X(A, i)$ of X; see Proposition 6.7.5. In particular $\pi_n X$ is determined by the object $\Lambda(X)$ in $\mathbf{Bypes}(\mathbf{M}(m,n), M_\#)$; see Remark 3.2.4.

Proof of Theorem 6.8.2 and Addendum 6.8.3. Consider Theorem 3.4.4 where we take

$$\mathbf{C} = \mathbf{Moore}(m,n) \cong \mathbf{M}(m,n). \qquad (*)$$

Then we have

$$\mathbf{spaces}_m^{r+1}(\mathbf{C}) = \mathbf{spaces}(m,n)_H$$

$$\mathbf{types}_m^r(\mathbf{C}) = \mathbf{types}(m,n)_H$$

and E, resp. F, in Theorem 3.4.4 coincide with the Moore functors $M^\#$, resp. $M_\#$ by use of the equivalence $(*)$. Hence we get the theorem and the addendum by Theorem 3.4.4. □

(6.8.4) Remark The classical Postnikov technique would use the detecting functor λ for the classification of spaces X in $\mathbf{types}(m,n)_H$. Hence X is obtained as the fibre of a map

$$X(A,i) \to K(\pi', n+1)$$

where $X(A,i) = P_{n-1}(X)$. This method, however, is not suitable for computation since it is very hard to compute the cohomology of the space $X(A,i)$. The detecting functor λ' in the theorem, however, needs only the Moore functor $M_\#$ which is easier to understand and for which more methods of computation are available.

6.9 The stable case with trivial 2-torsion

We consider a special case of the classification theorem 6.8.2 for which the Moore functors are completely determined by stable homotopy groups of spheres. Let

(6.9.1) $$\sigma_k = \varinjlim \pi_{n+k} S^n \cong \pi_{n+k} S^n \quad \text{for} \quad k < n-1$$

be the *stable k-stem* and let

$$\eta^*: \sigma_{k-1} \to \sigma_k$$

be induced by the Hopf element. Using η^* we define for an abelian group A the natural homomorphism $(r \geq 2)$

$$\eta: A * \sigma_{r-2} \oplus A \otimes \sigma_{r-1} \overset{q}{\twoheadrightarrow} A \otimes \sigma_{r-1} \xrightarrow{A \otimes \eta^*} A \otimes \sigma_r \overset{i}{\subset} A * \sigma_{r-1} \oplus A \otimes \sigma_r.$$

Here q is the projection and i is the inclusion and $A * B$ denotes the torsion product. Using this notation we define the following category.

(6.9.2) Definition A *stable r-sequence* S is a chain complex in **Ab** of the form

$$H \xrightarrow{b} \frac{A * \sigma_{r-1} \oplus A \otimes \sigma_r}{\eta\, \mathrm{image}(\delta)} \xrightarrow{\partial} R \xrightarrow{\delta} A * \sigma_{r-2} \oplus A \otimes \sigma_{r-1}$$

satisfying $\mathrm{image}(b) = \mathrm{kernel}(\partial)$. Moreover all groups are finitely generated and $A * \mathbb{Z}/2 = 0$ and $\mathrm{cok}(\partial) * \mathbb{Z}/2 = 0$. A morphism (φ, f, r) between such stable r-sequences consists of homomorphisms $\varphi \in \mathrm{Hom}(H, H')$, $f \in \mathrm{Hom}(A, A')$, and $r \in \mathrm{Hom}(R, R')$ compatible with b, ∂, and δ. We obtain natural equivalence relations $\overset{k}{\sim}$ and $\overset{b}{\sim}$ on morphisms in the same way as in Definition 3.6.1. Let $\mathbf{S}(r)$ be the full category of stable r-sequences for which H is free and let $\mathbf{s}(r)$ be the full category of stable r-sequences for which b is injective.

We now give an explicit result which is a split case of the classification theorem 6.8.2; see Theorem 3.6.3.

(6.9.3) Theorem *Let $m \geq 3$, $r \geq 2$, and $n = m + r < 2m - 2$. Then the homotopy types of $(m-1)$-connected $(n+1)$-dimensional finite CW-complexes X with $H_i(X) = 0$ for $m < i < n$ and $\mathbb{Z}/2 * H_* X = 0$ are in 1–1 correspondence with isomorphism classes of stable r-sequences in $\mathbf{S}(r)$.*

Let $\mathbf{spaces}(m,n)'_H$ be the full homotopy category of CW-complexes X as in the theorem and let $\mathbf{types}(m,n)'_H$ be the full homotopy category of n-types of such CW-complexes.

(6.9.4) Theorem *Let $m \geq 3$, $r \geq 2$, and $n = m + r < 2m - 2$. Then one has detecting functors*

$$\Lambda\colon \mathbf{spaces}(m,n)'_H \to \mathbf{S}(r)/\overset{k}{\sim}$$

$$\Lambda'\colon \mathbf{spaces}(m,n)'_H \to \mathbf{S}(r)/\overset{b}{\sim}$$

$$\lambda\colon \mathbf{types}(m,n)'_H \to \mathbf{s}(r)/\overset{k}{\sim}$$

$$\lambda'\colon \mathbf{types}(m,n)'_H \to \mathbf{s}(r)/\overset{b}{\sim}\,.$$

Here the categories on the right-hand side are purely algebraic additive categories, the sum being given by the direct sum of chain complexes. Also the categories on the left-hand side are additive categories since they are in the stable range. Moreover all functors in Theorem 6.9.4 are additive.

(6.9.5) Addendum Given a CW-complex $X = X_S$ corresponding to the stable r-sequence S in Definition 6.9.2 there is a commutative diagram with exact rows, $A = H_m X$,

$$\begin{array}{ccccccccccc}
H_{n+1}X & \longrightarrow & \Gamma_n X & \longrightarrow & \pi_n X & \to & H_n X & \longrightarrow & \Gamma_{n-1}X & \twoheadrightarrow & \pi_{n-1}X \\
\| & & \| & & \| & & \| & & \| & & \| \\
H & \to & \dfrac{A * \sigma_{r-1} \oplus A \otimes \sigma_r}{\eta \text{ image } \delta} & \to & \ker(\delta) & \to & \operatorname{cok}(\partial) & \to & A * \sigma_{r-2} \oplus A \otimes \sigma_{r-1} & \twoheadrightarrow & \operatorname{cok}(\delta).
\end{array}$$

The top row is Whitehead's exact sequence and the bottom row is deduced from the stable r-sequence S.

Proof of Theorem 6.9.4 We consider the Moore functors M_0, M_1, $M_\#$ on subcategories given by finitely generated abelian groups A, B with $\mathbb{Z}/2 * A = \mathbb{Z}/2 * B = 0$. Then we show in Theorem 6.12.15 below that the bype functor $M_\#$ is split and hence also the kype functor $M^\#$ is split. Moreover by Theorem 6.12.15 one has the natural isomorphisms

$$\pi_{n-1} M(A, m) = A * \sigma_{r-2} \oplus A \otimes \sigma_{r-1}$$

$$\pi_n M(A, n) = A * \sigma_{r-1} \oplus A \otimes \sigma_r.$$

Hence the Moore functors M_0, M_1 are given by

$$M_0(A, i) = \operatorname{kernel}(i\colon A * \sigma_{r-2} \oplus A \otimes \sigma_{r-1} \twoheadrightarrow \pi)$$

$$M_1(A, i) = \operatorname{cokernel}(\eta\colon M_0(A, i) \to A * \sigma_{r-1} \oplus A \otimes \sigma_r).$$

Here η is the restriction of η in (6.9.1). Now one readily checks that $\mathbf{S}(r) \cong \mathbf{S}(M_0, M_1)$ and $\mathbf{s}(r) \cong \mathbf{s}(M_0, M_1)$ so that the results above follow from Section 3.6. □

6.10 Moore spaces and Spanier–Whitehead duality

The Spanier–Whitehead dual of a Moore space $M(A, n)$ is again a Moore space in the case A is a finite abelian group. We here study the functorial properties of this duality between Moore spaces.

Recall that $\mathbf{A}_m^{n-m}$ denotes the full homotopy category of all finite CW-complexes $X = X_m^n$ with $\dim X \le n$ and trivial $(m-1)$-skeleton $X^{m-1} = *$. In the stable range $n < 2m - 1$ Spanier–Whitehead duality is a contravariant isomorphism of additive categories

(6.10.1) $$D\colon \mathbf{A}_m^{n-m} \xrightarrow{\cong} \mathbf{A}_m^{n-m}.$$

This functor carries X to $DX = X^*$ and carries the homotopy class $f \in [X, Y]$ to $Df = f^* \in [Y^*, X^*]$ such that

$$D: [X, Y] \cong [Y^*, X^*]$$

is an isomorphism of groups for $X, Y \in \mathbf{A}_m^{n-m}$. The isomorphism D satisfies $DD = \text{identity}$, that is $X^{**} = X$ and $f^{**} = f$. The definition of D depends on the choice of $(n+m)$-duality maps D_X: $X^* \wedge X \to S^{n+m}$; compare Section 5.2.

The dual of a sphere S^m is $DS^m = S^n$ and the dual of the Moore space $M(\mathbb{Z}/t, m)$ of the cyclic group $\mathbb{Z}/t$ is again such a Moore space,

$$DM(\mathbb{Z}/t, m) = M(\mathbb{Z}/t, n-1). \tag{6.10.2}$$

For the pseudo-projective plane $P_r = S^1 \cup_r e^2$ we have $\Sigma^{n-1} P_r = M(\mathbb{Z}/r, n)$. This yields the function

$$\Sigma^{m-1}: [P_r, P_t] \to [M(\mathbb{Z}/r, m), M(\mathbb{Z}/t, m)]$$

between sets of homotopy classes. For $\varphi \in \mathrm{Hom}(\mathbb{Z}/r, \mathbb{Z}/t)$ the element

$$\varphi = B(\varphi) \in [M(\mathbb{Z}/r, m), M(\mathbb{Z}/t, m)], \quad m \geq 3, \tag{6.10.3}$$

is uniquely determined by the following two properties. First $B(\varphi)$ induces φ in homology and second $B(\varphi)$ lies in the image of the function Σ^{m-1} above; see Theorem 1.4.4. We want to describe the dual of the map $B(\varphi)$. For this we use the canonical identification

$$\mathbb{Z}/t = \mathrm{Ext}(\mathbb{Z}/t, \mathbb{Z}) \tag{6.10.4}$$

which yields the isomorphism $\mathrm{Hom}(\mathbb{Z}/r, \mathbb{Z}/t) \cong \mathrm{Hom}(\mathbb{Z}/t, \mathbb{Z}/r)$ carrying φ up to $\varphi^{\#} = \mathrm{Ext}(\varphi, \mathbb{Z})$.

(6.10.5) Proposition *The duality map D_X with $X = M(\mathbb{Z}/t, m)$ and $t > 1$ can be chosen such that the isomorphism*

$$D: [M(\mathbb{Z}/r, m), M(\mathbb{Z}/t, m)] \cong [M(\mathbb{Z}/t, n-1), M(\mathbb{Z}/t, n-1)]$$

carries $B(\varphi)$ to $B(\varphi^{\#})$.

Proof Given the duality map D_X for all $t > 1$ one gets a derivation δ: ${}^F\mathbf{Cyc} \to \mathrm{Ext}(-, \otimes \mathbb{Z}/2)$ by setting with i and q as in (1) below

$$DB(\varphi) = B(\varphi^{\#}) + \delta(\varphi) i \eta_{n-1} q.$$

One can check by Theorem 1.4.8 that δ: $\mathrm{Hom}(\mathbb{Z}/r, \mathbb{Z}/t) \to \mathrm{Ext}(\mathbb{Z}/t, \mathbb{Z}/r \otimes \mathbb{Z}/2)$ is a homomorphism and that $\delta(\varphi) = \delta(\varphi^{\#})$. This shows that the derivation δ is completely determined by the values $\delta(\chi_r) \in \mathbb{Z}/2$ where χ_r: $\mathbb{Z}/2^{r+1} \to \mathbb{Z}/2^r$ is the projection, $r \geq 1$. We now alter the duality map D_X with $X = M(\mathbb{Z}/2^r, m)$ by the element $\varepsilon_r = \delta(\chi_r)\eta_{n+m}q$ where q: $X^* \wedge X \to S^{n+m+1}$ is the pinch map for the top cell. Then we see that these new duality maps D'_X yield a derivation δ' with

$$\delta'(\chi_r) = \delta(\chi_r) + \chi_r^* \varepsilon_{r+1} - (\chi_r)_* \varepsilon_r = 0.$$

In fact, we have $\delta(\chi_r) = (\chi_r)_* \varepsilon_r$ by definition of ε_r and $\chi_r^* \varepsilon_{r+1} = 0$. Hence also $\delta' = 0$ and the proposition is proved. □

For the Moore space $M(\mathbb{Z}/t, m) = S^m \cup_t e^{m+1}$ we have the *inclusion* i and the *pinch map* q such that

$$S^m \overset{i}{\rightarrowtail} M(\mathbb{Z}/t, m) \xrightarrow{q} S^{m+1} \tag{1}$$

is a cofibre sequence. The Spanier–Whitehead dual of the inclusion is the pinch map and the dual of the pinch map is the inclusion. Hence the cofibre sequence

$$S^n \xleftarrow{q = Di} M(\mathbb{Z}/t, n-1) \xleftarrow{i = Dq} S^{n-1} \tag{2}$$

is the dual of the sequence (1) above. This shows that D in (6.10.4) satisfies

$$D(B(\varphi) + i\eta_m q) = B(\varphi^{\#}) + i\eta_{n-1}q \tag{3}$$

where $\eta_m \in \pi_{m+1}S^m$ is the Hopf map. This formula determines the isomorphism D (Proposition 6.10.5) completely. We now consider more generally Moore spaces of abelian groups A.

Recall that for an abelian group A we have the group

(6.10.6) $$G(A) = [M(\mathbb{Z}/2, n), M(A, n)], \quad n \geq 3$$

together with the extension

$$A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \overset{\mu}{\twoheadrightarrow} A * \mathbb{Z}/2 \tag{1}$$

given by $A * \mathbb{Z}/2 \subset A \to A \otimes \mathbb{Z}/2$; see Definition 1.6.6. The extension is used for the definition of the category $\mathbf{G}$; objects in $\mathbf{G}$ are abelian groups and morphisms $A \to B$ are pairs (φ, ψ) with φ: $A \to B \in \mathbf{Ab}$ and ψ: $G(A) \to G(B) \in \mathbf{Ab}$ such that $\Delta(\varphi \otimes \mathbb{Z}/2) = \psi\Delta$ and $(\varphi * \mathbb{Z}/2)\mu = \mu\psi$. For the abelian group $\mathbf{G}(A, B)$ of such pairs (φ, ψ): $A \to B$ we have by Theorem 1.6.7 the isomorphism

$$[M(A, n), M(B, n)] = \mathbf{G}(A, B), \tag{2}$$

which in fact is given by an equivalence of categories $\mathbf{M}^n \cong \mathbf{G}$, $n \geq 3$. We define the group

$$\overline{G}(A) = [M(A,n), M(\mathbb{Z}/2,n)] \tag{6.10.7}$$

which plays a similar role to that of $G(A)$ in (6.10.6).

(6.10.8) Proposition *There is a natural isomorphism*

$$\overline{G}(A) = \mathrm{Hom}(G(A), \mathbb{Z}/4)$$

for which the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Ext}(A,\mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, equal] & \overline{G}(A) \arrow[r, two heads, "\mu"] \arrow[d, equal] & \mathrm{Hom}(A,\mathbb{Z}/2) \arrow[d, equal] \\
\mathrm{Hom}(A * \mathbb{Z}/2, \mathbb{Z}/4) \arrow[r, tail, "\Delta"] & \mathrm{Hom}(G(A),\mathbb{Z}/4) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A \otimes \mathbb{Z}/2, \mathbb{Z}/4).
\end{tikzcd}$$

In the bottom row we set $\Delta = \mathrm{Hom}(\mu, \mathbb{Z}/4)$ and $\mu = \mathrm{Hom}(\Delta, \mathbb{Z}/4)$.

Proof By (6.10.6)(2) we have

$$[M(A,n), M(\mathbb{Z}/2,n)] \cong \mathbf{G}(A, \mathbb{Z}/2) \cong \mathrm{Hom}(G(A), \mathbb{Z}/4)$$

where the right-hand isomorphism carries (φ, ψ) to ψ. In fact, ψ determines φ by the composite

$$\varphi\colon A \to A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \xrightarrow{\psi} \mathbb{Z}/4$$

which clearly maps to the subgroup $\mathbb{Z}/2$ of $\mathbb{Z}/4$. Compare also Lemma 8.2.7. □

We use the isomorphism in Proposition 6.10.8 as an identification.

(6.10.9) Proposition *For a finite abelian group A and $A^{\#} = \mathrm{Ext}(A, \mathbb{Z})$ there is a duality isomorphism of abelian groups*

$$\tau\colon G(A^{\#}) \cong \mathrm{Hom}(G(A), \mathbb{Z}/4)$$

for which the following diagram commutes.

$$\begin{tikzcd}
A^{\#} \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] \arrow[d, equal] & G(A^{\#}) \arrow[r, two heads, "\mu"] \arrow[d, equal, "\tau"] & A^{\#} * \mathbb{Z}/2 \arrow[d, equal] \\
\mathrm{Ext}(A,\mathbb{Z}/2) \arrow[r, tail, "\Delta"] & \mathrm{Hom}(G(A),\mathbb{Z}/4) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A,\mathbb{Z}/2).
\end{tikzcd}$$

The left- and right-hand side are the canonical isomorphisms; see also (6.11.4) below.

Proof We have by (6.10.1) and (6.10.7) the sequence of isomorphisms

$$\begin{tikzcd} G(A^{\#}) \arrow[d, dashed, "\tau"] & = & {[M(\mathbb{Z}/2,n), M(A^{\#},n)]} \arrow[d, "D", "\cong"'] \\ \mathrm{Hom}(G(A), Z/4) & = & {[M(A,m-1), M(\mathbb{Z}/2,m-1)].} \end{tikzcd}$$

Since D is compatible with Δ and μ by Proposition 6.10.5(2) we get the commutative diagram in the statements of the proposition. □

Let ${}^f\mathbf{Ab} \subset \mathbf{Ab}$ be the full subcategory of finite abelian groups. Then

(6.10.10) $$\#: {}^f\mathbf{Ab} \xrightarrow{\cong} {}^f\mathbf{Ab},\ A \mapsto A^{\#} = \mathrm{Ext}(A,\mathbb{Z})$$

is a contravariant isomorphism of categories with $(A^{\#})^{\#} = A$. On the other hand, we have the subcategory ${}^f\mathbf{M}^n \subset \mathbf{M}^n$ of Moore spaces of $M(A,n)$ with finite A. For this category we obtain by (6.10.1) the duality isomorphism

(6.10.11) $$D: {}^f\mathbf{M}^m \xrightarrow{\cong} {}^f\mathbf{M}^{n-1}$$

which carries $M(A,m)$ to $M(A^{\#},n-1)$; see (5.2.11). The isomorphism of categories $\mathbf{G} = \mathbf{M}^n$, $n \geq 3$, thus yields the composite

$$D: {}^f\mathbf{G} = {}^f\mathbf{M}^m \xrightarrow{\cong} {}^f\mathbf{M}^{n-1} = {}^f\mathbf{G}$$

where ${}^f\mathbf{G} \subset \mathbf{G}$ is the full category given by finite abelian groups A. This duality functor for ${}^f\mathbf{G}$ is computed in the next result.

(6.10.12) Theorem *The duality functor D: ${}^f\mathbf{G} \cong {}^f\mathbf{G}$ carries A to $A^{\#}$ and carries $(\varphi,\psi)\colon A \to B$ to the morphism $(\varphi^{\#}, \tau^{-1}\,\mathrm{Hom}(\psi,\mathbb{Z}/4)\tau)\colon B^{\#} \to A^{\#}$ where τ is given by Proposition* 6.10.9.

Proof Let $F\colon M(A,m) \to M(B,m)$ correspond to (φ,ψ) and let $(\varphi^{\#},\psi^{\#})$ correspond to $F^*\colon M(B^{\#},n-1) \to M(A^{\#},n-1)$. Then we compute $\psi^{\#}$: $G(B^{\#}) \to G(A^{\#})$ as follows. Let $\beta \in G(B^{\#}) = [M(\mathbb{Z}/2,n-1), M(B^{\#},n-1)]$. Then $\psi^{\#}(\beta) = F^* \circ \beta$. Now $F^*\beta = (\beta^*F)^*$ with $\beta^* = \tau(\beta)$ yields the result since $\beta^*F = \mathrm{Hom}(\psi,\mathbb{Z}/4)(\tau(\beta))$. □

6.11 Homotopy groups of Moore spaces in the stable range

We consider the homotopy groups and cohomotopy groups of Moore spaces

(6.11.1) $$\begin{aligned} \pi_n M(A,m) &= [S^n, M(A,m)], \\ \pi^m M(B,n-1) &= [M(B,n-1), S^m] \end{aligned}$$

in the stable range $m < n < 2m-1$. If A is finite abelian and $B = \operatorname{Ext}(A,\mathbb{Z}) = A^{\#}$ these homotopy groups are Spanier–Whitehead dual to each other, since we have the isomorphism

$$D\colon [S^n, M(A,m)] \cong [M(A^{\#}, n-1), S^m] \tag{1}$$

by Theorem 6.4.1. More generally we get the isomorphism

$$D\colon [X, M(A,m)] \cong [M(A^{\#}, n-1), X^*] \tag{2}$$

for $X \in \mathbf{A}_m^{n-m}$. Here the right-hand side is a homotopy group with coefficients in $A^{\#}$ which we described as a functor in Theorem 1.6.4. Hence we are able to obtain the dual result for the functor $M(A,n) \mapsto [X, M(A,n)]$. We first obtain the following dual of the universal coefficient sequence.

(6.11.2) Proposition *Let X be a CW-complex with* $\dim X < 2m-1$ *and let A be an abelian group. Moreover suppose that X is finite or that A is finitely generated. Then there is the binatural short exact sequence:*

$$A \otimes [X, S^m] \overset{\Delta}{\rightarrowtail} [X, M(A,m)] \overset{\mu}{\twoheadrightarrow} A * [X, S^{m+1}].$$

Here $A * B = \operatorname{Tor}_{\mathbb{Z}}(A, B)$ is the torsion product of abelian groups.

Proof Let $C \overset{d}{\rightarrowtail} D \twoheadrightarrow A$ be a short exact sequence where C and D are free abelian. Then the cofibre sequence

$$M(C,n) \xrightarrow{d} M(D,n) \to M(A,n) \to M(C,n+1) \to \cdots \tag{1}$$

induces for $\dim X < 2n-1$ the exact sequence

$$\begin{aligned} C \otimes [X, S^n] \xrightarrow{d\otimes 1} D \otimes [X, S^n] \to [X, M(A,n)] \\ \to C \otimes [X, S^{n+1}] \xrightarrow{d\otimes 1} D \otimes [X, S^{n+1}]. \end{aligned}$$

This sequence is obtained by applying the functor $[X, -]$ to the cofibre sequence (1). For this we use the fact that we have a natural isomorphism

$$[X, M(C,n)] = C \otimes [X, S^n] \tag{2}$$

in case C is free abelian. Here we need the assumption that X is finite or that A, and hence C, is finitely generated. $\square$

If A is a finite abelian group and $X \in \mathbf{A}_n^{m-n}$ we get by (6.11.1)(2) the commutative diagram of 'coefficient sequences':

$$\begin{array}{ccccc} A^{\#} \otimes \pi^n(X) & \overset{\Delta}{\rightarrowtail} & [X, M(A^{\#}, n)] & \overset{\mu}{\twoheadrightarrow} & A^{\#} * \pi^{n+1}(X) \\ \cong \downarrow D & & \cong \downarrow D & & \cong \downarrow D \\ \operatorname{Ext}(A, \pi_m X^*) & \overset{\Delta}{\rightarrowtail} & \pi_{m-1}(A, X^*) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(A, \pi_{m-1} X^*). \end{array} \tag{6.11.3}$$

Here the left-hand side and the right-hand side are given by the duality isomorphisms $\pi^n X \cong \pi_m X^*$, $\pi^{n+1}(X) \cong \pi_{m-1} X^*$ and by the binatural isomorphisms

$$(6.11.4) \qquad \begin{aligned} A^{\#} \otimes B &= \mathrm{Ext}(A, B), \\ A^{\#} * B &= \mathrm{Hom}(A, B) \end{aligned}$$

where A is finite, $A, B \in \mathbf{Ab}$.

Proof of (6.11.3) For a mapping cone C_f we obtain the dual $DC_f = C_g$ by a map g respresenting f^*. We can apply this to the mapping cone of d in Proposition 6.11.2(1). This now yields (6.11.3) since pinch map and inclusion behave as in Proposition 6.10.5(1), (2). □

Let $\mathbf{FM}^m$ be the subcategory of $\mathbf{M}^m$ consisting of Moore spaces $M(A, m)$ with A finitely generated. Then $\mathbf{FG}$ is the corresponding subcategory of $\mathbf{G}$. We consider the functor

$$(6.11.5) \qquad \pi^X \colon \mathbf{FG} = \mathbf{FM}^m \to \mathbf{Ab}$$

which carries A to $[X, M(A, m)]$. Here we assume that X is m-connected and $\dim X < 2m - 1$. The functor is the analogue of the functor $\pi_n(-, X)$ in (1.6.9). We can compute the functor π^X up to natural isomorphism in terms of the homomorphism

$$\eta\colon \pi^{m+1}(X) \otimes \mathbb{Z}/2 \to \pi^m(X) \tag{1}$$

induced by the Hopf map $\eta_m \in \pi_{m+1}(S^n)$, that is $\eta(\alpha \otimes 1) = \eta_m \alpha$ for $\alpha \in \pi^{m+1}(X) = [X, S^{m+1}]$.

(6.11.6) Definition We define for $A \in \mathbf{FAb}$ the abelian group $G(\eta, A)$ with $\eta\colon \pi \otimes \mathbb{Z}/2 \to \pi'$ by the push-out diagram:

$$\begin{tikzcd}
\mathrm{Ext}(\pi^{\#}, A \otimes \mathbb{Z}/2) \arrow[r, tail] \arrow[d, equal] & \mathbf{G}(\pi^{\#}, A) \arrow[r] \arrow[dd] & \mathrm{Hom}(\pi^{\#}, A) \arrow[dd, equal] \\
A \otimes \pi \otimes \mathbb{Z}/2 \arrow[d, "1 \otimes \eta"] \arrow[r, phantom, "\text{push}"] & & \\
A \otimes \pi' \arrow[r, tail, "\Delta"] & G(\eta, A) \arrow[r, "\mu"] & A * \pi.
\end{tikzcd}$$

Here we assume that π is a finite abelian group so that we get for $\pi^{\#} = \mathrm{Ext}(\pi, \mathbb{Z})$ the binatural identification in the diagram by (6.11.4). Moreover $\mathbf{G}(\pi^{\#}, A)$ is the group of morphisms $\pi^{\#} \to A$ in the category $\mathbf{G}$. The diagram is natural in $A \in \mathbf{FG}$ and hence yields a functor $G(\eta, -)\colon \mathbf{FG} \to \mathbf{Ab}$. The next result shows that this functor is naturally isomorphic to π^X in (6.11.5).

(6.11.7) Theorem *Let X be a finite CW-complex with* $\dim X < 2m-1$, $m \geq 3$, *and let $\eta\colon \pi^{m+1}(X) \otimes \mathbb{Z}/2 \to \pi^m(X)$ with $\eta(\alpha \otimes 1) = \eta_m \alpha$ be given by the Hopf map η_m. Moreover assume $\pi^{m+1}X$ is finite. Then one has for a finitely generated abelian group A an isomorphism*

$$[X, M(A,m)] \cong G(\eta, A)$$

*which is natural in $A \in$ **FG** and for which the following diagram commutes.*

$$\begin{tikzcd}
A \otimes \pi^m(X) \arrow[r, tail, "\Delta"] \arrow[d, equal] & {[X, M(A,m)]} \arrow[r, two heads, "\mu"] \arrow[d, "\cong"'] & A * \pi^{m+1}(X) \arrow[d, equal] \\
A \otimes \pi^m(X) \arrow[r, tail] & G(\eta, A) \arrow[r, two heads, "\mu"] & A * \pi^{m+1}(X).
\end{tikzcd}$$

The isomorphism is not natural in X. We point out that $\pi^{m+1}(X)$ in Definition 6.11.6 is automatically finite if X is $(m+1)$-connected. We can apply Theorem 6.11.7 in particular for the case that $X = S^n$ is a sphere. Thus we can compute $\pi_n M(A,m)$ for $n < 2m-1$ in terms of $\eta_*\colon \pi_n S^{m+1} \to \pi_n S^m$ only.

Proof of (6.11.7) Since we assume $\pi^{m+1}X$ to be finite the group $G(\eta, A)$ is well defined. If A is finite then the theorem is the Spanier–Whitehead dual of the result in Theorem 1.6.11. In fact in this case we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}(\pi^{\#}, A \otimes \mathbb{Z}/2) \arrow[r, tail] \arrow[d, equal] & \mathbf{G}(\pi^{\#}, A) \arrow[r] \arrow[dd, "\cong"', "D"] & \mathrm{Hom}(\pi^{\#}, A) \arrow[d, equal] \\
A \otimes \pi \otimes \mathbb{Z}/2 \arrow[d, equal] & & A * \pi \arrow[d, equal] \\
\mathrm{Ext}(A^{\#}, \pi \otimes \mathbb{Z}/2) \arrow[r, tail] & \mathbf{G}(A^{\#}, \pi) \arrow[r, two heads] & \mathrm{Hom}(A^{\#}, \pi)
\end{tikzcd}$$

where D is the duality isomorphism of Theorem 6.10.12. This diagram is natural in $A \in {}^f\mathbf{G}$. Now we can apply an argument as in the proof of Theorem 1.6.11. This proves the proposition for finite A; moreover it is easy to extend the isomorphism to the case of finitely generated A. □

(6.11.8) Corollary *Let X be a finite CW-complex with* $\dim X < 2m-1$, $m \geq 3$, *and $\pi^{m+1}X$ finite. Then the extension in Theorem* 6.11.7 *is split if and only if one of the following three conditions is satisfied:*

(*a*) *A has no direct summand $\mathbb{Z}/2$;*

(*b*) *$\pi^{m+1}X$ has no direct summand $\mathbb{Z}/2$;*

(*c*) *each element $\alpha \in \pi^{m+1}X$, generating a direct summand $\mathbb{Z}/2$, satisfies $\eta_m \alpha = 2\alpha'$ for some α'.*

Hence, if (a), (b), or (c) hold, one has an isomorphism of abelian groups (unnaturally)

$$[X, M(A,m)] \cong A * \pi^{m+1}(X) \oplus A \otimes \pi^m(X).$$

Proof If (a) or (b) is satisfied the top row in the diagram of Definition 6.11.6 is split; if (c) holds the bottom row in Definition 6.11.6 is still split, hence the corollary is a consequence of Theorem 6.11.7. □

6.12 Stable and principal maps between Moore spaces

We describe some properties of the stable homotopy groups ($m < n < 2m - 1$)

(6.12.1) $\quad \pi_n^{(m)}(A,B) = \pi(A,n,B,m) = [M(A,n), M(B,m)].$

We assume that A and B are finitely generated abelian groups. Using the universal coefficient sequences one has the following commutative diagram in which rows and columns are exact sequences and in which

(6.12.2)
$$\begin{aligned} \pi &= \pi_n S^m = \pi_{n+1} S^{m+1} \\ \pi_+ &= \pi_{n+1} S^m \\ \pi_- &= \pi_{n-1} S^m = \pi_n S^{m+1} \end{aligned}$$

are stable homotopy groups of spheres.

(6.12.3)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{Ext}(A, B \otimes \pi_+) \arrow[r, "\Delta_*"] \arrow[d, "\Delta_*"] & B \otimes \pi^m M(A,n) \arrow[r, "\mu_*"] \arrow[d, "\Delta"] & \mathrm{Hom}(A, B \otimes \pi) \arrow[r] \arrow[d, "\Delta_*"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(A, \pi_{n+1} M(B,m)) \arrow[r, "\Delta"] \arrow[d, "\mu_*"] & \pi(A,n,B,m) \arrow[r, "\mu"] \arrow[d, "\mu"] & \mathrm{Hom}(A, \pi_n M(B,m)) \arrow[r] \arrow[d, "\mu_*"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(A, B * \pi) \arrow[r, "\Delta_*"] \arrow[d] & B * \pi^{m+1} M(A,n) \arrow[r, "\mu_*"] \arrow[d] & \mathrm{Hom}(A, B * \pi_-) & \\
 & 0 & 0 & &
\end{tikzcd}$$

The column in the middle is obtained as a special case of Proposition 6.11.2.

Since A and B are finitely generated abelian groups we have natural isomorphisms:

$$(6.12.4)\qquad \begin{aligned} \operatorname{Ext}(A, B \otimes \pi_+) &= B \otimes \operatorname{Ext}(A, \pi_+) && (1)\\ \operatorname{Hom}(A, B \otimes \pi) &= B \otimes \operatorname{Hom}(A, \pi) && (2)\\ \operatorname{Ext}(A, B * \pi) &= B * \operatorname{Ext}(A, \pi) && (3)\\ \operatorname{Hom}(A, B * \pi_-) &= B * \operatorname{Hom}(A, \pi_-). && (4) \end{aligned}$$

Proof of (6.12.4) It is clear how to obtain these isomorphisms in the case that all groups involved are cyclic groups. In order to prove naturality we give the following definition of the isomorphisms in (6.12.4). Let $A_1 \rightarrowtail A_0 \twoheadrightarrow A$ and $B_1 \rightarrowtail B_0 \twoheadrightarrow B$ be short free resolutions of A and B respectively, where A_0, A, B_0, B_1 are finitely generated. For (2) let

$$\psi\colon B \otimes \operatorname{Hom}(A, \pi) \to \operatorname{Hom}(A, B \otimes \pi)$$

be defined by $\psi(b \otimes \varphi) = \varphi_b$ with $\varphi_b(a) = b \otimes \varphi(a)$. We obtain (1) be the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(A_1, B \otimes \pi_+) \arrow[r, phantom, "\overset{\psi}{=}"] \arrow[d] & B \otimes \operatorname{Hom}(A_1, \pi_+) \arrow[d] \\ \operatorname{Ext}(A, B \otimes \pi_+) & B \otimes \operatorname{Ext}(A, \pi_+). \arrow[l] \end{tikzcd}$$

Next we get (3) by the commutative diagram

$$\begin{tikzcd} B_1 \otimes \operatorname{Ext}(A, \pi) \arrow[r, phantom, "="] \arrow[d, phantom, "\cup"] & \operatorname{Ext}(A, B_1 \otimes \pi) \\ B * \operatorname{Ext}(A, \pi) & \operatorname{Ext}(A, B * \pi). \arrow[l] \arrow[u] \end{tikzcd}$$

Similarly we get (4) by the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(A, B_1 \otimes \pi_-) \arrow[r, phantom, "\overset{\psi}{=}"] & B_1 \otimes \operatorname{Hom}(A, \pi_-) \arrow[d, phantom, "\cup"] \\ \operatorname{Hom}(A, B * \pi_-) \arrow[u] \arrow[r] & B * \operatorname{Hom}(A, \pi_-). \end{tikzcd}$$ □

We use the isomorphisms in (6.12.4) as identifications. This way we get the

(6.12.5) Lemma *Diagram* (6.12.3) *commutes. In fact, the top row and the bottom row are induced by the* (Δ, μ)*-extension for* $\pi^m M(A, n)$ *and*

$\pi^{m+1}M(A,n)$ respectively. The left-hand column and the right-hand column are induced by the (Δ,μ)-extension for $\pi_{n+1}M(B,m)$ and $\pi_n M(B,m)$ respectively. Moreover diagram (6.12.3) *is natural in $M(A,n)\in \mathbf{M}^n$ and $M(B,m)\in \mathbf{M}^m$ respectively.*

(6.12.6) Definition Let π, π' be abelian groups and let $\eta\colon \pi\to\pi'$ be a homomorphism with $\eta(2\pi)=0$. We associate with η a homomorphism

$$\tau(\eta)\colon \mathrm{Hom}(A, B*\pi)\to \mathrm{Ext}(A, B\otimes\pi')$$

which is binatural for $A, B\in \mathbf{Ab}$. Let $\tau(\eta)$ be the composite

$$\begin{tikzcd}
\mathrm{Hom}(A, B*\pi) \arrow[r] \arrow[d, "\tau(\eta)"] & B*\mathrm{Hom}(A,\pi) \arrow[d, "\partial"] \\
\mathrm{Ext}(A, B\otimes\pi') & B\otimes \mathrm{Ext}(A,\pi'). \arrow[l]
\end{tikzcd}$$

Here the horizontal arrows are defined as in the proof of (6.12.4); they are isomorphisms if A and B are finitely generated. Moreover ∂ is the boundary in the six-term exact sequence of homological algebra induced by the extension

$$\mathrm{Ext}(A,\pi')\rightarrowtail G(A,\eta)\twoheadrightarrow \mathrm{Hom}(A,\pi)$$

in Definition 1.6.10.

Now consider again diagram (6.12.3). We want to describe the kernel of Δ_* in $\mathrm{Ext}(A, B\otimes\pi_+)$ and the image of μ_* in $\mathrm{Hom}(A, B*\pi_-)$. The Hopf maps induce homomorphisms, see (6.12.2),

$$\eta_-\colon \pi_-\to\pi,\ \eta_- = \eta_{n-1}^* = (\eta_m)_*$$
$$\eta_+\colon \pi\to\pi_+,\ \eta_+ = \eta_n^* = (\eta_m)_*$$

which satisfy the condition on η in Definition 6.12.6 so that $\tau(\eta_-)$ and $\tau(\eta_+)$ are defined. Now the exact sequences in diagram (6.12.3) show by Definition 6.11.6 and Theorem 1.6.11 respectively:

(6.12.7) Lemma *The kernel of Δ_* in* $\mathrm{Ext}(A, B\otimes\pi_+)$ *is the image of $\tau(\eta_+)$ and the image of μ_* in* $\mathrm{Hom}(A, B*\pi_-)$ *is the kernel of $\tau(\eta_-)$.*

In the lemma we again assume that A and B are finitely generated. The operator $\tau(\eta)$ has a natural interpretation as a Toda bracket. To this end we recall the classical definition of Toda brackets as follows.

(6.12.8) Definition Let

$$\begin{tikzcd}
W \arrow[r, "\gamma"] \arrow[rr, bend left=40, ""{name=U}] & X \arrow[r, "\beta"] \arrow[rr, bend right=40, ""{name=D}] & Y \arrow[r, "\alpha"] & Z
\arrow[from=1-2, to=U, phantom, "\Uparrow H"]
\arrow[from=1-3, to=D, phantom, "\Downarrow G"]
\end{tikzcd}$$

be a diagram in **Top*** where H: $\beta\gamma \simeq 0$ and G: $\alpha\beta \simeq 0$ are homotopies. Then the map

$$\tau_{H,G}: \Sigma W \to Z$$

is defined by the addition of homotopies, that is $\tau_{H,G} = -\alpha H + G(I \times \gamma)$ where $I = [0,1]$ is the unit interval. The *Toda bracket* $\langle \alpha, \beta, \gamma \rangle$ is the subset of $[\Sigma W, Z]$ consisting of all $\tau_{H,G}$ with H: $\alpha\beta \simeq 0$ and G: $\beta\gamma \simeq 0$. If the group $[\Sigma W, Z]$ is abelian the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is a coset of the subgroup

$$G = \alpha_*[\Sigma W, Y] + (\Sigma\gamma)^*[\Sigma X, Z] \subset [\Sigma W, Z]$$

or equivalently $\langle \alpha, \beta, \gamma \rangle$ is an element of the quotient group $[\Sigma W, Z]/G$. The element $\langle \alpha, \beta, \gamma \rangle$ depends only on the homotopy classes of α, β, and γ. The subgroup G is called the 'indeterminancy' of the Toda bracket $\langle \alpha, \beta, \gamma \rangle$.

We consider the following example which is associated with stable maps between Moore spaces.

(6.12.9) Example Let A and B be abelian groups and let

$$\begin{aligned} &A_1 \overset{d_A}{\rightarrowtail} A_0 \overset{q}{\twoheadrightarrow} A, \\ &B_1 \overset{d_B}{\rightarrowtail} B_0 \twoheadrightarrow B \end{aligned} \tag{1}$$

be short free resolutions. For an abelian group π we have the exact sequence

$$B * \pi \overset{i}{\rightarrowtail} B_1 \otimes \pi \to B_0 \otimes \pi \twoheadrightarrow B \otimes \pi \tag{2}$$

where the homomorphism in the middle is $d_B \otimes \pi$. We now consider maps between Moore spaces $(m < n < 2m + 1)$

$$M(A_1, n) \xrightarrow{d_A} M(A_0, n) \xrightarrow{\alpha} M(B_1, m) \xrightarrow{d_B} M(B_0, m)$$

where α is an element

$$\alpha \in \mathrm{Hom}(A_0, B_1 \otimes \pi) = [M(A_0, n), M(B_1, m)] \tag{3}$$

with $\pi = \pi_n S^m$. We have $\alpha d_A \simeq 0$ iff $d_A^* \alpha = 0$ in $\mathrm{Hom}(A_1, B_1 \otimes \pi)$ and we have $d_B \alpha \cong 0$ iff $(d_B \otimes \pi)_* \alpha = 0$ in $\mathrm{Hom}(A_0, B_0 \otimes \pi)$. This shows that the Toda bracket $\langle d_A, \alpha, d_B \rangle$ is defined if and only if there is an element $a \in \mathrm{Hom}(A, B * \pi)$ with

$$\alpha = iaq: A_0 \twoheadrightarrow A \xrightarrow{a} B * \pi \rightarrowtail B_1 \otimes \pi. \tag{4}$$

Moreover $\langle d_A, iaq, d_B\rangle$ is an element in the group

$$\mathrm{Ext}(A, B\otimes\pi') = [M(A_1, n+1), M(B_0 m)]/G \tag{5}$$

with $\pi' = \pi_{n+1}S^m$. Here the indeterminancy G is the sum of the images of

$$(d_B\otimes\pi')_*: \mathrm{Hom}(A_1, B_1\otimes\pi') \to \mathrm{Hom}(A_1, B_0\otimes\pi') \quad \text{and}$$

$$(d_A)^*: \mathrm{Hom}(A_0, B_0\otimes\pi') \to \mathrm{Hom}(A_1, B_0\otimes\pi')$$

so that the definition of Ext yields equation (5). Hence the Toda bracket $\tau(a) = \langle d_A, iaq, d_B\rangle$ defines the function

$$\tau: \mathrm{Hom}(A, B*\pi) \to \mathrm{Ext}(A, B\otimes\pi') \tag{6}$$

which is a homomorphism and natural in $A, B\in\mathbf{Ab}$.

(6.12.10) Proposition *Let*

$$\eta = \eta_n^*: \pi = \pi_n S^m \to \pi' = \pi_{n+1}S^m$$

be induced by the Hopf map η_n with $m < n < 2m-1$. Then the Toda bracket τ in Example 6.12.9 *coincides with the natural transformation $\tau(\eta)$ in Definition* 6.12.6. *Here we restrict τ and $\tau(\eta)$ to finitely generated abelain groups.*

The proposition yields a further interpretation of the operators $\tau(\eta_-)$ and $\tau(\eta_+)$ in Lemma 6.12.7. We shall not make use of the result in Proposition 6.12.10 so that we can omit here its somewhat elaborate proof. We do not know whether the restriction to finitely generated abelian groups is necessary.

Next we want to study 'principal maps' between Moore spaces. To this end we recall the definition of mapping cone and principal map; see Baues [AH]. For a pointed space U let CU be the reduced cone of U, that is CU is the quotient space $CU = I\times U/(I\times * \cup\{1\}\times U)$. The *mapping cone* C_g of a map $g: U\to V$ in $\mathbf{Top}^*$ is defined by the push-out diagram

$$\begin{tikzcd} CU \arrow[r, "\pi_g"] & C_g \\ U \arrow[u, "i_0"] \arrow[r, "g"] & V \arrow[u, "i_g"'] \end{tikzcd}$$

with $i_0(x) = (0, x)$ for $x\in U$.

(6.12.11) Definition Consider a diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{H}{\Leftarrow}"] & U \arrow[d, "g"] \\ Y \arrow[r, "v"] & V \end{tikzcd} \tag{1}$$

in the category **Top*** of pointed spaces where H: $vf \simeq gu$ is a homotopy. Then the triple (u, v, H) yields the map between mapping cones,

$$C(u, v, H): C_f \to C_g. \tag{2}$$

This map carries $y \in Y$ to $i_g v(y)$ and carries $(t, x) \in CX$, $0 \leq t \leq \frac{1}{2}$, to $i_g H(2t, x)$. Moreover (2) carries $(t, x) \in CX$ with $\frac{1}{2} \leq t \leq 1$ to $\pi_g(2t-1, u(x))$. We call this map and any map homotopic to (2) a *principal map*. Let

$$\mathrm{PRIN}(f, g) \subset [C_f, C_g] \tag{3}$$

be the subset of all homotopy classes represented by principal maps. The properties of principal maps are studied in Baues [AH].

Again let $A_1 \rightarrowtail{} \xrightarrow{d_A} A_0 \twoheadrightarrow A$ and $B_1 \rightarrowtail{} \xrightarrow{d_B} B_0 \twoheadrightarrow B$ be short free resolutions of A and B respectively. Then the Moore spaces $M(A, n)$ and $M(B, m)$ are mapping cones of the maps

$$d_A: M(A_1, n) \to M(A_0, n) \quad \text{and} \quad d_B: M(B_1, m) \to M(B_0, m)$$

respectively. Hence we get by Definition 6.12.11 the subset

(6.12.12) $\qquad \mathrm{PRIN}(d_A, d_B) \subset [M(A, n), M(B, m)]$

of principal maps between Moore spaces. This subset is a subgroup which is natural for maps a: $M(A, n) \to M(A', n)$ and b: $M(B, m) \to M(B', m)$ since such maps are always principal and since the composition of principal maps is principal. We consider d_A as a chain complex concentrated in degree 0 and 1. Similary

$$d_B \otimes \pi: B_1 \otimes \pi \to B_0 \otimes \pi$$

is a chain complex concentrated in degree 0 and 1. Hence we have the abelian group of homotopy classes of chain maps $[d_A, d_B \otimes \pi]$ which in an obvious way yields a bifunctor in $A, B \in$ **Ab**.

(6.12.13) Lemma *For all abelian groups A, B, π one has an isomorphism of abelian groups*

$$[d_A, d_B \otimes \pi] = \mathrm{Ext}(A, B * \pi) \oplus \mathrm{Hom}(A, B \otimes \pi).$$

This isomorphism is natural in A, B, π provided A or B are finitely generated or π is a field.

Proof The classification of chain maps yields the natural short exact sequence

$$\mathrm{Ext}(A, B * \pi) \rightarrowtail [d_A, d_B \otimes \pi] \twoheadrightarrow \mathrm{Hom}(A, B \otimes \pi)$$

which is split as a sequence of abelian groups. For finitely generated A or B we obtain a natural splitting by the commutative diagram (see (6.12.4))

$$\begin{tikzcd}
{[d_A, \pi] \otimes [\mathbb{Z}, d_B]} \arrow[r, phantom, "\cong"] \arrow[d, "s"] & \mathrm{Hom}(A, \pi) \otimes B \arrow[d, "\cong"'] \\
{[d_A, d_B \otimes \pi]} \arrow[r] & \mathrm{Hom}(A, B \otimes \pi).
\end{tikzcd}$$

Here π and $\mathbb{Z}$ denote chain complexes concentrated in degree 0. The function s carries $\{\xi\} \otimes \{a\}$ to the homotopy class of the composite of chain maps

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & B_1 \otimes \pi \arrow[d] \\
A_0 \arrow[r, "\xi"'] & \pi = \mathbb{Z} \otimes \pi \arrow[r, "a \otimes \pi"'] & B_0 \otimes \pi .
\end{tikzcd}$$

If π is a field we consider the following diagram where $\mathbb{Z}[B]$ is the free abelian group generated by B. Moreover d_B is obtained by $B_1 \rightarrowtail B_0 = \mathbb{Z}[B] \overset{p}{\twoheadrightarrow} B$.

$$\begin{tikzcd}
K(A, B, \pi) \arrow[r, phantom, "\subset"] \arrow[d, "\theta"] \arrow[dr, phantom, "\text{push}"] & {[d_A, \mathbb{Z}[B] \otimes \pi]} \arrow[r, phantom, "="] \arrow[d] & \mathrm{Hom}(A, \mathbb{Z}[B] \otimes \pi) \arrow[d, "(p \otimes \pi)_*"] \\
\mathrm{Ext}(A, B * \pi) \arrow[r, phantom, "\subset"] & {[d_A, d_B \otimes \pi]} \arrow[r] & \mathrm{Hom}(A, B \otimes \pi).
\end{tikzcd}$$

Here $K(A, B, \pi) = \mathrm{kernel}(p \otimes \pi)_*$ is a functor in A, B, π. Since π is a field $p \otimes \pi$ is split surjective and therefore the subdiagram push is a push-out diagram. One can now use naturality of θ to show that θ is trivial. Hence by naturality of the diagram we obtain a natural splitting which, if A or B are finitely generated, coincides with s above. □

(6.12.14) Theorem *Let A and B be finitely generated abelian groups and let η_- and η_+ be given as in Lemma 6.12.7 with $m < n < 2m - 1$. Then one has the following two exact sequences which are natural in A and $B \in \mathbf{G}$.*

$$\mathrm{PRIN}(d_A, d_B) \rightarrowtail \pi(A, n, B, m) \xrightarrow{\bar{\mu}} \mathrm{Hom}(A, B * \pi_-) \xrightarrow{\tau(\eta_-)} \mathrm{Ext}(A, B \otimes \pi)$$

$$\mathrm{Hom}(A, B * \pi) \xrightarrow{\tau(\eta_+)} \mathrm{Ext}(A, B \otimes \pi_+) \xrightarrow{\bar{\Delta}} \mathrm{PRIN}(d_A, d_B) \overset{\lambda}{\twoheadrightarrow} [d_A, d_B \otimes \pi].$$

Here $\bar{\mu} = \mu_* \mu$ and $\bar{\Delta} = \Delta_* \Delta$ are given by the operators in (6.12.3). Moreover λ carries the principal map $C(u, v, H)$ to the chain map $d_A \to d_B \otimes \pi$ given by u and v. The homotopy class of this chain map is well defined by the homotopy class of $C(u, v, H)$. The results in chapter V of Baues [AH] yield a proof of Theorem 6.12.14.

(6.12.15) Theorem *Let $m < n < 2m - 2$ and let A and B be direct sums of cyclic groups with $A * \mathbb{Z}/2 = 0$ and $B * \mathbb{Z}/2 = 0$. Then one has an isomorphism*

$$\theta \colon [M(A,n), M(B,m)] \cong \mathrm{Ext}(A, B \otimes \pi_+) \oplus \mathrm{Ext}(A, B * \pi)$$
$$\oplus \mathrm{Hom}(A, B \otimes \pi) \oplus \mathrm{Hom}(A, B * \pi_-)$$

which is natural in A and B. Here $\pi_+ = \pi_{n+1} S^m$, $\pi = \pi_n S^m$, and $\pi_- = \pi_{n-1} S^m$ are stable homotopy groups of spheres as in (6.12.2). Equivalently all rows and columns in (6.12.3) are short exact and naturally split provided A and B satisfy the assumptions above.

As usual we write $A = (\mathbb{Z}/k)a$ if A is a cyclic group of order k with generator $a \in A$. An element $\alpha \in \pi$ in a finite abelian group π generates the subgroup $(\mathbb{Z}/|\alpha|)\alpha$ where $|\alpha|$ is the order of α. A subset $E \subset \pi$ is a *basis* if $|\alpha|$ is a prime power for $\alpha \in E$ and

$$\bigoplus_{\alpha \in E} (\mathbb{Z}/|\alpha|)\alpha \to \pi$$

is an isomorphism. For the proof of Theorem 6.12.15 we introduce the following elements.

(6.12.16) Notation Let $E_r \subset \sigma_r$ be a basis of the stable group $\sigma_r = \pi_n S^m$, $n = m + r < 2n - 1$. Moreover let $\alpha \in E_r$ be an element of order $|\alpha| = p^{e(\alpha)}$ where p is a prime. We choose for α stable maps $\varepsilon(\alpha)$, $\xi(\alpha)$, $\eta(\alpha)$, and $\rho(\alpha)$ between elementary Moore spaces with the properties below, where $A = \mathbb{Z}/p^{e(\alpha)}$ and where i is the inclusion and q is the pinch map.

$$\begin{aligned}
\varepsilon_0^0(\alpha) &= \alpha \colon S^n \to S^m \\
\varepsilon^0(\alpha) &= i\alpha \colon S^n \to S^m \to M(A,m) \\
\varepsilon_0(\alpha) &= \alpha q \colon M(A, n-1) \to S^n \to S^m \\
\varepsilon(\alpha) &= i\alpha q \colon M(A, n-1) \to S^n \to S^m \to M(A,m).
\end{aligned} \tag{1}$$

We choose a map $\xi^0(\alpha)\colon S^{n+1} \to M(A,m)$ with $q_* \xi^0(\alpha) = \Sigma\alpha$ and let

$$\xi(\alpha) = \xi^0(\alpha) q \colon M(A,n) \to S^{n+1} \to M(A,m). \tag{2}$$

Let $\eta_0(\alpha)$: $M(A,n) \to S^m$ with $i^*\eta_0(\alpha) = \alpha$ be the Spanier–Whitehead dual of $\xi^0(\alpha)$ and let

$$\eta(\alpha) = i\eta_0(\alpha)\colon M(A,n) \to S^m \to M(A,m). \tag{3}$$

Moreover we denote by $\rho(\alpha)$ a map

$$\rho(\alpha)\colon M(A,n+1) \to M(A,m) \tag{4}$$

satisfying $i^*\rho(\alpha) = \xi^0(\alpha)$ and $q_*\,\rho(\alpha) = \Sigma\eta_0(\alpha)$. The element $\rho(\alpha)$ exists if and only if for $A = B = (\mathbb{Z}/p^{e(\alpha)})\alpha \subset \sigma_r$ we have

$$A = \mathrm{Hom}(A, A * A) \subset \mathrm{image}\,\overline{\mu} \subset \mathrm{Hom}(A, A * \sigma_r),$$

see Theorem 6.12.14. In particular, if $|\alpha| \neq 2$ the element $\rho(\alpha)$ exists. Moreover for $|\alpha| \neq 2$ we choose the elements (1)–(4) to be elements of order $|\alpha|$; for $|\alpha| = 2$ we choose such elements with minimal order. Now let B, B' be $\mathbb{Z}$ or cyclic groups of prime power order and let $\chi \in \mathrm{Hom}(B, A)$ and $\chi' \in \mathrm{Hom}(A, B')$ be the canonical generators which yield maps between Moore spaces, also denoted by χ and χ' respectively, see Theorem 1.4.4.

(6.12.17) Lemma *Appropriate compositions of χ, χ' and elements* (1)–(4) *above generate the abelian group $[M(B,d), M(B',d')]$ for $d < 2d' - 2$.*

(6.12.18) Lemma *If $|\alpha|$ is odd the element $\rho(\alpha)$ is Spanier–Whitehead self-dual, that is $D\rho(\alpha) = \rho(\alpha)$.*

Proof Since $\xi^0(\alpha)$ is the dual of $\eta_0(\alpha)$ we see that $\rho(\alpha) - D\rho(\alpha)$ is in the image of $\overline{\Delta} = \Delta\Delta_*$ in Theorem 6.12.14, that is, there is $\overline{\alpha} \in \mathrm{Ext}(A, A \otimes \pi_+)$ with $A = \mathbb{Z}/|\alpha|$ such that $\rho(\alpha) - D\rho(\alpha) = \overline{\Delta}(\overline{\alpha})$. This implies

$$D\Delta(\overline{\alpha}) = D(\rho(\alpha) - D\rho(\alpha)) = D\rho(\alpha) - \rho(\alpha) = -\overline{\Delta}(\overline{\alpha}).$$

On the other hand, $\overline{\Delta}(\overline{\alpha}) = i\overline{\alpha}q$:

$$M(A,n+1) \xrightarrow{q} S^{n+2} \xrightarrow{\overline{\alpha}} S^m \xrightarrow{i} M(A,m)$$

is self-fual, that is $D(i\overline{\alpha}q) = i\overline{\alpha}q$, so that we get $2\overline{\Delta}(\overline{\alpha}) = 0$. If $|\alpha|$ is odd this implies $\overline{\alpha} = 0$. □

Proof of Theorem 6.12.15 We have a basis $E_{r+1} \subset \pi_+$, $E_r \subset \pi$, resp. $E_{r-1} \subset \pi_-$ in the groups of the theorem. Then there is a unique natural isomorphism θ in the theorem which carries (for $A, B \in \{\mathbb{Z}, \mathbb{Z}/p^{e(\alpha)}\}$ and $\alpha \in E_{r+1}, E_r, E_{r-1}$) the elements chosen in (6.12.16) to the corresponding basis elements of the right-hand side in 6.12.15 given by α. Hence the isomorphism θ is determined by the choice of elements in (6.12.16). □

6.13 Quadratic $\mathbb{Z}$-modules

We here introduce the 'quadratic algebra' which is needed for the metastable range of homotopy theory; for a more extensive treatment see Baues [QF]. A *ringoid* **R** is a category for which all morphism sets are abelian groups and for which composition is bilinear (a ringoid is also called a 'pre-additive category' or an '**Ab**-category'). A *biproduct* in a ringoid is a diagram (see Mac Lane [C])

$$(6.13.1) \qquad X \underset{r_1}{\overset{i_1}{\rightleftarrows}} X \vee Y \underset{i_2}{\overset{r_2}{\rightleftarrows}} Y$$

which satisfies $r_1 i_1 = 1$, $r_2 i_2 = 1$, and $i_1 r_1 + i_2 r_2 = 1$. Both sums and products in a ringoid are canonically equipped with the structure of a biproduct. An *additive category* is a ringoid in which biproducts exist. Clearly the category **Ab** of abelian groups is an additive category with biproducts given by direct sums $A \oplus B$ of abelian groups. A functor F: $\mathbf{B} \to \mathbf{S}$ between ringoids is *additive* if

$$(6.13.2) \qquad F(f+g) = F(f) + F(g)$$

for morphisms $f, g \in \mathbf{R}(X, Y)$. Moreover we say that F is *quadratic* if the function Δ defined by

$$(6.13.3) \qquad \Delta(f,g) = F(f+g) - F(f) - F(g)$$

is bilinear. It is clear that an additive functor carries a biproduct to a biproduct.

Let $\mathbf{Add}(\mathbb{Z})$ be the full subcategory in **Ab** consisting of finitely generated free abelian groups. Additive functors F: $\mathbf{Add}(\mathbb{Z}) \to \mathbf{Ab}$ are in 1–1 correspondence with abelian groups; the correspondence is given by $F \mapsto F(\mathbb{Z})$. In fact one readily obtains the following equivalence of categories.

(6.13.4) Lemma *The category of additive functors* $\mathbf{Add}(\mathbb{Z}) \to \mathbf{Ab}$ *with natural transformations as morphisms is equivalent to the category* **Ab**. *The equivalence carries F to $F(\mathbb{Z})$ and the inverse of the equivalence carries an abelian group A to the functor* $\otimes A$: $\mathbf{Add}(\mathbb{Z}) \to \mathbf{Ab}$, $B \mapsto B \otimes A$, *given by the tensor product of abelian groups.*

We now introduce 'quadratic $\mathbb{Z}$-modules' which are in 1–1 correspondence with quadratic functors $\mathbf{Add}(\mathbb{Z}) \to \mathbf{Ab}$. In this sense a quadratic $\mathbb{Z}$-module is just the 'quadratic analogue' of an abelian group. Moreover quadratic $\mathbb{Z}$-modules allow precisely the quadratic generalization of Lemma (6.13.4) above; see Theorem 6.13.12.

(6.13.5) Definition A *quadratic $\mathbb{Z}$-module*

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

is a pair of abelian groups M_e, M_{ee} together with homomorphisms H, P which satisfy the equations

$$PHP = 2P \quad \text{and} \quad HPH = 2H.$$

A morphism $f\colon M \to N$ between quadratic $\mathbb{Z}$-modules is a pair of homomorphisms $f_e\colon M_e \to N_e$, $f_{ee}\colon M_{ee} \to N_{ee}$ which commute with H and P respectively. Let $\mathbf{QM}(\mathbb{Z})$ be the category of quadratic $\mathbb{Z}$-modules. This is an abelian category.

For a quadratic $\mathbb{Z}$-module M we define the *involution*

$$T = HP - 1\colon M_{ee} \to M_{ee}. \tag{6.13.6}$$

Then the equations for H and P in Definition 6.13.5 are equivalent to $PT = P$ and $TH = H$. We get $TT = 1$ since $1 + T = HP = HPT = T + T^2$. We define for $n \in \mathbb{Z}$ the function

$$\begin{aligned} &n_*\colon M_e \to M_e \\ &n_*(x) = nx + (n(n-1)/2)PH(x), \quad x \in M_e. \end{aligned} \tag{6.13.7}$$

One can check that $(n \cdot m)_* = n_* m_*$ and $(n+m)_* = n_* + m_* + nmPH$. Let $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$ by the cyclic group of order $n \geq 1$. We call M a *quadratic* $(\mathbb{Z}/n)$*-module* if $n \cdot M_{ee} = 0$ and $n_* M_e = 0$. Let $\mathbf{QM}(\mathbb{Z}/n) \subset \mathbf{QM}(\mathbb{Z})$ be the full subcategory of quadratic $(\mathbb{Z}/n)$-modules.

We identify a quadratic $\mathbb{Z}$-module M satisfying $M_{ee} = 0$ with the abelian group M_e. This yields the full inclusion of categories

$$\mathbf{Ab} \subset \mathbf{QM}(\mathbb{Z}) \tag{6.13.8}$$

which carries an abelian group A to the quadratic $\mathbb{Z}$-module $(A \to 0 \to A)$ which we also denote by A. Moreover we have the following canonical functors on $\mathbf{QM}(\mathbb{Z})$.

(6.13.9) Definition There is a *duality* functor $D\colon \mathbf{Qm}(\mathbb{Z}) \to \mathbf{QM}(\mathbb{Z})$ with $D(M)$ given by the interchange of the roles of H and P respectively, that is

$$D(M) = ((DM)_e \xrightarrow{H^D} (DM)_{ee} \xrightarrow{P^D} (DM)_e) \tag{1}$$

with $(DM)_e = M_{ee}$, $(DM)_{ee} = M_e$, $H^D = P$, and $P^D = H$. Clearly $DD(M) = M$. Moreover an additive functor A: $\mathbf{Ab} \to \mathbf{Ab}$ induces a functor A: $\mathbf{QM}(\mathbb{Z}) \to \mathbf{QM}(\mathbb{Z})$. Here we define $A(M)_{ee} = A(M_{ee})$ and $A(M)_e = A(M_e)$ with H and P given by $A(H)$ and $A(P)$ respectively. For example the functor $\otimes C$: $\mathbf{Ab} \to \mathbf{Ab}$, $C \in \mathbf{Ab}$, carries M to

$$M \otimes C = (M_e \otimes C \xrightarrow{H \otimes 1} M_{ee} \otimes C \xrightarrow{P \otimes 1} M_e \otimes C). \tag{2}$$

This tensor product should not be confused with the quadratic tensor product $C \otimes M = C \otimes_{\mathbb{Z}} M$ defined in Definition 6.13.13 below.

The following construction yields many examples of quadratic $\mathbb{Z}$-modules.

(6.13.10) Definition Let F: $\mathbf{R} \to \mathbf{Ab}$ be a quadratic functor and let $X \vee Y$ be a biproduct in $\mathbf{R}$. The *quadratic cross-effect* $F(X \mid Y)$ is defined by the image group, see (6.13.3),

$$F(X \mid Y) = \mathrm{image}\{\Delta(i_1 r_1, i_2 r_2): F(X \vee Y) \to F(X \vee Y)\}. \tag{1}$$

If $\mathbf{R}$ is an additive category this yields the biadditive functor

$$F(\,|\,): \mathbf{R} \times \mathbf{R} \to \mathbf{Ab}. \tag{2}$$

Moreover we have the isomorphism

$$\psi: F(X) \oplus F(Y) \oplus F(X \mid Y) \cong F(X \vee Y) \tag{3}$$

which is given by $F(i_1), F(i_2)$ and the *inclusion* i_{12}: $F(X \mid Y) \subset F(X \vee Y)$. Let r_{12} be the *retraction* of i_{12} obtained by ψ^{-1} and the projection to $F(X \mid Y)$. For the biproduct $X \vee X$ one has maps $\mu = i_1 + i_2$: $X \to X \vee X$ and $\nabla = r_1 + r_2$: $X \vee X \to X$. They yield homomorphism H and P with

$$F\{X\} = (F(X) \xrightarrow{H} F(X \mid X) \xrightarrow{P} F(X)) \tag{4}$$

by $H = r_{12} F(\mu)$ and $P = F(\nabla) i_{12}$. Now we derive from $f + g = \nabla(f \vee g)\mu$ the formula

$$F(f+g) = F(f) + F(g) + PF(f \mid g)H \tag{5}$$

or equivalently $\Delta(f, g) = PF(f \mid g)H$.

(6.13.11) Proposition *Let F: $\mathbf{R} \to \mathbf{Ab}$ be a quadratic functor where $\mathbf{R}$ is an additive category. Then $F\{X\}$, $X \in \mathbf{R}$, is a well-defined quadratic $\mathbb{Z}$-module and $X \mapsto F\{X\}$ defines a functor $\mathbf{R} \to \mathbf{QM}(\mathbb{Z})$.*

Proof We define the interchange map

$$t = i_2 r_1 + i_1 r_2: X \vee X \to X \vee X.$$

Then $t\mu = \mu$ and $\nabla t = \nabla$. Moreover t induces an isomorphism

$$T: F(X \mid X) \to F(X \mid X)$$

with $F(t) i_{12} = i_{12} T$ and $r_{12} F(t) = T r_{12}$. Hence we get $TH = H$ and $PT = P$.

Moreover we obtain $HP = 1 + T$ be applying F to the commutative diagram in **R**

$$\begin{array}{ccccc} X \vee X & \xrightarrow{\nabla} & X & \xrightarrow{\mu} & X \vee X \\ \Big\downarrow{\scriptstyle \mu \vee \mu} & & & & \Big\downarrow{\scriptstyle \nabla \vee \nabla} \\ X \vee X \vee X \vee X & & \xrightarrow[1 \vee T \vee 1]{} & & X \vee X \vee X \vee X. \end{array}$$

Here we use the biadditivity of $F(\,|\,)$. □

The significance of quadratic $\mathbb{Z}$-modules is now described by the following quadratic generalization of Lemma 6.13.4; see Baues [QF].

(6.13.12) Theorem *The category of quadratic functors* **Add**($\mathbb{Z}$) → **Ab** *with natural transformations as morphisms is equivalent to the category* **QM**($\mathbb{Z}$) *of quadratic $\mathbb{Z}$-modules. The equivalence carries F to $F\{\mathbb{Z}\}$ and the inverse of the equivalence carries the quadratic $\mathbb{Z}$-module M to the functor $\otimes_{\mathbb{Z}} M$:* **Add**($\mathbb{Z}$) → **Ab**, $A \mapsto A \otimes_{\mathbb{Z}} M$, *given by the quadratic tensor product below.*

Hence we have a 1–1 correspondence between quadratic functors F: **Add**($\mathbb{Z}$) → **Ab** and quadratic $\mathbb{Z}$-modules. In particular any quadratic functor F: **Add**($\mathbb{Z}$) → **Ab** is completely determined (up to isomorphism) by the fairly simple algebraic data of a quadratic $\mathbb{Z}$-module $M = F\{\mathbb{Z}\}$. Theorem 6.13.12 is one of the reasons to study the following quadratic tensor product and the corresponding quadratic Hom functor.

(6.13.13) Definition Let A be an abelian group and let M be a quadratic $\mathbb{Z}$-module. Then the *quadratic tensor product* $A \otimes_{\mathbb{Z}} M$ is the abelian group generated by the symbols $a \otimes m$, $[a,b] \otimes n$ with $a, b \in A$, $m \in M_e$, $n \in M_{ee}$. The relations are:

$(a+b) \otimes m = a \otimes m + b \otimes m + [a,b] \otimes H(m)$;

$[a,a] \otimes n = a \otimes P(n)$;

$a \otimes m$ is linear in m;

$[a,b] \otimes n$ is linear in a, b and n respectively.

These relations imply

$$[a,b] \otimes n = [b,a] \otimes T(n) \qquad (*)$$

where $T = HP - 1$ is the involution on M_{ee}. The quadratic tensor product is a functor

$$\otimes_{\mathbb{Z}} : \mathbf{Ab} \times \mathbf{QM}(\mathbb{Z}) \to \mathbf{Ab}.$$

Induced functions

$$f \otimes g : A \otimes_{\mathbb{Z}} M \to A' \otimes_{\mathbb{Z}} M'$$

are defined by $(f \otimes g)(a \otimes m) = (fa) \otimes (g_e m)$ and $(f \otimes g)([a,b] \otimes n) = [fa, fb] \otimes (g_{ee} n)$. We point out that the quadratic tensor product is compatible with direct limits in **Ab** and **QM**($\mathbb{Z}$) respectively. We also write $A \otimes M = A \otimes_{\mathbb{Z}} M$.

Proof of (∗)

$$\begin{aligned}
[b,a] \otimes T(n) &= [b,a] \otimes HP(n) - [b,a] \otimes n \\
&= (b+a) \otimes P(n) - b \otimes P(n) - a \otimes P(n) - [b,a] \otimes n \\
&= [b+a, b+a] \otimes n - [b,b] \otimes n - [a,a] \otimes n - [b,a] \otimes n \\
&= [a,b] \otimes n.
\end{aligned}$$

□

(6.13.14) Definition Again let A be an abelian group and let M be a quadratic $\mathbb{Z}$-module. A *quadratic form* $\alpha = (\alpha_e, \alpha_{ee})\colon A \to M$ is a pair of functions

$$\alpha_e\colon A \to M_e, \qquad \alpha_{ee}\colon A \times A \to M_{ee}$$

with the following properties ($a, b \in A$):

$$\begin{aligned}
&\alpha_e(a+b) = \alpha_e(a) + \alpha_e(b) + P\alpha_{ee}(a,b) \\
&\alpha_{ee}(a,a) = H\alpha_e(a) \\
&\alpha_{ee} \text{ is } \mathbb{Z}\text{-bilinear.}
\end{aligned}$$

These properties imply

$$\alpha_{ee}(a,b) = T\alpha_{ee}(b,a) \tag{$*$}$$

where $T = HP - 1$ is the involution on M_{ee}. Let $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ be the set of all quadratic forms $A \to M$. This is an abelian group by

$$(\alpha_e, \alpha_{ee}) + (\beta_e, \beta_{ee}) = (\alpha_e + \beta_e, \alpha_{ee} + \beta_{ee}).$$

Hence we obtain the *quadratic* Hom *functor*

$$\mathrm{Hom}_{\mathbb{Z}}\colon \mathbf{Ab}^{\mathrm{op}} \times \mathbf{QM}(\mathbb{Z}) \to \mathbf{Ab}.$$

Induced functions are given by the formula $\mathrm{Hom}(f, g)(\alpha) = \beta$ with $\beta_e = g_e \alpha_e f$ and $\beta_{ee} = g_{ee} \alpha_{ee}(f \times f)$.

Proof of (∗)

$$\begin{aligned}
T\alpha_{ee}(b,a) &= HP\alpha_{ee}(b,a) - \alpha_{ee}(b,a) \\
&= H(\alpha_e(b+a) - \alpha_e(b) - \alpha_e(\alpha)) - \alpha_{ee}(b,a) \\
&= \alpha_{ee}(b+a, b+a) - \alpha_{ee}(b,b) - \alpha_{ee}(a,a) - \alpha_{ee}(b,a) \\
&= \alpha_{ee}(a,b).
\end{aligned}$$

□

Restricted to the subcategory $\mathbf{Ab} \subset \mathbf{QM}(\mathbb{Z})$ the quadratic tensor product and the quadratic Hom functor coincides with the classical (linear) tensor product and Hom functor respectively. The next lemma is well known in the linear case.

(6.13.15) Lemma *Let C be a finitely generated free abelian group. Then one has for ${}^{\#}C = \mathrm{Hom}(C, \mathbb{Z})$ the isomorphism*

$$\chi : ({}^{\#}C) \otimes_{\mathbb{Z}} M \cong \mathrm{Hom}_{\mathbb{Z}}(C, M)$$

which is natural in $C \in \mathbf{Add}(\mathbb{Z})$ and $M \in \mathbf{QM}(\mathbb{Z})$.

Proof We define χ as follows. For $a, b \in {}^{\#}C$ let $\chi(a \otimes m) = \alpha = (\alpha_e, \alpha_{ee})$ be given as follows $(x, y \in C)$

$$\alpha_e(x) = a(x)m + (a(x)(a(x) - 1)/2)PH(m),$$

$$\alpha_{ee}(x, y) = a(x)a(y)H(m).$$

Moreover let $\chi([a, b] \otimes n) = \beta = (\beta_e, \beta_{ee})$ be defined by

$$\beta_e(x) = a(x)b(x)P(n),$$

$$\beta_{ee}(x, y) = a(x)b(y)n + a(y)b(x)Tn. \qquad \square$$

The lemma shows the next result which is an addendum to Theorem 6.13.12.

(6.13.16) Proposition *For each quadratic functor F: $\mathbf{Add}(\mathbb{Z}) \to \mathbf{Ab}$ one has a canonical natural isomorphism*

$$F(C) = C \otimes_{\mathbb{Z}} M, \quad C \in \mathbf{Add}(\mathbb{Z}).$$

Here $M = F\{\mathbb{Z}\}$ is a quadratic $\mathbb{Z}$-module. For each quadratic contravariant functor F: $\mathbf{Add}(\mathbb{Z}) \to \mathbf{Ab}$ one has a canonical natural isomorphism

$$F(C) = \mathrm{Hom}_{\mathbb{Z}}(C, M), \quad C \in \mathbf{Add}(\mathbb{Z}),$$

with $M = F^{\#}\{\mathbb{Z}\}$. Here $F^{\#}$ is the covariant functor defined by $F^{\#}(C) = F({}^{\#}C)$ with ${}^{\#}C = \mathrm{Hom}(C, \mathbb{Z})$.

The ring $\mathbb{Z}$ of integers in the discussion above can be replaced by any ring R or even by a small ringoid $\mathbf{R}$; this is done in Baues [QF]. For example, we get for the ring $R = \mathbb{Z}/n$ the following result corresponding to Theorem 6.13.12. Let $\mathbf{Add}(\mathbb{Z}/n)$ be the full subcategory of $\mathbf{Ab}$ consisting of finitely generated free $\mathbb{Z}/n$-modules.

(6.13.17) Theorem *The category of quadratic functors* **Add**$(\mathbb{Z}/n) \to$ **Ab** *with natural transformation as morphisms is equivalent to the category* **QM**$(\mathbb{Z}/n)$ *of quadratic* $\mathbb{Z}/n$*-modules. The equivalence carries* F *to* $F\{\mathbb{Z}/n\}$ *and the inverse of the equivalence carries the quadratic* $\mathbb{Z}/n$*-module* M *to the functor* $\otimes_{\mathbb{Z}} M$: **Add**$(\mathbb{Z}/n) \to$ **Ab**, $A \mapsto A \otimes_{\mathbb{Z}} M$ *given by the quadratic tensor product.*

We observe that the quadratic tensor product $A \otimes_{\mathbb{Z}} M$ and the quadratic $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ are both additive in M and quadratic in A. The quadratic cross-effects are given as follows. We obtain the inclusion

(6.13.18) $$A \otimes B \otimes M_{ee} = (A \mid B) \otimes_{\mathbb{Z}} M \overset{i_{11}}{\rightarrowtail} (A \oplus B) \otimes_{\mathbb{Z}} M$$

by $i_{12}(a \otimes b \otimes m) = [i_1 a, i_2 b] \otimes m$. Moreover we get the induced maps

$$A \otimes B \otimes M_{ee} \xrightarrow{T} B \otimes A \otimes M_{ee}, \tag{1}$$

$$A \otimes_{\mathbb{Z}} M \xrightarrow{H} A \otimes A \otimes M_{ee} \xrightarrow{P} A \otimes_{\mathbb{Z}} M. \tag{2}$$

Using the involution $T = HP - 1$ on M_{ee} they are defined by

$$\begin{aligned}
H(a \otimes m) &= (a \otimes a) \otimes H(m),\\
H([a,b] \otimes n) &= (a \otimes b) \otimes n + (b \otimes a) \otimes T(n),\\
T(a \otimes b \otimes n) &= b \otimes a \otimes T(n)\\
P(a \otimes b \otimes n) &= [a,b] \otimes n.
\end{aligned}$$

Clearly (2) is the quadratic $\mathbb{Z}$-module $\{A\} \otimes_{\mathbb{Z}} M$ defined in Definition 6.13.10(4). On the other hand, we have the projection

(6.13.19)
$$\mathrm{Hom}(A \otimes B, M_{ee}) = \mathrm{Hom}_{\mathbb{Z}}(A \mid B, M) \xleftarrow{r_{12}} \mathrm{Hom}_{\mathbb{Z}}(A \oplus B, M)$$

which carries $\alpha = (\alpha_e, \alpha_{ee})$ to β: $A \otimes B \to M_{ee}$ with $\beta(a \otimes b) = \alpha_{ee}(i_1 a, i_2 b)$. Now the structure maps for the cross-effect (6.13.19) are the homomorphisms

$$\mathrm{Hom}(A \otimes B, M_{ee}) \xrightarrow{T} \mathrm{Hom}(B \otimes A, M_{ee}), \tag{1}$$

$$\mathrm{Hom}_{\mathbb{Z}}(A, M) \xrightarrow{H} \mathrm{Hom}(A \otimes A, M_{ee}) \xrightarrow{P} \mathrm{Hom}_{\mathbb{Z}}(A, M). \tag{2}$$

Again using the involution $T = HP - 1$ on M_{ee} they are defined by

$$\begin{aligned}
(T\beta)(a \otimes b) &= T(\beta(b \otimes a))\\
(H\alpha)(a \otimes b) &= \alpha_{ee}(a,b) + T\alpha_{ee}(b,a)\\
(P\beta)_e(a) &= H\beta(a \otimes a)\\
(P\beta)_{ee}(a,b) &= \beta(a \otimes b).
\end{aligned}$$

Here (2) is the quadratic $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\{A\}, M)$ defined in Definition 6.13.10(4). Any quadratic functor F: $\mathbf{R} \to \mathbf{Ab}$ satisfies by Definition 6.13.10(3) the formula

$$(6.13.20) \qquad F(X_1 \vee \cdots \vee X_r) = \bigoplus_i F(X_i) \oplus \bigoplus_{i<j} F(X_i \mid X_j).$$

Using this formula we get by (6.13.18) and (6.13.19) similar formulas for $(A_1 \oplus \cdots \oplus A_r) \otimes_{\mathbb{Z}} M$ and $\mathrm{Hom}_{\mathbb{Z}}(A_1 \oplus \cdots \oplus A_r, M)$. Since $\mathbb{Z} \otimes_{\mathbb{Z}} M = M_e$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) = M_e$ we in particular get the following isomorphisms of abelian groups where $\pi^n = \pi \oplus \cdots \oplus \pi$ denotes the n-fold direct sum.

$$(\mathbb{Z}^n) \otimes_{\mathbb{Z}} M = (M_e)^n \oplus (M_{ee})^{n(n-1)/2} = \mathrm{Hom}(\mathbb{Z}^n, M).$$

Not every quadratic functor F: $\mathbf{Ab} \to \mathbf{Ab}$ is of the form $A \mapsto A \otimes_{\mathbb{Z}} M$. We always have, however, the canonical natural transformation

$$(6.13.21) \qquad \lambda \colon A \otimes_{\mathbb{Z}} F\{\mathbb{Z}\} \to F(A), \quad A \in \mathbf{Ab},$$

defined as follows. For $a \in A$ let $\bar{a}$: $\mathbb{Z} \to A$ be the homomorphism with $\bar{a}(1) = a$. Then we get for $m \in F(\mathbb{Z})$ and $n \in F(\mathbb{Z} \mid \mathbb{Z})$ the formulas $\lambda(a \otimes m) = F(\bar{a})(m)$ and $\lambda([a,b] \otimes n) = PF(\bar{a} \mid \bar{b})(n)$. By (6.13.20) and (6.13.16) the map λ is an isomorphism if A is a finitely generated free abelian group. We call λ the *tensor approximation* of the quadratic functor F: $\mathbf{Ab} \to \mathbf{Ab}$. Similarly we obtain a *Hom-approximation* of any quadratic functor G: $\mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$. This is a natural transformation

$$(6.13.22) \qquad \lambda \colon G(A) \to \mathrm{Hom}_{\mathbb{Z}}(A, G\{\mathbb{Z}\}), \quad A \in \mathbf{Ab},$$

which is an isomorphism if A is finitely generated and free. Here $G\{\mathbb{Z}\}$ is defined for $\mathbf{R} = \mathbf{Ab}^{\mathrm{op}}$ as in Definition 6.13.10; equivalently we have $G\{\mathbb{Z}\} = G^{\#}\{\mathbb{Z}\}$ with $G^{\#}$ as in Proposition 6.13.16.

We now consider the important classical quadratic functors

$$(6.13.23) \qquad \otimes^2, S^2, \Lambda^2, \hat{\otimes}^2, P^2, \Gamma \colon \mathbf{Ab} \to \mathbf{Ab}$$

which appear frequently in the literature. Here $\otimes^2$ is the *tensor square* defined by the tensor product in $\mathbf{Ab}$

$$\otimes^2(A) = A \otimes A. \tag{1}$$

The functor S^2 is the *symmetric square* given by

$$S^2(A) = A \otimes A / \{a \otimes b - b \otimes a \sim 0\}. \tag{2}$$

Moreover Λ^2 is the *exterior square*

$$\Lambda^2(A) = A \otimes A / \{a \otimes a \sim 0\} \tag{3}$$

and

$$\hat{\otimes}^2(A) = A \otimes A / \{a \otimes b + b \otimes a \sim 0\}. \tag{4}$$

Next P^2 is the *quadratic construction* defined by the quotient

$$P^2(A) = \Delta(A)/\Delta^3(A) \tag{5}$$

where $\Delta(A)$ is the augmentation ideal in the group ring $\mathbb{Z}[A]$ and $\Delta^3(A)$ its third power. We can define Whitehead's Γ-*functor* as a quotient

$$\Gamma(A) = P^2(A)/\{\tilde{\gamma}(a) - \tilde{\gamma}(-a) \sim 0\} \tag{6}$$

where $\tilde{\gamma}$: $A \to P^2(A)$ carries a to the element represented by $a - 1 \in \Delta(A)$. The composite γ: $A \to P^2(A) \twoheadrightarrow \Gamma(A)$ coincides with the universal quadratic map γ in (1.2.1). We say that a function f: $A \to B$ between abelian groups is *weak quadratic* if

$$[a,b]_f = f(a+b) - f(a) - f(b) \tag{7}$$

is bilinear for $a, b \in A$. Moreover f is *quadratic* if in addition $f(-a) = f(a)$ for $a, b \in A$. The function $\tilde{\gamma}$ is universal weak quadratic; that is, each weak quadratic function $f: A \to B$ admits a unique factorization $f = f^{\square}\tilde{\gamma}$ where $f^{\square}: P_2(A) \to B$ is a homomorphism. On the other hand γ is universal quadratic, that is, each quadratic function $f: A \to B'$ admits a unique factorization $f = f^{\square}\gamma$ where $f^{\square}$: $\Gamma(A) \to B$ is a homomorphism.

Each quadratic functor F: **Ab** $\to$ **Ab** determines the quadratic $\mathbb{Z}$-module $F\{\mathbb{Z}\}$ by Definition 6.13.10(4). For the functors F in (6.13.22) we in particular get the following isomorphisms of quadratic $\mathbb{Z}$-modules:

$$\begin{aligned} \otimes^2\{\mathbb{Z}\} &\cong \mathbb{Z}^{\otimes} = (\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}) \\ P^2\{\mathbb{Z}\} &\cong \mathbb{Z}^P = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z}). \end{aligned} \tag{8}$$

Here $\mathbb{Z}^P = D\mathbb{Z}^{\otimes}$ is the dual of $\mathbb{Z}^{\otimes}$; see Definition 6.13.9. Moreover we get

$$\begin{aligned} S^2\{\mathbb{Z}\} &\cong \mathbb{Z}^S = (\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}) \\ \Gamma\{\mathbb{Z}\} &\cong \mathbb{Z}^{\Gamma} = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z}) \end{aligned} \tag{9}$$

which are again dual quadratic $\mathbb{Z}$-modules. Next we obtain

$$\begin{aligned} \Lambda^2\{\mathbb{Z}\} &\cong \mathbb{Z}^{\Lambda} = (0 \to \mathbb{Z} \to 0) \\ \mathrm{id}\{\mathbb{Z}\} &= \mathbb{Z} = (\mathbb{Z} \to 0 \to \mathbb{Z}). \end{aligned} \tag{10}$$

Here id is the identity functor of **Ab** and $\mathbb{Z}$ is the quadratic $\mathbb{Z}$-module given by the inclusion (6.13.8). Hence $\mathbb{Z}$ is the dual of $\mathbb{Z}^{\Lambda}$. Finally we get

$$\hat{\otimes}^2\{\mathbb{Z}\} \cong P(1) = (\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2). \tag{11}$$

The following result gives a new interpretation of the classical functors above.

(6.13.24) Theorem *For each functor in* (6.13.22) *the tensor product approximation is an isomorphism. Hence for $A \in$* **Ab** *one has natural isomorphisms*

$$\begin{aligned}
\otimes^2(A) &= A \otimes \mathbb{Z}^{\otimes},\\
S^2(A) &= A \otimes \mathbb{Z}^{S},\\
\Lambda^2(A) &= A \otimes \mathbb{Z}^{\Lambda},\\
\hat{\otimes}^2(A) &= A \otimes P(1),\\
P^2(A) &= A \otimes \mathbb{Z}^{P},\\
\Gamma(A) &= A \otimes \mathbb{Z}^{\Gamma}.
\end{aligned}$$

The torsion functor F: **Ab** $\to$ **Ab** with $F(A) = A * A$, however, is a functor for which the tensor approximation is not an isomorphism, in fact $F\{Z\} = 0$ in this case.

The theorem above raises the question of classifying all indecomposable quadratic $\mathbb{Z}$-modules. For this recall that an object X in an additive category is indecomposable if X admits no isomorphism $X \cong A \oplus B$ with $A \neq 0$ and $B \neq 0$. It is an interesting problem of representation theory to classify all indecomposable quadratric $\mathbb{Z}$-modules which are finitely generated as abelian groups. There is the following result where we say that a quadratic $\mathbb{Z}$-module M is of *cyclic type* if M_e and M_{ee} are cyclic groups.

(6.13.25) Proposition *The quadratic $\mathbb{Z}$ modules below together with their duals furnish a complete list of indecomposable quadratic $\mathbb{Z}$-modules of cyclic type. Let $s, t \geq 1$ and let $C = \mathbb{Z}$ or $C = \mathbb{Z}/p^i$ where $p =$ prime, $i \geq 1$.*

M_e	$\xrightarrow{H}$	M_{ee}	$\xrightarrow{P}$	M_e	
C	$\xrightarrow{0}$	0	$\xrightarrow{0}$	C	
C	$\xrightarrow{1}$	C	$\xrightarrow{2}$	C	
$\mathbb{Z}$	$\xrightarrow{2^{t-1}}$	$\mathbb{Z}/2^t$	$\xrightarrow{0}$	$\mathbb{Z}$	
$\mathbb{Z}/2^s$	$\xrightarrow{2^{t-1}}$	$\mathbb{Z}/2^t$	$\xrightarrow{0}$	$\mathbb{Z}/2^s$	$s+t>1$
$\mathbb{Z}/2^s$	$\xrightarrow{2^{t-1}}$	$\mathbb{Z}/2^t$	$\xrightarrow{2^{s-1}}$	$\mathbb{Z}/2^s$	$s+t>1$
$\mathbb{Z}/2^{s+1}$	$\xrightarrow{1}$	$\mathbb{Z}/2^s$	$\xrightarrow{2}$	$\mathbb{Z}/2^{s+1}$	
$\mathbb{Z}/2^{s+1}$	$\xrightarrow{2^{s-1}+1}$	$\mathbb{Z}/2^s$	$\xrightarrow{2}$	$\mathbb{Z}/2^{s+1}$	$s>1$

With respect to the two tensor products of Definitions 6.13.13 and 6.13.9(2) we have the following rule.

(6.13.26) Proposition *For a quadratic $\mathbb{Z}$-module M and for abelian groups A, B we have the natural isomorphism*

$$A \otimes_{\mathbb{Z}} (M \otimes B) = (A \otimes_{\mathbb{Z}} M) \otimes B.$$

Here $\otimes_{\mathbb{Z}}$ is the quadratic tensor product and $M \otimes B$ is defined in Definition 6.13.9(2). Moreover $\otimes$ on the right-hand side denotes the usual tensor product of abelian groups.

6.14 Quadratic derived functors

In this section we associate with a quadratic functor F a quadratic chain functor F_*. The definition of F_* is motivated by properties of homotopy groups of Moore spaces which we exploit in the following section (6.15). The chain functor F_* is used here for the definition of derived functors. The Γ-chain functor Γ_* in Definition 6.2.5 is a special case of F_*. We obtain two 'quadratic torsion products', $A *' M$ and $A *'' M$, which are derived from the quadratic tensor product in Section 6.13. For a further discussion of such quadratic derived functors we refer the reader to Baues [QF].

Let $\mathbf{R}$ be a ringoid with a zero object denoted by 0. A *chain complex* $X_* = (X_*, d)$ in $\mathbf{R}$ is a sequence of maps in $\mathbf{R}$

$$\cdots \to X_n \xrightarrow{d} X_{n-1} \xrightarrow{d} \cdots \quad (n \in \mathbb{Z}) \tag{6.14.1}$$

with $dd = 0$. A chain map $F: X_* \to Y_*$ is given by maps $F = F_n: X_n \to Y_n$ with $dF = Fd$ and a chain homotopy $\alpha: F \simeq G$ is given by maps $\alpha = \alpha_n$: $X_{n-1} \to Y_n$ with $-F_n + G_n = \alpha_n d + d\alpha_{n+1}$. Let $\mathbf{Chain(R)}$ be the category of chain complexes in $\mathbf{R}$ and let $\mathbf{Chain(R)}/\simeq$ be its homotopy category. A chain complex X_* is *concentrated* in degree $n, n+1, \ldots, m$ with $n \leq m$ if $X_i = 0$ for $i < n$ and $i > m$. We also need the category $\mathbf{Pair(R)}$ of *pairs* in $\mathbf{R}$; objects are morphisms d_A: $A_1 \to A_0$ in $\mathbf{R}$ and morphisms are pairs $F = (F_1, F_0)$ for which the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{F_1} & B_1 \\ {\scriptstyle d_A}\downarrow & & \downarrow{\scriptstyle d_B} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \tag{6.14.2}$$

commutes. Hence $\mathbf{Pair(R)}$ is a full subcategory of $\mathbf{Chain(R)}$ consisting of chain complexes concentrated in degree 0 and 1. A homotopy $\alpha: F \simeq G$ of maps $F, G: d_A \to d_B$ in **Pair(R)** is a map $\alpha: A_0 \to B_1$ with $-F_1 + G_1 = \alpha d_A$ and $-F_0 + G_0 = d_B \alpha$.

(6.14.3) Definition Let **R** be an additive category and let F: **R** $\to$ **Ab** be a quadratic functor. Then we define the induced *quadratic chain functor*

$$F_*: \mathbf{Pair(R)} \to \mathbf{Chain}_{\mathbb{Z}} = \mathbf{Chain(Ab)}. \tag{1}$$

For an object $d_A: A_1 \to A_0$ in **Pair(R)** let $F_*(d_A)$ be the following chain complex of abelian grops concentrated in degree $0, 1, 2$

$$\begin{array}{ccccc} F(A_1 | A_1) & \xrightarrow{\partial_2} & F(A_1) \oplus F(A_1 | A_0) & \xrightarrow{\partial_1} & F(A_0) \\ \| & & \| & & \| \\ F_2(d_A) & & F_1(d_A) & & F_0(d_A). \end{array}$$

The boundary maps ∂_1, ∂_2 are given by P in the quadratic $\mathbb{Z}$-modules $F\{A_1\}$ and $F\{A_0\}$ respectively, see Definition 6.13.10(4), namely

$$\partial_1 = (F(d_A), PF(d_A | A_0)), \tag{2}$$

$$\partial_2 = (P, -F(A_1 | d_A)). \tag{3}$$

One readily checks $\partial_1 \partial_2 = 0$. In Baues [QF] (6.4) we show:

(6.14.4) Theorem *The quadratic chain functor F_* in Definition* 6.14.3 *induces a functor*

$$F_*: \mathbf{Pair(R)}/\simeq \;\to \mathbf{Chain}_{\mathbb{Z}}/\simeq$$

between homotopy categories.

We now consider the case **R** = **Ab**. Using short free resolutions we obtain the full and faithful functor

(6.14.5) $$i: \mathbf{Ab} \to \mathbf{Pair(Ab)}/\simeq$$

as follows. We choose for each abelian group a short exact sequence

$$A_1 \overset{d_A}{\rightarrowtail} A_0 \twoheadrightarrow A$$

where A_0 and A_1 are free abelian. For a homomorphism $\varphi: A \to B$ we choose a map F_φ: $d_A \to d_B$ in **Pair(Ab)** which induces φ. Then the functor i carries A to d_A and carries φ to the homotopy class $\{F_\varphi\}$ which depends only on φ. Using (6.14.5) and Theorem 6.14.4 we are now ready to define derived functors of a quadratic functor **Ab** $\to$ **Ab**.

(6.14.6) Definition Let $F: \mathbf{Ab} \to \mathbf{Ab}$ be a quadratic functor. Then one obtains the *derived functors*

$$L_t F: \mathbf{Ab} \to \mathbf{Ab} \quad (t = 0, 1, 2)$$

as follows. We define $L_t F$ by the homology group $(L_t F)(A) = H_t(F_* d_A)$ where $d_A = iA$ is chosen as in (6.14.5). That is, $L_t F$ is the composite of the functors

$$\mathbf{Ab} \xrightarrow{i} \mathbf{Pair(Ab)}/\simeq \xrightarrow{F_*} \mathbf{Chain}_{\mathbb{Z}}/\simeq \xrightarrow{H_t} \mathbf{Ab}.$$

The remarks (6.5), (7.5) in Baues [QF] show that $L_t F$ coincides with the derived functor considered by Dold and Puppe [HN].

We now obtain derived functors of the quadratic tensor product as follows.

(6.14.7) Definition Let A be an abelian group and let M be a quadratic $\mathbb{Z}$-module. Then we have the quadratic functor

$$\otimes M: \mathbf{Ab} \to \mathbf{Ab}$$

which carries A to the quadratic tensor product $(\otimes M)(A) = A \otimes_{\mathbb{Z}} M$. Hence the derived functors $L_t(\otimes M)$ are defined by Definition 6.14.6 above. We call

$$A *'' M = (L_2(\otimes M))A \tag{1}$$

and

$$A *' M = (L_1(\otimes M))A \tag{2}$$

the *quadratic torsion products*. Using (6.14.11) below we get $\otimes M = L_0(\otimes M)$, that is

$$A \otimes M = (L_0(\otimes M))A \tag{3}$$

is the quadratic tensor product. The quadratic tensor products are obtained more explicitly by the following definition where $d_A: A_1 \rightarrowtail A_0$ is a short free resolution of A. Consider the chain complex $(\otimes M)_* d_A$:

$$A_1 \otimes A_1 \otimes M_{ee} \xrightarrow{\partial_2} A_1 \otimes_{\mathbb{Z}} M \oplus A_1 \otimes A_0 \otimes M_{ee} \xrightarrow{\partial_1} A_0 \otimes_{\mathbb{Z}} M \tag{4}$$

which is defined by the boundary operators

$$\partial_2 = (P, -A_1 \otimes d_A \otimes M_{ee}) \quad \text{and} \quad \partial_1 = (d_A \otimes_{\mathbb{Z}} M, P(d_A \otimes A_0 \otimes M_{ee})).$$

Here P is given as in (6.13.18)(2). Then we have

$$A \otimes M = \mathrm{cokernel}(\partial_1) \tag{5}$$

$$A *' M = \mathrm{kernel}(\partial_1)/\mathrm{image}(\partial_2) \tag{6}$$

$$A *'' M = \mathrm{kernel}(\partial_2). \tag{7}$$

All functors in (5), (6), (7) are additive in M and quadratic in A. The quadratic cross-effects are:

(6.14.8)
$$(A \mid B) \otimes M = A \otimes B \otimes M_{ee} \tag{1}$$
$$(A \mid B) *' M = H_1(d_A \otimes d_B, M_{ee}) \tag{2}$$
$$(A \mid B) *'' M = A * B * M_{ee}. \tag{3}$$

Here $d_A \otimes d_B$ is the tensor product of chain complexes. The Künneth formula yields a natural exact sequence

$$(A * B) \otimes M_{ee} \rightarrowtail H_1(d_A \otimes d_B, M_{ee}) \twoheadrightarrow (A \otimes B) * M_{ee} \tag{4}$$

which is split (unnaturally). There is a natural isomorphism

$$H_1(d_A \otimes d_B, M_{ee}) = \mathrm{Trip}(A, B, M_{ee}) \tag{5}$$

where the right-hand side is the triple torsion product of Mac Lane [TT]. Moreover one has a natural injective homomorphism

(6.14.9) $$A *'' M \rightarrowtail A * A * M_{ee}.$$

The results in Remark 6.2.8 and Theorem 6.2.9 are proved in 7.7 and 7.8 of Baues [QF].

We shall need the following '*right exactness*' of the quadratic tensor product. Let $M_1 \xrightarrow{i} M_0 \xrightarrow{q} M \to 0$ be an exact sequence of quadratic $\mathbb{Z}$-modules in **QM**($\mathbb{Z}$) and let A be an abelian group. Then the induced sequence

(6.14.10) $$A \otimes_{\mathbb{Z}} M_1 \xrightarrow{1 \otimes i} A \otimes_{\mathbb{Z}} M_0 \xrightarrow{1 \otimes q} A \otimes_{\mathbb{Z}} M \to 0$$

is exact. Here, however, $1 \otimes i$ need not be injective in case i is injective. Moreover let $A_1 \xrightarrow{d} A_0 \xrightarrow{q} A \to 0$ be an exact sequence of abelian groups. Then the induced sequence

(6.14.11) $$A_1 \otimes M \oplus A_1 \otimes A_0 \otimes M_{ee} \xrightarrow{\partial_1} A_0 \otimes M \xrightarrow{q \otimes 1} A \otimes M \to 0$$

is exact where $\partial_1 = (d \otimes M, P(d \otimes A_0 \otimes M_{ee}))$. For example for $M = \mathbb{Z}^{\Gamma}$ we obtain the exact sequence in Lemma 1.2.8 by (6.13.24).

We point out that for any quadratic functor F: **Ab** $\to$ **Ab** the derived functors $L_t F$, $t \geq 1$, depend only on the restriction of F to the subcategory of free abelian groups. If F commutes with direct limits we have $F(A_0) = A_0 \otimes M$ for any free abelian group A_0. Here $M = F\{\mathbb{Z}\}$ is the quadratic $\mathbb{Z}$-module in Definition 6.13.10(4). Hence we obtain in this case

$$\begin{cases} (L_1 F)(A) = A *' M \\ (L_2 F)(A) = A *'' M \end{cases} \tag{6.14.12}$$

so that the quadratic torsion products suffice to describe the derived functors of Dold and Puppe. The equations in (6.14.8) hold for any quadratic functor F if A is finitely generated.

6.15 Metastable homotopy groups of Moore spaces

We consider homotopy groups $\pi_m M(A,n)$ of a Moore space $M(A,n)$, $n \geq 2$. In the stable range $m < 2n-1$ these groups are computable in terms of stable homotopy groups of spheres. We are here mainly interested in the metastable range $m < 3n-1$. In general it is an unsolved problem to describe the groups $\pi_m M(A,n)$ only in terms of properties of homotopy groups of spheres. In addition one has the problem of describing these groups as a functor on the category $\mathbf{M}^n$ of Moore spaces in degree n. For the stable range we describe partial solutions in Section 6.6; moreover for $m = n+2$, we obtain complete solutions in Chapters 8, 9, and 11.

More generally we shall deal with the homotopy groups

$$\pi_m^K M(A,n) = [\Sigma^m K, M(A,n)] \tag{6.15.1}$$

where $\Sigma^m K$ is the m-fold suspension of a CW-complex K and $m \geq 2$ so that (6.15.1) is an abelian group. Clearly for $K = S^0$ this is the homotopy group $\pi_m M(A,n)$. The group (6.15.1) yields the functor

$$\pi_m^K : \mathbf{M}^n \to \mathbf{Ab} \tag{1}$$

where the homotopy category $\mathbf{M}^n$, $n \geq 3$, of Moore spaces is an additive category with the sum given by

$$M(A,n) \vee M(B,n) = M(A \oplus B, n). \tag{2}$$

The left distributivity law of homotopy theory shows that the functor π_m^K is additive for $\dim(\Sigma^m K) < 2n-1$. Moreover the functor π_m^K is quadratic for $\dim(\Sigma^m K) < 3n-2$. We now consider the quadratic cross-effect of this functor. To this end we need for CW-complexes X, Y the *Whitehead product map*

$$w : \Sigma X \wedge Y \to \Sigma X \vee \Sigma Y \tag{3}$$

where $X \wedge Y = X \times Y / X \vee Y$ is the smash product. This map induces on homotopy sets the operation

$$[\Sigma X, U] \times [\Sigma Y, U] \to [\Sigma(X \wedge Y), U] \tag{4}$$

which carries (α, β) to the *Whitehead product* $[\alpha, \beta] = w^*(\alpha, \beta)$. For the inclusions $i_1\colon \Sigma X \subset \Sigma X \vee \Sigma Y$ and $i_2\colon \Sigma Y \subset \Sigma X \vee \Sigma Y$ we thus have $[i_1, i_2] = w$. We define the space

$$M(A \mid B, n) = \Sigma M(A, n-1) \wedge M(B, n-1). \tag{5}$$

Since $M(A, n) = \Sigma M(A, n-1)$ we thus have, as a special case of (3), the Whitehead product map

$$[i_1, i_2]\colon M(A \mid B, n) \to M(A, n) \vee M(B, n). \tag{6}$$

Now the Hilton–Milnor theorem shows that in the metastable case the functor π_m^K has the following cross-effect.

(6.15.2) Lemma *For* $\dim(\Sigma^m K) < 3n - 1$ *there is a binatural isomorphism*

$$\pi_m^K M(A \mid B, n) = \pi_m^K(M(A, n) \mid M(B, n))$$

which carries α *to the composite* $[i_1, i_2]\alpha$. *Using this isomorphism the quadratic* $\mathbb{Z}$*-module* $\pi_m^K\{M(A, n)\}$ *in Definition* 6.13.10(4) *coincides with*

$$\pi_m^K M(A, n) \xrightarrow{H} \pi_m^K M(A \mid B, n) \xrightarrow{P} \pi_m^K M(A, n).$$

Here $H = \gamma_2$ *is the* Hopf invariant *and* $P = [1, 1]_*$ *is induced by the* Whitehead product $[1, 1]$ *where* 1 *is the identity of* $M(A, n)$, *that is* $P(\alpha) = [1, 1]\alpha$. *Moreover* $T = -(\Sigma T_{21})_*$ *is induced by the interchange map* T_{21}.

This lemma yields many interesting examples of quadratic $\mathbb{Z}$-modules. In particular we get for spheres the quadratic $\mathbb{Z}$-modules $(m < 3n - 2)$

$$\pi_m\{S^n\} = (\pi_m S^n \xrightarrow{H} \pi_m S^{2n-1} \xrightarrow{P} \pi_m S^n). \tag{6.15.3}$$

Here H is the classical Hopf invariant for homotopy groups of spheres and $P = [\iota_n, \iota_n]_*$ is induced by the Whitehead square $[\iota_n, \iota_n]$ of the identity $\iota_n \in \pi_n S^n$. The operators H and P in (6.15.3) are known in many cases. For example we get by inspection of Toda's book [CM] the following list. Let $\oplus$ be the direct sum in $\mathbf{QM}(\mathbb{Z})$, that is

$$M \oplus N = (M_e \oplus N_e \xrightarrow{H \oplus H} M_{ee} \oplus N_{ee} \xrightarrow{P \oplus P} M_e \oplus N_e)$$

and recall that an abelian group A yields the quadratic $\mathbb{Z}$-module $A = (A \to 0 \to A)$.

(6.15.4) List of $\pi_m\{S^n\}$:

(n,m)	$\pi_m S^n \xrightarrow{H} \pi_m S^{2n-1} \xrightarrow{P} \pi_m S^n$
$(2,3)$	$\mathbb{Z}^\Gamma$
$(3,5)$	$\mathbb{Z}^\Lambda \oplus \mathbb{Z}/2$
$(3,6)$	$(\mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/4) \oplus \mathbb{Z}/3$
$(4,7)$	$(\mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{1,0} \mathbb{Z} \xrightarrow{2,-1} \mathbb{Z} \oplus \mathbb{Z}/4) \oplus \mathbb{Z}/3$
$(4,8)$	$\mathbb{Z}^P \otimes \mathbb{Z}/2$
$(4,9)$	$\mathbb{Z}^P \otimes \mathbb{Z}/2$
$(5,9)$	$(\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2)$
$(5,10)$	$\mathbb{Z}^S \otimes \mathbb{Z}/2$
$(5,11)$	$\mathbb{Z}^\Lambda \otimes \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$(5,12)$	$(\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/8 \xrightarrow{0} \mathbb{Z}/2) \oplus \mathbb{Z}^\Lambda \otimes \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/5$
$(6,11)$	$\mathbb{Z}^S$
$(6,12)$	$\mathbb{Z}^\Lambda \otimes \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$(6,13)$	$(\mathbb{Z}/4 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4) \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/5$
$(6,14)$	$\mathbb{Z}^\Gamma \otimes \mathbb{Z}/8 \oplus \mathbb{Z}^S \otimes \mathbb{Z}/3 \oplus \mathbb{Z}/2$
$(6,15)$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

Similarly we get as in (6.15.4) the quadratic $\mathbb{Z}$-module

$$\pi_m^K\{S^n\} = (\pi_m^K S^n \xrightarrow{H} \pi_m^K S^{2n-1} \xrightarrow{P} \pi_m^K S^n)$$

for $\dim(\Sigma^m K) < 3n-1$; see Lemma 6.15.2. Using the quadratic tensor product in (6.13.13) one has the following crucial result.

(6.15.5) Theorem *Let A be a free abelian group and let* $\dim(\Sigma^m K) < 3n-1$ *with $m, n \geq 2$. Then there is an isomorphism*

$$\pi_m^K M(A,n) = A \otimes_{\mathbb{Z}} \pi_m^K\{S^n\}$$

which is natural in A and K.

Proof Since both sides are compatible with direct limits it is enough to consider finitely generated free abelian groups A. For these the proposition is a special case of Proposition 6.13.16. □

We can extend the isomorphism in Theorem 6.15.5 in an appropriate way to all abelian groups. For this we define a functor Γ_m^K as follows.

(6.15.6) Definition Let K be a finite dimensional CW-complex and $m, n \geq 2$. Then we obtain a functor

$$\Gamma_m^K(-, n)\colon \mathbf{Ab} \to \mathbf{Ab}$$

which carries an abelian group A to the abelian group $\Gamma_m^K(A, n)$. If A is free abelian we have $\Gamma_m^K(A, n) = \pi_m^K M(A, n)$. If A is not free abelian we choose a short free resolution $A_1 \overset{d}{\rightarrowtail} A_0 \twoheadrightarrow A$ which yields the cofibre sequence

$$M(A_1, n) \xrightarrow{d} M(A_0, n) \xrightarrow{i_A} M(A, n)$$

where $M(A, n)$ is the mapping cone of $d = d_A$. Consider the maps

$$\pi_m^K M(A_0, n) \xleftarrow{(d,1)} \pi_m^K M(A_1 \oplus A_0, n) \xrightarrow{(0,1)_*} \pi_m^K M(A_0, n)$$

where 1 is the identity of $M(A_0, n)$ and 0 is the rivial map. Then we get the quotient group

$$\Gamma_m^K(A, n) = \pi_m^K M(A_0, n)/(d, 1)_* \mathrm{kernel}(0, 1)_*$$

which defines the functor above. As in (6.14.5) let $F_\varphi = (F_1, F_0)\colon d_A \to d_B$ be a pair map which induces $\varphi\colon A \to B$. Then F_φ induces the homomorphism φ_*: $\Gamma_m^K(A, n) \to \Gamma_m^K(B, n)$ which carries the coset of $\xi \in \pi_m^K M(A_0, n)$ to the coset of $F_0 \xi$.

(6.15.7) Lemma *The induced map φ_* is well defined.*

Proof We have to check that φ_* does not depend on the choice of F_φ. If $F'_\varphi = (F'_1, F'_0)$ is also a map which induces φ we have a homotopy $\alpha\colon F_\varphi \simeq F'_\varphi$ and hence $F'_0 = F_0 + d_B \alpha$. Now the left distributivity law of homotopy theory (see Section A.9 in the appendix) yields the formula

$$F'_0 \xi = (F_0 + d_B \alpha)\xi = F_0 \xi + d_B \alpha \xi - \sum_{n \geq 2} c_n(F_0, d_B \alpha)\gamma_n f$$

which shows that $F'_0 \xi - F_0 \xi$ is an element of $(d_B, 1)_*$ kernel$(0, 1)_*$. □

The next result is a corollary of Theorem 6.15.5.

(6.15.8) Corollary *Let A be an abelian group and let* $\dim(\Sigma^m K) < 3n - 1$ *with* $m, n \geq 2$. *Then there is an isomorphism*

$$\Gamma_m^K(A, n) = A \otimes_{\mathbb{Z}} \pi_m^K\{S^n\}$$

which is natural in A and K.

The corollary indicates that it should be possible to generalize the quadratic tensor product in such a way that an isomorphism as in (6.13.7) is also available if $\dim(\Sigma^m K) \geq 3n-1$. Then, however, $\Gamma_m^K(A,n)$ need not be quadratic in A.

Proof of (15.8) Since π_m^k is quadratic we have the commutative diagram

$$\begin{array}{ccc} \pi_m^K M(A_0,n) \oplus \pi_m^K M(A_1 \mid A_0,n) = \ker(0,1)_* & \xrightarrow{(d,1)_*} & \pi_m^K M(A_0,n) \\ \| & & \| \\ A_0 \otimes \pi_m^K\{S^n\} \oplus A_1 \otimes A_0 \otimes \pi_m^K S^{2n-1} & \xrightarrow{\partial_1} & A_0 \otimes \pi_m^K\{S^n\}. \end{array}$$

Here ∂_1 coincides with ∂_1 in (6.14.11). Hence quadratic right exactness in (6.14.11) yields the result. □

(6.15.9) Notation Let $k \geq 0$ and $n \geq 2$. We obtain as a special case the functor

$$\Gamma_n^k\colon \mathbf{Ab} \to \mathbf{Ab}.$$

Here we set $\Gamma_n^k(A) = \Gamma_n^K(A,n)$ where $K = S^k$ is the k-sphere. Hence (6.15.8) yields for $k < 2n-1$ the natural isomorphism

$$\Gamma_n^k(A) = A \otimes \pi_{n+k}\{S^n\}.$$

Now the list in (6.15.4) makes it easy to identify these functors. For example we get for $m = 11$, $n = 5$, $k = 6$ the natural isomorphism

$$\begin{aligned} \Gamma_5^6(A) &= A \otimes (\mathbb{Z}^\wedge \otimes \mathbb{Z}/2 \oplus \mathbb{Z}/2) \\ &= \Lambda^2(A) \otimes \mathbb{Z}/2 \oplus A \otimes \mathbb{Z}/2 \end{aligned}$$

where $\Lambda^2(A) = A \otimes \mathbb{Z}^\wedge$ is the exterior square; see Proposition 6.13.26. For $k = 0$ we get $\Gamma_n^0(A) = A$ and for $k = 1$ we have

$$\Gamma_n^1(A) = \begin{cases} \Gamma(A) & \text{for } n = 2 \\ A \otimes \mathbb{Z}/2 & \text{for } n \geq 3. \end{cases}$$

This is Whitehead's functor. Whitehead proved that an $(n-1)$-connected CW-space X satisfies $\Gamma_{n+1}X = \Gamma_n^1(H_n X)$, $n \geq 2$. We compute in Chapter 11 the functor Γ_2^2 which is not quadratic. This is needed for the computation of the homotopy group $\pi_4 M(A,2)$; see Theorem 11.1.9.

The relevance of the functor Γ_m^K in Definition 6.15.6 arises by the following result.

(6.15.10) Theorem *Let K be finite dimensional and let $n, m \geq 2$. Then there is a homomorphism*

$$\lambda\colon \Gamma_m^K(A,n) \to \pi_m^K M(A,n)$$

which is natural in $M(A,n)$ and K and which maps surjectively to the image of the map

$$i\colon \pi_m^K M(A_0,n) \to \pi_m^K M(A,n)$$

induced by the inclusion i in Definition 6.15.6.

Proof Let $X_1 = M(A_1,n)$ and $X_0 = M(A_0,n)$ such that $M(A,n)$ is given by the push-out diagram

$$\begin{tikzcd} CX_1 \arrow[r, "\pi_d"] & M(A,n) \\ X_1 \arrow[u, phantom, sloped, "\subset"] \arrow[r, "d"'] & X_0 \arrow[u, phantom, sloped, "\subset"] \arrow[u, phantom, "\;\; i"] \end{tikzcd}$$

where CX_1 is the cone of X_1. This yields the following commutative diagram with exact rows

$$\begin{tikzcd} & \pi_{m+1}^K(CX_1 \vee X_0, X_1 \vee X_0) \arrow[r, phantom, "\overset{\partial}{\cong}"] \arrow[d, "(\pi_d,1)_*"] & \pi_m^K(X_1 \vee X_0)_2 \arrow[d, "(d,1)_*"] & \\ \pi_{n+1}^K M(A,n) \arrow[r, "j"] & \pi_{m+1}^K(M(A,n), X_0) \arrow[r, "\partial"] & \pi_m^K X_0 \arrow[r, "i"] \arrow[d] & \pi_m^K M(A,n) \\ & & \Gamma_m^K(A,n). \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Here we set

$$\pi_m^K(X_1 \vee X_0)_2 = \text{kernel}\{(0,1)_*\colon \pi_m^K(X_1 \vee X_0) \to \pi_m^k(X_0)\}.$$

The boundary ∂ in the homotopy exact sequence of a pair yields the isomorphism in the top row. Hence $(d,1)_*$ carries kernel $(0,1)_*$ to the image of ∂ which is the kernel of i. Therefore the factorization λ of i exists. □

Using the diagram in the proof we define the functor

$$\begin{aligned} &\Gamma T_m^K(-,n)\colon \mathbf{Ab} \to \mathbf{Ab}, \\ &\Gamma T_m^K(A,n) = (\pi_d,1)_* \partial^{-1}\,\text{kernel}(d,1)_*. \end{aligned} \tag{6.15.11}$$

One has the inclusion

$$\Gamma T_m^K(A,n) \subset j\pi_{m+1}^K M(A,n) \tag{1}$$

which is natural in $M(A,n)$. Using Theorem 6.15.10 we see that

$$\Gamma_{m+1}^K(A,n) \xrightarrow{\lambda} \pi_{m+1}^K M(A,n) \xrightarrow{j} j\pi_{m+1}^K M(A,n) \to 0 \tag{2}$$

is exact, so that $j\pi_{m+1}^K M(A,n)$ can be identified with the cokernel of λ in Theorem 6.15.10.

(6.15.12) Lemma *The functor* (6.15.11) *is well defined.*

Proof Let $\overline{\varphi}\colon M(A,n)\to M(B,n)$ be a realization of $\varphi\colon A\to B$. Let $y\in \pi_{m+1}^K M(A,n)$ with $jy\in \Gamma T_m^K(A,n)$. Then there exists $x\in\pi_m^K(X_1\vee X_0)_2$ with

$$(d_A,1)_* x = 0 \tag{1}$$

and

$$E_d(x)=jx \quad \text{where} \quad E_d=(\pi_d,1)_*\partial^{-1}. \tag{2}$$

Here E_d is the 'functional suspension' considered in (II. §11) of Baues [AH]. Using proposition (II.12.3) in Baues [AH] we get $\overline{\overline{\varphi}}=\overline{\varphi}=i_B\alpha$, $\alpha\in \operatorname{Ext}(A,\Gamma_n^1 B)$,

$$\begin{aligned}\overline{\overline{\varphi}}_* y &= y^*(\overline{\varphi}+i_B\alpha)\\ &= y^*\overline{\varphi}+(Ex)^*(i_B\alpha,i_BF_0)\end{aligned}$$

where Ex is the partial suspension of x and where $F_0\colon M(A_0,n)\to M(B_0,n)$ is the restriction of $\overline{\varphi}$. Since $j(i_B)_*=0$ is trivial, we see that $j\overline{\overline{\varphi}}_* y = j\overline{\varphi}_* y = \varphi_* jy$ and hence $\varphi_*(jy)$ does not depend on the choice of the realization $\overline{\varphi}$. □

If A is free abelian we know that λ in (6.15.11) is an isomorphism and hence $\Gamma T_m^K(A,n)=0$ in this case. We are now ready to state the main theorem in this section.

(6.15.13) Theorem *Let K be a CW-complex with* $\dim(\Sigma^m K)<3n-1$ *and $m,n\geq 2$ and let A be an abelian group. Then there is an exact sequence*

$$0\to \Gamma T_m^K(A,n)\xrightarrow{i} j\pi_{m+1}^K M(A,n)\xrightarrow{h} A*''\pi_{m-1}^K\{S^n\}$$
$$\xrightarrow{\delta}\Gamma_m^K(A,n)\xrightarrow{\lambda}\pi_m^K M(A,n)\xrightarrow{j} j\pi_m^K M(A,n)\to 0.$$

Moreover we have for $\dim(\Sigma^m K)<3n-2$ *the isomorphisms*

$$\Gamma_m^K(A,n)\cong A\otimes\pi_m^K\{S^n\}$$

$$\Gamma T_m^K(A,n)\cong A*'\pi_m^K\{S^n\}.$$

All morphisms are natural in A and K. The quadratic torsion products $*'$ and $*''$ are defined in Section 6.14.

We point out that the case $\dim(\Sigma^m K) = 3n-2$ in the theorem is the first case outside the 'quadratic range'. In this case $\Gamma_m^K(A,n)$ is not quadratic in A and we have

(6.15.14) $\quad \Gamma T_m^K(A,n) = L_1\Gamma_m^K(A,n) \quad \text{for} \quad \dim \Sigma^m K \le 3n-2.$

Here L_1 is the derived functor in the sense of Dold and Puppe [HN]. In fact the theorem has an extension to arbitrary dimensions of $\Sigma^m K$ by a spectral sequence of W. Dreckmann. The E_2-term of this spectral sequence consists of the derived functors $L_j\Gamma_m^K(-,n)$, $j \ge 0$, with $L_0\Gamma_m^K(A,n) = \Gamma_m^K(A,n)$. The spectral sequence converges to $\pi_m^K M(A,n)$ and is natural in $M(A,n)$ and K. We now take $K = S^0$ in Theorem 6.15.13. Then we see by the inclusion (6.14.9),

$$A *'' \pi_{m-1}\{S^n\} \subset A * A * \pi_{m-1}(S^{2n-1}),$$

that $A *'' \pi_{m-1}\{S^n\} = 0$ is trivial for $m \le 2n$. Hence we obtain the following special cases.

(6.15.15) Corollary *For $n \ge 2$ one has the short exact sequences ($k < 2n-1$)*

$$0 \to A \otimes \pi_k S^n \to \pi_k M(A,n) \to A * \pi_{k-1}(S^n) \to 0$$
$$0 \to A \otimes \pi_{2n-1}\{S^n\} \to \pi_{2n-1} M(A,n) \to A * \pi_{2n-2}(S^n) \to 0$$
$$0 \to \Gamma_n^n(A) \to \pi_{2n} M(A,n) \to A *' \pi_{2n-1}\{S^n\} \to 0$$

with $\Gamma_n^n(A) = A \otimes \pi_{2n}\{S^n\}$ for $n \ge 3$. Moreover one has the exact sequence

$$L_2\Gamma_n^n(A) \to \Gamma_n^{n+1}(A) \xrightarrow{\lambda} \pi_{2n+1} M(A,n) \to L_1\Gamma_n^n(A) \to 0.$$

Here we have $L_1\Gamma_n^n(A) = A *' \pi_{2n}\{S^n\}$ and $L_2\Gamma_n^n(A) = A *'' \pi_{2n}\{S^n\}$ for $n \ge 3$ and $\Gamma_n^{n+1}(A) = A \otimes \pi_{2n+1}\{S^n\}$ for $n > 3$. All sequences are natural in $M(A,n) \in \mathbf{M}^n$.

The naturality implies that the exact sequence of Theorem 6.15.13 induces a corresponding exact sequence for cross-effects. Restricting to the quadratic range $\dim(\Sigma^m K) < 3n-2$ we hence get the following corollary where $j\pi_m^K(A \mid B, n)$ is the image of the map

$$j\colon \pi_m^K M(A \mid B, n) \to \pi_m^K(M(A \mid B, n), M(A_0 \mid B_0, n))$$

given by the pair $\Sigma i_A \wedge i_B$; see Definition 6.15.6 and (6.15.1)(5).

(6.15.16) Corollary *Let $\dim(\Sigma^m K) < 3n-2$ with $m, n \ge 2$ and let A and B be abelian groups. Then there is the following exact sequence of cross-effects of the functors in Theorem 6.15.13.*

$$0 \to \mathrm{Trp}(A, B, \pi_m^K S^{2n-1}) \to j\pi_{m+1}^K M(A \mid B, n) \to A * B * \pi_{m-1}^K S^{2n-1}$$
$$\xrightarrow{\partial} A \otimes B \otimes \pi_m^K S^{2n-1} \to \pi_m^K M(A \mid B, n) \to j\pi_m^K M(A \mid B, n) \to 0.$$

Here we use the formulas for quadratic cross-effects in (6.14.8); in particular Trp is the triple torsion product of Mac Lane; see (6.14.8)(5). We leave it to the reader to write down the cross-effect sequences for Corollary 6.15.15.

Proof of Theorem 6.15.13 The proof relies on the exact EHP-sequence for mapping cones obtained in Theorem A.6.9. For this we use the fact that the Moore space $M(A,n) = C_d$ is the mapping cone of a map

$$d\colon X_1 = M(A_1,n) \to X_0 = M(A_0,n)$$

where $X_1 = \Sigma X_1'$ with $X_1' = M(A_1, n-1)$. The following commutative diagram extends the diagram in the proof of Theorem 6.15.10. The operator E_m is $(\pi_d, 1)_* \partial^{-1}$ in Theorem 6.15.10.

$$\begin{tikzcd}
\mathrm{kernel}(d,1)_* \arrow[d] & \subset & \pi_m^K(X_1 \vee X_0)_2 \arrow[d, "E_m"'] \arrow[dr, "(d,1)_*"] & \\
j\pi_{m+1}^K C_d & \subset & \pi_{m+1}^K(C_d, X_0) \arrow[r, "\partial"] \arrow[d, "H_m"'] & \pi_m^K(X_0) \\
 & & \pi_{m-1}^K(\Sigma X_1' \wedge X_1') \arrow[d, "P_{m-1}"'] & \\
 & & \pi_{m-1}^K(X_1 \vee X_0). &
\end{tikzcd}$$

Here $P_{m-1} = [i_1, i_1 - i_0 d]_*$ is induced by the Whitehead product

$$[i_1, i_1 - i_0 d]\colon \Sigma X_1' \wedge X_1' \to X_1 \vee X_0 \tag{1}$$

where i_ε is the inclusion of X_ε in $X_1 \vee X_0$ for $\varepsilon = 0, 1$. If $\dim(\Sigma^m X) < 3n-2$ then the kernel of E_m is the image of P_m. For $\dim(\Sigma^m X) = 3n-2$ the kernel of E_m is the image of

$$P_m\colon \pi_m^K(\Sigma X_1' \wedge X_1' \vee X_1)_2 \to \pi_m^K(X_1 \vee X_0)_2 \tag{2}$$

with $P_m = ([i_1, i_1 - i_0 d], -i_1)_*$. Hence we get by (6.15.11)

$$\Gamma T_m^K(A,n) = E_m\,\mathrm{kernel}(d,1)_* = \mathrm{kernel}(d,1)_* / \mathrm{image}\, P_m. \tag{3}$$

This formula can be used to compute the functor $\Gamma T_m^K(A,n)$ in Theorem 6.15.10. In fact for $\dim(\Sigma^m K) < 3n-2$ we can identify $(d,1)_*$ with ∂_1 as in the proof of (6.15.8); similarly we can identify P_m with ∂_2 in Definition 6.14.7. This shows

$$\Gamma T_m^K(A,n) = A *' \pi_m^K\{S^n\} \quad \text{for} \quad \dim(\Sigma^m K) < 3n-2. \tag{4}$$

Finally an easy diagram chase yields the exact sequence in Theorem 6.15.13 since the row and the column of the diagram above are exact. In fact the map e is induced by E_m and h is induced by H_m since the kernel of $P_{m-1} = \partial_2$ is

$$\text{kernel } P_{m-1} = A *'' \pi_{m-1}^K\{S^n\} = \text{image } H_m \tag{5}$$

by Definition 6.14.7(7). Moreover δ is induced by

$$\partial : \text{image } H_m \to \text{cokernel}(d,1)_*$$

in the diagram where the cokernel of $(d,1)_*$ is $\Gamma_m^K(A,n)$. The maps λ and j are considered in (6.15.11). □

Theorem 6.15.13 is slightly more general than the corresponding result (9.5) in Baues [QF] where we deal only with the quadratic part of Theorem 6.15.13. For the classification of 1-connected 5-dimensional homotopy types in Chapter 12 we also need the non-quadratic part of Theorem 6.15.13.

7

THE HOMOTOPY CATEGORY OF $(n-1)$-CONNECTED $(n+1)$-TYPES

We have to consider the hierarchy of categories and functors ($n \geq 2$)

$$\mathbf{types}_n^0 \xleftarrow{P} \mathbf{types}_n^1 \xleftarrow{P} \mathbf{types}_n^2 \leftarrow \cdots \tag{1}$$

where $\mathbf{types}_n^k$ is the full category of $(n-1)$-connected $(n+k)$-types, that is, of CW-spaces Y with $\pi_i Y = 0$ for $i < n$ and $i > n+k$. The functor P is the Postnikov functor which carries an $(n+k)$-type to its $(n+k-1)$-type, $k \geq 1$. Since $(n-1)$-connected n-types are the same as Eilenberg–Mac Lane spaces $K(A,n)$, we can identify them with abelian groups. In fact one has an equivalence of categories, see (6.1.1),

$$\pi_n : \mathbf{types}_n^0 \xrightarrow{\sim} \mathbf{Ab}. \tag{2}$$

From this point of view $(n-1)$-connected $(n+k)$-types are natural objects of higher complexity extending abstract abelian groups. Following up this idea J.H.C. Whitehead looked for a purely algebraic equivalent of an $(n-1)$-connected $(n+k)$-type, $k > 0$. An important requirement for such an algebraic system is 'realizability' in three senses. In the first instance this means that there is an $(n-1)$-connected $(n+k)$-type which is in the appropriate relation to a given one of these algebraic systems, just as there is an Eilenberg–Mac Lane space $K(A,n)$ whose nth homotopy group is isomorphic to the given abelian group A. The second kind is the realizability of morphisms between such algebraic systems by maps between the corresponding $(n+k)$-types, and the third kind is the uniqueness up to homotopy of such realizations of a given morphism. Thus we are searching for a category $\mathbf{C}$ of algebraic models equivalent to the category $\mathbf{types}_n^k$ as achieved in (2) above for $k = 0$. Given such a category $\mathbf{C}$ the computation of bype and kype functors on $\mathbf{C}$ would give us then algebraic models of homotopy types by use of the classification theorem in Chapter 3. J.H.C. Whitehead classified the objects in $\mathbf{types}_n^1$ by homomorphisms

$$\Gamma_n^1(A) \to B. \tag{3}$$

A suitable category of algebraic models, equivalent to $\mathbf{types}_n^1$, however, is not given in the literature. Using results on the category $\mathbf{M}^n$ of Moore spaces of degree n we shall describe such algebraic categories. This will be important for the classification of $(n-1)$-connected $(n+3)$-dimensional homotopy types, $n \geq 2$, in the following chapters.

7.1 A linear extension for $\mathbf{types}_n^1$

We show that the homotopy category $\mathbf{types}_n^1$ of $(n-1)$-connected $(n+1)$-types can be described as a linear extension of the category $\Gamma\mathbf{Ab}_n$ consisting of quadratic functions for $n=2$ and of stable quadratic functions for $n \geq 3$.

Recall that a function $\eta: A \to B$ between abelian groups is quadratic if $\eta(-a)=\eta(a)$ for $a \in A$ and if $[a,b]_\eta = \eta(a+b)-\eta(a)-\eta(b)$ is bilinear in $a,b \in A$. The universal quadratic function $\gamma: A \to \Gamma A$ has the property that there is a unique homomorphism $\eta^\square: \Gamma A \to B$ with $\eta^\square \gamma = \eta$. This way we identify a quadratic function $\eta: A \to B$ and a homomorphism $\eta^\square: \Gamma A \to B$. Let $\Gamma\mathbf{Ab}$ be the *category of quadratic functions*. Objects are quadratic functions and morphisms $\varphi = (\varphi_0, \varphi_1): \eta \to \eta'$ are pairs of homomorphisms for which the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi_0} & A' \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ B & \xrightarrow{\varphi_1} & B' \end{array} \tag{7.1.1}$$

commutes or for which equivalently $(\eta')^\square \Gamma(\varphi_0) = \varphi_1 \eta^\square$. A quadratic function $\eta: A \to B$ is *stable* if $[a,b]_\eta = 0$ for all $a,b \in A$, that is, η is a homomorphism with $\eta(2a)=0$ so that a stable function η can be identified with a homomorphism $\eta^\square: A \otimes \mathbb{Z}/2 \to B$. Let $\mathbf{S}\Gamma\mathbf{Ab} = \Gamma\mathbf{Ab}_n$ $(n \geq 3)$ be the full subcategory of $\Gamma\mathbf{Ab} = \Gamma\mathbf{Ab}_2$ consisting of stable functions. We have the full inclusion $\mathbf{Ab} \subset \Gamma\mathbf{Ab}$ which carries the abelian group A to the universal quadratic map γ_A: $A \to \Gamma A$. We also have the full inclusion $\mathbf{Ab} \subset \mathbf{S}\Gamma\mathbf{Ab}$ which carries A to the universal stable quadratic map $\sigma\gamma_A: A \to A \otimes \mathbb{Z}/2$. Here $\sigma\gamma_A$ is the quotient map.

(7.1.2) Remark The functor $\Gamma_n^1: \mathbf{Ab} \to \mathbf{Ab}$ is given by $\Gamma_2^1 = \Gamma$ and $\Gamma_n^1 = \otimes \mathbb{Z}/2$ for $n \geq 3$. The Grothendieck construction of the bifunctor

$$\mathrm{Hom}(\Gamma_n^1, -): \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

is the following category. Objects are homomorphisms f: $\Gamma_n^1(A) \to B$ with $A, B \in \mathbf{Ab}$ and morphisms (φ_0, φ_1): $f \to g$ are pairs of homomorphisms in $\mathbf{Ab}$ with $g\Gamma_n^1(\varphi_0) = \varphi_1 f$. There is a canonical isomorphism of categories

$$\Gamma\mathbf{Ab}_n = \mathbf{Gro}\big(\mathrm{Hom}(\Gamma_n^1, -)\big)$$

which carries the quadratic function η to $\eta^\square$.

Let $\eta \in \pi_3 S^2$ be the Hopf map and let $\eta_n \in \pi_{n+1} S^n$ be the $(n-2)$-fold suspension of η. Then for any space X in $\mathbf{Top}^*$ the induced function η_n^*: $\pi_n(X) \to \pi_{n+1}(X)$, $\eta_n^*(\alpha) = \alpha \circ \eta_n$, is quadratic; moreover η_n^* is stable for $n \geq 3$. Since η_n^* is natural in X we thus obtain the functor

$$k_n: \mathbf{types}_n^1 \to \Gamma\mathbf{Ab}_n$$

which carries an $(n-1)$-connected $(n+1)$-type X to $k_n(X) = \eta_n^*$ and which carries a map $f: X \to Y$ in $\mathbf{types}_n^1$ to the induced map $(\pi_n(f), \pi_{n+1}(f))$ between homotopy groups.

(7.1.3) Proposition *The functor k_n is a detecting functor, $n \geq 2$.*

Proof For X in $\mathbf{types}_n^1$ let $A = \pi_n X$ and $B = \pi_{n+1} X$. Then the Postnikov decomposition of X shows that X is the fibre of a classifying map

$$k_X: K(A,n) \to K(B, n+2)$$

which is the first *k-invariant* of X. The homotopy class of k_X is an element

$$k_X \in [K(A,n), K(B,n+2)] = H^{n+2}(K(A,n), B) = \mathrm{Hom}(\Gamma_n^1(A), B)$$

and thus k_X corresponds to a quadratic map $A \to B$ which actually is η_n^*. The second isomorphism is obtained by the universal coefficient theorem for cohomology since we have isomorphisms $(n \geq 2)$ $H_n K(A,n) = A$, $H_{n+1} K(A,n) = 0$, and $H_{n+2} K(A,n) = \Gamma_n^1(A)$. Hence each quadratic map $\eta \in \mathrm{Hom}\,(\Gamma_n^1 A, B)$ is realizable by a space X in types_n^1 with classifying map $k_X = \eta$. Moreover each morphism $\varphi: k_X \to k_Y$ in $\Gamma\mathbf{Ab}_n$ corresponds to a homotopy commutative diagram

$$\begin{array}{ccc} K(A,n) & \xrightarrow{\varphi_0} & K(A',n) \\ {\scriptstyle k_X}\big\downarrow & & \big\downarrow{\scriptstyle k_Y} \\ K(B,n+2) & \xrightarrow{\varphi_1} & K(B',n+2) \end{array}$$

which thus yields a principal map between fibre spaces, $\bar{\varphi}: X \to Y$, which realizes φ. □

(7.1.4) Remark In Definition 2.5.8 we describe a detecting functor $(n \geq 2)$

$$\lambda: \mathbf{types}_n^1 \to \mathbf{Gro}\big(\mathrm{Hom}(\Gamma_n^1, -)\big)$$

which by the identification of categories in Remark 7.1.2 coincides with the detecting functor k_n in Proposition 7.1.3 above; see also Theorem 6.4.1.

(7.1.5) Definition For each abelian group A we have the Eilenberg–Mac Lane space $K(A,n)$. Extending this notation we introduce *quadratic spaces* $K(\eta, n)$ as follows. Let $n \geq 2$ and let η: $A \to B$ be a quadratic function which is stable for $n \geq 3$. Then we write $X = K(\eta, n)$ if X is an $(n-1)$-connected $(n+1)$-type for which isomorphisms $A \cong \pi_n X$, $B \cong \pi_{n+1} X$ are fixed such that

$$\eta: A = \pi_n X \xrightarrow{\eta_n^*} \pi_{n+1} X = B$$

coincides with η_n^*. The proposition above shows that the homotopy type of $K(\eta, n)$ is well defined by η. Moreover each morphism $\varphi = (\varphi_0, \varphi_1)$: $\eta \to \eta'$ in $\Gamma\mathbf{Ab}$ has a realization $\bar{\varphi}$: $K(\eta, n) \to K(\eta', n)$ with $k_n \bar{\varphi} = \varphi$. Here $\bar{\varphi}$ is not uniquely determined by φ. The loop space of an Eilenberg–Mac Lane space is $\Omega K(A, n) = K(A, n-1)$, $n \geq 2$. Similarly we have for $n \geq 3$

$$\Omega K(\eta, n) = K(\eta, n-1)$$

where η is stable. Hence $K(\eta, 2)$ is a loop space if and only if η is stable. In the next result we classify maps.

(7.1.6) Proposition *Let η: $A \to B$ and η': $A' \to B'$ be quadratic functions which are stable for $n \geq 3$. Then we have for $n \geq 2$ the exact sequence*

$$\operatorname{Ext}(A, B') \overset{+}{\rightarrowtail} [K(\eta, n), K(\eta', n)] \overset{k_n}{\twoheadrightarrow} \Gamma\mathbf{Ab}(\eta, \eta').$$

This is an exact sequence of abelian groups if η' is stable. For $n = 2$ the group $\operatorname{Ext}(A, B')$ *acts freely on the set* $[K(\eta, 2), K(\eta', 2)]$ *with orbits given by k_2. Moreover for $n \geq 2$ we have the linear distributivity law*

$$(\bar{\psi} + \beta)(\bar{\varphi} + \alpha) = \bar{\psi}\bar{\varphi} + \psi_*(\alpha) + \varphi^*(\beta).$$

Here $\bar{\psi}\bar{\varphi}$ is the composite

$$K(\eta, n) \xrightarrow{\bar{\varphi}} K(\eta', n) \xrightarrow{\bar{\psi}} K(\eta'', n)$$

and $\alpha \in \operatorname{Ext}(A, B')$, $\beta \in \operatorname{Ext}(A', B'')$ *and we set* $\psi_* = (\psi_1)_*$, $\varphi^* = (\varphi_0)^*$.

We point out that the loop space functor Ω is compatible with the exact sequence in the proposition. Therefore we get a sequence of functors:

(7.1.7) $$\mathbf{types}_2^1 \xleftarrow{\Omega} \mathbf{types}_3^1 \underset{\sim}{\xleftarrow{\Omega}} \mathbf{types}_4^1 \underset{\sim}{\leftarrow} \cdots$$

where Ω on $\mathbf{types}_3^1$ is full and faithful and where Ω: $\mathbf{types}_n^1 \to \mathbf{types}_{n-1}^1$ is an equivalence of categories for $n \geq 4$. Hence we see that $\mathbf{types}_n^1$ for $n \geq 3$ is equivalent to the full subcategory of $\mathbf{types}_2^1$ consisting of all $K(\eta, n)$ for which η is stable. Moreover the proposition shows that we have a linear extension of categories

(7.1.8) $$E \overset{+}{\rightarrowtail} \mathbf{types}_n^1 \overset{k_n}{\twoheadrightarrow} \Gamma\mathbf{Ab}_n$$

where E is the bimodule on $\Gamma\mathbf{Ab}$ given by $E(\eta, \eta') = \operatorname{Ext}(A, B')$ for η': $A \to B$, η: $A' \to B'$. The functor k_n carries $K(\eta, n)$ to η.

For the group $\mathfrak{E}(K(\eta, n))$ of homotopy equivalences of the space $K(\eta, n)$ we obtain by (7.1.8) the extension of groups

(7.1.9) $$\operatorname{Ext}(A, B) \rightarrowtail \mathfrak{E}(K(\eta, n)) \twoheadrightarrow \operatorname{Aut}(\eta)$$

where $\mathrm{Aut}(\eta)$ is the group of automorphisms of the object η in $\Gamma\mathbf{Ab}$ with $\mathrm{Aut}(\eta) \subset \mathrm{Aut}(A) \times \mathrm{Aut}(B)$. The extension (7.1.9) is split if A is cyclic or $\mathrm{Ext}(A, \Gamma_n^1 A) * \mathbb{Z}/2 = 0$.

Proof of Proposition 7.1.6 We have the fibre sequence $(n \geq 2)$

$$\xrightarrow{\Omega\eta'} K(B', n+1) \to K(\eta', n) \xrightarrow{p} K(A', n) \xrightarrow{\eta'} K(B', n+2)$$

and the action

$$\mu: K(\eta', n) \times K(B', n+1) \to K(\eta', n)$$

on the fibre. For $X = K(\eta, n)$ we thus obtain the action

$$\mu_*: [X, K(\eta', n)] \times H^{n+1}(X, B') \to [X, K(\eta', n)]$$

which carries $(x, \bar{\alpha})$ to $\mu_*(x, \bar{\alpha})$. The projection $p: X = K(\eta, n) \to (A, n)$ induces the inclusion (see for example (3.3) in Baues [MHH])

$$p^*: \mathrm{Ext}(A, B') = H^{n+1}(K(A, n), B') \rightarrowtail H^{n+1}(X, B')$$

so that we can define the action in the proposition by

$$x + \alpha = \mu^*(x, p^*\alpha) \quad \text{for} \quad \alpha \in \mathrm{Ext}(A, B').$$

By (V.10.7)–(V.10.9) in Baues [AH] we see that all elements in

$$[K(\eta, n), K(\eta', n)] = \mathbf{PRIN}(\eta, \eta')$$

are given by principal maps between fibre spaces. Therefore the proposition is a very special case of (V.10.19) in Baues [AH]. □

For an abelian group A we have the universal quadratic function $\gamma_A = \gamma_A^2$: $A \to \Gamma A$ and the universal stable quadratic function $\sigma\gamma_A = \gamma_A^n$: $A \to A \otimes \mathbb{Z}/2$ which is the quotient map, $n \geq 3$. The $(n+1)$-type of a Moore space $M(A, n)$ is $K(\gamma_A^n, n)$ for $n \geq 2$ and we have

$$p_{n+1}: M(A, n) \to P_{n+1}M(A, n) = K(\gamma_A^n, n)$$

inducing the isomorphisms $\pi_n M(A, n) = A$ and $\pi_{n+1} M(A, n) = \Gamma_n^1(A)$. The Postnikov functor P_{n+1} yields a full embedding of homotopy categories

$$P_{n+1}: \mathbf{M}^n \subset \mathbf{types}_n^1$$

which is compatible with the linear extension in (7.1.8) and Section 1.3. That is, for $n \geq 2$ we have the commutative diagram of linear extensions

$$(7.1.10) \qquad \begin{array}{ccccc} \mathrm{Ext}(-, \Gamma_n^1) & \overset{+}{\rightarrowtail} & \mathbf{M}^n & \twoheadrightarrow & \mathbf{Ab} \\ \cap & & \cap & & \cap \\ E & \overset{+}{\rightarrowtail} & \mathbf{types}_n^1 & \twoheadrightarrow & \Gamma\mathbf{Ab}_n \end{array}$$

where the full inclusion $\mathbf{Ab} \subset \Gamma\mathbf{Ab}_n$ carries A to γ_A^n. In fact, we can describe the category $\mathbf{types}_n^1$ completely in terms of the category $\mathbf{M}^n$; for this we introduce the enriched category of Moore spaces in the next section.

(7.1.11) Remark For a CW-space X and $n \geq 2$ we have the natural isomorphism of groups

$$H^n(X, A) = [X, K(A, n)]$$

where the left-hand side is the reduced singular cohomology with coefficients in A and where $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$. Now let η be a stable quadratic function, that is a homomorphism η: $A \otimes \mathbb{Z}/2 \to B$. Then we define for a CW-space X the *cohomology*

$$H^n(X, \eta) = [X, K(\eta, n)], \quad n \geq 2,$$

with coefficients in η. This is an abelian group and a functor in X. The fibre sequence for $K(\eta, n)$ yields the following exact sequence of abelian groups where Sq_A^2: $H^m(X, A) \to H^{m+2}(X, A \otimes \mathbb{Z}/2)$ is the *Steenrod square* (which is trivial for $m = 1$):

$$H^{n-1}(X, A) \xrightarrow{\eta_* Sq_A^2} H^{n+1}(X, B) \to H^n(X, \eta)$$
$$\to H^n(X, A) \xrightarrow{\eta_* Sq_A^2} H^{n+2}(X, B).$$

One can check that $\eta_* Sq_A^2$ is also induced by the map η: $K(A, m) \to K(B, m+2)$ given by η. The cohomology $H^n(X, \eta)$ is not functorial in η but for φ: $\eta \to \eta'$ a map $\bar{\varphi}$: $K(\eta, n) \to K(\eta', n)$ induces a homomorphism $\bar{\varphi}_*$: $H^n(X, \eta) \to H^n(X, \eta')$ which is not well defined by φ. We clearly have

$$H^n(K(\eta, n), \eta') = [K(\eta, n), K(\eta', n)].$$

This group can also be computed by the proposition above. An algebraic description of this group is given in Theorem 7.2.9 below. The cohomology groups $H^n(X, \eta)$ were recently studied in quite different terms by Bullejos, Carrasco, and Cegarra [CC].

7.2 The enriched category of Moore spaces

The full homotopy category $\mathbf{M}^n$ of Moore spaces $M(A, n)$ is studied in Chapter 1. We here use the category $\mathbf{M}^n$ to obtain an algebraic description of the category $\mathbf{types}_n^1$ of $(n-1)$-connected $(n+1)$-types, $n \geq 2$. The linear extension of categories

(7.2.1) $$\mathrm{Ext}(-, \Gamma_n^1) \overset{+}{\rightarrowtail} \mathbf{M}^n \xrightarrow{H_n} \mathbf{Ab}$$

is given by the homology functor H_n and by the action $+$ of $\mathrm{Ext}(A, \Gamma_n^1(B))$ on the set of homotopy classes $[M(A, n), M(B, n)]$ where $\Gamma_n^1(B) = \pi_{n+1} M(B, n)$;

compare Section 1.3. We use the action in the following definition of the 'enriched' category of Moore spaces.

(7.2.2) Definition The *enriched category* $\Gamma\mathbf{M}^n$ *of Moore spaces* is defined as follows $(n \geq 2)$. Objects are pairs $(M(A,n), \eta)$ where η: $A \to B$ is a quadratic function which is stable for $n \geq 3$. A morphism

$$\{\varphi_1, \zeta, \overline{\varphi}_0\}: (M(A,n), \eta) \to (M(A',n), \eta')$$

in $\Gamma\mathbf{M}^n$, with η': $A' \to B'$, is represented by a tuple

$$\varphi_1 \in \mathrm{Hom}(B, B')$$

$$\zeta \in \mathrm{Ext}(A, B')$$

$$\overline{\varphi}_0 \in [M(A,n), M(A',n)]$$

where $\overline{\varphi}_0$ induces φ_0: $A \to A'$ in homology such that (φ_0, φ_1): $\eta \to \eta'$ is a morphism in $\Gamma\mathbf{Ab}$, that is $\eta' \varphi_0 = \varphi_1 \eta$. The equivalence class $\{\varphi_1, \zeta, \overline{\varphi}_0\}$ is given by the equivalence relation

$$(\varphi_1, \zeta, \overline{\varphi}_0) \sim (\varphi_1, \zeta + \eta'_* \delta, \overline{\varphi}_0 - \delta) \quad \text{for} \quad \delta \in \mathrm{Ext}(A, \Gamma_n^1 A').$$

Here we use the homomorphism η': $\Gamma_n^1 A' \to B'$ given by η' and we use the action $+$ of $\mathrm{Ext}(A, \Gamma_n^1 A')$ on the set $[M(A,n), M(A',n)]$ which yields $\overline{\varphi}_0 - \delta$. We define the composition in $\Gamma\mathbf{M}^n$ by

$$\{\psi_1, \zeta', \psi_0\}\{\varphi_1, \zeta, \overline{\varphi}_0\} = \left\{\psi_1\varphi_1, \psi_{1*}\zeta + \varphi_0^*\zeta', \overline{\psi}_0\overline{\varphi}_0\right\}.$$

Here $\overline{\psi}_0\overline{\varphi}_0$ is the composition of maps between Moore spaces. For $n \geq 3$ we obtain an additive structure of $\Gamma\mathbf{M}^n$ by the formula

$$\{\varphi_1, \zeta, \overline{\varphi}_0\} + \{\varphi'_1, \zeta', \overline{\varphi}'_0\} = \{\varphi_1 + \varphi'_1, \zeta + \zeta', \overline{\varphi}_0 + \overline{\varphi}'_0\}.$$

This shows that $\Gamma\mathbf{M}^n$ is an additive category for $n \geq 3$. Moreover the suspension yields the canonical isomorphism $\Gamma\mathbf{M}^n \cong \Gamma\mathbf{M}^{n+1}$ for $n \geq 3$.

We have the full inclusion of categories

$$\mathbf{M}^n \subset \Gamma\mathbf{M}^n \tag{7.2.3}$$

which carries $M(A,n)$ to $(M(A,n), \gamma_A)$ where γ_A: $A \to \Gamma_n^1 A$ is the universal quadratic map (stable for $n \geq 3$). Moreover the inclusion carries a map $\overline{\varphi}_0$ in $\mathbf{M}^n$ to the map $\{\Gamma(\varphi_0), 0, \overline{\varphi}_0\}$ in $\Gamma\mathbf{M}^n$. The linear extension (7.2.1) has the following generalization for the category $\Gamma\mathbf{M}^n$.

(7.2.4) Lemma *There is a linear extension of categories* $(n \geq 3)$

$$E \overset{+}{\rightarrowtail} \Gamma\mathbf{M}^n \xrightarrow{k_n} \Gamma\mathbf{Ab}_n$$

where E is the bimodule on $\Gamma\mathbf{Ab}$ *given by* $E(\eta, \eta') = \mathrm{Ext}(A, B')$ *as in* (7.1.8). *Moreover k_n is the forgetful functor which carries the object* $(M(A,n), \eta)$ *to* η.

The action + of E is given in the obvious way by $\{\varphi_1, \zeta, \overline{\varphi}_0\} + \alpha = \{\varphi_1, \zeta + \alpha, \overline{\varphi}_0\}$.

The lemma is readily obtained by use of the linear extension (7.2.1). Moreover a cocycle for the extension (7.2.1) yields a cocycle which determines the extension $\Gamma\mathbf{M}^n$ in the lemma.

(7.2.5) Definition Let $\mathbf{spaces}_n$ be the full subcategory of $(n-1)$-connected CW-spaces in $\mathbf{Top}^*/\simeq$. We define a functor $(n \geq 2)$

$$K_n\colon \mathbf{spaces}_n \to \Gamma\mathbf{M}^n$$

as follows. For an $(n-1)$-connected CW-space X let $A = H_nX = \pi_nX$ and let $\eta = \eta_n^*\colon A = \pi_nX \to \pi_{n+1}X = B$ be induced by the suspended Hopf map. Then K_n carries X to the object $(M(A,n), \eta)$. We now choose for each X a map

$$\alpha\colon M(A,n) \to X \tag{1}$$

which induces the identity $H_n(\alpha)$ of $A = H_nX$. For a map $F\colon X \to X'$ between $(n-1)$-connected CW-spaces we thus obtain the diagram

$$\begin{array}{ccc} M(A,n) & \xrightarrow{\alpha} & X \\ {\scriptstyle\overline{\varphi}_0}\downarrow & & \downarrow{\scriptstyle F} \\ M(A',n) & \xrightarrow{\alpha} & X' \end{array} \tag{2}$$

where $\overline{\varphi}_0$ realizes the homomorphism $\varphi_0 = H_nF\colon A \to A'$. Thus the diagram commutes if we apply the homology functor H_n, but $\overline{\varphi}_0$ in general cannot be chosen such that the diagram actually commutes in $\mathbf{Top}^*/\simeq$. Let $\eta'\colon A' = H_nX' = \pi_nX' \to B = \pi_{n+1}X'$ be given by X'. Then

$$O(\overline{\varphi}_0) = \Delta^{-1}(F\alpha - a\overline{\varphi}_0) \in \operatorname{Ext}(A, B') \tag{3}$$

is determined by the universal coefficient sequence for $[M(A,n), X'] = \pi_n(A, X')$. We now define the functor K_n on $F\colon X \to X'$ by

$$K_n(F) = \{\varphi_1, O(\overline{\varphi}_0), \overline{\varphi}_0\}\colon (M(A,n), \eta) \to (M(A',n), \eta') \tag{4}$$

where $\varphi_1 = \pi_{n+1}(F)$.

(7.2.6) Lemma K_n *is a well-defined functor.*

Proof The definition of $K_n(F)$ depends on the choice of $\overline{\varphi}_0$. A different choice is of the form $\overline{\varphi}_0 - \delta$ with $\delta \in \operatorname{Ext}(A, \Gamma_n^1 A')$. Now we get

$$\begin{aligned} O(\overline{\varphi}_0 - \delta) &= \Delta^{-1}(F\alpha - \alpha(\overline{\varphi}_n - \delta)) \\ &= O(\overline{\varphi}_0) + \eta_*\delta. \end{aligned}$$

This shows that we have an equivalence

$$(\varphi_1, O(\overline{\varphi}_0), \overline{\varphi}_0) \sim (\varphi_1, O(\overline{\varphi}_0 - \delta), \overline{\varphi}_0 - \delta)$$

and therefore $K_n(F)$ is well defined. Now it is easy to check that K_n is a well-defined functor which clearly depends on the choice of the maps α above. □

We now restrict the functor K_n to the category of $(n-1)$-connected $(n+1)$-types.

(7.2.7) Theorem *The functor K_n above yields for $n \geq 2$ an equivalence of categories*

$$K_n : \mathbf{types}_n^1 \xrightarrow{\sim} \Gamma\mathbf{M}^n.$$

Proof The functor K_n induces a map between linear extensions of categories

$$\begin{array}{ccccc} E & \overset{+}{\rightarrowtail} & \mathbf{types}_n^1 & \xrightarrow{k_n} & \Gamma\mathbf{Ab}_n \\ \| & & \downarrow{\scriptstyle K_n} & & \| \\ E & \overset{+}{\rightarrowtail} & \Gamma\mathbf{M}^n & \xrightarrow[k_n]{} & \Gamma\mathbf{Ab}_n \end{array}$$

This implies that K_n is an equivalence of categories. □

The theorem shows that we can use all results on the category $\mathbf{M}^n$ for computation in the category $\mathbf{types}_n^1$. In particular since we have algebraic models of $\mathbf{M}^n$ we obtain in this way algebraic categories equivalent to the category $\mathbf{types}_n^1$. For example we have for $n \geq 3$ the equivalence of categories $\mathbf{M}^n = \mathbf{G}$ where $\mathbf{G}$ is the algebraic category in Section 1.6. Using this equivalence we obtain in the same way as in Definition 7.2.2 the enriched category $\Gamma\mathbf{G}$ as follows.

(7.2.8) Definition Recall that we have for each abelian group A the extension in $\mathbf{Ab}$

$$A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \overset{\mu}{\twoheadrightarrow} A * \mathbb{Z}/2$$

associated with $A * \mathbb{Z}/2 \subset A \to A \otimes \mathbb{Z}/2$. Objects in the *enriched category* $\Gamma\mathbf{G}$ are stable quadratic functions η: $A \to B$ or equivalently homomorphisms η: $A \otimes \mathbb{Z}/2 \to B$ with $A, B \in \mathbf{Ab}$. Let η': $A' \to B'$ be a further object in $\Gamma\mathbf{G}$. A morphism

$$\{\varphi_1, \zeta, \varphi_0, \bar{\varphi}_0\} : \eta \to \eta' \tag{1}$$

in $\Gamma\mathbf{G}$ is represented as follows. Let $\zeta \in \mathrm{Ext}(A, B')$ and let $\varphi_1, \varphi_0, \overline{\varphi}_0$ be homomorphisms in $\mathbf{Ab}$ for which the following diagram commutes.

$$\begin{array}{ccccccc} B & \xleftarrow{\eta} & A \otimes \mathbb{Z}/2 & \overset{\Delta}{\rightarrowtail} & G(A) & \overset{\mu}{\twoheadrightarrow} & A * \mathbb{Z}/2 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0 \otimes 1} & & \downarrow{\scriptstyle \overline{\varphi}_0} & & \downarrow{\scriptstyle \varphi_0 * 1} \\ B' & \xleftarrow[\eta']{} & A' \otimes \mathbb{Z}/2 & \underset{\Delta}{\rightarrowtail} & G(A') & \underset{\mu}{\twoheadrightarrow} & A' * \mathbb{Z}/2 \end{array} \tag{2}$$

Then the equivalence class $\{\varphi_1, \zeta, \varphi_0, \overline{\varphi}_0\}$ is given by the relation

$$(\varphi_1, \zeta, \varphi_0, \overline{\varphi}_0) \sim (\varphi_1, \zeta + \eta'_* \delta, \varphi_0, \overline{\varphi}_0 - \Delta\delta\mu) \tag{3}$$

for $\delta \in \mathrm{Ext}(A, A' \otimes \mathbb{Z}/2) = \mathrm{Hom}(A * \mathbb{Z}/2, A' \otimes \mathbb{Z}/2)$. Composition is defined by

$$\{\varphi_1, \zeta, \varphi_0, \overline{\varphi}_0\} \circ \{\varphi'_1, \zeta', \varphi'_0, \overline{\varphi}'_0\} = \{\varphi_1\varphi'_1, (\varphi_1)_* \zeta + (\varphi_0)^* \zeta', \varphi_0\varphi'_0, \overline{\varphi}_0\overline{\varphi}'_0\}.$$

Also we obtain the structure of an additive category for $\Gamma\mathbf{G}$ as in Definition 7.2.2 by

$$\{\varphi_1, \zeta, \varphi_0, \overline{\varphi}_0\} + \{\varphi''_1, \zeta'', \varphi''_0, \overline{\varphi}''_0\} = \{\varphi_1 + \varphi''_1, \zeta + \zeta'', \varphi_0 + \varphi''_0, \overline{\varphi}_0 + \overline{\varphi}''_0\}.$$

Recall that $\mathbf{S\Gamma Ab}$ is the category of stable quadratic functions. We obtain as in (7.1.8) a linear extension of categories.

$$E \overset{+}{\rightarrowtail} \Gamma\mathbf{G} \overset{\phi}{\twoheadrightarrow} \mathbf{S\Gamma Ab}. \tag{4}$$

Here ϕ is the identity on objects and carries (1) to $(\varphi_1, \varphi_0)\colon \eta \to \eta'$ in $\mathbf{S\Gamma Ab}$. The action of E is given for $\alpha \in \mathrm{Ext}(A, B') = E(\eta, \eta')$ by

$$\{\varphi_1, \zeta, \varphi_0, \overline{\varphi}_0\} + \alpha = \{\varphi_1, \zeta + \alpha, \varphi_0, \overline{\varphi}_0\}.$$

One readily checks that (4) is a well-defined linear extension. We derive from Theorem 7.2.7 the next result which provides us with an algebraic model of the category $\mathbf{types}^1_n$ for $n \geq 3$.

(7.2.9) Theorem *For $n \geq 3$ one has equivalences of additive categories*

$$\mathbf{types}^1_n = \Gamma\mathbf{M}^n = \Gamma\mathbf{G}$$

which are compatible with the linear extensions in the proof of Theorem 7.2.7 and Definition 7.2.8(4).

Using the equivalence we identify a map $K(\eta, n) \to K(\eta', n)$, $n \geq 3$, with an algebraic morphism $\eta \to \eta'$ in $\Gamma\mathbf{G}$.

8

ON THE HOMOTOPY CLASSIFICATION OF $(n-1)$-CONNECTED $(n+3)$-DIMENSIONAL POLYHEDRA, $n \geq 4$

J.H.C. Whitehead in 1948 classified $(n-1)$-connected $(n+2)$-dimensional homotopy types. Since then it has been a challenging problem to consider the next step in the classification of $(n-1)$-connected $(n+3)$-dimensional homotopy types. Various authors have worked on this problem in the stable range $n \geq 4$, for example Shiraiwa [HT], Uehara [HT], Chang [PI], [HT], [AS], [NH], and Chow [HG]. They use a complicated method of primary and secondary cohomology operations; compare Section 8.5 below. We here apply the new method of boundary invariants which, via the classification theorem 3.4.4, yields fairly simple classifying data for $(n-1)$-connected $(n+3)$-dimensional homotopy types. In this chapter we deal only with the stable case $n \geq 4$. In Chapters 9 and 12 we describe the considerably more intricate unstable cases, $n = 3$ and $n = 2$.

8.1 Algebraic models of $(n-1)$-connected $(n+3)$-dimensional homotopy types, $n \geq 4$

We introduce the purely algebraic category of A^3-systems. Then we formulate the main result of this chapter which shows that A^3-systems are algebraic models of certain homotopy types. Recall that we have for an abelian group A the short exact sequence

$$(8.1.1) \qquad A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \overset{\mu}{\twoheadrightarrow} A * \mathbb{Z}/2$$

associated with

$$\tau_A \colon A * \mathbb{Z}/2 \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} A \otimes \mathbb{Z}/2.$$

Here i is the inclusion and p is the projection. The abelian extension (8.1.1) is determined up to equivalence by the property $\Delta^{-1}(2 \cdot \mu^{-1}(x)) = \tau_A(x)$ for $x \in A * \mathbb{Z}/2$. For each homomorphism $\varphi\colon A \to B$ there is a homomorphism $\bar{\varphi}\colon G(A) \to G(B)$ compatible with Δ and μ in (8.1.1), that is $\mu\bar{\varphi} = (\varphi * \mathbb{Z}/2)\mu$

and $\Delta(\varphi \otimes \mathbb{Z}/2) = \overline{\varphi}\Delta$. We have the dual extension, see Lemma 8.2.7,

(8.1.2)

$$\begin{array}{ccccc} \mathrm{Ext}(A,\mathbb{Z}/2) & \overset{\Delta}{\rightarrowtail} & \mathrm{Hom}(G(A),\mathbb{Z}/4) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A,\mathbb{Z}/2) \\ \| & & \| & & \| \\ \mathrm{Hom}(A * \mathbb{Z}/2,\mathbb{Z}/4) & \xrightarrow{\mu} & \mathrm{Hom}(G(A),\mathbb{Z}/4) & \xrightarrow{\Delta^*} & \mathrm{Hom}(A \otimes \mathbb{Z}/2,\mathbb{Z}/4) \end{array}$$

Here the bottom row is obtained by applying the functor $\mathrm{Hom}(-,\mathbb{Z}/4)$ to extension (8.1.1). The isomorphism at the left-hand side is given by Lemma 1.6.3. We now use the extension (8.1.1) and (8.1.2) for the definition of the following extensions $G(\eta)$ and $\overline{G}(A,\eta)$ respectively.

(8.1.3) (A) Definition For a homomorphism η: $H \otimes \mathbb{Z}/2 \to L$ in **Ab** let $G(\eta)$ be defined by the push-out diagram in **Ab**

$$\begin{array}{ccccc} L \otimes \mathbb{Z}/2 & \overset{\Delta}{\rightarrowtail} & G(\eta) & \overset{\mu}{\twoheadrightarrow} & H * \mathbb{Z}/2 \\ {\scriptstyle \eta \otimes 1}\uparrow & \text{push} & \uparrow{\scriptstyle \overline{\eta}} & & \| \\ H \otimes \mathbb{Z}/2 & \rightarrowtail & G(H) & \twoheadrightarrow & H * \mathbb{Z}/2 \end{array}$$

where the bottom row is given by (8.1.1). Hence $G(\eta)$ in the short exact top row is the abelian extension associated with the homomorphism

$$\tau_L \eta \tau_H : H * \mathbb{Z}/2 \subset H \twoheadrightarrow H \otimes \mathbb{Z}/2 \xrightarrow{\eta} L * \mathbb{Z}/2 \subset L \twoheadrightarrow L \otimes \mathbb{Z}/2.$$

(8.1.3) (B) Definition Recall that objects in the category **G** are abelian groups A and morphisms $\overline{\varphi} = (\varphi, \overline{\varphi})$: $A \to B$ are pairs

$$(\varphi, \overline{\varphi}) \in \mathrm{Hom}(A, B) \times \mathrm{Hom}(G(A), G(B))$$

compatible with Δ and μ in (8.1.1); see Section 1.6. Next let

$$\mathbf{S\Gamma Ab}' \subset \mathbf{S\Gamma Ab}$$

be the full subcategory of stable quadratic functions η: $H \otimes \mathbb{Z}/2 \to L$ for which there exists a factorization η: $H \otimes \mathbb{Z}/2 \rightarrowtail G(H) \to L$; see (7.1.1).

Morphisms (ψ_1, ψ_0): $\eta \to \eta'$ with η': $H' \otimes \mathbb{Z}/2 \to L'$ are pairs $(\psi_1, \psi_0) \in \mathrm{Hom}(L, L') \oplus \mathrm{Hom}(H, H')$ compatible with η and η'. We define an algebraic functor

$$\bar{G}: \mathbf{G}^{\mathrm{op}} \times \mathbf{S\Gamma Ab}' \to \mathbf{Ab}$$

as follows. If A or H is finitely generated let $\bar{G}(A, \eta)$ be given by the push-out diagram in **Ab**

$$\begin{tikzcd}
\mathrm{Ext}(A, L) \arrow[r, "\Delta"] & \bar{G}(A, \eta) \arrow[r, "\mu"] & \mathrm{Hom}(A, H \otimes \mathbb{Z}/2) \arrow[r] & 0 \\
\mathrm{Ext}(A, H \otimes \mathbb{Z}/2) \arrow[u, "\eta_*"] \arrow[ur, phantom, "\text{push}"] \arrow[d, equal] & & & \\
\mathrm{Ext}(A, \mathbb{Z}/2) \otimes H \arrow[r] & \mathrm{Hom}(G(A), \mathbb{Z}/4) \otimes H \arrow[r] \arrow[uu, "\bar{\eta}_*"] & \mathrm{Hom}(A, \mathbb{Z}/2) \otimes H \arrow[r] \arrow[uu, equal] & 0
\end{tikzcd}$$

Here the bottom row is obtained by applying the functor $-\otimes H$ to the extension (8.1.2). Induced homomorphisms are defined by

$$(\varphi, \bar{\varphi})^* = \mathrm{Ext}(\varphi, L) \oplus \mathrm{Hom}(\bar{\varphi}, \mathbb{Z}/4) \otimes H$$

$$(\psi_1, \psi_0)_* = \mathrm{Ext}(A, \psi_1) \oplus \mathrm{Hom}(G(A), \mathbb{Z}/4) \otimes \psi_0.$$

(8.1.3) (C) Addendum In the general case we have to use the following more intricate definition of the group $\bar{G}(A, \eta)$ which canonically coincides with the definition (8.1.3) (B) if A or H are finitely generated. For $n \geq 0$ let $\mathbb{Z}/n[H]$ be the free $\mathbb{Z}/n$-module generated by the set H. We have the canonical map

$$p_n: \mathbb{Z}/n[H] \to H \otimes \mathbb{Z}/n$$

which carries $\Sigma_i \alpha_i[x_i]$ with $\alpha_i \in \mathbb{Z}/n, x_i \in H$ to the corresponding sum $\Sigma_i x_i \otimes \alpha_i$. Clearly p_n is a surjective homomorphism which is natural in $H \in \mathbf{Ab}$. Let $K(H)$ be the kernel

$$K(H) = \mathrm{kernel}\{\mathbb{Z}/4[H] \xrightarrow{p} \mathbb{Z}/2[H] \xrightarrow{p_2} H \otimes \mathbb{Z}/2\}$$

where p is reduction modulo 2. Naturality shows that K is a functor $\mathbf{Ab} \to \mathbf{Ab}$. We now define the natural transformation

$$\theta_H: K(H) * \mathbb{Z}/2 \to \mathrm{cok}(\tau_H)$$

where τ_H is defined as in (8.1.1). For $y = \Sigma_i \alpha_i[x_i] \in K(H) * \mathbb{Z}/2$ with $\alpha_i \in \mathbb{Z}$ there is $x \in H$ such that $\Sigma_i \alpha_i x_i = 2x$ in H. Now θ carries y to the element in $\operatorname{cok}(\tau_H)$ represented by x. One readily checks that θ is a well-defined homomorphism natural in $H \in \mathbf{Ab}$. In fact, θ_H is obtained by the following commutative diagram

$$\begin{tikzcd}
K(H) \arrow[r, tail] \arrow[d, "\theta''"] & \mathbb{Z}/4[H] \arrow[d, "\theta'"] \arrow[dr, "p_2 p"] & \\
\operatorname{cok}\tau_H \arrow[r, tail] & H \otimes \mathbb{Z}/4 \arrow[r, two heads] & H \otimes \mathbb{Z}/2
\end{tikzcd}$$

in which the bottom row is short exact. The map θ' carries $[h]$ to $h \otimes 1$ and the restriction of θ'' to $K(H) * \mathbb{Z}/2$ is θ_H. For abelian groups A, H we define $\Delta_* \operatorname{Ext}(A, H \otimes \mathbb{Z}/2)$ by the image of

$$\Delta_* : \operatorname{Ext}(A, H \otimes \mathbb{Z}/2) \to \operatorname{Ext}(A, G(H))$$

induced by Δ in (8.1.1). Clearly $\Delta_* \operatorname{Ext}(A, H \otimes \mathbb{Z}/2)$ is functorial in $A, H \in$ **Ab**. If K is a $\mathbb{Z}/2$-vector space we get

$$\Delta_* \operatorname{Ext}(K, H \otimes \mathbb{Z}/2) = \operatorname{Hom}(K, \operatorname{cok}\tau_H).$$

Using $K = K(H) * \mathbb{Z}/2$ and θ_H above we thus obtain a homomorphism

$$\begin{cases} \theta : \operatorname{Hom}(A * \mathbb{Z}/2, K(H) * \mathbb{Z}/2) \to \Delta_* \operatorname{Ext}(A, H \otimes \mathbb{Z}/2) \\ \theta(\alpha) = \alpha^* \theta_H \end{cases}$$

which is natural in A and H. We are now ready to define $\bar{G}(A, \eta)$ by the push-out diagram:

$$\begin{tikzcd}
\operatorname{Hom}(A * \mathbb{Z}/2, K(H) * \mathbb{Z}/2) \arrow[r, tail, "j"] \arrow[d, "\theta"] & \operatorname{Hom}(G(A), \mathbb{Z}/4[H]) \arrow[r, two heads, "\Delta^*"] \arrow[dd] & \operatorname{Hom}(A \otimes \mathbb{Z}/2, \mathbb{Z}/4[H]) \arrow[d, equal] \\
\Delta_* \operatorname{Ext}(A, H \otimes \mathbb{Z}/2) \arrow[d, "\eta_*"] \arrow[r, phantom, "\text{push}"] & & \operatorname{Hom}(A, \mathbb{Z}/2[H]) \arrow[d, "(p_2)_*"] \\
\operatorname{Ext}(A, L) \arrow[r, tail, "\Delta"] & \bar{G}(A, \eta) \arrow[r, two heads, "\mu"] & \operatorname{Hom}(A, H \otimes \mathbb{Z}/2)
\end{tikzcd}$$

Here η_* is well defined since η factors through Δ: $H \otimes \mathbb{Z}/2 \to G(H)$. The inclusion j carries α to the composition

$$j(\alpha) : G(A) \xrightarrow{\mu} A * \mathbb{Z}/2 \xrightarrow{\alpha} K(H) * \mathbb{Z}/2 \subset \mathbb{Z}/4[H].$$

We observe that image(j) = kernel$(p_2)_* \Delta^*$, so that the bottom row is short exact since $(p_2)_* \Delta^*$ is surjective. Induced maps on $\overline{G}(A, \eta)$ are now defined by

$$(\varphi, \overline{\varphi})^* = \mathrm{Ext}(\varphi, L) \oplus \mathrm{Hom}(\overline{\varphi}, \mathbb{Z}/4[H])$$
$$(\psi_1, \psi_0)_* = \mathrm{Ext}(A, \psi_1) \oplus \mathrm{Hom}(G(A), \mathbb{Z}/4[\psi_0]).$$

This completes the definition of the functor $\overline{G}$ in Definition 8.1.3 (B).

Using the notation on the groups $G(\eta)$ and $\overline{G}(A, \eta)$ in Definitions 8.1.3 we are now ready to define algebraic models of $(n-1)$-connected $(n+3)$-dimensional homotopy types which we call A^3-systems.

(8.1.4) Definition An A^3*-system*

$$S = (H_0, H_2, H_3, \pi_1, b_2, \eta, b_3, \beta) \tag{1}$$

is a tuple consisting of abelian groups H_0, H_2, H_3, π_1 and elements

$$\begin{aligned} &b_2 \in \mathrm{Hom}(H_2, H_0 \otimes \mathbb{Z}/2), \\ &\eta \in \mathrm{Hom}(H_0 \otimes \mathbb{Z}/2, \pi_1) \\ &b_2 \in \mathrm{Hom}(H_3, G(\eta)), \\ &\beta \in \overline{G}(H_2, \eta_\#). \end{aligned} \tag{2}$$

Here $\eta_\# = q\Delta(\eta \otimes 1)$ is the composition

$$\eta_\#: H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta \otimes 1} \pi_1 \otimes \mathbb{Z}/2 \xrightarrow{\Delta} G(\eta) \overset{q}{\twoheadrightarrow} \mathrm{cok}(b_3) \tag{3}$$

where q is the quotient map for the cokernel of b_3. These elements satisfy the following conditions (4) and (5). The sequence

$$H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1 \tag{4}$$

is exact and β satisfies

$$\mu(\beta) = b_2 \tag{5}$$

where μ is the operator on $\overline{G}$ in Definition 8.1.3. A morphism

$$(\varphi_0, \varphi_2, \varphi_3, \varphi_\pi, \varphi_\Gamma): S \to S' \tag{6}$$

between A^3-systems is a tuple of homomorphisms

$$\begin{cases} \varphi_i: H_i \to H_i' & (i = 0, 2, 3) \\ \varphi_\pi: \pi_1 \to \pi_1' \\ \varphi_\Gamma: G(\eta) \to G(\eta') \end{cases}$$

such that the following diagrams (7), (8), (9) commute and such that the equation 10 holds.

$$\begin{tikzcd} H_2 \arrow[r, "b_2"] \arrow[d, "\varphi_2"] & H_0 \otimes \mathbb{Z}/2 \arrow[r, "\eta"] \arrow[d, "\varphi_0 \otimes 1"] & \pi_1 \arrow[d, "\varphi_\pi"] \\ H_2' \arrow[r, "b_2'"'] & H_0' \otimes \mathbb{Z}/2 \arrow[r, "\eta'"] & \pi_1' \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} \pi_1 \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] \arrow[d, "\varphi_\pi \otimes 1"] & G(\eta) \arrow[r, two heads, "\mu"] \arrow[d, "\varphi_\Gamma"] & H_0 * \mathbb{Z}/2 \arrow[d, "\varphi_0 * 1"] \\ \pi_1' \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] & G(\eta') \arrow[r, two heads, "\mu"] & H_0' * \mathbb{Z}/2 \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} H_3 \arrow[r, "b_3"] \arrow[d, "\varphi_3"] & G(\eta) \arrow[d, "\varphi_\Gamma"] \\ H_3' \arrow[r, "b_3'"'] & G(\eta') \end{tikzcd} \tag{9}$$

Hence φ_Γ induces φ_Γ: $\operatorname{cok}(b_3) \to \operatorname{cok}(b_3')$ such that $(\varphi_0, \varphi_\Gamma)$: $q\Delta(\eta \otimes 1) \to q\Delta(\eta' \otimes 1)$ is a morphism in $\mathbf{S\Gamma Ab}'$ which induces $(\varphi_0, \varphi_\Gamma)_*$ as in Definition 8.1.3. We have

$$(\varphi_0, \varphi_\Gamma)_*(\beta) = (\varphi_2, \overline{\varphi}_2)^*(\beta') \tag{10}$$

in $\overline{G}(H_2, q\Delta(\eta' \otimes 1))$. In (10) we choose $\overline{\varphi}_2$ for φ_2 as in (8.1.1). The right-hand side of (10) does not depend on the choice of $\overline{\varphi}_2$.

An A^3-system S as above is *free* if H_3 is free abelian, and S is *injective* if b_3: $H_3 \to G(\eta)$ is injective. Let A^3-**System** resp. A^3-**system** be the full category of free, resp. injective, A^3-systems. We have the canonical forgetful functor

$$\phi: A^3\text{-}\mathbf{System} \to A^3\text{-}\mathbf{system} \tag{11}$$

which replaces b_3: $H_3 \to G(\eta)$ by the inclusion $b_3(H_3) \subset G(\eta)$ of the image of b_3. One readily checks that this forgetful ϕ is full and representative.

(8.1.5) Definition We associate with an A^3-system S as in Definition 8.1.4 the exact Γ-*sequence*

$$H_3 \xrightarrow{b_3} G(\eta) \to \pi_2 \to H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1 \to H_1 \to 0.$$

Here $H_1 = \operatorname{cok}(\eta)$ is the cokernel of η and the extension

$$\operatorname{cok}(b_3) \rightarrowtail \pi_2 \twoheadrightarrow \ker(b_2) \tag{1}$$

is obtained by the element β in Definition 8.1.4, that is, the group $\pi_2 = \pi(\beta_\dagger)$ is given by the extension element $\beta_\dagger \in \mathrm{Ext}(\ker(b_2), \mathrm{cok}(b_3))$ defined by

$$\beta_\dagger = \Delta^{-1}(j, \bar{j})^*(\beta). \tag{2}$$

Here j: $\ker(b_2) \subset H_2$ is the inclusion. The element $\beta_\dagger$ does not depend on the choice of $(j, \bar{j})$ in $\mathbf{G}$. Compare (2.6.7).

Recall that $\mathbf{spaces}_n^3$ denotes the full homotopy category of $(n-1)$-connected $(n+3)$-dimensional CW-spaces X and that $\mathbf{types}_n^2$ is the full homotopy category of $(n-1)$-connected $(n+2)$-types. We have the Postnikov functor

$$P: \mathbf{spaces}_n^3 \to \mathbf{types}_n^2$$

which carries X to its $(n+2)$-type.

(8.1.6) Theorem *In the stable range $n \geq 4$ there are detecting functors:*

$$\Lambda': \mathbf{spaces}_n^3 \to A^3\text{-}\mathbf{System}$$

$$\lambda': \mathbf{types}_n^2 \to A^3\text{-}\mathbf{system}.$$

Moreover there is a natural isomorphism $\phi\lambda(X) = \lambda' P(X)$ for the forgetful functor ϕ in Definition 8.1.4(11) *and for the Postnikov functor P above.*

Remark Theorem 8.1.6 is an application of the detecting functors Λ', λ' in the classification theorem 3.4.4. These detecting functors are defined by boundary invariants. There should also be a similar result to Theorem 8.1.6 above concerning the detecting functors Λ, λ in Theorem 3.4.4 given by k-invariants. It turns out, however, that the computation of boundary invariants is simpler than the corresponding computation of k-invariants. The analogue of Theorem 8.1.6 for k-invariants remains an open problem though for $\pi_{n+1} = 0$ we have discussed the functors Λ, λ in Theorem 3.6.5; for $\pi_{n+1} = 0$ Theorem 8.1.6 above corresponds exactly to the detecting functor Λ' in Theorem 3.6.5. The result there was based on the computation of $H_{n+3}K(B,n)$ and $H_{n+2}(A, K(B,n))$. In Theorem 8.1.6 we treat the case $\pi_{n+1}X \neq 0$ which is based on the computation of $H_{n+3}K(\eta, n)$ and $\Gamma_{n+1}(A, K(\eta, n))$ in Section 8.3 below.

Let S be a free A^3-system. The detecting functor Λ' in Theorem 8.1.6 shows that there is a unique $(n-1)$-connected $(n+3)$-dimensional homotopy type $X = X_S, n \geq 4$, with $\Lambda'(X) \cong S$. Then the Γ-sequence for S is the top row of the following commutative diagram

(8.1.7)

$$\begin{array}{ccccccccccccc}
H_3 & \to & G(\eta) & \to & \pi_2 & \to & H_2 & \to & H_0 \otimes \mathbb{Z}/2 & \xrightarrow{\eta} & \pi_1 & \twoheadrightarrow & H_1 \\
\| & & \| & & \downarrow\cong & & \| & & \| & & \| & & \| \\
H_{n+3}X & \to & \Gamma_{n+2}X & \to & \pi_{n+2}X & \to & H_{n+2}X & \to & \Gamma_{n+1}X & \longrightarrow & \pi_{n+1}X & \twoheadrightarrow & H_{n+1}X
\end{array}$$

The bottom row is Whitehead's certain exact sequence for X. The diagram describes a weak natural isomorphism of exact sequences. This shows that the homotopy group $\pi_{n+2}(X)$ is completely understood in terms of the A^3-system S.

(8.1.8) Example For an abelian group A let S_A be the unique A^3-system with $H_0 = A$ and $H_1 = H_2 = H_3 = 0$. That is

$$S_A = \begin{pmatrix} H_0, H_2, H_3, \pi_1, b_2, \eta, b_3, \beta \\ A, 0, 0, A \otimes \mathbb{Z}/2, 0, 1, 0, 0 \end{pmatrix}.$$

Then the space X with $\Lambda'(X) \cong S_A$ is the Moore space $X = M(A, n), n \geq 4$.

As an application we now derive from Theorem 8.1.6 the following result on maps into spheres.

(8.1.9) Theorem *Let X_S be the $(n-1)$-connected $(n+3)$-dimensional homotopy type associated with the A^3-system S in Definition* 8.1.4 *with $n \geq 4$. Then a homomorphism φ_0: $H_0 = H_n(X_S) \to H_n(S^n) = \mathbb{Z}$ is realizable by a map $X_S \to S^n$ if and only if there exist homomorphisms*

$$\varphi_\pi : \pi_1 = \pi_{n+1}(X_S) \to \mathbb{Z}/2 \quad \text{and} \quad \varphi_\Gamma : \Gamma(\eta) = \Gamma_{n+2}(X_S) \to \mathbb{Z}/2$$

such that $\varphi_\pi \eta = \varphi_0 \otimes \mathbb{Z}/2$, $(\varphi_0 \otimes 1) b_2 = 0$, $\varphi_\Gamma \Delta = \varphi_\pi \otimes 1$, $\varphi_\Gamma b_3 = 0$, *and* $(\varphi_0, \varphi_\Gamma)_*(\beta) = 0$.

Proof The conditions show that

$$(\varphi_0, 0, 0, \varphi_\pi, \varphi_\Gamma): S \to S_{\mathbb{Z}}$$

is a morphism between A^3-systems which is realizable by a map $X_S \to S^n$ since Λ' in Theorem 8.1.6 is a detecting functor; see Example 8.1.8. □

Remark The problem treated in Theorem 8.1.9 has an old tradition in algebraic topology, starting with a theorem of Hopf. Many authors considered maps from an $(n+k)$-dimensional polyhedron into the sphere S^n for $k = 0,1,2$. An explicit criterion like in Theorem 8.1.9 for $k=3$ was not achieved in the literature. Clearly Theorem 8.1.9 is only a simple application of the classification theorem 8.1.6 since more generally this theorem can be used to decide what homology homomorphisms $H_*(X_S) \to H_*(X_{S'})$ are realizable by a map $X_S \to X_{S'}$.

The *stable* nth *homotopy group* of a space X is the direct limit

$$\pi_n^S(X) = \lim\{\pi_n X \to \pi_{n+1}\Sigma X \to \pi_{n+2}\Sigma^2 X \to \cdots\}$$

given by the suspension homomorphism $\Sigma\colon \pi_{n+k}(\Sigma^k X) \to \pi_{n+k+1}(\Sigma^{k+1}X)$ for $k \geq 0$. As a simple application of Theorem 8.1.6 and (8.1.7) we now obtain a well-known result; see for example G.W. Whitehead [RA]:

(8.1.10) Proposition *The third stable homotopy group of the real projective space $\mathbb{R}P_\infty$ is $\pi_3^S(\mathbb{R}P_\infty) = \mathbb{Z}/8$.*

This result follows also from the proof of the well known Kahn–Priddy theorem. For the proof we consider the following explicit example of an A^3-system; see also Section 12.6 below.

(8.1.11) Example Ler $\mathbb{R}P_4$ be the real projective 4-space. Then the space

$$X = \Sigma^{n-1}\mathbb{R}P_4 \quad \text{with} \quad n \geq 4, \tag{1}$$

is an $(n-1)$-connected $(n+3)$-dimensional complex for which the A^3-system $\Lambda'(X) = S$ is given by

$$S = \begin{pmatrix} H_0, H_2, H_3, \pi_1, b_2, \eta, b_3, \beta \\ \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, 1, 0, \Delta(1) \end{pmatrix}. \tag{2}$$

Hence S determines the homotopy type of $X_S = X = \Sigma^{n-1}\mathbb{R}P_4$. We derive S from the following facts. It is well known that the homology groups of X are $H_n X = \mathbb{Z}/2 = H_{n+2}X$ and that $H_i X = 0$ otherwise, $i > 0$. Since the generator $y \in H_2(\mathbb{R}P_4, \mathbb{Z}/2)$ has a non-trivial cup square $y \cup y \in H_4(\mathbb{R}P_4, \mathbb{Z}/2)$ the space X has a non-trivial Steenrod square. Hence X is not a one-point union of Moore spaces. Moreover, for the real projective 3-space $\mathbb{R}P_3$ we know, for $n \geq 3$,

$$\Sigma^{n-1}\mathbb{R}P_3 \simeq M(\mathbb{Z}/2, n) \vee S^{n+2}, \tag{3}$$

compare (IV.A.11) in Baues [CH]. This does not hold for $n=2$. We now prove (2). By (3) we know that $b_2 = 0$. Since $H_1 = 0$ we obtain $\pi_1 = \mathbb{Z}/2$ and $\eta = 1$. Hence $G(\eta) = \mathbb{Z}/4$. Since $H_3 = 0$ also $b_3 = 0$. Now β is in the image of

$$\Delta\colon \mathrm{Ext}(H_2, \Gamma(\eta)) = \mathbb{Z}/2 \to \Gamma(H_2, \Delta(\eta \otimes 1))$$

since $\mu(\beta)=b_2=0$. Moreover $\beta\neq 0$ since X is not a wedge of Moore spaces. Hence $\beta=\Delta(1)$. Therefore the extension in Definition 8.1.5(1)

$$G(\eta)=\mathbb{Z}/4 \rightarrowtail \pi_2 \twoheadrightarrow H_2=\mathbb{Z}/2$$

is non-trivial and thus $\pi_{n+2}X=\mathbb{Z}/8$. This proves Proposition 8.1.10.

8.2 On $\pi_{n+2}M(A,n)$

Moore functors are dual to the Eilenberg–Mac Lane functors; see Chapter 6. While many computations on Eilenberg–Mac Lane functors can be found in the literature there is only a little known on Moore functors. In this section we describe explicit examples of Moore functors which are needed for the proof of the classification theorem 8.1.6. We compute the homotopy groups $\pi_{n+2}M(A,n)$ and $\pi_{n+1}(A,M(B,n))$ of Moore spaces and we determine the functorial properties of these groups, $n\geq 4$.

Let $\mathbf{M}^n$ be the full homotopy category of Moore spaces $M(A,n)$. Morphisms in $\mathbf{M}^n$ are homotopy classes of maps $\bar{\varphi}$: $M(A,n)\to M(B,n)$. For $n\geq 3$ we have the algebraic category $\mathbf{G}$ which is equivalent to $\mathbf{M}^n$. Objects in $\mathbf{G}$ are abelian groups and morphisms $A\to B$ are proper homomorphisms (φ,ψ): $G(A)\to G(B)$ where $G(A)$ is part of the extension (8.1.1); compare Section 1.6. There is for each A an isomorphism, compatible with Δ and μ,

$$G(A)=\pi_n(\mathbb{Z}/2,M(A,n)); \tag{8.2.1}$$

see (1.6.4). Using this isomorphism we obtain the equivalence of categories

$$\mathbf{M}^n \xrightarrow{\sim} \mathbf{G} \quad \text{for} \quad n\geq 3 \tag{8.2.2}$$

which carries $M(A,n)$ to A and carries $\bar{\varphi}$ to (φ,ψ) with $\varphi=H_n\bar{\varphi}$ and $\psi=\pi_n(\mathbb{Z}/2,\bar{\varphi})$. Hence a proper homomorphism (φ,ψ): $G(A)\to G(B)$ determines a unique element

$$\bar{\varphi}\in[M(A,n),M(B,n)]$$

with $H_n\bar{\varphi}=\varphi$ and $\pi_n(\mathbb{Z}/2,\bar{\varphi})=\psi$. We use (8.2.2) as an identification of categories. The *Moore functor*

$$\pi_m(-,n)\colon \mathbf{G}=\mathbf{M}^n\to\mathbf{Ab} \tag{8.2.3}$$

carries A to $\pi_m(A,n)=\pi_m M(A,n)$ and carries (φ,ψ): $A\to B$ to $\pi_m(\bar{\varphi})$. Moreover there is the *Moore bifunctor*

$$\pi_m^{(n)}\colon \mathbf{G}^{\mathrm{op}}\times\mathbf{G}=(\mathbf{M}^m)^{\mathrm{op}}\times\mathbf{M}^n\to\mathbf{Ab} \tag{8.2.4}$$

which carries (A,B) to the homotopy group

$$\pi_m^{(n)}(A,B)=\pi_m(A,M(B,n))=[M(A,m),M(B,n)].$$

Induced maps are defined as in (8.2.4). Since **G** and **Ab** are algebraic categories it should be possible to obtain purely algebraic descriptions of these Moore functors. For small values of m one has the following examples of such algebraic functors. For $m = n$ the functor $\pi_n(-, n)$ carries A to A and (φ, ψ) to φ. For $m = n+1$, $n \geq 3$, the functor $\pi_{n+1}(-, n)$ carries A to $A \otimes \mathbb{Z}/2$ and (φ, ψ) to $\varphi \otimes \mathbb{Z}/2$. We get up to a canonical natural isomorphism the Moore functor $\pi_{n+2}(-, n)$, $n \geq 4$, as follows.

(8.2.5) Theorem *For $n \geq 4$ the functor $\pi_{n+2}(-, n)$ carries A to $G(A)$ and (φ, ψ) to ψ, that is, there is an isomorphism $\pi_{n+2}(M(A, n)) \cong G(A)$ of groups which is natural in $A \in$ **G**.*

Proof Let $\xi \in \pi_{n+2}M(\mathbb{Z}/2, n) = \mathbb{Z}/4, n \geq 4$, be a generator. Then ξ induces an isomorphism

$$\xi^*: G(A) = \pi_n(\mathbb{Z}/2, M(A, n)) \cong \pi_{n+2}M(A, n) \tag{1}$$

which is natural in $M(A, n) \in \mathbf{M}^n$. In fact for the pinch map q the composite

$$q\xi = \eta_{n+1}: S^{n+2} \to M(\mathbb{Z}/2, n) \to S^{n+1} \tag{2}$$

is the Hopf map. This shows that the following diagram commutes

$$\begin{tikzcd} \mathrm{Ext}(\mathbb{Z}/2, A \otimes \mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, equal] & \pi_n(\mathbb{Z}/2, M(A, n)) \arrow[r, two heads, "\mu"] \arrow[d, "\xi^*"] & \mathrm{Hom}(\mathbb{Z}/2, A) \arrow[d, equal] \\ A \otimes \mathbb{Z}/2 \arrow[r, tail] & \pi_{n+2}M(A, n) \arrow[r, two heads] & A * \mathbb{Z}/2 \end{tikzcd} \tag{3}$$

Here the top row is the universal coefficient sequence and the bottom row is induced by the inclusion of the n-skeleton $M(A, n)^n \subset M(A, n)$; compare Proposition 6.11.2. The commutativity of the diagram shows that ξ^* is an isomorphism, hence we obtain the theorem by (8.2.1). □

Next we consider the *cohomotopy group*

(8.2.6) $$\pi_{n+1}(A, S^n) = [M(A, n+1), S^n], \quad n \geq 3.$$

This group, as a functor in $A \in$ **G**, can be characterized as follows.

(8.2.7) Lemma *For $n \geq 3$ there is an isomorphism*

$$\theta: \pi_{n+1}(A, S^n) = \mathrm{Hom}(G(A), \mathbb{Z}/4)$$

*which is natural in $A \in$ **G**, that is $\theta \circ (\bar{\varphi})^* = \mathrm{Hom}(\psi, \mathbb{Z}/4) \circ \theta$; see* (8.2.2).

Moreover the isomorphism θ makes the following diagram with short exact rows commutative.

(8.2.8)

$$\begin{tikzcd}
\mathrm{Ext}(A,\mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, equal] & \pi_{n+1}(A,S^n) \arrow[r, two heads, "\mu"] \arrow[d, equal] & \mathrm{Hom}(A,\mathbb{Z}/2) \arrow[d, equal] \\
\mathrm{Hom}(A*\mathbb{Z}/2,\mathbb{Z}/4) \arrow[r, tail] & \mathrm{Hom}(G(A),\mathbb{Z}/4) \arrow[r] & \mathrm{Hom}(A\otimes\mathbb{Z}/2,\mathbb{Z}/4)
\end{tikzcd}$$

The top row is the universal coefficient sequence and the bottom row is given by (8.1.2), hence this is a topological interpretation of the exact sequence in (8.1.2).

Proof of Lemma 8.2.7 We can derive the result from Theorem 1.6.4. Here we give an independent proof which defines the isomorphism θ explicitly as follows. Since we are in the stable range we may assume $n \geq 4$. Let

$$\eta\colon M(\mathbb{Z}/2,n+1) \to S^n$$

be the Spanier–Whitehead dual of ξ in Theorem 8.2.5. Then the composite

$$\eta_n\colon S^{N+1} \xrightarrow{i} M(\mathbb{Z}/2,n+1) \xrightarrow{\eta} S^n,$$

where i is the inclusion of the bottom sphere, is the Hopf map. This shows that η induces an isomorphism η_*

$$\begin{tikzcd}
\pi_{n+1}(A,S^n) & {[M(A,n+1),M(\mathbb{Z}/2,n+1)]} \arrow[l, "\eta_*"', "\cong"] \arrow[d, equal] \\
\mathrm{Hom}(G(A),G(\mathbb{Z}/2)) \arrow[r, equal] & \mathbf{G}(A,\mathbb{Z}/2)
\end{tikzcd}$$

where $G(\mathbb{Z}/2)=\mathbb{Z}/4$. □

Finally we consider examples of Moore bifunctors $\pi_m^{(n)}$. We clearly have binatural isomorphisms $(A,B\in\mathbf{G}, n\geq 3)$

$$\pi_{n-1}^{(n)}(A,B)=\mathrm{Ext}(A,B) \qquad (8.2.9)$$

$$\pi_n^{(n)}(A,B)=\mathbf{G}(A,B)$$

where $\mathbf{G}(A,B)$ is the abelian group of morphisms $A\to B$ in $\mathbf{G}$. The next result yields an algebraic characterization of $\pi_{n+1}^{(n)}$, $n\geq 4$.

(8.2.10) Theorem *Let Δ_G: $\mathbf{G}\to\mathbf{S\Gamma Ab}'$ be the functor which carries B to the inclusion $\Delta_G(B)=\Delta$: $B\otimes\mathbb{Z}/2\subset G(B)$ and let $n\geq 4$. Then there is an isomorphism*

$$[M(A,n+1),M(B,n)]=\pi_{n+1}^{(n)}(A,B)=\overline{G}(A,\Delta_G(B))$$

which is natural in $A, B \in \mathbf{G}$ *and which is compatible with* Δ *and* μ *in the universal coefficient sequence. Here* $\overline{G}$ *is the functor in Definition* 8.1.3(*B*) *and Addendum* 8.1.3(*C*).

We point out that the natural isomorphism in Theorem 8.2.10 is available for all abelian groups A, B.

Proof of Theorem 8.2.10 If A or B are finitely generated we obtain the following commutative diagram which is natural in $M(A, n+1) \in \mathbf{M}^{n+1}$ and $M(B, n) \in \mathbf{M}^n$.

$$\begin{tikzcd}
\operatorname{Ext}(A, \mathbb{Z}/2) \otimes B \arrow[r, "\Delta \otimes 1"] \arrow[d, equal] & \pi_{n+1}(A, S^n) \otimes B \arrow[r, two heads, "\mu \otimes 1"] \arrow[dd, "k"] & \operatorname{Hom}(A, \mathbb{Z}/2) \otimes B \arrow[dd, equal] \\
\operatorname{Ext}(A, B \otimes \mathbb{Z}/2) \arrow[d, "\Delta_*"] & & \\
\operatorname{Ext}(A, \pi_{n+2} M(B, n)) \arrow[r, tail, "\Delta"] & \pi_{n+1}(A, M(B, n)) \arrow[r, two heads, "\mu"] & \operatorname{Hom}(A, B \otimes \mathbb{Z}/2)
\end{tikzcd}$$

Here k is given by composition, that is for $a \in \pi_{n+1}(A, S^n)$ and $b \in B = \pi_n M(B, n)$ we have $k(a \otimes b) = b \circ a$. The top row, induced by (8.2.8), is exact though $\Delta \otimes 1$ need not be injective. Moreover the bottom row is the universal coefficient sequence. Since the rows are exact the left-hand square of the diagram is a push-out diagram of abelian groups.

Using (8.2.8) this push-out diagram corresponds exactly to Definition 8.1.3(B) with $\eta = \Delta_G(B)$. Hence we obtain the isomorphism in the theorem. If A and B are arbitrary abelian groups we use the following construction. For a space Y and the set B we have the one-point union

$$Y[B] = \bigvee_{b \in B} Y_b \quad \text{with} \quad Y_b = Y$$

which is a functor in Y and in B. The map η: $P_2^{n+2} = M(\mathbb{Z}/2, n+1) \to S^n$ in the proof of Lemma 8.2.7 induces the map

$$\eta\colon M(\mathbb{Z}/2[B], n+1) = P_2^{n+2}[B] \xrightarrow{\eta[B]} S^n[B] = M(\mathbb{Z}[B], n)$$

which is natural in B. Moreover we have the inclusion

$$i\colon M(\mathbb{Z}[B], n) \subset M(B, n)$$

which is natural in $\mathbf{M}^n$, that is $\bar{\varphi}i = i(S^n[\varphi])$. We now consider the following diagram which is again natural in $M(A, n+1)$ and $M(B, n)$.

$$\begin{tikzcd} & & \mathrm{Hom}(G(A), \mathbb{Z}/4[B]) \arrow[d, equal] & \\ K(A,B) \arrow[d, "\eta_\#"'] & \subset & \pi_{n+1}(A, M(\mathbb{Z}/2[B], n+1)) \arrow[r, "\mu"] \arrow[d, "(i\eta)_*"] & \mathrm{Hom}(A, \mathbb{Z}/2[B]) \arrow[d, "(p_2)_*"] \\ \mathrm{Ext}(A, G(B)) \arrow[rr, tail, "\Delta"] & \text{push} & \pi_{n+1}(A, M(B,n)) \arrow[r, "\mu"] & \mathrm{Hom}(A, B \otimes \mathbb{Z}/2) \end{tikzcd}$$

Here $K(A, B) = \mathrm{kernel}(p_2)_* \mu$ is a functor in $A, B \in \mathbf{Ab}$ and we identify $G(B) = \pi_{n+2} M(B, n)$ by Theorem 8.2.5. The map $i\eta$ above induces $(p_2)_*$ such that the diagram commutes. We observe that $(p_2)_* \mu$ can be identified with the map $(p_2)_* \Delta^*$ in the diagram of Addendum 8.1.3(C). This yields the identification of bifunctors

$$K(A, B) = \mathrm{Hom}(A * \mathbb{Z}/2, K(B) * \mathbb{Z}/2)$$

via the inclusion j in Addendum 8.1.3(C). For the computation of $\eta_\#$ we first apply the natural map

$$Q = \lambda\colon \pi_{n+1}(A, M(B, n)) = \mathrm{PRIN}(d_A, d_B) \to [d_A, d_B \otimes \mathbb{Z}/2]$$

to the diagram above; see Theorem 6.12.14 and (8.2.12) below. The second part in the proof of Lemma 6.12.13 shows that the composite

$$K(A, B) \xrightarrow{\eta_\#} \mathrm{Ext}(A, G(B)) \xrightarrow{\mu_*} \mathrm{Ext}(A, B * \mathbb{Z}/2)$$

is trivial and hence $\eta_\#$ admits a factorization

$$\eta_\#\colon K(A, B) \xrightarrow{\theta} \Delta_* \mathrm{Ext}(A, B \otimes \mathbb{Z}/2) = \ker(\mu_*).$$

It remains to show that for $A = K(B) * \mathbb{Z}/2$ the element $\eta_\#(1) = \theta(1)$ coincides with the homomorphism θ_B in Addendum 8.1.3(C). For this consider $A = \mathbb{Z}/2$ and $y \in K(\mathbb{Z}/2, B)$ given by $y \in K(B) * \mathbb{Z}/2$ as in Addendum 8.1.3(C). Then we have a subgroup $j\colon B' \subset B$ generated by $x_i \in B$ and we have $y = j_* y'$. Now $\theta(y')$ can be computed by Definition 8.1.3(B). This in fact shows that $\theta(1) = \theta_B$. □

We have the commutative diagram

(8.2.11)

$$\begin{tikzcd} \mathrm{Ext}(A, G(B)) \arrow[r, tail, "\Delta"] \arrow[d, equal] & \bar{G}(A, \Delta(B)) \arrow[r, two heads, "\mu"] \arrow[d, equal] & \mathrm{Hom}(A, B \otimes \mathbb{Z}/2) \arrow[d, equal] \\ \mathrm{Ext}(A, \pi_{n+2} M(B, n)) \arrow[r, tail] & \pi_{n+1}(A, M(B, n)) \arrow[r, two heads] & \mathrm{Hom}(A, \pi_{n+1} M(B, n)) \end{tikzcd}$$

The left-hand side is the isomorphism given by Theorem 8.2.5. We now apply the push-out diagram (6.6.7)(ii) to the bottom row of (8.2.11). This yields a connection of the Moore bifunctor $\pi_{n+1}^{(n)}$ with the Eilenberg–Mac Lane bifunctor $H_{n+2}^{(n)}$, namely one has the binatural push-out diagram $(A, B \in \mathbf{G})$

(8.2.12)
$$\begin{array}{ccccc} \mathrm{Ext}(A, G(B)) & \rightarrowtail & \pi_{n+1}^{(n)}(A,B) & \twoheadrightarrow & \mathrm{Hom}(A, B\otimes \mathbb{Z}/2) \\ \downarrow{\scriptstyle \mu_*} & \text{push} & \downarrow{\scriptstyle Q} & & \| \\ \mathrm{Ext}(A, B * \mathbb{Z}/2) & \rightarrowtail & H_{n+2}^{(n)}(A,B) & \twoheadrightarrow & \mathrm{Hom}(A, B\otimes \mathbb{Z}/2) \end{array}$$

Here the bottom is naturally split since $\mu\Delta = 0$. Hence we can identify $(n \geq 4)$

(8.2.13) $$H_{n+2}^{(n)}(A,B) = \mathrm{Ext}(A, B * \mathbb{Z}/2) \oplus \mathrm{Hom}(A, B \otimes \mathbb{Z}/2).$$

The operator Q coincides with Q in Corollary 6.6.8 and $H_{n+2}^{(n)}$ is the Eilenberg–Mac Lane functor which we already know to be split since $H_{(n)}^{n+3}$ is split; see $r = 2$, $m \geq 4$ in (6.3.9).

8.3 The group Γ_{n+2} of an $(n-1)$-connected space, $n \geq 4$

Whitehead's Γ-groups $\Gamma_n X$ appear in the certain exact sequence. For a CW-complex X they are defined by the image

(8.3.1) $$\Gamma_m X = \mathrm{image}\{\pi_m X^{m-1} \to \pi_m X^m\}$$

where X^m is the m-skeleton of X. If X is $(n-1)$-connected, then clearly $\Gamma_m X = 0$ for $m \leq n$. Whitehead computed the first non-vanishing group

(8.3.2) $$\Gamma_{n+1}(X) = H_n(X) \otimes \mathbb{Z}/2, \quad n \geq 3.$$

Here we do the next step and compute $\Gamma_{n+2}(X)$ for $n \geq 4$ Moreover we compute the Γ-group with coefficients $\Gamma_{n+1}(A, X)$ and we describe the functorial properties of these groups. The groups $\Gamma_{n+2}(X)$ and $\Gamma_{n+1}(A, X)$ depend only on the $(n+1)$-type $P_{n+1}(X)$ of the $(n-1)$-connected space X. This $(n+1)$-type is of the form

(8.3.3) $$P_{n+1}(X) = K(\eta, n),$$

where $K(\eta, n)$ is the quadratic space associated with the quadratic map

$$\eta = (\eta_n)^*\colon \pi_n(X) \to \pi_{n+1}(X),$$

compare Definition 7.1.5. For $n \geq 3$ this is a stable quadratic map given by a homomorphism $\eta\colon \pi_n(X) \otimes \mathbb{Z}/2 \to \pi_{n+1}(X)$. The Postnikov map $X \to K(\eta, n)$ induces a natural isomorphism

(8.3.4) $$\begin{cases} \Gamma_{n+2} X = \Gamma_{n+2} K(\eta, n) \\ \Gamma_{n+1}(A, X) = \Gamma_{n+1}(A, K(\eta, n)). \end{cases}$$

Hence as an abelian group $\Gamma_{n+2}(X)$ and $\Gamma_{n+1}(A,X)$ are determined by η and (A,η) respectively. The computation of the functor Γ_{n+2} is based on the following short exact sequence; see Theorem 5.3.7.

(8.3.5) Proposition *Let $n \geq 4$ and let X be $(n-1)$-connected. Then one has the natural short exact sequence*

$$\pi_{n+1}(X) \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} \Gamma_{n+2}(X) \overset{\mu}{\twoheadrightarrow} H_n(X) * \mathbb{Z}/2$$

where $\Delta(\alpha \otimes 1) = \alpha\eta_{n+1}$ is induced by the Hopf map η_{n+1}.

Proof We may assume that X is a CW-complex with $\dim(X) \leq n+3$ and $X^{n-1} = *$. Using Lemma (I.7.5) in Baues [CH] we obtain a map $g\colon A \to B$ where A and B are both one-point unions of n-spheres and $(n+1)$-spheres such that the mapping cone of g is homotopy equivalent to X. Moreover $H_n g$ is injective. Since we are in the stable range we thus have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
\pi_{n+2}A & \xrightarrow{g_*} & \pi_{n+2}B & \longrightarrow & \pi_{n+2}C_g & \longrightarrow & \pi_{n+1}A & \xrightarrow{g_*} & \pi_{n+1}B \\
\| & & \| & & \cup & & \cup & & \cup \\
\pi_{n+2}A & \xrightarrow[q_1]{} & \pi_{n+2}B & \longrightarrow & \Gamma_{n+2}X & \longrightarrow & H_n(A) \otimes \mathbb{Z}/2 & \xrightarrow[g_2]{} & H_n(B) \otimes \mathbb{Z}/2
\end{array}$$

The cokernel of $g_1 = g_*$ is $\pi_{n+1}(X) \otimes \mathbb{Z}/2$ and the kernel of $g_2 = H_n(g) \otimes 1$ is $H_n(X) * \mathbb{Z}/2$. This yields the required exact sequence which is natural since Δ is natural. □

(8.3.6) Corollary *Let $n \geq 4$ and let X be an $(n-1)$-connected space with $B = H_n(X)$. Moreover let $\beta\colon M(B,n) \to X$ be a map such that $H_n(\beta)$ is the identity on B. Then one has the commutative diagram with short exact rows*

$$\begin{array}{ccccc}
\pi_{n+1}(X) \otimes \mathbb{Z}/2 & \overset{\Delta}{\rightarrowtail} & \Gamma_{n+2}(X) & \overset{\mu}{\twoheadrightarrow} & H_n(X) * \mathbb{Z}/2 \\
\uparrow{\scriptstyle \eta_n^*} & & \uparrow{\scriptstyle \beta_*} & & \| \\
B \otimes \mathbb{Z}/2 & \overset{\Delta}{\rightarrowtail} & \pi_{n+2}M(B,n) & \overset{\mu}{\twoheadrightarrow} & B * \mathbb{Z}/2
\end{array}$$

where η_n^* carries $b \otimes 1$ with $b \in B = \pi_n X$ to $(b\eta_n) \otimes 1$.

The bottom row is, on the one hand, the exact sequence in Theorem 8.2.5(3); on the other hand it is the exact sequence in Proposition 8.3.5 for $X = M(B,n)$. We observe that the diagram in Corollary 8.3.6 is a push-out diagram of abelian groups which determines the group $\Gamma_{n+2}(X)$. The isomorphism ξ_* in Theorem 8.2.5(1) has the following generalization.

(8.3.7) Theorem *Let X be an $(n-1)$-connected space. Then there is a natural isomorphism ξ^* for which the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{Z}/2, \pi_{n+1}X) \arrow[r, tail, "\Delta"] \arrow[d, equal] & \pi_n(\mathbb{Z}/2, X) \arrow[r, two heads, "\mu"] \arrow[d, "\xi^*", "\cong"'] & \mathrm{Hom}(\mathbb{Z}/2, \pi_n X) \arrow[d, equal] \\
\pi_{n+1}(X) \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] & \Gamma_{n+2}(X) \arrow[r, "\mu"] & H_n(X) * \mathbb{Z}/2
\end{tikzcd}$$

Here $\pi_n(\mathbb{Z}/2, X)$ is the homotopy group with coefficients in $\mathbb{Z}/2$ and the top row is the universal coefficient sequence. The left- and the right-hand side denote the canonical identifications. Moreover, we obtain for η in (8.3.3) *the isomorphism of groups*

$$\theta: G(\eta) = \Gamma_{n+2}(X)$$

which is compatible with Δ and μ. Here $G(\eta)$ is the group in Definition 8.1.3(A).

Proof The isomorphism ξ^* carries $\alpha: M(\mathbb{Z}/2, n) \xrightarrow{\alpha'} X^{n+1} \subset X$ to the element of $\Gamma_{n+2}X$ given by the composite

$$S^{n+2} \xrightarrow{\xi} M(\mathbb{Z}/2, n) \xrightarrow{\alpha'} X^{n+1}$$

As in the proof of Theorem 8.2.5 we see that the diagram in Theorem 8.3.7 commutes. Moreover we define θ by $\theta\Delta = \Delta$ and $\theta\bar{\eta} = \beta_* \xi^*$ where ξ^* is the isomorphism in Theorem 8.2.5(3) and where we use β_* in Corollary 8.3.6. Compare also Theorem 1.6.11 where $G(\mathbb{Z}/2, \eta) = G(\eta)$. □

(8.3.8) Remark By putting $X = K(B, n)$ in Corollary 8.3.6 we readily get, for $n \geq 4$,

$$H_{n+3}K(B, n) = \Gamma_{n+2}K(B, n) = B * \mathbb{Z}/2$$

and the operator

$$Q: \pi_{n+2}M(B, n) = G(B) \twoheadrightarrow H_{n+3}K(B, n) = B * \mathbb{Z}/2$$

in Theorem 6.6.6 is surjective and coincides with μ in Corollary 8.3.6.

We use the next result for the computation of $\Gamma_{n+1}(A, X)$. Here β in Corollary 8.3.6 induces an isomorphism $\Gamma_{n+1}(\beta)$.

(8.3.9) Lemma *Let X and β be given as in Corollary 8.3.6. Then one has the commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}(A,\Gamma_{n+2}X) \arrow[r, tail, "\Delta"] & \Gamma_{n+1}(A,X) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A,\Gamma_{n+1}X) \\
\mathrm{Ext}(A,\pi_{n+2}M(B,n)) \arrow[r, tail, "\Delta"] \arrow[u, "\mathrm{Ext}(A,\beta_*)"] \arrow[ur, phantom, "\text{push}"] & \pi_{n+1}(A,M(B,n)) \arrow[r, two heads, "\mu"] \arrow[u, "\beta_*"] & \mathrm{Hom}(A,B\otimes\mathbb{Z}/2) \arrow[u, equal, "(\Gamma_{n+1}\beta)_*"]
\end{tikzcd}$$

The top row is the universal coefficient sequence (Definition 2.2.3) which is natural in X. The bottom row is the universal coefficient for $X = M(B,n)$ which was algebraically characterized in Theorem 8.2.10. Since the diagram in Lemma 8.3.9 is a push-out diagram we obtain by composing this diagram and diagram (8.2.11) the isomorphism

$$\overline{G}(A,\Delta(\eta\otimes 1)) \cong \Gamma_{n+1}(A,X). \tag{8.3.10}$$

Here we use the composite

$$\pi_n(X)\otimes\mathbb{Z}/2 \xrightarrow{\eta\otimes 1} \pi_{n+1}(X)\otimes\mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(\eta).$$

The left-hand side of (8.3.10) is defined by the bifunctor $\overline{G}$ in Definition 8.1.3. By putting $X = K(B,n)$ in (8.3.10) one has $\eta = 0$ and hence one has the binatural isomorphism

$$\begin{aligned} H_{n+2}(A,K(B,n)) &= \Gamma_{n+1}(A,K(B,n)) \\ &= \mathrm{Ext}(A,B*\mathbb{Z}/2)\oplus\mathrm{Hom}(A,B\otimes\mathbb{Z}/2). \end{aligned}$$

Here the second equation is a consequence of (8.3.10). We discussed this already in (8.2.13). We now study the functorial properties of Theorem 8.3.7 and (8.3.10). For this we need the operator $\xi\mapsto\xi_\#$ given as follows.

(8.3.11) Definition Let A, B, R be abelian groups. Then one has a homomorphism

$$\mathrm{Ext}(A,B) \to \mathrm{Hom}(A*R, B\otimes R),\ \xi\mapsto\xi_\#,$$

which is natural in A and B. If $A_1 \rightarrowtail A_0 \twoheadrightarrow A$ is a short free resolution of A then $\xi\in\mathrm{Ext}(A,B)$ is represented by $\xi_1\in\mathrm{Hom}(A_1,B)$ and $\xi_\#$ is the composite

$$\xi_\#\colon A*R \subset A_1\otimes R \xrightarrow{\xi_1\otimes R} B\otimes R.$$

Let $B \rightarrowtail E \twoheadrightarrow A$ be an extension of abelian groups representing ξ. Then there is the classical *six-term exact sequence*

$$0 \to B * R \to E * R \to A * R \xrightarrow{\xi_\#} B \otimes R \to E \otimes R \to A \otimes R \to 0$$

where $\xi_\#$ is the boundary operator. Using the inclusion Δ: $\mathrm{Ext}(B, \pi_{n+1}X) \rightarrowtail \pi_n(B, X)$ we define for $\beta \in \pi_n(B, X)$ and $\xi \in \mathrm{Ext}(B, \pi_{n+1}X)$ the *difference homomorphism*

$$(\beta + \Delta\xi)_* - \beta_* : \pi_{n+2}M(B, n) \to \Gamma_{n+2}(X) \qquad (*)$$

where β_* is the same as in Corollary 8.3.6. On the other hand we have

$$(\beta + \Delta\xi)_* - \beta_* : \pi_{n+1}(A, M(B, n)) \to \Gamma_{n+1}(A, X) \qquad (**)$$

where β_* is the same as in Lemma 8.3.9.

(8.3.12) Proposition *The homomorphism* $(*)$ *coincides with the composite*

$$\pi_{n+2}M(B, n) \overset{\mu}{\twoheadrightarrow} B * \mathbb{Z}/2 \xrightarrow{\xi_\#} \pi_{n+1}(X) \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} \Gamma_{n+2}(X).$$

Moreover $(**)$ *coincides with the composite*

$$\pi_{n+1}(A, M(B, n)) \xrightarrow{Q_1} \mathrm{Ext}(A, B * \mathbb{Z}/2)$$
$$\xrightarrow{(\Delta\xi_\#)_*} \mathrm{Ext}(A, \Gamma_{n+2}X) \xrightarrow{\Delta} \Gamma_{n+1}(A, X)$$

where Q_1 *is the first coordinate of* Q *in* (8.2.12); *see* (8.2.13).

We leave the proof as an exercise; compare the more sophisticated unstable version of Proposition 8.3.12 in Theorem 11.4.7 below.

8.4 Proof of the classification theorem 8.1.6

We apply the classification theorem 3.4.4 where we set $r = 2$ and $\mathbf{C} = \mathbf{types}^1_n = \Gamma\mathbf{G}, n \geq 4$; compare 7.2.9. Since $\mathbf{spaces}^3_n = \mathbf{spaces}^3_n(\mathbf{C})$ we obtain the detecting functor

$$\Lambda' : \mathbf{spaces}^3_n \to \mathbf{Bypes}(\Gamma\mathbf{G}, F) \qquad (1)$$

where F is the following bype functor on $\Gamma\mathbf{G}$. For η: $H_0 \otimes \mathbb{Z}/2 \to \pi_1$ let

$$F_1(\eta) = G(\eta) \qquad (1)$$

as in (8.1.3) and

$$F_0(\eta) = \mathrm{kernel}(\eta: H_0 \otimes \mathbb{Z}/2 \to \pi_1). \qquad (3)$$

Moreover we define the bype functor F on $\Gamma\mathbf{G}$ by the following pull-back diagram where we use the functor $\overline{G}$ in (8.3.10) and where we use the inclusion j: $F_0(\eta) \subset H_0 \otimes \mathbb{Z}/2$.

$$\begin{array}{ccccc} \operatorname{Ext}(A, \Gamma(\eta)) & \overset{\Delta}{\rightarrowtail} & \overline{G}(A, \Delta(\eta \otimes 1)) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(A, H_0 \otimes \mathbb{Z}/2) \\ \| & & \uparrow & \text{pull} & \uparrow j_* \\ \operatorname{Ext}(A, F_1\eta) & \overset{\Delta}{\rightarrowtail} & F(A, \eta) & \overset{\mu}{\twoheadrightarrow} & \operatorname{Hom}(A, F_o\eta) \end{array} \tag{4}$$

Now Lemma 8.3.9 and the classification theorem 3.4.4 yield the detecting functor Λ' in (1). We claim that we have a forgetful functor

$$\phi: \mathbf{Bypes}(\Gamma\mathbf{G}, F) \to \mathbf{A^3\text{-}System} \tag{5}$$

which is also a detecting functor. This yields the result in Theorem 8.1.6 by use of the detecting functor $\phi\Lambda'$. The functor ϕ is essentially the identity on objects; compare for this the definition of F-bypes in Section 3.2 and the definition of A^3-systems in Definition 8.1.4. Let φ_i: $H_i \to H_i'$ be homomorphisms. Then ϕ carries the morphism $(\chi, \varphi_2, \varphi_3)$ in $\mathbf{Bypes}(\Gamma\mathbf{G}, F)$ with $\chi = \{\varphi_\pi, \zeta, \varphi_0, \overline{\varphi}_0\}$: $\eta \to \eta'$ in $\Gamma\mathbf{G}$ to the morphism $\varphi = (\varphi_0, \varphi_2, \varphi_3, \varphi_\pi, \varphi_\Gamma)$ in **$\mathbf{A^3}$-System** with

$$\varphi_\Gamma = \chi_*: \Gamma = G(\eta) \to \Gamma' = G(\eta'). \tag{6}$$

We have only to check that ϕ is a full functor, that is, for φ_Γ there always exists χ such that $\varphi_\Gamma = \xi_*$ as in (6). For this we consider the diagram

$$\begin{array}{ccc} G(\eta) & \overset{(\bar{\eta}, \Delta)}{\twoheadleftarrow} & G(H_0) \oplus \pi_1 \otimes \mathbb{Z}/2 \\ \varphi_\Gamma \downarrow & & \downarrow (\bar{\varphi}_0, \varphi_\pi, \zeta)_* \\ G(\eta') & \overset{(\bar{\eta}', \Delta)}{\twoheadleftarrow} & G(H_0') \oplus \pi_1' \otimes \mathbb{Z}/2 \end{array} \tag{7}$$

where $(\bar{\varphi}_0, \varphi_\pi, \zeta)_*$ has the coordinates

$$\begin{aligned} \overline{\varphi}_0&: \ G(H_0) \to G'(H_0'), \\ \varphi_\pi \otimes 1&: \ \pi_1 \otimes \mathbb{Z}/2 \to \pi_1' \otimes \mathbb{Z}/2, \\ \zeta_\# \mu&: \ G(H_0) \xrightarrow{\mu} H_0 * \mathbb{Z}/2 \to \pi_1' \otimes \mathbb{Z}/2. \end{aligned}$$

The definition in (8.3.11) shows that the right-hand side of (7) induces χ_* in (6). We claim that for φ_Γ there is $(\overline{\varphi}_0, \varphi_\pi, \zeta)$ such that diagram (7) commutes. To see this we first choose a map $\overline{\varphi}_0$: $G(H_0) \to G(H_0')$ compatible with φ_0. Then $\overline{\varphi}_0 \oplus \varphi_\pi \otimes 1$ induces a map φ_Γ': $G(\eta) \to G(\eta')$ compatible with Δ and μ. Since also φ_Γ is compatible with Δ and μ in Definition 8.1.4(8) there is

$$\delta \in \operatorname{Hom}(H_0 * \mathbb{Z}/2, \pi_1' \otimes \mathbb{Z}/2)$$

with $\varphi_\Gamma = \varphi'_\Gamma + \Delta\delta\mu$. Since the operation $\zeta \mapsto \zeta_\#$ is given by the surjection

$$\operatorname{Ext}(H_0, \pi'_1) \xrightarrow{q_*} \operatorname{Ext}(H_0, \pi'_1 \otimes \mathbb{Z}/2) = \operatorname{Hom}(H_0 * \mathbb{Z}/2, \pi'_1 \otimes \mathbb{Z}/2)$$

we see that there exists ζ with $\zeta_\# = \delta$. This implies that diagram (7) commutes for $(\overline{\varphi}_0, \varphi_\pi, \zeta)$. Therefore ϕ is a full functor and hence a detecting functor. Similarly we get the detecting functor λ' in Theorem 8.1.6 by use of λ' in Theorem 3.4.4. □

8.5 Adem operations

Algebraic data which classify homotopy types are by no means well determined or unique. It is, however, suitable to search for such data which directly refer to basic invariants like homology groups and homotopy groups. For example, the A^3-systems in Section 8.1 which classify $(n-1)$-connected $(n+3)$-dimensional polyhedra X use the homology groups $H_* X$ of X with $\mathbb{Z}$-coefficients, and the homotopy groups $\pi_{n+1}X, \pi_{n+2}X$ can easily be computed by the A^3-system associated with X.

On the other hand, there is a classical approach to classifying homotopy types X by cohomology groups with coefficients in various abelian groups and by cohomology operations. It is a kind of old belief that at least in the stable range it is possible to find such cohomological data which classify homotopy types. The use of cohomology requires the restriction to spaces with finitely generated homology groups. Given such cohomological data it is then a hard problem to determine the homotopy groups $\pi_n X$ for $n < \dim X$ in terms of these data; see for example (5.3.5).

Actually only a very few complete results are known on the classification of homotopy types via cohomology and cohomology operations. J.H.C. Whitehead [HT] for example used Steenrod squares for the classification of $(n-1)$-connected $(n+2)$-dimensional polyhedra, $n \geq 3$. Such Steenrod squares are primary cohomology operations. It is not possible to classify $(n-1)$-connected $(n+3)$-dimensional polyhedra only by primary cohomology operations. In fact for the double Hopf map $\eta_n^2\colon S^{n+2} \to S^n$ we have the mapping cone

$$X = S^n \cup_{\eta_n^2} e^{n+3}$$

and all primary cohomology operations on X are trivial. It is a classical result of Adem that there is a non-trivial secondary cohomology operation for X showing that η_n^2 is essential. This suggested the classification of $(n-1)$-connected $(n+3)$-dimensional homotopy types, $n \geq 4$, by primary and secondary cohomology operations; compare the papers of Shiraiwa, Chang, and Chow. The results turned out to be very intricate and unclear. Here we follow the work of my student S. Jäschke [AO] who shows that indeed a classification via primary and secondary cohomology operations is possible. For this one needs the notion of an A^3-cohomology system which is a modification of the concept of Chang and which relies on secondary cohomology operations of Adem type.

(8.5.1) Notation We describe some fundamental concepts of homotopy theory, concerning extensions, coextensions, lifts, and colifts, respectively. Consider maps

$$A \xrightarrow{f} B \xrightarrow{g} Y \quad \text{with} \quad gf \simeq 0 \quad \text{in} \quad \mathbf{Top}^*. \tag{1}$$

Given the null homotopy H: $gf \simeq 0$ we obtain the following maps depending on H

$$C_f \xrightarrow{\bar{g}} Y, \quad \textit{extension} \text{ of } g, \tag{2}$$

$$\Sigma A \xrightarrow{\bar{f}} C_g, \quad \textit{coextension} \text{ of } f, \tag{3}$$

$$A \xrightarrow{\bar{f}} P_g, \quad \textit{lift} \text{ of } f, \tag{4}$$

$$P_f \xrightarrow{\bar{g}} \Omega Y, \quad \textit{colift} \text{ of } g. \tag{5}$$

Here $G_g = Y \cup_g CB$ is the mapping cone given by the push-out

$$\begin{tikzcd} CB \arrow[r, "\pi_g"] & C_g \\ B \arrow[u, "i_0"] \arrow[r, "g"'] \arrow[ur, phantom, "\text{push}"] & Y \arrow[u, "i_g"'] \end{tikzcd} \tag{6}$$

where $CB = I \times B / I \times * \cup \{1\} \times X$ is the cone on X. On the other hand, $P_g = B \times_g WY$ is the fibre of g given by the pull-back

$$\begin{tikzcd} P_g \arrow[r, "\pi_g"] \arrow[d, "q_g"'] \arrow[dr, phantom, "\text{pull}"] & WY \arrow[d, "q_0"] \\ B \arrow[r, "g"'] & Y \end{tikzcd} \tag{7}$$

where WY is the contractible path object of Y, that is, $WY = \{\sigma \in Y^I, \sigma(1) = *\}$ and $q_0(\sigma) = \sigma(0)$. A null homotopy H: $gf \simeq 0$ as above can be identified with maps

$$H: CA \to Y \quad \text{and} \quad \bar{H}: A \to WY \tag{8}$$

respectively. The extension $\bar{g}$ is the map with $\bar{g}i_g = g$ and $\bar{g}\pi_g = H$; the lift $\bar{f}$ is the map with $q_g\bar{f} = f$ and $\pi_g\bar{f} = \bar{H}$. Moreover for the suspension ΣA and the loop space ΩY we obtain the coextension $\bar{f}$ and the colift $\bar{g}$ above follows. Consider the commutative diagrams

$$\begin{tikzcd} CA \arrow[r, "C(f)"] & CB \arrow[r, "\pi_g"] & C_g \\ A \arrow[u] \arrow[r] \arrow[d] & B \arrow[u] \arrow[r] & Y \arrow[u] \\ CA \arrow[urr, "H"'] & & \end{tikzcd} \tag{9}$$

$$\begin{tikzcd} P_f \arrow[r, "\pi_f"] \arrow[d] & WB \arrow[r, "W(g)"] \arrow[d] & WY \arrow[d] \\ A \arrow[r] \arrow[drr, "\bar{H}"'] & B \arrow[r] & Y \\ & & WY \arrow[u] \end{tikzcd} \tag{10}$$

These diagrams yield the maps

$$\bar{f}\colon \Sigma A \approx CA \cup_A CA \xrightarrow{(\pi_g C(f),\, i_g H)} C_g \tag{11}$$

$$\bar{g}\colon P_f \xrightarrow{(W(g)\pi_f,\, \bar{H})} WY \times_Y WY \approx \Omega Y. \tag{12}$$

Here the homeomorphisms are chosen such that

$$\Sigma f \simeq q_0\bar{f}\colon \Sigma A \to C_g \to \Sigma B \tag{13}$$

$$\Omega g \simeq \bar{g}i_0\colon \Omega B \subset P_f \to \Omega Y. \tag{14}$$

The map q_0 is the pinch map $C_q \to C_g/Y \approx \Sigma B$ and i_0 is the inclusion of the fibre $\Omega B \approx (q_g)^{-1}(*) \subset P_f$. More generally extension and coextension can be defined in any cofibration category and lift and colift are the strictly dual constructions in a fibration category; see Baues [AH].

The Steenrod operations are homomorphisms

(8.5.2) $$Sq^n\colon H^k(X,\mathbb{Z}/2) \to H^{k+n}(X,\mathbb{Z}/2)$$

which are natural in X. They determine up to homotopy maps

$$Sq^n\colon K(\mathbb{Z}/2,k) \to K(\mathbb{Z}/2,k+n) \tag{1}$$

which induce (8.5.2) via the isomorphism $H^m(X,\pi)=[X,K(\pi,m)]$. The operator Sq^1 coincides with the Bockstein homomorphism associated with the exact sequence $\mathbb{Z}/2 \xrightarrow{\mu_4^2} \mathbb{Z}/4 \xrightarrow{\mu_2^4} \mathbb{Z}/2$ so that

$$\to H^n(X,\mathbb{Z}/2) \xrightarrow{\mu_4^2} H^n(X,\mathbb{Z}/4) \xrightarrow{\mu_2^4} H^n(X,\mathbb{Z}/2) \xrightarrow{Sq^1} H^{n+1}(X,\mathbb{Z}/2) \xrightarrow{\mu_4^2} \tag{2}$$

is a long exact sequence; in particular $\mu_2^4 Sq^1=0$ and $\mu_4^2 Sq^1=0$. Hence also $Sq^1\mu_2^0=0$ where $\mu_2^0\colon \mathbb{Z}\to\mathbb{Z}/2$ is the quotient map. The *Adem relations*

$$\text{(8.5.3)} \qquad \begin{cases} Sq^3 = Sq^1Sq^2 \\ Sq^3Sq^1 + Sq^2Sq^2 = 0 \end{cases}$$

give rise to the following composites gf which are null homotopic (compare for example Mosher and Tangora [CO]).

$$K(\mathbb{Z}/2,n) \xrightarrow{(Sq^1,Sq^2)} K(\mathbb{Z}/2,n+1)\times K(\mathbb{Z}/2,n+2) \xrightarrow{(Sq^3,Sq^2)} K(\mathbb{Z}/2,n+4) \tag{1}$$

$$K(\mathbb{Z}/2,n) \xrightarrow{(Sq^2Sq^1,Sq^2)} K(\mathbb{Z}/2,n+3)\times K(\mathbb{Z}/2,n+2) \xrightarrow{(Sq^1,Sq^2)} K(\mathbb{Z}/2,n+4) \tag{2}$$

$$K(\mathbb{Z},n) \xrightarrow{Sq^2\mu_2^0} K(\mathbb{Z}/2,n+2) \xrightarrow{Sq^2} K(\mathbb{Z}/2,n+4) \tag{3}$$

$$K(\mathbb{Z}/2,n) \xrightarrow{Sq^2} K(\mathbb{Z}/2,n+2) \xrightarrow{\mu_4^2Sq^2} K(\mathbb{Z}/4,n+4) \tag{4}$$

$$K(\mathbb{Z}/4,n) \xrightarrow{Sq^2\mu_2^4} K(\mathbb{Z}/2,n+2) \xrightarrow{Sq^2} K(\mathbb{Z}/2,n+4). \tag{5}$$

Given a composite $gf\colon A\to B\to Y$, $gf\simeq 0$, as in (1)–(5) we choose a colift $\bar{g}$ of g as in (8.5.1) and we define the *secondary operation* ϕ associated with $gf\simeq 0$ as follows: Consider the diagram

$$\begin{tikzcd} & P_f \arrow[r, "\bar{g}"] \arrow[d] & \Omega Y & \\ X \arrow[ur, "\bar{u}"] \arrow[r, "u"] & A \arrow[r, "f"] & B \arrow[r, "g"] & Y \end{tikzcd}$$

where $fu\simeq 0$. Then we choose a lift $\bar{u}$ of u and $\bar{g}\bar{u}$ represents $\phi(u)$. Hence given $\{u\}\in[X,A]$ with $f_*\{u\}=0$ we obtain the well-defined subset

$$\phi\{u\}\subset[X,\Omega Y]$$

consisting of all composites $\bar{g}\bar{u}$ where $\bar{g}$ and $\bar{u}$ are given by homotopies

$fu \simeq 0$ and $gf \simeq 0$ respectively. For the maps in (1)–(5) above this subset is actually a coset of a subgroup of $[X, \Omega Y]$. This way one derives from (1)–(5) the following well-defined *Adem operations* (1)′–(5)′

$$H^n(X,\mathbb{Z}/2) \supset \ker(Sq^1) \cap \ker(Sq^2) \xrightarrow{\phi'} \frac{H^{n+3}(X,\mathbb{Z}/2)}{\operatorname{im}(Sq^3+Sq^2)} \tag{1$'$}$$

$$H^n(X,\mathbb{Z}/2) \supset \ker(Sq^2) \cap \ker(Sq^2Sq^1) \xrightarrow{\phi''} \frac{H^{n+3}(X,\mathbb{Z}/2)}{\operatorname{im}(Sq^1+Sq^2)} \tag{2$'$}$$

$$H^n(X,\mathbb{Z}) \supset \ker(Sq^2\mu_2^0) \xrightarrow{\phi_2^0} \frac{H^{n+3}(X,\mathbb{Z}/2)}{\operatorname{im}(Sq^2)} \tag{3$'$}$$

$$H^n(X,\mathbb{Z}/2) \supset \ker(Sq^2) \xrightarrow{\phi_4^2} \frac{H^{n+3}(X,\mathbb{Z}/4)}{\operatorname{im}(\mu_4^2 Sq^2)} \tag{4$'$}$$

$$H^n(X,\mathbb{Z}/4) \supset \ker(Sq^2\mu_2^4) \xrightarrow{\phi_2^4} \frac{H^{n+3}(X,\mathbb{Z}/2)}{\operatorname{im}(Sq^2)} \tag{5$'$}$$

where $\operatorname{im}(Sq^r+Sq^s) = \operatorname{im}(Sq^r) + \operatorname{im}(Sq^s)$ is the sum of subgroups. Originally Adem used 'chain maps' for the definition of such operations. In (3) we also write $Sq^2\mu_2^0 = Sq_{\mathbb{Z}}^2$ and ϕ_2^0 coincides with the operation used in Theorem 5.3.8(b). One can check that the following diagrams commute, where the arrows i and q denote inclusions and quotient maps respectively

$$\begin{array}{ccc} \ker(Sq^2\mu_2^0) & \xrightarrow{\phi_2^0} & H^{n+3}(X,\mathbb{Z}/2)/\operatorname{im}(Sq^2) \\ \downarrow{\scriptstyle \mu_2^0} & & \downarrow{\scriptstyle q} \\ \ker(Sq^2)\cap\ker(Sq^1) & \xrightarrow{\phi'} & H^{n+3}(X,\mathbb{Z}/2)/\operatorname{im}(Sq^3+Sq^2) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle q} \\ \ker(Sq^2)\cap\ker(Sq^2Sq^1) & \xrightarrow{\phi''} & H^{n+3}(X,\mathbb{Z}/2)/\operatorname{im}(Sq^1+Sq^2) \end{array} \tag{8.5.4}$$

$$\begin{array}{ccc} \ker(Sq^2\mu_2^4) & \xrightarrow{\phi_2^4} & H^{n+3}(X,\mathbb{Z}/2)/\operatorname{im}(Sq^2) \\ \downarrow{\scriptstyle \mu_2^4} & & \downarrow{\scriptstyle q} \\ \ker(Sq^2)\cap\ker(Sq^1) & \xrightarrow{\phi'} & H^{n+3}(X,\mathbb{Z}/2)/\operatorname{im}(Sq^3+Sq^2) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mu_4^2} \\ \ker(Sq^2) & \xrightarrow{\phi_4^2} & H^{n+3}(X,\mathbb{Z}/4)/\operatorname{im}(\mu_4^2 Sq^2) \end{array} \tag{8.5.5}$$

We are now ready to introduce the notion of an A^3-cohomology system which describes further properties of the Adem operations ϕ_4^2 and ϕ_2^4.

(8.5.6) Definition An A^3*-cohomology system* is a tuple

$$S = (H^*(0), H^*(2), H^*(4), \Delta(2), \Delta(4), \mu(2), \mu(4), \mu_4^2, \mu_2^4, Sq_0^1, Sq^2, \phi_4^2, \phi_2^4)$$

with the following properties.

(a) $H^*(0), H^*(2), H^*(4)$ are graded finitely generated abelian groups concentrated in degree 0, 1, 2, 3, and $H^0(0)$ is free abelian.

(b) $\mu_4^2\colon H^*(2) \to H^*(4)$ and $\mu_2^4\colon H^*(4) \to H^*(2)$ are homomorphisms of degree 0.

(c) $\Delta(2), \Delta(4)$ and $\mu(2), \mu(4)$ are homorphisms of degree 0 and 1 respectively for which the following diagram commutes

$$\begin{tikzcd}
H^*(0) \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta(2)"] \arrow[d, "1 \otimes \mu_4^2"] & H^*(2) \arrow[r, "\mu(2)"] \arrow[d, "\mu_4^2"] & H^*(0) * \mathbb{Z}/2 \arrow[d, "1 * \mu_4^2"] \\
H^*(0) \otimes \mathbb{Z}/4 \arrow[r, tail, "\Delta(4)"] \arrow[d, "1 \otimes \mu_2^4"] & H^*(4) \arrow[r, "\mu(4)"] \arrow[d, "\mu_2^4"] & H_*(0) * \mathbb{Z}/4 \arrow[d, "1 * \mu_2^4"] \\
H^*(0) \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta(2)"] & H^*(2) \arrow[r, "\mu(2)"] & H^*(0) * \mathbb{Z}/2
\end{tikzcd}$$

The rows are split short exact sequences and splittings can be chosen which extend the diagram commutatively, hence $\mu_2^4 \mu_4^2 = 0$.

(d) $Sq_0^1\colon H^*(2) \to H^*(0)$ and $Sq^2\colon H^*(2) \to H^*(2)$ are homomorphisms of degree 1 and 2 respectively.

(e) For $\tau = 2, 4$ we define $\mu_\tau^0\colon H^*(0) \to H^*(\tau)$ by $\mu_\tau^0(x) = \Delta(\tau)(x \otimes 1)$ and we set

$$Sq^1 = \mu_2^0 Sq_0^1.$$

With this notation the following diagram commutes and has exact rows.

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(0) \arrow[r, "2"] \arrow[d, "\mu_2^0"] & H^i(0) \arrow[r, "\mu_2^0"] \arrow[d, "\mu_4^0"] & H^i(2) \arrow[r, "Sq_0^1"] \arrow[d, equal] & H^{i+1}(0) \arrow[r, "2"] \arrow[d, "\mu_2^0"] & \cdots \\
\cdots \arrow[r] & H^i(2) \arrow[r, "\mu_4^2"] & H^i(4) \arrow[r, "\mu_2^4"] & H^i(2) \arrow[r, "Sq^1"] & H^{i+1}(2) \arrow[r, "\mu_4^2"] & \cdots
\end{tikzcd}$$

(f)
$$H^0(2) \supset \ker(Sq^2) \xrightarrow{\phi_4^2} H^3(4)/\mathrm{im}(\mu_4^2 Sq^2)$$

$$H^0(4) \supset \ker(Sq^2\mu_2^4) \xrightarrow{\phi_2^4} H^3(2)/\mathrm{im}(Sq^2)$$

are homomorphisms of degree 3 such that the following three diagrams commute.

$$\begin{array}{ccc}
\ker(Sq^2\mu_2^4) & \xrightarrow{\phi_2^4} & H^3(2)/\mathrm{im}(Sq^2) \\
\downarrow{\scriptstyle \mu_2^4} & & \downarrow{\scriptstyle \mu_4^2} \\
\ker(Sq^2) & \xrightarrow{\phi_4^2} & H^3(4)/\mathrm{im}(\mu_4^2 Sq^2)
\end{array}$$

$$\begin{array}{ccc}
\ker(Sq^2) & \xrightarrow{\phi_4^2} & H^3(4)/\mathrm{im}(\mu_4^2 Sq^2) \\
\downarrow{\scriptstyle Sq^1} & & \downarrow{\scriptstyle \mu_2^4} \\
H^1(2) & \xrightarrow{Sq^2} & H^3(2)
\end{array}$$

$$\begin{array}{ccc}
H^0(2) & \xrightarrow{Sq^2} & H^2(2) \\
\downarrow{\scriptstyle \mu_4^2} & & \downarrow{\scriptstyle Sq^1} \\
\ker(Sq^2\mu_2^4) & \xrightarrow{\phi_2^4} & H^3(2)/\mathrm{im}(Sq^2)
\end{array}$$

A *morphism* $f\colon S \to S'$ between A^3-cohomology systems is given by a tuple of homomorphisms of degree 0

$$f(0)\colon H^*(0) \to H^*(0)'$$

$$f(2)\colon H^*(2) \to H^*(2)'$$

$$f(4)\colon H^*(4) \to H^*(4)'$$

such that f is compatible with all operators and diagrams above. Let A^3-**cohomology** be the corresponding category. This is an additive category with the obvious notion of direct sum of A^3-cohomology systems given by the direct sum of abelian groups.

Recall that $\mathbf{A}_n^3$ is the full homotopy category of $(n-1)$-connected $(n+3)$-dimensional polyhedra which are finite.

(8.5.7) Theorem *Let $n \geq 4$. One has a detecting functor*

$$\lambda\colon \mathbf{A}_n^3 \longrightarrow A^3\textbf{-cohomology}$$

which carries a space X to the A^3-cohomology system $\lambda(X)$ given by $H^i(\tau) = H^{n+i}(X, \mathbb{Z}/\tau)$ for $\tau = 0, 2, 4$ and $i \in \mathbb{Z}$. Moreover $\Delta(\tau)$ and $\mu(\tau)$ are operators of the universal coefficient sequence and μ_2^4, μ_4^2, Sq_0^1, Sq^2 are defined in (8.5.2) *and* ϕ_4^2, ϕ_2^4 *are the Adem operations in* (8.5.3)(4)′, (5)′. *The functor* λ *is an additive functor.*

Hence isomorphism types of A^3-cohomology systems are in 1–1 correspondence with homology types of finite $(n-1)$-connected $(n+3)$-dimensional polyhedra, $n \geq 4$. The theorem was essentially obtained by Chang. Jäschke [AO] wrote down a complete proof. It shows that the classification via cohomology operations is considerably more complicated than the classification via boundary invariants in Section 8.1. The examples in (10.2.16) show that the pair of operations ϕ_2^4, ϕ_4^2 is needed for the classification. The classical Adem operations ϕ', ϕ'', ϕ_2^0 do not suffice to detect all possible homotopy types in $\mathbf{A}_n^3$.

9

ON THE HOMOTOPY CLASSIFICATION OF 2-CONNECTED 6-DIMENSIONAL POLYHEDRA

In this chapter we describe algebraic models which characterize the homotopy types of 2-connected 6-dimensional polyhedra. Such polyhedra are in the metastable range so that diverse features of 'quadratic algebra' are involved in the classification. We proceed in a similar way as in Chapter 8 where we classified $(n-1)$-connected $(n+3)$-dimensional homotopy types which are in the stable range $n \geq 4$. We apply boundary invariants which, via the classification theorem 3.4.4, yield algebraic classifying data for 2-connected 6-dimensional homotopy types.

9.1 Algebraic models of 2-connected 6-dimensional homotopy types

We introduce the purely algebraic category of A_3^3-systems. The main result of this chapter shows that A_3^3-systems are algebraic models which classify 2-connected 6-dimensional homotopy types. Recall that we have the *exterior square* which is the functor

(9.1.1)
$$\Lambda^2\colon \mathbf{Ab} \to \mathbf{Ab}$$

defined by $\Lambda^2(A) = A \otimes A/\{a \otimes a \sim 0\}$. Let H and L be abelian groups. A *quadratic Λ-map* is a homomorphism

$$\lambda\colon H \otimes \mathbb{Z}/2 \oplus \Lambda^2(H) \to L$$

which admits a factorization

$$\lambda\colon H \otimes \mathbb{Z}/2 \oplus \Lambda^2(H) \xrightarrow{\Delta \oplus 1} G(H) \oplus \Lambda^2(H) \to L.$$

Here we use (9.1.2) below. Let $\Lambda\mathbf{Ab}$ be the category of such maps; objects are quadratic Λ-maps and morphisms $(\psi_1, \psi_0)\colon \lambda \to \lambda'$ are pairs $(\psi_1, \psi_0) \in \mathrm{Hom}(L, L') \oplus \mathrm{Hom}(H, H')$ for which the diagram

$$\begin{tikzcd}
H \otimes \mathbb{Z}/2 \oplus \Lambda^2(H) \arrow[r, "(\psi_0)_*"] \arrow[d, "\lambda"] & H' \otimes \mathbb{Z}/2 \oplus \Lambda^2(H') \arrow[d, "\lambda'"] \\
L \arrow[r, "\psi_1"'] & L'
\end{tikzcd}$$

commutes. Here we set $(\psi_0)_* = \psi_0 \otimes \mathbb{Z}/2 \oplus \Lambda^2(\psi_0)$. We have for any abelian group A the extensions

(9.1.2) $$A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \overset{\mu}{\twoheadrightarrow} A * \mathbb{Z}/2$$

(9.1.3) $$\mathrm{Ext}(A,\mathbb{Z}/2) \overset{\Delta}{\rightarrowtail} \mathrm{Hom}(G(A),\mathbb{Z}/4) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A,\mathbb{Z}/2)$$

as in (8.1.1) and (8.1.2). We use (9.1.2) for the definition of morphisms in the category **G**; see Section 1.6. Recall that **QM**($\mathbb{Z}$) denotes the category of quadratic $\mathbb{Z}$-modules; see Section 6.13. We introduce a functor

(9.1.4) $$\Lambda_1 : \mathbf{G}^{\mathrm{op}} \to \mathbf{QM}(\mathbb{Z})$$

as follows. Let ∂: Hom $(A,\mathbb{Z}/2) \to \mathrm{Ext}(A,\mathbb{Z})$ be the connecting homomorphism induced by the exact sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$. Then we define for an object A in **G**

$$\Lambda_1(A) = \big(\mathrm{Hom}(G(A),\mathbb{Z}/4) \xrightarrow{\partial\mu} \mathrm{Ext}(A,\mathbb{Z}) \xrightarrow{0} \mathrm{Hom}(G(A),\mathbb{Z}/4)\big)$$

where $H = \partial\mu$ is given by μ and ∂ above and where $P = 0$ is trivial. The functor Λ_1 carries a morphism $(\varphi,\overline{\varphi})$: $A \to B$ in **G** to the map $(\varphi,\overline{\varphi})^*$: $\Lambda_1(A) \to \Lambda_1(B)$ in **QM**($\mathbb{Z}$) given by Hom $(\overline{\varphi},\mathbb{Z}/4)$ and Ext $(\varphi,\mathbb{Z})$.

We observe that the exact sequence (9.1.3) induces the following short exact sequence of quadratic $\mathbb{Z}$-modules

(9.1.5) $$\Lambda_0(A) \overset{\Delta}{\rightarrowtail} \Lambda_1(A) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A,\mathbb{Z}/2)$$

which is natural in $A \in \mathbf{G}$. Here we set

$$\begin{array}{ccccccc}
\Lambda_0(A) & = & (\mathrm{Ext}(A,\mathbb{Z}/2) & \xrightarrow{0} & \mathrm{Ext}(A,\mathbb{Z}) & \xrightarrow{0} & \mathrm{Ext}(A,\mathbb{Z}/2)) \\
\downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle\Delta} \\
\Lambda_1(A) & = & (\mathrm{Hom}(G(A),\mathbb{Z}/4) & \xrightarrow{H} & \mathrm{Ext}(A,\mathbb{Z}) & \xrightarrow{P} & \mathrm{Hom}(G(A),\mathbb{Z}/4)) \\
\downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \downarrow & & \downarrow{\scriptstyle\mu} \\
\mathrm{Hom}(A,\mathbb{Z}/2) & = & (\mathrm{Hom}(A,\mathbb{Z}/2) & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}(A,\mathbb{Z}/2))
\end{array}$$

The quadratic $\mathbb{Z}$-module corresponding to an abelian group D is given by $D = (D \to 0 \to D)$. Hence one has the isomorphism of quadratic $\mathbb{Z}$-modules.

(9.1.6) $$\Lambda_0(A) = \mathrm{Ext}(A,\mathbb{Z}/2) \oplus \mathbb{Z}^{\Lambda} \otimes \mathrm{Ext}(A,\mathbb{Z})$$

where $\mathbb{Z}^\Lambda = (0 \to \mathbb{Z} \to 0)$ is the quadratic $\mathbb{Z}$-module associated with the exterior square functor Λ^2 in (9.1.1). We now apply the quadratic tensor product which is a functor

$$\otimes : \mathbf{Ab} \times \mathbf{QM}(\mathbb{Z}) \to \mathbf{Ab}.$$

Then (9.1.6) shows that for an abelian group B we have the isomorphism in the top row of the following diagram

(9.1.7)

$$\begin{array}{ccc}
B \otimes \Lambda_0(A) & = & B \otimes \mathrm{Ext}(A, \mathbb{Z}/2) \oplus \Lambda^2(B) \otimes \mathrm{Ext}(A, \mathbb{Z}) \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon_1 \oplus \varepsilon_2} \\
\mathrm{Ext}(A, B \otimes \mathbb{Z}/2 \oplus \Lambda^2 B) & = & \mathrm{Ext}(A, B \otimes \mathbb{Z}/2) \oplus \mathrm{Ext}(A, \Lambda^2 B)
\end{array}$$

Here $\varepsilon_1, \varepsilon_2$ on the right-hand side are the *evaluation maps* with $\varepsilon_1(x \otimes y) = (x \otimes \mathbb{Z}/2)_*(y)$ and $\varepsilon_2(u \otimes v) = u_*(v)$ where $x: \mathbb{Z} \to B$ and $u: \mathbb{Z} \to \Lambda^2(B)$ denote the homomorphisms given by $x \in B$ and $u \in \Lambda^2(B)$ respectively. We are now ready for the definition of the bifunctor $\tilde{G}$ which replaces the bifunctor $\overline{G}$ in Definition 8.1.3.

(9.1.8) (A) Definition We define a functor

$$\tilde{G} : \mathbf{G}^{\mathrm{op}} \times \Lambda\mathbf{Ab} \to \mathbf{Ab}$$

as follows. If A or H are finitely generated let $\tilde{G}(A, \lambda)$ with λ as in (9.1.1) be given by the push-out diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Ext}(A, L) & \xrightarrow{\Delta} & \tilde{G}(A, \lambda) & \xrightarrow{\mu} & \mathrm{Hom}(A, H \otimes \mathbb{Z}/2) \to 0 \\
\uparrow{\scriptstyle \lambda_*} & \text{push} & & & \| \\
\mathrm{Ext}(A, H \otimes \mathbb{Z}/2 \oplus \Lambda^2 H) & & \uparrow{\scriptstyle \bar{\lambda}_*} & & \| \\
\uparrow{\scriptstyle \varepsilon} & & & & \| \\
H \otimes \Lambda_0(A) & \xrightarrow{H \otimes \Delta} & H \otimes \Lambda_1(A) & \xrightarrow{H \otimes \mu} & H \otimes \mathrm{Hom}(A, \mathbb{Z}/2) \to 0
\end{array}$$

Here the bottom row is obtained by applying the quadratic tensor product $H \otimes$ to the short exact sequence (9.1.5). The bottom row need not be short exact. For a morphism $(\varphi, \overline{\varphi}): A \to B$ in $\mathbf{G}$ we obtain

$$(\varphi, \overline{\varphi})^* : \tilde{G}(B, \lambda) \to \tilde{G}(A, \lambda)$$

$$(\varphi, \overline{\varphi})^* = \mathrm{Ext}(\varphi, L) \oplus H \otimes \Lambda_1(\varphi, \overline{\varphi})$$

and for a morphism $(\psi_1, \psi_0)\colon \lambda \to \lambda'$ in $\Lambda\mathbf{Ab}$ as in (9.1.1) we get

$$(\psi_1, \psi_0)_* : \Lambda_1(A, \lambda) \to \Lambda_1(A, \lambda')$$

$$(\psi_1, \psi_0)_* = \operatorname{Ext}(A, \psi_1) \oplus \psi_0 \otimes \overline{G}(A).$$

(9.1.8) (B) Definition In general we have to use the following intricate definition of the group $\tilde{G}(A, \lambda)$ which canonically coincides with Definition (9.1.8) (A) if A or H are finitely generated. Here we use notation as in Addendum 8.1.3 (C). The short exact sequence of quadratic $\mathbb{Z}$-modules in (9.1.5) yields for $A = \mathbb{Z}/2$ the following commutative diagram

$$\begin{tikzcd} K(H) \arrow[r, tail] \arrow[d, "\theta''"] & \mathbb{Z}/4[H] \arrow[d, "\theta'"] \arrow[dr, "p_2 p"] & \\ \operatorname{cok}(\tau_H) \oplus \Lambda^2 H \arrow[r, tail] & H \otimes \Lambda_1(\mathbb{Z}/2) \arrow[r] & H \otimes \mathbb{Z}/2 \end{tikzcd}$$

in which the bottom row is short exact. Here the homomorphism θ' carries $[x]$ with $x \in H$ to $\theta'[x] = x \otimes 1$ where $1 \in \mathbb{Z}/4$. For this recall that $\Lambda_1(\mathbb{Z}/2) = (\mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/4)$. Let

$$\bar{\theta}_H : K(H) * \mathbb{Z}/2 \to \operatorname{cok}(\tau_H) \oplus \Lambda^2 H$$

be the restriction of θ''. One can check that $\bar{\theta}_H = (\theta_H, \gamma_H p)$ is given by θ_H in Addendum 8.1.3 (C) and by γ_H in Definition 6.2.13(B). We now obtain $\tilde{G}(A, \lambda)$ by the following push-out diagram

$$\begin{tikzcd} \operatorname{Hom}(A * \mathbb{Z}/2, K(H) * \mathbb{Z}/2) \arrow[r, tail, "j"] \arrow[d, "\theta"] & \operatorname{Hom}(G(A), \mathbb{Z}/4[H]) \arrow[dd] \arrow[ddr, "(p_2)_* \Delta^*"] & \\ \Delta_* \operatorname{Ext}(A, H \otimes \mathbb{Z}/2 \oplus \Lambda^2 H) \arrow[d, "\lambda_*"] \arrow[r, phantom, "\text{push}"] & & \\ \operatorname{Ext}(A, L) \arrow[r, tail, "\Delta"] & \tilde{G}(A, \lambda) \arrow[r, two heads, "\mu"] & \operatorname{Hom}(A, H \otimes \mathbb{Z}/2) \end{tikzcd}$$

Compare the diagram in Addendem 8.1.3 (C). The natural transformation θ is defined by

$$\theta(\alpha) = \alpha^* \bar{\theta}_H$$

with

$$\bar{\theta}_H \in \Delta_* \operatorname{Ext}(K, H \otimes \mathbb{Z}/2 \oplus \Lambda^2 H) = \operatorname{Hom}(K, \operatorname{Cok}(\tau_H) \oplus \Lambda^2 H)$$

where $K = K(H) * \mathbb{Z}/2$. This completes the definition of the bifunctor $\tilde{G}$ in Definition 9.1.8 (A).

Using the notation on the group $G(\eta)$ in Definition 8.1.3 and the group $\tilde{G}(A, \lambda)$ in Definition 9.1.8 above we are now ready to define algebraic models of 2-connected 6-dimensional homotopy types which we call A_3^3-systems.

(9.1.9) Definition An *A_3^3-system*

$$S = (H_3, H_5, H_6, \pi_4, b_5, \eta, b_6, \beta) \tag{1}$$

is a tuple consisting of abelian groups H_3, H_5, H_6, π_4 and elements

$$\begin{aligned} &b_5 \in \mathrm{Hom}(H_5, H_3 \otimes \mathbb{Z}/2) \\ &\eta \in \mathrm{Hom}(H_3 \otimes \mathbb{Z}/2, \pi_4) \\ &b_6 \in \mathrm{Hom}(H_6, G(\eta) \oplus \Lambda^2 H_3) \\ &\beta \in \tilde{G}(H_5, \eta_\square). \end{aligned} \tag{2}$$

Here $\eta_\square = q(\Delta(\eta \otimes 1) \oplus \Lambda^2 H_3)$ is the composite

$$\eta_\square : H_3 \otimes \mathbb{Z}/2 \oplus \Lambda^2 H_3 \to G(\eta) \oplus \Lambda^2 H_3 \overset{q}{\twoheadrightarrow} \mathrm{cok}(b_6) \tag{3}$$

where q is the quotient map for the cokernel of b_6 and where the first arrow, $\Delta(\eta \otimes 1) \oplus \Lambda^2 H_3$, is induced by $\Delta(\eta \otimes 1)$: $H_3 \otimes \mathbb{Z}/2 \to \pi_4 \otimes \mathbb{Z}/2 \to G(\eta)$; compare the definition of $G(\eta)$ in Definition 8.1.3. These elements satisfy the following conditions (4) and (5). The sequence

$$H_5 \xrightarrow{b_5} H_3 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_4 \tag{4}$$

is exact and β satisfies

$$\mu(\beta) = b_5 \tag{5}$$

where μ is the operator in Definition 9.1.8. A morphism

$$(\varphi_3, \varphi_5, \varphi_6, \varphi_\pi, \varphi_\Gamma): S \to S' \tag{6}$$

between A_3^3-systems is a tuple of homomorphisms

$$\begin{cases} \varphi_i : H_i \to H_i' & (i = 3,5,6) \\ \varphi_\pi : \pi_4 \to \pi_4' \\ \varphi_\Gamma : G(\eta) \to G(\eta') \end{cases}$$

such that the following diagrams (7), (8), and (9) commute and such that equation (10) holds.

$$\begin{tikzcd} H_5 \arrow[r, "b_5"] \arrow[d, "\varphi_5"] & H_3 \otimes \mathbb{Z}/2 \arrow[r, "\eta"] \arrow[d, "\varphi_3 \otimes 1"] & \pi_4 \arrow[d, "\varphi_\pi"] \\ H_5' \arrow[r, "b_5'"'] & H_3' \otimes \mathbb{Z}/2 \arrow[r, "\eta'"'] & \pi_4' \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} \pi_4 \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] \arrow[d, "\varphi_\pi \otimes 1"] & G(\eta) \arrow[r, "\mu"] \arrow[d, "\varphi_\Gamma"] & H_3 * \mathbb{Z}/2 \arrow[d, "\varphi_3 * 1"] \\ \pi_4' \otimes \mathbb{Z}/4 \arrow[r, tail, "\Delta"] & G(\eta') \arrow[r, "\mu"] & H_3' * \mathbb{Z}/2 \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} H_6 \arrow[r, "b_6"] \arrow[d, "\varphi_6"] & G(\eta) \oplus \Lambda^2 H_3 \arrow[d, "\varphi_\Gamma \oplus \Lambda^2(\varphi_3)"] \\ H_6' \arrow[r, "b_6'"'] & G(\eta') \oplus \Lambda^2 H_3' \end{tikzcd} \tag{9}$$

Hence $\varphi_\Gamma \oplus \Lambda^2(\varphi_3)$ induces $\varphi_\Gamma \oplus \Lambda^2(\varphi_3)$: $\operatorname{cok}(b_6) \to \operatorname{cok}(b_6')$ such that $(\varphi_3, \varphi_\Gamma \oplus \Lambda^2(\varphi_3))$: $\eta_\square \to \eta'_\square$ is a morphism in Λ**Ab** which induces $(\varphi_3, \varphi_\Gamma \oplus \Lambda^2(\varphi_3))_*$ as in Definition 9.1.8. We have

$$(\varphi_3, \varphi_\Gamma \oplus \Lambda^2(\varphi_3))_*(\beta) = (\varphi_3, \overline{\varphi}_3)^*(\beta') \tag{10}$$

in $\bar{G}(H_3, \eta'_\square)$. Here we choose $\overline{\varphi}_3$ for φ_3 as in (8.1.1). The right-hand side of (10) does not depend on the choice of $\overline{\varphi}_3$; compare Definition 9.1.8. An A_3^3-system S as above is *free* if H_6 is free abelian, and S is *injective* if b_6: $H_6 \to \Gamma(\eta) \oplus \Lambda^2(H_3)$ is injective. Let A_3^3-**System**, resp. A_3^3-**system**, be the full category of free, resp. injective, A_3^3-systems. We have the canonical forgetful functor

$$\phi\colon A_3^3\text{-}\mathbf{System} \to A_3^3\text{-}\mathbf{system} \tag{11}$$

which replaces b_6 by the inclusion $b_6(H_6) \subset G(\eta) \oplus \Lambda^2(H_3)$ of the image of b_6. One readily checks that the forgetful functor ϕ is full and representative.

(9.1.10) Definition We associate with an A_3^3-system S as in Definition 9.1.9 the exact *Γ-sequence*

$$H_6 \xrightarrow{b_6} G(\eta) \oplus \Lambda^2(H_3) \to \pi_5 \to H_5 \xrightarrow{b_5} H_3 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_4 \to H_4 \to 0.$$

Here $H_4 = \operatorname{cok}(\eta)$ and the extension

$$\operatorname{cok}(b_6) \rightarrowtail \pi_5 \twoheadrightarrow \ker(b_5) \tag{1}$$

is obtained by the element β in Definition 9.1.9, that is the group $\pi_5 = \pi(\beta_\dagger)$ is given by the extension element $\beta_\dagger \in \mathrm{Ext}(\ker(b_5), \mathrm{cok}(b_6))$ defined by

$$\beta_\dagger = \Delta^{-1}(j, \bar{j})^*(\beta). \tag{2}$$

Here $j\colon \ker(b_5) \subset H_5$ is the inclusion. The element $\beta_\dagger$ does not depend on the choice of $(j, \bar{j})$ in $\mathbf{G}$. Compare (2.6.7).

Recall that $\mathbf{spaces}_3^3$ is the full homotopy category of 2-connected 6-dimensional CW-spaces X and that $\mathbf{types}_3^2$ is the full homotopy category of 2-connected 5-types. We have the Postnikov functor

$$P\colon \mathbf{spaces}_3^3 \to \mathbf{types}_3^2$$

which carries X to its 5-type.

(9.1.11) Theorem *There are detecting functors*

$$\Lambda'\colon \mathbf{spaces}_3^3 \to A_3^3\text{-}\mathbf{Systems}$$

$$\lambda'\colon \mathbf{types}_3^2 \to A_3^3\text{-}\mathbf{systems}.$$

Moreover there is a natural isomorphism $\phi\Lambda'(X) = \lambda' P(X)$ for the forgetful functor ϕ in Definition 9.1.9 (11) *and for the Postnikov functor P above.*

Let S be a free A_3^3-system. The detecting functor Λ' in Theorem 9.1.11 shows that there is a unique 2-connected 6-dimensional homotopy type $X = X_S$ with $\Lambda'(X) \cong S$. Then the Γ-sequence for S is the top row of the following commutative diagram

(9.1.12)

$$\begin{tikzcd}
H_6 \arrow[r] \arrow[d, equal] & G(\eta) \oplus \Lambda^2(H_3) \arrow[r] \arrow[d, equal] & \pi(\beta_\dagger) \arrow[r] \arrow[d, "="'] & H_5 \arrow[r] \arrow[d, equal] & H_3 \otimes \mathbb{Z}/2 \arrow[r, "\eta"] \arrow[d, equal] & \pi_4 \arrow[r, two heads] \arrow[d, equal] & H_4 \arrow[d, equal] \\
H_6 X \arrow[r] & \Gamma_5 X \arrow[r] & \pi_5 X \arrow[r] & H_5 X \arrow[r] & \Gamma_4 X \arrow[r] & \pi_4 X \arrow[r, two heads] & H_4 X
\end{tikzcd}$$

The bottom row is Whitehead's certain exact sequence for X. The diagram describes a weak natural isomorphism of exact sequences.

Proof of Theorem 9.1.11 Using similar arguments as in Section 8.4 one gets the theorem by the results in the following sections 9.2 and 9.3. In particular one has the canonical forgetful functor

$$\phi\colon \mathbf{Bypes}(\Gamma\mathbf{G}, F) \to A_3^3\text{-}\mathbf{Systems}$$

which is a detecting functor so that Λ' in the statement of the theorem is the composite of ϕ and the detecting functor Λ' in the classification theorem 3.4.4. □

Remark In Theorem 6.4.1 we describe the detecting functor

$$\Lambda'\colon \mathbf{spaces}(3,5)_\pi \to \mathbf{Bypes}(\mathbf{Ab}, H_5^{(3)}) \tag{1}$$

where $H_5^{(3)}$ is the Eilenberg–Mac Lane functor which we compute in 9.2.11 below and which satisfies

$$H_5^{(3)}(A,B) = \bar{G}(A,\lambda). \tag{2}$$

Here $\lambda\colon B \otimes \mathbb{Z}/2 \oplus \Lambda^2 B \to B * \mathbb{Z}/2 \oplus \Lambda^2 B$ is given by $\lambda = 0 \oplus \Lambda^2 B$; see Remark 9.2.12. One readily checks that the functor Λ' in Theorem 9.1.11 actually yields a commutative diagram

$$\begin{array}{ccc} \mathbf{spaces}_3^3 & \xrightarrow{\Lambda'} & A_3^3\text{-}\mathbf{Systems} \\ \cup & & \cup \\ \mathbf{spaces}(3,5)_\pi & \xrightarrow{\Lambda'} & \mathbf{Bypes}(\mathbf{Ab}, H_5^{(3)}) \end{array}$$

In fact, bypes in $\mathbf{Bypes}(\mathbf{Ab}, H_5^{(3)})$ can be identified via (2) with free A_3^3-systems for which $\pi_4 = 0$. In Theorem 6.4.1 we also describe the detecting functor

$$\Lambda\colon \mathbf{spaces}(3,5)_\pi \to \mathbf{Kypes}\big(\mathbf{Ab}, H_{(3)}^6\big)$$

by use of k-invariants. The corresponding detecting functor Λ for $\mathbf{spaces}_3^3$ which also uses k-invariants, however, is not known.

(9.1.13) Definition We define the *suspension* functor

$$\Sigma\colon A_3^3\text{-}\mathbf{Systems} \to A^3\text{-}\mathbf{Systems} \tag{1}$$

as follows. First we obtain the natural map Σ between quadratic $\mathbb{Z}$-modules given by

$$\begin{array}{ccccccc} \Lambda_1(A) & = \big(\mathrm{Hom}(G(A),\mathbb{Z}/4) & \xrightarrow{\partial\mu} & \mathrm{Ext}(A,\mathbb{Z}) & \xrightarrow{0} & \mathrm{Hom}(G(A),\mathbb{Z}/4)\big) \\ \Sigma\downarrow & \downarrow 1 & & \downarrow & & \downarrow 1 \\ \mathrm{Hom}(G(A),\mathbb{Z}/4) & = \big(\mathrm{Hom}(G(A),\mathbb{Z}/4) & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}(G(A),\mathbb{Z}/4)\big) \end{array}$$

Now let S be an A_3^3-system as in Definition 9.1.9. Then we have for $\eta_\square$ the commutative diagram

$$\begin{array}{lcccc}
\eta_\square : H_3 \otimes \mathbb{Z}/2 \oplus \Lambda^2 H_3 & \to & G(\eta) \oplus \Lambda^2 H_3 & \overset{q}{\twoheadrightarrow} & \mathrm{cok}(b_6) \\
\quad\downarrow p_1 & & \downarrow p_1 & & \downarrow \sigma \\
\eta_\# : \quad H_3 \otimes \mathbb{Z}/2 & \to & G(\eta) & \overset{q}{\twoheadrightarrow} & \mathrm{cok}(p_1 b_6)
\end{array}$$

where p_1 is the projection and where $\eta_\# = q\Delta(\eta \otimes 1)$ as in Definition 8.1.4(3). Using Σ and σ above we obtain the homomorphism

$$\Sigma : \tilde{G}(H_3, \eta_\square) \to \tilde{G}(H_3, \eta_\#)$$

$$\Sigma = \mathrm{Ext}(H_3, \sigma) \oplus H_3 \otimes \Sigma. \tag{2}$$

For this compare the definition of the bifunctors $\tilde{G}$ and $\overline{G}$ respectively. The suspension functor Σ above is now defined by $\Sigma(S) = S'$ where S' is the A^3-system given by

$$S' = (H_3, H_5, H_6, \pi_4, b_5, \eta, p_1 b_6, \Sigma(\beta)). \tag{3}$$

Moreover the suspension functor is the identity on morphisms.

We now obtain for $n \geq 4$ the following diagram of functors

(9.1.14)
$$\begin{array}{ccc}
\textbf{spaces}_3^3 & \xrightarrow{\Sigma^{n-3}} & \textbf{spaces}_n^3 \\
\downarrow \Lambda' & & \downarrow \Lambda' \\
A_3^3\textbf{-Systems} & \xrightarrow{\Sigma} & A^3\textbf{-Systems}
\end{array}$$

Here Σ^{n-3} is the $(n-3)$-fold suspension functor for spaces and Σ is the algebraic suspension functor in Definition 9.1.13. Moreover Λ' denotes the detecting functor in Theorems 9.1.11 and 8.1.6 respectively.

(9.1.15) Theorem *Diagram* (9.1.14) *commutes up to a natural isomorphism, that is, for a 2-connected 6-dimensional polyhedron X one has an isomorphism of A^3-systems*

$$\Lambda'(\Sigma^{n-3}X) \cong \Sigma(\Lambda' X)$$

which is natural in X, $n \geq 4$.

Since by the Freudenthal suspension theorem the functor Σ^{n-3} is representative, also the algebraic functor Σ is representative; this also can be readily seen by the definition of Σ. As an application we get

(9.1.16) Corollary *Let X be a 2-connected 6-dimensional polyhedron with H_3X cyclic. Then the homotopy type of X is determined by the suspension ΣX, that is $\Sigma X \simeq \Sigma Y$ implies $X \simeq Y$.*

Proof Since H_3X is cyclic we have $\Lambda^2 H_3 X = 0$. Hence $p_1 b_6 = b_6$ in S' of Definition 9.1.13 (3). Moreover Σ in Definition 9.1.13 (2) is an isomorphism since Σ is compatible with Δ and μ. □

Similarly we derive from (9.1.12) the

(9.1.17) Corollary *Let X be a 2-connected polyhedron with H_3X cyclic. Then the suspension Σ: $\pi_5 X \cong \pi_6 \Sigma X$ is an isomorphism.*

For example we get by Proposition 8.1.10

(9.1.18) $$\pi_5(\Sigma^2 \mathbb{R}P_\infty) = \mathbb{Z}/8.$$

9.2 On $\pi_5 M(A,3)$

We compute the homotopy groups $\pi_5 M(A,3)$ and $\pi_4(A, M(B,3))$ of Moore spaces and we determine the functorial properties of these groups. As special cases of (8.2.3) we consider the functors

(9.2.1) $$\pi_5(-,3)\colon \mathbf{G} = \mathbf{M}^3 \to \mathbf{Ab}$$

(9.2.2) $$\pi_4^{(3)}\colon \mathbf{G}^{\mathrm{op}} \times \mathbf{G} = (\mathbf{M}^4)^{\mathrm{op}} \times \mathbf{M}^3 \to \mathbf{Ab}$$

which carry A to $\pi_5 M(A,3)$ and (A,B) to $\pi_4(A, M(B,3))$ respectively. We want to describe the functors above up to a canonical natural isomorphism by purely algebraic functors defined via the algebraic structure of the category $\mathbf{G}$.

(9.2.3) Lemma *One has a natural short exact sequence*

$$A \otimes \mathbb{Z}/2 \oplus \Lambda^2(A) \overset{\Delta}{\rightarrowtail} \pi_5 M(A,3) \overset{\mu}{\twoheadrightarrow} A * \mathbb{Z}/2.$$

Moreover the suspension Σ yields the following short exact sequence which is naturally split.

$$\Lambda^2(A) \overset{\Delta}{\rightarrowtail} \pi_5 M(A,3) \overset{\Sigma}{\twoheadrightarrow} \pi_6 M(A,4)$$

Proof We apply the second sequence in Corollary 6.15.15 for $n = 3$. Since $\pi_5\{S^n\} = \mathbb{Z}/2 \oplus \mathbb{Z}^\Lambda$ we get $A \otimes \pi_5\{S^n\} = A \otimes \mathbb{Z}/2 \oplus \Lambda^2(A)$. Moreover

$\pi_4(S^n) = \mathbb{Z}/2$ so that the first sequence of the lemma is a special case of Corollary 6.15.15. Next the Freudenthal suspension theorem shows that the second sequence is short exact. Moreover this sequence is naturally split since the composite $\Sigma\pi_4 M(A,2) \subset \pi_5 M(A,3) \xrightarrow{\Sigma} \pi_6 M(A,4)$ is an isomorphism. □

The lemma yields by Theorem 8.2.5 the following result.

(9.2.4) Theorem *The functor* $\pi_5(-,3)$: $\mathbf{G} \to \mathbf{Ab}$ *carries* A *to* $G(A) \oplus \Lambda^2(A)$ *and carries* (φ, ψ) *to* $\psi \oplus \Lambda^2(\varphi)$, *that is, there is an isomorphism* $\pi_5(M(A,3)) = G(A) \oplus \Lambda^2(A)$ *which is natural in* $A \in \mathbf{G}$.

This is the purely algebraic description of the functor $\pi_5(-,3)$ in (9.2.1). Let $\mathbf{Add}(\mathbb{Z})$ be the category of finitely generated free abelian groups. We consider the quadratic functor

$$\pi_4^A : \mathbf{Add}(\mathbb{Z}) \to \mathbf{Ab} \tag{9.2.5}$$

which carries B to the homotopy group with coefficients A, $\pi_4(A, M(B,3))$. This functor is determined by the quadratic $\mathbb{Z}$-module

$$\pi_4^A\{S^3\} = \left(\pi_4(A,S^3) \xrightarrow{H} \pi_4(A,S^5) \xrightarrow{P} \pi_4(A,S^3)\right) \tag{1}$$

as in Definition 6.13.10 (4). Here we have

$$\pi_4(A,S^5) = \mathrm{Ext}(A,\mathbb{Z}) \tag{2}$$

$$\pi_4(A,S^3) = \mathrm{Hom}(G(A),\mathbb{Z}/4). \tag{3}$$

Both isomorphisms are natural in $A \in \mathbf{G}$; compare Lemma 8.2.7. For the quadratic $\mathbb{Z}$-module $\Lambda_1(A)$ in (9.1.4) we obtain the following topological interpretation:

(9.2.6) Proposition *There is an isomorphism* $\pi_4^A\{S^3\} = \Lambda_1(A)$ *of quadratic* $\mathbb{Z}$*-modules which is natural in* $A \in \mathbf{G}$

Proof Using the isomorphism η_* in the proof of Lemma 8.2.7 we see that each element $x \in \pi_4(A,S^3)$ is a composite

$$x: M(A,4) \xrightarrow{y} M(\mathbb{Z}/2,4) \xrightarrow{\eta} S^3$$

where y is a suspended map. Hence H in (9.2.5) (1) is determined by the left distributivity law $(\alpha, \beta \in \pi_3 U)$

$$\begin{aligned} x^*(\alpha+\beta) &= y^*\eta^*(\alpha+\beta) = y^*(\eta^*\alpha + \eta^*\beta + [\alpha,\beta]\gamma_2\eta) \\ &= x^*\alpha + x^*\beta + [\alpha,\beta](\gamma_2\eta)y \end{aligned}$$

with $H(x) = (\gamma_2\eta)y$. Here γ_2 is the James–Hopf invariant and $[\ ,\ ]$ is the Whitehead product; compare Lemma 6.15.2. For the computation of $\gamma_2\eta$ we consider the adjoint $\bar{\eta}$: $M(\mathbb{Z}/2,3) \to \Omega S^3 \simeq J(S^2)$ of η where $J(S^2)$ is the infinite reduced product of James with 4-skeleton $S^2 \cup_w e^4$. Here $w = [\iota_2, \iota_2]$ is the Whitehead square. Hence $\bar{\eta}$ yields a map $\bar{\eta}$: $S^3 \cup_2 e^4 \to S^2 \cup_w e^4$ which extends the Hopf map η_2: $S^3 \to S^2$. Since $2\eta_2 = w$ we see that $\bar{\eta}$ is a principal map between mapping cones associated with

$$\begin{array}{ccc} S^3 & \xrightarrow{1} & S^3 \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle w} \\ S^3 & \xrightarrow[\eta_2]{} & S^2 \end{array}$$

This implies that $q = \gamma_2\eta$: $M(\mathbb{Z}/2,4) \to S^5$ is the pinch map. In fact $\gamma_2\eta$ is the composite of $\bar{\eta}$ and the James map $J(S^2) \to J(S^4)$, which is of degree 1 in dimension 4. Therefore H in (9.2.5) (1) carries $x = \eta y$ to $qy \in \operatorname{Ext}(A,\mathbb{Z})$. Using the isomorphism (9.2.5) (3) the element x corresponds to $\psi \in \operatorname{Hom}(G(A),\mathbb{Z}/4)$ with $\psi = \pi_4(\mathbb{Z}/2, y)$ and $\mu(\psi) = H_4(y) \in \operatorname{Hom}(A,\mathbb{Z}/2)$. Let (y_1, y_0): $d_A \to d_2$ be a chain map representing $H_4(y) = \mu(\psi)$. Then $qy \in \operatorname{Ext}(A,\mathbb{Z})$ is represented by $y_1 \in \operatorname{Hom}(A_1,\mathbb{Z})$. Equivalently we have $qy = \partial\mu(\psi)$ and hence we get $H = \partial\mu$ in $\Lambda_1(A)$. We clearly have $P = 0$ since S^3 is an H-space. □

We are now able to characterize the functor $\pi_4^{(3)}$ in (9.2.2) similarly as in Theorem 8.2.10.

(9.2.7) Theorem *Let Δ_G: $\mathbf{G} \to \Lambda\mathbf{Ab}$ be the functor which carries $B \in \mathbf{G}$ to the inclusion $\Delta_G(B)$: $B \otimes \mathbb{Z}/2 \oplus \Lambda^2(B) \subset G(H) \oplus \Lambda^2(B)$ which is a Λ-quadratic map. Then there is an isomorphism*

$$[M(A,4), M(B,3)] = \pi_4^{(3)}(A,B) = \bar{G}(A, \Delta_G(B))$$

which is natural in $A, B \in \mathbf{G}$ and which is compatible with Δ and μ in the universal coefficient sequence. Here $\bar{G}$ is the bifunctor in Definition 9.1.8.

We point out that the isomorphism in Theorem 9.2.7 is available for all abelian groups A, B.

Proof of Theorem 9.2.7 If A or B are finitely generated we obtain the following commutative diagram which is natural in $M(A,4) \in \mathbf{M}^4$ and $M(B,3) \in \mathbf{M}^3$.

$$\begin{array}{ccccc} B \otimes \Lambda_0(A) & \xrightarrow{1\otimes\Delta} & B \otimes \pi_4^A\{S^3\} & \xrightarrow{1\otimes\mu} & B \otimes \operatorname{Hom}(A,\mathbb{Z}/2) \\ \downarrow{\scriptstyle \Delta_*\varepsilon} & & \downarrow{\scriptstyle k} & & \| \\ \operatorname{Ext}(A, \pi_5(B,3)) & \xrightarrow{\Delta} & \pi_4(A, M(B,3)) & \xrightarrow{\mu} & \operatorname{Hom}(A, B \otimes \mathbb{Z}/2) \end{array}$$

Using Proposition 9.2.6 as an identification the top row is defined by applying the quadratic tensor product to the short exact sequence (9.1.5); compare also Definition 9.1.8 (1). We define k in the diagram by composition of maps, that is, for $b, b' \in B = \pi_n M(B,n)$, $a \in \pi_4(A, S^3)$, and $c \in \pi_4(A, S^5)$ let

$$k(b \otimes a) = b \circ a \quad \text{and} \quad k([b,b'] \otimes c) = [b,b'] \circ c.$$

On the right-hand side $[b,b']$ denotes the Whitehead product. The bottom row is the universal coefficient sequence where we have the identification $\pi_4 M(B,3) = B \otimes \mathbb{Z}/2$ and

$$\pi_5 M(B,3) = \pi_5(B,3) = G(B) \oplus \Lambda^2(B);$$

see Theorem 9.2.4. We obtain by $\Delta\colon B \otimes \mathbb{Z}/2 \rightarrowtail G(B)$ and ε in (9.1.7) the composite $\Delta_* \varepsilon$,

$$B \otimes \Lambda_0(A) \xrightarrow{\varepsilon} \mathrm{Ext}(A, B \otimes \mathbb{Z}/2 \oplus \Lambda^2 B) \xrightarrow{\Delta_*} \mathrm{Ext}(A, \pi_5(B,3))$$

where $\Delta_* = \mathrm{Ext}(A, \Delta \oplus \Lambda^2 B)$. Now one can check that the diagram above commutes. Since the rows are exact this diagram is actually a push-out diagram. This proves the theorem if A or B are finitely generated. In the general case we proceed similarly as in the proof of Theorem 8.2.10. □

We have the commutative diagram

(9.2.8)

$$\begin{tikzcd}
\mathrm{Ext}(A, G(B) \oplus \Lambda^2 B) \arrow[r, tail, "\Delta"] \arrow[d, equal] & \tilde{G}(A, \Delta_G(B)) \arrow[r, two heads, "\mu"] \arrow[d, equal] & \mathrm{Hom}(A, B \otimes \mathbb{Z}/2) \arrow[d, equal] \\
\mathrm{Ext}(A, \pi_5 M(B,3)) \arrow[r, tail] & \pi_4(A, M(B,3)) \arrow[r, two heads] & \mathrm{Hom}(A, \pi_4 M(B,3))
\end{tikzcd}$$

The left-hand side is the isomorphism given by Theorem 9.2.4. The diagram is the metastable analogue of the corresponding stable result in (8.2.11).

We now apply the push-out diagram (6.6.7) (ii) to the bottom row of (9.2.8). This yields the connection of the Moore bifunctor $\pi_4^{(3)}$ with the Eilenberg–Mac Lane functor $H_5^{(3)}$. For the operator Q, $n = 3$, in Theorem 6.6.6 one has the simple description

(9.2.9)

$$\begin{tikzcd}
\pi_5(B,3) = \pi_5 M(B,3) \arrow[r, "Q"] \arrow[d, equal] & H_6 K(B,3) = H_6(B,3) \arrow[d, equal] \\
G(B) \oplus \Lambda^2(B) \arrow[r, two heads, "\mu \oplus 1"] & B * \mathbb{Z}/2 \oplus \Lambda^2 B
\end{tikzcd}$$

where the bottom row is induced by the map μ in (9.1.2); see Theorem 9.3.5

below. Using (6.6.7) (ii) and (9.2.8) we now obtain the binatural push-out diagram with short exact rows

(9.2.10)

$$\begin{array}{ccccc}
\mathrm{Ext}(A,G(B)\oplus\Lambda^2B) & \overset{\Delta}{\rightarrowtail} & \pi_4^{(3)}(A,B) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A,B\otimes\mathbb{Z}/2) \\
\downarrow{\scriptstyle \mathrm{Ext}(A,Q)} & \text{push} & \downarrow{\scriptstyle Q} & & \| \\
\mathrm{Ext}(A,H_6(B,3)) & \overset{\Delta}{\rightarrowtail} & H_5^{(3)}(A,B) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(A,H_5(B,3))
\end{array}$$

This push-out diagram in connection with Theorem 9.2.7 leads to the following computation of the Eilenberg–Mac Lane functor $H_5^{(3)}$; compare (6.3.9), $m=3$.

(9.2.11) Theorem *There is a binatural isomorphism* $(A,B\in\mathbf{Ab})$

$$H_5^{(3)}(A,B)=L_\#(A,B)\oplus\mathrm{Ext}(A,B*\mathbb{Z}/2)$$

where $L_\#$ *is the bifunctor in Definition* 6.2.13. *Moreover the isomorphism is compatible with* Δ *and* μ.

Proof We combine the push-out (9.2.10) and Theorem 9.2.7. For this we need the composite

$$\mathrm{Ext}(A,\mu\oplus 1)\Delta_*\varepsilon=\mathrm{Ext}(A,(\mu\Delta)\oplus 1)\oplus\varepsilon_2 \tag{1}$$

where $\mu\Delta=0$. Hence $H_5^3(A,B)$ is the direct sum of $\mathrm{Ext}(A,B*\mathbb{Z}/2)$ and the push-out of the top row in the following diagram

$$\begin{array}{ccccc}
\mathrm{Ext}(A,\Lambda^2B) & \xleftarrow{(0,\varepsilon_2)} & B\otimes\Lambda_0(A) & \xrightarrow{\Delta\otimes 1} & B\otimes\Lambda_1(A) \\
\| & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle B\otimes(\mu,1)} \\
\mathrm{Ext}(A,\Lambda^2B) & \longleftarrow & (\Lambda^2B)\otimes\mathrm{Ext}(A,\mathbb{Z}) & \longrightarrow & B\otimes L(A)
\end{array} \tag{2}$$

Here $(\mu,1)\colon \Lambda_1(A)\to L(A)$ is the natural map between quadratic $\mathbb{Z}$-modules given by

$$\begin{array}{ccccccc}
\Lambda_1(A) & = & (\mathrm{Hom}(G(A),\mathbb{Z}/4) & \xrightarrow{\partial\mu} & \mathrm{Ext}(A,\mathbb{Z}) & \xrightarrow{0} & \mathrm{Hom}(G(A),\mathbb{Z}/4)) \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \mu} \\
L(A) & = & (\mathrm{Hom}(A,\mathbb{Z}/2) & \xrightarrow{\partial} & \mathrm{Ext}(A,\mathbb{Z}) & \xrightarrow{0} & \mathrm{Hom}(A,\mathbb{Z}/2))
\end{array}$$

compare the definition of $\Lambda_1(A)$ in (9.1.4). We now observe that the push-out of the top row of (2) is via diagram (2) isomorphic to the push out of the bottom row of (2). This follows since

$$\mathrm{Ext}(A,\mathbb{Z}/2)\overset{\Delta}{\rightarrowtail}\Lambda_1(A)\twoheadrightarrow L(A) \tag{3}$$

is a short exact sequence of quadratic $\mathbb{Z}$-modules and since the quadratic tensor product is left exact. Now the push-out of the bottom row of (2) is $L_{\#}(A, B)$. Here we assume that A or B are finitely generated. □

(9.1.2) Remark Using the quadratic Λ-map

$$\lambda = 0 \oplus \Lambda^2 B\colon B \otimes \mathbb{Z}/2 \oplus \Lambda^2 B \to B * \mathbb{Z}/2 \oplus \Lambda^2 B$$

we get the binatural isomorphism

$$L_{\#}(A, B) = \tilde{G}(A, \lambda).$$

Here the right-hand side is defined in Definition 9.1.8.

9.3 Whitehead's group Γ_5 of a 2-connected space

We here compute the groups $\Gamma_5 X$ and $\Gamma_4(A, X)$ of a 2-connected space X and we determine the functorial properties of these groups; we proceed similarly as in the stable case; see Section 8.3. Since the groups depend only on the 4-type of X we have for $\eta = \eta_3^*\colon \pi_3(X) \otimes \mathbb{Z}/2 \to \pi_4 X$

(9.3.1) $$\Gamma_5(X) = \Gamma_5 K(\eta, 3)$$

and

(9.3.2) $$\Gamma_4(A, X) = \Gamma_5(A, K(\eta, 3)).$$

The computation of these groups is based on the next result.

(9.3.3) Proposition *Let X be a 2-connected space. Then one has the natural exact sequence*

$$\pi_4(X) \otimes \mathbb{Z}/2 \oplus \Lambda^2(H_3 X) \overset{\Delta}{\rightarrowtail} \Gamma_5 X \overset{\mu}{\twoheadrightarrow} H_3(X) * \mathbb{Z}/2.$$

Here Δ is given by the Hopf map η_3 and by the Whitehead product [,], that is $\Delta(\alpha \otimes 1) = \alpha\eta_3$ for $\alpha \in \pi_4(X)$ and $\Delta(x \wedge y) = [x, y]$ for $x, y \in \pi_3 X = H_3 X$.

Proof The proof is similar to the proof of Proposition 8.3.5. We again consider the mapping cone C_g with $n = 3$. Here, however, we use the exact EHP sequence for the mapping cone C_g; see Theorem A.6.9. □

(9.3.4) Corollary *Let X be a 2-connected space. Then one has a short exact sequence*

$$\Lambda^2(H_3 X) \overset{\Delta}{\rightarrowtail} \Gamma_5(X) \overset{\Sigma}{\twoheadrightarrow} \Gamma_6(\Sigma X)$$

which is naturally split. Here Σ is the suspension operator and Δ is defined as in Proposition 9.3.3.

Proof The suspension maps the sequence in Lemma 9.2.3 surjectively to the sequence in Proposition 8.3.5. Moreover $\Sigma\colon \pi_4 X \to \pi_5 \Sigma X$ is an isomorphism. Hence we obtain the required exact sequence. Now let $\beta\colon M(B,3) \to X$ be a map which induces the identity $H_3\beta$ with $B = H_3 X$. Then Lemma 9.2.3 and Proposition 9.3.3 yield the push-out diagram

$$\begin{tikzcd}
\pi_4 X \otimes \mathbb{Z}/2 \oplus \Lambda^2 B \arrow[r, tail, "\Lambda"] & \Gamma_5 X \\
B \otimes \mathbb{Z}/2 \oplus \Lambda^2 B \arrow[u, "\eta_n^* \oplus 1"] \arrow[r, tail, "\Delta \oplus 1"] & \pi_6 M(B,4) \oplus \Lambda^2 B \cong \pi_5 M(B,3) \arrow[u, "\beta_*"']
\end{tikzcd}$$

This shows that $\Lambda^2 B$ is a direct summand of $\Gamma_5 X$. Moreover Q in Theorems 6.6.6 and 6.6.11 yields a natural retraction of Δ in the Corollary. □

More generally than in Theorem 9.2.4 we now get the following result which is an unstable version of Theorem 8.3.7.

(9.3.5) Theorem *Let X be a 2-connected space. Then there is a natural commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{Z}/2, \pi_5 X) \oplus \Lambda^2 \pi_3 X \arrow[r, tail, "\Delta \oplus 1"] \arrow[d, equal] & \pi_4(\mathbb{Z}/2, X) \oplus \Lambda^2 \pi_3 X \arrow[r, "\mu \oplus 0"] \arrow[d, equal, "(\zeta^*, w)"] & \mathrm{Hom}(\mathbb{Z}/2, \pi_3 X) \arrow[d, equal] \\
\pi_5(X) \otimes \mathbb{Z}/2 \oplus \Lambda^2 H_3 X \arrow[r, tail, "\Delta"] & \Gamma_5(X) \arrow[r, "\mu"] & H_3(X) * \mathbb{Z}/2
\end{tikzcd}$$

Here the top row is given by the universal coefficient sequence for $\pi_4(\mathbb{Z}/2, X)$. The left- and the right-hand side denote the canonical identifications. Moreover, we obtain for η in (9.3.1) *the isomorphism of groups*

$$\theta\colon G(\eta) \oplus \Lambda^2(H_3 X) = \Gamma_5(X)$$

which is compatible with Δ and μ. Here $G(\eta)$ is the group in Definition 8.1.3 (A).

Proof The isomorphism (ζ^*, w) is defined by the map ζ^* in Theorem 8.3.7 and by the Whitehead product w in Proposition 9.3.3 above. □

(9.3.6) Remark By setting $X = K(B,3)$ we readily get

$$H_6 K(B,3) = \Gamma_5 K(B,3) = B * \mathbb{Z}/2 \oplus \Lambda^2 B.$$

Moreover since β is used for the definition of the operator Q in Theorem

6.6.6, we deduce from the diagram in the proof of Corollary 9.3.4 that diagram (9.2.9) commutes.

We now study the group $\Gamma_4(A, X)$ in a similar way as in Lemma 8.3.9.

(9.3.7) Lemma *Let X be 2-connected with $B = H_3X$ and let β: $M(B,3) \to X$ be a map which induces the identity $H_3(\beta)$. Then one has the commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}(A, \Gamma_5 X) \arrow[r, tail, "\Delta"] & \Gamma_4(A, X) \arrow[r, "\mu"] & \mathrm{Hom}(A, \Gamma_4 X) \\
\mathrm{Ext}(A, \pi_5 M(B,3)) \arrow[u, "\mathrm{Ext}(A,\beta_*)"] \arrow[r, tail, "\Delta"] & \pi_4(A, M(B,3)) \arrow[u, "\beta_*"] \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, B \otimes \mathbb{Z}/2) \arrow[u, equal, "(\Gamma_4\beta)_*"]
\end{tikzcd}$$

The top row is the universal coefficient sequence (Definition 2.2.3) which is natural in X. For $X = M(B, 3)$ the top row yields the bottom row which also coincides with the exact sequence in (9.2.8). Since the diagram is a push-out diagram we can compute the group $\Gamma_4(A, X)$ by use of (9.2.8) and Theorem 9.3.5.

A homomorphism η: $B \otimes \mathbb{Z}/2 \to \pi$ yields the quadratic Λ-map

(9.3.8) $\quad \lambda = \Delta(\eta \oplus 1) \oplus \Lambda^2 B$: $B \otimes \mathbb{Z}/2 \oplus \Lambda^2 B \to G(\eta) \oplus \Lambda^2 B = L$

given by $\Delta(\eta \otimes 1)$: $B \otimes \mathbb{Z}/2 \to \pi \otimes \mathbb{Z}/2 \to G(\eta)$.

(9.3.9) Theorem *For the abelian group A and for η: $B \otimes \mathbb{Z}/2 \to \pi$ one has the isomorphism*

$$\theta': \tilde{G}(A, \Delta(\eta \otimes 1) \oplus \Lambda^2 B) = \Gamma_4(A, K(\eta, 3))$$

which is compatible with Δ and μ.

Compare the isomorphism in (8.3.10). The theorem extends Theorem 9.2.7.

10

DECOMPOSITION OF HOMOTOPY TYPES

In this chapter we describe explicit results on the classification of homotopy types. For example we give a complete list of $(n-1)$-connected $(n+3)$-dimensional homotopy types X, $n \geq 4$, for which all homology groups $H_i(X)$ are cyclic. We also obtain a complete list of all homotopy types X for which $\pi_n X$, $\pi_{n+1} X$, $\pi_{n+2} X$, $n \geq 4$, are cyclic groups and for which $\pi_i X$ is trivial otherwise. These results are obtained by describing the corresponding indecomposable homotopy types. The indecomposable $(n-1)$-connected $(n+3)$-dimensional polyhedra were found in Baues and Hennes [HC]. The complete solutions of classification problems in this chapter show that the prospect of homotopy types is not so dim.

10.1 The decomposition problem in representation theory and topology

Let **C** be a category with an initial object $*$ and assume sums, denoted by $A \vee B$, exist in **C**. An object X in **C** is *decomposable* if there exists an isomorphism $X \cong A \vee B$ in **C** where A and B are not isomorphic to $*$. Hence an object X is *indecomposable* if $X \cong A \vee B$ implies $A \cong *$ or $B \cong *$. A *decomposition* of X is an isomorphism

(10.1.1) $$X \cong A_1 \vee \cdots \vee A_n,\ n < \infty,$$

in **C** where A_i is indecomposable for all $i \in \{1, \ldots, n\}$. The decomposition of X is *unique* if $B_1 \vee \cdots \vee B_m \cong X \cong A_1 \vee \cdots \vee A_n$ implies that $m = n$ and that there is a permutation σ with $B_{\sigma_i} \cong A_i$. A morphism f in **C** is *indecomposable* if the object f is indecomposable in the category **Pair(C)**. The objects of **Pair(C)** are the morphisms of **C** and the morphisms $f \to g$ in **Pair(C)** are the pairs (α, β) of morphisms in **C** with $g\alpha = \beta f$. The sum of f and g is the morphism $f \vee g = (i_1 f, i_2 g)$. The *decomposition problem* in **C** can be described by the following task: find a complete list of indecomposable isomorphism types in **C** and describe the possible decompositions of objects in **C**. We now consider various examples and solutions of such decomposition problems. These examples originated in representation theory and topology.

First let R be a ring and let **C** be a full category of R-modules (satisfying some finiteness restraint). The initial object in **C** is the trivial module 0 and the sum in **C** is the direct sum of modules, denoted by $M \oplus N$. With respect to the decomposition problem for modules in **C**, Gabriel states in the introduction of [IR]: 'The main and perhaps hopeless purpose in representation theory is to find an efficient general method for constructing the indecomposable objects by means of simple objects, which are supposed to be given'. Various results on such decomposition problems are outlined in Gabriel [IR]. We shall use only the following examples.

(10.1.2) Example For $R = \mathbb{Z}$ let $\mathbf{C}$ be the category of finitely generated abelian groups. In this case the indecomposable objects are well known; they are given by the cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/p^i$ where p is a prime and $i \geq 1$.

(10.1.3) Example Let k be a field and let R be the quotient ring $R = k\langle X,Y\rangle/(X^2,Y^2)$. Here (X^2,Y^2) stands for the ideal generated by X^2 and Y^2 in the free associative algebra $k\langle X,Y\rangle$ in the variables X and Y. Let $\mathbf{C}$ be the full category of R-modules which are finite dimensional as k-vector spaces. C.M. Ringel [RT] gave a complete list of indecomposable objects in $\mathbf{C}$. These objects are characterized by certain words which are partially of a similar nature as the words used in Section 10.2 below.

(10.1.4) Example In topology we also consider graded rings like the Steenrod algebra and graded modules like the homology or cohomology of a space. Let $R = \mathfrak{A}_p$ be the mod p Steenrod algebra and let $k \geq 0$. We consider the category $\mathbf{C}$ of all graded R-modules H for which H_i is a finite $\mathbb{Z}/p$-vector space and for which $H_i = 0$ for $i < 0$ and $i > k$. It is a hard problem to compute the indecomposable objects of $\mathbf{C}$; only for $k \leq 4p - 5$ is the answer known by the work of Henn [CP]. In fact, Henn's result is closely related to the result of Ringel in Example 10.1.3 above; to see this we consider the case $p = 2$. The restriction $k \leq 3$ then implies that the $\mathfrak{A}$-module structure of H is completely determined by Sq_1 and Sq_2 with $Sq_1Sq_1 = 0$ and $Sq_2Sq_2 = 0$. Hence, forgetting degrees, the module H is actually a module over the ring $\mathbb{Z}/2\langle X,Y\rangle/(X^2,Y^2)$ with $X = Sq_1$, $Y = Sq_2$ and such modules were classified by Ringel.

Next we describe the fundamental decomposition problem of homotopy theory. Let $\mathbf{Top}^*/\simeq$ be the homotopy category of pointed topological spaces. The set of morphisms $X \to Y$ in $\mathbf{Top}^*/\simeq$ is the set of homotopy classes $[X,Y]$. Isomorphisms in $\mathbf{Top}^*/\simeq$ are called homotopy equivalences and isomorphism types in $\mathbf{Top}^*/\simeq$ are homotopy types. Let $\mathbf{A}_n^k$ be the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of finite $(n-1)$-connected $(n+k)$-dimensional CW-complexes; the objects of $\mathbf{A}_n^k$ are also called A_n^k-polyhedra, see J.H.C. Whitehead [HT]. The suspension Σ gives us the sequence of functors

$$\mathbf{A}_1^k \xrightarrow{\Sigma} \mathbf{A}_2^k \to \cdots \to \mathbf{A}_n^k \xrightarrow{\Sigma} \mathbf{A}_{n+1}^k \to \cdots \tag{10.1.5}$$

which is the *k-stem of homotopy categories*. The Freudenthal suspension theorem shows that for $k+1<n$ the functor Σ: $\mathbf{A}_n^k \to \mathbf{A}_{n+1}^k$ is an equivalence of categories; moreover for $k+1=n$ this functor is full and a 1–1 correspondence of homotopy types. We say that the homotopy types of $\mathbf{A}_n^k$ are *stable* if $k+1 \le n$; the morphisms of $\mathbf{A}_n^k$, however, are stable if $k+1<n$.

The computation of the k-stem is a classical and principal problem of homotopy theory which, in particular, was studied for $k \le 2$ by J.H.C. Whitehead [SC], [HT], [CE]. The k-stem of homotopy groups of spheres, denoted by $\pi_{n+k}(S^n)$, $n \ge 2$, is now known for fairly large k; for example one can find a complete list for $k \le 19$ in Toda's book [CM]. The k-stem of homotopy types, however, is still mysterious even for very small k. The initial object of the category $\mathbf{A}_n^k$ is the point $*$ and the sum in $\mathbf{A}_n^k$ is the one-point union of spaces. The suspension Σ in (10.1.5) carries a sum to a sum and Σ: $\mathbf{A}_n^k \to \mathbf{A}_{n+1}^k$ yields a 1–1 correspondence of indecomposable homotopy types for $k+1 \le n$. As in the case of modules we use a finiteness restraint: we consider the decomposition problem in the stable k-stem of homotopy categories only for finite (or equivalently compact) CW-complexes.

The following results on the decomposition problem in the category $\mathbf{A}_n^k$ are known. Recall that the *elementary Moore spaces* of $\mathbf{A}_n^k$ are the spheres S^m, $n \le m \le n+k$, and the Moore spaces $M(\mathbb{Z}/p^i, m)$ where p^i is a prime power and $n \le m < n+k$. These are indecomposable objects in $\mathbf{A}_n^k$. The next result is classical and follows by use of the Hurewicz theorem from Example (10.1.2).

(10.1.6) Proposition (*A*) *For $n \ge 1$ the sphere S^n is the only indecomposable homotopy type of $\mathbf{A}_n^0$, and each object in $\mathbf{A}_n^0$ has a unique decomposition.*

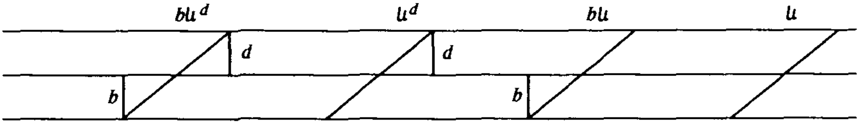
(*B*) *Let $n \ge 2$. The elementary Moore spaces of $\mathbf{A}_n^1$ are the only indecomposable homotopy types in $\mathbf{A}_n^1$ and each object in $\mathbf{A}_n^1$ has a unique decomposition.*

It is known that there are 2-dimensional complexes in $\mathbf{A}_1^1$ which admit different decompositions, see for example Dyer and Sieradski [TH]. Next we consider the decomposition problem in the category $\mathbf{A}_n^2$, $n \ge 3$. For this we define in the following list the elementary complexes X of Chang which are mapping cones of the corresponding attaching maps in the list. Let i_1, resp. i_2 be the inclusions of S^{n+1}, resp. S^n, into the one-point union $S^{n+1} \vee S^n$ and let η_n be the Hopf map and p, q be powers of 2.

(10.1.7) Elementary complexes of Chang

X	attaching map
$X(\eta) = S^n \cup e^{n+2}$	η_n: $S^{n+1} \to S^n$
$X(\eta q) = S^n \vee S^{n+1} \cup e^{n+2}$	$qi_1 + i_2\eta_n$: $S^{n+1} \to S^{n+1} \vee S^n$
$X(_p\eta) = S^n \cup e^{n+1} \cup e^{n+2}$	(η_n, p): $S^{n+1} \vee S^n \to S^n$
$X(_p\eta q) = S^n \vee S^{n+1} \cup e^{n+1} \cup e^{n+2}$	$(qi_1 + i_2\eta_n, pi_2)$: $S^{n+1} \vee S^n \to S^{n+1} \vee S^n$

These complexes are also discussed in the books of Hilton [IH], [HT]. Our notation of the elementary Chang complexes above in terms of the 'words' $\eta, \eta q, {}_p\eta, {}_p\eta q$ is compatible with the notation on elementary A_n^3-complexes in Section 10.2. These words can also be realized by the following graphs where vertical edges are associated with numbers p, q and where the edge connecting level 0 to 2 is denoted by η.



Equivalently these are all subgraphs (or subwords) of ${}_p\eta q$ which contain η. In Section 10.2 we shall describe the elementary A_n^3-polyhedra by subgraphs (or subwords) of more complicated graphs.

(10.1.8) Theorem of Chang [HI] *The elementary Moore spaces and the elementary Chang complexes above are the only indecomposable homotopy types in $\mathbf{A}_n^2$, $n \geq 3$, and each homotopy type in $\mathbf{A}_n^2$ has a unique decomposition.*

Proof We use Theorem 3.5.6 which shows that each homotopy type X in $\mathbf{A}_n^2$, $n \geq 3$, is given by a $\otimes \mathbb{Z}/2$-sequence

$$H_1 \xrightarrow{b} H \otimes \mathbb{Z}/2 \xrightarrow{i} \pi \xrightarrow{h} H_0 \to 0 \tag{1}$$

with $H = H_n X$, $H_0 = H_{n+1} X$, $H_1 = H_{n+2} X$ finitely generated, H_1 free abelian, and $\pi = \pi_{n+1} X$. The elementary Moore spaces and the elementary Chang complexes yield the $\otimes \mathbb{Z}/2$-sequences in the following list where p and q are powers of 2 and l is a power of an odd prime.

X	H_1	$\xrightarrow{b}$	$H \otimes \mathbb{Z}/2$	$\xrightarrow{i}$	π	$\xrightarrow{h}$	H_0	H
S^n	0	$\longrightarrow$	$\mathbb{Z}/2$	$\xrightarrow{1}$	$\mathbb{Z}/2$	$\longrightarrow$	0	$\mathbb{Z}$
S^{n+1}	0	$\longrightarrow$	0	$\longrightarrow$	$\mathbb{Z}$	$\xrightarrow{1}$	$\mathbb{Z}$	0
S^{n+2}	$\mathbb{Z}$	$\longrightarrow$	0	$\longrightarrow$	0	$\longrightarrow$	0	0
$M(\mathbb{Z}/q, n)$	0	$\longrightarrow$	$\mathbb{Z}/2$	$\xrightarrow{1}$	$\mathbb{Z}/2$	$\longrightarrow$	0	$\mathbb{Z}/q$
$M(\mathbb{Z}/q, n+1)$	0	$\longrightarrow$	0	$\longrightarrow$	$\mathbb{Z}/q$	$\xrightarrow{1}$	$\mathbb{Z}/q$	0
$M(\mathbb{Z}/l, n)$	0	$\longrightarrow$	0	$\longrightarrow$	0	$\longrightarrow$	0	$\mathbb{Z}/l$
$M(\mathbb{Z}/l, n+1)$	0	$\longrightarrow$	0	$\longrightarrow$	$\mathbb{Z}/l$	$\xrightarrow{1}$	$\mathbb{Z}/l$	0
$X(\eta)$	$\mathbb{Z}$	$\xrightarrow{1}$	$\mathbb{Z}/2$	$\longrightarrow$	0	$\longrightarrow$	0	$\mathbb{Z}$
$X(\eta q)$	0	$\longrightarrow$	$\mathbb{Z}/2$	$\xrightarrow{2}$	$\mathbb{Z}/2q$	$\xrightarrow{1}$	$\mathbb{Z}/q$	$\mathbb{Z}$
$X({}_p\eta)$	$\mathbb{Z}$	$\xrightarrow{1}$	$\mathbb{Z}/2$	$\longrightarrow$	0	$\longrightarrow$	0	$\mathbb{Z}/p$
$X({}_p\eta q)$	0	$\longrightarrow$	$\mathbb{Z}/2$	$\xrightarrow{2}$	$\mathbb{Z}/2q$	$\xrightarrow{1}$	$\mathbb{Z}/q$	$\mathbb{Z}/p$

We have to show that each sequence (1) is a direct sum of sequences as in the list. Then the additive detecting functor Λ' in Theorem 3.5.6 yields the proposition in Theorem 10.1.8. We choose a basis $B(\pi) \subset \pi$ where $B(\pi)$ yields a decomposition of π as in Example 10.1.2. For $x \in B(\pi)$ we get $hx \in H_0$ and the orders satisfy either $|hx| = |x| \leq \infty$ or $|x| = 2|hx| = 2^{q+1}$. Moreover the elements x in the second case yield a system of generators $2^q x$ of image i. The elements hx with $x \in B(\pi)$ and $hx \neq 0$ form a basis of H_0. We choose a basis $B(H_1) \subset H_1$ and $B(H \otimes \mathbb{Z}/2) \subset H \otimes \mathbb{Z}/2$ such that i carries elements of $B(H \otimes \mathbb{Z}/2)$ to elements of the form $2^q x$, $x \in B(\pi)$, and such that b carries an element $y \in B(H_1)$ to an element in $B(H \otimes \mathbb{Z}/2)$ or to 0. Using these generators all homomorphisms b, i, h are given by diagonal matrices. Finally we choose a basis $B(H \otimes \mathbb{Z}/2)$. These generators now yield a direct sum decomposition of (1) such that each summand is one of the sequences in the list above. Moreover this decomposition is unique in the sense of (10.1.1). □

Spanier–Whitehead duality yields the following equations which we easily derive from the definitions:

(10.1.9) Proposition *The Spanier–Whitehead duality functor D: $\mathbf{A}_n^2 \cong \mathbf{A}_n^2$ satisfies*

$$DX(\eta) = X(\eta)$$

$$DX(\eta q) = X({}_q\eta)$$

$$DX({}_p\eta) = X(\eta p)$$

$$DX({}_p\eta q) = X({}_q\eta p).$$

Hence the Spanier–Whitehead duality turns the graphs in (10.1.7) *around by* 180°; *see also Definition* 10.2.3 *and Theorem* 10.2.10 *below.*

10.2 The indecomposable $(n-1)$-connected $(n+3)$-dimensional polyhedra, $n \geq 4$

We here describe all elementary $(n-1)$-connected $(n+3)$-dimensional polyhedra in terms of certain words, or graphs. This extends the results of Chang in Theorem 10.1.8. The fundamental results in this section are deduced from the classification of stable $(n-1)$-connected $(n+3)$-dimensional polyhedra in Chapter 8; compare Baues and Hennes [HC].

For the description of the indecomposable objects in $\mathbf{A}_n^3$, $n \geq 4$, we use certain words. Let L be a set, the elements of which are called 'letters'. A word with letters in L is an element in the free monoid generated by L. Such a word a is written $a = a_1 a_2 \cdots a_n$ with $a_i \in L$, $n \geq 0$; for $n = 0$ this is the empty word ϕ. Let $b = b_1 \cdots b_k$ be a word. We write $w = \cdots b$ if there is a word a with $w = ab$, similarly we write $w = b \cdots$ if there is a word c with $w = bc$ and we write $w = \cdots b \cdots$ if there exist words a and c with $w = abc$. A *subword* of an infinite sequence $\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots$ with $a_i \in L$, $i \in \mathbb{Z}$, is a finite connected subsequence $a_n a_{n+1} \cdots a_{n+k}$, $n \in \mathbb{Z}$. For the word $a = a_1 \cdots a_n$ we define the word $-a = a_n a_{n-1} \cdots a_1$ by reversing the order in a.

(10.2.1) Definition We define a collection of finite words $w = w_1 w_2 \cdots w_k$. The letters w_i of w are the symbols ξ, η, ε or natural numbers t, s_i, r_i, $i \in \mathbb{Z}$, which are powers of 2. We write the letters s_i as upper indices, the letters r_i as lower indices, and the letter t in the middle of the line since we have to distinguish between these numbers. For example $\eta 4 \xi^2 \eta_8$ is such a word with $t = 4$, $r_1 = 8$, $s_1 = 2$. A basic sequence is defined by

$$\xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\cdots. \tag{1}$$

This is the infinite product $a(1)a(2)\cdots$ of words $a(i) = \xi^{s_i}\eta_{r_i}$, $i \geq 1$. A *basic word* is any subword of (1). A central sequence is defined by

$$\cdots {}^{s_{-2}}\xi_{r_{-2}}\eta^{s_{-1}}\xi_{r_{-1}}\eta t \xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\cdots. \tag{2}$$

A *central word* w is any subword of (2) containing the number t. Whence a central word w is of the form $w = atb$ where $-a$ and b are basic words. An ε-sequence is defined by

$$\cdots {}^{s_{-2}}\xi_{r_{-2}}\eta^{s_{-1}}\xi_{r_{-1}}\varepsilon^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\cdots. \tag{3}$$

An *ε-word* w is any subword of (3) containing the letter ε; again we can write $w = a\varepsilon b$ where $-a$ and b are basic words. A *general word* is a basic word, a central word, or an ε-word.

A general word w is called *special* if w contains at least one of the letters ξ, η, or ε and if the following conditions (i), D(i), (ii), and D(ii) are satisfied in case $w = a\varepsilon b$ is an ε-word. We associate with b the tuple

$$s(b) = (s_1^b, s_2^b, \dots) = \begin{cases} (s_1, \dots, s_m, \infty, 1, 1, \dots) & \text{if } b = \cdots \xi \\ (s_1, \dots, s_m, 1, 1, 1, \dots) & \text{otherwise} \end{cases}$$

$$r(b) = (r_1^b, r_2^b, \dots) = \begin{cases} (r_1, \dots, r_l, \infty, 1, 1, \dots) & \text{if } b = \cdots \eta \\ (r_1, \dots, r_l, 1, 1, 1, \dots) & \text{otherwise} \end{cases}$$

where $s_1 \cdots s_m$ and $r_1 \cdots r_l$ are the words of upper indices and lower indices respectively given by b. In the same way we get $s(-a) = (s_1^{-a}, s_2^{-a}, \dots)$ and $r(-a) = (r_1^{-a}, r_2^{-a}, \dots)$ with $s_i^{-a} \in \{s_{-i}, \infty, 1\}$ and $r_i^{-a} \in \{r_{-i}, \infty, 1\}$, $i \in \mathbb{N}$. The conditions in question on the ε-word $w = a\varepsilon b$ are:

(i) $$b = \phi \Rightarrow a \neq \xi_2$$

D(i) $$a = \phi \Rightarrow b \neq {}^2\eta.$$

Moreover if $a \neq \phi$ and $b \neq \phi$ we have

(ii)
$$s_1 = 2 \Rightarrow r_{-1} \geq 4$$

and

$$(2r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \dots, -s_i^b, r_i^b, \dots)$$
$$< (r_1^{-a}, -s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s_3^{-a}, \dots, r_i^{-a}, -s_i^{-a}, \dots)$$

D(ii)
$$r_{-1} = 2 \Rightarrow s_1 \geq 4$$

and

$$(-s_1^b, r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \dots, -s_i^b, r_i^b, \dots)$$
$$< (-2 \cdot s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s_3^{-a}, \dots, r_i^{-a}, -s_i^{-a}, \dots).$$

The index i runs through $i = 2, 3, \dots$ as indicated. In (ii) and D(ii) we use the lexicographical ordering $<$ from the left, that is $(n_1, n_2, \dots) < (m_1, m_2, \dots)$ if and only if there is $t \geq 1$ with $n_j = m_j$ for $j < t$ and $n_t < m_t$.

Finally we define a *cyclic word* by a pair (w, φ) where w is a basic word of the form $(p \geq 1)$

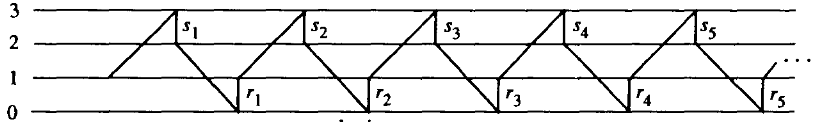
$$w = \xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2} \cdots \xi^{s_p}\eta_{r_p} \tag{4}$$

and where φ is an automorphism of a finite dimensional $\mathbb{Z}/2$-vector space $V = V(\varphi)$. Two cyclic words (w, φ) and (w', φ') are *equivalent* if w' is a cyclic permutation of w, that is,

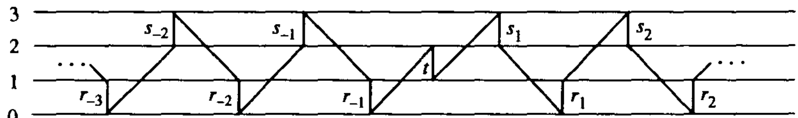
$$w' = \xi^{s_t}\eta_{r_t} \cdots \xi^{s_p}\eta_{r_p}\xi^{s_1}\eta_{r_1} \cdots \xi^{s_{t-1}}\eta_{r_{t-1}},$$

and if there is an isomorphism $\Psi\colon V(\varphi) \cong V(\varphi')$ with $\varphi = \Psi^{-1}\varphi'\Psi$. A cyclic word (w, φ) is a *special cyclic word* if φ is an indecomposable automorphism and if w is not of the form $w = w'w' \cdots w'$ where the right-hand side is a j-fold power of a word w' with $j > 1$.

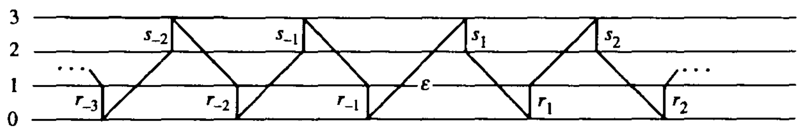
The sequences (1), (2), (3) can be visualized by the infinite graphs sketched below. The letters s_i, resp. r_i, correspond to vertical edges connecting the levels 2 and 3, resp. the levels 0, 1. The letters η, resp. ξ, correspond to diagonal edges connecting the levels 0 and 2, resp. the levels 1 and 3. Moreover ε connects the levels 0 and 3 and t the levels 1 and 2. We identify a general word with the connected finite subgraph of the infinite graphs below. Therefore the *vertices of level i* of a general word are defined by the vertices of level i of the corresponding graph, $i \in \{0,1,2,3\}$. We also write $|x| = i$ if x is a vertex of level i.



basic sequence



central sequence



ε-sequence

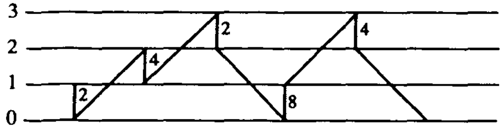
(10.2.2) Remark There is a simple rule which creates exactly all graphs corresponding to general words. Draw in the plane $\mathbb{R}^2$ a connected finite graph of total height at most 3 that alternatingly consists of vertical edges of height one and diagonal edges of height 2 or 3. Moreover endow each vertical edge with a power of 2. An *equivalence relation* on such graphs is generated by reflection at a vertical line. One readily checks that the equivalence classes of such graphs are in 1–1 correspondence to all general words.

(10.2.3) Definition Let w be a basic word, a central word, or an ε-word. We obtain the *dual word* $D(w)$ by reflection of the graph w at a horizontal line and by using the equivalence defined in Remark 10.2.2. Then $D(w)$ is again a basic word, a central word, or an ε-word respectively. Clearly the reflection replaces each letter ξ in w by the letter η and vice versa; moreover it turns a lower index into an upper index and vice versa. We define the *dual cyclic word* $D(w, \varphi)$ as follows. For the cyclic word (w, φ) in (10.2.1) (4) let $D(w, \varphi) = (w', (\varphi^*)^{-1})$. Here we set

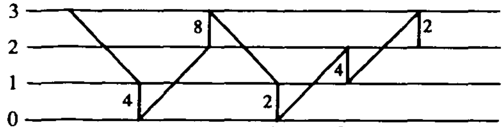
$$w' = \xi^{r_1}\eta_{s_2}\xi^{r_2}\cdots\eta_{s_p}\xi^{r_p}\eta_{s_1}$$

and we set $\varphi^* = \mathrm{Hom}(\varphi, \mathbb{Z}/2)$ with $V(\varphi^*) = \mathrm{Hom}(V(\varphi), \mathbb{Z}/2)$. Up to a cyclic permutation w' is just $D(w)$ defined above. We point out that the dual words $D(w)$ and $D(w, \varphi)$ are special if and only if w and (w, φ) respectively are special.

As an example we have the special words $w = {}_2\eta 4\xi^2\eta_8\xi^4\eta$ and $D(w) = \xi_4\eta^8\xi_2\eta 4\xi^2$ which are dual to each other. They correspond to the graphs



$w = {}_2\eta 4\xi^2\eta_8\xi^4\eta$



$D(w) = \xi_4\eta^8\xi_2\eta 4\xi^2$

Hence the dual graph $D(w)$ is obtained by rotating the graph of w.

We are going to construct certain A_n^3-polyhedra, $n \geq 4$, associated with the words in Definition 10.2.1. To this end we first define the homology of a word.

(10.2.4) Definition Let w be a general word and let $r_\alpha \cdots r_\beta$ and $s_\mu \cdots s_\nu$ be the words of lower indices and of upper indices respectively given by w. We define the *torsion groups* of w by

$$T_0(w) = \mathbb{Z}/r_\alpha \oplus \cdots \oplus \mathbb{Z}/r_\beta, \tag{1}$$

$$T_1(w) = \mathbb{Z}/t \quad \text{if } w \text{ is a central word,} \tag{2}$$

$$T_2(w) = \mathbb{Z}/s_\mu \oplus \cdots \oplus \mathbb{Z}/s_\nu, \tag{3}$$

and we set $T_i(w) = 0$ otherwise. We define the *integral homology* of w by

$$H_i(w) = \mathbb{Z}^{L_i(w)} \oplus T_i(w) \oplus \mathbb{Z}^{R_i(w)}. \tag{4}$$

Here $\beta_i(w) = L_i(w) + R_i(w)$ is the *Betti number* of w; this is the number of end-points of the graph w which are vertices of level i and which are not vertices of vertical edges; we call such vertices x *spherical vertices* of w. Let $L(w)$, resp. $R(w)$, be the *left*, resp. *right*, spherical vertex of w in case they occur. Now we set $L_i(w) = 1$ if $|L(w)| = i$ and $R_i(w) = 1$ if $|R(w)| = i$; moreover $L_i(w) = 0$ and $R_i(w) = 0$ otherwise.

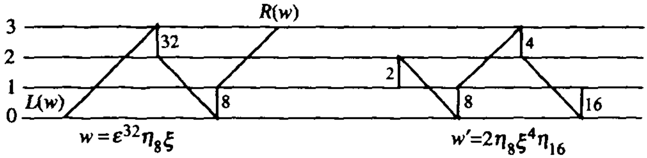
Using the equation (4) we have specified an *ordered basis* B_i of $H_i(w)$. We point out that

$$\beta_0(w) + \beta_1(w) + \beta_2(w) + \beta_3(w) \leq 2. \tag{5}$$

For a cyclic word (w, φ) we set

$$H_i(w, \varphi) = \bigoplus_{v} T_i(w) \tag{6}$$

where $v = \dim V(\varphi)$ and where the right-hand side is the v-fold direct sum of $T_i(w)$. As an example we consider the special words



The homology of these words is:

	$w = \varepsilon^{32}\eta_8\,\xi$	$w' = 2\eta_8\,\xi^4\eta_{16}$
H_3	$\mathbb{Z}$	0
H_2	$\mathbb{Z}/32$	$\mathbb{Z}/4$
H_1	0	$\mathbb{Z}/2$
H_0	$\mathbb{Z}\oplus\mathbb{Z}/8$	$\mathbb{Z}/8\oplus\mathbb{Z}/16$

Here w has two spherical vertices while w' has no spherical vertex. We point out that the numbers 2^k attached to vertical edges correspond to cyclic groups $\mathbb{Z}/2^k$ in the homology. We describe many further examples in Sections 10.5 and 10.6.

For the construction of polyhedra $X(w)$ associated with words w we use the following generators; see Definition 11.6.3 below.

(10.2.5) Generators of homotopy groups Let r, s be powers of 2. We have the Hopf maps

$$\eta = \eta_n\colon S^{n+1}\to S^n,\qquad \xi = \eta_{n+1}\colon S^{n+2}\to S^{n+1},\qquad \varepsilon = \eta_n^2\colon S^{n+2}\to S^n.$$

We use the composites

$$\eta = i\eta_n\colon S^{n+1}\to M(\mathbb{Z}/r, n),\qquad \xi = \eta_{n+1}q\colon M(\mathbb{Z}/r, n+1)\to S^{n+1}$$

which are $(2n+1)$-dual. Moreover we have the $(2n+2)$-dual groups, $n \geq 4$

$$[S^{n+2}, M(\mathbb{Z}/r, n)] = \begin{cases}\mathbb{Z}/4\xi_2 & \text{for } r = 2\\ \mathbb{Z}/2\xi_r + \mathbb{Z}/2\varepsilon_r & \text{for } r \geq 4\end{cases}$$

$$[M(\mathbb{Z}/s, n+1), S^n] = \begin{cases}\mathbb{Z}/4\eta^2 & \text{for } s = 2\\ \mathbb{Z}/2\eta^s + \mathbb{Z}/2\varepsilon^s & \text{for } s \geq 4\end{cases}$$

where $\varepsilon_r = i\eta_n^2$ and $\varepsilon^s = \eta_n^2 q$ and $\xi_r = \chi_r^2\xi_2$ and $\eta^s = \eta^2\chi_2^s$. Next we use

$$[M(\mathbb{Z}/s, n+1), M(\mathbb{Z}/r, n)] = \begin{cases}\mathbb{Z}/2\xi_2^2 \oplus \mathbb{Z}/2\eta_2^2 & \text{for } s = r = 2\\ \mathbb{Z}/4\xi_2^s \oplus \mathbb{Z}/2\eta_2^s & \text{for } s \geq 4, r = 2\\ \mathbb{Z}/2\xi_r^2 \oplus \mathbb{Z}/4\eta_r^2 & \text{for } s = 2, r \geq 4\\ \mathbb{Z}/2\xi_r^s \oplus \mathbb{Z}/2\eta_r^s \oplus \mathbb{Z}/2\varepsilon_r^s & \text{otherwise.}\end{cases}$$

Hence we have $\xi_r^s = \chi_r^2 \xi_2 q$, $\eta_r^s = i\eta^2\chi_2^s$, and $\varepsilon_r^s = i\eta_n^2 q$. We have the $(2n+2)$-dualities $D(\xi_r^s) = \eta_s^r$ and $D(\varepsilon_r^s) = \varepsilon_s^r$.

(10.2.6) Definition Let $n \geq 4$ and let w be a general word. We define the A_n^3-polyhedron $X(w) = C_f$ by the mapping cone C_f of a map $f = f(w)$: $A \to B$ where

$$\begin{cases} A = M(H_3, n+2) \vee M(H_2, n+1) \vee S_c^{n+1} \\ B = M(H_0, n) \vee S_c^{n+1} \vee S_b^{n+1}. \end{cases} \tag{1}$$

Here $H_i = H_i(w)$ is the homology group in Definition 10.2.4 above. We set $S_c^{n+1} = S^{n+1}$ if w is a central word and we set $S_c^{n+1} = *$ otherwise; moreover we set $S_b^{n+1} = S^{n+1}$ if w is a basic word of the form $w = \xi \cdots$ and we set $S_b^{n+1} = *$ otherwise. We now obtain the attaching map

$$f = f(w)\colon M(H_3, n+2) \vee M(H_2, n+1) \vee S_c^{n+1} \to M(H_0, n) \vee S_c^{n+1} \vee S_b^{n+1} \tag{2}$$

as follows. We first describe B and A in (1) as one-point unions of elementary Moore spaces. For each letter r_δ of $r_\alpha \cdots r_\beta$ (see Definition 10.2.4) we have the inclusion

$$j(r_\delta)\colon M(\mathbb{Z}/r_\delta, n) \subset B. \tag{3}$$

Moreover for each spherical vertex x of w with $|x| \leq 1$ we have the inclusion

$$j(x)\colon S^{n+|x|} \subset B. \tag{4}$$

This is the inclusion of S_b^{n+1} if $|x| = 1$. The space B is exactly the one-point union of the subspaces (3), (4) and of j_c: $S_c^{n+1} \subset B$. Next we consider the space A in (1). For each letter s_τ of $s_\mu \cdots s_\nu$ (see Definition 10.2.4) we have the inclusion

$$j(s_\tau)\colon M(\mathbb{Z}/s_\tau, n+1) \subset A. \tag{5}$$

Moreover for each spherical vertex x of w with $|x| \geq 2$ we have the inclusion

$$j(x)\colon S^{n+|x|-1} \subset A. \tag{6}$$

The space A is exactly the one-point union of the subspaces (5), (6) and of j_c: $S_c^{n+1} \subset A$.

We now define $f = f(w)$ by the following equations. For a letter s_τ as above and for $\delta = \tau - 1$ we set

$$fj(s_\tau) = \begin{cases} j(r_\delta)\xi_{r_\delta}^{s_\tau} + j(r_\tau)\eta_{r_\tau}^{s_\tau} & \text{if } w = \cdots {}_{r_\delta}\xi^{s_\tau}\eta_{r_\tau} \cdots \\ j(r_\delta)\eta_{r_\delta}^{s_\tau} + j(r_\tau)\xi_{r_\tau}^{s_\tau} & \text{if } w = \cdots {}_{r_\delta}\eta^{s_\tau}\xi_{r_\tau} \cdots \\ j_c\eta_{n+1}q + j(r_1)\eta_{r_1}^{s_1} & \text{if } w = \cdots t\xi^{s_1}\eta_{r_1} \cdots \text{ and } \tau = 1 \\ j(r_{-1})\varepsilon_{r_{-1}}^{s_1} + j(r_1)\eta_{r_1}^{s_1} & \text{if } w = \cdots r_{-1}\varepsilon^{s_1}\eta_{r_1} \cdots \text{ and } \tau = 1. \end{cases} \tag{7}$$

The first equation also holds if the letters r_δ are empty, that is if $w = \xi^{s_\tau}\eta \cdots$ or if $w = \cdots \xi^{s_\tau}\eta$ respectively. In this case we set $j(r_\delta) = j(x)$, if $x = L(w)$, resp. $j(r_\tau) = j(y)$, if $y = R(w)$; see Definition 10.2.4. We use a similar convention for the other equations in (7). Using (2) and (7) we see that $fj(s_\tau)$ is well defined for all general words w. Next we define $fj(x)$ where x is a spherical vertex of w with $|x| \geq 2$.

$$fj(x) = \begin{cases} j(r_\alpha)\xi_{r_\alpha} & \text{if } w = \xi_{r_\alpha} \ldots, |x| = 3, x = L(w) \\ j(r_\alpha)i\eta_n & \text{if } w = \eta_{r_\alpha} \ldots, |x| = 2, x = L(w) \\ j(r_\beta)\xi_{r_\beta} & \text{if } w = \cdots {}_{r_\beta}\xi, |x| = 3, x = R(w) \\ j(r_{-1})i\eta_n^2 & \text{if } w = \cdots {}_{r_{-1}}\varepsilon, |x| = 3, x = R(w). \end{cases} \tag{8}$$

Using (8) and (2) the element $fj(x)$ is well defined for all general words w. Finally we define fj_c by

$$fj_c = \begin{cases} j(r_{-1})i\eta_n + j_c(t\iota) & \text{if } w = \cdots {}_{r_{-1}}\eta t \cdots \\ j(x)\eta_n + j_c(t\iota) & \text{if } w = \eta t \ldots, x = L(w) \\ j_c(t\iota) & \text{if } w = t \cdots . \end{cases} \tag{9}$$

Here ι is the identity of S^{n+1}. This completes the definition of $f = f(w)$ and hence the definition of $X(w) = C_f$.

We point out that the construction of $f(w)$ follows exactly the pattern given by the word w or the graph of w. For this we subdivide the graph of w by a horizontal line between levels 1 and 2; all edges crossing this line are summands in the attaching map $f(w)$. For example consider the graphs $\varepsilon^{32}\eta_8\xi$, $2\eta_8\xi^4\eta_{16}$, and ${}_2\eta 4\xi^2\eta_8\xi^4\eta$ above. Then we get

$$f(\varepsilon^{32}\eta_8\xi) = \begin{tikzcd} M(\mathbb{Z}/32, n+1) \vee \arrow[d, "\varepsilon"] \arrow[dr, "\eta"] & S^{n+2} \arrow[d, "\xi"] \\ S^n & \vee M(\mathbb{Z}/8, n) \end{tikzcd}$$

$$f(2\eta_8\xi^4\eta_{16}) = \begin{tikzcd} S^{n+1} \vee \arrow[d, "2"] \arrow[dr, "i\eta_n"] & & M(\mathbb{Z}/4, n+1) \arrow[dl, "\xi"] \arrow[dr, "\eta"] & \\ S^{n+1} & \vee M(\mathbb{Z}/8, n) & & \vee M(\mathbb{Z}/16, n) \end{tikzcd}$$

$$f({}_2\eta 4\xi^2\eta_8\xi^4\eta) = \begin{tikzcd} S^{n+1} \arrow[d, "i\eta_n"] \arrow[dr, "4"] & \vee M(\mathbb{Z}/2, n+1) \vee \arrow[d, "\xi"] \arrow[dr, "\eta"] & M(\mathbb{Z}/4, n+1) \arrow[d, "\xi"] \arrow[dr, "\eta"] & \\ M(\mathbb{Z}/2, n) \vee & S^{n+1} & \vee M(\mathbb{Z}/8, n) & \vee S^n \end{tikzcd}$$

Here ξ, η, ε are the corresponding generators in (10.2.5).

(10.2.7) Definition Let $n \geq 4$ and let (w, φ) be a cyclic word. We define the A_n^3-polyhedron $X(w, \varphi) = C_f$ by the mapping cone of a map $f = f(w, \varphi)$ where

$$f\colon M(H_2, n+1) \to M(H_0, n) \tag{1}$$

with $H_i = H_i(w, \varphi)$; see Definition 10.2.4 (6). For $u \in \{1, \dots, v\}$ we have the inclusion ($m = n, n+1$ and $i = 0, 2$)

$$j_u\colon M(T_i(w), m) \subset M(H_i, m) \tag{2}$$

by the direct sum decomposition in Definition 10.2.4 (6). Moreover we have for each letter r_δ and s_τ of $r_1 \cdots r_p$ and $s_1 \cdots s_p$ (see Definition 10.2.1 (4)) the inclusions

$$j(r_\delta)\colon M(\mathbb{Z}/2^{r_\delta}, n) \subset M(T_0(w), n), \tag{3}$$

$$j(s_\tau)\colon M(\mathbb{Z}/2^{s_\tau}, n+1) \subset M(T_2(w), n+1). \tag{4}$$

Compare (10.2.5) (3) and (5). We choose a basis $\{b_1, \dots, b_v\}$ of the vector space $V(\varphi)$ and we define $\varphi_u^e \in \{0, 1\}$ by $\varphi(b_u) = \sum_{e=1}^{v} \varphi_u^e b_e$. This yields a definition of f by the following formulas (5) and (6).

$$fj_u j(s_\tau) = j_u\left[j(r_\delta)\xi_{r_\delta}^{s_\tau} + j(r_\tau)\eta_{r_\tau}^{s_\tau}\right] \tag{5}$$

If $w = \cdots {}_{r_\delta}\xi^{s_\tau}\eta_{r_\tau} \cdots$, $\tau \in \{2, \dots, p\}$, and $\delta = \tau - 1$; see Definition 10.2.1 (4). Moreover we set

$$fj_u j(s_1) = j_u j(r_1)\eta_{r_1}^{s_1} + \sum_{e=1}^{v} \varphi_u^e j_e j(r_p)\xi_{r_p}^{s_1}. \tag{6}$$

The spaces $X(w)$ and $X(w, \varphi)$ are constructed in such a way that the integral homology is given by

(10.2.8) $\qquad H_{n+i}X(w) = H_i(w),\ H_{n+i}X(w, \varphi) = H_i(w, \varphi)$

where we use the homology of the words w and (w, φ) in Definition 10.2.4.

The next result solves the decomposition problem in the category $\mathbf{A}_n^3$, $n \geq 4$. This result generalizes the theorem of Chang (10.1.8) for the next dimension. Its proof, however, is considerably more intricate than the fairly direct proof of Theorem 10.1.8 above.

(10.2.9) Decomposition theorem *Let $n \geq 4$. The elementary Moore spaces in $\mathbf{A}_n^3$, the complexes $X(w)$ where w is a special word, and the complexes $X(w, \varphi)$ where (w, φ) is a special cyclic word furnish a complete list of all indecomposable homotopy types in $\mathbf{A}_n^3$. For two complexes X, X' in this list there is a homotopy equivalence $X \simeq X'$ if and only if there are equivalent special cyclic words*

$(w, \varphi) \sim (w', \varphi')$ with $X = X(w, \varphi)$ and $X' = X(w', \varphi')$. Moreover each homotopy type in $\mathbf{A}_n^3$ has a unique decomposition.

The proof of the decomposition theorem relies on the classification theorem (8.1.6). Given this theorem one can solve the decomposition problem in the algebraic category of A^3-systems. For this an intricate generalization of the representation theory of Ringel and Henn is needed; see (10.1.3) and (10.1.4). We refer the reader to Baues and Hennes [HC] for the complete proof of the decomposition theorem. Spanier–Whitehead duality of indecomposable complexes in $\mathbf{A}_n^3$ is completely clarified by the next result.

(10.2.10) Theorem *Let $n \geq 5$. For a general word w and for a cyclic word (w, φ) let Dw and $D(w, \varphi)$ be the dual words defined in Definition* 10.2.3. *Then $X(Dw)$ is the Spanier–Whitehead $(2n+3)$-dual of $X(w)$ and $X(D(w, \varphi))$ is the Spanier–Whitehead $(2n+3)$-dual of $X(w, \varphi)$.*

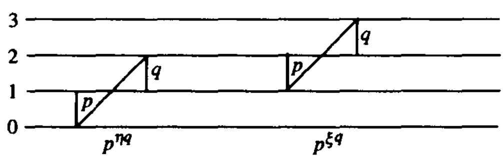
Proof The result essentially follows from the careful choice of generators in (10.2.5) which is compatible with Spanier–Whitehead duality. This implies that there are $(2n+2)$-dualities $f(w)^* = f(Dw)$ and $f(w, \varphi)^* = f(D(w, \varphi))$. Hence the proposition is a consequence of the fact that Spanier–Whitehead duality carries a mapping cone of f to the mapping cone of $D(f)$. □

We point out that $X(w)$ in (10.2.5) coincides with the corresponding elementary complex in (10.1.7) if w is one of the words η, ηq, ${}_p\eta$, ${}_p\eta q$. Moreover the suspensions of such complexes are given by

(10.2.11)

$$\Sigma X(\eta) = X(\xi), \qquad \Sigma X(\eta q) = X(\xi^q), \qquad \Sigma X({}_p\eta) = X(p\xi),$$
$$\Sigma X({}_p\eta q) = X({}_p\xi^q).$$

The words ${}_p\eta q$ and $p\xi^q$ correspond to the two possible subgraphs in a central sequence which both look like the graph in (10.1.7). Hence the Chang complexes yield only the following elementary A_n^3-polyhedra



This precisely describes the embedding of indecomposable A_m^2-polyhedra $(m = n, n+1)$ into the much larger set of indecomposable A_n^3-polyhedra. In a similar way we expect horrendous complexity if one considers the embedding of indecomposable A_m^3-polyhedra $(m = \mathrm{n}, n+1)$ into the unknown set of indecomposable A_n^4-polyhedra. As a corollary of the decomposition theorem we get the following surprising result.

(10.2.12) Theorem *Let $n \geq 4$ and let X be an $(n-1)$-connected $(n+3)$-dimensional finite polyhedron with Betti numbers $\beta_i(X)$. If*

$$2 < \beta_n(X) + \beta_{n+1}(X) + \beta_{n+2}(X) + \beta_{n+3}(X)$$

or if $H_{n+1}(X)$ contains the direct sum of two cyclic groups then X is decomposable.

Proof A general word has at most two spherical vertices and hence an indecomposable homotopy type X in $\mathbf{A}_n^3$ satisfies $\dim(H_* X \otimes \mathbb{Q}) \leq 2$. Moreover $H_{n+1}w$ is non-trivial only for central words w and general words $w = \xi \cdots$ and in these cases $H_{n+1}w$ is $\mathbb{Z}/2^t$ or $\mathbb{Z}$ respectively. Hence, X is indecomposable, implies $H_{n+1}X$ is cyclic of prime power order or $\mathbb{Z}$. □

(10.2.13) The A^3-system of $X(w)$ By Theorem 8.1.6 we have the detecting functor

$$\Lambda' \colon \mathbf{spaces}_n^3 \to A^3\textbf{-System}.$$

This is an additive functor between additive categories. Hence the indecomposable complexes $X(w)$ in the decomposition theorem 10.2.9 correspond via Λ' to indecomposable A^3-systems

$$S(w) = \Lambda' X(w). \tag{1}$$

We here compute the A^3-system

$$S(w) = (H_0, H_2, H_3, \pi_1, b_2^w, \eta^w, b_3^w, \beta^w) \tag{2}$$

explicitly in terms of w. Here we have the homology

$$H_i = H_i(w) = H_{n+i}(X(w)), \quad i \in \{0,1,2,3\}, \tag{3}$$

as defined in Definition 10.2.4 and we have the homotopy groups

$$\pi_i = \pi_i(w) = \pi_{n+i}(X(w)), \quad i \in \{1,2\}, \tag{4}$$

which are part of the exact Γ-sequence of $S(w)$ given by w as in Definition 8.1.5:

$$H_3 \xrightarrow{b_3^w} G(\eta^w) \to \pi_2 \to H_2 \xrightarrow{b_2^w} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^w} \pi_1 \xrightarrow{h} H_1 \to 0. \tag{5}$$

This sequence is isomorphic to the corresponding part of Whitehead's exact sequence for $X(w)$; see (8.1.7). We first describe b_2^w; for this we denote the basis of $H_i(w)$ in Definition 10.2.4 (4) by

$$\begin{aligned} B_0 &= \{L_0(w), r_\alpha, \ldots, r_\beta, R_0(w)\} && \subset H_0 \\ B_1 &= \{L_1(w), t, R_1(w)\} && \subset H_1 \\ B_2 &= \{L_2(w), s_\mu, \ldots, s_\nu, R_2(w)\} && \subset H_2 \\ B_3 &= \{L_3(w), R_3(w)\} && \subset H_3. \end{aligned} \tag{6}$$

We also write $L_0(w) = r_{\alpha-1}$, $R_0(w) = r_{\beta+1}$ and we set $L_2(w) = s_{\mu-1}$ and $R_2(w) = s_{\nu+1}$ in case there are spherical vertices of w; see Definition 10.2.4. With this notation we define

$$b_2^w : H_2 \to H_0 \otimes \mathbb{Z}/2 \tag{7}$$

$$b_2^w(s_\tau) = \begin{cases} r_\delta \otimes 1 & \text{if } w = \cdots {}_{r_\delta}\eta^{s_\tau} \cdots,\ \delta = \tau - 1, \\ r_\tau \otimes 1 & \text{if } w = \cdots {}^{s_\tau}\eta_{r_\tau} \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Hence b_2^w carries basis elements to basis elements or to the trivial element so that the cokernel of b_2^w is

$$\operatorname{cok}(b_2^w) = \oplus\{(\mathbb{Z}/2)\bar{r}_\tau, r_\tau \otimes 1 \notin \text{image } b_2^w\}. \tag{8}$$

We now define $\pi_1(w)$ by

$$\pi_1 = \begin{cases} (\mathbb{Z}/2t)\bar{t} \oplus \operatorname{cok}(b_2^w)/(\mathbb{Z}/2)\bar{r}_{-1} & \text{if } w = \cdots {}_{r_{-1}}\eta t \cdots \\ \overline{H}_1 \oplus \operatorname{cok}(b_2^w) & \text{otherwise} \end{cases} \tag{9}$$

with $\overline{H}_1 = H_1$. The homomorphism η^w in (5) is given by the composite

$$\eta^w : H_0 \otimes \mathbb{Z}/2 \twoheadrightarrow \operatorname{cok}(b_2^w) \overset{i}{\subset} \pi_1 \tag{10}$$

where the inclusion i carries the basis element $\bar{r}_{-1}$ to $t \cdot \bar{t}$ if $w = \cdots r_{-1}\eta t$ and carries $\bar{r}_\tau$ to $\bar{r}_\tau$ otherwise. Clearly h in (5) is trivial on the second summand of $\pi_1(w)$ and satisfies $h(\bar{t}) = t$ and $h \mid \overline{H}_1 = \mathrm{id}$ respectively. Now (10) induces the homomorphism

$$\eta^w \otimes 1 : H_0 \otimes \mathbb{Z}/2 \to \pi_1 \otimes \mathbb{Z}/2 \tag{11}$$

which carries $r_{-1} \otimes 1 \notin \text{image } b_2^w$ to the trivial element if $w = \cdots r_{-1}\eta t$ and carries $r_\tau \otimes 1 \notin \text{image } b_2^w$ to $\bar{r}_\tau \otimes 1$ otherwise. Hence $\eta^w \otimes 1$ carries basis elements to basis elements or to the trivial element. Using (11) we obtain the group $G(\eta^w)$ together with the short exact sequence

$$\pi_1 \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(\eta^w) \xrightarrow{\mu} H_0 * \mathbb{Z}/2 \tag{12}$$

by the push-out of Definition 8.1.3. The basis of $H_0 * \mathbb{Z}/2$ is $\{r_\tau/2, \alpha \le \tau \le \beta\}$ and we have the basis elements

$$\overline{r_\tau/2} \in G(\eta^w) \quad \text{with}\, \mu(\overline{r_\tau/2}) = r_\tau/2. \tag{13}$$

Here the order of $\overline{r_\tau/2}$ is 4 if the number r_τ is 2 and if $(\eta^w \otimes 1)(r_\tau \otimes 1) \neq 0$; in this case one has

$$2\overline{r_\tau/2} = \Delta(\eta^w \otimes 1)(r_\tau \otimes 1). \tag{14}$$

Otherwise the order of $\overline{r_\tau/2}$ is 2. A complete basis of $G(\eta^w)$ is given by all elements (13) and all Δ-images of basis elements in $\pi_1(w)\otimes\mathbb{Z}/2$ which are not of the form (14). This describes the short exact sequence (12) completely. We now obtain

$$b_3^w\colon H_3 \to G(\eta^w). \tag{15}$$

Here $L_3(w)$ is a spherical vertex, and hence a basis element in H_3, if $w=\xi\cdots$ and in this case we get

$$b_3^w(L_3(w)) = \overline{r_\alpha/2} \quad \text{if} \quad w = \xi_{r_\alpha}\cdots .$$

On the other hand, $R_3(w)$ is a spherical vertex if $w = \cdots\xi$ or $w=\varepsilon$ and in these cases we get

$$b_3^w(R_3(w)) = \begin{cases} \overline{r_\beta/2} & \text{if } w = \cdots{}_{r_\beta}\xi \\ \Delta[\bar r_{-1}\otimes 1] & \text{if } w = \cdots{}_{r_{-1}}\varepsilon. \end{cases}$$

These formulas define b_3^w. Given b_3^w we obtain the composite homomorphism

$$\eta_\#^w = q\Delta(\eta^w\otimes 1)\colon H_0\otimes\mathbb{Z}/2 \to \pi_1\otimes\mathbb{Z}/2 \to G(\eta^w) \to \operatorname{cok}(b_3^w) \tag{16}$$

where q is the quotient map. As in Definition 8.1.3 we obtain the group $\bar G(H_2,\eta_\#^w)$ by the following push-out diagram

$$\begin{tikzcd}
\operatorname{Ext}(H_2,\operatorname{cok} b_3^w) \arrow[r, tail, "\Delta"] & \bar G(H_2,\eta_\#^w) \arrow[r, two heads, "\mu"] & \operatorname{Hom}(H_2, H_0\otimes\mathbb{Z}/2) \\
\operatorname{Ext}(H_2, H_0\otimes\mathbb{Z}/2) \arrow[u, "(\eta_\#^w)_*"] \arrow[ur, phantom, "\text{push}"] & & \\
\operatorname{Ext}(H_2,\mathbb{Z}/2)\otimes H_0 \arrow[u, equal] \arrow[r, "\Delta\otimes 1"] & \operatorname{Hom}(G(H_2),\mathbb{Z}/4)\otimes H_0 \arrow[uu, "\nabla"] \arrow[r, two heads, "\mu\otimes 1"] & \operatorname{Hom}(H_2,\mathbb{Z}/2)\otimes H_0 \arrow[uu, equal]
\end{tikzcd}$$

Finally we define the element β^w in (2) by

$$\beta^w = \Delta(q_*\beta_1^w) + \nabla(\beta_2^w) \in \bar G(H_2,\eta_\#^w) \tag{17}$$

where

$$\beta_1^w \in \operatorname{Ext}(H_2, G(\eta^w))$$

$$\beta_2^w \in \operatorname{Hom}(G(H_2),\mathbb{Z}/4)\otimes H_0$$

are the following elements. Given the canonical direct sum decomposition of $G(H_2)$, see Proposition 1.6.5, the element β_2^w is the canonical 'lift' with

$$(\mu\otimes 1)\beta_2^w = b_2^w. \tag{18}$$

That is, via (7) the element b_2^w is a sum of basis elements and β_2^w is exactly the same sum of the corresponding basis elements in $\mathrm{Hom}(G(H_2), \mathbb{Z}/4) \otimes H_0$. Here only $\mathrm{Hom}(G(\mathbb{Z}/2), \mathbb{Z}/4) \otimes \mathbb{Z}/r = \mathbb{Z}/4$ with $r \leq 4$ is a non-split extension of $\mathrm{Hom}(Z/2, \mathbb{Z}/2) \otimes \mathbb{Z}/r = \mathbb{Z}/2$ and in this case the basis element of $\mathbb{Z}/4$ corresponds to the basis element of $\mathbb{Z}/2$. Hence the element β_2^w corresponds to all η-summands in the attaching map $f(w) \mid M(H_2, n+1)$ in Definition 10.2.6. Next we define β_1^w similarly by all ξ-summands and ε-summands in the attaching map $f(w) \mid M(H_2, n+1)$. Let $\mathbb{Z}s_\tau$ be the subgroup of H_2 generated by the basis element $s_\tau \in B_2$. Then the inclusion $j_{s_\tau}: \mathbb{Z}/s_\tau \subset H_2$ induces

$$(j_{s_\tau})^*: \mathrm{Ext}(H_2, G(\eta^w)) \to \mathrm{Ext}(\mathbb{Z}s_\tau, G(\eta^w)) = \mathbb{Z}s_\tau \otimes G(\eta^w).$$

Now we define β_1^w by the coordinates

$$(j_{s_\tau})^*\beta_1^w = \begin{cases} s_\tau \otimes \overline{r_\delta/2} & \text{if } w = \cdots {}_{r_\delta}\xi^{s_\tau} \cdots,\ \delta = \tau - 1 \\ s_\tau \otimes \overline{r_\tau/2} & \text{if } w = \cdots {}^{s_\tau}\xi_{r_\tau} \\ s_1 \otimes \Delta(\bar{t} \otimes 1) & \text{if } w = \cdots t\xi^{s_1} \cdots,\ \tau = 1 \\ s_1 \otimes \Delta\left(\overline{r_{-1}} \otimes 1\right) & \text{if } w = \cdots {}_{r_{-1}}\varepsilon^{s_1} \cdots,\ \tau = 1. \end{cases} \tag{19}$$

This completes the definition of β^w. Using Addendum 2.6.5 and the attaching map $f(w)$ in Definition 10.2.6 we see that β^w is actually the boundary invariant of $X(w)$. As in Definition 8.1.5 the element β^w yields the extension element $\{\pi_2(w)\}$ and hence we are now able to compute $\pi_2(w) = \pi_{n+2}X(w)$.

Now let (w, φ) be a cyclic word and let $H_* = H_*(w, \varphi) = H_* X(w, \varphi)$ be the homology, with $H_3 = H_1 = 0$. Moreover since b_2 is surjective also $\pi_1 X(w, \varphi) = 0$. Hence we get the Γ-sequence

$$0 \to H_0 * \mathbb{Z}/2 \to \pi_2 \to H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \to 0 \tag{20}$$

and the boundary invariant

$$\beta \in \overline{G}(H_2, \eta_*) = \mathrm{Ext}(H_2, H_0 * \mathbb{Z}/2) \oplus \mathrm{Hom}(H_2, H_0 \otimes \mathbb{Z}/2) \tag{21}$$

with $\beta = (\beta_1, b_2)$. Similarly as above b_2 is given by the η-summands in the attaching map $f(w)$, and β_1 is given by the ξ-summands in the attaching map $f(w)$; see Definition 10.2.7 (5), (6). This way we obtain the A^3-system $S(w, \varphi) \cong \Lambda' X(w, \varphi)$. The isomorphism (21) is natural in H_2 hence the extension

$$H_0 * \mathbb{Z}/2 \rightarrowtail \pi_2 \twoheadrightarrow \ker(b_2)$$

is given by $\{\pi_2\} = j^*(\beta_1)$ where $j: \ker(b_2) \subset H_2$ is the inclusion.

(10.2.14) Steenrod squares for $X(w)$ For finite $(n-1)$-connected $(n+3)$-dimensional polyhedra X we have the Steenrod squaring operations

(a) $$Sq_2\colon H_2(2) \to H_0(2)$$

(b) $$Sq_2\colon H_3(2) \to H_1(2)$$

where $H_i(2) = H_{n+i}(X, \mathbb{Z}/2)$ is the homology with coefficients in $\mathbb{Z}/2$. For cohomology groups $H^i(2) = H^{n+i}(X, \mathbb{Z}/2) = \mathrm{Hom}(H_i(2), \mathbb{Z}/2)$ one has the dual operations

(a)′ $$Sq^2\colon H^0(2) \to H^3(2)$$

(b)′ $$Sq^2\colon H^1(2) \to H^3(2)$$

which are given by $Sq^2 = \mathrm{Hom}(Sq_2, \mathbb{Z}/2)$; see (5.2.15). If $X = X(w)$ is given by a word we have a basis of $H_i = H_{n+i}(X)$ and the isomorphism

$$H_i(2) = H_i \otimes \mathbb{Z}/2 \oplus H_{i-1} * \mathbb{Z}/2$$

yields a basis of $H_i(2)$; see Definition 10.2.4. In terms of this basis

(a) $$Sq_2\colon H_2 \otimes \mathbb{Z}/2 \oplus H_1 * \mathbb{Z}/2 \to H_0 \otimes \mathbb{Z}/2$$

is determined in the obvious way by the letters η in the word w. That is the restriction $H_2 \otimes \mathbb{Z}/2 \to H_0 \otimes \mathbb{Z}/2$ is defined as b_2^w in (10.2.13) (7) and the restriction $H_1 * \mathbb{Z}/2 \to H_0 \otimes \mathbb{Z}/2$ carries the generator $t/2 \in H_1 * \mathbb{Z}/2$ to $r_{-1} \otimes 1$ if $w = \cdots{}_{r_{-1}}\eta t \cdots$, where r_{-1} denotes the spherical vertex if $w = \eta t \cdots$. Similarly

(b) $$Sq_2\colon H_3 \otimes \mathbb{Z}/2 \oplus H_2 * \mathbb{Z}/2 \to H_1 \otimes \mathbb{Z}/2 \oplus H_0 * \mathbb{Z}/2$$

is determined by the letters ξ in the word w. For $X = X(w, \varphi)$ we obtain Sq_2 by similar formulas as in the definition of the attaching map $f(w, \varphi)$ in Definition 10.2.7; in fact, in this case we have $\beta = (\beta_1 = Sq_2, b_2 = Sq_2)$ where β is defined as in (10.2.13) (21) with $\mathrm{Ext}(H_2, H_0 * \mathbb{Z}/2) = \mathrm{Hom}(H_2 * \mathbb{Z}/2, H_0 * \mathbb{Z}/2)$.

(10.2.15) Adem operations for $X(w)$ We first consider $w = {}_r\varepsilon^s, {}_r\xi^s, {}^s\eta_r$ where r and s are powers of two or empty ($=0$). Then the Adem operations

$\phi', \phi'', \phi_2^0, \phi_4^2, \phi_2^4$ in Section 8.5 are computed in the following table.

	$X({}_r\varepsilon^s)$	$r \neq 0$ $X({}_r\xi^s)$	$s \neq 0$ $X({}^s\eta_r)$
ϕ'	$\begin{cases} 0, (r,s) = (2, \geq 0) \\ \chi_2^2, \text{ otherwise} \end{cases}$	0	0
ϕ''	$\begin{cases} 0, (r,s) = (\geq 0, 2) \\ \chi_2^2, \text{ otherwise} \end{cases}$	0	0
ϕ_2^0	$\begin{cases} \chi_2^0, (r,s) = (0, \geq 0) \\ 0, \text{ otherwise} \end{cases}$	0	$\begin{cases} \chi_2^0, (r,s) = (0,2) \\ 0, \text{ otherwise} \end{cases}$
ϕ_4^2	$\begin{cases} 0, (r,s) = (\geq 0, 2) \\ \chi_4^2, \text{ otherwise} \end{cases}$	$\begin{cases} \chi_2^2, (r,s) = (2, \geq 0) \\ 0, \text{ otherwise} \end{cases}$	0
ϕ_2^4	$\begin{cases} 0, (r,s) = (2, \geq 0) \\ \chi_2^4, \text{ otherwise} \end{cases}$	0	$\begin{cases} \chi_2^2, (r,s) = (\geq 0, 2) \\ 0, \text{ otherwise} \end{cases}$

Here $\chi_m^n \in \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m)$ is the canonical generator. In the general case we can compute the Adem operations in the same way for $X(w)$ and $X(w, \varphi)$ by using the attaching map f in Definitions 10.2.6 and 10.2.7 and the table above.

(10.2.16) Example For the indecomposable space $X({}^2\xi_2\varepsilon^8)$ we have

$$\begin{aligned} \phi' &= 0 \quad \text{since } \ker(Sq^1) = 0, \\ \phi'' &= 0 \quad \text{since } \ker(Sq^2Sq^1) = 0, \\ \phi_2^0 &= 0 \quad \text{by (10.2.15)}, \\ \phi_2^4 &= 0 \quad \text{by (10.2.15)}, \end{aligned}$$

and only $\phi_4^2 \neq 0$. On the other hand, the indecomposable space $X({}_8\varepsilon^2\eta_2)$ satisfies

$$\begin{aligned} \phi' &= 0 \quad \text{since } \mathrm{im}(Sq^3) = H^3(2), \\ \phi'' &= 0 \quad \text{since } \mathrm{im}(Sq^1) = H^3(2), \\ \phi_2^0 &= 0 \quad \text{by (10.2.15)}, \\ \phi_4^2 &= 0 \quad \text{by (10.2.15)}, \end{aligned}$$

and only $\phi_2^4 \neq 0$. These examples show that the classical $\mathbb{Z}/2$-Adem operations ϕ', ϕ'', ϕ_2^0 do not suffice to classify $(n-1)$-connected $(n+3)$-

dimensional polyhedra. One has to use both operations ϕ_4^2 and ϕ_2^4. This is done in the definition of A^3-cohomology systems in Section 8.5.

10.3 The $(n-1)$-connected $(n+3)$-dimensional polyhedra with cyclic homology groups, $n \geq 4$

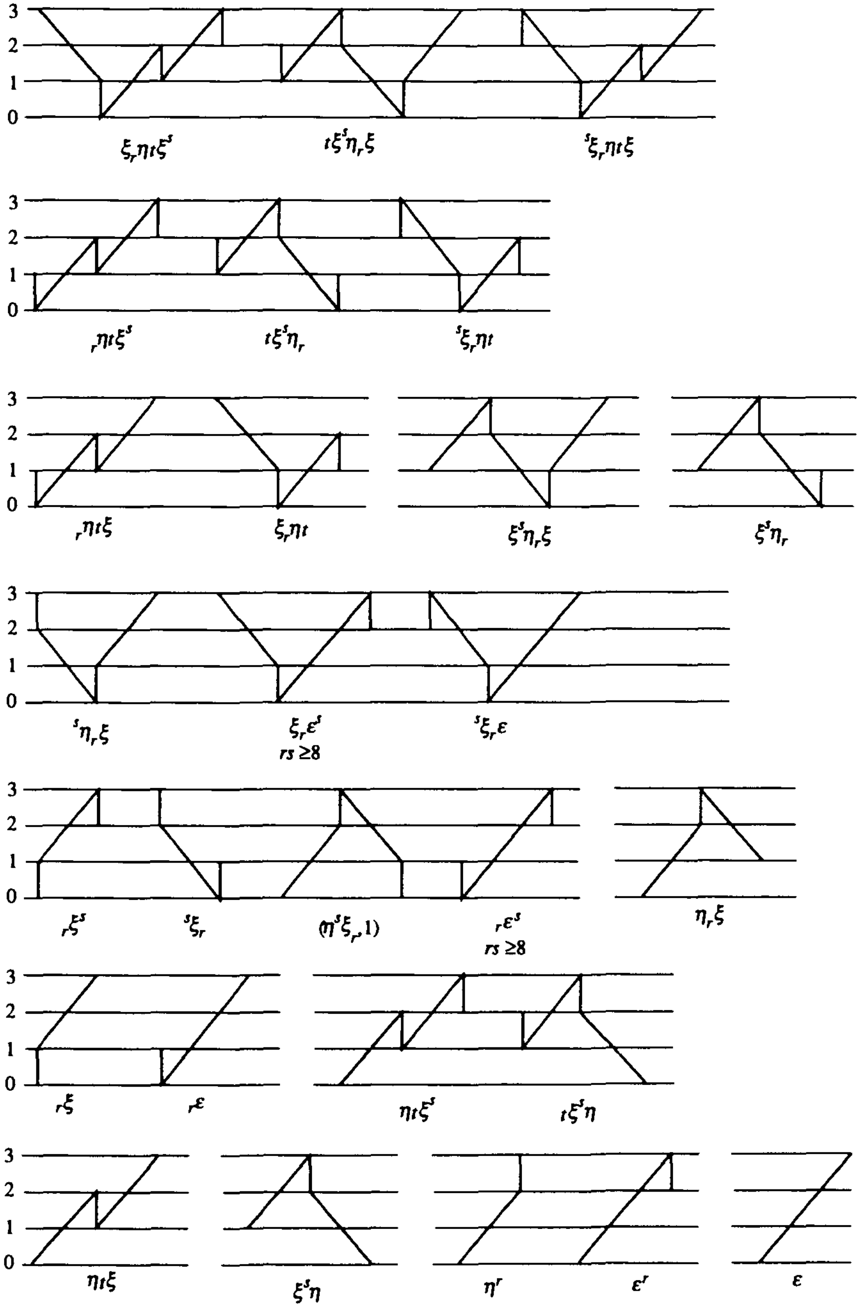
We here give an application of the classification theorem 10.2.9. We describe explicitly all indecomposable $(n-1)$-connected $(n+3)$-dimensional homotopy types X, $n \geq 4$, for which all homology groups $H_i X$ are cyclic, $i \geq 0$. Let $H_* = (H_0, H_1, H_2, H_3)$ be a tuple of finitely generated abelian groups with H_3 free abelian and let $N(H_*)$ be the number of all indecomposable homotopy types X as above, with homology groups $H_{n+i}(X) \cong H_i$ for $i \in \{0,1,2,3\}$.

(10.3.1) Theorem *Let $n \geq 4$. The indecomposable $(n-1)$-connected $(n+3)$-dimensional homotopy types X, for which all homology groups $H_i(X)$ are cyclic, are exactly the elementary Moore spaces in $\mathbf{A}_n^3$, the elementary Chang complexes in* (10.2.11), *and the spaces $X(w)$ where w is one of the words in the following list.*

The list describes all w of the theorem ordered by the homology $H_* \cong H_*(X(w))$. Below we also describe all graphs of such words w; the attaching map for $X(w)$ is obtained by Definitions 10.2.6 and 10.2.7. Let r, t, s be powers of 2.

$H_* = (H_0$	H_1	H_2	$H_3)$	$N(H_*)$	w with $H_* X(w) \cong H_*$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	$\mathbb{Z}$	3	$\xi_r \eta t \xi^s$, $t\xi^s \eta_r \xi$, ${}^s\xi_r \eta t \xi$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	0	3	${}_r\eta t \xi^s$, $t\xi^s \eta_r$, ${}^s\xi_r \eta t$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	0	$\mathbb{Z}$	2	${}_r\eta t \xi$, $\xi_r \eta t$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	$\mathbb{Z}$	1	$\xi^s \eta_r \xi$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	0	1	$\xi^s \eta_r$
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	$\mathbb{Z}$	$\begin{cases} 2, & r=s=2 \\ 3, & rs \geq 8 \end{cases}$	${}^s\eta_r \xi$, ${}^s\xi_r \varepsilon$ and $\xi_r \varepsilon^s$ for $rs \geq 8$
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	0	$\begin{cases} 3, & r=s=2 \\ 4, & rs \geq 8 \end{cases}$	${}_r\xi^s$, ${}^s\eta_r$, $(\eta\, {}^s\xi_r, 1)$, and ${}_r\varepsilon^s$ for $rs \geq 8$
$\mathbb{Z}/r$	0	$\mathbb{Z}$	$\mathbb{Z}$	1	$\eta_r \xi$
$\mathbb{Z}/r$	0	0	$\mathbb{Z}$	2	${}_r\xi$, ${}_r\varepsilon$
$\mathbb{Z}$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	0	2	$\eta t \xi^s$, $t\xi^s \eta$
$\mathbb{Z}$	$\mathbb{Z}/t$	0	$\mathbb{Z}$	1	$\eta t \xi$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/s$	0	1	$\xi^s \eta$
$\mathbb{Z}$	0	$\mathbb{Z}/s$	0	2	η^r, ε^r
$\mathbb{Z}$	0	0	$\mathbb{Z}$	1	ε

All words in the list are special words, except the word $(\eta\,{}^s\xi_r, 1)$ which is a special cyclic word associated with the automorphism 1 of $\mathbb{Z}/2$.



(10.3.2) Remark Let $n \geq 4$ and let $H_* = (H_0, H_1, H_2, H_3)$ be a tuple of cyclic groups with $H_3 \in \{\mathbb{Z}, 0\}$. Then it is easy to describe, by use of Theorem 10.3.1, all $(n-1)$-connected homotopy types X with $H_{n+i}(X) = H_i$ for $0 \leq i \leq 3$ and $\dim X \leq n+3$. In fact all such homotopy types are in a canonical way one-point unions of the indecomposable homotopy types in Theorem 10.3.1. For example for $H_* = (\mathbb{Z}/6, \mathbb{Z}/2, \mathbb{Z}/2, 0)$ there exist exactly nine such homotopy types X which are

$$M(\mathbb{Z}/6, n) \vee M(\mathbb{Z}/2, n+1) \vee M(\mathbb{Z}/2, n+2)$$

$$M(\mathbb{Z}/6, n) \vee X(2\xi^2)$$

$$M(\mathbb{Z}/3, n) \vee X({}_2\eta 2) \vee M(\mathbb{Z}/2, n+2)$$

$$M(\mathbb{Z}/3, n) \vee X({}_2\xi^2) \vee M(\mathbb{Z}/2, n+1)$$

$$M(\mathbb{Z}/3, n) \vee X\left({}^2\eta_2\right) \vee M(\mathbb{Z}/2, n+1)$$

$$M(\mathbb{Z}/3, n) \vee X(\eta^2\xi_2, 1) \vee M(\mathbb{Z}/2, n+1)$$

$$M(\mathbb{Z}/3, n) \vee X({}_2\eta 2\xi^2)$$

$$M(\mathbb{Z}/3, n) \vee X(2\xi^2\eta_2)$$

$$M(\mathbb{Z}/3, n) \vee X\left({}^2\xi_2\eta 2\right).$$

It is easy to compute the homotopy groups $\pi_n, \pi_{n+1}, \pi_{n+2}$ of these spaces; see (10.2.13). Similarly we see that there are 24 homotopy types X for $H_* = (\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z})$. We leave this as an exercise; compare the list in Baues [HC], p. 24.

10.4 The decomposition problem for stable types

A simply connected CW-space X is of *finite type* if equivalently (a) or (b) are satisfied: (a) all homology groups of X are finitely generated, (b) all homotopy groups of X are finitely generated. Let $\mathbf{a}_n^k$ be the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of CW-spaces X of finite type with trivial homotopy groups $\pi_i X = 0$ for $i < n$ and $i > n+k$. Hence the objects of $\mathbf{a}_n^k$ are $(n-1)$-connected $(n+k)$-types of finite type which we also call $\mathbf{a}_n^k$-*types*; compare the notation in (10.1.5). The loop space functor gives us the sequence of homotopy categories $(n \geq 2)$

(10.4.1) $$\mathbf{a}_1^k \xleftarrow{\Omega} \mathbf{a}_2^k \leftarrow \cdots \leftarrow \mathbf{a}_{n-1}^k \xleftarrow{\Omega} \mathbf{a}_n^k \leftarrow$$

which is Eckmann–Hilton dual to the k-stem of homotopy categories in (10.1.5). For $k+2<n$ the functor Ω: $\mathbf{a}_n^k \to \mathbf{a}_{n-1}^k$ is an equivalence of categories, and for $k+2=n$ this functor is full and faithful but not representative. We have the Postnikov functor

$$P_{n+k}: \mathbf{A}_n^{k+1} \to \mathbf{a}_n^k \tag{10.4.2}$$

which is full and representative; see (2.5.2). This is the restriction of the Postnikov functor in (3.4.6) which is compatible with the detecting functors in the classification theorem (3.4.4). In the stable range $k+2<n$ we have the commutative diagram

$$\begin{tikzcd} \mathbf{A}_n^{k+1} \arrow[d, "P_{n+k}"'] & \mathbf{A}_{n-1}^{k+1} \arrow[l, "\Sigma", "\approx"'] \arrow[d, "P_{n+k-1}"] \\ \mathbf{a}_n^k \arrow[r, "\Omega", "\approx"'] & \mathbf{a}_{n-1}^k \end{tikzcd}$$

Moreover we get:

(10.4.3) Lemma *In the stable range $k+2<n$ the Postnikov functor P_{n+k}: $\mathbf{A}_n^{k+1} \to \mathbf{a}_n^k$ is an additive functor between additive categories. The biproduct in $\mathbf{A}_n^{k+1}$ is the one-point of spaces and the biproduct in $\mathbf{a}_n^k$ is the product of spaces. In particular, one has a canonical isomorphism*

$$P_{n+k}(X \vee Y) = P_{n+k}(X) \times P_{n+k}(Y)$$

for $X, Y \in \mathbf{A}_n^{k+1}$.

The 'decomposition problem for stable types' asks for the complete classification of indecomposable objects in the additive category $\mathbf{a}_n^k$, $k+2<n$. There is a relationship between the decomposition problem in $\mathbf{A}_n^{k+1}$ and $\mathbf{a}_n^k$ respectively. For this we recall the following classical 'theorem on trees of homotopy types' due to J.H.C. Whitehead [SH]; see also II.§6 in Baues [CH].

(10.4.4) Theorem *Let X, Y be two finite m-dimensional CW-complexes, $m \geq 2$, and assume X and Y have the same $(m-1)$-type, that is $P_{m-1}X = P_{m-1}Y$. Then there exist natural numbers A, B such that the one-point unions*

$$X \vee \bigvee_A S^m \simeq Y \vee \bigvee_B S^m$$

are homotopy equivalent.

The theorem shows that each $(m-1)$-type Q determines a connected tree $HT(Q, m)$ which we call the *tree of homotopy types* of (Q, m). The vertices of this tree are the homotopy types $\{X\}$ of finite m-dimensional CW-complexes with $P_{m-1}(X) \simeq Q$. The vertex $\{X\}$ is connected by an edge to the vertex $\{Y\}$ if Y has the homotopy type of $X \vee S^m$. The *roots* of this tree are the homotopy types $\{X\}$ as above which do not admit a decomposition $X \simeq X' \vee S^m$.

(10.4.5) Example We describe simply connected 4-dimensional CW-complexes X_1, X_2 with $X_1 \not\simeq X_2$ but $X_1 \vee S^4 \simeq X_2 \vee S^4$. Let

$$X_1 = (S^2 \cup_5 e^3), \qquad X_2 = (S^2 \cup_5 e^3) \cup_{2\eta} e^4$$

where $\eta = \eta_2$ in the Hopf map. Using the detecting functor of Theorem 3.5.6 we see that

$$\Lambda(X_1) = \left(\mathbb{Z} \xrightarrow{1} \Gamma(\mathbb{Z}/5) \to 0 \to 0, H_0 = \mathbb{Z}/5\right)$$

$$\Lambda(X_2) = \left(\mathbb{Z} \xrightarrow{2} \Gamma(\mathbb{Z}/5) \to 0 \to 0, H_0 = \mathbb{Z}/5\right)$$

where $\Gamma(\mathbb{Z}/5) = \mathbb{Z}/5$. A homotopy equivalence f: $X_1 \simeq X_2$ would give us a commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1"] \arrow[d, "\pm 1"'] & \mathbb{Z}/5 \arrow[d, "\Gamma(f_*)"] \\ \mathbb{Z} \arrow[r, "2"] & \mathbb{Z}/5 \end{tikzcd}$$

Here $\Gamma(f_*)$ has to be a square number (see Corollary 1.2.9) and hence $\Gamma(f_*) \equiv \pm 1$ modulo 5. This yields the contradiction, so that $X_1 \not\simeq X_2$. On the other hand, the following commutative diagram shows by Theorem 3.5.6 that $X_1 \vee S^4 \simeq X_2 \vee S^4$,

$$\begin{tikzcd} \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "{(0,1)}"] \arrow[d, "{\begin{pmatrix} 2, & 1 \\ 5, & 3 \end{pmatrix}}"'] & \mathbb{Z}/5 \arrow[d, "1"] \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "{(0,2)}"] & \mathbb{Z}/5 \end{tikzcd}$$

Let $\mathrm{Ind}(\mathbf{A}_n^{k+1})$ and $\mathrm{Ind}(\mathbf{a}_n^k)$ be the sets of indecomposable homotopy types in $\mathbf{A}_n^{k+1}$ and $\mathbf{a}_n^k$ respectively. Then Theorem 10.4.4 gives us the following comparison between indecomposable objects in $\mathbf{A}_n^{k+1}$ and $\mathbf{a}_n^k$.

(10.4.6) Theorem *Let $k+2<n$. Then the Postnikov functor P_{n+k} yields a surjection between sets*

$$P_{n+k}: \operatorname{Ind}(\mathbf{A}_n^{k+1}) - \{S^{n+k+1}\} \twoheadrightarrow \operatorname{Ind}(\mathbf{a}_n^k)$$

and for $Q \in \operatorname{Ind}(\mathbf{a}_n^k)$ the inverse image $P_{n+k}^{-1}\{Q\}$ is the set of roots in $HT(Q, n+k)$.

There is only one root in $HT(Q, n+k)$ in case the objects in $\mathbf{A}_n^{k+1}$ have a unique decomposition. Therefore the theorem of Chang (10.1.8) and the decomposition theorem 10.2.9 show:

(10.4.7) Theorem *For $k = 0, 1, 2$ the Postnikov functor P_{n+k} yields a bijection* $(n \geq k+2)$

$$P_{n+k}: \operatorname{Ind}(\mathbf{A}_n^{k+1}) - \{S^{n+k+1}\} \approx \operatorname{Ind}(\mathbf{a}_n^k).$$

Hence the decomposition theorem 10.2.9 for $\mathbf{A}_n^3$ actually also solves the decomposition problem in $\mathbf{a}_n^2$. We say that $K(A, n)$ is an *elementary* Eilenberg–Mac Lane space if A is an elementary cyclic group, that is $A = \mathbb{Z}$ or $A = \mathbb{Z}/p^i$ where p^i is a prime power. Using Example 10.1.2 and Proposition 10.1.6 we get for $k = 0$:

(10.4.8) Proposition *Let $n \geq 2$. The elementary Eilenberg–Mac Lane spaces are the only indecomposable homotopy types in $\mathbf{a}_n^0$ and each object in $\mathbf{a}_n^0$ has a unique decomposition. Moreover, the Postnikov functor $\mathbf{A}_n^1 \to \mathbf{a}_n^0$ carries an elementary Moore space to the corresponding elementary Eilenberg–Mac Lane space; see Proposition* 10.1.6.

For $k = 1$ the situation is more complicated.

(10.4.9) Definition Let p, q be powers of 2 and $n \geq 3$. Then there is a unique indecomposable space $K(\mathbb{Z}, \mathbb{Z}/q, n)$ in $\mathbf{a}_n^1$ with homotopy groups $\pi_n = \mathbb{Z}$ and $\pi_{n+1} = \mathbb{Z}/q$. Moreover there is a unique indecomposable space $K(\mathbb{Z}/p, \mathbb{Z}/q, n)$ in $\mathbf{a}_n^1$ with homotopy groups $\pi_n = \mathbb{Z}/p$ and $\pi_{n+1} = \mathbb{Z}/q$. In fact, $K(\mathbb{Z}, \mathbb{Z}/q, n) = K(\eta, n)$ and $K(\mathbb{Z}/p, \mathbb{Z}/q, n) = K(\eta', n)$ where η: $\mathbb{Z} \to \mathbb{Z}/q$ and η': $\mathbb{Z}/p \to \mathbb{Z}/q$ are the unique non-trivial stable quadratic functions, that is, $\eta(1) = q/2$ and $\eta'(1) = q/2$. We call $K(\mathbb{Z}, \mathbb{Z}/q, n)$ and $K(\mathbb{Z}/p, \mathbb{Z}/q, n)$ the *elementary Chang types*.

In addition to the theorem of Chang (10.1.8) we now get:

(10.4.10) Proposition *Let $n \geq 3$. The elementary Eilenberg–Mac Lane spaces in $\mathbf{a}_n^1$ and the elementary Chang types are the only indecomposable homotopy* *types in* $\mathbf{a}_n^1$ *and each object in* $\mathbf{a}_n^1$ *has a unique decomposition. Moreover, the bijection in Theorem* 10.4.7,

$$P_{n+1}: \operatorname{Ind}(\mathbf{A}_n^2) - \{S^{n+2}\} \approx \operatorname{Ind}(\mathbf{a}_n^1),$$

is given by the following list where we use the elementary Chang complexes in Theorem 10.1.8. *Let p and q be powers of* 2.

X	$P_{n+1}X$
S^n	$K(\mathbb{Z}, \mathbb{Z}/2, n)$
S^{n+1}	$K(\mathbb{Z}, n+1)$
$M(\mathbb{Z}/p, n)$	$K(\mathbb{Z}/p, \mathbb{Z}/2, n)$
$M(\mathbb{Z}/q, n+1)$	$K(\mathbb{Z}/q, n+1)$
$X(\eta)$	$K(\mathbb{Z}, n)$
$X({}_p\eta)$	$K(\mathbb{Z}/p, n)$
$X(\eta q)$	$K(\mathbb{Z}, \mathbb{Z}/2q, n)$
$X({}_p\eta q)$	$K(\mathbb{Z}/p, \mathbb{Z}/2q, n)$

Moreover P_{n+1} carries an elementary Moore space of odd primes in $\mathbf{A}_n^2$ to the corresponding elementary Eilenberg–Mac Lane space.

10.5 The $(n-1)$-connected $(n+2)$-types with cyclic homotopy groups, $n \geq 4$

We describe explicitly all indecomposable $(n-1)$-connected $(n+2)$-types X, $n \geq 4$, for which all homotopy groups $\pi_n X, \pi_{n+1} X, \pi_{n+2} X$ are cyclic. We use the bijection of Theorem 10.4.7 and the computation of homotopy groups $\pi_{n+2} X$ via A^3-systems in (10.2.13). The elementary Eilenberg–Mac Lane spaces and the elementary Chang types in Definition 10.4.9 have cyclic homotopy groups. They correspond to spaces $X(w)$ as follows.

(10.5.1) Theorem *The elementary Eilenberg–Mac Lane spaces and the elementary Chang types in* $\mathbf{a}_n^2$, $n \geq 4$, *correspond via the bijection*

$$P_{n+2}^{-1}: \operatorname{Ind}(\mathbf{a}_n^2) \approx \operatorname{Ind}(\mathbf{A}_n^3) - \{S^{n+3}\}$$

to the indecomposable homotopy types in $\mathbf{A}_n^3$ *described in the following list.*

Here P_{n+2}^{-1} carries odd elementary Eilenberg–Mac Lane spaces to the corresponding odd elementary Moore spaces. Let r, s, t be powers of 2.

Y	$P_{n+2}^{-1}(Y) \in \mathbf{A}_n^3$
$K(\mathbb{Z}, n+2)$	S^{n+2}
$K(\mathbb{Z}, n+1)$	$X(\xi)$
$K(\mathbb{Z}, n)$	$X({}^2\eta)$
$K(\mathbb{Z}/s, n+2)$	$M(\mathbb{Z}/s, n+2)$
$K(\mathbb{Z}/t, n+1)$	$X(t\xi)$
$K(\mathbb{Z}/r, n)$	$X({}^2\eta_r \xi)$
$K(\mathbb{Z}, \mathbb{Z}/s, n+1)$	$\begin{cases} S^{n+1}, & s=2 \\ X(\xi^{s'}), & s=2s' \geq 4 \end{cases}$
$K(\mathbb{Z}/t, \mathbb{Z}/s, n+1)$	$\begin{cases} M(\mathbb{Z}/t, n+1), & s=2 \\ X(t\xi^{s'}), & s=2s' \geq 4 \end{cases}$
$K(\mathbb{Z}, \mathbb{Z}/t, n)$	$\begin{cases} X(\varepsilon), & t=2 \\ X(\eta t'\xi), & t=2t' \geq 4 \end{cases}$
$K(\mathbb{Z}/r, \mathbb{Z}/t, n)$	$\begin{cases} X({}_2\xi), & t=2, r=2 \\ X(\xi_r \varepsilon), & t=2, r \geq 4 \\ X(\xi_r \eta t'\xi), & t=2t' \geq 4 \end{cases}$

The space $P_{n+2}^{-1}(Y)$ describes the $(n+3)$-skeleton of Y up to a one-point union of spheres S^{n+3}, that is

$$Y^{n+3} \simeq P_{n+2}^{-1}(Y) \vee \bigvee_A S^{n+3} \tag{10.5.2}$$

where A is an appropriate number ≥ 0. Part of the list above corresponds to Proposition 10.4.10; see (10.2.11).

Proof of Theorem 10.5.1 The right-hand side of the list describes indecomposable objects, hence we have only to show that these objects have the appropriate homotopy groups. This is done by the A^3-systems in (10.2.13). For example we have to show that $X = X(\xi_r \eta t'\xi)$ satisfies $\pi_n X = \mathbb{Z}/r$, $\pi_{n+1} X = \mathbb{Z}/2t'$, and $\pi_{n+2} X = 0$. We obtain $\pi_{n+1} X$ by (10.2.13) (9) and we get $\pi_{n+2} X = \pi_2$ by the exact sequence

$$H_3 = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{b_3} G(\eta^w) \to \pi_2 \to H_2 = 0$$

where b_3 is surjective by (10.2.13) (15). We leave it to the reader to check the other cases. $\square$

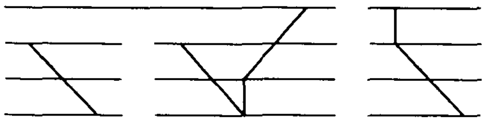
Let $\pi_* = (\pi_0, \pi_1, \pi_2)$ be a tuple of finitely generated abelian groups and let $N(\pi_*)$ be the number of all indecomposable homotopy types X with homotopy groups $\pi_{n+i}(X) \cong \pi_i$ for $i = 0, 1, 2$ and $\pi_j(X) = 0$ otherwise, $n \geq 4$.

(10.5.3) Theorem *Let $n \geq 4$. The indecomposable $(n-1)$-connected $(n+2)$-types X for which all homotopy groups $\pi_i(X)$ are cyclic, are exactly the elementary Eilenberg–Mac Lane spaces in $\mathbf{a}_n^2$, the elementary Chang types in Theorem* 10.5.1, *and the spaces $P_{n+2}X(w)$ where w is one of the words in the following list.*

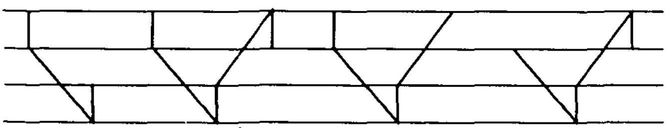
The list describes all w of the theorem ordered by the homotopy groups $\pi_* \cong \pi_* X(w)$. Below we describe also all graphs of such words w. Let $r, t, s \geq 2$ be powers of 2 and for $t, s \geq 4$ let $2t' = t$ and $2s' = s$.

$\pi_* = (\pi_0$	π_1	$\pi_2)$	$N(\pi_*)$	w with $\pi_* X(w) \cong \pi_*$
$\mathbb{Z}$	0	$\mathbb{Z}$	1	η
$\mathbb{Z}/r$	0	$\mathbb{Z}$	1	$\eta_r \xi$
$\mathbb{Z}$	0	$\mathbb{Z}/s$	1	${}^{2s}\eta$
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	3	$\begin{cases} {}^{2}\eta_r \text{ for } s = 2,\ {}^{2}\eta_r \xi^{s'} \text{ for } s = 2s' \geq 4 \\ {}^{2s}\eta_r \xi,\ (\eta\,{}^{s}\xi_r, 1) \end{cases}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/s$	1	$\xi^s \eta$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	1	$\xi^s \eta_r \xi$
$\mathbb{Z}$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	1	$\begin{cases} P_{n+2}S^n,\ t = s = 2 \\ \eta t',\ t = 2t' \geq 4,\ s = 2 \\ \varepsilon^{s'},\ t = 2,\ s = 2s' \geq 4 \\ \eta t' \xi^{s'},\ t = 2t' \geq 4,\ s = 2s' \geq 4 \end{cases}$
$\mathbb{Z}/r$	$\mathbb{Z}/t$ $t \geq 4$	$\mathbb{Z}/s$ $s \geq 4$	2	$\begin{cases} \xi_r \eta t' \xi^{s'},\ {}^{s'}\xi_r \eta t' \xi \\ \text{with } t = 2t',\ s = 2s' \end{cases}$
$\mathbb{Z}/r$	$\mathbb{Z}/t$ $t \geq 4$	$\mathbb{Z}/2$	1	$\xi_r \eta t',\ t = 2t'$
$\mathbb{Z}/r$ $r \geq 4$	$\mathbb{Z}/2$	$\mathbb{Z}/s$ $s \geq 4$	2	$\begin{cases} {}^{s'}\xi_r \varepsilon \text{ with} \\ \xi_r \varepsilon^{s'},\ s = 2s' \end{cases}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/s$ $s \geq 4$	2	$\begin{cases} P_{n+2}M(\mathbb{Z}/2, n) \text{ for } s = 4 \text{ and} \\ {}^{s'}\xi_2 \varepsilon \text{ for } s = 2s' \geq 4, \text{ and} \\ {}_2\xi^{s''} \text{ for } s = 4s'' \geq 8 \end{cases}$
$\mathbb{Z}/r$ $r \geq 4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	2	$\begin{cases} {}_r\varepsilon \text{ and} \\ {}_r\xi \end{cases}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	1	${}_2\varepsilon$

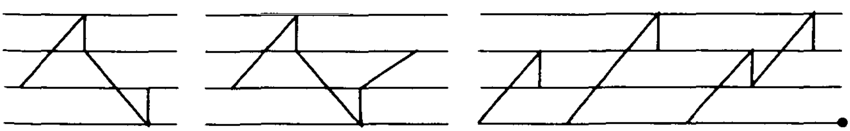
For all tuples of cyclic groups $\pi_* = (\pi_0, \pi_1, \pi_2)$, $\pi_0 \neq 0$, $\pi_2 \neq 0$ which are not in the list we have $N(\pi_*) = 0$. If $\pi_0 = 0$ or π_2-0 we use the list in Theorem 10.5.1. All words in the list are special words, except the word $(\eta\xi_r, 1)$ which is a special cyclic word associated with the identity automorphism 1 of $\mathbb{Z}/2$. We now show the list of graphs:



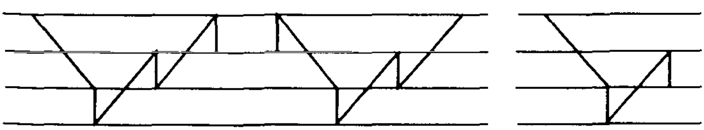
η $\eta_r\xi$ ${}^{2s}\eta$



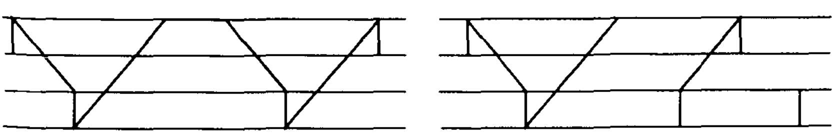
${}^{2}\eta_r$ ${}^{2}\eta_r\xi^{s'}$ ${}^{2s}\eta_r\xi$ $(\eta^s\xi_r, 1)$



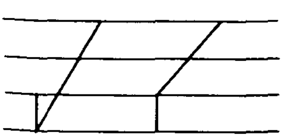
$\xi^s\eta$ $\xi^s\eta_r\xi$ η' $\varepsilon^{s'}$ $\eta'\xi^{s'}$ S^n



$\xi_r\eta'\xi^{s'}$ ${}^{s'}\xi_r\eta'\xi$ $\xi_r\eta'$



${}^{s'}\xi_r\varepsilon$ $\xi_r\varepsilon^{s'}$ ${}^{s'}\xi_2\varepsilon$ ${}_2\xi^{s''}$ $M(Z_2, n)$



${}_r\varepsilon$ ${}_r\xi$

Here s and t satisfy the conditions described in the list of words above.

(10.5.4) Remark Let $n \geq 4$ and let $\pi_* = (\pi_0, \pi_1, \pi_2)$ be a tuple of cyclic groups. Then it is easy to describe, by use of Theorem 10.5.3, all homotopy types X with $\pi_{n+i}(X) \cong \pi_i$ for $i = 0, 1, 2$ and $\pi_j X = 0$ for $j < n$ and $j > n+2$. In fact all such homotopy types are in a canonical way products of the indecomposable homotopy types in Theorem 10.5.3. For example for $\pi_* = (\mathbb{Z}/6, \mathbb{Z}/2, \mathbb{Z}/2)$ there exist exactly seven such homotopy types X, which are

$$K(\mathbb{Z}/6, n) \times K(\mathbb{Z}/2, n+1) \times K(\mathbb{Z}/2, n+2)$$

$$K(\mathbb{Z}/6, n) \times K(\mathbb{Z}/2, \mathbb{Z}/2, n+1)$$

$$K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, \mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n+1)$$

$$K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, n+1) \times P_{n+2} X\left({}^2\eta_2\right)$$

$$K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, n+1) \times P_{n+2} X\left({}^4\eta_2\,\xi\right)$$

$$K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, n+1) \times P_{n+2} X(\eta^2 \xi_2, 1)$$

$$K(\mathbb{Z}/3, n) \times P_{n+2} X({}_2\varepsilon).$$

It is clear how to compute the homology H_n, H_{n+1}, and H_{n+2} of these spaces and, in fact, we can easily describe the A^3-system of these spaces. We leave it to the reader to consider other cases, for example for $\pi_* = (\mathbb{Z}_4, \mathbb{Z}_{10}, \mathbb{Z})$ there exist exactly three homotopy types X with $\pi_* \cong \pi_* X$.

Proof of Theorem 10.5.3 Let $\pi_* = (\pi_0, \pi_1, \pi_2)$ be a tuple of cyclic groups 0, $\mathbb{Z}$, or $\mathbb{Z}/2^k$ and assume $\pi_0 \neq 0$ and $\pi_2 \neq 0$. We want to describe all indecomposable X in $\mathbf{A}_n^3$ with $\pi_{n+i}(X) \cong \pi_i$ for $i = 0, 1, 2$. We clearly have $\pi_0 = H_0$ and the exact sequence

$$H_3 \to G(\eta^w) \to \pi_2 \to H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^w} \pi_1 \to H_1 \to 0 \tag{1}$$

where $H_0 \otimes \mathbb{Z}/2 = \mathbb{Z}/2$ and we have the extension

$$\pi_1 \otimes \mathbb{Z}/2 \rightarrowtail G(\eta^w) \twoheadrightarrow H_0 * \mathbb{Z}/2. \tag{2}$$

Moreover π_2 is determined by β as in (10.2.13).

First case, $\pi_1 = 0$

Then also $H_1 = 0$ and $b_2 \neq 0$; moreover H_2 is cyclic or $H_2 = \text{cyclic} \oplus \mathbb{Z}/2$ since π_2 is cyclic. We have $G(\eta^w) = H_0 * \mathbb{Z}/2$ since $\pi_1 = 0$. Hence $H_3 = \mathbb{Z}$ or $H_3 = 0$. The only special cyclic word with these properties is $X = X(\eta \xi_r, 1)$

satisfying $\pi_0 = \mathbb{Z}/r$, $\pi_2 = \mathbb{Z}/s$; see (10.2.13) (21) for the computation of π_2. An ε-word w for $X = X(w)$ is not possible since $b_2 \neq 0$, also a central word is not possible since $H_1 = 0$. Hence it suffices to consider basic words with the homological properties above. The basic words w are

$$\eta,\quad {}^s\eta,\quad \eta_r,\quad {}^s\eta_r,\quad \eta_r\xi,\quad \eta_r\xi^2,\quad {}^s\eta_r\xi,\quad {}^s\eta_r\xi^2 \quad \text{and} \quad {}^2\eta_r\xi^s. \tag{3}$$

We obtain $\pi_2 = \pi_{n+2}X(w)$ by the remark following (10.2.13) (19). For w in (3) we get $(2s' = s)$

$$\pi_2 = \mathbb{Z},\quad \mathbb{Z}/s',\quad \mathbb{Z}\oplus\mathbb{Z}/2,\quad \mathbb{Z}/s'\oplus\mathbb{Z}/2,\quad \mathbb{Z},\quad \mathbb{Z}\oplus\mathbb{Z}/4,\quad \mathbb{Z}/s',$$
$$\mathbb{Z}/s'\oplus\mathbb{Z}/4 \quad \text{and} \quad \mathbb{Z}/2s \quad \text{respectively.}$$

Hence the cyclic cases of π_2 are those in the list with $\pi_1 = 0$. For this we replace s' resp. $2s$ by s. This completes the first case.

Second case, $\pi_1 \neq 0$ and $b_2 \neq 0$

Since $b_2 \neq 0$ we have $\eta^w = 0$ and $\pi_1 = H_1 \neq 0$ is cyclic and $G(\eta^w) = \pi_1 \otimes \mathbb{Z}/2 \oplus H_0 * \mathbb{Z}/2$. If $H_1 = \mathbb{Z}/t$ then w has to be a central word, but since $b_2 \neq 0$ and H_0 cyclic, this is not possible. If $H_1 = \mathbb{Z}$, w has to be a basic word of the form $\xi\,{}^s\eta$ for $H_0 = \mathbb{Z}$, or one of $\xi\,{}^s\eta_r$ and $\xi\,{}^s\eta_r\xi$ and $\xi\,{}^s\eta_r\xi^2$ and $\xi\,{}^2\eta_r\xi^s$ for $H_0 = \mathbb{Z}/r$. However for $H_0 = \mathbb{Z}/r$ we have $H_3 \neq 0$ since $G(\eta^w)$ is not cyclic. Hence only $\xi\,{}^s\eta_r\xi$ and $\xi\,{}^s\eta$ remain. For $X(\xi\,{}^s\eta)$ we have $\pi_1 = H_1 = \mathbb{Z} = H_0$. π_2 is a non-trivial extension and hence $\pi_2 = \mathbb{Z}/s$. For $X(\xi\,{}^s\eta_r\xi)$ we have $\pi_1 = H_1 = \mathbb{Z}$ and $H_0 = \mathbb{Z}/r$ and π_2 is a non-trivial extension with $\pi_2 = \mathbb{Z}/s$. This completes the second case.

Third case, $\pi_1 \neq 0$ and $b_2 = 0$ and $H_1 \neq 0$

Then H_2 is cyclic and we see that π_1 is a non-trivial extension of H_1 by $\mathbb{Z}/2$ and hence w has to be a central word with $H_1 = \mathbb{Z}/t$. Moreover for $H_1 \neq 0$ and $H_0 = \mathbb{Z}/r$ we have $H_3 \neq 0$ since $G(\eta^w)$ is not cyclic. Hence we get the possible words ηt, $\eta t\xi$, $\eta t\xi^s$, $\xi_r\eta t$, $\xi_r\eta t\xi$, $\xi_r\eta t\xi^s$, or ${}^s\xi_r\eta t\xi$. Here $\eta t\xi$ and $\xi_r\eta t\xi$ appear in the list of Theorem 10.5.1 with $\pi_2 = 0$. For ηt we get $\pi_1 = \mathbb{Z}/2t$ and $\pi_2 = \mathbb{Z}/2$. For $\eta t\xi^s$ we get $\pi_1 = \mathbb{Z}/2t$ and $\pi_2 = \mathbb{Z}/2s$. For $\xi_r\eta t$ we get $\pi_1 = \mathbb{Z}/2t$ and $\pi_2 = \mathbb{Z}/2$. For $\xi_r\eta t\xi^s$ we get $\pi_1 = \mathbb{Z}/2t$ and $\pi_2 = \mathbb{Z}/2s$. For ${}^s\xi_r\eta t\xi$ we get $\pi_1 = \mathbb{Z}/2t$ and $\pi_2 = \mathbb{Z}/2s$. This completes the third case.

Fourth case, $\pi_1 \neq 0$ and $b_2 = 0$ and $H_1 = 0$

Then H_2 is cyclic and $\eta^w\colon \mathbb{Z}/2 \cong \pi_1$ is an isomorphism. Now w has to be a basic word of the form ${}_r\xi, {}_r\xi^s$ (since $b_2 = 0$) or an ε-word of the form ε, ${}_r\varepsilon$,

ε^s, ${}_r\varepsilon^s$ $(rs \geq 8)$, $\xi_r\varepsilon$ $(r \neq 2)$, $\xi_r\varepsilon^s$ $(rs \geq 8)$, ${}^s\xi_r\varepsilon$ We compute $\pi_2(w) = \pi_{n+2}X(w)$ as follows: $\pi_2({}_r\xi) = 0$ for $r = 2$ and $= \mathbb{Z}/2$ for $r \geq 4$, $\pi_2({}_r\xi^s) = \mathbb{Z}/4s$ for $r = 2$ and non-cyclic otherwise, $\pi_2(\varepsilon) = 0$, $\pi_2({}_r\varepsilon) = \mathbb{Z}/2$, $\pi_2(\varepsilon^s) = \mathbb{Z}/2s$, $\pi_2({}_r\varepsilon^s) = \mathbb{Z}/2 \oplus \mathbb{Z}/2s$, $\pi_2(\xi_r\varepsilon) = 0$ for $r \neq 2$. $\pi_2(\xi_r\varepsilon^s) = \mathbb{Z}/2s$, $\pi_2({}^s\xi_r\varepsilon) = \mathbb{Z}/2s$. This completes the fourth case.

Final case

We finally have to consider indecomposable spaces X in $\mathbf{A}_n^3$ which are not of the form $X(w)$, $X(w, \varphi)$ and for which $\pi_n X \neq 0$, $\pi_{n+1}X$, $\pi_{n+2}X \neq 0$ are cyclic. The only possibilities for X are the elementary Moore spaces S^n and $M(\mathbb{Z}/2, n)$. □

(10.5.5) Remark on k-invariants Let $n \geq 4$ and let $\pi_* = (\pi_0, \pi_1, \pi_2)$ be a tuple of abelian groups. The classical approach to classify homotopy types Y with homotopy groups $\pi_{n+i}(Y) \cong \pi_i$ for $i = 0, 1, 2$ and $\pi_j Y = 0$ for $j < n$, $j > n+2$ uses the Postnikov tower and the k-invariants of Y. For this we first choose a homomorphism

$$\eta \colon \pi_0 \otimes \mathbb{Z}/2 \to \pi_1$$

and then we choose an element

$$k \in H^{n+3}(K(\eta, n), \pi_2)$$

Then there is a unique homotopy type $Y(\eta, k)$ with k-invariants η and k respectively. We have the split short exact sequence

$$\begin{array}{ccccc}
\mathrm{Ext}(H_{n+2}K(\eta,n),\pi_2) & \overset{\Delta}{\rightarrowtail} & H^{n+3}(K(\eta,n),\pi_2) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(H_{n+3}K(\eta,n),\pi_2) \\
\| & & & & \| \\
\mathrm{Ext}(\ker(\eta),\pi_2) & & & & \mathrm{Hom}(G(\eta),\pi_2)
\end{array}$$

where $G(\eta)$ is determined by η as in Definition 8.1.3 and $\ker(\eta) \subset \pi_0 \otimes \mathbb{Z}/2$. The split exact sequence, however, does not show us how the group of homotopy equivalences $\mathscr{E}(K(\eta, n))$ acts on the direct sum $\mathrm{Ext}(\ker(\eta), \pi_2) \oplus \mathrm{Hom}(G(\eta), \pi_2)$. This action is needed if we want to classify the homotopy types Y above since for $\alpha \in \mathscr{E}(K(\eta, n))$ the spaces $Y(\eta, k)$ and $Y(\eta, \alpha^* k)$ are homotopy equivalent. Using the 'theorem on k-invariants' (2.5.10) we have relations between the Postnikov invariant k and the exact sequence

$$H_3 \to G(\eta) \xrightarrow{k_*} \pi_2 \to H_2 \to \pi_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1 \to H_1 \to 0$$

which is Whitehead's Γ-sequence for the $(n+2)$-skeleton X of $Y(\eta,k)$. Here $k_* = \mu(k)$ is given by μ above and the element

$$k_\dagger = \Delta^{-1}q_*(k) \in \mathrm{Ext}(\ker(\eta), \mathrm{cok}(k_*))$$

determines the extension

$$\mathrm{cok}(k_*) \rightarrowtail H_2 \twoheadrightarrow \ker(\eta)$$

given by the exact sequences. We can apply these facts to get some hold on the k-invariant of the spaces $Y = P_{n+2}X(w)$ described in the list of Theorem 10.5.3. For example for

$$w = \xi_r \eta t' \xi^{s'}, \qquad t = 2t', \qquad s = 2s',$$

we have $\pi_0 = \mathbb{Z}/r$, $\pi_1 = \mathbb{Z}/t$, $\pi_2 = \mathbb{Z}/s$ and

$$\eta = \eta^w : \mathbb{Z}/2 \overset{t'}{\rightarrowtail} \mathbb{Z}/t$$

is the inclusion. Hence μ above is an isomorphism and the k-invariant is

$$\begin{aligned} &k = \mu^{-1}(k_*) \in H^{n+3}(K(\eta,n),\pi_2) \cong \mathrm{Hom}(G(\eta),\pi_2)\\ &k_* : G(\eta) = \pi_1 \otimes \mathbb{Z}/2 \oplus \pi_0 * \mathbb{Z}/2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/s = \pi_2. \end{aligned}$$

Here k^* is trivial on $\pi_0 * \mathbb{Z}/2 = \mathbb{Z}/2$ and is the inclusion $s' : \mathbb{Z}/2 \rightarrowtail \mathbb{Z}/s$ on $\pi_1 \otimes \mathbb{Z}/2 = \mathbb{Z}/2$. This follows from the A^3-system associated with w in (10.2.13). For $w' = {}^{s'}\xi_r \eta t' \xi$ we obtain $\eta = \eta^{w'}$ as above and $k = \mu^{-1}(k_*)$ where k_* is trivial on $\pi_1 \otimes \mathbb{Z}/2$ and the inclusion on $\pi_0 * \mathbb{Z}/2$. By Theorem 10.5.3 the spaces $Y = P_{n+2}X(w)$ and $Y' = P_{n+2}X(w')$ are the only indecomposable homotopy types which realize $\pi_* = (\mathbb{Z}/r, \mathbb{Z}/t, \mathbb{Z}/s)$, $t, s \geq 4$. The A_n^3-polyhedra $X(w)$ and $X(w')$ correspond to the $(n+3)$-skeleton of Y and Y' respectively; see (10.5.2).

10.6 Example: the truncated real projective spaces $\mathbb{R}P_{n+4}/\mathbb{R}P_n$

The real projective space RP_∞ is a CW-complex with n-skeleton $\mathbb{R}P_n$ and with exactly one cell in each dimension. Hence the quotient spaces

(10.6.1) $$P_n^3 = \mathbb{R}P_{n+3}/\mathbb{R}P_{n-1}$$

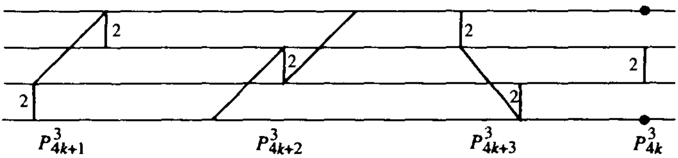
are $(n-1)$-connected $(n+3)$-dimensional spaces. We here show how these spaces fit into the classification. For $n \geq 4$ the spaces P_n^3 are stable; moreover since the homology groups of P_n^3 are cyclic, the 2-connected 6-dimensional spaces $P_3^3, \Sigma P_2^3, \Sigma^2 P_1^3$ with $P_1^3 = \mathbb{R}P_4$ are determined by their stabilization;

see Corollary 9.1.16. Hence it suffices to consider the stabilization of P_n^3, $n \geq 1$, and the simply connected 5-dimensional spaces $\Sigma(\mathbb{R}P_4)$ and $\mathbb{R}P_5/S^1$ where $S^1 = \mathbb{R}P_1$. We say that two finite CW-complexes X, Y are *stably* (Σ^r)*-equivalent* if there exists a homotopy equivalence $\Sigma^n X \simeq \Sigma^m Y$ for some $n, m \geq 0$ with $|n - m| = r$.

(10.6.2) Theorem *One has stable equivalences, $n \geq 1$,*

$$P_n^3 \sim \begin{cases} X({}_2\xi^2) & \text{for } n \equiv 1(4) \\ X(\eta 2\xi) & \text{for } n \equiv 2(4) \\ X({}^2\eta_2) & \text{for } n \equiv 3(4) \\ S^n \vee S^{n+3} \vee M(\mathbb{Z}/2, n+1) & \text{for } n \equiv 0(4). \end{cases}$$

Hence the graphs of these stable spaces are ($k \geq 0$)



where P_{4k}^3 with $k \geq 1$ is a one-point union of Moore spaces. For the stable homotopy groups $\pi_n^S X = \lim \pi_{n+k} \Sigma^k X$ we get the list below which we derive from the equivalence in Theorem 10.6.2. Let

$$P_n^\infty = \mathbb{R}P_\infty / \mathbb{R}P_{n-1}. \tag{10.6.3}$$

Then we have for $k \leq 2$ the isomorphism $\pi_{n+k}^S(P_n^\infty) = \pi_{n+k}^S(P_n^3)$ and this group is computed in the following table.

(10.6.4)

$n \geq 1$	$\pi_n^S(P_n^\infty)$	$\pi_{n+1}^S(P_n^\infty)$	$\pi_{n+2}^S(P_n^\infty)$
$n = 4k + 1$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$
$n = 4k + 2$	$\mathbb{Z}$	$\mathbb{Z}/4$	0
$n = 4k + 3$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$n = 4k$	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

Proof of (10.6.4) The case $n = 4k + 1$ is considered in Example 8.1.11 or in Theorem 10.5.3. The case $n = 4k + 2$ is obtained by Theorem 10.5.1 since $P_{n+2} X(\eta t \xi) = K(\mathbb{Z}, \mathbb{Z}/2t, n)$, $t \geq 2$. Moreover for $n = 4k + 3$ see again Theorem 10.5.3. $\square$

Proof of Theorem 10.6.2 Let $u \in H^2(\mathbb{R}P_\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2$ be the generator. Then the n-fold cup product $u^n \in H^n(\mathbb{R}P_\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2$ is a generator and the Steenrod square Sq^i satisfies

$$Sq^i(u^n) = \binom{n}{i} u^{n+i}, \tag{1}$$

compare 2.4 in Steenrod and Epstein [CO]. The formula also yields the action of Sq^2 on $H^*(P_n^3, \mathbb{Z}/2)$. Now one readily checks that the equivalences in Theorem 10.6.2 correspond to isomorphisms of homology groups compatible with the action of Sq^2. The classification of homotopy types with cyclic homology groups in Theorem 10.3.1 and (10.2.14) now shows that the first three equivalences hold. The case P_{4k}^3 is more complicated since Sq^2 acts trivially. Hence we get either an equivalence as in the theorem or

$$P_{4k}^3 \overset{?}{\sim} M(\mathbb{Z}/2, 4k+1) \vee X(\varepsilon). \tag{2}$$

Considering P_{4k}^3 as a Thom space shows readily that (2) does not hold. We also refer to Davis and Mahowald [CS] where all truncated spaces P_n^k are classified up to stable equivalence. □

We now consider the unstable spaces $\Sigma\mathbb{R}P_4$ and $\mathbb{R}P_5/S^1$ which are simply connected and 5-dimensional.

(10.6.5) Theorem *The space $\Sigma\mathbb{R}P_4$ is the mapping cone of*

$$\xi_2^2 + \gamma_2^2 : \Sigma^2 P_2 \to \Sigma P_2. \tag{1}$$

Moreover the space $\mathbb{R}P_5/S^1$ is the mapping cone of

$$(i_3\eta_3 + [i_3, i_2], 2i_3 + i_2\eta_2) : S^4 \vee S^3 \to S^3 \vee S^2. \tag{2}$$

Here i_3 and i_2 are the inclusions of S^3 and S^2 respectively and $[i_3, i_2]$ is the Whitehead product. In (1) we use the generators ξ_2^2, γ_2^2 defined in (11.5.16) with $\Sigma(\gamma_2^2) = 0$. We point out that $\Sigma\mathbb{R}P_4 = T_1(4)$ is one of the Brown–Gitler spaces in Goerss, Lannes, and Morel [VW].

Proof of Theorem 10.6.5 We know that $\Sigma\mathbb{R}P_4$ is stably $X({}_2\xi^2)$, therefore the attaching map has to be $f = \xi_2^2 + \delta\gamma_2^2$; see (11.5.9), with $\delta \in \{+1, -1\}$. In Baues [CH] IV.A.11 we have seen that $\Sigma\mathbb{R}P_3$ is the mapping cone of $(2\eta_2, -2)$: $S^3 \vee S^2 \to S^2$. This shows that $\delta = +1$. On the other hand, we proved in Baues [CH] IV.A.9 that $\mathbb{R}P_4/S^1$ is the mapping cone of

$$i_2\eta_2 + 2i_3 : S^3 \to S^2 \vee S^3 \simeq \mathbb{R}P_3/S^1.$$

We therefore get the following Γ-sequence for $\mathbb{R}P_5/S^1$:

$$\begin{array}{ccccccccccccc}
H_5 & \to & \Gamma_4 & \to & \pi_4 & \to & H_4 & \to & \Gamma H_2 & \xrightarrow{\eta} & \pi_3 & \to & H_3 & \to 0 \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & \\
\mathbb{Z} & & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & & \mathbb{Z}/2 & & 0 & & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & & \mathbb{Z}/2 &
\end{array}$$

Since $H_2 = \mathbb{Z}$ we get

$$\Gamma_4 = \Gamma_4(\mathbb{R}P_5/S^1) = \Gamma_2^2(\eta) = \mathbb{Z}/2(i_3\eta_3) \oplus \mathbb{Z}/2[i_3, i_2]$$

compare Section 11.3. Now b_5: $H_5 \to \Gamma_4$ is non-trivial since $\mathbb{R}P_5/S^1$ is stably $X(\eta 2\xi)$. In fact this implies $b_5(1) = i_3\eta_3 + \delta[i_3, i_2]$ with $\sigma \in \{1, -1\}$. Since there is a non-trivial cup product ($\mathbb{Z}/2$-coefficients) $H_3(2) \otimes H_2(2) \to H_5(2)$ we see that $\delta = +1$. The boundary invariant β is trivial since $H_4 = 0$ and hence the homotopy type of $\mathbb{R}P_5/S^1$ is determined by the exact sequence above. This yields the required homotopy equivalence for $\mathbb{R}P_5/S^1$. □

(10.6.6) Corollary *We obtain the homotopy groups $\pi_3(\Sigma\mathbb{R}P_4) = \mathbb{Z}/2$ and $\pi_4(\Sigma\mathbb{R}P_4) = \mathbb{Z}/4$ and $\pi_3(\mathbb{R}P_5/S^1) = \mathbb{Z}$ and $\pi_4(\mathbb{R}P_5/S^1) = \mathbb{Z}/2$.*

Proof The Γ-sequence for $\mathbb{R}P_5/S^1$ is described in the proof of Theorem 10.6.5. We now consider the Γ-sequence for $\Sigma\mathbb{R}P_4$:

$$\begin{array}{ccccccccccccc}
H_5 & \to & \Gamma_4 & \to & \pi_4 & \xrightarrow{0} & H_4 & \to & \Gamma H_2 & \xrightarrow{\eta} & \pi_3 & \to & H_3 & \to 0 \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & \\
0 & & \mathbb{Z}/4 & \cong & \mathbb{Z}/4 & & \mathbb{Z}/2 & \rightarrowtail & \mathbb{Z}/4 & \twoheadrightarrow & \mathbb{Z}/2 & & 0 &
\end{array}$$

Here b_4 is non-trivial and hence $\Gamma_4 \cong H_4$ is an isomorphism. Moreover we have in (11.3.7) the isomorphism η_*: $\Gamma_2^2(H_2) \cong \Gamma_2^2(\eta) = \mathbb{Z}/2$ so that α_*: $\mathbb{Z}/4 = \pi_4 M(H_2, 2) \cong \Gamma_4$ is an isomorphism

(10.6.7) Corollary $[\Sigma\mathbb{R}P_4, S^2] \cong \mathbb{Z}/2$. *We leave this as an exercise, use* (11.5.25).

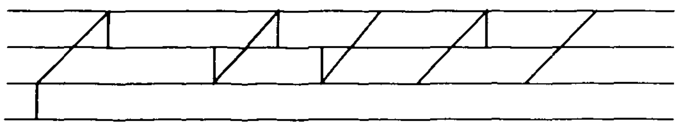
10.7 The stable equivalence classes of 4-dimensional polyhedra and simply connected 5-dimensional polyhedra

In the decomposition theorem 10.2.9 we determine for $n \geq 4$ the set $\mathfrak{X}$ of homotopy types of finite $(n-1)$-connected $(n+3)$-dimensional polyhedra. The homotopy types $\{\Sigma^{n-1}X\}$, where X is a finite connected 4-dimensional polyhedron, form a subset $\mathfrak{X}(4) \subset \mathfrak{X}$. Moreover the homotopy types $\{\Sigma^{n-2}Y\}$, where Y is a finite 1-connected 5-dimensional polyhedron, form a subset $\mathfrak{X}(5)$ with

$$\mathfrak{X}(4) \subset \mathfrak{X}(5) \subset \mathfrak{X}. \tag{10.7.1}$$

We now describe these subsets explicitly in terms of the spaces $X(w)$ and $X(w, \varphi)$ used in the decomposition theorem.

(10.7.2) Definition Let $r, t, s \geq 2$ be powers of 2. The *4-dimensional words* are ${}_r\xi^s$ with $r \geq s$, and $t\xi^s, t\xi, \xi^s, \xi$. The corresponding graphs are



The next result describes the set $\mathfrak{X}(4)$ in terms of 4-dimensional words.

(10.7.3) Theorem *Let $n \geq 4$ and let X be a finite connected 4-dimensional CW-complex. Then there is a decomposition (unique up to permutation)*

$$\Sigma^{n-1}X \simeq X_1 \vee \cdots \vee X_k.$$

Here the complexes X_i for $i = 1, \ldots, k$ are elementary Moore spaces or spaces $X(w)$ where w is a 4-dimensional word. Moreover for each 4-dimensional word w there is a finite connected 4-dimensional complex $A(w)$ with $\Sigma^{n-1}A(w) \simeq X(w)$.

Compare (V.A.4) in Baues [CH] where we call the spaces $A(w)$ 'elementary cup square spaces'.

(10.7.4) Definition Let w be a special word as defined in Definition 10.2.1. We say that w is a *5-dimensional special word* if w satisfies the following properties (1) and (2):

(1) $w \neq \eta^s \cdots$ and $w \neq \cdots {}^s\eta$;

(2) for each subword of the form ${}_r\eta^s$ or ${}^s\eta_r$ of w (that is $w = \cdots {}_r\eta^s \cdots$ or $w = \cdots {}^s\eta_r \cdots$) we have $2r \leq s$.

Moreover a special cyclic word (w, φ) in Definition 10.2.1 is 5-dimensional if w satisfies (2).

Now the next result describes the set $\mathfrak{X}(5)$ in (10.7.1) in terms of such 5-dimensional words.

(10.7.5) Theorem *Let $n \geq 4$ and let X be a finite 1-connected 5-dimensional CW-complex. Then there is a decomposition (unique up to permutation)*

$$\Sigma^{n-2}X \simeq X_1 \vee \cdots \vee X_k.$$

Here the complexes X_i for $1 = 1,\ldots,k$ are elementary Moore spaces or spaces $X(w)$ and $X(w,\varphi)$ where w and (w,φ) are 5-dimensional special words and 5-dimensional special cyclic words respectively. Moreover for each 5-dimensional special word w and for each 5-dimensional special cyclic word (w,φ) there exist finite 1-connected 5-dimensional CW-complexes $B(w)$ and $B(w,\varphi)$ respectively such that $\Sigma^{n-2}B(w) \simeq X(w)$ and $\Sigma^{n-2}B(w,\varphi) \simeq X(w,\varphi)$.

Proof The existence of $B(w)$ and $B(w,\varphi)$ follows from Theorem 11.6.6 below by desuspension of the attaching map $f(w)$ and $f(w,\varphi)$ in Definitions 10.2.6 and 10.2.7. Now let X be a finite 1-connected 5-dimensional CW-complex. Then the decomposition theorem 10.2.9 yields an equivalence $\Sigma^{n-2}X \simeq X_1 \vee \cdots \vee X_k$ where the X_i are elementary Moore spaces or $X(w)$ or $X(w,\varphi)$. We have to show that w satisfies (1) and (2) in Definition 10.7.4. For this we use the Pontrjagin square p for which we have the following natural commutative diagram of homomorphisms; see (1.5.3) in Baues [CH] and J.H.C. Whitehead [CE].

$$\begin{array}{ccc}
\Gamma H^2(X,A) & \xrightarrow{\quad p \quad} & H^4(X,\Gamma A) \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_*} \\
H^2(X,A)\otimes \mathbb{Z}/2 & & H^4(X,A\otimes\mathbb{Z}/2) \\
\| & & \| \\
H^2(X,\mathbb{Z}/2)\otimes A & \xrightarrow{Sq^2\otimes A} & H^4(X,\mathbb{Z}/2)\otimes A
\end{array}$$

Here A is any abelian group of coefficients. Now the commutativity of this diagram implies that w has to satisfy conditions (1) and (2) in Definition 10.7.4. In fact, assume $w = \cdots {}_r\eta^s \cdots$. Then r and s correspond to basis elements (see Definition 10.2.4)

$$e_r \in H_n(\Sigma^{n-2}X), \quad e_s \in H_{n+2}(\Sigma^{n-2}X)$$

of order r and s respectively. For $n = 2$ we obtain by e_r the dual basis element $e_r^* \in H^2(X,A)$ and $e_s \in H_4(X)$ determines a direct summand $\mathrm{Hom}(\mathbb{Z}/s,\Gamma A) \subset H^4(X,\Gamma A)$ with $A = \mathbb{Z}/r$, $\Gamma(A) = \mathbb{Z}/2r$. Now by (10.2.14) $w = \cdots {}_r\eta^s \cdots$ implies that $(Sq^2\otimes A)\sigma(e_r^*) \neq 0$ and therefore the coordinate $\varphi \in \mathrm{Hom}(\mathbb{Z}/s,\Gamma A)$ of $p(\gamma e_r^*) \in H^4(X,\Gamma A)$ satisfies $\sigma_*(\varphi) \neq 0$. This implies $s \geq 2r$. Here γ and σ are the functions in (1.2.1), (1.2.2). In a similar way we see that w satisfies (1) in (10.7.4). □

11

HOMOTOPY GROUPS IN DIMENSION 4

The computation of the homotopy groups $\pi_n X, n \geq 1$, of a connected space X is a fundamental problem of algebraic topology. It is well known how to determine the fundamental group $\pi_1 X$ in terms of the attaching maps of 2-cells in X. For $n \geq 2$ we may assume that X is simply connected since $\pi_n X$ coincides with the corresponding homotopy group of the universal cover of X. For a simply connected space the Hurewicz theorem shows that $\pi_2 X$ is isomorphic to the homology $H_2 X$ and that the Hurewicz homomorphism $\pi_3 X \to H_3 X$ is surjective. J.H.C. Whitehead considered the homotopy group $\pi_3(X)$ and showed that one has an exact sequence

$$H_4 X \xrightarrow{b_4} \Gamma(H_2 X) \xrightarrow{\eta} \pi_3 X \to H_3 X \to 0.$$

For this he computed the group $\Gamma_3(X)$ by the natural formula $\Gamma_3(X) = \Gamma(H_2 X)$. The corresponding computation of $\pi_4(X)$, however, was not achieved in the literature. In this chapter we compute $\Gamma_4 X = \Gamma_4(\eta)$ in terms of the homomorphism $\eta\colon \Gamma(H_2 X) \to \pi_3 X$ so that $\pi_4 X$ is now embedded in the exact sequence

$$\to H_5 X \xrightarrow{b_5} \Gamma_4(\eta) \to \pi_4 X \to H_4 X \xrightarrow{b_4} \cdots$$

which extends the sequence of J.H.C. Whitehead above. The formula $\Gamma_4(X) = \Gamma_4(\eta)$ relies on the computation of the homotopy group $\pi_4 M(A,2)$ of a Moore space of degree 2. Here A is an arbitrary abelian group. The results in this chapter are crucial for the classification of simply connected 5-dimensional homotopy types in Chapter 12 below where we also study the functorial properties of $\Gamma_4(\eta)$.

11.1 On $\pi_4(M(A,2))$

Let A be an abelian group. In this section we embed the homotopy group $\pi_4 M(A,2)$ in a natural short exact sequence. For this we need the following algebraic functors which carry abelian groups to abelian groups. First we recall from Definition 6.2.7 the definition of the *Γ-torsion* $\Gamma T(A)$ of A. If $d_A\colon A_1 \to A_0$ is a short free resolution of A, then we have

(11.1.1) $$\Gamma T(A) = \mathrm{kernel}(\delta_1)/\mathrm{image}(\delta_2)$$

with

$$A_1 \otimes A_1 \xrightarrow{\delta_2} \Gamma(A_1) \oplus A_1 \otimes A_0 \xrightarrow{\delta_1} \Gamma(A_0)$$

given by $\delta_1 = (\Gamma(d_A),[d_A,1])$ and $\delta_2 = ([1,1], -1 \otimes d_A)$. Let $\Gamma_*(d_A)$ be the chain complex given by (δ_2, δ_1) and which is concentrated in degree $0,1,2$; see Definition 6.2.5.

(11.1.2) Proposition *We have $\Gamma T(A) = A * \mathbb{Z}/2$ if A is cyclic, and*

$$\Gamma T(A \oplus B) = \Gamma T(A) \oplus \Gamma T(B) \oplus A * B$$

These formulas easily allow the computation of $\Gamma T(A)$ for all direct sums of cyclic groups A since ΓT is compatible with direct limits.

Proof of Proposition 11.1.2: Clearly $\Gamma T(\mathbb{Z}) = 0$. For $A = \mathbb{Z}/n$ let $d_A = n\colon \mathbb{Z} \to \mathbb{Z}$ so that $\Gamma_* d_A$ is

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

with $\delta_2(1) = (2,0) - (0,n)$ and $\delta_1(x,y) = n^2x + 2ny$. This shows $\Gamma T(\mathbb{Z}/n) = \mathbb{Z}/n * \mathbb{Z}/2$. Next we prove the cross-effect formula $\Gamma T(A \mid B) = A * B$. For this we consider the cross-effect $\Gamma_*(d_A \mid d_B)$ in the chain complex $\Gamma_*(d_A \oplus d_B)$. We readily see that $\Gamma_*(d_A \mid d_B)$ is given by

$$A_1 \otimes B_1 \oplus B_1 \otimes A_1 \xrightarrow{\partial_2} A_1 \otimes B_1 \oplus (A_1 \otimes B_0 \oplus B_0 \otimes A_1) \xrightarrow{\partial_1} A_0 \otimes B_0.$$

We map this chain complex to the chain complex $d_A \otimes B$ concentrated in degree 0 and 1. Let $q\colon B_0 \to B$ be the map of the resolution of B. Then we map $\Gamma_*(d_A \mid d_B)$ to $d_A \otimes B$ by $1 \otimes q$ on $A_0 \otimes B_0 = \Gamma_0(d_A \mid d_B)$ and by $(0, 1 \otimes q, 0)$ on $\Gamma_1(d_A \mid d_B)$. This chain map $h\colon \Gamma_*(d_A \mid d_B) \to d_A \otimes B$ induces isomorphisms in homology in degree 0 and 1. Hence

$$\Gamma T(A \mid B) = H_1 \Gamma_*(d_A \mid d_B) = H_1(d_A \otimes B) = A * B.$$

Compare the more general proof in Baues [QF] 7.3. □

Whitehead's Γ-functor is endowed with natural homomorphisms, see Section 1.2,

$$A \otimes \mathbb{Z}/2 \xleftarrow{\sigma} \Gamma A \xrightarrow{H} A \otimes A \tag{11.1.3}$$

given by $\sigma(\gamma(a)) = a \otimes 1$ and $H(\gamma(a)) = a \otimes a$ for $a \in A$. Similarly we obtain natural homomorphisms

$$A * \mathbb{Z}/2 \xleftarrow{\sigma} \Gamma T A \xrightarrow{H} A * A \tag{11.1.4}$$

as follows. We define chain maps

$$d_A \otimes \mathbb{Z}/2 \xleftarrow{\sigma} \Gamma_* d_A \xrightarrow{h} d_A \otimes A$$

by $\sigma_0 = \sigma$, $h_0 = (1 \otimes q)H$, $\sigma_1 = (\sigma, 0)$, and $h_1 = (0, 1 \otimes q)$. These chain maps induce (11.1.3) and (11.1.4) in homology. We have for $a, b, d \in A$ with $ha = hb = 0$ and $2d = 0$ the formulas

$$H\tau_h(a,b) = \tau_h(a,b) - \tau_h(b,a)$$

$$H\gamma(d) = \tau_2(d,d)$$

$$\sigma\tau_h(a,b) = 0$$

$$\sigma\gamma(d) = d.$$

Here $\gamma(d)$, $\tau_h(a,b) \in \Gamma T(A)$ are the generators in the proof of Theorem 6.2.9. We may assume that h is a power of a prime. Next we need the following notation on Lie algebras.

(11.1.5) Definition Let $T(A,1)$ be the free graded tensor algebra generated by the abelian group A where A is concentrated in degree 1. Thus $T(A,1)$ is a graded $\mathbb{Z}$-module for which

$$T(A,1)_n = A \otimes \cdots \otimes A = A^{\otimes n} \tag{1}$$

is the n-fold tensor product of A. We define the structure of a graded Lie algebra on $T(A,1)$ by

$$[x,y] = xy - (-1)^{|x||y|}yx \tag{2}$$

for $x, y \in T(A,1)$. Let $L(A,1)$ be the sub-Lie algebra generated by A in $T(A,1)$ and let $L_n(A,1) = L(A,1) \cap A^{\otimes n}$. Clearly, $L_n(A,1)$ is a functor which carries abelian groups to abelian groups. For $L_3(A,1)$ we have a further characterization as follows: consider the triple Lie bracket homomorphism

$$[[1,1],1]\colon A \otimes A \otimes A \to A \otimes A \otimes A \tag{3}$$

which carries $a \otimes b \otimes c$ to $[[a,b],c] = (a \otimes b + b \otimes a) \otimes c - c \otimes (a \otimes b + b \otimes a)$.

Lemma *The image of* $[[1,1],1]$ *is* $L_3(A,1)$ *and the kernel of* $[[1,1],1]$ *is the subgroup* W_3 *in* $A \otimes A \otimes A$ *which is generated by the following elements*

(a) $a \otimes b \otimes c + c \otimes a \otimes b + b \otimes c \otimes a$

(b) $a \otimes b \otimes c - b \otimes a \otimes c$

(c) $a \otimes a \otimes a$.

Therefore we have the natural isomorphisms

$$L_3(A,1) = [[A,A],A] = A \otimes A \otimes A/W_3. \tag{4}$$

Here $[[A,A],A]$ denotes the image of $[[1,1],1]$ above. Moreover we derive from Whitehead's quadratic functor Γ the following functor Γ_2^2: **Ab** $\to$ **Ab**

(11.1.6) Definition Let $\Gamma_2^2(A)$ be the quotient group

$$\Gamma_2^2(A) = (\Gamma(A) \otimes \mathbb{Z}/2 \oplus \Gamma(A) \otimes A)/M(A)$$

where $M(A)$ is the subgroup generated by the elements

$$\begin{cases} \gamma(x) \otimes x & (1) \\ [x,y] \otimes 1 + \gamma(x) \otimes y + [y,x] \otimes x & (2) \end{cases}$$

with $x, y \in A$. A homomorphism $\varphi: A \to B$ induces $\Gamma_2^2(A) \to \Gamma_2^2(B)$ by $\Gamma(\varphi) \otimes \mathbb{Z}/2 \oplus \Gamma(\varphi) \otimes \varphi$. The relation (2) implies that

$$[x,x'] \otimes y + [y,x] \otimes x' + [y,x'] \otimes x \in M(A). \tag{3}$$

For this replace x in (2) by $x + x' \in A$.

(11.1.7) Lemma *There is a natural isomorphism*

$$\Gamma_2^2(A) = \Gamma(A) \otimes \mathbb{Z}/2 \oplus L_3(A,1)$$

which carries $u \otimes 1$ with $u \in \Gamma(A)$ to the equivalence class of $u \otimes 1$ in $\Gamma_2^2(A)$ and which carries $[[x,y],z] \in L_3(A,1)$ to the equivalence class of $[x,y] \otimes z$.

We define a homomorphism

(11.1.8) $$\Delta: \Gamma_2^2(A) \to \pi_4 M(A,2)$$

as follows. Using the identification $A = \pi_2 M(A,2)$ and $\Gamma(A) = \pi_3 M(A,2)$ the function Δ carries $u \otimes 1 \in \Gamma(A) \otimes \mathbb{Z}/2$ to the composite $u\eta_3$ where $\eta_3: S^4 \to S^3$ is the Hopf map. Moreover Δ maps $[[x,y],z] \in L_3(A,1)$ to the triple Whitehead product $[[x,y],z] \in \pi_4 M(A,2)$ and maps $u \otimes x \in \Gamma(A) \otimes A$ to the Whitehead product $[u,x] \in \pi_4 M(A,2)$. The relation (1), (2), and (3) in Definition 11.1.6 correspond to the following classical formulas so that Δ is well defined:

(1) the equation $[\eta_3, \iota_2] = 0$ where $\iota_2 \in \pi_2 S^2$ is the generator;

(2) the Barcus–Barratt formula which for $a, b \in \pi_2 X$ yields the equation $[a\eta_3, b] = [a,b]\eta_4 - [[b,a],a]$;

(3) the Jacobi identity for Whitehead products which for $a, b, c \in \pi_2 X$ yields $[[a,b],c]+[[c,a],b]+[[b,c],a]=0$.

(11.1.9) Theorem *There is a natural short exact sequence* ($A \in \mathbf{Ab}$)

$$\Gamma_2^2(A) \overset{\Delta}{\rightarrowtail} \pi_4 M(A,2) \overset{\mu}{\twoheadrightarrow} \Gamma T(A).$$

Moreover $\Delta L_3(A,1)$ *is a direct summand of* $\pi_4 M(A,2)$, *unnaturally.*

In the proof of Theorem 11.1.9 we need the following notation for mapping cones. Let $f\colon X_1 \to X_0$ be a map in $\mathbf{Top}^*$ and let C_f be the mapping cone of f. Hence C_f is the push-out

$$\begin{array}{ccc} CX_1 & \xrightarrow{\pi_f} & C_f \\ \cup & & \cup i_f \\ X_1 & \xrightarrow[f]{} & X_0 \end{array} \tag{11.1.10}$$

where CX_1 is the cone of X_1. Let

$$\pi_n(X_1 \vee X_0)_2 = \text{kernel}(0,1)_*\colon \pi_n(X_1 \vee X_0) \to \pi_n(X_0).$$

Then we obtain the following commutative diagram with exact rows

$$\begin{tikzcd} & \pi_{n+1}(CX_1 \vee X_0, X_1 \vee X_0) \arrow[r, "\partial", "\approx"'] \arrow[d, "(\pi_f,1)_*"] & \pi_n(X_1 \vee X_0)_2 \arrow[d, "(f,1)_*"] & \\ \to \pi_{n+1}C_f \arrow[r] & \pi_{n+1}(C_f, X_0) \arrow[r, "\partial"] & \pi_n(X_0) \arrow[r] & \pi_n(C_f) \end{tikzcd}$$

The bottom row is the exact homotopy sequence of the pair (C_f, X_0). We call

$$E_f' = (\pi_f, 1)_* \partial^{-1}\colon \pi_n(X_1 \vee X_0)_2 \to \pi_{n+1}(C_f, X_0)$$

the *functional suspension*. The operator E_f' is part of the EHP-sequence in Section A.6 in the appendix; see also Baues [AH] and Baues [OT].

Proof of Theorem 11.1.9 Compare the proof of Theorem 6.15.13. Let

$$f\colon X_1 = M(A_1,2) \to X_0 = M(A_0,2) \tag{1}$$

be a map which induces the resolution $d_A\colon A_1 \to A_0$, that is $d_A = H_2(f)$. Then the mapping cone of f is the Moore space $C_f = M(A,2)$. We now obtain the commutative diagram

$$\begin{tikzcd} & \pi_3(X_1 \vee X_0)_2 \arrow[r, phantom, "="] \arrow[d, "E_f'"'] & \Gamma(A_1) \oplus A_1 \otimes A_0 \arrow[d, "\delta_1"] \\ \pi_4 M(A,2) \arrow[r, "j"'] & \pi_4(C_f, X_0) \arrow[r, "\partial"'] & \pi_3 X_0 = \Gamma A_0 \end{tikzcd} \tag{2}$$

where δ_1 is the operator in (11.1.1). The EHP sequence shows that kernel(E'_f) = image(δ_2). Hence we obtain a well-defined homomorphism.

$$\mu = (E'_f)^{-1} j \colon \pi_4 M(A,2) \to \Gamma T(A) = \frac{\operatorname{kernel}(\delta_1)}{\operatorname{image}(\delta_2)}. \tag{3}$$

This homomorphism is surjective since the EHP sequence shows that E'_f is surjective. Moreover

$$\begin{aligned} \operatorname{kernel}(\mu) &= \operatorname{kernel}(j) \\ &= \operatorname{image}(\pi_4 X_0 \to \pi_4 M(A,2)). \end{aligned} \tag{4}$$

We have to show that Δ in (11.1.8) induces an isomorphism

$$\Delta \colon \Gamma_2^2(A) \to \operatorname{kernel}(\mu). \tag{5}$$

For this we consider the diagram

$$\begin{tikzcd} \pi_4(X_1 \vee X_0)_2 \arrow[d, "E'_f"'] \arrow[dr, "(f,1)_*"] & & \\ \pi_5(C_f, X_0) \arrow[r, "\partial"'] & \pi_4(X_0) \arrow[r] & \pi_4 M(A,2) \end{tikzcd}$$

where the EHP sequence shows that E'_f is surjective. Hence we get

$$\operatorname{kernel}(\mu) = \pi_4(X_0)/(f,1)_* \pi_4(X_1 \vee X_0)_2. \tag{6}$$

Now we can use the Hilton–Milnor formula for $\pi_4(X_0)$ and $\pi_4(X_1 \vee X_0)$ which shows by (6) that (5) is an isomorphism; we omit the somewhat tedious computations. One readily checks this way that Δ in (5) is surjective. For this injectivity of Δ in (5) we can use Lemma 11.1.7 and the following commutative diagram where γ_3 is the James–Hopf invariant.

$$\begin{tikzcd} \Gamma(A) \otimes \mathbb{Z}/2 \arrow[r, "\eta_3^*"] \arrow[dr, "0"'] & \pi_4 M(A,2) \arrow[d, "\gamma_3"] & L_3(A,1) \arrow[l, "w"'] \arrow[d, "i"] \\ & \pi_4 \Sigma M_A \wedge M_A \wedge M_A \arrow[r, phantom, "="] & \otimes^3 A \end{tikzcd}$$

Here i is the inclusion and w is the triple Whitehead product. The diagram implies that w is injective and that the injectivity of Δ in (5) follows from the injectivity of η_3^*. In 11.1.17 we show that $\Delta L_3(A,1)$ is always a direct summand. □

We consider the $(n-2)$-fold suspension operator Σ^{n-2} which for $n \geq 4$ is part of the following commutative diagram

(11.1.11)
$$\begin{tikzcd} \Gamma_2^2(A) \arrow[r, tail] \arrow[d, "\bar{\sigma}"] & \pi_4 M(A,2) \arrow[r, two heads] \arrow[d, "\Sigma^{n-2}"] & \Gamma T(A) \arrow[d, "\sigma"] \\ A \otimes \mathbb{Z}/2 \arrow[r, tail] & \pi_{n-2} M(A,n) \arrow[r, two heads] & A * \mathbb{Z}/2 \end{tikzcd}$$

Here σ on $\Gamma T(A)$ is defined in (11.1.4) and $\bar{\sigma}$ on $\Gamma_2^2(A)$ is defined by (11.1.3) with $\bar{\sigma}(\Gamma A \otimes A) = 0$. Both maps σ and $\bar{\sigma}$ are surjective so that also Σ^{n-2} is surjective. If A is cyclic then σ and $\bar{\sigma}$ are isomorphisms. Hence we get

(11.1.12) Proposition *For a cyclic group A one has the isomorphism $(n \geq 4)$*

$$\Sigma^{n-2}: \pi_4 M(A,2) \cong \pi_{n+2} M(A,n).$$

This also shows that the sequence in Theorem 11.1.9 *in general does not split since we know the group $\pi_{n+2} M(A,n) = G(A)$ by Theorem* 8.2.5.

We introduce two new functors

(11.1.13)
$$\pi_4', \pi_4'': \mathbf{M}^2 \to \mathbf{Ab}$$

given by the natural quotient groups

$$\pi_4' M(A,2) = \pi_4 M(A,2)/\Delta\Gamma(A) \otimes \mathbb{Z}/2, \tag{1}$$

$$\pi_4'' M(A,2) = \pi_4 M(A,2)/\Delta L_3(A,1). \tag{2}$$

Here we use Lemma 11.1.7 and Theorem 11.1.9. The exact sequence in Theorem 11.1.9 induces the natural short exact sequences

$$L_3(A,1) \overset{\Delta}{\rightarrowtail} \pi_4' M(A,2) \overset{\mu}{\twoheadrightarrow} \Gamma T(A) \tag{3}$$

$$\Gamma(A) \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} \pi_4'' M(A,2) \overset{\mu}{\twoheadrightarrow} \Gamma T(A). \tag{4}$$

We shall prove that the extension (3) is split for all abelian groups A. Using (3) and (4) we obtain the natural pull-back diagram

$$\begin{tikzcd} \pi_4 M(A,2) \arrow[r, two heads, "q"] \arrow[d, "q"'] \arrow[dr, phantom, "\text{pull}"] & \pi_4' M(A,2) \arrow[d, "\mu"] \\ \pi_4'' M(A,2) \arrow[r, "\mu"] & \Gamma T(A) \end{tikzcd} \tag{5}$$

in which q denotes the quotient map. This pull-back diagram shows that the functor π_4 on $\mathbf{M}^2$ is completely determined by the functors π_4' and π_4'' on $\mathbf{M}^2$.

The functor π_4' has the following natural interpretation. Let $\hat{\Omega}M(A,2)$ be the universal cover of the loop space $\Omega M(A,2)$. Then one has the natural Hurewicz homomorphism

$$\pi_4 M(A,2) \cong \pi_3 \Omega M(A,2) = \pi_3 \hat{\Omega} M(A,2) \overset{h}{\twoheadrightarrow} H_3 \hat{\Omega} M(A,2). \tag{6}$$

Here the third homology H_3 of $\hat{\Omega}M(A,2)$ is isomorphic to $\pi_4' M(A,2)$. In fact, there is an isomorphism

$$\pi_4' M(A,2) \cong H_3 \hat{\Omega} M(A,2) \tag{7}$$

induced by hq^{-1} where q is the quotient map in (5).

Proof of (7) The map h in (6) is embedded in Whitehead's exact sequence of $X = \hat{\Omega}M(A,2)$

$$\Gamma H_2 X \xrightarrow{j} \pi_3 X \xrightarrow{h} H_2 X \to 0.$$

Here $H_2 X = \pi_3 M(A,2) = \Gamma(A)$ and j coincides with the composite

$$j\colon \Gamma A \twoheadrightarrow \Gamma(A) \otimes \mathbb{Z}/2 \rightarrowtail \pi_4 M(A,2) = \pi_3 X.$$

Hence the kernel of h is $\Gamma(A) \otimes \mathbb{Z}/2$ and therefore the isomorphism (7) is well defined. □

We now compute the group $\pi_4' M(A,2)$ for any abelian group A. For the Moore space $M(A,2) = \Sigma M_A$ we have the James–Hopf invariant

$$\gamma_3\colon \pi_4 M(A,2) \to \pi_4 \Sigma M_A \wedge M_A \wedge M_A = \otimes^3 A$$

which satisfies $\gamma_3 \Delta \Gamma(A) \otimes \mathbb{Z}/2 = 0$. Hence γ_3 induces the following commutative diagram in **Ab** with short exact rows.

(11.1.14)
$$\begin{array}{ccccc}
L_3(A,1) & \overset{\Delta}{\rightarrowtail} & \pi_4' M(A,2) & \overset{\mu}{\twoheadrightarrow} & \Gamma T(A) \\
\| & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \bar{\gamma}_3} \\
L_3(A,1) & \overset{i}{\rightarrowtail} & \otimes^3 A & \overset{q}{\twoheadrightarrow} & \otimes^3 A / L_3(A,1)
\end{array}$$

Compare the proof of Theorem 11.1.9. This diagram is a pull-back diagram which determines the group $\pi_4' M(A,2)$ via the operator $\bar{\gamma}_3$. This operator is not natural in A. For the computation of $\bar{\gamma}_3$ we need the James–Hopf invariant γ_2.

$$\text{(11.1.15)}\qquad \begin{array}{ccc} [\Sigma P_2, \Sigma M_A] & \xrightarrow{\gamma_2} & [\Sigma P_2, \Sigma M_A \wedge M_A] \\ \downarrow \mu & & \| \\ A \supset \mathrm{Hom}(\mathbb{Z}/2, A) & & \mathrm{Ext}(\mathbb{Z}/2, \otimes^2 A) \twoheadleftarrow \otimes^2 A \end{array}$$

For $d \in A$ with $2d = 0$ let $\bar{\gamma}_2(d) \in \otimes^2 A$ be an element which represents $\gamma_2(\bar{d})$ with $\bar{d} \in \mu^{-1}(d)$. As usual we identify an element $a \in A$ with $ha = 0$ with an element $a \in \mathrm{Hom}(\mathbb{Z}/h, A)$ carrying $1 \in \mathbb{Z}/h$ to a.

(11.1.16) Theorem *Let h be a power of a prime and let $a, b, d \in A$ with $ha = hb = 0$ and $2d = 0$. Then we have $\bar{\gamma}_3(\tau_h(a, b)) = 0$ if h is odd and we get in $\otimes^3 A / L_3(A, 1)$ the formulas*

$$\bar{\gamma}_3(\tau_h(a, b)) = (h/2)[a + b, a \otimes b] \quad \text{for } h \text{ even},$$

$$\bar{\gamma}_3(\gamma(d)) = [d, \bar{\gamma}_2(d)].$$

Here we use the Lie bracket in $T(A, 1)$; see Definition 11.1.5.

Proof Let $\bar{a}, \bar{b} \in [\Sigma P_h, \Sigma M_A]$, $\bar{d} \in [\Sigma P_2, \Sigma M_A]$ be elements which realize a, b, and d respectively. Then we have for the generators $\xi_2, \xi_{h,h}$ in Section 11.5 the formulas

$$\mu\big([1,1](\bar{a} \# \bar{b})\xi_{h,h}\big) = \tau_h(a, b) \tag{1}$$

$$\mu\big(\bar{d}\xi_2\big) = \gamma(d). \tag{2}$$

Here $[1,1]\colon \Sigma M_A \wedge M_A \to \Sigma M_A$ is the Whitehead square; compare the notation in (11.5.10) below. If h is odd let $\xi_{h,h} \in \pi_4 \Sigma P_h \wedge P_h = \mathbb{Z}/h * \mathbb{Z}/h$ be the canonical generator. Now the James–Hopf invariant γ_3 satisfies the following distributivity laws:

$$\begin{aligned} &\gamma_3\big([1,1](\bar{a} \# \bar{b})\xi_{h,h}\big) \\ &\quad = (\gamma_3[1,1])(\bar{a} \# \bar{b})\xi_{h,h} \\ &\quad = (\Sigma T_{221} + \Sigma T_{121} - \Sigma T_{112} - \Sigma T_{212})\bar{a} \# \bar{b}\xi_{h,h} \\ &\quad = \bar{b} \# \bar{b} \# \bar{a}(\Sigma T_{221})\xi_{h,h} + \bar{a} \# \bar{b} \# \bar{a}(\Sigma T_{121})\xi_{h,h} \\ &\qquad - \bar{a} \# \bar{a} \# \bar{b}(\Sigma T_{212})\xi_{h,h} - \bar{b} \# \bar{a} \# \bar{b}(\Sigma T_{212})\xi_{h,h} \\ &\quad = (h(h-1)/2)(b \otimes b \otimes a + a \otimes b \otimes a - a \otimes a \otimes b - b \otimes a \otimes b). \end{aligned}$$

Here we have in $\otimes^3 A$ the equation

$$a \otimes b \otimes b = b \otimes b \otimes a + [[a,b],b].$$

Hence we have proved the first formula in Theorem 11.1.16. Moreover we get

$$\gamma_3(\bar{d}\xi_2) = (\bar{d}\#\gamma_2(\bar{d}) + \gamma_2(\bar{d})\#\bar{d})\xi_{2,2}.$$

This yields the second equation in Theorem 11.1.16 since $2d = 0$. The equations above for γ_3 are obtained by the results in Section 11.5 below and the general distributivity laws in Section A.10. □

We observe that $2\bar{\gamma}_3 = 0$. As an application we obtain the following result which we already mentioned in Theorem 11.1.9.

(11.1.17) Theorem *For any abelian group A the extension for $\pi_4' M(A,2)$ in (11.1.13)(3) is split. This implies that $L(A,1)_3$ is a direct summand of $\pi_4 M(A,2)$.*

Proof For any abelian group A we have the commutative diagram

$$\begin{tikzcd} & L_3(A,1) \arrow[dl, "i"'] \arrow[dr, "3"] & \\ \otimes^3 A \arrow[rr, "{[[1,1],1]}"'] & & L_3(A,1) \end{tikzcd}$$

where 3 denotes multiplication by 3. In fact, for $x, y, \in A$ we get

$$\begin{aligned} &[[1,1],1]i[[x,y],z] \\ &\quad = [[1,1],1](xyz + yxz - zxy - zyx) \\ &\quad = [[x,y],z] + [[y,x],z] - [[z,x],y] - [[z,y],x] \\ &\quad = 3[[x,y],z] \end{aligned}$$

Here we use $[y,x] = [x,y]$ and the Jacobi identity. The diagram shows that the extension in the bottom row of (11.1.14) yields an element

$$\{\otimes^3 A\} \in \operatorname{Ext}(\otimes^2 A/L_3(A,1), L_3(A,1))$$

of order 3, that is $3\{\otimes^2 A\} = 0$. On the other hand, the pull-back diagram (11.1.14) shows that the extension in the top row satisfies

$$\{\pi_4' M(A,2)\} = (\bar{\gamma}_3)^*\{\otimes^3 A\} \in \operatorname{Ext}(\Gamma T(A), L_3(A,1)).$$

Here $\bar{\gamma}_3$ by Theorem 11.1.16 satisfies $2\bar{\gamma}_3 = 0$. Hence we get $\{\pi_4' M(A,2)\} = 0$ and therefore the extension for $\pi_4' M(A,2)$ is split. □

(11.1.18) Definition We define for any abelian group A the *retraction*

$$r\colon \pi_4' M(A,2) \to L_3(A,1)$$

of the inclusion Δ by the formula

$$r(x) = [[1,1],1]\gamma_3(x) - 2\gamma_3(x). \tag{1}$$

Here we use the James–Hopf invariant γ_3 in (11.1.14). Since $2\bar{\gamma}_3 = 0$ we see that $2\gamma_3(x) \in L_3(A,1) \subset \otimes^3 A$. Moreover for $y \in L_3(A,1)$ we get

$$\begin{aligned} r\Delta(y) &= [[1,1],1]\gamma_3\Delta(y) - 2\gamma_3\Delta(y) \\ &= [[1,1],1]i(y) - 2y \\ &= 3y - 2y = y \end{aligned}$$

so that r is indeed a retraction for Δ. Here we need the commutative diagram in the proof of Theorem 11.1.17. Using the retraction above we obtain the isomorphism

$$(\mu, r)\colon \pi_4' M(A,2) \cong \Gamma T(A) \oplus L_3(A,1). \tag{2}$$

In case A is a direct sum of cyclic groups this isomorphism, however, is not compatible with the direct sum decomposition given by the Hilton–Milnor formula.

Next we compute the group $\pi_4'' M(A,2)$. For this we choose for each abelian groups A a map

(16.1.19) $$\bar{q}\colon M(A,2) \to M(A \otimes \mathbb{Z}/2, 2)$$

which induces the quotient map $q\colon A \to A \otimes \mathbb{Z}/2$ in homology. If A is a direct sum of cyclic groups we choose $\bar{q}$ as a suspension. The map $\bar{q}$ induces the commutative diagram

(11.1.20)

$$\begin{tikzcd} \Gamma(A) \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] \arrow[d, "q_*"', "\cong"] & \pi_4'' M(A,2) \arrow[r, two heads, "\mu"] \arrow[d, "\bar{q}_*"] \arrow[dr, phantom, "\text{pull}"] & \Gamma T(A) \arrow[d, "q_*"] \\ \Gamma(A \otimes \mathbb{Z}/2) \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] & \pi_4'' M(A \otimes \mathbb{Z}/2, 2) \arrow[r, two heads, "\mu"] & \Gamma T(A \otimes \mathbb{Z}/2) \end{tikzcd}$$

The rows of this diagram are the short exact sequences given (11.1.13)(4). Since q_* on the left-hand side is an isomorphism this diagram is a pull-back diagram. For a basis B of the $\mathbb{Z}/2$ vector space $A \otimes \mathbb{Z}/2$ we get

$$M(A \otimes \mathbb{Z}/2, 2) = \bigvee_B \Sigma P_2.$$

Since $\pi_4 \Sigma P_2 = \mathbb{Z}/4$ and $\pi_4 \Sigma P_2 \wedge P_2 = \mathbb{Z}/4$, see Section 11.5, the Hilton–Milnor formula in Remark 11.5.3 below shows that $\pi_4''\left(\bigvee_B \Sigma P_2 \right) = U$ is a free $\mathbb{Z}/4$ module generated by the disjoint union

$$B \cup \{(b,b'), b < b', b, b' \in B\}$$

where we choose an ordering of B. This basis of U is also a basis of the $\mathbb{Z}/2$-vector space $\Gamma T(A \otimes \mathbb{Z}/2)$. Therefore (11.1.20) and Theorem 11.1.17 and (11.1.13)(5) yield the next result which determines the abelian group $\pi_4 M(A,2)$ for any $A \in \mathbf{Ab}$.

(11.1.21) Theorem *For any abelian group A we obtain the abelian group $\pi_4 M(A,2)$ by a pull-back diagram:*

$$\begin{tikzcd}
\pi_4 M(A,2) \arrow[rr] \arrow[d] & \text{pull} & U \arrow[d, "q'"] \\
L_3(A,1) \oplus \Gamma T(A) \arrow[r, "p_2"'] & \Gamma T(A) \arrow[r, "q_*"'] & \Gamma T(A \otimes \mathbb{Z}/2)
\end{tikzcd}$$

Here p_2 is the projection and $q: A \to A \otimes \mathbb{Z}/2$ is the quotient map. Moreover U is a free $\mathbb{Z}/4$-module for which there is an isomorphism $\theta: U \otimes \mathbb{Z}/2 = \Gamma T(A \otimes \mathbb{Z}/2)$. This isomorphism defines $q' = \theta_q: U \twoheadrightarrow U \otimes \mathbb{Z}/2 = \Gamma T(A \otimes \mathbb{Z}/2)$.

11.2 On $\pi_3(A, M(B,2))$

We now consider the homotopy group $\pi_3(A, M(B,2))$ of a Moore space $M(B,2)$ with coefficients in A where $A, B \in \mathbf{Ab}$. We have the universal coefficient sequence which is compatible with the suspension operator:

(11.2.1)

$$\begin{tikzcd}
\mathrm{Ext}(A, \pi_4 M(B,2)) \arrow[r, tail, "\Delta"] \arrow[d, "\Sigma^2_*"] & \pi_3(A, M(B,2)) \arrow[r, two heads, "\mu"] \arrow[d, "\Sigma^2"] & \mathrm{Hom}(A, \Gamma B) \arrow[d, "\sigma_*"] \\
\mathrm{Ext}(A, \pi_6 M(B,4)) \arrow[r, tail, "\Delta"] & \pi_5(A, M(B,4)) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, B \otimes \mathbb{Z}/2)
\end{tikzcd}$$

If $B = \mathbb{Z}/k$ is a cyclic group we derive from Proposition 11.1.12 that Σ^2_* is an isomorphism. Hence in this case the diagram is a pull-back of abelian groups. In (8.2.12) we computed the bottom row of (11.2.1), hence (11.2.1) yields the group $\pi_3(A, M(\mathbb{Z}/k, 2))$ as a functor in $M(A,3) \in \mathbf{M}^3$. More generally we can compute the group $\pi_3(A, M(B,2))$ as follows.

(11.2.2) Definition For an abelian group B we have the surjective operator

$\mu: \pi_3(\Gamma B, M(B,2)) \twoheadrightarrow \mathrm{Hom}(\Gamma B, \Gamma B)$ as in the top row of (11.2.1) where we set $A = \Gamma B$. A *generalized Hopf map*

$$\eta_B: M(\Gamma B, 3) \to M(B, 2)$$

is a map which satisfies $\mu(\eta_B) = 1_{\Gamma B}$ where $1_{\Gamma B}$ is the identity of ΓB. If B is free abelian then η_B is well defined up to homotopy; in fact $\eta_{\mathbb{Z}} = \eta_2$ is the Hopf map for $B = \mathbb{Z}$.

A generalized Hopf map η_B induces the commutative diagram

(11.2.3)

$$\begin{tikzcd}
\mathrm{Ext}(A, \Gamma(B) \otimes \mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, "\Delta_*"] \arrow[dr, phantom, "\text{push}"] & \pi_3(A, M(\Gamma B, 3)) \arrow[r, "\mu"] \arrow[d, "(\eta_B)_*"] & \mathrm{Hom}(A, \Gamma B) \arrow[d, equal] \\
\mathrm{Ext}(A, \pi_4 M(B,2)) \arrow[r, tail, "\Delta"] & \pi_3(A, M(B,2)) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(A, \Gamma B)
\end{tikzcd}$$

which is a push-out of abelian groups. Here Δ_* is induced by Δ in (11.1.8). We know the group

(11.2.4) $$\pi_3(A, M(\Gamma B, 3)) = \mathbf{G}(A, \Gamma B)$$

by use of the category $\mathbf{G}$ in Definition 11.6.6. Hence the push-out diagram (11.2.3) determines the extension in the bottom row completely. For example, if B is cyclic then ΓB has no direct summand $\mathbb{Z}/2$ and hence the top row of (11.2.3) is split and therefore also the bottom row of (11.2.3) is split. We have the composite

$$\mathrm{Ext}(B, \Gamma_2^2 A) \xrightarrow{\Delta_*} \mathrm{Ext}(B, \pi_4 M(A,2)) \overset{\Delta}{\rightarrowtail} \pi_3(B, M(A,2))$$

where $\Gamma_2^2 A = \Gamma(A) \otimes \mathbb{Z}/2 \oplus L_3(A,1)$. As in (11.1.13) we introduce two new functors

(11.2.5) $$\pi_3', \pi_3'': (\mathbf{M}^3)^{\mathrm{op}} \times \mathbf{M}^2 \to \mathbf{Ab}$$

$$\pi_3'(B, M(A,2)) = \pi_3(B, M(A,2))/\Delta\Delta_* \mathrm{Ext}(B, \Gamma(A) \otimes \mathbb{Z}/2)$$

$$\pi_3''(B, M(A,2)) = \pi_3(B, M(A,2))/\Delta\Delta_* \mathrm{Ext}(B, L_3(A,1)).$$

Here $\Delta\Delta_*$ for π_3' need not be injective while $\Delta\Delta_*$ for π_3'' is always injective since $L_3(A,1)$ is a direct summand of $\pi_4 M(A,2)$. The operator $\lambda = Q$ in Addendum 11.4.5 below, see also Section 6.6, yields the following natural diagrams in $\mathbf{Ab}$ in which all rows are short exact.

$$\mathrm{Ext}(B, L_3(A,1)) \overset{\Delta}{\rightarrowtail} \pi_3'(B, M(A,2)) \overset{\lambda}{\twoheadrightarrow} \Gamma T_{\#}(B, A). \qquad (1)$$

This sequence is split for all A, B in **Ab** (unnaturally) as follows from the push-out diagram (11.2.3) and the fact that (11.1.13)(3) is split for all A.

$$\begin{tikzcd}
\mathrm{Ext}(B,\pi_4''M(A,2)) \arrow[r, tail, "\Delta"] \arrow[d, "\mathrm{Ext}(B,\mu)"] & \pi_3''(B,M(A,2)) \arrow[r, "\mu"] \arrow[d, "\lambda"] & \mathrm{Hom}(B,\Gamma A) \arrow[d, equal] \\
\mathrm{Ext}(B,\Gamma T(A)) \arrow[r, tail] & \Gamma T_{\#}(B,A) \arrow[r, "\mu"] & \mathrm{Hom}(B,\Gamma A)
\end{tikzcd} \tag{2}$$

$$\mathrm{Ext}(B,\Gamma(A)\otimes\mathbb{Z}/2)/K \overset{\Delta}{\rightarrowtail} \pi_3''(B,M(A,2)) \overset{\lambda}{\twoheadrightarrow} \Gamma T_{\#}(B,A) \tag{3}$$

Here K is the image of the boundary homomorphism

$$\partial\colon \mathrm{Hom}(B,\Gamma T(A)) \to \mathrm{Ext}(B,\Gamma(A)\otimes\mathbb{Z}/2) \tag{4}$$

associated by the extension (11.1.13)(4) for $\pi_4''M(A,2)$. The exact sequence (3) is a consequence of (2). As in (11.1.13)(5) we have the binatural pull-back diagram

$$\begin{tikzcd}
\pi_3(B,M(A,2)) \arrow[r, two heads, "q"] \arrow[d, "q"] & \pi_3''(B,M(A,2)) \arrow[d, "\lambda"] \\
\pi_3'(B,M(A,2)) \arrow[r, "\lambda"] & \Gamma T_{\#}(B,A)
\end{tikzcd} \tag{5}$$

where q denotes the quotient map. This shows that the induced maps on $\pi_3(B,M(A,2))$ are completely determined by the functors π_3' and π_3'' in (11.2.5).

Similarly to (11.1.13) we have the following natural interpretation of the functor π_3' above. Again let $\hat{\Omega}M(A,2)$ be the universal cover of the loop space of $M(A,2)$. Then we have the Hurewicz map with coefficients

$$\begin{aligned}
\pi_3(B,M(A,2)) &= \pi_2(B,\Omega M(A,2)) \\
&= \pi_2(B,\hat{\Omega}M(A,2)) \xrightarrow{h} H_2(B,\hat{\Omega}M(A,2)).
\end{aligned} \tag{6}$$

Here the right-hand side is the pseudo-homology with coefficients in B. We obtain the isomorphism

$$hq^{-1}\colon \pi_3'(B,M(A,2)) \cong H_2(B,\hat{\Omega}M(A,2)) \tag{7}$$

which is natural in $M(A,2)$ and B. A proof of (7) is deduced from the generalized Γ-sequence with coefficients in the same way as in (11.1.13)(7).

11.3 On $\Gamma_4 X$ and $\Gamma_3(B, X)$

We here study the groups $\Gamma_4 X$ and $\Gamma_3(B, X)$ of a space X but we do not yet determine the functorial properties of these groups. Recall that J.H.C. Whitehead described the group $\Gamma_3 X$ by use of the functor Γ, that is

$$\Gamma_3 X \cong \Gamma(\pi_2 X). \tag{11.3.1}$$

Hence $\Gamma_3 X$ depends only on the second homotopy group of X. We know that $\Gamma_4 X$ (as an abelian group) depends only on the quadratic function

$$\eta_X = \eta_2^* : \pi_2 X \to \pi_3 X \tag{11.3.2}$$

induced by the Hopf map $\eta_2 \in \pi_3 S^2$. In fact, η_X determines the 3-type $K(\eta_X, 2)$ of the universal covering $\hat{X}$ of X, see Proposition 7.1.3, and hence η_X determines $\Gamma_4 X = \Gamma_4 \hat{X} = \Gamma_4 K(\eta_X, 2)$ as an abelian group. We now show how to compute $\Gamma_4 X$ in terms of the function η_X.

(11.3.3) Definition Recall that $\Gamma\mathbf{Ab}$ is the category of quadratic functions; see (7.1.1). Objects are quadratic functions η: $A \to B$ between abelian groups which we can identity with homomorphisms $\eta^\square : \Gamma(A) \to B$. We now define the functor

$$\Gamma_2^2 : \Gamma\mathbf{Ab} \to \mathbf{Ab} \tag{1}$$

which generalizes the functor Γ_2^2 in Definition 11.1.6. The functor Γ_2^2 carries the object η to the abelian group $\Gamma_2^2(\eta)$ given by the push-out diagram

$$\begin{tikzcd} B \otimes \mathbb{Z}/2 \oplus B \otimes A \arrow[r, two heads, "q"] & \Gamma_2^2(\eta) \\ \Gamma(A) \otimes \mathbb{Z}/2 \oplus \Gamma(A) \otimes A \arrow[u, "\eta^\square \otimes \mathbb{Z}/2 \oplus \eta^\square \otimes A"] \arrow[r, two heads, "q"] \arrow[ur, phantom, "\text{push}"] & \Gamma_2^2(A) \arrow[u, "\eta_*"'] \end{tikzcd} \tag{2}$$

Here the bottom row is given by the quotient map in the definition of $\Gamma_2^2(A)$. A map $\varphi = (\varphi_0, \varphi_1): \eta \to \eta'$ in $\Gamma\mathbf{Ab}$ with $\eta'\varphi_0 = \varphi_1\eta$ induces $\varphi_* : \Gamma_2^2(\eta) \to \Gamma_2^2(\eta')$ by $\varphi_1 \otimes \mathbb{Z}/2 \oplus \varphi_1 \otimes \varphi_0$ where φ_0: $A \to A'$ and φ_1: $B \to B'$. We can describe $\Gamma_2^2(\eta)$ also by the quotient

$$\Gamma_2^2(\eta) = (B \otimes \mathbb{Z}/2 \oplus B \otimes A)/M(\eta) \tag{3}$$

where $M(\eta)$ is the subgroup generated by the following elements with $x, y, z \in A$:

$$\begin{cases} (\eta x) \otimes x & (4) \\ [x, y]_\eta \otimes 1 + (\eta x) \otimes y + [y, x]_\eta \otimes x. & (5) \end{cases}$$

Here we set $[x,y]_\eta = \eta(x+y) - \eta(y) \in B$ and $1 \in \mathbb{Z}/2$ is the generator. Clearly the kernel of q in the top row of (2) coincides with the subgroup $M(\eta)$; this follows by the corresponding relations (1) and (2) in (11.1.6). If $B = \Gamma A$ and if $\eta = \gamma_A$ is the universal quadratic function then $\eta^\square = 1$ is the identity of ΓA and hence $\Gamma_2^2(\gamma_A) = \Gamma_2^2(A)$ in this case by (2). Since η_X in (11.3.2) is natural in X we obtain by (1) the functor

$$\mathbf{Top}^*/\simeq \to \mathbf{Ab}, \qquad X \mapsto \Gamma_2^2(\eta_X) \tag{6}$$

which carries the homotopy category of pointed spaces to abelian groups.

(11.3.4) Theorem *Let X be a pointed space. Then one has the short exact sequence*

$$\Gamma_2^2(\eta_X) \overset{\Delta}{\rightarrowtail} \Gamma_4 X \overset{\mu}{\twoheadrightarrow} \Gamma T(\pi_2 X)$$

which is natural in X.

Here we have

$$\Gamma_2^2(\eta_X) = (\pi_3(X) \otimes \mathbb{Z}/2 \oplus \pi_3(X) \otimes \pi_2(X))/M(\eta_X)$$

and Δ carries $x \otimes 1$ to $\Delta(x \otimes 1) = x\eta_4$ and $x \otimes y$ to the Whitehead product $\Delta(x \otimes y) = [x,y]$ where $x \in \pi_3 X, y \in \pi_2 X$. We point out that for $\eta = \eta_X$ the element $[x,y]_\eta = [x,y] \in \pi_3(X)$ is the Whitehead product of $x, y \in \pi_2(X)$. Therefore the relations (4), (5) in Definition 11.3.3 which generate $M(\eta_X)$ correspond to the well-known formulas (11.1.8)(1)(2). This shows that the homomorphism Δ in (11.1.4) is well defined and natural. Injectivity of Δ in Theorem 11.3.4 shows that the group $\Gamma_2^2(\eta_X)$ is the subgroup

$$\text{(11.3.5)} \qquad \Gamma_2^2(\eta_X) = \eta_3^* \pi_3 X + [\pi_3 X, \pi_2 X] \subset \Gamma_4 X$$

where $+$ is the sum of subgroups (not the direct sum).

We derive from Theorem 11.3.4 the next result on the operator Q in Theorem 6.6.6. For this we simply set $X = K(A,2)$.

(11.3.5) Corollary *We have $H_5 K(A,2) = \Gamma_4 K(A,2) = \Gamma T(A)$ and*

$$Q\colon \pi_4 M(A,2) \twoheadrightarrow H_5 K(A,2) = \Gamma T(A)$$

is surjective and coincides with μ.

Clearly for $X = M(A,2)$ the exact sequence in Theorem 11.3.4 coincides with Theorem 11.1.9. Therefore the naturality of the sequence in Theorem 11.3.4 yields for a simply connected space X the following commutative diagram.

$$\text{(11.3.7)} \qquad \begin{array}{ccccc} \Gamma_2^2(\eta_X) & \overset{\Delta}{\rightarrowtail} & \Gamma_4 X & \overset{\mu}{\twoheadrightarrow} & \Gamma T(H_2 X) \\ \eta_* \uparrow & \text{push} & \uparrow \alpha_* & & \| \\ \Gamma_2^2(A) & \overset{\Delta}{\rightarrowtail} & \pi_4 M(A,2) & \overset{\mu}{\twoheadrightarrow} & \Gamma T(A) \end{array}$$

Here we have $A = H_2 X$ and $\alpha: M(A,2) \to X$ is a map which induces the identity $1 = H_2(\alpha)$ on A. Moreover for $\eta = \eta_X$ the induced map η_* in (11.3.7) coincides with η_* in Definition 11.3.3(2). Since diagram (11.3.7) is a push-out of abelian groups we can compute the abelian group $\Gamma_4 X$ by use of $\pi_4 M(A,2)$. We determined the extension in the bottom row of (11.3.7) for any abelian group A in Theorem 11.1.21. Therefore the push-out diagram (11.3.7) shows how to compute the abelian group $\Gamma_4 X$ only in terms of η_X.

In a similar way we can compute the group $\Gamma_3(B,X)$ for a simply connected space X since we have the commutative diagram

(11.3.8)
$$\begin{tikzcd}
\mathrm{Ext}(B,\Gamma_4 X) \arrow[r, tail] & \Gamma_3(B,X) \arrow[r, two heads] & \mathrm{Hom}(B,\Gamma_3 X) \\
\mathrm{Ext}(B,\pi_4 M(A,2)) \arrow[r, tail] \arrow[u, "\mathrm{Ext}(B,\alpha_*)"] & \pi_3(B,M(A,2)) \arrow[r, two heads] \arrow[u, "\alpha_*"] & \mathrm{Hom}(B,\Gamma A) \arrow[u, equal]
\end{tikzcd}$$

induced by α above. This is a push-out diagram of abelian groups and we know the extension in the bottom row by (11.2.3). This yields also the extension in the top row since we know how to compute α_* in (11.3.7).

In the rest of this section we prove Theorem 11.3.4.

Proof of Theorem 11.3.4 Let $H_i = H_i X$ and $\pi_i = \pi_i X$ and let α be given as in (11.3.7). We consider the homomorphism

$$\overline{M}: \pi_3 \otimes \mathbb{Z}/2 \oplus \pi_3 \otimes \pi_2 \to \Gamma_4 X$$

which satisfies $\overline{M}(x \otimes 1) = x\eta_4$ and $\overline{M}(y \otimes z) = [y,z]$ for $x, y \in \pi_3, z \in \pi_2$. Only the 4-skeleton of X is involved in the definition of $\Gamma_4 X$. Therefore we may assume that X is a CW-complex with $X^1 = *$ and $\dim X = 4$. Let $C_* = C_* X$ be the cellular chain complex with cycles $Z_i = \ker(d: C_i \to C_{i-1})$ and boundaries $B_i = dC_{i+1}$. Let $t: B_2 \to C_3$ be a splitting of d so that $C_3 = tB_2 \oplus Z_3$. Then there is a map

(1) $$g: A = M(C_4,3) \vee M(tB_2,2) \to B = M(Z_3,3) \vee M(C_2,2)$$

such that the mapping cone C_g of g is homotopy equivalent to X, see I.7.5 in Baues [CH]. The map $H_2 g$ coincides with the inclusion $tB_2 \to C_2$ given by d. Moreover the map $H_3 g$ coincides with the map $d: C_4 \to Z_3$ given by d. Hence g is determined up to homotopy by the triple $(C,t,\bar{g})$ where $\bar{g}: C_4 \to \Gamma(C_2)$ is a homomorphism given by the coordinate $M(C_4,3) \to M(C_2,2)$ of g. We consider the commutative diagram

(2)
$$\begin{tikzcd}
\pi_5(C_g,B) \arrow[r, "\partial"] & \pi_4 B \arrow[r, "i"] & \pi_4 C_g \arrow[r, "j"] & \pi_4(C_g,B) \arrow[r, "\partial"] & \pi_3 B \\
\pi_4(A \vee B)_2 \arrow[u, "E_g'"] \arrow[ur, "(g,1)_*"'] & & & \pi_3(A \vee B)_2 \arrow[u, "E_g'"] \arrow[ur, "(g,1)_*"'] &
\end{tikzcd}$$

the top row of which is the exact sequence of the pair (C_g, B). The operator $E'_g = (\pi_g, 1)_* \partial^{-1}$ is the functional suspension in (11.1.10).

(3) Lemma *Both homomorphisms E'_g in (2) are surjective*

Proof The result is clear by Corollary A.6.3 for $n = 3$ since A is 1-connected ($a = 2$). For $n = 4$ we consider the exact sequence in Theorem A.6.9 which is the row in the following commutative diagram

$$\begin{tikzcd}
 & & & \ker(g,1)_* \arrow[r, two heads] \arrow[d, tail] & j\pi_4 C_g \arrow[d, tail] \\
\pi_4(A \vee B)_2 \arrow[r, "E'_g"] & \pi_5(C_g, B) \arrow[r, "H_g"] & \pi_3(A \wedge A') \arrow[r, "P'_g"] & \pi_3(A \vee B)_2 \arrow[r, two heads, "E'_g"] \arrow[dr, "(g,1)_*"'] & \pi_4(C_g, B) \arrow[d, "\partial"] \\
 & & & & \pi_3(B)
\end{tikzcd}$$

Here we have $A = \Sigma A'$. We show below that P'_g is injective, therefore $H_g = 0$ and thus E'_g on the left-hand side is surjective.

By definition of A and of $P'_g = [i_1, i_1 - i_2 g]_*$ the following diagram commutes

$$\begin{tikzcd}
\pi_3(A \wedge A') \arrow[r, "P'_g"] \arrow[d, equal] & \pi_3(A \vee B)_2 \arrow[d, equal] \\
tB_2 \otimes tB_2 \arrow[r, "P''_g"] & C_4 \oplus \Gamma(tB_2) \oplus tB_2 \otimes C_2
\end{tikzcd}$$

with $P''_g = (0, [1,1], -1 \otimes d)$. Since d is injective also $1 \otimes d$ is injective and thus P'_g is injective. □

(4) Corollary *We have the short exact sequence*

$$V_g \rightarrowtail \pi_4 C_g \twoheadrightarrow U_g$$

where $V_g = i\pi_4 B = \pi_4 B/(g,1)_* \pi_4(A \vee B)_2$ *and*

$$U_g = j\pi_4 C_g = \ker(g,1)_* / P'_g \pi_3(A \wedge A').$$

(5) Lemma *For $\pi_i = \pi_i C_g$ we have $V_g = i\pi_4 B = (\eta_3)^* \pi_3 + [\pi_3, \pi_2]$. This is the image of $\overline{M}$ above.*

Proof of (5) Compare diagram (6) below. We know that $\pi_3 B \to \pi_3 C_g$ and $\pi_2 B \to \pi_2 C_g$ are surjective. Therefore Image $\overline{M} \subset V_g$. On the other hand, we have by the Hilton–Milnor theorem the surjection

$$\overline{M} = \overline{M}_B \colon \pi_3 B \otimes \mathbb{Z}_2 \oplus \pi_3 B \otimes \pi_2 B \twoheadrightarrow \pi_4 B.$$

This shows $V_g \subset$ Image $\overline{M} \subset \pi_4 C_g$. □

We consider the diagram

$$(6)\qquad \begin{tikzcd} \pi_3 C_g \otimes \mathbb{Z}_2 \oplus \pi_3 C_g \otimes \pi_2 C_g \arrow[r, "\overline{M}"] & \pi_4 C_g \\ \pi_3 B \otimes \mathbb{Z}_2 \oplus \pi_3 B \otimes \pi_2 B \arrow[u, "\bar{i}"] \arrow[r, two heads, "\overline{M}_B"] & \pi_4 B \arrow[u, "i_4"] \end{tikzcd}$$

Here $\bar{i} = i_3 \otimes \mathbb{Z}_2 + i_3 \otimes i_2$ is surjective where $i_n\colon \pi_n B \to \pi_n C_g$ is induced by $B \subset C_g$.

(7) Lemma $\ker \overline{M} = \bar{i} \ker \overline{M}_B$.

This is the crucial fact which we use for the proof of Theorem 11.3.4.

Proof of (7) Consider the following commutative diagram with the notation below.

$$(a)\qquad \begin{tikzcd} \pi_3 B \otimes \mathbb{Z}_2 \oplus \pi_3 B \otimes \pi_2 B \arrow[r, "\overline{M}_B"] & \pi_4 B \\ G \arrow[u, "L"] \arrow[r, dashed] \arrow[d, phantom, "\cap"] & \pi_4(A \vee B)_2 \arrow[u, "(g,1)_*"'] \arrow[d, phantom, "\cap"] \\ \pi_3(A \vee B) \otimes \mathbb{Z}_2 \oplus \pi_3(A \vee B) \otimes \pi_2(A \vee B) \arrow[r, two heads, "\overline{M}_{A \vee B}"'] \arrow[uu, bend left, "\overline{(g,1)_*}"] & \pi_4(A \vee B) \arrow[uu, bend right, "(g,1)_*"'] \end{tikzcd}$$

Here $\overline{(g,1)_*} = (g,1)_* \otimes \mathbb{Z}_2 \oplus (g,1)_* \otimes (g,1)_*$. Since

$$\ker i_4 = (g,1)_* \pi_4(A \vee B)_2$$

the lemma follows from

$$(b)\qquad \bar{i}L = 0 \quad \text{if} \quad \overline{M}_{A \vee B}(G) = \pi_4(A \vee B)_2.$$

In fact, we have by (6) the inclusion $\bar{i} \ker \overline{M}_B \subset \ker \overline{M}$. The inclusion $\ker \overline{M} \subset \bar{i} \ker \overline{M}_B$ follows from (b) by a diagram chase in (6) and (a). We obtain G in (a) and (b) by

$$G = (\pi_3(A \vee B)_2 \otimes \mathbb{Z}_2) \oplus (\pi_3(A \vee B)_2 \otimes \pi_2(A \vee B)) \oplus (\pi_3 B \otimes \pi_2 A).$$

Now $\bar{i}L = 0$ follows from the following exact sequences:

$$\pi_3(A \vee B)_2 \xrightarrow[(g,1)_*]{} \pi_3(B) \xrightarrow[i]{} \pi_3 C_g,$$

$$\pi_2(A) \xrightarrow[g_*]{} \pi_2(B) \xrightarrow[i]{} \pi_2 C_g. \qquad \square$$

Next we show:

(8) Lemma *The kernel of $\overline{M}$ is generated by the elements in Definition* 11.3.3(4), (5).

Proof of (8) For $\overline{M}_B$ we have the following description as a direct sum of homomorphisms:

$$\pi_3 B \otimes \mathbb{Z}_2 \oplus \pi_3 B \otimes \pi_2 B \xrightarrow{\overline{M}_B} \pi_4 B$$

$$\overline{M}_B = \oplus \begin{cases} Z_3 \otimes \mathbb{Z}_2 & \xrightarrow{1} Z_3 \otimes \mathbb{Z}_2 \\ Z_3 \otimes C_2 & \xrightarrow{1} Z_3 \otimes C_2 \\ \pi_3 B_0 \otimes \mathbb{Z}_2 \oplus \pi_3 B_0 \otimes \pi_2 B_0 & \xrightarrow[\overline{M}_{B_0}]{} \pi_4 B_0. \end{cases}$$

Here $B_0 = M(C_2, 2)$ and $\overline{M}_B = 1 \oplus 1 \oplus \overline{M}_{B_0}$. This shows that

(9) $$\ker \overline{M}_B = \ker \overline{M}_{B_0}.$$

Now (7) and (9) show that it is enough to prove (8) for $X = B_0$. This can be done by use of the Hilton–Milnor theorem. In fact, we choose a basis Z in C_2 with

(10) $$X = B_0 = M(C_2, 2) = \bigvee_Z S^2.$$

Moreover, we choose an ordering of the basis Z. Then $\pi_4 X$ is the sum of the following cyclic groups with generators as indicated and $\eta = \eta_2 \in \pi_3 S^2$,

(11) $$\begin{aligned} \mathbb{Z}_2 &= \pi_4(S^2) \ni a\eta(\Sigma\eta) && \text{with } a \in Z, b \in Z \\ \mathbb{Z}_2 &= \pi_3(S^3) \ni [a, b](\Sigma\eta) && \text{with } a < b, \\ \mathbb{Z} &= \pi_4(S^4) \ni [a, [b, a]] && \text{with } b \le a. \end{aligned}$$

On the other hand, $\pi_3 X = \Gamma(C_2)$ is the free abelian group generated by $\gamma(a)$ and $[a, b]$ with $a < b$. Clearly, $\pi_2 X = C_2$ is the free abelian group generated by the elements in Z. We have to compute the kernel of

$$\overline{M}: \Gamma(C_2) \otimes \mathbb{Z}_2 \oplus \Gamma(C_2) \otimes C_2 \to \pi_4 X$$

(12) $$\begin{cases} \gamma(a) \otimes 1 & \mapsto a\eta(\Sigma_\eta) \\ [a, b] \otimes 1 & \mapsto [a, b]\Sigma_\eta \ (a < b) \\ & \\ \gamma(a) \otimes b & \mapsto [a\eta, b] \\ [a, b] \otimes c & \mapsto [[a, b], c] \ (a < b). \end{cases}$$

The Barcus–Barratt formula yields

(13) $$[a\eta, b] = [a, b](\Sigma\eta) + [a, [b, a]].$$

Since $[\eta, i_2] = 0$ in $\pi_4(S^2)$ we have $[a\eta, a] = 0$. Now (11), (12), and (13) imply that ker $\overline{M}$ is actually generated by the elements in Definition 11.3.3(4), (5). This is seen by expressing the values of $\overline{M}$ in (12) in terms of the generators in (11). For this we use (13) and the Jacobi identity for Whitehead products. This completes the proof of (8). □

For the proof of Theorem 11.3.4 we show that we have the commutative diagram

(14)
$$\begin{tikzcd}
\Gamma_2^2(\eta_X) \arrow[r, tail] \arrow[d, equal] & \Gamma_4 C_g \arrow[r, two heads] \arrow[d, tail] & \Gamma T(H_2) \arrow[d, tail] \\
V_g \arrow[r, tail] & \pi_4 C_g \arrow[r, two heads] \arrow[d, "h"] & U_g \arrow[d, "h"] \\
 & \ker b_4 \arrow[r, equal] & \ker b_4
\end{tikzcd}$$

in which each row and each column is a short exact sequence; compare Theorem 11.3.4 and (4). By (4) we have

(15) $$U_g = \ker(g,1)_* / P'_g \pi_3(A \wedge A')$$

where we use the following homomorphisms

$$\begin{tikzcd}
\pi_3(A \wedge A') \arrow[r, "P'_g"] \arrow[d, equal] & \pi_3(A \vee B)_2 \arrow[r, "(g,1)_*"] \arrow[d, equal] & \pi_3(B) \arrow[d, equal] \\
tB_2 \otimes tB_2 \arrow[r] & C_4 \oplus \Gamma(tB_2) \oplus tB_2 \otimes C_2 \arrow[r] & Z_3 \oplus \Gamma C_2 .
\end{tikzcd}$$

Here we have

$$P'_g = (0, [1,1], -1 \otimes d_3)$$

$$(g,1)_* = (d_4, \bar{g}) \oplus (\Gamma d_3, [d_3, 1]).$$

This shows by definition of $\Gamma T(H_2)$ in (11.1.1) that the kernel of $h: U_g \to \ker b_4$ is $\Gamma T(H_2)$. Recall that b_4 makes the diagram

$$\begin{tikzcd}
C_4 \arrow[r, "\bar{g}"] & \Gamma(C_2) \arrow[d, "p"] \\
H_4 \arrow[u, phantom, "\cup"] \arrow[r, "b_4"'] & \Gamma(H_2)
\end{tikzcd}$$

commute where $\ker p = \mathrm{im}\,(\Gamma d_3, [d_3, 1])$ and where $\bar{g}$ is defined by g in (1). □

11.3A Appendix: Nilization of $\Gamma_4 X$

We consider the functor

$$\text{(11.3A.1)} \qquad \text{nil}: \mathbf{Gr} \to \mathbf{Gr}$$

which carries a group G into its (second) *nilization* $\text{nil}(G) = G/\Gamma_3 G$ where $\Gamma_3 G$ is the subgroup of G generated by triple commutators in G. Clearly the *abelianization* satisfies $A = G^{\text{ab}} = (\text{nil } G)^{\text{ab}}$. For a free group G we have the natural short exact sequence

$$\text{(11.3A.2)} \qquad \Lambda^2(A) \overset{w}{\rightarrowtail} \text{nil}(G) \overset{\text{ab}}{\twoheadrightarrow} A$$

where ab is the abelianization and where w is the commutator map with $w(\{x\} \wedge \{y\}) = -x - y + x + y$ for $x, y \in \text{nil}(G)$ and $\{x\} = \text{ab}(x)$. Clearly the exact sequence (11.3A.2) fits into the commutative diagram of short exact rows of groups

$$\text{(11.3A.3)} \qquad \begin{tikzcd} {[G,G]} \arrow[r, tail] \arrow[d, "\text{nil}"] & G \arrow[r, two heads, "\text{ab}"] \arrow[d, "\text{nil}"] & A \arrow[d, equal] \\ \Lambda^2(A) \arrow[r, tail] & \text{nil}(G) \arrow[r, two heads] & A \end{tikzcd}$$

where nil is the quotient map and where $[G, G]$ is the commutator subgroup of G.

Now let $G = GX$ be the simplicial loop group of Kan; see (11.4.7). Then $A = AX$ is the abelianization of GX as in Theorem 1.4.8 and in this case (11.3A.3) is a diagram of simplicial groups which induces the natural operator

$$\text{(11.3A.4)} \qquad \Gamma_n(X) = \pi_{n-1}[G, G] \xrightarrow{\text{nil}} \pi_{n-1}\Lambda^2(A) = \Gamma_n^{\text{nil}}(X)$$

which we call the *nilization* of $\Gamma_n(X)$. Here we assume that X is simply connected. For $n \le 4$ we compute the nilization $\Gamma_n^{\text{nil}}(X)$ as follows. We have the exact sequence

$$\text{(11.3A.5)} \qquad \Gamma(H_2 X) \xrightarrow{\eta^{\square}} \pi_3 X \xrightarrow{h} H_3 X \to 0$$

which is part of Whitehead's Γ-sequence. Here $\eta^{\square}$ is induced by $\eta = \eta_X$ in (11.3.2). We therefore can identify the third homology $H_3 X$ with the cokernel of $\eta_X^{\square}$. The definition of $\Gamma_2^2(\eta_X)$ in Definition 11.3.3 shows that the sequence (11.3A.5) induces the exact sequence

(11.3A.6)

$$\Gamma_2^2(H_2 X) \xrightarrow{\eta_*} \Gamma_2^2(\eta_X) \xrightarrow{h_*} H_3 X \otimes \mathbb{Z}/2 \oplus H_3 X \otimes H_2 X \to 0$$

where h_* is given by $h \otimes \mathbb{Z}/2 \oplus h \otimes H_2 X$. Using h_* we obtain the following push-out diagram which defines $\bar{\Gamma}_4^{\mathrm{nil}}(X)$

(11.3A.7)
$$\begin{tikzcd} \Gamma_2^2(\eta_X) \arrow[r, tail] \arrow[d, "h_*"] \arrow[dr, phantom, "\text{push}"] & \Gamma_4(X) \arrow[r, two heads] \arrow[d, "h"] & \Gamma T(H_2 X) \arrow[d, equal] \\ H_3 X \otimes \mathbb{Z}/2 \oplus H_3 X \otimes H_2 X \arrow[r, tail] & \bar{\Gamma}_4^{\mathrm{nil}}(X) \arrow[r, two heads] & \Gamma T(H_2 X) \end{tikzcd}$$

The push-out (11.3.7) and (11.3A.6) show that the bottom row splits (unnaturally). Moreover we have by (11.3A.6) the natural exact sequence

(11.3A.8)
$$\Gamma_2^2(H_2 X) \xrightarrow{\Delta\eta_*} \Gamma_4(X) \xrightarrow{h} \bar{\Gamma}_4^{\mathrm{nil}}(X) \to 0.$$

(11.3A.9) Theorem *Let X be simply connected. Then the nilization*

$$\mathrm{nil}_* : \Gamma_n(X) \to \Gamma_n^{\mathrm{nil}}(X)$$

is an isomorphism for $n \le 3$ and for $n = 4$ one has a natural isomorphism θ for which the diagram

$$\begin{tikzcd} & \Gamma_4^{\mathrm{nil}}(X) \arrow[dd, equal, "\theta"] \\ \Gamma_4(X) \arrow[ur, "\mathrm{nil}_*"] \arrow[dr, "h"'] & \\ & \bar{\Gamma}_4^{\mathrm{nil}}(X) \end{tikzcd}$$

commutes. In particular nil_* *on $\Gamma_4(X)$ is surjective and the kernel is determined by* (11.3*A*.8).

We think of diagram (11.3A.7) as the unstable analogue of the diagram in Theorem 5.3.7(a), so that $\bar{\Gamma}_4^{\mathrm{nil}}(X)$ is the unstable analogue of the group $H_3(X, \mathbb{Z}/2)$. Indeed the suspension σ yields the commutative diagram

(11.3A.10)
$$\begin{tikzcd} H_3 X \otimes \mathbb{Z}/2 \oplus H_3 X \otimes H_2 X \arrow[r, tail] \arrow[d, "{(1,0)}"] & \bar{\Gamma}_4^{\mathrm{nil}} X \arrow[r, two heads] \arrow[d, "\sigma"] & \Gamma T(H_2 X) \arrow[d, "\sigma"] \\ H_3 X \otimes \mathbb{Z}/2 \arrow[r, tail, "\Delta"] & H_3(X, \mathbb{Z}/2) \arrow[r, two heads, "\mu"] & H_2(X) * \mathbb{Z}/2 \end{tikzcd}$$

Here the bottom row is the universal coefficient sequence. All vertical arrows are surjective. The unstable analogue of the diagram in Theorem 5.3.8(a) is the following commutative diagram for the secondary boundary b_5 of X

(11.3A.11)
$$\begin{tikzcd} \ker Sq' \arrow[r, tail] \arrow[d, "\phi'"] & H_5X \arrow[d, "b_5"] \arrow[dr, "Sq'"] & \\ \dfrac{\Gamma_2^2(H_2X)}{\ker(\eta_*)} \arrow[r, tail, "\Delta"'] & \Gamma_4 X \arrow[r, two heads, "h"'] & \bar{\Gamma}_4^{\mathrm{nil}} X \end{tikzcd}$$

This diagram defines the natural maps Sq' and ϕ' for simply connected spaces X. The stabilization of diagram (11.3A.11) is the diagram in Theorem 5.3.8(a). Therefore Sq' is a desuspension of the Steenrod square $Sq_2^{\mathbb{Z}}$ and ϕ' is a desuspension of the secondary Adem operation ϕ_*, that is $\sigma Sq' = Sq_2^{\mathbb{Z}}$ and $\sigma\phi' = \phi_*$.

11.4 On $H_3(B, K(A,2))$ and difference homomorphisms

We determine the pseudo-homology $H_3(B, K(A,2))$ and we describe the connection of this group with $\pi_3(B, M(A,2))$. This also leads to the computation of difference homomorphisms for induced maps on $\pi_4 M(A,2)$ and $\pi_3(B, M(A,2))$. The quadratic functor Γ: **Ab** → **Ab** yields the bifunctor

(11.4.1) $$\Gamma T_{\#}: \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

which carries a pair of abelian groups (B, A) to the set of homotopy classes of chain maps

$$\Gamma T_{\#}(B, A) = [d_B, \Gamma_* d_A]. \tag{1}$$

Here $d_A: A_1 \rightarrowtail A_0$ is a short free resolution of A which is a chain complex concentrated in degree 0 and 1. Moreover $\Gamma_* d_A$ is the chain complex in (11.1.1); see also Definition 6.2.11. We have $\Gamma T_{\#}(\mathbb{Z}, A) = \Gamma(A)$. In fact, since $H_0\Gamma_* d_A = \Gamma(A)$ and $H_1\Gamma_* d_A = \Gamma T(A)$ we get the binatural short exact sequence

$$\mathrm{Ext}(B, \Gamma T(A)) \overset{\Delta}{\rightarrowtail} \Gamma T_{\#}(B, A) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(B, \Gamma(A)) \tag{2}$$

which is split (unnaturally). Here μ carries a chain map in (1) to its induced map in homology. We now consider the pseudo-homology $H_3(B, K(A,2))$ of an Eilenberg–Mac Lane space $K(A,2)$ given by the group of homotopy classes of chain maps,

$$H_3(B, K(A,2)) = [C_* M(B,3), C_* K(A,2)],$$

where C_* is the singular chain complex.

(11.4.2) Theorem *There is a binatural isomorphism*

$$H_3(B, K(A,2)) \cong \Gamma T_\#(B, A)$$

for which the following diagram of short exact sequences commutes.

$$\begin{tikzcd}
\mathrm{Ext}(B, H_5K(A,2)) \arrow[r, tail, "\Delta"] \arrow[d, equal] & H_3(B, K(A,2)) \arrow[r, two heads, "\mu"] \arrow[d, equal] & \mathrm{Hom}(B, H_4K(A,2)) \arrow[d, equal] \\
\mathrm{Ext}(B, \Gamma T(A)) \arrow[r, tail, "\Delta"] & \Gamma T_\#(B, A) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(B, \Gamma(A))
\end{tikzcd}$$

Here we use the isomorphisms $\Gamma(A) = H_4K(A,2)$ and $H_5K(A,2) = \Gamma T(A)$. In the proof of Theorem 11.4.2 we use the surjection Q in Corollary 6.6.8 and we obtain a new interpretation of Q by use of twisted maps between mapping cones. To this end we describe the following concept of twisted maps which generalize the principal maps in Definition 6.12.11.

(11.4.3) Definition Let $f: X \to Y$ and $g: U \to V$ be maps in **Top***. A *twisted map*

$$F = C(u, v, H, G): C_f \to C_g \tag{1}$$

between the mapping cones of f and g is obtained as follows. Consider a diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{H}{\Rightarrow}"] & U \vee V \arrow[r, "{(0,1)}"] \arrow[d, "{(g,1)}"] & V \\
Y \arrow[r, "v"'] & V &
\end{tikzcd} \tag{2}$$

together with homotopies $H: vf \simeq (g,1)u$ and $(0,1)u \simeq 0$. Then there is a pair map

$$G: (CX, X) \to (CU \vee V, U \vee V)$$

extending u. Here CU is the cone on U. Using the inclusion $i_g: V \subset C_g$ and the projection $(\pi_g, 1): CU \vee V \to C_g$, see (11.1.10), for the mapping cone C_g we get F in (1) by the formulas:

$$\begin{aligned}
F(y) &= i_g v(y) && \text{for } y \in Y \\
F(t,x) &= i_g H(2t, x) && \text{for } 0 \le t \le 1/2, x \in X \\
F(t,x) &= (\pi_g, 1)G(2t-1, x) && \text{for } 1/2 \le t \le 1, x \in X.
\end{aligned}$$

Let

$$\mathrm{TWIST}(f, g) \subset [C_f, C_g] \tag{3}$$

be the subset of all homotopy classes represented by twisted maps. Moreover let

$$\mathrm{Twist}(f,g) \subset [X, U \vee V]_2 \times [Y, V] \tag{4}$$

be the set of all pairs of homotopy classes $\{u\},\{v\}$ as in (2) with $(g,1)_*\{u\} = f^*\{v\}$ and $(0,1)_*\{u\} = 0$. We say that F in (1) is *associated* to the pair $(\{u\},\{v\}) \in \mathrm{Twist}(f,g)$. The properties of twisted maps are studied in Baues [AH], Chapter V.

The Moore spaces $M(A,2)$, $M(B,3)$ are mapping cones of $d_A^2\colon M(A_1,2) \to M(A_0,2)$ and $d_B^3\colon M(B_1,3) \to M(B_0,3)$ respectively. Now it is easy to see that

$$\text{(11.4.4)} \qquad \mathrm{Twist}(d_B^3, d_A^2) = \mathbf{Chain}(d_B, \Gamma_* d_A)$$

where the right-hand side is the set of chain maps $d_B \to \Gamma_* d_A$. In addition to Theorem 11.4.2 we show the

(11.4.5) Addendum *There is a commutative diagram*

$$\begin{array}{ccc} [M(B,3), M(A,2)] & \overset{Q}{\twoheadrightarrow} & H_4(B, K(A,2)) \\ \Vert & & \Vert \\ \mathrm{TWIST}(d_B^3, d_A^2) & \overset{\lambda}{\twoheadrightarrow} & [d_A, \Gamma_* d_B] \quad = \Gamma T_\#(A,B) \end{array}$$

where λ carries a twisted map F associated to (x,y) to the homotopy class of the chain map (x,y) given by (11.4.4).

Proof of Addendum 11.4.5 and Theorem 11.4.2 The arguments in (V.§7) of Baues [AH] show that all maps $M(B,3) \to M(A,2)$ are homotopic to twisted maps. Moreover (V.7.17) in Baues [AH] shows that λ is well defined. The push-out diagram in Corollary 6.6.8 yields the kernel of Q. Since λ fits into a push-out diagram of the same kind we see that $\mathrm{kernel}(\lambda) = \mathrm{kernel}(Q)$. We now define the isomorphism in Theorem 11.4.2 by λQ^{-1}. □

We need the following purely algebraic notation.

(11.4.6) Definition Let A, π, R be abelian groups and let $\xi \in \mathrm{Ext}(A,\pi)$ be represented by $\xi_1 \in \mathrm{Hom}(A_1,\pi)$ where $d_A\colon A_1 \to A_0$ is a short free resolution of A. Then we obtain the composite

$$\xi_\#\colon A * R \subset A_1 \otimes R \xrightarrow{\xi_1 \otimes R} \pi \otimes R \tag{1}$$

which depends only on ξ; see also Definition 8.3.11. Moreover we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{Ext}(B, A * R) \arrow[d, tail, "\Delta"] \arrow[dr, "\mathrm{Ext}(B,\xi_\#)"] & \\ {[d_B, d_A \otimes R]} \arrow[r, "\xi_\#"'] & \mathrm{Ext}(B, \pi \otimes R) \end{tikzcd} \tag{2}$$

Here $\xi_\#$ in the bottom row is defined as follows. For a chain map $F: d_B \to d_A \otimes R$ the element $\xi_\#\{F\}$ is represented by the composite

$$B_1 \xrightarrow{F_1} A_1 \otimes R \xrightarrow{\xi_1 \otimes R} \pi \otimes R.$$

The inclusion Δ in (2) is the usual one which carries $\{b\}$ represented by $b \in \mathrm{Hom}(B_1, A * R)$ to the chain map which is 0 in degree 0 and which is $B_1 \xrightarrow{b} A * R \subset A_1 \otimes R$ in degree 1. Hence diagram 2 commutes. If A and B are finitely generated we have a natural retraction

$$r: [d_B, d_A \otimes R] \to \mathrm{Ext}(B, A * R) \tag{3}$$

of Δ in (4); see Lemma 6.12.13. In this case $\xi_\#$ in the bottom row of (2) is simply $\mathrm{Ext}(B, \xi_\#)r$. Using Addendum (11.4.5) we are able to prove the unstable analogue of the deviation formula in Proposition 8.3.12. For this consider the elements

$$\alpha, \alpha + \Delta\xi \in [M(A,2), X]$$

with $\xi \in \mathrm{Ext}(A, \pi_3 X)$. Here $\alpha + \Delta\xi$ is defined by $\Delta: \mathrm{Ext}(A, \pi_3 X) \rightarrowtail \pi_2(A, X)$. We obtain the *difference homomorphisms* $(\alpha + \Delta\xi)_* - \alpha_*$

$$\Gamma_4(\alpha + \Delta\xi) - \Gamma_4\alpha: \pi_4 M(A,2) \to \Gamma_4 X$$

$$\Gamma_3(B, \alpha + \Delta\xi) - \Gamma_3(B, \alpha): \pi_3(B, M(A,2)) \to \Gamma_3(B, X).$$

Here $\Gamma_4\alpha = \alpha_*$ is the same as in (11.3.7). Let $\pi_i = \pi_i X$ and let $a = \pi_2(\alpha) \in \mathrm{Hom}(A, \pi_2 X)$ be induced by α.

(11.4.7) Theorem *The difference homomorphisms above yield the following commutative diagrams* (11.4.8) *and* (11.4.9).

$$\begin{tikzcd} \pi_4 M(A,2) \arrow[r, "\mu"] \arrow[dd, "(\alpha+\Delta\xi)_* - \alpha_*"] & \Gamma T(A) \arrow[r, "{(\sigma, H)}"] & A * (\mathbb{Z}/2 \oplus A) \arrow[d, "\xi_*"] \\ & & \pi_3 \otimes (\mathbb{Z}/2 \oplus A) \arrow[d, "a_*"] \\ \Gamma_4(X) & \Gamma_2^2(\eta_X) \arrow[l, tail, "\Delta"'] & \pi_3 \otimes (\mathbb{Z}/2 \oplus \pi_2) \arrow[l, "q"'] \end{tikzcd} \tag{11.4.8}$$

Here we use (σ, H) *in* (11.1.4) *and the quotient map* q *in Definition* 11.3.3(3); *moreover we set* $a_* = \pi_3 \otimes (\mathbb{Z}/2 \oplus a)$.

(11.4.9)

$$\begin{array}{ccccc} \pi_3(B, M(A,2)) & \xrightarrow{\lambda} & \Gamma T_\#(B,A) & \xrightarrow{(\sigma,h)_*} & [d_B, d_A \otimes (\mathbb{Z}/2 \oplus A)] \\ \Big\downarrow{\scriptstyle (\alpha+\Delta\xi)_* - \alpha_*} & & & & \Big\downarrow{\scriptstyle \xi_\#} \\ & & & & \mathrm{Ext}(B, \pi_3 \otimes (\mathbb{Z}/2 \oplus A)) \\ & & & & \Big\downarrow{\scriptstyle \mathrm{Ext}(B, \Delta q a_*)} \\ \Gamma_3(B,X) & & \xleftarrow{\quad\Delta\quad} & & \mathrm{Ext}(B, \Gamma_4 X) \end{array}$$

Here $(\sigma,h)_*$ *is induced by the chain map* $(\sigma,h)\colon \Gamma_* d_A \to d_A \otimes (\mathbb{Z}/2 \oplus A)$, *see* (11.1.4). *The surjective homomorphism* λ *is defined in Addendum* 11.4.5. *If* A *and* B *are finitely generated we can use the retraction* r *in Definition* 11.4.6(3) *for* $\xi_\#$.

Proof of Theorem 11.4.7 For the mapping cone $C_d = M(A,2), d = d_A^2$, we know that each element $x \in \pi_4(C_d)$ is functional in the sense that $j(x) = (\pi_d, 1)_* \partial^{-1}(\zeta) = E'_d(\zeta)$, see (11.1.10). Hence II.12.3 in Baues [AH] shows

$$-\alpha_* x + (\alpha + \Delta\xi)_* x = (E\zeta)^*(\xi_1, \alpha_0) \tag{1}$$

where $\alpha_0 = \alpha \mid M(A_0, 2)$ and where $\xi_1 \colon M(A_1, 3) \to X$ represents ξ. Moreover E is the partial suspension for which the following diagram commutes

$$\begin{array}{ccc} \zeta \in \pi_3(X_1 \vee X_0)_2 & \xrightarrow{E} & \pi_4(\Sigma X_1 \vee X_0)_2 \\ \| & & \| \\ \Gamma A_1 \oplus A_1 \otimes A_0 & \xrightarrow{\sigma \oplus 1} & A_1 \otimes \mathbb{Z}/2 \oplus A_1 \otimes A_0 \end{array} \tag{2}$$

Hence (1) shows that the first diagram of Theorem 11.4.7 commutes.

Next we know by Addendum 11.4.5 that $F \in \pi_3(B, M(A,2))$ is a twisted map associated to $(u, v) \in \mathrm{Twist}(d_B^3, d_A^2) = \mathbf{Chain}(d_B, \Gamma_* d_A)$. Hence we have by II.12.7 and V.3.12 (3) in Baues [AH] the formula

$$-\alpha_* x + (\alpha + \Delta\xi)_* x = (Eu)^*(\xi_1, \alpha_0). \tag{3}$$

Now one can check as in (2) that Eu is given by $(\sigma, \tau)\lambda(F)$ with $\lambda(F) = \{(u, v)\}$. This yields the commutativity of the second diagram in Theorem 11.4.7. □

11.5 Elementary homotopy groups in dimension 4

Let **PCyc⁰** be the full subcategory of **Ab** consisting of elementary cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/p^t$ where p^t is a prime power. We also write $\mathbb{Z}=\mathbb{Z}/0$. The *elementary Moore spaces* are $(d \geq 2)$

$$(11.5.1) \qquad M(\mathbb{Z}/n,d)=\Sigma^{d-1}P_n \quad \text{with} \quad \mathbb{Z}/n \in \mathbf{PCyc}^0.$$

Here $P_n=S^1 \cup_n e^2$ is the pseudo-projective plane for n not equal to 0 and $P_0=S^1$. The quotient space $X \wedge Y=X \times Y/X \vee Y$ is the *smash product* for pointed CW-complexes X,Y. The *elementary homotopy groups* which we consider in this section are for $\mathbb{Z}/k,\mathbb{Z}/n,\mathbb{Z}/m \in \mathbf{PCyc}^0$ the groups

(11.5.2)

$$\pi_4\Sigma P_n, \quad \pi_4\Sigma P_n \wedge P_m, \quad \pi_3(\mathbb{Z}/k,\Sigma P_n), \quad \pi_3(\mathbb{Z}/k,\Sigma P_n \wedge P_m).$$

The elementary homotopy groups arise in the following application of the Hilton–Milnor theorem.

(11.5.3) Remark Let A and B be direct sums of cyclic groups

$$A=\bigoplus_i (\mathbb{Z}/a_i)\alpha_i \quad \text{and} \quad B=\bigoplus_j (\mathbb{Z}/b_j)\beta_j$$

with $\mathbb{Z}/a_i,\mathbb{Z}/b_j \in \mathbf{PCyc}^0$. Then we obtain the Moore spaces

$$M(A,2)=\bigvee_i \Sigma P_{a_i}, \qquad M(B,3)=\bigvee_s \Sigma^2 P_{b_s}$$

as one-point unions of elementary Moore spaces. The inclusions $\alpha_i\colon \Sigma P_{a_i} \subset M(A,2)$ yield the Whitehead product $[\alpha_i,\alpha_j]\colon \Sigma P_{a_i} \wedge P_{a_j} \to M(A,2)$. Then the Hilton-Milnor theorem shows

$$\pi_4 M(A,2)=\Big(\bigoplus_i \pi_4\Sigma P_{a_i}\Big) \oplus \Big(\bigoplus_{i<j} \pi_4\Sigma P_{a_i} \wedge P_{a_j}\Big) \oplus L_3(A,1).$$

The isomorphism is given by $\pi_4(\alpha_i)$ on $\pi_4\Sigma P_{a_i}$, by $\pi_4[\alpha_i,\alpha_j]$ on $\pi_4\Sigma P_{a_i} \wedge P_{a_j}$, and by Δ in Theorem 11.1.9 on $L_3(A,1)$. Similarly we get

$$\pi_3(B,M(A,2))=\Big(\bigoplus_i \pi_3(B,\Sigma P_{a_i})\Big) \oplus \Big(\bigoplus_{i<j} \pi_3\big(B,\Sigma P_{a_i} \wedge P_{a_j}\big)\Big)$$
$$\oplus \operatorname{Ext}(B,L_3(A,1)).$$

This shows that the groups $\pi_4 M(A,2)$ and $\pi_3(B,M(A,2))$ are completely determined by the elementary groups in (11.5.2).

(11.5.4) Elementary exact sequences Let $\mathbb{Z}/n, \mathbb{Z}/m \in \mathbf{PCyc}^0$. Then we have short exact sequences:

$$\Gamma(\mathbb{Z}/n) \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} \pi_4 \Sigma P_n \overset{\mu}{\twoheadrightarrow} \Gamma T(\mathbb{Z}/n) = \mathbb{Z}/n * \mathbb{Z}/2 \tag{1}$$

$$\mathbb{Z}/n \otimes \mathbb{Z}/m \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} \pi_4 \Sigma P_n \wedge P_m \overset{\mu}{\twoheadrightarrow} \mathbb{Z}/n * \mathbb{Z}/m \tag{2}$$

$$\mathrm{Ext}(B, \pi_4 \Sigma P_n) \overset{\Delta}{\rightarrowtail} \pi_3(B, \Sigma P_n) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(B, \Gamma(\mathbb{Z}/n)) \tag{3}$$

$$\mathrm{Ext}(B, \pi_4 \Sigma P_n \wedge P_m) \overset{\Delta}{\rightarrowtail} \pi_3(B, \Sigma P_n \wedge P_m) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(B, \mathbb{Z}/n \otimes \mathbb{Z}/m). \tag{4}$$

Here (1) is a special case of Theorem 11.1.9 and (2) is part of Whitehead's exact sequence $\Gamma_4 X \to \pi_4 X \to H_4 X$ for $X = \Sigma P_n \wedge P_m$. Moreover (3) and (4) are for $B \in \mathbf{Ab}$ the universal coefficient sequences. We solve the extension problems for the elementary exact sequences as follows.

(11.5.5) Theorem *Let $\mathbb{Z}/n, \mathbb{Z}/m \in \mathbf{PCyc}^0$. Then* (1) *is non-split if and only if $n = 2$. Moreover* (2) *is non-split if and only if $n = m = 2$. Next* (3) *is split for all $B \in \mathbf{Ab}$. For $B = \mathbb{Z}/k \in \mathbf{PCyc}^0$ the sequence* (4) *is non-split if and only if $k = 2$ and*

$$(n, m) \in \{(2,0), (0,2), (2^t, 2), (2, 2^t), t > 1\}.$$

(11.5.6) Remark Let A and B be direct sums of cyclic groups as in Remark 11.5.3. Then the extension in Theorem 11.1.9,

$$\Gamma_2^2(A) \overset{\Delta}{\rightarrowtail} \pi_4 M(A,2) \overset{\mu}{\twoheadrightarrow} \Gamma T(A),$$

coincides via Remark 11.5.3 with the direct sum of elementary exact sequences. Hence Theorem 11.5.5 also solves the extension problem for $\pi_4 M(A,2)$. Similarly Theorem 11.5.5 solves the extension problem for

$$\mathrm{Ext}(B, \pi_4 M(A,2)) \overset{\Delta}{\rightarrowtail} \pi_3(B, M(A,2)) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(B, \Gamma A)$$

since this sequence via Remark 11.5.3 is a direct sum of elementary exact sequences.

Proof of Theorem 11.5.5 We obtain the result on (1) by Proposition 11.1.12 and Theorem 8.2.5. Moreover for (2) we observe that $\pi_4 X$ with $X = \Sigma P_n \wedge P_m$ is in the stable range so that we can apply (5.3.5) where we use the Cartan formula for the computation of $Sq^2(i)$. Next the splitting of (3) is proved in (11.2.4). Finally using Remark 11.5.3 we solve the extension problem for (5.4)(4) by the push-out diagram (11.2.3). Here we also can use Theorem 1.6.11. □

Theorem 11.5.5 determines the elementary homotopy groups (11.5.2) as abelian groups completely.

(11.5.7) Remark The space $P_n \wedge P_m$ is also the mapping cone of $n \wedge 1: S^1 \wedge P_m \to S^1 \wedge P_m$ where $n: S^1 \to S^1$ is a map of degree n and where 1 is the identity of P_m with $n, m > 0$. The identity I_m of ΣP_m with m a prime power satisfies

$$n \wedge 1 = n \cdot I_m = 0 \quad \text{in} \quad [\Sigma P_m, \Sigma P_m]$$

if $m = 2$ and $(4, n) = 4$ or if m not equal to 2 and $(m, n) = m$; see Corollary 1.4.10. In these cases we get the homotopy equivalence

$$P_n \wedge P_m \simeq \Sigma P_m \vee \Sigma^2 P_m .$$

For $n = 2 = m$ there is no such decomposition since then $P_2 \wedge P_2$ is the mapping cone of

$$(2i_2, i_2\eta_2 + 2i_3): S^2 \vee S^3 \to S^2 \vee S^3.$$

Here i_t $(t = 2,3)$ is the inclusion $S^t \subset S^2 \vee S^3$ and $\eta_2 \in \pi_3 S^2$ is the Hopf map. Compare IV.A.13 in Baues [CH].

For the elementary Moore spaces we use the inclusion and the pinch map

$$(11.5.8) \qquad S^d \xrightarrow{i = i_n} \Sigma^{d-1} P_n \xrightarrow{q = q_n} S^{d+1}$$

and we use the inclusion

$$i = i_{nm} = \Sigma i_n \wedge i_m : S^3 \subset \Sigma P_n \wedge P_m .$$

Let $\eta_r \in \pi_{r+1} S^r$ be the *Hopf map* with $\Sigma \eta_r = \eta_{r+1}$ for $n \geq 2$. We write $C = (\mathbb{Z}/t)x$ if C is a cyclic group isomorphic to $\mathbb{Z}/t$ with generator $x, t \geq 0$. We know that

$$\pi_3 S^2 = \mathbb{Z}\eta_2 , \pi_{r+1} S^r = (\mathbb{Z}/2)\eta_r \quad \text{for } r \geq 3,$$

$$\pi_4 S^2 = (\mathbb{Z}/2)\eta_2^2 \quad \text{with } \eta_2^2 = \eta_3\eta_2 ,$$

$$\pi_3 \Sigma P_n = \mathbb{Z}/(n^2, 2n) i\eta_2 \quad \text{and} \quad \pi_4 \Sigma^2 P_n = \mathbb{Z}/(n,2) i\eta_3 ,$$

$$[\Sigma^2 P_k, S^2] = \mathbb{Z}/(k,2)\eta_2^2 q \quad \text{and} \quad [\Sigma^2 P_k, S^3] = \mathbb{Z}/(k,2)\eta_3 q,$$

$$[\Sigma^2 P_k, \Sigma^2 P_n] = \mathbf{G}(\mathbb{Z}/k, \mathbb{Z}/n);$$

see Theorem 1.6.7. Here we assume that n and k are powers of primes and (a,b) denotes the greatest common divisor of $a, b \in \mathbb{Z}$. These groups together with the groups in the next theorem yield a complete list of all elementary homotopy groups in (11.5.2). For this we point out that the essential features of the elementary homotopy groups in (11.5.2) only arise when k, n, m are 0 or powers of 2. This follows from the short exact sequences (11.5.4) above.

(11.5.9) Theorem *Let n, m, k be powers of 2. Then one has generators $\varepsilon, \gamma, \xi, \eta$ for which the following equations hold. For the definition of these generators see* (11.5.16) *below.*

$$\pi_4 \Sigma P_n = \begin{cases} (\mathbb{Z}/4)\xi \quad (2\xi = \varepsilon) & \text{for } n = 2 \\ (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/2)\varepsilon & \text{for } n \geq 4. \end{cases}$$

Here we write $\xi = \xi_n$ and $\varepsilon = \varepsilon_n = i\eta_2^2$ is given by the double Hopf map.

$$\pi_4(\Sigma P_n \wedge P_m) = \begin{cases} (\mathbb{Z}/4)\xi \quad (2\xi = \varepsilon) & \text{for } n = m = 2 \\ (\mathbb{Z}/r)\xi \oplus (\mathbb{Z}/2)\varepsilon & \text{for } n \geq 4 \text{ or } m \geq 4. \end{cases}$$

Here $\mathbb{Z}/r = \mathbb{Z}/n * \mathbb{Z}/m$ is the cyclic group of order $r = (n, m)$. We write $\xi = \xi_{n,m}$ and $\varepsilon = \varepsilon_{n,m} = i\eta_3$ is obtained by the Hopf element.

$$[\Sigma^2 P_k, \Sigma P_n] = \begin{cases} (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/2)\gamma \quad (\varepsilon = 0) & \text{for } k = n = 2 \\ (\mathbb{Z}/4)\xi \oplus (\mathbb{Z}/4)\gamma \quad (\varepsilon = 2\xi) & \text{for } k \geq 4,\ n = 2 \\ (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/g)\gamma \oplus (\mathbb{Z}/2)\varepsilon & \text{for } k \geq 2,\ n \geq 4. \end{cases}$$

Here $\mathbb{Z}/g = \mathrm{Hom}(\mathbb{Z}/k, \Gamma(\mathbb{Z}/n))$ is the cyclic group of order $g = (2n, k)$ and we write $\gamma = \gamma_n^k$. The elements ξ and ε are given by $\xi = \xi_n^k = \xi_n q$ and $\varepsilon = \varepsilon_n^k = i\eta_2^2 q$.

$$[\Sigma^2 P_k, \Sigma P_n \wedge P_m]$$

$$= \begin{cases} (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/2)\eta \quad (\varepsilon = 0) & \text{for } (k, n, m) = (2,2,2) \\ (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/4)\eta \quad (\varepsilon = 2\eta) & \text{for } (k, n, m) = (2, \geq 4, 2), (2, 2, \geq 4) \\ (\mathbb{Z}/4)\xi \oplus (\mathbb{Z}/2)\eta \quad (\varepsilon = 2\xi) & \text{for } (k, n, m) = (\geq 4, 2, 2) \\ (\mathbb{Z}/e)\xi \oplus (\mathbb{Z}/d)\eta \oplus (\mathbb{Z}/2)\varepsilon & \text{otherwise}. \end{cases}$$

Here $\mathbb{Z}/e = \mathrm{Ext}(\mathbb{Z}/k, \mathbb{Z}/n * \mathbb{Z}/m)$ and $\mathbb{Z}/d = \mathrm{Hom}(\mathbb{Z}/k, \mathbb{Z}/n \otimes \mathbb{Z}/m)$ and we write $\eta = \eta_{n,m}^k$. The elements ξ and ε are given by $\xi = \xi_{n,m}^k = \xi_{n,m} q$ and $\varepsilon = \varepsilon_{n,m}^k = i\eta_3 q$.

We need the following notation and facts; see Appendix A. Let A and B be finite dimensional connected CW-complexes with base point and let

$$T_{21}: A \wedge B \to B \wedge A \tag{11.5.10}$$

be the *interchange map*. Moreover for maps $f: \Sigma A \to \Sigma A', g: \Sigma B \to \Sigma B'$ between suspensions let $f \# g$ be the composite

$$\Sigma A \wedge B \xrightarrow{f \wedge B} \Sigma A' \wedge B \xrightarrow{\Sigma T_{21}} \Sigma B \wedge A' \xrightarrow{g \wedge A'} \Sigma B' \wedge A' \xrightarrow{\Sigma T_{21}} \Sigma A' \wedge B'. \quad (1)$$

Hence $f\#g=\Sigma f'\wedge g'$ if $f=\Sigma f'$ and $g=\Sigma g'$. Moreover for $\alpha\in[\Sigma A,X]$ and $\beta\in[\Sigma B,X]$ let

$$[\alpha,\beta]\in[\Sigma A\wedge B,X] \tag{2}$$

be the *Whitehead product*. We have the interchange rule

$$[\alpha,\beta]=-(\Sigma T_{21})^*[\beta,\alpha] \tag{3}$$

and for $a\in[A,A']$, $b\in[B,B']$, $v\in[X,X']$ we have naturality:

$$[(\Sigma a)^*\alpha,(\Sigma b)^*\beta]=(\Sigma a\wedge b)^*[ga,\beta]$$

$$v_*[\alpha,\beta]=[v_*\alpha,v_*\beta]. \tag{4}$$

Moreover let $\Delta=\Delta_A\colon A\to A\wedge A$ be the reduced diagonal with $\Delta(x)=x\wedge x$ for $x\in A$. Then one has

$$[\alpha,\beta_1+\beta_2]=[\alpha,\beta_2]+[\alpha,\beta_1]+[[\alpha,\beta_1],\beta_2]\Sigma(A\wedge\Delta_B). \tag{5}$$

$$[\alpha_1+\alpha_2,\beta]=[\alpha_1,\beta]-[\alpha_2,[\alpha_1,\beta]]\Sigma(\Delta_A\wedge B)+[\alpha_2,\beta]. \tag{6}$$

We also write $A\wedge\Delta_B=T_{122}$ and $\Delta_A\wedge B=T_{112}$.

Let $A^{\wedge n}=A\wedge A\wedge\cdots\wedge A$ be the n-fold smash product of A. For $\varphi\in[\Sigma B,\Sigma A]$ let

$$\gamma_n(\varphi)\in[\Sigma B,\Sigma A^{\wedge n}] \tag{7}$$

be the nth *James–Hopf invariant* (defined with respect to the lexicographical ordering from the left). The James–Hopf invariant is natural in A and B. As an example one has $\gamma_2(\eta_2)=1\in\pi_3S^3$ for the Hopf map η_2 and $2\eta_2=[1,1]$. We need the following formulas for the *Whitehead square*

$$[1,1]\in[\Sigma A\wedge A,\Sigma A] \tag{8}$$

which is the Whitehead product given by the identity 1 of ΣA. If A is $(r-1)$-connected and $\dim(A)<4r-3$ we have

$$\gamma_2[1,1]=-\Sigma T_{21}+\Sigma T_{12}\in[\Sigma A^{\wedge 2},\Sigma A^{\wedge 2}] \tag{9}$$

$$\gamma_3[1,1]=\Sigma T_{221}+\Sigma T_{121}-\Sigma T_{112}-\Sigma T_{212}\in[\Sigma A^{\wedge 2},\Sigma A^{\wedge 3}]. \tag{10}$$

Here T_{12} is the identity of $A^{\wedge 2}$ and

$$T_{221}(a_1\wedge a_2)=a_2\wedge a_2\wedge a_1,T_{121}(a_1\wedge a_2)=a_1\wedge a_2\wedge a_1,$$

and $T_{212}(a_1\wedge a_2)=a_2\wedge a_1\wedge a_2$ for $a_1,a_2\in A$. See Section A.10.

(11.5.11) Theorem *The James–Hopf invariants*

$$\gamma_3\colon \pi_4(\Sigma P_n) \to \pi_4(\Sigma P_n \wedge P_n \wedge P_n) = \mathbb{Z}/n$$

$$\gamma_3\colon \left[\Sigma^2 P_k, \Sigma P_n\right] \to \left[\Sigma^2 P_k, \Sigma P_n \wedge P_n \wedge P_n\right] = \mathrm{Ext}(\mathbb{Z}/k, \mathbb{Z}/n)$$

are trivial for all $n, k \geq 0$.

Proof We may assume that n and k are powers of 2. Let $\chi_n^2 \in [\Sigma P_2, \Sigma P_n]$ be defined as in (11.5.16). Then we have the following commutative diagram with short exact columns

$$\begin{tikzcd}
 & \pi_4 S^2 \arrow[r, "\gamma_3"] \arrow[d] & \pi_4 S^4 = \mathbb{Z} \arrow[d] \\
\pi_4 \Sigma P_2 \arrow[r, "(\chi_n^2)_*"] \arrow[d] & \pi_4 \Sigma P_n \arrow[r, "\gamma_3"] \arrow[d] & \pi_4 \Sigma P_n \wedge P_n \wedge P_n \\
\Gamma T(\mathbb{Z}/2) \arrow[r, phantom, "="] & \Gamma T(\mathbb{Z}/n) \arrow[ur] &
\end{tikzcd}$$

Since $\gamma_3(\chi_n^2)_* = \chi_n^2 \# \chi_n^2 \# \chi_n^2 \gamma_3$ it is hence enough to prove $\gamma_3 \pi_4 \Sigma P_2 = 0$. Now γ_3 is induced by the James map

$$\Omega \Sigma P_2 \simeq JP_2 \xrightarrow{\gamma} JP_2 \wedge P_2 \wedge P_2 \simeq \Omega \Sigma P_2 \wedge P_2 \wedge P_2$$

and we have the commutative diagram

$$\begin{tikzcd}
\pi_3 JP_2 \arrow[r, "h"] \arrow[d, "\gamma_*"'] & H_3 CJ\rho(P_2) \arrow[r] & H_3 JP_2 \arrow[d, "\gamma_3"] \\
\pi_3 JP_2 \wedge P_2 \wedge P_2 & = & H_3 JP_2 \wedge P_2 \wedge P_2
\end{tikzcd}$$

Here we use as in Baues [CH] (III.6.6) the Hurewicz map for the universal covering $\hat{J}P_2$ of JP_2:

$$h\colon \pi_3 JP_2 = \pi_3 \hat{J}P_2 \twoheadrightarrow H_3 \hat{J}P_2 = H_3 CJ\rho(P_2)$$

which is surjective since $\hat{J}P_2$ is 1-connected. Hence it is sufficient to prove that the composite

$$H_3 CJ\rho(P_2) \xrightarrow{p} H_3 JP_2 \xrightarrow{\gamma_*} H_3 JP_2 \wedge P_2 \wedge P_2$$

is trivial. This is an exercise. Here $CJ\rho(P_2)$ is a completely algebraic chain complex; the map p corresponds to the projection $\hat{J}P_2 \to JP_2$ of the universal covering. Next we consider γ_3 on $[\Sigma^2 P_k, \Sigma P_n]$. We obtain the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{Z}_k, \pi_4 \Sigma P_n) \arrow[r, "\gamma_{3*}"'] \arrow[d, tail] & \mathrm{Ext}(\mathbb{Z}_k, \pi_4 \Sigma P_n \wedge P_n \wedge P_n) \arrow[d, equal] \\
{[\Sigma^2 P_k, \Sigma P_n]} \arrow[r, "\gamma_3"'] \arrow[d] & {[\Sigma^2 P_k, \Sigma P_n \wedge P_n \wedge P_n]} \arrow[d, equal] \\
\mathrm{Hom}(\mathbb{Z}_k, \Gamma \mathbb{Z}_n) \arrow[r, dashed, "\bar{\gamma}_3"'] & \mathrm{Ext}(\mathbb{Z}_k, \mathbb{Z}_n)
\end{tikzcd}$$

Here γ_3 is trivial on $\pi_4(\Sigma P_n)$. Therefore there exists the factorization $\bar{\gamma}_3$ which is natural in $\mathbb{Z}_k$ and in $\mathbb{Z}_n$. For $n < k$ the inclusion $\chi : \mathbb{Z}_n \to \mathbb{Z}_k$ yields the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}_k, \Gamma \mathbb{Z}_n) \arrow[r, "\bar{\gamma}_3"'] \arrow[d, "\chi^*"] & \mathrm{Ext}(\mathbb{Z}_k, \mathbb{Z}_n) \arrow[d, "\chi^*", "\cong"'] \\
\mathrm{Hom}(\mathbb{Z}_n, \Gamma \mathbb{Z}_n) \arrow[r, "\bar{\gamma}_3"] & \mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}_n)
\end{tikzcd}$$

Next consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{Z}_k, \pi_4 \Sigma P_n \wedge P_n) \arrow[r, "{[1,1]_* = 0}"] \arrow[d, tail] & \mathrm{Ext}(\mathbb{Z}_k, \pi_4 \Sigma P_n) \arrow[d, tail] \\
{[\Sigma^2 P_k, \Sigma P_n \wedge P_n]} \arrow[r, "{[1,1]_*}"] \arrow[d] & {[\Sigma^2 P_k, \Sigma P_n]} \arrow[d] \\
\mathrm{Hom}(\mathbb{Z}_k, \mathbb{Z}_n \otimes \mathbb{Z}_n) \arrow[r, two heads, "{[1,1]_*}"'] \arrow[ur, "W"] & \mathrm{Hom}(\mathbb{Z}_k, \Gamma \mathbb{Z}_n)
\end{tikzcd}$$

Here $[1,1]_*$ is trivial on $\pi_4 \Sigma P_n \wedge P_n$ by Lemma 11.5.15 so that the map W is defined. We observe that for $k < n$ the map $[1,1]_*$ in the bottom row is surjective. Hence $\bar{\gamma}_3 = 0$ is a consequence of the fact that $\gamma_3[1,1]$ by (11.5.10) (10) induces the trivial map on $[\Sigma^2 P_k, \Sigma P_n \wedge P_n]$; see Lemma 11.5.27. □

We now consider the generators $\varepsilon, \gamma, \xi, \eta$ in Theorem 11.5.9. First we recall that the identity $I_2 = \chi_2^2$ of ΣP_2 satisfies

(11.5.12) $$[\Sigma P_2, \Sigma P_2] = (\mathbb{Z}/4)\chi_2^2 \quad \text{with} \quad 2\chi_2^2 = i\eta_2 q.$$

We choose the generators ξ_2 and $\xi_{2,2}$ as in the following theorem.

(11.5.13) Theorem *Let ξ_2 be a generator of*

$$\pi_4 \Sigma P_2 = (\mathbb{Z}/4)\xi_2. \tag{1}$$

Then ξ_2 *induces the isomorphism of groups*

$$\xi_2^*: [\Sigma P_2, \Sigma P_2] \cong \pi_4 \Sigma P_2. \tag{2}$$

Moreover the James–Hopf invariant

$$\gamma_2: \pi_4 \Sigma P_2 \cong \pi_4 \Sigma P_2 \wedge P_2 \tag{3}$$

is an isomorphism. Let $\xi_{2,2} = \gamma_2 \xi_2$. *Then the following formulas are satisfied*:

$$q\xi_2 = \eta_3 \quad \text{and} \quad 2\xi_2 = i\eta_2^2 \quad \text{and} \quad 2\xi_{2,2} = i\eta_3 \tag{4}$$

$$(\Sigma T_{21})_* \xi_{2,2} = \xi_{2,2} \quad \text{and} \quad [1,1]_* \xi_{2,2} = 0. \tag{5}$$

Proof The map ξ_2^* in (2) is a homomorphism since the suspension Σ in $\xi_2^* = \Sigma^{-1}(\Sigma\xi_2)^*\Sigma$ is an isomorphism. Hence ξ_2^* is an isomorphism with $\xi_2^*\chi_2^2 = \xi_2$. Since $q\xi_2 = \eta_3$ by (11.5.4) (1) we get

$$2\xi_2 = \xi_2^*(2\chi_2^2) = \xi_2^* i\eta_2 q = i\eta_2 q\xi_2 = i\eta_2\eta_3 = i\eta_2^2.$$

This also implies

$$2\xi_{2,2} = \gamma_2(2\xi_2) = \gamma_2(i\eta_2\eta_3) = i\eta_3$$

so that (4) is proved. We obtain (3) by the commutative diagram of elementary exact sequences

(11.5.14)
$$\begin{array}{ccccc} \pi_3(\Sigma P_2) \otimes \mathbb{Z}/2 & \overset{\Delta}{\rightarrowtail} & \pi_4 \Sigma P_2 & \overset{\mu}{\twoheadrightarrow} & \Gamma T(\mathbb{Z}/2) \\ \cong \downarrow \gamma_2 \otimes 1 & & \downarrow \gamma_2 & & \cong \downarrow h \\ \pi_3(\Sigma P_2 \wedge P_2) \otimes \mathbb{Z}/2 & \overset{\Delta}{\rightarrowtail} & \pi_4 \Sigma P_2 \wedge P_2 & \overset{\mu}{\twoheadrightarrow} & \mathbb{Z}/2 * \mathbb{Z}/2 \end{array}$$

Moreover (5) is a consequence of the next lemma. □

(11.5.15) Lemma *For* $n \geq 0$ *the intercharge map* T_{21} *induces the identity*

$$\mathrm{id} = (\Sigma T_{21})_*: \pi_4 \Sigma P_n \wedge P_n \to \pi_4 \Sigma P_n \wedge P_n.$$

Moreover the Whitehead square induces the trivial map

$$0 = [1,1]_*: \pi_4 \Sigma P_n \wedge P_n \to \pi_4 \Sigma P_n.$$

Proof We consider the following commutative diagram obtained by applying the suspension to Whitehead's exact sequence with n even

$$\begin{array}{ccccc} \mathbb{Z}/2n = \Gamma(H_2 P_n \wedge P_n) & \overset{i}{\rightarrowtail} & \pi_3 P_n \wedge P_n & \overset{h}{\twoheadrightarrow} & H_3 P_n \wedge P_n = \mathbb{Z}/n \\ \downarrow \sigma & & \downarrow \Sigma & & \| \\ \mathbb{Z}/2 = \mathbb{Z}/n \otimes \mathbb{Z}/2 & \overset{i}{\rightarrowtail} & \pi_4 \Sigma P_n \wedge P_n & \overset{h}{\twoheadrightarrow} & H_4 \Sigma P_n \wedge P_n \end{array}$$

Now $T_{21}: P_n \wedge P_n \to P_n \wedge P_n$ induces the identity on $H_3 P_n \wedge P_n$ and on $\Gamma(H_2 P_n \wedge P_n)$. Therefore there is $\alpha: \mathbb{Z}/n \to \mathbb{Z}/2n$ such that

$$\pi_3(T_{21}) = \mathrm{id} + i\alpha h.$$

This implies $\pi_4(\Sigma T_{21}) = \mathrm{id} + i(\sigma\alpha)h$ where, however, $\sigma\alpha = 0$. Hence the first proposition is proved. We now prove $0 = [1,1]_*$ as follows. Since the generator $\iota_2 \in \pi_2 S^2$ satisfies $[[\iota_2, \iota_2], \iota_2] = 0$ we see that $[[1,1],1]\beta = 0$ for $\beta \in \pi_4 \Sigma P_n \wedge P_n \wedge P_n$. Therefore we obtain for $\alpha \in \pi_4 \Sigma P_n$ the equation $(2I_n)_* \alpha = 2\alpha - [1,1]\gamma_2(\alpha)$ where $I_n = \chi_n^n$ is the identity of ΣP_n. Here we use (11.5.11) and $\mathrm{id} = (\Sigma T_{21})_*$ above. Now let $n = 2$ and $\alpha = \xi_2$. Then we get by Theorem 11.5.13(4) and (11.5.12).

$$(2I_2)_* \xi_2 = i\eta_2 q \xi_2 = i\eta_2\eta_3 = 2\xi_2.$$

This implies $[1,1]\gamma_2 \xi_2 = 0$ and hence $0 = [1,1]_*$ for $n = 2$. Next let $n = 2^t$, $t > 1$, and let $\chi: \mathbb{Z}/n \to \mathbb{Z}/2$ be the surjective homomorphism. We choose $\bar{\chi}: P_n \to P_2$ which induces $\chi = \pi_1(\bar{\chi})$. Now we obtain the commutative diagram of elementary exact sequences

$$\begin{tikzcd}
 & \pi_4 \Sigma P_n \wedge P_n \arrow[d, "{[1,1]_*}"'] & \pi_4 S^3 \arrow[d, equal] \\
\mathbb{Z}/2 \arrow[r, tail, "\Delta"] \arrow[d, equal] & \pi_4 \Sigma P_n \arrow[r, two heads, "\mu"] \arrow[ur, "q_*"] \arrow[d, "(\Sigma\bar{\chi})_*"'] \arrow[dr, phantom, "\text{pull}"] & \mathbb{Z}/2 \arrow[d, "0"] \\
\mathbb{Z}/2 \arrow[r, tail] & \pi_4 \Sigma P_2 \arrow[r, two heads] & \mathbb{Z}/2
\end{tikzcd}$$

Here $[1,1]_* = 0$ is trivial. In fact $q_*[1,1]_* = [q,q] = 0$ is trivial since S^3 is an H-space and $(\Sigma\bar{\chi})_*[1,1]_* = [1,1]_*(\Sigma\bar{\chi} \wedge \bar{\chi})_*$ is trivial since $0 = [1,1]_*$ for $n = 2$. □

(11.5.16) Definition of the generators Let k, n, m be powers of primes. The canonical generator

$$\chi = \chi_n^k \in \mathrm{Hom}(\mathbb{Z}/k, \mathbb{Z}/n) = \mathbb{Z}/k * \mathbb{Z}/n$$

carries 1 to $n/(k,n)$ where (k,n) is the greatest common divisor. Let

$$\chi_n^k = B_2(\chi_n^k) \in [\Sigma P_k, \Sigma P_n] \tag{1}$$

be given by B_2 in Theorem 1.4.4 so that χ_n^k induces χ in homology. In the following the context always shows whether χ_n^k denotes a homomorphism or a homotopy class. We choose $\xi_2 \in \pi_4 \Sigma P_2$ as in Theorem 11.5.13 and we set

$$\xi_n = (\chi_n^2)_* \xi_2 \in \pi_4 \Sigma P_n. \tag{2}$$

Then $\mu(\xi_n) \in \Gamma T(\mathbb{Z}/n) = \mathbb{Z}/2 * \mathbb{Z}/n$ is the generator. Moreover let $\xi_{2,2} = \gamma_2(\xi_2)$ as in Theorem 11.5.13. For (n,m) not equal to $(2,2)$ let $\xi_{n,m} \in \pi_4 \Sigma P_n \wedge P_m$ be the unique element for which $\mu(\xi_{n,m}) \in \mathbb{Z}/n * \mathbb{Z}/m$ is the canonical generator and for which

$$(\chi_2^n \# \chi_2^m)_* \xi_{n,m} = 0. \tag{3}$$

See (11.5.10) (1) for the definition of $\chi_2^n \# \chi_2^m$. Here $\chi_2^n \# \chi_2^m$ induces a pull-back diagram between elementary exact sequences (11.5.4) (2) so that $\xi_{n,m}$ is well defined since $H_4\, \chi_2^n \# \chi_2^m = \chi * \chi = 0$ for (n,m) not equal to $(2,2)$. Using $\xi_{n,m}$ we define $\xi_{n,m}^k = \xi_{n,m} q$ as in Theorem 11.5.9. Next we obtain $\eta_{n,m}^k \in [\Sigma^2 P_k, \Sigma P_n \wedge P_m]$ by

$$\eta_{n,m}^k = \begin{cases} i_n \# \chi_m^k & \text{if } m = (n,m) \\ -(\Sigma T_{21})_* \eta_{m,n}^k & \text{otherwise.} \end{cases} \tag{4}$$

Here $i_n: S^2 \subset \Sigma P_n$ is the inclusion and we use the product # in (11.5.10)(1). One readily checks that $\mu(\eta_{n,m}^k) \in \operatorname{Hom}(\mathbb{Z}/k, \mathbb{Z}/n \otimes \mathbb{Z}/m)$ is the canonical generator. In (4) the numbers n, m or k may also be 0.

We define $\gamma_n^k \in [\Sigma^2 P_k, \Sigma P_n]$, n even, by setting

$$\gamma_n^k = \gamma_n^{2n}(\Sigma \chi_{2n}^k) \tag{5}$$

where γ_n^{2n} is the following generalized Hopf map; see Definition 11.2.2. The element $\gamma_n^{2n} \in [\Sigma^2 P_{2n}, \Sigma P_n]$ is the unique element which is represented by a twisted map; see (11.4.4), associated with the diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{[i_1, i_2]} & S^2 \vee S^2 \\ {\scriptstyle 2n}\downarrow & & \downarrow{\scriptstyle (n,1)} \\ S^3 & \xrightarrow{\eta_2} & S^2 \end{array} \tag{6}$$

and which satisfies

$$\gamma_2(\gamma_n^{2n}) = \eta_{n,n}^{2n} - \xi_{n,n}^{2n}. \tag{7}$$

Here γ_2 is the James–Hopf invariant. Using (6) we see that γ_n^{2n} is a generalized Hopf map and $\mu(\gamma_n^k) \in \operatorname{Hom}(\mathbb{Z}/k, \Gamma(\mathbb{Z}/n))$ with $\Gamma(\mathbb{Z}/n) = \mathbb{Z}/2n$ is the canonical generator. If n is odd let $P = 2: \mathbb{Z}/n \to \mathbb{Z}/n$ be the isomorphism which is multiplication by 2. This isomorphism yields the homotopy equivalence

$$B_3(2): \Sigma^2 P_n \simeq \Sigma^2 P_n$$

the homotopy inverse of which is $B_3(1/2)$; see Corollary 1.4.6. Then we set

$$\gamma_n^n = [i_n, 1] B_3(1/2): \Sigma^2 P_n \to \Sigma P_n \tag{8}$$

where $[i_n, 1]: \Sigma S^1 \wedge P_n \to \Sigma P_n$ is the Whitehead product of the inclusion $i_n: \Sigma S^1 \subset \Sigma P_n$ and the identity 1 of ΣP_n. Again γ_n^n is a generalized Hopf map and $\gamma_n^k = \gamma_n^n(\Sigma \chi_n^k)$ yields the canonical generator $\mu(\gamma_n^k)$. Compare the proof of Lemma 11.5.19 below. For n odd we obtain the James–Hopf invariant of γ_n^n by the formula

$$\gamma_2(\gamma_n^n) = \eta_{n,n}^n - \xi_{n,n}^n B_3(1/2). \tag{9}$$

This follows from (A.10.2)(h) and (11.5.20)(1) below.

(11.5.17) Lemma *The generalized Hopf map γ_n^{2n} is well defined by the conditions in* (11.5.16)(6), (7).

Proof Using the exact sequence (11.2.5)(3) and its cross-effect sequence we obtain the following commutative diagram; for this recall that $L_3(A,1) = 0$ if $A = \mathbb{Z}/n$ is cyclic. Let k, n be powers of 2.

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{Z}/k, \pi_3(\Sigma P_n) \otimes \mathbb{Z}/2)/K \arrow[r, tail] \arrow[d, "(\gamma_2 \otimes 1)_*", "\cong"'] & {[\Sigma^2 P_k, \Sigma P_n]} \arrow[r, two heads, "\lambda"] \arrow[d, "\gamma_2"] & \Gamma T_*(\mathbb{Z}/k, \mathbb{Z}/n) \arrow[d, "h_*"] \\
\mathrm{Ext}(\mathbb{Z}/k, \pi_3(\Sigma P_n \wedge P_n) \otimes \mathbb{Z}/2)/K \arrow[r, tail] & {[\Sigma^2 P_k, \Sigma P_n \wedge P_n]} \arrow[r, two heads, "\lambda"] \arrow[dr, "C_*"'] & {[d_{\mathbb{Z}/k}, A \otimes d_A]} \arrow[d, equal] \\
 & & {[C_* \Sigma^2 P_k, C_* \Sigma P_n \wedge P_n]}
\end{tikzcd} \tag{1}$$

Here we have $K \cong \mathbb{Z}/2$ if $k = n = 2$ and $K = 0$ otherwise. The map h_* is induced by h in (11.1.4) and C_* denotes cellular chains. Since $(\gamma_2 \otimes 1)_*$ is an isomorphism this diagram is a pull-back.

Now let $k = 2n$. Then Addendum 11.4.5 shows that $\lambda(\gamma_n^{2n})$ is the chain map associated with diagram 11.5.16 and one can check

$$\lambda(\eta_{n,n}^{2n} - \xi_{n,n}^{2n}) = h_* \lambda(\gamma_n^{2n}). \tag{2}$$

For this we consider chain maps $F = (F_1, F_0)$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "F_1"] \arrow[d, "2n"'] & \mathbb{Z} \otimes \mathbb{Z}/n \arrow[d, "n \otimes 1 = 0"] \\
\mathbb{Z} \arrow[r, "F_0"] & \mathbb{Z} \otimes \mathbb{Z}/n
\end{tikzcd} \tag{3}$$

Then $h_* \lambda(\gamma_n^{2n})$ is represented by $F = (1,1)$ and $\lambda(\eta_{n,n}^{2n})$ is represented by $F = (1,2)$ and $\lambda(\xi_{n,n}^{2n})$ is represented by $F = (0,1)$. This proves (2) and therefore the pull-back (1) shows that λ_n^{2n} is well defined. □

Next we consider the James–Hopf invariants

$$\gamma_2\colon \pi_4(\Sigma P_n) \to \pi_4(\Sigma P_n \wedge P_n)$$
$$\gamma_2\colon [\Sigma^2 P_k, \Sigma P_n] \to [\Sigma^2 P_k, \Sigma P_n \wedge P_n].$$

They satisfy on generators the following equations:

(11.5.18) Lemma *Let k, n be powers of 2. Then*

$$\gamma_2(\varepsilon_n) = \varepsilon_{n,n}$$
$$\gamma_2(\varepsilon_n^k) = \varepsilon_{n,n}^k$$
$$\gamma_2(\xi_n) = (n/2)\xi_{n,n}$$
$$\gamma_2(\xi_n^k) = (n/2)\xi_{n,n}^k$$
$$\gamma_2(\gamma_n^k) = \frac{2(n,k)}{(2n,k)}\eta_{n,n}^k - \frac{k}{(2n,k)}\xi_{n,n}^k.$$

Proof We first check the formula on $\gamma_2(\xi_n)$. We have $\gamma_2(\xi_n) = \gamma_2(\chi_n^2)_* \xi_2 = (\chi_n^2 \# \chi_n^2)_* \gamma_2 \xi_2 = (\chi_n^2 \# \chi_n^2)\xi_{2,2}$. Hence we get for $n > 2$

$$(\chi_2^n \# \chi_2^n)_* \gamma_2(\xi_n) = 0$$

since for $n > 2$ we have $\chi_2^n \chi_n^2 = 0$. On the other hand, $\mu\gamma_2(\xi_n) = n/2 \in \mathbb{Z}/n * \mathbb{Z}/n = \mathbb{Z}/n$ since for the canonical generator $\chi \in \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/n)$ we have $\chi(1) = n/2$ and hence

$$\mu\gamma_2(\xi_n) = \mu(\chi_n^2 \# \chi_n^2)_* \xi_{2,2} = (\chi * \chi)\mu(\xi_{2,2})$$

with $\mu(\xi_{2,2}) = 1$ and $\chi * \chi = \chi$. Next we check the formula on $\gamma_2(\gamma_n^k)$. We have

$$\gamma_2(\gamma_n^k) = \gamma_2(\gamma_n^{2n})(\Sigma \chi_{2n}^k)$$
$$= \eta_{nn}^{2n}(\Sigma \chi_{2n}^k) - \xi_{n,n}^{2n}(\Sigma \chi_{2n}^k)$$

and now we can use Lemma 11.5.24(f). This yields the result since $\varepsilon_{2,2}^2 = 0$. □

In addition to Lemma 11.5.15 we now consider for the Whitehead square $[1,1]$ the induced homomorphism

$$[1,1]_*:[\Sigma^2 P_k, \Sigma P_n \wedge P_n] \to [\Sigma^2 P_k, \Sigma P_n]$$

On generators we get the formulas:

(11.5.19) Lemma *Let k and n be powers of 2. Then*

$$[1,1]_* \varepsilon^k_{n,n} = 0$$

$$[1,1]_* \xi^k_{n,n} = 0$$

$$[1,1]_* \eta^k_{n,n} = \frac{(2n,k)}{(n,k)}\gamma^k_n + \delta^k_n \varepsilon^k_n.$$

Here let $\delta^k_n = 1$ for $n=4, k \in \{2,4\}$, and $\delta^k_n = 0$ otherwise. In particular we get for the generalized Hopf map γ^{2n}_n the formula

$$2\gamma^{2n}_n = [1,1]\eta^{2n}_{n,n}.$$

Proof The first two equations are consequences of Lemma 11.5.15. Since $\eta^k_{n,n} = i_n \# \chi^k_n$ we have $[1,1]_*\ \eta^k_{n,n} = [i_n, 1]\ (1 \wedge \chi^k_n)$ where $i_n: S^2 \subset \Sigma P_n$ is the inclusion. Here the Whitehead product $[i_n, 1] \in [\Sigma S^1 \wedge P_n, \Sigma P_n]$ is a twisted map associated with the right-hand square of the diagram

$$\begin{array}{ccccc}
S^3 & \xrightarrow{k/(k,n)} & S^3 & \xrightarrow{[i_1,i_2]} & S^2 \vee S^2 \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle (n,1)} \\
S^3 & \xrightarrow{n/(k,n)} & S^3 & \xrightarrow{2\eta_2} & S^2
\end{array}$$

The diagram represents $\lambda([i_n, 1](1 \wedge \chi^k_n)) \in \Gamma T_\#(\mathbb{Z}/k, \mathbb{Z}/n)$. Using (11.5.16)(5), (6) we therefore have

$$\lambda([1,1]_* \eta^k_{n,n}) = \frac{(2n,k)}{(n,k)} \lambda(\gamma^k_n).$$

Using the pull-back diagram in Lemma 11.5.17(1) it remains to show that the James–Hopf invariants coincide. By Lemma 11.5.18 we have

$$\gamma_2\left(\frac{(2n,k)}{(n,k)}\gamma^k_n\right) = \frac{(2n,k)}{(n,k)}\gamma_2(\gamma^k_n) = 2\eta^k_{n,n} - \frac{k}{(n,k)}\xi^k_{n,n}.$$

On the other hand, we obtain by (11.5.10)(9)

$$\gamma_2([1,1]\eta^k_{n,n}) = (\gamma_2[1,1])\eta^k_{n,n}$$

$$= -(\Sigma T_{21})\eta^k_{n,n} + \eta^k_{n,n} = 2\eta^k_{n,n} - \frac{k}{(n,k)}\xi^k_{n,n} + \delta^k_n \varepsilon^k_{n,n}$$

where in the last equation we have used the next result (Lemma 11.5.20). □

(11.5.20) Lemma *Let k, n, m be powers of 2. Then the interchange map T_{21} on $P_n \wedge P_m$ induces the isomorphism*

$$(\Sigma T_{21})_* : [\Sigma^2 P_k, \Sigma P_n \wedge P_m] \xrightarrow{\cong} [\Sigma^2 P_k, \Sigma P_m \wedge P_n]$$

which on generators is given by the following equations:

$$(\Sigma T_{21})_* \varepsilon^k_{n,m} = \varepsilon^k_{m,n}$$

$$(\Sigma T_{21})_* \xi^k_{n,m} = \xi^k_{m,n}$$

$$(\Sigma T_{21})_* \eta^k_{n,m} = -\eta^k_{m,n} \quad \text{for} \quad m \text{ not equal to } n$$

$$(\Sigma T_{21})_* \eta^k_{n,n} = -\eta^k_{n,n} + \frac{k}{(n,k)}\xi^k_{n,n} + \delta^k_n \varepsilon^k_{n,n}.$$

Here let $\delta^k_n = 1$ for $n = 4$ and $k \in \{2,4\}$ and $\delta^k_n = 0$ otherwise.

Proof The first two equations follow from Lemma 11.5.15 and the third equation is a consequence of Definition (11.5.16)(4). It remains to prove the fourth equation. One readily checks that $\lambda: [\Sigma^2 P, \Sigma P_n \wedge P_n] \to [d_{\mathbb{Z}/n}, A \otimes d_A]$ with $A = \mathbb{Z}/n$ carries both sides of the equation

$$(1 + \Sigma T)_* \eta^n_{n,n} = \xi^n_{n,n} + \delta_n \varepsilon^n_{n,n} \tag{1}$$

to the same element. Hence the exact sequence in the bottom row of Lemma 11.5.17(1) shows that for appropriate $\delta_n \in \mathbb{Z}$ equation (1) holds. We set $\delta_2 = 0$ since $\varepsilon^2_{2,2} = 0$. Since

$$\eta^n_{n,n}(\Sigma \chi^k_n) = \eta^k_{n,n} \tag{2}$$

by (11.5.16)(4) and since

$$\xi^n_{n,n}(\Sigma \chi^k_n) = \xi_{n,n}(q\Sigma\chi^k_n) = \frac{k}{(n,k)}\xi^k_{n,n} \tag{3}$$

we see that (1) implies $\delta^k_n = \delta_n \cdot k/(n,k)$ by a similar argument as in (3) where we replace ξ by ε. We have to compute δ_n in (1) for $n > 2$. For this we compute in $[\Sigma^2 P_n, \Sigma P_2 \wedge P_2]$

$$(\chi_2^n \# \chi_n^n)_*(1+\Sigma T)_* \eta_{n,n}^n \tag{4}$$

$$= (1+\Sigma T)_*(\chi_2^n \# \chi_2^n)_* \eta_{n,n}^n$$

$$= (1+\Sigma T)_* \eta_{2,2}^2 (\Sigma \chi_2^n) \tag{5}$$

$$= \xi_{22}^2(\Sigma \chi_2^n) = \xi_{22}(q \Sigma \chi_2^n) \tag{6}$$

$$= (n/2)\xi_{2,2}^n = (n/4)\varepsilon_{2,2}^n \quad \text{for} \quad n > 2.$$

Here (5) is a consequence of the definition of $\eta_{n,n}^n$ and (6) follows from (1) since $\varepsilon_{2,2}^2 = 0$. On the other hand, we get by (1),

$$(4) = (\chi_2^n \# \chi_2^n)_*(\xi_{n,n}^n + \delta_n \varepsilon_{n,n}^n)$$

$$= (\chi_2^n \# \chi_2^n)_*(\xi_{n,n} q) + \delta_n \varepsilon_{2,2}^n$$

$$= \delta_n \varepsilon_{2,2}^n. \tag{7}$$

Here (7) is a consequence of (11.5.16)(3). Hence we have proved that $\delta_n \equiv n/4$ modulo 2 for $n > 2$ since $\varepsilon_{2,2}^n \neq 0$ for $n > 2$. Hence we get $\delta_n^k \equiv nk/4(n,k)$ for $n > 2$ and $\delta_2^k = 0$. □

(11.5.21) Lemma *Let n, m be powers of 2 and let*

$$i_{n,m} \colon \Sigma^2 P_m = \Sigma S^1 \wedge P_m \subset \Sigma P_n \wedge P_m$$

be the inclusion given by $S^1 \subset P_n$. Then we have in $[\Sigma^2 P_m, \Sigma P_n \wedge P_m]$

$$i_{n,m} = \begin{cases} \eta_{n,m}^m & \text{if } m \le n \\ \eta_{n,m}^m + \xi_{n,m}^m & \text{if } m > n. \end{cases}$$

Proof We use the short exact sequence, $A = \mathbb{Z}/n$, $B = \mathbb{Z}/m$

$$\begin{tikzcd} \mathrm{Ext}(\mathbb{Z}/k, \pi_3(\Sigma P_n \wedge P_m) \otimes \mathbb{Z}/2)/K \arrow[r, tail] & {[\Sigma^2 P_k, \Sigma P_n \wedge P_m]} \arrow[r, "\lambda"] \arrow[dr, "C_*"'] & {[d_{\mathbb{Z}/k}, d_A \otimes B]} \arrow[d, equal] \\ & & {[C_* \Sigma P_k, C_* \Sigma P_n \wedge P_m]} \end{tikzcd}$$

where $K \cong \mathbb{Z}/2$ if $k = n = m = 2$ and $K = 0$ otherwise. For $A = B$ this is the bottom row in Lemma 11.5.17(1). Let $k = m$. By definition of $\eta_{n,m}^m$ the equation in (11.5.21) is true if $m = (n,m)$; see (11.5.16)(4) where χ_m^m is the identity. Now assume $m > n$. Then we have by (11.5.16)(4)

$$\eta_{n,m}^m = -(\Sigma T_{21})_* \eta_{m,n}^m = -(\Sigma T_{21})_* i_{m,n}(\Sigma \chi_n^m) \tag{1}$$

where $\eta^n_{m,n} = i_{m,n}\colon \Sigma^2 P_n \subset \Sigma P_m \wedge P_n$ is the inclusion. One can check that λ above satisfies $\lambda(i_{n,m}) = \lambda(\eta^m_{n,m} + \xi^m_{n,m})$. Moreover we have

$$(\chi^n_n \# \chi^m_n) i_{n,m} = i_{n,n}(\Sigma \chi^m_n) = \eta^m_{n,n}. \tag{2}$$

On the other hand, we have by (1)

$$\begin{aligned}
&(\chi^n_n \# \chi^m_n)(\eta^m_{n,m} + \xi^m_{n,m}) \\
&\quad = -(\Sigma T_{21})(\chi^m_n \# \chi^n_n) i_{m,n}(\Sigma \chi^m_n) + (\chi^n_n \# \chi^m_n)\xi^m_{n,m} && (3)\\
&\quad = -(\Sigma T_{21}) i_{n,n}(\Sigma \chi^n_n) + (\chi^n_n \# \chi^m_n)\xi^m_{n,m} \\
&\quad = -(\Sigma T_{21})\eta^m_{n,n} + (\chi^n_n \# \chi^m_n)\xi^m_{n,m} && (4)\\
&\quad = \eta^m_{n,n} - \frac{m}{(m,n)}\xi^m_{n,n} + (\chi^n_n \# \chi^m_n)\xi^m_{n,m} && (5)\\
&\quad = \eta^m_{n,n}. && (6)
\end{aligned}$$

In (4) we use (2) and in (5) we use Lemma 11.5.20, Moreover (6) is a consequence of Lemma 11.5.24(b) below since $N(\chi^n_n * \chi^m_n)' = m/(m,n) = m/n$ for $m > n$. Since $\chi^n_n \# \chi^m_n$ induces an isomorphism for the kernel of λ above this completes the proof of Lemma 11.5.21. □

For $\varphi \in \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m)$ let

(11.5.22) $$\varphi = B_2(\varphi) \in [\Sigma P_n, \Sigma P_m]$$

be defined by Theorem 1.4.4. The context below always shows clearly whether φ denotes a homomorphism or a homotopy class. Recall that $B_2(\varphi)$ is the suspension of a principal map $P_n \to P_m$ inducing φ.

Now let m, n, r, s, k, t be powers of 2 and let $\varphi \in \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/s)$, $\psi \in \mathrm{Hom}(\mathbb{Z}/m, \mathbb{Z}/r)$, and $\tau \in \mathrm{Hom}(\mathbb{Z}/t, \mathbb{Z}/k)$. Then we have induced homomorphisms

(a) $$\varphi_*\colon \pi_4 \Sigma P_n \to \pi_4 \Sigma P_s$$

(b) $$(\varphi \# \psi)_*\colon \pi_4 \Sigma P_n \wedge P_m \to \pi_4 \Sigma P_s \wedge P_r$$

(c) $$\varphi_*\colon [\Sigma^2 P_k, \Sigma P_n] \to [\Sigma^2 P_k, \Sigma P_s]$$

(d) $$(\varphi \# \psi)_*\colon [\Sigma^2 P_k, \Sigma P_n \wedge P_m] \to [\Sigma^2 P_k, \Sigma P_s \wedge P_r]$$

(e) $$(\Sigma\tau)^*\colon [\Sigma^2 P_k, \Sigma P_n] \to [\Sigma^2 P_t, \Sigma P_n]$$

(f) $$(\Sigma\tau)^*\colon [\Sigma^2 P_k, \Sigma P_n \wedge P_m] \to [\Sigma^2 P_t, \Sigma P_n \wedge P_m].$$

These homomorphisms are computed on generators by the following result. For this we use the notation:

(11.5.23) Definition Let $A=(\mathbb{Z}/n)x$ and $B=(\mathbb{Z}/m)y$ be cyclic groups with fixed generators x and y respectively and let $\varphi \in \mathrm{Hom}(A,B)$. We choose a number $N(\varphi)$ which satisfies $\varphi(x)=N(\varphi)y$ and let $\bar{N}(\varphi)$ be the unique number with $0 \le \bar{N}(\varphi) < m$ and $\varphi(x)=\bar{N}(\varphi)\cdot y$. Moreover we define $M(\varphi)$ by $nN(\varphi)=mM(\varphi)$. For example $\mathbb{Z}/n, \mathbb{Z}/n \otimes \mathbb{Z}/m$, and $\mathbb{Z}/n * \mathbb{Z}/m$ are cyclic groups with canonical fixed generators.

(11.5.24) Lemma *The homomorphisms of Lemma* 11.5.21 *satisfy the following equations* (a)–(f).

(a)
$$\varphi_* \varepsilon_n = N(\varphi)\varepsilon_s$$
$$\varphi_* \xi_n = M(\varphi)M(\varphi)\xi_s.$$

(b)
$$(\varphi\#\psi)_* \varepsilon_{n,m} = N(\varphi\otimes\psi)\varepsilon_{s,r}$$
$$(\varphi\#\psi)_* \xi_{n,m} = \bar{N}(\varphi * \psi)\xi_{s,r}.$$

(c)
$$\varphi_* \varepsilon_n = N(\varphi)\varepsilon_s^k$$
$$\varphi_* \xi_n^k = M(\varphi)M(\varphi)\xi_s^k.$$
$$\varphi_* \gamma_n^k = N(\varphi)M(\varphi)\frac{(2s,k)}{(2n,k)}\gamma_s^k.$$

(d)
$$(\varphi\#\psi)_* \varepsilon_{n,m}^k = N(\varphi\otimes\psi)\varepsilon_{s,r}^k$$
$$(\varphi\#\psi)_* \xi_{n,m}^k = \bar{N}(\varphi * \psi)\xi_{s,r}^k.$$
$$(\varphi\#\psi)_*\eta_{n,m}^k = \begin{cases} N(\varphi)(\eta_{s,r}^r + \delta_{s,r}\xi_{s,r}^r)\Sigma(\psi\chi_m^k) & \text{for } m \le n \\ N(\psi)(\eta_{s,r}^s - \delta_{r,s}\xi_{s,r}^s)\Sigma(\varphi\chi_n^k) & \text{for } m > n. \end{cases}$$

Here we set $\delta_{s,r}=1$ *for* $r>s$ *and* $\delta_{s,r}=0$ *for* $r \ge s$. *The right-hand side can be computed by the formulas in* (f) *below.*

(e)
$$(\Sigma\tau)^*\varepsilon_n^k = M(\tau)\varepsilon_n^\tau$$
$$(\Sigma\tau)^*\xi_n^k = M(\tau)\xi_n^t$$
$$(\Sigma\tau)^*\gamma_n^k = N(\tau)\frac{(t,2n)}{(k,2n)}\gamma_n^t.$$

(f)
$$(\Sigma\tau)^*\varepsilon_{n,m}^k = M(\tau)\varepsilon_{n,m}^t$$
$$(\Sigma\tau)^*\xi_{n,m} = M(\tau)\xi_{n,m}^t$$
$$(\Sigma\tau)^*\eta_{n,m}^k = N(\tau)\frac{(t,a)}{(k,a)}\eta_{n,m}^t + \delta_a^{k,t}(N(\tau)(N(\tau)-1)/2)\varepsilon_{n,m}^t.$$

Here we set $a = \min(m, n)$ *and we set* $\delta_a^{k,t} = 1$ *for* $t = a = 2$ *and* $k \leq 4$ *and* $\delta_a^{k,t} = 0$ *otherwise.*

Proof All formulas for ε-terms are clear since $B_2(\varphi)$ has degree $N(\varphi)$ on the bottom sphere S^2 and has degree $M(\varphi)$ on the 3-cell. Moreover one readily checks the equation

$$\varphi\chi_n^2 = \chi_s^2(M(\varphi)\chi_2^2)$$

for homomorphisms. For $a = M(\varphi)$ we have in $[\Sigma P_2, \Sigma P_2]$ the equations; see Theorem 1.4.8

$$\begin{aligned} B_2(a\chi_2^2) &= a\chi_2^2 + \frac{a(a-1)}{2} i\eta_2 q \\ &= a\chi_2^2 + a(a-1)\chi_2^2 = a^2\chi_2^2 \end{aligned}$$

where we use (11.5.12). Since B_2 is a functor we thus obtain the second formula of (a). Now we prove the second formula of (b). The operator μ carries both sides of the formula to the same element. Next we obtain the coefficient of $\varepsilon_{s,r}$ by applying $\chi_2^s \# \chi_2^r$; see (11.5.16) (2). We have $\chi_2^s\varphi = \varphi_2\chi_2^n$ with $\varphi_2 = \varphi \otimes \mathbb{Z}/2$ in **Ab** and hence we get the element

$$(\chi_2^s \# \chi_2^r)_*(\varphi \# \psi)_*\xi_{n,m} = (\varphi_2 \# \psi_2)_*(\chi_2^n \# \chi_2^m)\xi_{n,m}$$

which is trivial for (n, m) not equal to $(2,2)$ and which is $(\varphi_2 \# \psi_2)_*\xi_{2,2}$ for $(n, m) = (2,2)$. Here we have $\varphi_2 \# \psi_2 = 0$ for (s, r) not equal to $(2,2)$ and $(n, \mathrm{m}) = (2,2)$ since then $\varphi_2 = 0$ or $\psi_2 = 0$. On the other hand, the element $(\chi_2^s \# \chi_2^r)(\bar{N}(\varphi * \psi)\xi_{s,r})$ is trivial for (s, r) not equal to $(2,2)$ and is $\bar{N}(\varphi * \psi)\xi_{2,2}$ for $(s, r) = (2,2)$. For $(s, r) = (2,2)$ however we get $\varphi * \psi = 0$ for (n, m) not equal to $(2,2)$. This completes the proof of the second formula of (b).

It is enough to prove the third equation in (c) for $k = 2n$ since we can apply (e) for k not equal to $2n$. Hence let $k = 2n$. One readily checks for λ in the top row of Lemma 11.5.17(1) the equation

$$\lambda(\varphi_*\gamma_n^{2n}) = \varphi_*\lambda(\gamma_n^{2n}) = \alpha\lambda(\gamma_s^{2n})$$

with $\alpha = N(\varphi)^2(n, s)/s$. Using the pull-back diagram of Lemma 11.5.17(1) it hence remains to show that James–Hopf invariants satisfy

$$\gamma_2(\varphi_*\gamma_n^{2n}) = \alpha\gamma_2(\gamma_s^{2n}).$$

On the one hand, we have

$$\begin{aligned} \gamma_2(\varphi_*\gamma_n^{2n}) &= (\varphi\#\varphi)_*\gamma_2(\gamma_n^{2n}) = (\varphi\#\varphi)_*(\eta_{n,n}^{2n} - \xi_{n,n}^{2n}) \\ &= N(\varphi)\eta_{ss}^s\Sigma(\varphi\chi_n^{2n}) - \bar{N}(\varphi * \varphi)\xi_{s,s}^{2n}. \end{aligned}$$

Here we use (d). For $\overline{\varphi} = \beta \chi_{2s}^{2n}$ with $\beta = N(\varphi)(n,s)/s$ we have $\varphi \chi_n^{2n} = \chi_s^{2s} \overline{\varphi}$ in **Ab** and one checks as in the proof of (e) that $\Sigma\overline{\varphi} = \Sigma(\beta\chi_{2s}^{2n}) = \beta\Sigma\chi_{2s}^{2n}$. Hence we get

$$\gamma_2(\varphi_* \gamma_n^{2n}) = \beta N(\varphi)\eta_{ss}^{s}(\Sigma\chi_s^{2s})(\Sigma\chi_{2s}^{2n}) - \overline{N}(\varphi * \varphi)\xi_{s,s}^{2n}$$

with $\beta N(\varphi) = \alpha$. On the other hand, we have

$$\begin{aligned}\gamma_2(\alpha\gamma_s^{2n}) &= \alpha(\gamma_s^{2s}\Sigma\chi_{2s}^{2n})\\ &= \alpha(\eta_{s,s}^{2s} - \xi_{s,s}^{2s})\Sigma\chi_{2s}^{2n}\\ &= \alpha(\eta_{s,s}^{s}\Sigma\chi_s^{2s})\Sigma\chi_{2s}^{2n} - N(\varphi)M(\varphi)\xi_{s,s}^{2n}.\end{aligned}$$

Here we use (f) and the definition of $\eta_{s,s}^{2s}$. Hence it remains to show

$$(N(\varphi)M(\varphi) - \overline{N}(\varphi * \varphi))\xi_{s,s}^{2n} = 0. \tag{*}$$

Here we have $\overline{N}(\varphi * \varphi) \equiv N(\varphi)M(\varphi)$ modulo s, hence (*) holds for $s > 2$ since then $2\xi_{s,s}^{2n} = 0$. For $s = 2$ and φ not equal to 0 we have $N(\varphi) = 1$, $M(\varphi) = n/2$, and $\overline{N}(\varphi * \varphi) = 0$ for $n > 2$. Hence (*) holds for $s > 2, n > 2$. Now it is clear that (*) holds for $s = n = 2$. This completes the proof of (c).

For the proof of (d) we may assume $m = (n,m)$; see (11.5.16)(4). Then we obtain $\eta_{n,m}^k$ by the composite in the top row of the commutative diagram

$$\begin{tikzcd} \eta_{n,m}^k\colon \Sigma^2 P_k \arrow[r, "\Sigma\chi_m^k"] \arrow[dr, "N(\varphi)\cdot\Sigma(\chi_m^k)"'] & \Sigma^2 P_m \arrow[r, "i_{n,m}"] \arrow[d, "N(\varphi)\cdot\Sigma\psi"] & \Sigma P_n \wedge P_m \arrow[d, "\varphi\#\psi"] \\ & \Sigma^2 P_r \arrow[r, "i_{s,r}"'] & \Sigma P_s \wedge P_r \end{tikzcd}$$

Here $i_{n,m}$ and $i_{s,r}$ are the canonical inclusions. Let $\delta_{s,r} = \delta = 0$ if $r \le s$ and let $\delta = 1$ for $r > s$. Then Lemma 11.5.21 and the diagram show

$$(\varphi\#\psi)\eta_{n,m}^k = N(\varphi)\cdot(\eta_{s,r}^r + \delta\xi_{s,r}^r)\Sigma(\psi\chi_m^k).$$

This yields the equations in (d).

For the proof of (e) we only consider $(\Sigma\tau)^*\gamma_n^k$. We have

$$(\Sigma\tau)^*\gamma_n^k = \gamma_n^{2n}(\Sigma\chi_{2n}^k)(\Sigma\tau) = \gamma_n^{2n}\Sigma(\chi_{2n}^k\tau)$$

where $\chi_{2n}^k\tau = \alpha\chi_{2n}^t$ with $\alpha = N(\tau)(t,2n)/(k,2n)$. Moreover we have by Theorem 1.4.8

$$\Sigma(\alpha\chi_{2n}^t) = \alpha(\Sigma\chi_{2n}^t) + \binom{\alpha}{2}\frac{t}{2}\left(\frac{2n}{(t,2n)}\right)^2 i\eta_3 q.$$

Here the η_3-term vanishes for $t > 2$ since $2\eta_3 = 0$. Moreover for $t = 2$ we get $\binom{\alpha}{2} n^2 i\eta_3 q = 0$. This shows that actually $\Sigma(\alpha\chi^t_{2n}) = \alpha\Sigma\chi^t_{2n}$ and the third formula of (e) is proven.

For the proof of (f) we only consider the element $(\Sigma\tau)^*\eta^k_{n,m}$ for $m \leq n$. Then we have

$$\begin{aligned}(\Sigma\tau)^*\eta^k_{n,m} &= i_{n,m}(\Sigma\chi^k_m(\Sigma\tau))\\ &= i_{n,m}\Sigma(\chi^k_m\tau).\end{aligned}$$

Now $\chi^k_m\tau = \alpha\chi^t_m$ in **Ab** with $\alpha = N(\tau)(t,m)/(k,m)$. Therefore we get by Theorem 1.4.8.

$$\Sigma(\chi^k_m\tau) = \alpha\Sigma\chi^t_m + \frac{\alpha(\alpha-1)}{2}\cdot\frac{t}{2}N^2 i\eta_3 q$$

with $N = N(\chi^t_m) = m/(t,m)$. This shows $(\Sigma\tau)^*\eta^k_{n,m} = \alpha\eta^t_{n,m} + \beta\varepsilon^t_{n,m}$ where

$$\beta \equiv (mt/2(t,m))(\alpha(\alpha-1)/2) \quad \text{modulo } 2.$$

Hence we get

$$\beta \equiv \begin{cases} N(\tau)(N(\tau)-1)/2 & \text{for } t = m = 2, k \geq 4\\ 0 & \text{otherwise}\end{cases}$$

□

Next we consider the pinch map $q\colon \Sigma P^n \to S^3$ which induces

$$\begin{aligned}q_*&\colon \pi_4\Sigma P_n \to \pi_4 S^3 = (\mathbb{Z}/2)\eta_3\\ q_*&\colon [\Sigma^2 P_k, \Sigma P_n] \to [\Sigma^2 P_k, S^3] = \mathbb{Z}/(k,2)\eta_3 q.\end{aligned}$$

(11.5.25) Lemma

$$\begin{aligned}q_*\varepsilon_n &= 0\\ q_*\xi_n &= (n/2)\eta_3\\ q_*\varepsilon^k_n &= 0\\ q_*\xi^k_n &= (n/2)\eta_3 q\\ q_*\gamma^k_n &= 0.\end{aligned}$$

Proof We have $q_*\xi_2 = \eta_3$ by Theorem 11.5.13 and hence $q_*\xi_n = q_*\chi_n^2\xi_2 = M(\chi_n^2)q\xi_2$ where $M(\chi_n^2) = n/2$. Moreover $q_*\gamma_n^k = 0$ by (11.5.16) (6). □

The reduced diagonal $\Delta: P_n \to P_n \wedge P_n$ satisfies

(11.5.26) $$\Delta = (n(n-1)/2)iq: P_n \to S^2 \to P_n \wedge P_n.$$

Therefore we can apply Lemma 11.5.25 for the computation of $(\Sigma\Delta)_*$ on $\pi_4\Sigma P_n$ and $(\Sigma^2 P_k, \Sigma P_n]$. Moreover we have to compute the induced maps

$$\Sigma(1\wedge\Delta)_*: \pi_4(\Sigma P_n \wedge P_m) \to \pi_4(\Sigma P_n \wedge P_m \wedge P_m) = \mathbb{Z}/n \otimes \mathbb{Z}/m$$

$$\Sigma(1\wedge\Delta)_*: [\Sigma^2 P_k, \Sigma P_n \wedge P_m] \to [\Sigma^2 P_k, \Sigma P_n \wedge P_m \wedge P_m]$$
$$= \mathbb{Z}/k \otimes \mathbb{Z}/n \otimes \mathbb{Z}/m$$

and similarly $\Sigma(\Delta \wedge 1)_*$ where Δ is the reduced diagonal on P_m and P_n respectively. Let $1_n \in \mathbb{Z}/n$ be the canonical generator.

(11.5.27) Lemma *Let k, m, n be powers of 2. Then $\Sigma(1\wedge\Delta)_*$ and $\Sigma(\Delta\wedge 1)_*$ carry the elements $\varepsilon_{m,n}$, $\varepsilon_{m,n}^k$, $\xi_{m,n}$, $\xi_{m,n}^k$ (m not equal to n) to 0. Moreover*

$$\Sigma(1\wedge\Delta)_*\xi_{n,n} = \Sigma(\Delta\wedge 1)_*\xi_{n,n} = (n/2)1_n \otimes 1_n$$

$$\Sigma(1\wedge\Delta)_*\xi_{n,n}^k = \Sigma(\Delta\wedge 1)_*\xi_{n,n}^k = (n/2)1_k \otimes 1_n \otimes 1_n$$

$$\Sigma(\Delta\wedge 1)_*\eta_{n,m}^k = \delta_{n,m}\frac{nk}{2(k,n)}1_k \otimes 1_n \otimes 1_m$$

$$\Sigma(1\wedge\Delta)_*\eta_{n,m}^k = (1-\delta_{n,m})\frac{mk}{2(k,m)}1_k \otimes 1_n \otimes 1_m$$

where $\delta_{n,m} = 1$ for $m > n$ and $\delta_{n,m} = 0$ for $m \le n$.

Proof We have the commutative diagram

$$\begin{array}{ccccc}
\pi_4(\Sigma P_n \wedge P_m) & \xrightarrow{h} & H_4\Sigma P_n \wedge P_m & = & \mathbb{Z}/n * \mathbb{Z}/m \\
\downarrow{\scriptstyle (\Sigma q\wedge 1)_*} & & \downarrow & & \cap \\
\pi_4(\Sigma S^2 \wedge P_m) & = & H_4(\Sigma S^2 \wedge P_m) & = & \mathbb{Z}/m
\end{array}$$

where h is the Hurewicz map and q is the pinch map. Hence we obtain $\Sigma(\Delta\wedge 1)_*\xi_{n,m}$ by (11.5.26) since $h(\xi_{n,m})$ is the canonical generator. This shows that the composite

$$\mathbb{Z}/n * \mathbb{Z}/m \subset \mathbb{Z}/m \xrightarrow{n/2} \mathbb{Z}/m \twoheadrightarrow \mathbb{Z}/n \otimes \mathbb{Z}/m$$

carries the canonical generator to $\Sigma(\Delta \wedge 1)_* \xi_{n,m}$. This yields the result on $\xi_{n,m}$ and $\xi^k_{n,m}$. Next we consider $\eta^k_{n,m}$ for $m \le n$ so that by (11.5.26)

$$\Sigma(\Delta \wedge 1)_* \eta^k_{n,m} = 0$$

$$\Sigma(1 \wedge \Delta)_* \eta^k_{n,m} = (m/2) M(\chi^k_m) 1_k \otimes 1_n \otimes 1_m$$

since $\eta^k_{n,m} = i_{n,m} \Sigma \chi^k_m$; see Lemma 11.5.21. For $m > n$ we use (11.5.16)(4). □

(11.5.28) Remark Let A and B be direct sums of cyclic groups and for $\varphi \in \mathrm{Hom}(A,B)$ let $s\varphi: M(A,2) \to M(B,2)$ be a map which induces φ. Then the formulas in this section and in Section A.10 allow explicit computations of the induced maps

$$(s\varphi)_*: \pi_4 M(A,2) \to \pi_4 M(B,2)$$

$$(s\varphi)_*: [\Sigma^2 P_k, M(A,2)] \to [\Sigma^2 P_k, M(A,2)]$$

on generators in Remark 11.5.3 and Theorem 11.5.9. For this we need, in particular, the left distributivity law of Theorem A.10.2(b).

11.6 The suspension of elementary homotopy groups in dimension 4

Let X and Y be pointed spaces. We say that the set of homotopy classes $[X,Y]$ is *stable* if the suspension yields bijections for $n \ge 1$

$$\Sigma: [\Sigma^{n-1}X, \Sigma^{n-1}Y] \approx [\Sigma^n X, \Sigma^n Y].$$

For example we get

(11.6.1) Proposition *The groups $\pi_4 \Sigma P_n, \pi_4 \Sigma P_n \wedge P_m, [\Sigma^2 P_k, \Sigma P_n \wedge P_m]$, and $[\Sigma^3 P_k, \Sigma^2 P_n]$ are stable for $k, n, m \ge 0$.*

We want to describe the suspension on elementary homotopy groups in dimension 4. By Proposition 11.6.1 we only have to consider

$$\Sigma: [\Sigma^2 P_k, \Sigma P_n] \to [\Sigma^3 P_k, \Sigma^2 P_n]. \tag{11.6.2}$$

For this we define generators in $[\Sigma^3 P_k, \Sigma^2 P_n]$ as follows.

(11.6.3) Definition Let k, n be powers of 2 and $r \ge 2$. Then $\varepsilon^k_n = \varepsilon$ is the composite

$$\varepsilon^k_n: \Sigma^r P_k \xrightarrow{q} S^{r+2} \xrightarrow{\eta^2_r} S^r \xrightarrow{i} \Sigma^{r-1} P_n$$

where q is the pinch map and i is the inclusion. We also set $\varepsilon_n = i\eta_r^2$ and $\varepsilon^k = \eta_r^2 q$. Now we choose generators

$$\xi_2 \in [S^4, \Sigma P_2] \cong \mathbb{Z}/4,$$

see Theorem 11.5.13 and

$$\eta^2 \in [\Sigma^3 P_2, S^3] \cong \mathbb{Z}/4$$

which are stably Spanier–Whitehead dual to each other. Then we obtain the composites

$$\xi_n^k \colon \Sigma^r P_k \xrightarrow{q} S^{r+2} \xrightarrow{\Sigma^{r-2}\xi_2} \Sigma^{r-1} P_2 \xrightarrow{\chi_n^2} \Sigma^{r-1} P_n, r \geq 2,$$

$$\eta_n^k \colon \Sigma^r P_k \xrightarrow{\chi_2^k} \Sigma^r P_2 \xrightarrow{\Sigma^{r-3}\eta_2} S^r \xrightarrow{i} \Sigma^{r-1} P_n, r \geq 3,$$

and we set $\xi_n = \chi_n^2(\Sigma^{r-2}\xi_2)$ and $\eta^k = (\Sigma^{r-3}\eta_2)\chi_2^k$. Here $\chi_n^k = B_r(\chi)$ is given by the canonical generator $\chi = \chi_n^k \in \mathrm{Hom}(\mathbb{Z}/k, \mathbb{Z}/n)$; see Corollary 1.4.6. We also write $\varepsilon_n^k = \varepsilon_n^k(r)$, $\xi_n^k = \xi_n^k(r)$, and $\eta_n^k = \eta_n^k(r)$ if we want to specify the dimension r. Hence $\varepsilon_n^k(r)$ and $\xi_n^k(r)$ are just suspensions of the corresponding elements in Theorem 11.5.9 and (11.5.16). The element $\eta_n^k(r)$, $r \geq 3$, however, is a new type of element which is Spanier–Whitehead dual to ξ_k^n. If k or n is odd we have $[\Sigma^3 P_k, \Sigma^2 P_n] = 0$ for $k, n \geq 0$. If $n = 0$ we get the Spanier–Whitehead dual of $[S^4, \Sigma^2 P_k]$ in Theorem 11.5.9 given by

$$(11.6.4) \qquad [\Sigma^3 P_k, S^3] = \begin{cases} (\mathbb{Z}/4)\eta \text{ with } 2\eta = \varepsilon & k = 2 \\ (\mathbb{Z}/2)\eta \oplus (\mathbb{Z}/2)\varepsilon & k = 2^t \geq 4 \end{cases}$$

where $\eta = \eta^k$ and $\varepsilon = \varepsilon^k$. Moreover the suspension

$$\Sigma \colon [\Sigma^2 P_k, S^2] = \mathbb{Z}/(k,2)\eta_2^2 q \to [\Sigma^3 P_k, S^3]$$

carries the generator $\eta_2^2 q$ to ε^k and hence η^k is not in the image of Σ. Moreover we get

(11.6.5) Theorem *Let k and n be powers of 2. Then $\xi = \xi_n^k$, $\varepsilon = \varepsilon_n^k$, $\eta = \eta_n^k$ satisfy*

$$[\Sigma^3 P_k, \Sigma^2 P_n]$$

$$= \begin{cases} (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/2)\eta & \text{with } \varepsilon = 0, \quad (k,n) = (2,2) \\ (\mathbb{Z}/4)\xi \oplus (\mathbb{Z}/2)\eta & \text{with } \varepsilon = 2\xi, \quad (k,n) = (\geq 4, 2) \\ (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/4)\eta & \text{with } \varepsilon = 2\eta, \quad (k,n) = (2, \geq 4) \\ (\mathbb{Z}/2)\xi \oplus (\mathbb{Z}/2) \oplus (\mathbb{Z}/2)\varepsilon, & \quad (k,n) = (\geq 4, \geq 4). \end{cases}$$

(11.6.6) Theorem *Let k and n be powers of 2. Then the suspension $\Sigma:[\Sigma^2 P_k, \Sigma P_n] \to [\Sigma^3 P_k, \Sigma^2 P_n]$ carries ε_n^k to ε_n^k and ξ_n^k to ξ_n^k and satisfies*

$$\Sigma(\gamma_n^k) = \begin{cases} 0 & k \le n \\ \eta_2^4 + \delta\varepsilon_2^4 & (k,n) = (4,2), \delta \in \{0,1\} \\ \eta_n^k & \text{otherwise.} \end{cases}$$

We do not know whether $\delta = 0$ or $\delta = 1$ for $(k,n) = (4,2)$. Hence for $k > n$ the suspension is surjective while for $k \le n$ the suspension in Theorem 11.6.6 is not surjective.

Proof of Theorems 11.6.5 and 11.6.6 The definition of η_n^k shows that the diagram

$$\begin{tikzcd} \Sigma^3 P_k \arrow[r, "\eta_n^k"] \arrow[d, "i"'] & \Sigma^2 P_n \arrow[d, "i"] \\ S^4 \arrow[r, "\eta_3"] & S^3 \end{tikzcd}$$

homotopy commutes where η_3 is the Hopf element. This follows by duality from $q\xi_2 = \eta_3$ in Theorem 11.5.13. Hence the operator

$$\mu: \left[\Sigma^3 P_k, \Sigma^2 P_n\right] \twoheadrightarrow \mathrm{Hom}(\mathbb{Z}/k, \pi_4 \Sigma^2 P_n) = \mathbb{Z}/k * (\mathbb{Z}/n \otimes \mathbb{Z}/2)$$

carries η_n^k to the generator. Hence using stability, Theorem 11.6.5 is a consequence of Theorem 8.2.10; compare also Theorem 9.2.7. Next we prove the formula for $\Sigma\gamma_n^k$ in Theorem 11.6.6. For $k \le n$ we obtain $\Sigma\gamma_n^k = 0$ by Lemma 11.5.19. We now consider $\Sigma\gamma_n^{2n}$. Here $\Sigma\gamma_n^{2n}$ is a principal map associated with

$$\begin{tikzcd} S^4 \arrow[r, "0"] \arrow[d, "2n"'] & S^3 \arrow[d, "n"] \\ S^4 \arrow[r, "\eta_3"] & S^3 \end{tikzcd}$$

This implies that $\Sigma\gamma_n^{2n} = \eta_n^{2n} + \delta_n \varepsilon_n^{2n}$ with $\delta_n \in \{0,1\}$. Since $(\chi_{2n}^k)^* \varepsilon_n^{2n} = 0$ for $k > 2n$ we hence obtain $\Sigma\gamma_n^k = \eta_n^k$ for $k > 2n$. Next we get

$$\begin{cases} \chi_2^n \eta_n^{2n} = \eta_2^{2n}, \quad \chi_2^n \varepsilon_n^{2n} = \varepsilon_2^{2n}, \\ \chi_2^n \gamma_n^{2n} = \gamma_2^{2n} \end{cases}$$

by 11.5.24. This implies for $n > 2$

$$\begin{aligned} \delta_n \varepsilon_2^{2n} &= (\chi_2^n)_*(\Sigma\gamma_n^{2n} - \eta_n^{2n}) \\ &= \Sigma\gamma_2^{2n} - \eta_2^{2n} = 0 \end{aligned}$$

so that $\delta_n = 0$ for $n > 2$. □

12

ON THE HOMOTOPY CLASSIFICATION OF SIMPLY CONNECTED 5-DIMENSIONAL POLYHEDRA

The classical result of J.H.C. Whitehead on simply connected 4-dimensional homotopy types relies on the computation of the group $\Gamma_3 X$ which fortunately has the simple description

$$\Gamma_3 X = \Gamma(\pi_2 X)$$

in terms of the Γ-functor. A homomorphism $\Gamma(A) \to B$ is given by a quadratic function $\eta: A \to B$ which is the algebraic equivalent of a simply connected 3-type, denoted by $K(\eta, 2)$. The homotopy classes of maps

$$K(\eta, 2) \to K(\eta', 2),$$

however, do not coincide with the obvious algebraic maps $\eta \to \eta'$ between quadratic functions in the category $\mathbf{\Gamma Ab}$. In fact, the homotopy category $\mathbf{types}_2^1$ of simply connected 3-types is a complicated linear extension of the category $\mathbf{\Gamma Ab}$. We therefore introduce the diagram of functors

$$\begin{tikzcd} \mathbf{types}_2^1 \arrow[rr, "G"] \arrow[dr] & & \mathbf{\Gamma Ab(C)} \arrow[dl] \\ & \mathbf{\Gamma Ab} & \end{tikzcd}$$

where $\mathbf{\Gamma Ab(C)}$ is an algebraic category which via G is a better approximation of the category $\mathbf{types}_2^1$ than $\mathbf{\Gamma Ab}$. Our classification of simply connected 5-dimensional homotopy types X relies on the computation of the group

$$\Gamma_4(X) = \Gamma_4 K(\eta, 2) = \bar{\Gamma}_4(\eta)$$

as a functor in X or in $\eta = \eta_X \in \mathbf{\Gamma Ab(C)}$. The algebra needed to describe the functor $\bar{\Gamma}_4$ is somewhat bizarre. Given the functor $\bar{\Gamma}_4$ and also the bifunctor $\bar{\Gamma}_3$ with

$$\Gamma_3(H, K(\eta, 2)) = \bar{\Gamma}_3(H, \eta)$$

we are able to describe algebraic models of simply connected 5-dimensional homotopy types X for which $H_2 X$ is finitely generated. We also consider such homotopy types for which $H_2 X$ is uniquely 2-divisible or free abelian.

12.1 The groups $G(q, A)$

Recall that $P_q = S^1 \cup_q e^2$ is the pseudo-projective plane of degree q which yields the Moore space $M(\mathbb{Z}/q, n) = \Sigma^{n-1}P_q, n \geq 2$. Let X be a space. In this section and in Section 12.2 we consider the homotopy group

(12.1.1) $$\pi_n(\mathbb{Z}/q, X) = \left[\Sigma^{n-1}P_q, X\right]$$

with coefficients in the cyclic group $\mathbb{Z}/q$. Let ${}^f\mathbf{Cyc}$ be the full subcategory of **Ab** consisting of finite cyclic groups. Then (12.1.1) yields the functor

$$\pi_n(-, X): {}^f\mathbf{Cyc}^{\mathrm{op}} \to \mathbf{Gr} \tag{1}$$

which carries $\mathbb{Z}/q$ to the group (12.1.1) and which carries a homomorphism $\varphi: \mathbb{Z}/q \to \mathbb{Z}/t$ to the induced homomorphism

$$\varphi^* = (B_n\varphi)^*: \pi_n(\mathbb{Z}/t, X) \to \pi_n(\mathbb{Z}/q, X) \tag{2}$$

given by the functor B_n in Corollary 1.4.6. The group $\pi_2(\mathbb{Z}/q, X)$ in general is not abelian; the universal coefficient sequence, however, yields the central extension of groups ($n \geq 2$)

(12.1.2) $$\mathrm{Ext}(\mathbb{Z}/q, \pi_{n+1}X) \overset{\Delta}{\rightarrowtail} \pi_n(\mathbb{Z}/q, X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(\mathbb{Z}/q, \pi_n X)$$

which is natural in $\mathbb{Z}/q$. We use the following notation. For a small category $\mathbf{C}$ a $\mathbf{C}$-group is the same as the functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Gr}$. Let $\mathbf{C}$-groups be the category of such functors; morphism are natural transformations. Hence (12.1.2) is a central extension of ${}^f\mathbf{Cyc}$-groups.

Let $S^n \xrightarrow{i_n} \Sigma^{n-1}P_q \xrightarrow{q_{n+1}} S^{n+1}$ be the inclusion and pinch map respectively. Then Δ in (12.1.2) carries $1 \otimes x \in \mathbb{Z}/q \otimes \pi_{n+1}X = \mathrm{Ext}(\mathbb{Z}/q, \pi_{n+1}X)$ to $\Delta(1 \otimes x) = q_{n+1}^*(x)$. Moreover μ in (12.1.2) carries an element $y \in \pi_n(\mathbb{Z}/q, X)$ to $(i_n)^*y \in \mathbb{Z}/q * \pi_n X = \mathrm{Hom}(\mathbb{Z}/q, \pi_n X)$. Commutators in the group $\pi_2(\mathbb{Z}/q, Z)$ satisfy the following rule

(12.1.3) Proposition *For $x, y \in \pi_2(\mathbb{Z}/q, X)$ we have the formula*

$$-x - y + x + y = (q(q-1)/2)q_3^*[i_2^*, i_2^*y]$$

where $[-, -]: \pi_2 X \otimes \pi_2 X \to \pi_3 X$ denotes the Whitehead product.

This result is originally due to Barratt [TG].

Proof of Proposition 12.1.3 For pointed CW-complexes A, B let $A \wedge B = A \times B / A \vee B$ be the smash product and let $\Delta: A \to A \wedge A$ be the reduced diagonal. The commutator satisfies the formula

$$-x - y + x + y = (\Sigma\Delta)^*[x, y]$$

where $[x, y] \in [\Sigma P_q \wedge P_q, X]$ is the (generalized) Whitehead product; compare Baues [CC]. By (III.D.20) in Baues [CH] we know that the reduced diagonal Δ of P_q is part of the homotopy commutative diagram

$$\begin{array}{ccc} P_q & \xrightarrow{\Delta} & P_q \wedge P_q \\ {\scriptstyle q_2}\big\downarrow & & \big\uparrow{\scriptstyle i_1 \wedge i_1} \\ S^2 & \xrightarrow{\upsilon} & S^1 \wedge S^1 = S^2 \end{array}$$

Here q_2 is the pinch map and i_1 is the inclusion and υ is a map of degree $q(q-1)/2$. This yields the result. □

We now introduce a purely algebraic construction of a group denoted by $G(q, A)$. For this we use the properties of Whitehead's functor Γ in Section 1.2. The topological meaning of the group $G(q, A)$ is described in Theorem 12.1.6 below.

(12.1.4) Definition Let A be an abelian group. We have the natural homomorphism between $\mathbb{Z}/2$-vector spaces

$$H: \Gamma(A) \otimes \mathbb{Z}/2 = \Gamma(A \otimes \mathbb{Z}/2) \otimes \mathbb{Z}/2 \to \otimes^2(A \otimes \mathbb{Z}/2) \tag{1}$$

with $H(\gamma(a) \otimes 1) = (a \otimes 1) \otimes (a \otimes 1)$. This homomorphism is injective and hence admits a *retraction homomorphism*

$$r: \otimes^2(A \otimes \mathbb{Z}/2) \to \Gamma(A) \otimes \mathbb{Z}/2 \tag{2}$$

with $rH = id$. For example, given a basis E of the $\mathbb{Z}/2$-vector space $A \otimes \mathbb{Z}/2$ and a well ordering $<$ on E we can define a retraction r on basis elements by the formula $(b, b' \in E)$

$$r(b \otimes b') = \begin{cases} \gamma(b) \otimes 1 & \text{for } b = b' \\ [b, b'] \otimes 1 & \text{for } b > b' \\ 0 & \text{for } b < b'. \end{cases} \tag{3}$$

Now let $q \geq 1$ and let

$$j_A: \mathrm{Hom}(\mathbb{Z}/q, A) = A * \mathbb{Z}/q \subset A \overset{p}{\twoheadrightarrow} A \otimes \mathbb{Z}/2 \tag{4}$$

be given by the projection p with $p(x) = x \otimes 1$. Also let

$$p_A: \Gamma(A) \otimes \mathbb{Z}/2 \overset{p}{\twoheadrightarrow} \Gamma(A) \otimes \mathbb{Z}/2 \otimes \mathbb{Z}/q = \mathrm{Ext}(\mathbb{Z}/2 \otimes \mathbb{Z}/q, \Gamma(A))$$
$$\xrightarrow{p^*} \mathrm{Ext}(\mathbb{Z}/q, \Gamma(A)) \tag{5}$$

be defined by the indicated projections p. Then we obtain the homomorphism

$$\begin{cases} \Delta_r\colon \operatorname{Hom}(\mathbb{Z}/q, A) \otimes \operatorname{Hom}(\mathbb{Z}/q, A) \to \operatorname{Ext}(\mathbb{Z}/q, \Gamma A) \\ \Delta_r = p_A r(j_A \otimes j_A) \end{cases} \tag{6}$$

which depends on the choice of the retraction r in (2). Clearly Δ_r is not natural in A since r cannot be chosen to be natural. However one can easily check that Δ_r is natural for homomorphisms $\varphi\colon \mathbb{Z}/q \to \mathbb{Z}/t$ between cyclic groups, that is

$$\Delta_r(\varphi^* \otimes \varphi^*) = \varphi^* \Delta_r. \tag{7}$$

We now define a group

$$G_r(q, A) = \operatorname{Hom}(\mathbb{Z}/q, A) \times \operatorname{Ext}(\mathbb{Z}/q, \Gamma(A)) \tag{8}$$

where the group law on the right-hand side is given by the *cocycle* Δ_r, that is

$$(a, b) + (a', b') = (a + a', b + b' + \Delta_r(a \otimes a')). \tag{9}$$

This yields a functor

$$G_r(-, A)\colon {}^f\mathbf{Cyc}^{\mathrm{op}} \to \mathbf{Gr} \tag{10}$$

which carries Z/q to $G_r(q, A)$. For $\varphi\colon \mathbb{Z}/q \to \mathbb{Z}/t$ we define $\varphi^*\colon G_r(t, A) \to G_r(q, A)$ by $\varphi^* = \operatorname{Hom}(\varphi, A) \times \operatorname{Ext}(\varphi, \Gamma(A))$. It is clear that φ^* is a group homomorphism. In addition we get a central extension

$$\operatorname{Ext}(\mathbb{Z}/q, \Gamma(A)) \xrightarrow{\Delta} G_r(q, A) \overset{\mu}{\twoheadrightarrow} \operatorname{Hom}(\mathbb{Z}/q, A) \tag{11}$$

with $\Delta(b) = (0, b)$ and $\mu(a, b) = a$. This extension is natural in $\mathbb{Z}/q$ so that $G_r(-, A)$ is a central extension in the functor category ${}^f\mathbf{Cyc}$-groups in the same way as in (12.1.2). The next result shows that the extension $G_r(-, A)$ does not depend on the choice of the retraction r.

(12.1.5) Lemma *For two retractions r_1, r_2 in Definition* 12.1.4(2) *there is an isomorphism of groups*

$$\bar{\chi}\colon G_{r_1}(q, A) \cong G_{r_2}(q, A)$$

which is natural in $\mathbb{Z}/q$ and which is compatible with Δ and μ, that is $\bar{\chi}\Delta = \Delta$ and $\mu\chi = \mu$. We therefore omit r and write $G(q, A) = G_r(q, A)$.

The lemma is also a consequence of Theorem 12.1.6 below. Since the proof of this theorem is rather sophisticated we first give an independent algebraic proof of the lemma.

Proof of Lemma 12.1.5 Let $v: \otimes^2(A \otimes \mathbb{Z}/2) \to \Lambda^2(A \otimes \mathbb{Z}/2)$ be the quotient map for the exterior square. Then the retractions r_1, r_2 yield a unique homomorphism

$$m: \Lambda^2(A \otimes \mathbb{Z}/2) \to \Gamma(A) \otimes \mathbb{Z}/2 \quad \text{with} \quad mv = r_2 - r_1. \tag{1}$$

Let the homomorphism

$$\delta: \Gamma(A \otimes \mathbb{Z}/2) \to \Lambda^2(A \otimes \mathbb{Z}/2) \tag{2}$$

be defined on a $\mathbb{Z}/2$-basis E of $A \otimes \mathbb{Z}/2$, namely $\delta\gamma e = 0$ and $\delta[e, e'] = e \wedge e'$ for $e, e' \in E$. One readily checks that δ is well defined and that

$$\delta[a, b] = a \wedge b \quad \text{for all} \quad a, b \in A \otimes \mathbb{Z}/2.$$

We obtain by δ the quadratic function

$$\delta_A = p_A m \delta\gamma j_A: \mathrm{Hom}(\mathbb{Z}/q, A) \to \mathrm{Ext}(\mathbb{Z}/q, \Gamma(A)) \tag{3}$$

where we use p_A and j_A in Definition 12.1.4. We again observe similarly as in Definition 12.1.4(7) that δ_A is actually natural in $\mathbb{Z}/q$. We use δ_A for the definition of the isomorphism $\bar{\chi}$ in Lemma 12.1.5. We define the bijection

$$\bar{\chi}: \mathrm{Hom}(\mathbb{Z}/q, A) \times \mathrm{Ext}(\mathbb{Z}/q, \Gamma A) \to \mathrm{Hom}(\mathbb{Z}/q, A) \times \mathrm{Ext}(\mathbb{Z}/q, \Gamma A) \tag{4}$$

by $\bar{\chi}(a, b) = (a, b + \delta_A(a))$. Thus also $\bar{\chi}$ is natural in $\mathbb{Z}/q$ and $\bar{\chi}$ is an isomorphism of groups (see Definition 12.1.4(9)) since we have

$$\begin{aligned}
\Delta_{r_1}(a, a') + \delta_A(a + a') &= p_A r_1(j_A a \otimes j_A a') + p_A m \delta\gamma(j_A a + j_A a') \\
&= p_A(r_1 j_A a \otimes j_A a' + m\delta[j_A a, j_A a']) + \delta_A(a) + \delta_A(a') \\
&= p_A(r_1 + mv)(j_A a \otimes j_A a') + \delta_A(a) + \delta_A(a') \\
&= (p_A r_2 j_A \otimes j_A)(a \otimes a') + \delta_A(a) + \delta_A(a') \\
&= \Delta_{r_2}(a, a') + \delta_A(a) + \delta_A(a').
\end{aligned}$$

This completes the proof of Lemma 12.1.5. □

The next result shows that the algebraically defined group $G(q, A)$ in Lemma 12.1.5 is isomorphic to a homotopy group of a Moore space.

(12.1.6) Theorem *There is an isomorphism of groups*

$$\chi: G(q, A) \cong \pi_2(\mathbb{Z}/q, M(A, 2))$$

which is natural in $\mathbb{Z}/q$ and which is compatible with Δ and μ, that is $\chi\Delta = \Delta$ and $\mu\chi = \mu$.

Thus we can choose for each abelian group A a retraction r and an isomorphism χ as in the theorem. We will use this isomorphism as an identification. The group $G(q,A)$ in the theorem yields a purely algebraic description of the homotopy group $\pi_2(\mathbb{Z}/q, M(A,2))$. The suspension $\Sigma^{n-2}, n \geq 3$, induces a push-out diagram of groups

$$\begin{tikzcd} \mathrm{Ext}(\mathbb{Z}/q, \Gamma(A)) \arrow[r, tail, "\Delta"] \arrow[d, "\sigma_*"] & \pi_2(\mathbb{Z}/q, M(A,2)) \arrow[r, two heads, "\mu"] \arrow[d, "\Sigma^{n-2}"] & \mathrm{Hom}(\mathbb{Z}/q, A) \arrow[d, equal] \\ \mathrm{Ext}(\mathbb{Z}/q, A \otimes \mathbb{Z}/2) \arrow[r, tail, "\Delta"] & \pi_n(\mathbb{Z}/q, M(A,n)) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(\mathbb{Z}/q, A) \end{tikzcd} \tag{$*$}$$

Hence we can apply the theorem also for the computation of the group $\pi_n(\mathbb{Z}/q, M(A, n)), n \geq 3$. In particular, a cocycle for the group $\pi_n(\mathbb{Z}/q, M(A,n))$ is $\sigma_* \Delta_r$. The proof of Theorem 12.1.6 has several parts. We first consider commutators.

(12.1.7) Lemma *The commutator for $x, y \in G_r(q, A)$ satisfies the same formula as the commutator for $x, y \in \pi_2(\mathbb{Z}/q, M(A,2))$, namely*

$$-x-y+x+y = (q(q-1)/2)\Delta([\mu x, \mu y] \otimes 1).$$

Proof For $x = (a,b)$ and $y = (a',b')$ we get the commutator formula in $G_r(q, A)$ as follows:

$$\begin{aligned} -x-y+x+y &= \Delta(\Delta_r(a,a') - \Delta_r(a',a)) = \Delta p_A r(j_A \otimes j_A)(a \otimes a' + a' \otimes a) \\ &= \Delta p_A rH[j_A a, j_A a'] \\ &= \Delta p_A [j_A a, j_A a'] \\ &= (q(q-1)/2)\Delta([a,a'] \otimes 1). \end{aligned}$$

□

Next we consider a *splitting function* $s = s_q$ for μ:

(12.1.8) $$\pi_2(\mathbb{Z}/q, M(A,2)) \underset{s}{\overset{\mu}{\rightleftarrows}} \mathrm{Hom}(\mathbb{Z}/q, A),$$

that is $\mu s = id$. Such a splitting function yields the *cocycle*

$$\Delta_s\colon \mathrm{Hom}(\mathbb{Z}/q, A) \times \mathrm{Hom}(\mathbb{Z}/q, A) \to \mathrm{Ext}(\mathbb{Z}/q, \Gamma A)$$

with $\Delta_s(x,y) = \Delta^{-1}(s(x+y) - s(x) - s(y))$. We say that $s = s_q$ is *natural* in $\mathbb{Z}/q$ if for $\varphi \in \mathrm{Hom}(\mathbb{Z}/q, \mathbb{Z}/t)$ we have $s_q \varphi^* = B_2(\varphi)^* s_t$. Clearly for natural splitting functions also the cocycle Δ_s is natural in $\mathbb{Z}/q$.

(12.1.9) Proposition *Let A be a direct sum of cyclic groups. Then there exist splitting functions $s_q, q \geq 1$, which are natural in $\mathbb{Z}/q$.*

Proof Let $A = \oplus_i (\mathbb{Z}/a_i)\alpha_i$ with $i \in I, a_i \geq 0$, and let $<$ be an ordering of I. Then $x \in \mathrm{Hom}(\mathbb{Z}/q, A)$ is given by coordinates $x_i \in \mathrm{Hom}(\mathbb{Z}/q, \mathbb{Z}/a_i)$ with x_i non-trivial for only finitely many indices i. Hence we can define

$$s_q(x) = \sum_{i \in I}^{<} \alpha_i B_2(x_i)$$

where the sum is the ordered sum in the group $[\Sigma P_q, M(A,2)]$ and where $\alpha_i: M(\mathbb{Z}/a_i, 2) \to M(A,2)$ is the inclusion; see (1.5.2). We clearly have for $\varphi: \mathbb{Z}/t \to \mathbb{Z}/q$

$$\begin{aligned} B_2(\varphi)^* s_q(x) &= \sum_{i \in I} \alpha_i B_2(x_i) B_2(\varphi) \\ &= \sum_{i \in I} \alpha_i B_2(x_i \varphi) = s_q \varphi^*(x) \end{aligned}$$

where $(x_i \varphi) = (x\varphi)_i$ is the coordinate of $x\varphi$. In the first equation we use the fact that B_2 is a suspended map and in the second equation we use the functorial property of B_2. □

Proof of Theorem 12.1.6 We first assume that A is a direct sum of cyclic groups $(\mathbb{Z}/a_i)\alpha_i, i \in I$, as in the proof of Proposition (12.1.9). For the splitting function $s = s_q$ (which is natural in $\mathbb{Z}/q$) in this proof we get the cocycle Δ_s by

$$\begin{aligned} \Delta_s(x,y) &= s_q(x+y) - s_q(x) - s_q(y) \\ &= \left(\sum_i^{<} \alpha_i B_2(x_i + y_i) \right) - R \end{aligned}$$

with

$$\begin{aligned} R &= \sum_i^{<} \alpha_i B_2(x_i) + \sum_i^{<} \alpha_i B_2(y_i) \\ &= \sum_i^{<} (\alpha_i B_2(x_i) + \alpha_i B_2(y_i)) + q_0 \cdot \sum_{i>j} [n_i \alpha_i, m_j \alpha_j] \otimes 1. \end{aligned}$$

Here $q_0 = q(q-1)/2$ and $x = \Sigma_i n_i \alpha_i, y = \Sigma_j m_j \alpha_j$, that is $x_i(1) = n_i(1)$ and $y_j(1) = m_j \cdot 1$ with $n_i, m_j \in \mathbb{Z}$. Compare the commutator rule in Proposition 12.1.3. Since by Theorem 1.4.8

$$\alpha_i(B_2(x_i) + B_2(y_i)) = \alpha_i B_2(x_i + y_i) + q_0 n_i m_i \gamma(\alpha_i) \otimes 1$$

we thus get

$$\begin{aligned} \Delta_s(x,y) &= q_0 \left(\sum_{i>j} n_i m_j [\alpha_i, \alpha_j] + \sum_i n_i m_i \gamma(\alpha_1) \right) \otimes 1 \\ &= p_A r(j_A x \otimes j_A y) \end{aligned}$$

where r is the retraction (for the basis of $A \otimes \mathbb{Z}/2$ given by the elements $\alpha_i \otimes 1$). We can thus define the isomorphism

$$\chi: G_r(q, A) \cong \pi_2(\mathbb{Z}/q, M(A,2))$$

by $(x, b) \mapsto s_q(x) + \Delta(b)$. Next we consider the case when A is any abelian group. Then $A \otimes \mathbb{Z}/q$ is a bounded abelian group and hence a direct sum $A \otimes \mathbb{Z}/q = \oplus_i (\mathbb{Z}/a_i)\alpha_i$ of cyclic groups; compare for example Fuchs [I]. We now choose for the projection $p: A \to A \otimes \mathbb{Z}/q, q(\alpha) = \alpha \otimes 1$, a map $\bar{p}: M(A,2) \to M(A \otimes \mathbb{Z}/q, 2)$. This map yields the following pull-back diagram of abelian groups

$$\begin{array}{ccccc}
\operatorname{Ext}(\mathbb{Z}/q, \Gamma(A)) & \rightarrowtail & \pi_2(\mathbb{Z}/q, M(A,2)) & \xrightarrow{\mu} & \operatorname{Hom}(\mathbb{Z}/q, A) \\
\cong \downarrow \Gamma(p)_* & & \downarrow \bar{p}_* & & \downarrow p_* \\
\operatorname{Ext}(\mathbb{Z}/q, \Gamma(A \otimes \mathbb{Z}/q)) & \rightarrowtail & \pi_2(\mathbb{Z}/q, M(A \otimes \mathbb{Z}/q, 2)) & \xrightarrow{\mu} & \operatorname{Hom}(\mathbb{Z}/q, A \otimes \mathbb{Z}/q)
\end{array}$$

Here $\Gamma(p)_*$ is an isomorphism. Hence a splitting function s_q for $A \otimes \mathbb{Z}/q$ yields also a splitting function s_q for A. This splitting, however, depends on the choice of the basis in $A \otimes \mathbb{Z}/q$ above and hence we cannot use Proposition 12.1.9 for the naturality of s_q. For the naturality of an isomorphism

$$\chi = \chi_{(q)}: G_r(q, A) \cong \pi_2(\mathbb{Z}/q, M(A,2))$$

it is enough to consider powers $q = 2^t$ of 2. Using the basis of $A \otimes \mathbb{Z}/q$ above we obtain a retraction r_q and a splitting function s_q together with an isomorphism

$$\chi_q: G_{r_q}(q', A) \cong \pi_2(\mathbb{Z}/q', M(A,2)), \quad q' \leq q,$$

which is natural in $\mathbb{Z}/q'$ for $q' \leq q, q' = 2^{t'}$. We set $r = r_2$ and we define $\chi_{(q)}$ inductively as follows. For $q = 2$ we set $\chi_{(2)} = \chi_2$. Now assume $\chi_{(q')}$ is defined for $q' = 2^t \geq 2$. Then we obtain $\chi_{(q)}$ for $q = 2^{t+1}$ by the composition

$$\chi_{(q)} = \chi_q \bar{\chi}: G_r(q, A) \cong G_{r_q}(q, A) \cong \pi_2(\mathbb{Z}/q, M(A,2)).$$

Here $\bar{\chi}$ is the isomorphism given by $r_q - r$ as in the proof of Lemma 12.1.5. Hence $\bar{\chi}$ is natural in $\mathbb{Z}/q$ and $\chi_{(q)}$ is natural in $\mathbb{Z}/q'$ for $q' \leq q$. This completes the proof of Theorem 12.1.6. □

12.2 Homotopy groups with cyclic coefficients

We compute the homotopy groups $\pi_n(\mathbb{Z}/q, X)$ as functors in $\mathbb{Z}/q$. For this we need the following modification of the group $G(q, A)$ in Section 12.1.

(12.2.1) Definition Let $\eta: A \to B$ be a quadratic function which induces the homomorphism $\eta^\square: \Gamma(A) \to B$. Then the group $G(q,\eta)$ is defined by the product set

$$G(q,\eta) = \mathrm{Hom}(\mathbb{Z}/q, A) \times \mathrm{Ext}(\mathbb{Z}/q, B).$$

The group law

$$(a,b) + (a',b') = (a + a', b + b' + \eta^\square \Delta_r(a \otimes a'))$$

is given by the cocycle Δ_r in Definition 12.1.4. Since Δ_r is natural in $\mathbb{Z}/q$ we obtain in this way the functor

$$G(-,\eta): {}^f\mathbf{Cyc}^{\mathrm{op}} \to \mathbf{Gr}$$

which carries $\mathbb{Z}/q$ to $G(q,\eta)$. Moreover we have a central extension

$$\mathrm{Ext}(\mathbb{Z}/q, B) \overset{\Delta}{\rightarrowtail} G(\mathbb{Z}/q, \eta) \xrightarrow{\mu} \mathrm{Hom}(\mathbb{Z}/q, A)$$

of ${}^f\mathbf{Cyc}$-groups as in (12.1.2). Clearly the universal quadratic function $\gamma_A: A \to \Gamma(A)$ yields the group $G(q, A) = G(q, \gamma_A)$ in Definition 12.1.4.

(12.2.2) Theorem *For a space X in $\mathbf{Top}^*$ let $\eta = \eta_n^*: \pi_n X \to \pi_{n+1} X$ be the quadratic map induced by the Hopf map η_n, $n \geq 2$. Then one has an isomorphism of groups*

$$\pi_n(\mathbb{Z}/q, X) \cong G(q,\eta)$$

which is natural in $\mathbb{Z}/q \in {}^f\mathbf{Cyc}$. Moreover this is an isomorphism of central extensions compatible with Δ and μ.

For the proof of Theorem 12.2.2 we need the following notation on central push-outs.

(12.2.3) Definition The centre C of a group G is the subgroup of all elements $g \in G$ which commute with all other elements in G, that is $g \cdot x = x \cdot g$ for all $x \in G$. A homomorphism $\alpha: A \to G$ is *central* if A is abelian and if αA lies in the centre of G. A commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "c\text{-push}"] & G \arrow[d, "\bar\beta"] \\ B \arrow[r, "\bar\alpha"] & E \end{tikzcd}$$

of groups is a *central push out* if α and $\bar\alpha$ are central and if for any pair of homomorphisms

$$f: G \to L, \quad g: B \to L, \quad f\alpha = g\beta,$$

with g central, there is a unique homomorphism $\varphi: E \to G$ with $\varphi\bar{\beta} = f$ and $\varphi\bar{\alpha} = g$. We obtain $E = (G \times B)/\sim$ from the product group $G \times B$ by the equivalence relation $(x \cdot \alpha(a), y) \sim (x, \beta(a) \cdot y)$ with $x \in G, y \in B, a \in A$.

For example the diagram

$$\begin{array}{ccccc} \mathrm{Ext}(\mathbb{Z}/q, \Gamma(A)) & \overset{\Delta}{\rightarrowtail} & G(q, A) & \overset{\mu}{\twoheadrightarrow} & \mathrm{Hom}(\mathbb{Z}/q, A) \\ \downarrow{\scriptstyle (\eta^{\square})_*} & & \downarrow & & \| \\ \mathrm{Ext}(\mathbb{Z}/q, B) & \overset{\Delta}{\rightarrowtail} & G(q, \eta) & \xrightarrow{\mu} & \mathrm{Hom}(\mathbb{Z}/q, A) \end{array} \tag{12.2.4}$$

is a central push-out diagram.

Proof of Theorem 12.2.2 Let $f: Y \to X$ be the $(n-1)$-connected cover of X and for $A = \pi_n Y = \pi_n X$ let $g: M(A, n) \to Y$ be a map which induces an isomorphism $H_n(g) = \pi_n(g)$. Then we have the isomorphism

$$f_*: \pi_n(\mathbb{Z}/q, Y) \cong \pi_n(\mathbb{Z}/q, X)$$

and the push-out diagram $(B = \pi_{n+1})$

$$\begin{array}{ccccc} \mathrm{Ext}(\mathbb{Z}/q, \Gamma_n^1(A)) & \rightarrowtail & \pi_n(\mathbb{Z}/q, M(A, n)) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}/q, A) \\ \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle g_*} & & \| \\ \mathrm{Ext}(\mathbb{Z}/q, B) & \overset{\Delta}{\rightarrowtail} & \pi_n(\mathbb{Z}/q, Y) & \xrightarrow{\mu} & \mathrm{Hom}(\mathbb{Z}/q, A) \end{array}$$

This yields, via (12.2.4) and Theorem 12.1.6, the isomorphism in the proposition; see Theorem 12.1.6(*) for $n \geq 3$. Compare also the proof of Theorem 1.6.11 which, however, is only available for $n \geq 3$. □

Theorem 12.2.2 motivates the definition of the following algebraic category.

(12.2.5) Definition Let $\mathbf{C}$ be a full subcategory of ${}^f\mathbf{Cyc}$. We define the *$\mathbf{C}$-enriched category* $\Gamma\mathbf{Ab}(\mathbf{C})$ *of quadratic functions* as follows. Objects are quadratic functions which are also the objects in $\Gamma\mathbf{Ab}$; see (7.1.1) For quadratic functions $\eta: A \to B, \eta': A' \to B'$ a morphism

$$(\varphi_0, \varphi_1, F): \eta \to \eta'$$

in $\Gamma\mathbf{Ab}(\mathbf{C})$ is given by a morphism $(\varphi_0, \varphi_1): \eta \to \eta'$ in $\Gamma\mathbf{Ab}$ and by homomorphisms

$$F: G(\mathbb{Z}/q, \eta) \to G(\mathbb{Z}/q, \eta'), \quad \mathbb{Z}/q \in \mathbf{C},$$

for which the following diagram commutes and is natural in $\mathbb{Z}/q \in \mathbf{C}$.

$$\begin{tikzcd} \operatorname{Ext}(\mathbb{Z}/q, B) \arrow[r, tail, "\Delta"] \arrow[d, "(\varphi_1)_*"] & G(q,\eta) \arrow[r, two heads, "\mu"] \arrow[d, "F"] & \operatorname{Hom}(\mathbb{Z}/q, A) \arrow[d, "(\varphi_0)_*"] \\ \operatorname{Ext}(\mathbb{Z}/q, B') \arrow[r, tail, "\Delta"] & G(q,\eta') \arrow[r, two heads, "\mu"] & \operatorname{Hom}(\mathbb{Z}/q, A') \end{tikzcd}$$

Let $\mathbf{S\Gamma Ab(C)} = \mathbf{\Gamma Ab}_n\mathbf{(C)}$, $n \geq 3$, be the full subcategory of stable quadratic functions in $\mathbf{\Gamma Ab(C)} = \mathbf{\Gamma Ab}_2\mathbf{(C)}$. We obtain for $n \geq 2$ the functor

$$G = G_n \colon \mathbf{Top}^* \to \mathbf{\Gamma Ab}_n(\mathbf{C}). \tag{12.2.6}$$

The functor G_n carries a space X to the quadratic function

$$\eta_X = (\eta_n)^* \colon \pi_n(X) \to \pi_{n+1}(X).$$

Moreover G_n carries a map $f\colon X \to Y$ to the proper morphism $(\pi_n f, \pi_{n+1} f, F)\colon \eta_X \to \eta_Y$ given by

$$F\colon G(q,\eta_X) \cong \pi_n(\mathbb{Z}/q, X) \xrightarrow{f_*} \pi_n(\mathbb{Z}/q, Y) \cong G(q, \eta_Y)$$

with $\mathbb{Z}/q \in \mathbf{C}$. Here the isomorphisms are obtained by Theorem 12.2.2. Theorem 12.2.2 shows that G_n is a well-defined functor. The enriched category $\mathbf{\Gamma Ab}_n(\mathbf{C})$ is part of a linear extension of categories

$$N \overset{+}{\rightarrowtail} \mathbf{\Gamma Ab}_n(\mathbf{C}) \xrightarrow{\phi} \mathbf{\Gamma Ab}_n. \tag{12.2.7}$$

Here ϕ is the forgetful functor which is the identity on objects and which carries the morphism $(\varphi_0, \varphi_1, F)$ to (φ_0, φ_1). The natural system N is the bimodule on $\mathbf{\Gamma Ab}_n$ given by

$$N(\eta, \eta') = \operatorname{Nat}_{\mathbf{C}}(\operatorname{Hom}(-, A), \operatorname{Ext}(-, B'))$$

where $\operatorname{Hom}(-, A), \operatorname{Ext}(-, B')\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Gr}$ are abelian $\mathbf{C}$-groups and where Nat denotes the abelian group of natural transformations. For an element $x \in N(\eta, \eta')$ with

$$x\colon \operatorname{Hom}(\mathbb{Z}/q, A) \to \operatorname{Ext}(\mathbb{Z}/q, B'), \quad \mathbb{Z}/q \in \mathbf{C},$$

we obtain the action $+$ of N by

$$(\varphi_0, \varphi_1, F) + x = (\varphi_0, \varphi_1, F + \Delta x \mu).$$

Since $G(q, \eta)$ is part of a central extension we see that this is a well-defined action. We need the following natural transformation g between natural systems on $\mathbf{\Gamma Ab}_n$.

(12.2.8)

$$g\colon E(\eta, \eta') = \operatorname{Ext}(A, B') \to N(\eta, \eta') = \operatorname{Nat}_{\mathbf{C}}(\operatorname{Hom}(-, A), \operatorname{Ext}(-, B')).$$

The map g carries $x \in \operatorname{Ext}(A, B')$ to $g(x) = y \in N(\eta, \eta')$ where

$$\begin{cases} y\colon \operatorname{Hom}(\mathbb{Z}/q, A) \to \operatorname{Ext}(\mathbb{Z}/q, B'), & \mathbb{Z}/q \in \mathbf{C}, \\ \text{with } y(\varphi) = \varphi^*(x) \text{ for } \varphi^* = \operatorname{Ext}(\varphi, B'). \end{cases}$$

Compare (1.6.2). The next result on g is crucial for applications below.

(12.2.9) Lemma *Let A be a direct sum of cyclic groups in $\mathbf{C}$. Then g in* (12.2.8) *above is an isomorphism.*

This is a slight generalization of Lemma 1.6.3. We are now ready to compare the linear extension for $\mathbf{types}_n^1$ in (7.1.8) with the linear extension (12.2.7).

(12.2.10) Theorem *Let $n \geq 2$ and let $\mathbf{C}$ be a full subcategory of ${}^f\mathbf{Cyc}$. Then one has a map between linear extensions*

$$\begin{tikzcd} E \arrow[r, tail, "+"] \arrow[d, "g"] & \mathbf{types}_n^1 \arrow[r, "k_n"] \arrow[d, "G"] & \Gamma\mathbf{Ab}_n \arrow[d, equal] \\ N \arrow[r, tail, "+"] & \Gamma\mathbf{Ab}_n(\mathbf{C}) \arrow[r, "\phi"] & \Gamma\mathbf{Ab}_n \end{tikzcd}$$

Here g is the transformation in (12.2.8).

The functor G in the theorem is given by the functor G_n in (12.2.6).

Proof of Theorem 12.2.10 We have to show that the functor G is equivariant with respect to g, that is

$$G(f + \alpha) = G(f) + g(\alpha)$$

for $\theta \in \operatorname{Ext}(A, B')$ and $f \in [K(\eta, n), K(\eta', n)]$. For this we use the equivalence of categories

$$\mathbf{types}_n^1 = \Gamma\mathbf{M}^n$$

in Theorem 7.2.7. In fact, assume

$$f = \overline{\varphi} \in [M(A, 2), M(A', 2)] = [K(\gamma, 2), K(\gamma', 2)],$$

where γ, γ' are the universal quadratic maps. Then we have for $\alpha \in \operatorname{Ext}(A, \Gamma A')$

$$G(\overline{\varphi} + \alpha) = G(\overline{\varphi}) + g(\alpha)$$

This follows since for $\bar{x} \in G(q, A) = [\Sigma P_q, M(A,2)]$ we have the linear distributivity law in $\mathbf{M}^2$

$$(\bar{\varphi} + \alpha)\bar{x} = \bar{\varphi}\bar{x} + x^*(\alpha) = \bar{\varphi}\bar{x} + \Delta g(\alpha)\mu(\bar{x})$$

where $x = \mu\bar{x} = H_2\bar{x}$. Compare the proof of Theorem 1.6. 7. □

We immediately derive from Lemma 12.2.9 and Theorem 12.2.10 the following results:

(12.2.11) Theorem *The functor*

$$G\colon \mathbf{types}_n^1 \to \Gamma\mathbf{Ab}_n(\mathbf{C})$$

is full and faithful on the subcategory of all $K(\eta, n)$, $\eta\colon A \to B$, for which A is a direct sum of a free abelian group and of cyclic groups in $\mathbf{C}$.

(12.2.12) Corollary *Let $\eta\colon A \to B$ be a quadratic map where A is a direct sum of cyclic groups. Then the group of homotopy equivalences $\mathfrak{E}(K(\eta,2))$ is isomorphic to the group of automorphisms of η in the category $\Gamma\mathbf{Ab}({}^f\mathbf{Cyc})$. If $\eta = \gamma_A\colon A \to \Gamma A$ we have $\mathfrak{E}(K(\gamma_A,2)) = \mathfrak{E}(M(A,2))$.*

12.2A Appendix: Theories of cogroups and generalized homotopy groups

The functor

$$G_n\colon \mathbf{Top}^* \to \Gamma\mathbf{Ab}_n(\mathbf{C}), \tag{12.2A.1}$$

in (12.2.6) is a special case of a general concept of homotopy groups; see Theorem 12.2A.11 below. To see this we introduce the following general notation on theories. Recall that a *contravariant functor* F is the same as a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{K}$ where $\mathbf{C}^{\mathrm{op}}$ is the *opposite category* of $\mathbf{C}$.

(12.2A.2) Definition A *theory*, $\mathbf{T}$, is a small category with a zero object $*$ and with finite sums denoted by $X \vee Y$. Using $*$ we have zero morphisms $0\colon X \to * \to Y$ for all objects X, Y in $\mathbf{T}$. We say that $\mathbf{T}$ is a *single-sorted theory* generated by $X \in \mathrm{Ob}\mathbf{T}$ if all objects of $\mathbf{T}$ are finite sums of X. That is, any object Y of $\mathbf{T}$ is of the form

$$Y = \bigvee_{e \in E} X_e \quad \text{with} \quad X_e = X \quad \text{for} \quad e \in E$$

where E is a finite set. This is the zero object if E is empty. Single-sorted theories were introduced and studied by Lawvere [FS]. We need theories which are not single sorted; they are also considered in Barr and Wells [TT].

(12.2A.3) Definition Let **T** be a theory. A *cogroup* in **T** is an object X endowed with morphisms $\mu: X \to X \vee X, \nu: X \to X$ for which the following diagrams commute where $1 = 1_X$. The morphism μ is the *comultiplication* and ν is the *coinverse*.

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d, "\mu"] \arrow[dr, "1"] & \\ X & X \vee X \arrow[l, "(0,1)"] \arrow[r, "(1,0)"'] & X \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\mu"'] & X \vee X \arrow[d, "1 \vee \mu"] \\ X \vee X \arrow[r, "\mu \vee 1"'] & X \vee X \vee X \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} & X \arrow[dl, "0"'] \arrow[d, "\mu"] \arrow[dr, "0"] & \\ X & X \vee X \arrow[l, "(1,\nu)"] \arrow[r, "(\nu,1)"'] & X \end{tikzcd} \tag{3}$$

The cogroup (X, μ, ν) is *commutative* or *abelian* if the interchange map $T: X \vee X \to X \vee X$, with $Ti_1 = i_2$ and $Ti_2 = i_1$, satisfies $T\mu = \mu$ in **T**. We say that **T** is a *theory of cogroups* if each object in **T** is a cogroup and if the cogroup structure of a sum $X \vee Y$ is given by the composition

$$X \vee Y \xrightarrow{\mu \vee \mu} X \vee X \vee Y \vee Y = (X \vee Y) \vee (X \vee Y).$$

(12.2A.4) Definition Let **T** be a theory and let **Set*** be the category of pointed sets. A *model* G for the theory **T** is a functor

$$G: \mathbf{T}^{\mathrm{op}} \to \mathbf{Set^*} \tag{1}$$

which carries the zero object $*$ in **T** to the zero object $* = \{0\}$ in **Set*** and which carries sums in **T** to products in **Set***; that is,

$$G(X \vee Y) = G(X) \times G(Y) \tag{2}$$

is the product of sets where the inclusions $i_1: X \to X \vee Y$, $i_2: Y \to X \vee Y$ induce the projections $p_1 = G(i_1)$, $p_2 = G(i_2)$ of the product set $p_1(x, y) = x$, $p_2(x, y) = y$ for $x \in G(X)$, $y \in G(Y)$. Let **model(T)** be the category of such models for **T**. Morphisms between models are the natural transformations of the corresponding functors. One readily checks the

(12.2A.5) Lemma *Let* $\mathbf{T}$ *be a theory of cogroups and let* G *be a model for* $\mathbf{T}$. *Then* G *carries objects of* $\mathbf{T}$ *to groups. That is, for any object* X *in* $\mathbf{T}$ *the set* $G(X)$ *has the structure of a group which we write additively. For* $x, y \in G(X)$ *addition is given by* $x + y = \mu^*(x, y)$ *where we use* (2) *above. The negative of* x *is* $-x = \nu^*(x)$ *and the neutral element is the base point* $*$ *of* $G(X)$.

For example for each object Y in a theory $\mathbf{T}$ one gets the model

$$\begin{cases} \square_Y = \mathbf{T}(-,Y): \mathbf{T}^{\mathrm{op}} \to \mathbf{Set}^* & \text{with} \\ \square_Y(X) = \mathbf{T}(X,Y), & X \in Ob(\mathbf{T}). \end{cases}$$

The base point of the morphism set $\mathbf{T}(X,Y)$ is the zero map 0. The universal properties of sums show that $\square_Y$ satisfies Definition 12.2A.4(2). We call $\square_Y$ the *model presented by* Y. For any model $G \in \mathbf{model(T)}$ we have the *Yoneda lemma*

$$\mathbf{model(T)}(\square_Y, G) = G(Y), \qquad (*)$$

that is, the natural transformations $F: \square_Y \to G$ are in 1–1 correspondence with the elements $f \in G(Y)$. The correspondence carries F to $f = F(1_Y)$ with $1_Y \in \square_Y = \mathbf{T}(Y,Y)$. Moreover the Yoneda lemma shows that one has a faithful *Yoneda functor*

$$\square : \mathbf{T} \to \mathbf{model(T)}, \qquad (**)$$

which carries an object Y to $\square_Y = \mathbf{T}(-,Y)$ and which carries a morphism $g: Y \to Y'$ in $\mathbf{T}$ to the induced natural transformation $g_*: \mathbf{T}(-,Y) \to \mathbf{T}(-,T')$ given by $g_* f = g \circ f$ for $g \in \mathbf{T}(X,Y)$. Now let $\mathbf{T}$ be a theory of cogroups. Then $\mathbf{T}(X,Y)$ is a group. For a morphism $g: Y \to Y'$ the induced map

$$g_*: \mathbf{T}(X,Y) \to \mathbf{T}(X,Y'), \quad g_* h = gh, \qquad (1)$$

is a homomorphism between groups. For $f: X' \to X$ the induced map

$$f^*: \mathbf{T}(X,Y) \to \mathbf{T}(X',Y), \quad f^* h = hf, \qquad (2)$$

need not be a homomorphism. Here f^* is a homomorphism between groups for all Y if and only if the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\mu"] & X \arrow[d, "\mu"] \\ X' \vee X' \arrow[r, "f \vee f"'] & X \vee X \end{tikzcd} \qquad (3)$$

commutes. In this case we say that f is a *morphism of cogroups* in $\mathbf{T}$ or f is *linear*. In a similar way we see that for any model G the induced map

$f^*: G(X) \to G(X')$, in general, is not a homomorphism between groups; but f^* is a homomorphism if f is linear. Moreover we observe that for a morphism $\tau: G \to G'$ between models the induced map $\tau_X: G(X) \to G'(X)$ is always a homomorphism between groups.

We now are ready to define 'generalized homotopy groups'.

(12.2A.6) Definition Let $\mathfrak{X}$ be a class of spaces in $\mathbf{Top}^*$ and let $|\mathfrak{X}|$ be the full subcategory of the homotopy category $\mathbf{Top}^*/\simeq$ consisting of all finite one-points unions $X_1 \vee \cdots \vee X_r$ with $X_i \in \mathfrak{X}$ for $i = 1, \ldots, r$. Moreover let $\Sigma^n \mathfrak{X}$ be the class of n-fold suspensions $\Sigma^n \mathfrak{X} = \{\Sigma^n X; X \in \mathfrak{X}\}$. Then $[\Sigma^n \mathfrak{X}]$ is the homotopy category of all one-point unions

$$\Sigma^n(X_1 \vee \cdots \vee X_r) = \Sigma^n X_1 \vee \cdots \vee \Sigma^n X_r. \tag{1}$$

The one-point union is the sum in the category $[\Sigma^n \mathfrak{X}]$ for $n \geq 0$. Hence the category $[\Sigma^n \mathfrak{X}]$ is a theory, in fact, a theory of cogroups for $n \geq 1$ since the suspensions $\Sigma^n X$ are cogroups by the classical comultiplication $\Sigma^n X \to \Sigma^n X \vee \Sigma^n X$. Recall that $[X, Y]$ is the set of homotopy classes of base-point preserving maps $X \to Y$. The *generalized homotopy groups* or *$\mathfrak{X}$-homotopy groups* are the functors $(n \geq 0)$

$$\pi_n^{\mathfrak{X}}: \mathbf{Top}^*/\simeq \; \to \mathbf{model}[\Sigma^n \mathfrak{X}] \tag{2}$$

which carry a space X to the model $M_X = \pi_n^{\mathfrak{X}}(X)$ with

$$\begin{cases} M_X = [-, X]: [\Sigma^n \mathfrak{X}] \to \mathbf{Set}^* \quad \text{with} \\ M_X(\Sigma^n(X_1 \vee \cdots \vee X_r)) = [\Sigma^n(X_1 \vee \cdots \vee X_r), X]. \end{cases} \tag{3}$$

Clearly M_X carries a sum to a product. A map $f: X \to Y$ in $\mathbf{Top}^*/\simeq$ induces in the obvious way a natural transformation $f_*: M_X \to M_Y$.

(12.2A.7) Example Consider the class $\mathfrak{X} = \{S^0\}$ which consists only of the 0-sphere S^0. Then we have canonical isomorphisms of categories

$$\mathbf{model}[\{S^0\}] = \mathbf{Set}^*,$$

$$\mathbf{model}[\Sigma\{S^0\}] = \mathbf{Gr},$$

$$\mathbf{model}[\Sigma^n\{S^0\}] = \mathbf{Ab}, \quad n \geq 2,$$

where $\mathbf{Gr}$ and $\mathbf{Ab}$ are the categories of groups and abelian groups respectively. Moreover the functor $\pi_n^{\{S^0\}}$ can be identified with the classical homotopy groups

$$\pi_0^{\{S^0\}} = \pi_0: \mathbf{Top}^*/\simeq \; \to \mathbf{Set}^*,$$

$$\pi_1^{\{S^0\}} = \pi_1: \mathbf{Top}^*/\simeq \; \to \mathbf{Gr},$$

$$\pi_n^{\{S^0\}} = \pi_n: \mathbf{Top}^*/\simeq \; \to \mathbf{Ab}, \quad n \geq 2.$$

In this sense $\mathfrak{X}$-homotopy groups above generalize the well-known homotopy groups π_n with $\pi_n(X) = [\Sigma^n S^0, X]$.

(12.2A.8) Example For $\mathfrak{X} = \{S^1, S^2, \dots\}$ consisting of all spheres $S^n, n \geq 1$, the category $\mathbf{model}[\Sigma\,\mathfrak{X}]$ is the category of *π-modules* used by Stover [KS], Dwyer and Kan [πA]. Using the functor $\pi_n^{\mathfrak{X}}$ Stover describes a generalized Van Kampen theorem in terms of a spectral sequence; see also Artin and Mazur [VK] and Dreckmann [DH].

(12.2A.9) Example For $\mathfrak{X} = \{S^1_{\mathbb{Q}}, S^2_{\mathbb{Q}}, \dots\}$ consisting of all rational spheres $S^n_{\mathbb{Q}}, n \geq 1$, the category $\mathbf{model}(\Sigma\,\mathfrak{X})$ is equivalent to the category of graded rational Lie algebras concentrated in degree ≥ 1. This is the basic example for rational homotopy theory. In Baues [CC] we also consider $\mathbf{model}(\Sigma\,\mathfrak{X})$ for $\mathfrak{X} = \{S^1_R, \dots S^2_R, \dots\}$ consisting of all localized spheres $S^n_R, n \geq 1$, where $R \subset \mathbb{Q}$ is a subring with $\frac{1}{2}, \frac{1}{3} \in R$.

Let $\mathbf{C}$ be a full subcategory of $\mathbf{Cyc}$ and let

$$\mathfrak{P} = \mathfrak{P}(\mathbf{C}) = \left\{S^1, S^2, P_q; \mathbb{Z}/q \in \mathbf{C}\right\} \tag{12.2A.10}$$

be the class of spaces consisting of the 1-sphere, the 2-sphere, and the pseudo-projective planes P_q for which $\mathbb{Z}/q$ is an object in $\mathbf{C}$. In this case we get the following result which shows that the functor G_n in (12.2.6) can be identified with $\pi_{n-1}^{\mathfrak{P}}$.

(12.2A.11) Theorem *There is a canonical full inclusion of categories* ($n \geq 2$)

$$\Gamma\mathbf{Ab}_n(\mathbf{C}) \subset \mathbf{model}[\Sigma^{n-1}\,\mathfrak{P}]$$

such that the diagram of homotopy functors

$$\begin{array}{ccc} \mathbf{Top}^* & \xrightarrow{\pi_{n-1}^{\mathfrak{P}}} & \mathbf{model}[\Sigma^{n-1}\,\mathfrak{P}] \\ \| & & \cup \\ \mathbf{Top}^* & \xrightarrow{G_n} & \Gamma\mathbf{Ab}_n(\mathbf{C}) \end{array}$$

commutes.

Proof We observe by Theorem 12.2.11 that

$$G = G_n : [\Sigma^{n-1}\,\mathfrak{P}] \to \Gamma\mathbf{Ab}_n(\mathbf{C}) \tag{1}$$

is a full and faithful functor. Hence we obtain the functor

$$j\colon \mathbf{\Gamma Ab}_n(\mathbf{C}) \to \mathbf{model}[\Sigma^{n-1}\mathfrak{P}] \tag{2}$$

which carries η to the model

$$\begin{cases} M_\eta\colon [\Sigma^{n-1}\mathfrak{P}] \to \mathbf{Set}^* \\ M_\eta(Y) = [G(Y), \eta] = [Y, K(\eta, n)] \end{cases}$$

where $Y = \Sigma^{n-1}(X_1 \vee \cdots \vee X_r)$, $X_i \in \mathfrak{P}$, and where $[G(Y), \eta]$ is the set of morphisms $G(Y) \to \eta$ in $\mathbf{\Gamma Ab}_n(\mathbf{C})$. Since G in (1) carries sums to sums we see that M_n is a model and that the functor j in (2) is well defined. Now one can check that j is full and faithful and that the diagram in Theorem 12.2A.11 commutes. In fact, for $\eta\colon A \to B$ the model M_η satisfies

$$M_\eta(\Sigma^{n-1}S^1) = A,$$

$$M_\eta(\Sigma^{n-1}S^2) = B,$$

$$M_\eta\left(\Sigma^{n-1}P_q\right) = G(q, \eta). \qquad \square$$

(12.2A.12) Example Unsöld [AP] studied the generalized homotopy groups $\pi_n^{\mathfrak{X}}, n \geq 1$, for $\mathfrak{X} = \{S^2, S^3, S^4, \mathbb{C}P_2\}$ where $\mathbb{C}P_2$ is the *complex projective plane*. This leads to the classification of homotopy types of $(n-1)$-connected $(n+4)$-dimensional CW-complexes with torsion-free homology, $n \geq 3$.

12.3 The functor Γ_4

In Section 11.3 we computed $\Gamma_4(X)$ as an abelian group in terms of the quadratic function $\eta = \eta_X\colon \pi_2 X \to \pi_3 X$ induced by the Hopf map η_2. Here we describe the functorial properties of $\Gamma_4(X)$. For this it suffices to consider $X = K(\eta, 2)$ where $K(\eta, 2)$ is the 1-connected 3-type given by η. Let $\mathbf{PCyc}$ be the full subcategory of $\mathbf{Ab}$ consisting of cyclic groups $\mathbb{Z}/q$ where q is a power of a prime. We consider the diagram of functors

(12.3.1)
$$\begin{tikzcd} \mathbf{types}_2^1 \arrow[rr, "G"] \arrow[dr, "\Gamma_4"'] & & \mathbf{\Gamma Ab(PCyc)} \arrow[dl, "\bar{\Gamma}_4"] \\ & \mathbf{Ab} & \end{tikzcd}$$

Here $G = G_2$ is the functor in (12.2.8) which carries $K(\eta, 2)$ to η and Γ_4 is the functor in Whitehead's exact sequence.

(12.3.2) Theorem *There is a functor* $\bar{\Gamma}_4$ *in* (12.3.1) *together with a natural isomorphism* $\theta: \bar{\Gamma}_4 G \cong \Gamma_4$.

We shall describe the functor $\bar{\Gamma}_4$ purely algebraically by defining $\bar{\Gamma}_4(\eta)$ in terms of generators and relations. We have the short exact sequence

$$\Gamma_2^2(\eta) \xrightarrow{\Delta} \Gamma_4 K(\eta, 2) \overset{\mu}{\twoheadrightarrow} \Gamma T(A) \tag{12.3.3}$$

for $\eta: A \to B \in Ob(\Gamma\mathbf{Ab})$. The sequence is natural in $K(\eta,2) \in \mathbf{types}_2^1$; see Section 11.3. We first describe the torsion functor ΓT in terms of generators and relations, then we choose generators in $\Gamma_4 K(\eta, 2)$ which map via μ to generators in $\Gamma T(A)$. This leads to the definition of $\bar{\Gamma}_4$ below.

(12.3.4) Definition We define the functor

$$\bar{\Gamma}\bar{T}: \mathbf{Ab} \to \mathbf{Ab}$$

as follows. The group $\bar{\Gamma}\bar{T}$ is generated by the symbols

$$\begin{cases} \tilde{x} \otimes \xi_2, & \tilde{x} \in \mathrm{Hom}(\mathbb{Z}/2, A) \\ [\tilde{x}, \tilde{y}] \otimes \xi_{n,n}, & \tilde{x}, \tilde{y} \in \mathrm{Hom}(\mathbb{Z}/n, A) \end{cases} \tag{$*$}$$

with $\mathbb{Z}/n \in \mathbf{PCyc}$. Induced maps for $\varphi: A \to A' \in \mathbf{Ab}$ are defined by

$$\begin{cases} \varphi_*(\tilde{x} \otimes \xi_2) = (\varphi\tilde{x}) \otimes \xi_2 \\ \varphi_*([\tilde{x}, \tilde{y}] \otimes \xi_{n,n}) = [\varphi\tilde{x}, \varphi\tilde{y}] \otimes \xi_{n,n}. \end{cases} \tag{$**$}$$

The relations in question for the generators in $(*)$ are given by the following list (a)–(e):

(a) $[\tilde{x}, \tilde{x}] \otimes \xi_{n,n} = 0$;

(b) $[\tilde{x}, \tilde{x}] \otimes \xi_{n,n} = -[\tilde{y}, \tilde{x}] \otimes \xi_{n,n}$.
Let $\chi_n^k: \mathbb{Z}/k \to \mathbb{Z}/n$ be the canonical generator in $\mathrm{Hom}(\mathbb{Z}/k, \mathbb{Z}/n)$. Then:

(c) $[\tilde{x}, (\chi_n^{nm})^* \tilde{y}] \otimes \xi_{nm,nm} = [(\chi_{nm}^n)^* \tilde{x}, \tilde{y}] \otimes \xi_{n,n}$;

(d) $(\tilde{x} + \tilde{y}) \otimes \xi_2 = \tilde{x} \otimes \xi_2 + \tilde{y} \otimes \xi_2 + [\tilde{x}, \tilde{y}] \otimes \xi_{2,2}$;

(e) $[\tilde{x}, \tilde{y}] \otimes \xi_{n,n}$ is linear in $\tilde{x}$ and $\tilde{y}$.

(12.3.5) Lemma *There is a natural isomorphism*

$$\bar{\Gamma}\bar{T}(A) = \Gamma T(A).$$

Proof The isomorphism carries $\tilde{x} \otimes \xi_2$ to $\gamma(\tilde{x}) \in \Gamma(A * \mathbb{Z}/2)$ and carries $[\tilde{x}, \tilde{y}] \otimes \xi_{n,n}$ to $\tau_n(\tilde{x}, \tilde{y}) \in A * A$. Here we use the generators of $\Gamma T(A)$ in the proof of Theorem 6.2.7. □

We now use the isomorphism $G(n, \eta) = [\Sigma P_n, K(\eta, 2)]$ in Theorem 12.2.2 which is natural in $\mathbb{Z}/n \in$ **PCyc**. Moreover we use the diagram

(12.3.6)
$$\begin{tikzcd} \mathrm{Ext}(\mathbb{Z}/n, B) \arrow[r, tail, "\Delta"] & G(n,\eta) \arrow[r, two heads, "\mu"] \arrow[dr, "\mu"'] & \mathrm{Hom}(\mathbb{Z}/n, A) \arrow[d, tail] \\ B \arrow[u] \arrow[ur, "\Delta"] & & A \end{tikzcd}$$

where we identify $\mathrm{Ext}(\mathbb{Z}/n, B) = \mathbb{Z}/n \otimes B$ and $\mathrm{Hom}(\mathbb{Z}/n, A) = \mathbb{Z}/n * A \subset A$. The generators in Definition 12.3.4 correspond to the following composites which we consider as elements in $\Gamma_4 K(\eta, 2)$,

(12.3.7)
$$\begin{cases} x \otimes \xi_2 \colon S^4 \xrightarrow{\xi_2} \Sigma P_2 \xrightarrow{x} K(\eta, 2)^3 \\ [x, y] \otimes \xi_{n,n} \colon S^4 \xrightarrow{\xi_{n,n}} \Sigma P_n \wedge P_n \xrightarrow{[x,y]} K(\eta, 2)^3. \end{cases}$$

Here $[x, y]$ is the Whitehead product for $x, y \in G(n, \eta)$. The elements $\xi_2, \xi_{n,n}$ are the maps described in Section 11.5. One readily checks that μ in (12.3.3) satisfies

$$\begin{cases} \mu(x \otimes \xi_2) = \tilde{x} \otimes \xi_2 \\ \mu([x, y] \otimes \xi_{n,n}) = [\tilde{x}, \tilde{y}] \otimes \xi_{n,n} \end{cases}$$

where $\tilde{x} = \mu(x)$ is given by μ in (12.3.6). Now the exact sequence (12.3.3) and Lemma 12.3.5 show:

(12.3.8) Lemma *The abelian group $\Gamma_4 K(\eta, 2)$ is generated by elements $\Delta(\rho)$ with $\rho \in \Gamma_2^2(\eta)$ and the elements $x \otimes \xi_2$, $[x, y] \otimes \xi_{n,n}$ above.*

We want to describe the relations of the generators in Lemma 12.3.8. For this recall that

(12.3.9)
$$\Gamma_2^2(\eta) = (B \otimes \mathbb{Z}/2 \oplus B \otimes A)/\sim$$

is obtained by the equivalence relation in Definition 11.3.3. For $a \in A$, $b \in B$ we thus have the elements

$$\{b \otimes 1\}, \{b \otimes a\} \in \Gamma_2^2(\eta) \tag{1}$$

represented by $b \otimes 1 \in B \otimes \mathbb{Z}/2$ and $b \otimes a \in B \otimes A$ respectively. For $x, y \in G(n, \eta)$ we get $\tilde{x} = \mu(x), \tilde{y} = \mu(y) \in A$ by (12.3.6) and the quadratic function η yields

$$[\tilde{x}, \tilde{y}]_\eta = \eta(\tilde{x}, \tilde{y}) - \eta(\tilde{x}) - \eta(\tilde{y}) \in B \tag{3}$$

as in Definition 11.3.3(5). We now obtain the abelian group $\bar{\Gamma}_4(\eta)$ by describing generators and relations.

(12.3.10) Definition We define the algebraic functor $\bar{\Gamma}_4$ in (12.3.1) together with a natural short exact sequence

$$\Gamma_2^2(\eta) \xrightarrow{\Delta} \bar{\Gamma}_4(\eta) \overset{\mu}{\twoheadrightarrow} \Gamma T(A) \tag{$*$}$$

for $\eta\colon A \to B$. The group $\bar{\Gamma}_4(\eta)$ is generated by the symbols

$$\begin{cases} x \otimes \xi_2, & x \in G(2,\eta) \\ [x,y] \otimes \xi_{n,n}, & x,y \in G(n,\eta), \quad \mathbb{Z}/n \in \mathbf{PCyc} \\ \Delta(\rho), & \rho \in \Gamma_2^2(\eta). \end{cases} \tag{$**$}$$

Moreover Δ in $(*)$ carries ρ to $\Delta(\rho)$ and μ in $(*)$ is given by $(\mu\Delta(\rho) = 0$ and (12.3.7). A morphism

$$\chi = (\varphi_0, \varphi_1, F)\colon \eta \to \eta' \in \mathbf{\Gamma Ab(PCyc)}$$

as in Definition 12.2.5 induces the homomorphism

$$\bar{\Gamma}_4(\chi) = \chi_*\colon \bar{\Gamma}_4(\eta) \to \bar{\Gamma}_4(\eta')$$

defined on generators by

$$\begin{cases} \chi_*(x \otimes \xi_2) = (Fx) \otimes \xi_2 \\ \chi_*([x,y] \otimes \xi_{n,n}) = [Fx, Fy] \otimes \xi_{n,n} \\ \chi_*\Delta(\rho) = \Delta\Gamma_2^2(\varphi_0, \varphi_1)(\rho). \end{cases} \tag{$***$}$$

The relations in question for $\bar{\Gamma}_4(\eta)$ are given by the following list (a)–(f):

(a) $[x,x] \otimes \xi_{n,n} = 0$;

(b) $[x,y] \otimes \xi_{n,n} = -[y,x] \otimes \xi_{n,n}$;

(c) $[x,(\chi_n^{nm})^*y] \otimes \xi_{nm,nm} = [(\chi_{nm}^n)^*x, y] \otimes \xi_{n,n}$.
For this compare Definition 12.3.4(c) and the functorial properties of $G(n,\eta)$ in $\mathbb{Z}/n$ in Section 12.2.

(d) $(x+y) \otimes \xi_2 = x \otimes \xi_2 + y \otimes \xi_2 + [x,y] \otimes \xi_{2,2} + \Delta(\rho_1)$
where $\rho_1 = \{[\tilde{x},\tilde{y}]_\eta \otimes \tilde{y}\} \in \Gamma_2^2(\eta)$;

(e) $[x+y,z] \otimes \xi_{n,n} = [x,y] \otimes \xi_{n,n} + [y,z] \otimes \xi_{n,n} + \Delta(\rho_2)$
where $\rho_2 = 0$ if n is odd and $\rho_2 = (n/2)\{[\tilde{x},\tilde{y}]_\eta \otimes \tilde{y}\} \in \Gamma_2^2(\eta)$ if n is even;

(f) Δ in $(*)$ is an injective homomorphism and for $b \in B$ the element Δb given by (12.3.6) satisfies
$(\Delta b) \otimes \xi_2 = \Delta\{b \otimes 1\}$,
$[\Delta b, x] \otimes \xi_{n,n} = \Delta\{b \otimes \bar{x}\}$.

This completes the definition of the functor $\bar{\Gamma}_4$ in (12.3.1) and Theorem 12.3.2. The exactness of $(*)$ is a consequence of Lemma 12.3.5 since killing the elements $\Delta(\rho)$ in the relations above yields exactly the relations in Definition 12.3.4.

Proof of Theorem 12.3.2 We define the isomorphism

$$\theta\colon \bar{\Gamma}_4(\eta) = \Gamma_4 K(\eta, 2)$$

on generators in the obvious way by $\theta\Delta(\rho) = \Delta(\rho)$, see (12.3.3), $\theta(x \otimes \xi_2) = x \otimes \xi_2$, and $\theta[x, y] \otimes \xi_{n,n} = [x, y] \otimes \xi_{n,n}$, see (12.3.7). The formulas in Sections 11.5 and A.10 show that θ is a well-defined homomorphism compatible with Δ and μ; this shows that θ is an isomorphism since Definition 12.3.10$(*)$ is exact. In fact, θ is well defined since we use Lemma 11.5.15 for (a) and (b) and we use Lemma 11.5.24(b) for (c). Moreover (d) is obtained by Theorem A.10.2(b) since $[[x, y], y]T_{212}\xi_{2,2}$ corresponds to ρ_1; see Lemma 11.5.27. Next we get (e) by (11.5.10)(6), Lemma 11.5.27 since ρ_2 corresponds to $-[y, [x, z]]\Sigma(\Delta \wedge 1)\xi_{n,n} = [[x, z], y]\Sigma(\Delta \wedge 1)\xi_{n,n}$. Finally $q_*\xi_2 = \eta_3$ in (11.5.25) yields the first equation in (f) and the diagram in the proof of Lemma 11.5.27 yields the second equation in (f). □

12.4 The bifunctor Γ_3

In Section 11.3 we computed $\Gamma_3(H, X)$ as an abelian group in terms of the quadratic function $\eta = \eta_X\colon \pi_2 X \to \pi_3 X$ induced by the Hopf map η_2. Here we describe the functorial properties of $\Gamma_3(H, X)$ provided the homology $H_2 X$ is finitely generated. We proceed similarly as in Section 12.3. Again it suffices to consider $X = K(\eta, 2)$ where $K(\eta, 2)$ is the 1-connected 3-type given by η. Recall that we have the equivalence of categories

$$G\colon \mathbf{M}^3 = \mathbf{G}$$

where $\mathbf{M}^3$ is the homotopy category of Moore spaces in degree 3 and where $\mathbf{G}$ is the algebraic category in Theorem 1.6.7. We now consider the diagram of functors

$$\begin{tikzcd}[column sep=small] (\mathbf{M}^3)^{\mathrm{op}} \times \mathbf{types}_2^1 \arrow[rr, "G \times G"] \arrow[dr, "\Gamma_3"'] & & \mathbf{G}^{\mathrm{op}} \times \Gamma\mathbf{Ab}(\mathbf{PCyc}) \arrow[dl, "\Gamma_3"] \\ & \mathbf{Ab} & \end{tikzcd} \tag{12.4.1}$$

where we use the functor G in (12.3.1) which carries $K(\eta,2)$ to η. The functor Γ_3 carries $(M(H,3),K(\eta,2))$ to the abelian group $\Gamma_3(H,K(\eta,2))$ in Section 2.2.

(12.4.2) Theorem *There is a functor* $\bar{\Gamma}_3$ *in* (12.4.1) *together with a natural transformation* $\theta\colon \bar{\Gamma}_3(G\times G)\to\Gamma_3$ *such that*

$$\theta\colon \bar{\Gamma}_3(H,\eta)\to\Gamma_3(H,K(\eta,2))$$

is an isomorphism if the group A, *given by* $\eta\colon A\to B$, *is finitely generated.*

We shall describe the bifunctor $\bar{\Gamma}_3$ purely algebraically by defining $\bar{\Gamma}_3(H,\eta)$ in terms of generators and relations. For this we use the short exact sequence

$$\text{(12.4.3)}\qquad \operatorname{Ext}\big(H,\bar{\Gamma}_4(\eta)\big)\rightarrowtail^{\Delta}\Gamma_3(H,\,K(\eta,2))\overset{\mu}{\twoheadrightarrow}\operatorname{Hom}(H,\Gamma(A))$$

given by the universal coefficient sequence and by Theorem 12.3.2. The sequence is natural in $K(\eta,2)\in\mathbf{types}_2^1$ and $H\in\mathbf{G}=\mathbf{M}^3$. We first describe the bifunctor $(H,A)\mapsto\operatorname{Hom}(H,\Gamma(A))$ in terms of generators and relations, then we choose generators in $\Gamma_3(H,K(\eta,2))$ which map via μ to the generators in $\operatorname{Hom}(H,\Gamma(A))$. This leads to the definition of $\bar{\Gamma}_3$ below.

(12.4.4) Definition We define the bifunctor

$$\bar{\Gamma}\colon \mathbf{Ab}^{\mathrm{op}}\times\mathbf{Ab}\to\mathbf{Ab}$$

as follows. Let $\mathbf{PCyc}^0$ be the full subcategory of $\mathbf{Ab}$ consisting of $\mathbb{Z}=\mathbb{Z}/0$ and cyclic groups $\mathbb{Z}/q$ where q is a power of a prime. For $\mathbb{Z}/n$, $\mathbb{Z}/m$ in $\mathbf{PCyc}^0$ we have the homomorphisms

$$T\colon \mathbb{Z}/n\otimes\mathbb{Z}/m\cong\mathbb{Z}/m\otimes\mathbb{Z}/n$$

$$\Gamma(\mathbb{Z}/n)\xrightarrow{H}\mathbb{Z}/n\otimes\mathbb{Z}/n\xrightarrow{P}\Gamma(\mathbb{Z}/n)$$

where T is the interchange map and where H and $P=[1,1]$ are defined as in Section 1.2. The group $\bar{\Gamma}(H,A)$ is generated by the symbols

$$\begin{cases}\tilde{x}\otimes\gamma_n\otimes\tilde{a}\\ [\tilde{x},\tilde{y}]\otimes\eta_{n,m}\otimes\tilde{b}\end{cases}\qquad(*)$$

where $\tilde{x}\in\operatorname{Hom}(\mathbb{Z}/n,A)$ and $\tilde{a}\in\operatorname{Hom}(H,\Gamma(\mathbb{Z}/n))$, $\tilde{y}\in\operatorname{Hom}(\mathbb{Z}/m,A)$ and $\tilde{b}\in\operatorname{Hom}(H,\mathbb{Z}/n\otimes\mathbb{Z}/m)$ with $\mathbb{Z}/n,\mathbb{Z}/m\in\mathbf{PCyc}^0$. Induced maps are defined by

$$\begin{cases}\psi^*\varphi_*(\tilde{x}\otimes\gamma_n\otimes\tilde{a})=(\varphi\tilde{x})\otimes\gamma_n\otimes(\tilde{a}\psi)\\ \psi^*\varphi_*\big([\tilde{x},\tilde{y}]\otimes\eta_{n,m}\otimes\tilde{b}\big)=[\varphi\tilde{x},\varphi\tilde{y}]\otimes\eta_{n,m}\otimes\big(\tilde{b}\psi\big).\end{cases}\qquad(**)$$

The relations in question are:

(a) $[\tilde{x},\tilde{x}]\otimes\eta_{n,n}\otimes\tilde{b}=\tilde{x}\otimes\gamma_n\otimes(P\tilde{b})$;

(b) $[\tilde{x},\tilde{y}]\otimes\eta_{n,m}\otimes\tilde{b}=[\tilde{y},\tilde{x}]\otimes\eta_{m,n}\otimes(T\tilde{b})$;

(c) for the generator $\chi=\chi_n^k\in\mathrm{Hom}(\mathbb{Z}/k,\mathbb{Z}/n)$ we have

$$(\chi^*\tilde{x})\otimes\gamma_k\otimes\tilde{a}=\tilde{x}\otimes\gamma_n\otimes\Gamma(\chi)_*\tilde{a}$$

$$[\chi^*\tilde{x},\tilde{y}]\otimes\eta_{k,m}\otimes\tilde{b}=[\tilde{x},\tilde{y}]\otimes\eta_{n,m}\otimes(\chi\otimes 1)_*\tilde{b};$$

(d) $(\tilde{x}+\tilde{y})\otimes\gamma_n\otimes\tilde{a}=\tilde{x}\otimes\gamma_n\otimes\tilde{a}+\tilde{y}\otimes\gamma_n\otimes\tilde{a}+[\tilde{x},\tilde{y}]\otimes\eta_{n,n}\otimes(H\tilde{a})$;

(e) $[\tilde{x},\tilde{y}]\otimes\eta_{n,m}\otimes\tilde{b}$ is linear in $\tilde{x},\tilde{y}$;

(f) $\tilde{x}\otimes\gamma_n\otimes\tilde{a}$ and $[\tilde{x},\tilde{y}]\otimes\eta_{n,m}\otimes\tilde{b}$ are linear in $\tilde{a}$ and $\tilde{b}$ respectively.

(12.4.5) Lemma *There is a natural transformation*

$$\theta:\bar{\Gamma}(H,A)\to\mathrm{Hom}(H,\Gamma A)$$

which is an isomorphism if A is a finitely generated abelian group.

Proof We define θ by

$$\theta(\tilde{x}\otimes\gamma_n\otimes\tilde{a})=\Gamma(\tilde{x})\tilde{a}$$

$$\theta\big([\tilde{x},\tilde{y}]\otimes\eta_{n,m}\otimes\tilde{b}\big)=[\tilde{x},\tilde{y}]\tilde{b}$$

where $[\tilde{x},\tilde{y}]=[1,1](\tilde{x}\otimes\tilde{y})$ is defined as in (1.2.4). The proposition now is a very special case of 4.4. in Baues [QF]. □

We now use $G(n,\eta)$ in (12.3.6) and Theorem 12.2.2 and we set $G(0,\eta)=A$ for $n=0$. The generators in Definition 12.4.4 correspond to the following composites of maps which we consider as elements in $\Gamma_3(H,K(\eta,2))$; see (2.2.3)(2),

(12.4.6)

$$\begin{cases} x\otimes\gamma_n\otimes a\colon M(H,3)\xrightarrow{a}M(\Gamma(\mathbb{Z}/n),3)\xrightarrow{\gamma_n}\Sigma P_n\xrightarrow{x}K(\eta,2)^3\\ [x,y]\otimes\eta_{n,m}\otimes b\colon M(H,3)\xrightarrow{b}M(\mathbb{Z}/n\otimes\mathbb{Z}/m,3)\xrightarrow{\eta_{n,m}}\Sigma P_n\wedge P_m\xrightarrow{[x,y]}K(\eta,2)^3.\end{cases}$$

Here $[x,y]$ is the Whitehead product for $x\in G(n,\eta)$, $y\in G(m,\eta)$ and $\mathbb{Z}/n$, $\mathbb{Z}/m\in\mathbf{PCyc}^0$. The maps γ_n and $\eta_{n,m}$ are the generators in the corresponding elementary homotopy groups, see (11.5.16). For this we identify in the canonical way the groups $\Gamma(\mathbb{Z}/n)$ and $\mathbb{Z}/n\otimes\mathbb{Z}/m$ with the corresponding cyclic groups so that

$$M(\Gamma(\mathbb{Z}/n),3)=\Sigma^2P_k\quad\text{with}\quad k=(n^2,2n),\tag{1}$$

$$M(\mathbb{Z}/n\otimes\mathbb{Z}/m,3)=\Sigma^2P_k\quad\text{with}\quad k=(n,m).\tag{2}$$

Using these identifications we set $\gamma_n = \gamma_n^k$ and $\eta_{n,m} = \eta_{n,m}^k$. Clearly for $n = 0$ the map $\gamma_0\colon S^3 \to S^2$ is the Hopf map. One readily checks that μ in (12.4.3) carries the elements (12.3.7) to the corresponding elements in $\mathrm{Hom}(H, \Gamma A)$, that is

(12.4.7)
$$\begin{cases} \mu(x \otimes \gamma_n \otimes a) = \theta(\bar{x} \otimes \gamma_n \otimes \tilde{a}) \\ \mu([x,y] \otimes \eta_{n,m} \otimes b) = \theta\big([\bar{x},\bar{y}] \otimes \eta_{n,m} \otimes \tilde{b}\big). \end{cases}$$

Here we set $\bar{x} = \mu(x)$ as in (12.3.6), and we set $\mu(a) = \tilde{a}$ and $\mu(b) = \tilde{b}$ where we use μ in the universal coefficient sequence

(12.4.8)
$$\mathrm{Ext}(H, \mathbb{Z}/k \otimes \mathbb{Z}/2) \overset{\Delta}{\rightarrowtail} [M(H,3), \Sigma^2 P_k] \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H, \mathbb{Z}/k)$$

which is natural in $\mathbb{Z}/k \in$ **PCyc** since we can use B_3 in Corollary 1.4.6. Using G in (12.4.1) we identify $[M(H,3), \Sigma^2 P_k] = \mathbf{G}(H, \mathbb{Z}/k)$.

(12.4.9) Lemma *The elements* (12.4.6) *and the elements* $\Delta(\rho)$ *with* $\rho \in \mathrm{Ext}(H, \Gamma_4(\eta))$ *generate the abelian group* $\Gamma_3(H, K(\eta, 2))$ *provided the group* A *is finitely generated with* $\eta\colon A \to B$.

This is a consequence of (12.4.7), Lemma 12.4.5 and (12.4.3). Below we describe explicitly a complete set of relations for the generators in the lemma. We need the following notation. Given a homomorphism $f\colon H \to \mathbb{Z}/k$ in **Ab** and an element $u \in U = \Gamma_4(\eta)$ let

(12.4.10)
$$f^{\#}u = f^*(1 \otimes u) \in \mathrm{Ext}(H, \Gamma_4(\eta)).$$

Here $1 \otimes u \in \mathbb{Z}/k \otimes U = \mathrm{Ext}(\mathbb{Z}/k, U)$ is represented by u.

(12.4.11) Definition We define the algebraic bifunctor $\bar{\Gamma}_3$ in (12.4.1) together with a natural short exact sequence

$$\mathrm{Ext}(H, \Gamma_4(\eta)) \overset{\Delta}{\rightarrowtail} \bar{\Gamma}_3(H, \eta) \overset{\mu}{\twoheadrightarrow} \bar{\Gamma}(H, A) \qquad (*)$$

for $\eta\colon A \to B$. The group $\bar{\Gamma}_3(H, \eta)$ is generated by the symbols ($\mathbb{Z}/n, \mathbb{Z}/m \in$ **PCyc**)

$$\begin{cases} x \otimes \gamma_n \otimes a, \quad x \in G(n, \eta), \quad a \in \mathbf{G}(H, \Gamma(\mathbb{Z}/n)) \\ [x,y] \otimes \eta_{n,m} \otimes b, \quad x \in G(n,\eta), \quad y \in G(m, \eta), \quad b \in \mathbf{G}(H, \mathbb{Z}/n \otimes \mathbb{Z}/m) \\ \Delta(\rho), \quad \rho \in \mathrm{Ext}(H, \Gamma_4(\eta)). \end{cases}$$

$(**)$

Moreover Δ in $(*)$ carries ρ to $\Delta(\rho)$ and μ in $(*)$ is given by $\mu\Delta(\rho) = 0$ and (12.4.7). A morphism $\overline{\varphi} = (\varphi, \overline{\varphi})\colon H' \to H$ in **G** and a morphism $\chi = (\varphi_0, \varphi_1, F)\colon \eta \to \eta'$ in **ΓAb(PCyc)** induce the homomorphism

$$\overline{\varphi}^*\chi_* : \overline{\Gamma}_3(H, \eta) \to \overline{\Gamma}_3(H', \eta')$$

defined on generators by

$$\begin{cases} \overline{\varphi}^*\chi_*(\Delta(\rho)) = \Delta(\varphi^*\Gamma_4(\chi)_*(\rho)) \\ \overline{\varphi}^*\chi_*(x \otimes \gamma_n \otimes a) = (\chi_* x) \otimes \gamma_n \otimes (\overline{\varphi}^* a) \\ \overline{\varphi}^*\chi_*([x, y] \otimes \eta_{n,m} \otimes b) = [\chi_* x, \chi_* y] \otimes \eta_{n,m} \otimes (\overline{\varphi}^* b). \end{cases} \qquad (***)$$

Here we set $\chi_* x = Fx$ for $n > 0$ and $\chi_* x = \varphi_0 x$ for $n = 0$ and $x \in G(n, \eta)$. The relations in question for $\overline{\Gamma}_3(H, \eta)$ are given by the following list (a)–(g):

(a) $[x, x] \otimes \eta_{n,n} \otimes b = x \otimes \gamma_n \otimes (P_* b) + \Delta(\bar{b}^\# \rho_1)$ where $\rho_1 \in \Gamma_4(\eta)$ is given by

$$\rho_1 = \delta_n \Delta\{\eta(\tilde{x}) \otimes 1\}.$$

Here we set $\delta_n = 1$ for $n = 4$ and $\delta_n = 0$ otherwise.

(b) $[x, y] \otimes \eta_{n,m} \otimes b = [y, x] \otimes \eta_{m,n} \otimes (T_* b) + \Delta(\bar{b}^\# \rho_2)$ where $\rho_2 \in \Gamma_4(\eta)$ is given by

$$\rho_2 = -[y, x] \otimes \xi_{n,n} + \delta_n \Delta\{[\tilde{x}, \tilde{y}]_\eta \otimes 1\}$$

for $n = m > 0$ and $\rho_2 = 0$ otherwise. Here we use δ_n as in (a). We point out that with the identification in (12.4.6)(2) the map T_* is actually the identity.

(c) For the generator $\chi = \chi_n^k \in \mathrm{Hom}(\mathbb{Z}/k, \mathbb{Z}/n)$, n not equal to k, we have

$$(\chi^* x) \otimes \gamma_k \otimes a = x \otimes \gamma_n \otimes (\Gamma(\chi)_* a),$$

$$[\chi^* x, y] \otimes \eta_{k,m} \otimes b = [x, y] \otimes \eta_{n,m} \otimes ((\chi \otimes 1)_* b) + \Delta\big(\bar{b}^\# \rho_3\big).$$

Let $\delta_{n,m} = 1$ if n and m are powers of the same prime and $m > n$ and let $\delta_{n,m} = 0$ otherwise. The term $\rho_3 \in \Gamma_4(\eta)$ in the second equation is given by $\rho_3 = 0$ if $m = 0$. Moreover for m, n not equal to 0 let

$$\rho_3 = \begin{cases} \dfrac{n}{(k,n)} \delta_{n,m} [x, (\chi_m^n)^* y] \otimes \xi_{n,n} + \varepsilon & \text{for } k = 0 \text{ or } (k,m) = m \\ \dfrac{-k}{(k,n)} \delta_{m,n} [(\chi_n^m)^* x, y] \otimes \xi_{m,m} & \text{otherwise} \end{cases}$$

where

$$\varepsilon = \Delta\{[\tilde{x}, \tilde{y}]_\eta \otimes 1\} \in \Gamma_4(\eta)$$

if $m = 2$ and $n = 2k$ where k is a power of 2; moreover $\varepsilon = 0$ otherwise.

(d)

$$(x+y) \otimes \gamma_n \otimes a = x \otimes \gamma_n \otimes a + y \otimes \gamma_n \otimes a$$
$$+ [x, y] \otimes \eta_{n,n} \otimes (H_* a) + \Delta\big((\tau \tilde{a})^{\#} \rho_4\big).$$

Here $\tau: \Gamma(\mathbb{Z}/n) \to \Gamma(\mathbb{Z}/n)$ is the isomorphism $1/2$ if n is odd and is the identity otherwise. Moreover $\rho_4 \in \Gamma_4(\eta)$ is given by $\rho_4 = 0$ for $n = 0$ and $\rho_4 = -[x, y] \otimes \xi_{n,n} + \varepsilon$ for n not equal to 0 where $\varepsilon = (n/2)\Delta\{[\tilde{x}, \tilde{y}]_n \otimes \tilde{y}\}$ if n is a power of 2 and $\varepsilon = 0$ otherwise.

(e)

$$[x+y, z] \otimes \eta_{n,m} \otimes b = [x, z] \otimes \eta_{n,m} \otimes b + [y, z] \otimes \eta_{n,m} \otimes b + \Delta\big(\tilde{b}^{\#} \rho_5\big).$$

Here $\rho_5 \in \Gamma_4(\eta)$ is given by

$$\rho_5 = (n/2)\Delta\{[\tilde{x}, \tilde{z}]_\eta \otimes \tilde{y}\}$$

if $m = 0$ and n a power of 2 or if m, n are powers of 2 with $m > n$. Moreover $\rho_5 = 0$ otherwise.

(f) $x \otimes \gamma_n \otimes a$ and $[x, y] \otimes \eta_{n,m} \otimes b$ are linear in a and b respectively.

(g) Δ in $(*)$ is an injective homomorphism and Δ in (12.3.6) satisfies for $x' \in B, n > 0$,

$$(\Delta x') \otimes \gamma_n \otimes a = 0$$

$$[\Delta x', y] \otimes \eta_{n,m} \otimes b = \Delta\big(\tilde{b}^{\#} \rho_5\big)$$

where $\rho_5 \in \Gamma_4(\eta)$ is given by $\rho_5 = -\Delta[x' \otimes \tilde{y}\}$ if $m > n$ or $m = 0$; and $\rho_5 = 0$ otherwise. Finally Δ in (12.4.8) satisfies for $a' \in \mathrm{Ext}(H, \mathbb{Z}/n \otimes \mathbb{Z}/2)$

$$x \otimes \gamma_n \otimes (\Delta a') = \Delta(\rho_6)_*(a')$$

where $\rho_6: \mathbb{Z}/n \otimes \mathbb{Z}/2 \to \Gamma_4(\eta)$ carries the generator to $\Delta\{\eta(\tilde{x}\} \otimes 1\}$. Moreover for $b' \in \mathrm{Ext}(H, \mathbb{Z}/n \otimes \mathbb{Z}/m \otimes \mathbb{Z}/2)$ we have

$$[x, y] \otimes \eta_{n,m} \otimes (\Delta b') = \Delta(\rho_7)_*(b')$$

where $\rho_7: \mathbb{Z}/n \otimes \mathbb{Z}/m \otimes \mathbb{Z}/2 \to \Gamma_4(\eta)$ carries the generator to $\Delta\{[\tilde{x}, \tilde{y}]_\eta \otimes 1\}$.

This completes the definition of the functor $\bar{\Gamma}_3$ in (12.4.1) and Theorem 12.4.2. The exactness of $(*)$ above is a consequence of the definition of $\bar{\Gamma}$ in Definition 12.4.4 since killing the elements $\Delta(\rho)$ in the relations above yields precisely the relations in Definition 12.4.4.

(12.4.12) Remark We have the exact sequence

$$\mathrm{Ext}(H, \Gamma_2^2(\eta)) \to \bar{\Gamma}_3(H, \eta) \to \Gamma T_\#(H, A) \to 0$$

where we assume that A is finitely generated. Hence in this case we obtain by (12.4.11) also generators and relations for the bifunctor $\Gamma T_\#(H, A)$. For this we only need to kill all curly brackets in the relations of (12.4.11) above.

Proof of Theorem 12.4.2 We define the transformation

$$\theta \colon \bar{\Gamma}_3(H, \eta) \to \Gamma_3(H, K(\eta, 2))$$

on generators in the obvious way by $\theta\Delta(\rho) = \Delta(\rho)$; see (12.4.3), and (12.4.6). The formulas in Sections 11.5 and A.10 show that θ is a well-defined homomorphism compatible with Δ and μ. If A is finitely generated this shows by Lemma 12.4.5 that θ is an isomorphism since (12.4.11)(*) is short exact. In fact, θ is well defined since we use Lemma 11.5.19 for (a) and we use Lemma 11.5.20 and (11.5.10)(3) for (b). Next we obtain (c) by Lemma 11.5.24; for the second equation this is a somewhat nasty computation. We derive (d) from Theorem A.10.2(b), (11.5.16)(7), Lemma 11.5.24(e), Lemma 11.5.27. We derive (e) from (11.5.10)(6) and Lemma 11.5.27. Finally we obtain (g) by the definition of γ_n and $\eta_{n,m}$. □

12.5 Algebraic models of 1-connected 5-dimensional homotopy types

We introduce the algebraic category of A_2^3-systems and we describe a detecting functor from the homotopy category of simply connected 5-dimensional CW-spaces to the category of A_2^3-systems. Hence isomorphism types of A_2^3-systems are in 1–1 correspondence with 1-connected 5-dimensional homotopy types. The definition of A_2^3-systems here depends on the highly intricate functors Γ_4 and Γ_3 in Sections 12.3 and 12.4. In the following sections we describe special cases for which these functors are obtained more directly by algebraic constructions.

Recall that quadratic functions $\eta\colon A \to B$, or equivalently homomorphisms $\Gamma(A) \to B$, are the objects of the category $\mathbf{\Gamma Ab}$; see (7.1.1). Moreover $\mathbf{G}$ is the category equivalent to the homotopy category of Moore spaces $\mathbf{M}^n, n \geq 3$, in Section 1.6. The objects of $\mathbf{G}$ are abelian groups $H \in \mathbf{Ab}$.

We shall describe various examples of algebraic categories $\mathbf{K}$ and functors

$$\begin{aligned} &k\colon \mathbf{K} \to \mathbf{\Gamma Ab} \\ (12.5.1) \qquad &\Gamma_4\colon \mathbf{K} \to \mathbf{Ab} \\ &\Gamma_3\colon \mathbf{G}^{\mathrm{op}} \times \mathbf{K} \to \mathbf{Ab} \end{aligned}$$

together with a short exact sequence

$$\mathrm{Ext}(H, \Gamma_4(\eta)) \xrightarrow{\Delta} \Gamma_3(H, \eta) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H, \Gamma(A))$$

which is natural in $H \in \mathbf{G}$ and $\eta \in \mathbf{K}$. Here the objects η in $\mathbf{K}$ are also objects of $\mathbf{\Gamma Ab}$ and $k: Ob(\mathbf{K}) \subset Ob(\mathbf{\Gamma(Ab})$ is an inclusion. Given such data $(\mathbf{K}, k, \Gamma_4, \Gamma_3, \Delta, \mu)$ we introduce the following algebraic objects.

(12.5.2) Definition An A_2^3*-system over* $\mathbf{K}$

$$S = (H_2, H_4, H_5, \pi_3, b_4, \eta, b_5, \beta) \tag{1}$$

is a tuple consisting of abelian groups H_2, H_4, H_5, π_3 and elements

$$\begin{aligned} &b_4 \in \mathrm{Hom}(H_4, \Gamma(H_2)) \\ &\eta \in \mathrm{Hom}(\Gamma(H_2), \pi_3) \quad \text{with} \quad \eta \in Ob(\mathbf{K}) \\ &b_5 \in \mathrm{Hom}(H_5, \Gamma_4(\eta)) \\ &\beta \in \Gamma_3(H_4, \eta)/\Delta(b_5)_* \mathrm{Ext}(H_4, H_5). \end{aligned} \tag{2}$$

The elements satisfy the following conditions (3) and (4). The sequence

$$H_4 \xrightarrow{b_4} \Gamma H_2 \xrightarrow{\eta} \pi_3 \tag{3}$$

is exact and

$$\mu(\beta) = b_4 \tag{4}$$

where μ is the operator given by $\mu: \Gamma_3(H_4, \eta) \to \mathrm{Hom}(H_4, \Gamma(H_2))$ in (12.5.1) above. A morphism

$$(\varphi_2, \varphi_4, \varphi_5, \varphi_\pi, \varphi_\Gamma): S \to S' \tag{5}$$

between A_2^3-systems is a tuple of homomorphisms in $\mathbf{Ab}$

$$\begin{cases} \varphi_i: H_i \to H_i' & (i = 2, 4, 5) \\ \varphi_\pi: \pi_3 \to \pi_3 \\ \varphi_\Gamma: \Gamma_4(\eta) \to \Gamma_4(\eta') \end{cases}$$

such that the following properties are satisfied. The diagram

$$\begin{tikzcd} H_4 \arrow[r, "b_4"] \arrow[d, "\varphi_4"] & \Gamma(H_2) \arrow[r, "\eta"] \arrow[d, "\Gamma(\varphi_2)"] & \pi_3 \arrow[d, "\varphi_\pi"] \\ H_4' \arrow[r, "b_4"'] & \Gamma(H_2') \arrow[r, "\eta'"'] & \pi_3' \end{tikzcd} \tag{6}$$

commutes and also

$$\begin{tikzcd} H_5 \arrow[r, "b_5"] \arrow[d, "\varphi_5"] & \Gamma_4(\eta) \arrow[d, "\varphi_\Gamma"] \\ H_5' \arrow[r, "b_5'"'] & \Gamma_4(\eta') \end{tikzcd} \tag{7}$$

commutes. Moreover, there is a morphism $\chi: \eta \to \eta'$ in $\mathbf{K}$ with $k(\chi) = (\varphi_2, \varphi_\pi)$, see (12.5.1), and there is a morphism $(\varphi_4, \bar{\varphi}_4): H_4 \to H_4'$ in $\mathbf{G}$ such that

$$\varphi_\Gamma = \Gamma_4(\chi): \Gamma_4(\eta) \to \Gamma_4(\eta') \tag{8}$$

is induced by χ and such that

$$\chi_*(\beta) = (\varphi_4, \bar{\varphi}_4)^*(\beta') \tag{9}$$

in $\Gamma_3(H_4, \eta')/\Delta(b_5')_* \operatorname{Ext}(H_4, H_5')$. Here χ_* and $(\varphi_4, \bar{\varphi}_4)^*$ are the homomorphisms induced by the functor Γ_3. There is an obvious composition of morphisms so that the category of A_2^3-systems over $\mathbf{K}$ is well defined.

An A_2^3-system S as above is *free* if H_5 is free abelian and S is *injective* if $b_5: H_5 \to \Gamma_4(\eta)$ is injective. Let A_2^3-**Systems(K)** resp. A_2^3-**systems(K)** be the full subcategories of free, resp. injective, A_2^3-systems. We have the canonical forgetful functor

$$\phi: A_2^3\text{-}\mathbf{Systems(K)} \to A_2^3\text{-}\mathbf{systems(K)} \tag{10}$$

which replaces b_5 by the inclusion $b_5(H_5) \subset \Gamma_4(\eta)$. One readily checks that ϕ is full and representative.

(12.5.3) Definition We associate with an A_2^3-system S the exact Γ-*sequence*

$$H_5 \xrightarrow{b_5} \Gamma_4(\eta) \to \pi_4 \to H_4 \xrightarrow{b_4} \Gamma(H_2) \xrightarrow{\eta} \pi_3 \to H_3 \to 0.$$

Here $H_3 = \operatorname{cok}(\eta)$ is the cokernel of η and the extension

$$\operatorname{cok}(b_5) \rightarrowtail \pi_4 \twoheadrightarrow \ker(b_4)$$

is obtained by the element β in Definition 12.5.2(1), that is, the group $\pi_4 = \pi(\beta_\dagger)$ is given by the extension element

$$\beta_\dagger \in \mathrm{Ext}(\ker(b_4), \mathrm{cok}(b_5))$$

defined by

$$\beta_\dagger = \Delta^{-1}(j, \bar{j})^*(\beta)$$

Here $j: \ker(b_4) \subset H_4$ is the inclusion and $(j, \bar{j}): \ker(b_4) \to H_4$ is a morphism in $\mathbf{G}$ which induces $(j, \bar{j})^*$ via the functor Γ_3; compare (2.6.7).

Let $\mathbf{types}_2^1(\mathbf{K})$ be the full subcategory of the category of 1-connected 3-types consisting of all $K(\eta, 2)$ with $\eta \in \mathbf{K}$. We say that the structure (12.5.1) is *good* if there is a full functor

(12.5.4) $$\tau: \mathbf{types}_2^1(\mathbf{K}) \twoheadrightarrow \mathbf{K}$$

which carries $K(\eta, 2)$ to η such that the composite functors

$$\mathbf{types}_2^1(\mathbf{K}) \xrightarrow{\tau} \mathbf{K} \xrightarrow{\Gamma_4} \mathbf{Ab}$$

$$(\mathbf{M}^3)^{\mathrm{op}} \times \mathbf{types}_2^1(\mathbf{K}) \xrightarrow{G \times \tau} \mathbf{G}^{\mathrm{op}} \times \mathbf{K} \xrightarrow{\Gamma_3} \mathbf{Ab}$$

are naturally isomorphic to the homotopy functors Γ_4 and Γ_3 respectively, that is

(a) $$\Gamma_4 K(\eta, 2) = \Gamma_4(\eta)$$

and

(b) $$\Gamma_3(H, K(\eta, 2)) = \Gamma_3(H, \eta)$$

and these isomorphisms are compatible with Δ and μ in (12.5.1) and Definition 2.2.3(3). Moreover the composite functor

$$\mathbf{types}_2^1(\mathbf{K}) \xrightarrow{\tau} \mathbf{K} \xrightarrow{k} \Gamma\mathbf{Ab}$$

is naturally isomorphic to the functor k_2 which carries $K(\eta, 2)$ to η; see Proposition (7.1.3).

(12.5.5) Example Let $\mathbf{K}$ be the full subcategory

$$\mathbf{K} \subset \Gamma\mathbf{Ab}(\mathbf{PCyc})$$

consisting of all objects $\eta: A \to B$ for which A is finitely generated. Then we obtain by G in (12.3.1) the equivalence of categories

$$\tau: \mathbf{types}_2^1(\mathbf{K}) \xrightarrow{\sim} \mathbf{K}$$

where $\mathbf{types}_2^1(\mathbf{K})$ is the homotopy category of all $K(\eta, 2)$ with $\eta \in \mathbf{K}$. Moreover using the functors $\bar{\Gamma}_4$ and $\bar{\Gamma}_3$ in Sections 12.3 and 12.4 we obtain a good structure as above.

For a category $\mathbf{K}$ as in (12.5.1) or (12.5.4) let $\mathbf{spaces}_2^3(\mathbf{K})$ be the full homotopy category of 1-connected 5-dimensional CW-spaces X for which the quadratic function

$$\eta_X = (\eta_2)^*: \pi_2 X = H_2 X \to \pi_3 X$$

is an object in $\mathbf{K}$. Let $\mathbf{types}_2^2(\mathbf{K})$ be the corresponding category of 1-connected 4-types. We have the Postnikov functor

$$P: \mathbf{spaces}_2^3(\mathbf{K}) \to \mathbf{types}_2^2(\mathbf{K}) \tag{12.5.6}$$

which carries X to its 4-type. The next result can be applied to the example in (12.5.5) above.

(12.5.7) Classification theorem *Given a category* $\mathbf{K}$ *and a good structure as in* (12.5.4) *and* (12.5.1) *there are detecting functors*

$$\Lambda': \mathbf{spaces}_2^3(\mathbf{K}) \to A_2^3\text{-}\mathbf{Systems}(\mathbf{K})$$

$$\lambda': \mathbf{types}_2^2(\mathbf{K}) \to A_2^3\text{-}\mathbf{systems}(\mathbf{K}).$$

Moreover there is a natural isomorphism

$$\phi\Lambda'(X) = \lambda' P(X)$$

for the forgetful functor ϕ *in Definition* 12.5.2(10) *and the Postnikov functor* P *in* (12.5.6).

Proof Let $\mathbf{C} = \mathbf{types}_2^1(\mathbf{K})$ be the category in the classification theorem (3.4.4). The bype functor F on $\mathbf{C}$ defined in (3.4.3)(3), $n = 4$, leads via the good structure on $\mathbf{K}$ to the following bype functor F' on $\mathbf{K}$. Let

$$F': \mathbf{Ab}^{\mathrm{op}} \times \mathbf{K} \to \mathbf{Ab}$$

be defined by the pull-back diagram, $H \in \mathbf{Ab}$, $\eta \in \mathbf{K}$,

$$\begin{array}{ccccc}
\mathrm{Ext}(H, \Gamma_4(\eta)) & \rightarrowtail & \Gamma_3(H, \eta) & \twoheadrightarrow & \mathrm{Hom}(H, \Gamma(A)) \\
\Vert & & \cup & & \cup \\
\mathrm{Ext}(H, F_1'(\eta)) & \rightarrowtail & F'(H, \eta) & \twoheadrightarrow & \mathrm{Hom}(H, F_0'\eta)
\end{array}$$

where $F_1'(\eta) = \Gamma_4(\eta)$ and $F_0'(\eta) = \ker(\eta: \Gamma(A) \to B)$. Induced maps for F' are defined via the induced maps for the functor Γ_3 in (12.5.1) and (12.5.4). It is clear that we have detecting functors

$$\mathbf{Bypes}(\mathbf{C}, F) \xrightarrow{\tau} \mathbf{Bypes}(\mathbf{K}, F') \xrightarrow{\tau'} A_2^3\text{-}\mathbf{Systems}(\mathbf{K}).$$

The detecting functor τ is induced by the full functor τ in (12.5.4). Moreover the detecting functor τ' is the 'forgetful' functor. The functors τ and τ' are essentially the identity on objects and by definition they are both full functors. Now the classification theorem 3.4.4 yields the proposition of the theorem. □

Using the isomorphisms (12.5.4)(a), (b) as identification we define the detecting functor Λ' in Theorem 12.5.7 by

(12.5.8) $\quad \Lambda'(X) = (H_2X, H_4X, H_5X, \pi_3X, b_4X, \eta_X, b_5X, \beta(X))$

where b_4X, b_5X, η_X are part of Whitehead's exact Γ-sequence with $\Gamma_3(X) = \Gamma(H_2X)$ and where $\beta(X) = \beta_4(X)$ is the boundary invariant of X. Similarly we define the detecting functor λ'.

The detecting functor Λ' in Theorem 12.5.7 shows that for each free A_2^3-system S over $\mathbf{K}$ there is a unique 1-connected 5-dimensional homotopy type $X = X_S$ with $\Lambda'(X) \cong S$. Then the Γ-sequence for S in Definition 12.5.3 is the top row in the following commutative diagram

$$\begin{tikzcd}
H_5 \arrow[r] \arrow[d, equal] & \Gamma_4(\eta) \arrow[r] \arrow[d, equal] & \pi_4 \arrow[r] \arrow[d] & H_4 \arrow[r] \arrow[d, equal] & \Gamma(H_2) \arrow[r, "\eta"] \arrow[d, equal] & \pi_3 \arrow[r, two heads] \arrow[d, equal] & H_3 \arrow[d, equal] \\
H_5X \arrow[r] & \Gamma_4X \arrow[r] & \pi_4X \arrow[r] & H_4X \arrow[r] & \Gamma_3X \arrow[r] & \pi_3X \arrow[r, two heads] & H_3X
\end{tikzcd}$$

The bottom row is Whitehead's exact Γ-sequence of X. The diagram describes a weak natural isomorphism of exact sequences; see (3.2.5)(5). If we apply Theorem 12.5.7 to Example 12.5.5 we get the following crucial result.

(12.5.9) Classification theorem *Let*

$$\mathbf{K} \subset \mathbf{\Gamma Ab(PCyc)}$$

be the full subcategory of all η: $A \to B$ for which A is finitely generated and let an A_2^3-system over $\mathbf{K}$ *be defined by the functors $\bar{\Gamma}_4$ and $\bar{\Gamma}_3$ in Sections* 12.3 *and* 12.4. *Then there is a detecting functor from the full homotopy category of all simply connected 5-dimensional CW-spaces X with H_2X finitely generated to the category A_2^3-*$\mathbf{System(K)}$.

Hence this result yields in particular algebraic models of all homotopy types of finite polyhedra which are simply connected and of dimension ≤ 5. In the following sections we describe applications of Theorem 12.5.7 for which the functors Γ_4, Γ_3 are less complicated.

12.6 The case $\pi_3 X = 0$

We consider A_2^3-systems which correspond to simply connected 5-dimensional homotopy types X with $\pi_3 X = 0$. They turn out to be the same as certain bypes used in Theorem (6.4.1). Let $\mathbf{K} = \mathbf{Ab}$ be the category of abelian groups. Then we obtain the functors

$$
\begin{aligned}
&k: \mathbf{K} \to \Gamma\mathbf{Ab}\\
(12.6.1)\qquad &\Gamma_4 = \Gamma T: \mathbf{K} \to \mathbf{Ab}\\
&\Gamma_3: \mathbf{G}^{\mathrm{op}} \times \mathbf{K} \to \mathbf{Ab}^{\mathrm{op}} \times \mathbf{K} \xrightarrow{\Gamma T_{*}} \mathbf{Ab}
\end{aligned}
$$

as follows. Let k be the inclusion which carries an abelian group A to the quadratic function $\eta: A \to B$ for which $B = 0$ is trivial. Moreover $\Gamma_4 = \Gamma T$ is the Γ-torsion functor and Γ_3 is given by the projection $\mathbf{G} \to \mathbf{Ab}$ and by the torsion bifunctor ΓT_{*}. See Definition 6.2.11 where one also finds the short exact (Δ, μ)-sequence for ΓT_{*}. Hence the functors in (12.6.1) form a structure as in (12.5.1) and the A_2^3-systems over $\mathbf{K}$ are defined by (12.6.1). Using the equivalence

$$\tau: \mathbf{types}_2^1(\mathbf{K}) \xrightarrow{\sim} \mathbf{K}$$

which carries $K(\eta, 2) = K(A, 2)$ to A we see that this structure is good in the sense of (12.5.4); compare Section 6.3. Hence we can apply Theorem 12.5.7. For this we have the identification

$$A_2^3\text{-}\mathbf{Systems}(\mathbf{K}) = \mathbf{Bypes}(\mathbf{Ab}, \Gamma T_{*})$$

of categories so that the following corollary of Theorem 12.5.7 is also a consequence of Theorem 6.4.1.

(12.6.2) Theorem *Let A_2^3-systems be defined by the structure on $\mathbf{K} = \mathbf{Ab}$ in* (12.6.1) *above. Then there is a detecting functor from the homotopy category of simply connected 5-dimensional CW-spaces X with $\pi_3 X = 0$ to the category A_2^3-$\mathbf{Systems}(\mathbf{K})$.*

12.7 The case $H_2 X$ uniquely 2-divisible

We here consider simply connected 5-dimensional homotopy types X for which $H_2 X$ is uniquely 2-divisible. Our method is not the same as the approach by 'tame homotopy theory', see Dwyer [TH], Anick [HA], and Hess [MT]. The results in this section yield new insights into why tame homotopy theory works; moreover comparison of the results here with the case in Theorem 12.5.9 sheds light on the kind of complexity appearing outside the tame range.

An abelian group A is *uniquely 2-divisible* if multiplication by 2 is an isomorphism $2\colon A \cong A$; the inverse of this isomorphism is denoted by $1/2$. Let $\mathbb{Z}[1/2]$ be the smallest subring of the rationals $\mathbb{Q}$ containing $1/2 \in \mathbb{Q}$. Then A is uniquely 2-divisible if and only if A is a $\mathbb{Z}[1/2]$-module or equivalently iff $A * \mathbb{Z}/2 = A \otimes \mathbb{Z}/2 = 0$. Let $\mathbf{Ab}[1/2]$ be the full subcategory of $\mathbf{Ab}$ consisting of uniquely 2-divisible groups. Moreover let $\mathbf{M}^2[1/2]$ be the full homotopy category of Moore spaces $M(A,2)$ with $A \in \mathbf{Ab}[1/2]$.

(12.7.1) Proposition *There is a functor*

$$s\colon \mathbf{Ab}[1/2] \to \mathbf{M}^2[1/2]$$

which is a splitting of the homology functor H_2.

Proof Let $\varphi \in 2\,\mathrm{Hom}(A,B)$, say $\varphi = 2\psi$. We construct an element

$$s(\varphi) \in [M(A,2), M(B,2)] = G \tag{1}$$

as follows. We choose homotopy equivalences $M(A,2) = \Sigma X$, $M(B,2) = \Sigma Y$ for appropriate $X = M_A$ and $Y = M_B$. Then the map $-1_{\Sigma X}\colon \Sigma X \to \Sigma X$ induces the inverse, $-\alpha$, in the group $[\Sigma X, \Sigma Y] = G$. Now we set

$$s(\varphi) = \bar{\psi} - (-1_{\Sigma Y})\bar{\psi} \tag{2}$$

where $\bar{\psi} \in G$ is a map which realizes ψ. Then $s(\varphi)$, clearly, realizes $2\psi = \varphi$. Moreover, $s(\varphi)$ does not depend on the choice of $\bar{\psi}$. For this we first remark that we have the central extension of groups

$$\mathrm{Ext}(A, \Gamma B) \overset{i}{\rightarrowtail} G \twoheadrightarrow \mathrm{Hom}(A,B) \tag{3}$$

with $i(\alpha) = 0 + \alpha$. Moreover, we have

$$\bar{\psi} + \alpha = \bar{\psi} + i(\alpha) \tag{4}$$

where $+$ on the left is the action and where $+$ on the right is the group structure of G. Now consider a second realization $\bar{\bar{\psi}}$ of ψ. Then clearly $\bar{\bar{\psi}} = \bar{\psi} + \alpha$ for appropriate α and we get with $-1 = -1_{\Sigma Y}$:

$$\bar{\bar{\psi}} - (-1)\bar{\bar{\psi}} = (\bar{\psi} + \alpha) - (-1)(\bar{\psi} + \alpha) \tag{5}$$

$$= (\bar{\psi} + \alpha) - [(-1)\bar{\psi} + (-1)_* \alpha] \tag{6}$$

$$= \bar{\psi} + i\alpha - i\alpha - (-1)\bar{\psi} = \bar{\psi} - (-1)\bar{\psi}. \tag{7}$$

In (7) we use (4) and the fact that

$$(-1)_* = \mathrm{id}\colon \Gamma B \to \Gamma B \quad \text{for} \quad -1\colon B \to B. \tag{8}$$

In (6) we use the linear distributivity law for the action $+$.

Next we show that s in (2) is actually a functor. Let $\varphi_0 \in \mathrm{Hom}(A_0, A)$ with $\varphi_0 = 2\psi_0$. Then we get:

$$\begin{aligned} s(\varphi) \circ s(\varphi_0) &= (\bar{\psi} - (-1)\bar{\psi})(\bar{\psi}_0 - (-1)\bar{\psi}_0) \\ &= (\bar{\psi} - (-1)\bar{\psi})\bar{\psi}_0 - (\bar{\psi} - (-1)\bar{\psi})(-1)\bar{\psi}_0. \end{aligned} \tag{9}$$

For $\alpha = 2\psi\psi_0$ we have the realization

$$\bar{\alpha} = (\bar{\psi} - (-1)\bar{\psi})\bar{\psi}_0.$$

Moreover, we get for the second summand in (9)

$$-(\bar{\psi} - (-1)\bar{\psi})(-1)\bar{\psi}_0 = -((-1)\bar{\psi} - \bar{\psi})\bar{\psi}_0 = -(-1)\bar{\alpha}. \tag{10}$$

Therefore (9) and (10) show

$$s(\varphi) \circ s(\varphi_0) = \bar{\alpha} - (-1)\bar{\alpha} = s(\varphi\varphi_0) \tag{11}$$

since $2\alpha = \varphi\varphi_0$. □

(12.7.2) Theorem *The composite functor*

$$\mathbf{Ab}[1/2] \xrightarrow{s} \mathbf{M}^2[1/2] \xrightarrow{\pi_4} \mathbf{Ab}$$

is naturally isomorphic to the functor which carries A to $\Gamma T(A) \oplus L_3(A, 1)$ and $\varphi: A \to B$ to $\Gamma T(\varphi) \oplus L_3(\varphi, 1)$. The composite functor

$$(\mathbf{M}^3)^{\mathrm{op}} \times \mathbf{Ab}[1/2] \xrightarrow{1 \times s} (\mathbf{M}^3)^{\mathrm{op}} \times \mathbf{M}^2[1/2] \xrightarrow{\pi_3} \mathbf{Ab}$$

is naturally isomorphic to the functor which carries $(M(B,3), A)$ to $\mathrm{Ext}(B, L_3(A, 1)) \oplus \Gamma T_*(A, B)$ and $(\bar{\psi}, \varphi)$ to $\mathrm{Ext}(\psi, L_3(\varphi, 1)) \oplus \Gamma T_*(\psi, \varphi)$. The isomorphisms are both compatible with the corresponding operators Δ and μ.

We do not describe the isomorphisms explicitly since we derive Theorem 12.7.2 from the following fact.

(12.7.3) Lemma $H^1(\mathbf{Ab}[1/2], \mathrm{Hom}(\Gamma T, L_3(-, 1))) = 0.$

Proof Let $\delta: \mathbf{Ab}[1/2] \to \mathrm{Hom}(\Gamma T, L_3(-, 1))$ be a derivation. We have to show that δ is an inner derivation. For $\varphi: A \to B$ we have $2_B\varphi = \varphi 2_A$ where $2_A: A \cong A$ is multiplication by 2. Hence we get $\delta(2_B\varphi) = \delta(\varphi 2_A)$ where

$$\begin{aligned}\delta(2_B\varphi) &= \varphi^*\delta(2_B) + (2_B)_*\delta(\varphi)\\ &= \varphi^*\delta(2_B) + 8\delta(\varphi)\\ \delta(\varphi 2_A) &= \varphi_*\delta(2_A) + (2_A)^*\delta(\varphi)\\ &= \varphi_*\delta(2_A) + 4\delta(\varphi).\end{aligned}$$

Hence we have

$$4\delta(\varphi) = \varphi_*\delta(2_A) - \varphi^*\delta(2_B)$$

or equivalently

$$\delta(\varphi) = \varphi_*(1/4)\delta(2_A) - \varphi^*(1/4)\delta(2_B)$$

and hence φ is an inner derivation. □

Proof of Theorem 12.7.2 Since A is uniquely 2-divisible also $\Gamma(A)$ is uniquely 2-divisible and hence $\pi_4 M(A,2) = \pi_4' M(A,2)$. By (11.1.17) the natural sequence

$$L_3(A,1) \rightarrowtail \pi_4 s(A) \twoheadrightarrow \Gamma T(A)$$

is split for each A. We choose a splitting s_A and obtain a derivation δ as in the proof of Lemma 12.7.3 by

$$\delta(\varphi) = \pi_4 s(\varphi)s_A - s_B\Gamma T(\varphi).$$

This derivation, by Lemma 12.7.3, is an inner derivation. Hence we can alter the splitting in such a way that we obtain a natural splitting. In a similar way one can prove the second part of Theorem 12.7.2. □

Now let $\mathbf{\Gamma Ab}[1/2]$ be the full subcategory of $\mathbf{\Gamma Ab}$ consisting of all quadratic functions $\eta: A \to B$ with $A \in \mathbf{Ab}[1/2]$. Moreover let $\mathbf{types}_2^1[1/2]$ be the full homotopy category of all $K(\eta,2)$ with $\eta \in \mathbf{\Gamma Ab}[1/2]$. Then there is a functor

(12.7.4) $$s: \mathbf{\Gamma Ab}[1/2] \to \mathbf{types}_2^1[1/2]$$

which is a splitting of the functor k_2 in (7.1.8). We obtain the functor s by the equivalence $K_2: \mathbf{types}_2^1 = \Gamma\mathbf{M}^2$ in Theorem 7.2.7 and by the functor s in Proposition 12.7.1, that is, s in (12.7.4) carries (φ_1, φ_0) to $K_2^{-1}\{\varphi_1, 0, s\varphi_0\}$. Using the splitting s and the linear extension (7.1.8) one obtains the equivalence of categories

(12.7.5) $$\tau: \mathbf{types}_2^1[1/2] \xrightarrow{\sim} \mathbf{\Gamma Ab}[1/2] \times E$$

where the right-hand side is the canonical split extension for the natural system E with $E(\eta,\eta')=\mathrm{Ext}(A,B')$; see Definition 1.1.9(d). A morphism in $\Gamma\mathbf{Ab}[1/2]\times E$ is given by a tuple $\chi=(\varphi_1,\varphi_0,\xi)\colon\eta\to\eta'$ where $(\varphi_1,\varphi_0)\colon \eta\to\eta'$ is a morphism in $\Gamma\mathbf{Ab}$ and where $\xi\in\mathrm{Ext}(A,B')$. Now let $\mathbf{K}=\Gamma\mathbf{Ab}[1/2]\times E$ and let

$$\begin{aligned} &k\colon \mathbf{K}\to\Gamma\mathbf{Ab}[1/2]\subset\Gamma\mathbf{Ab}\\ &\Gamma_4\colon \mathbf{K}\to\mathbf{Ab}\\ &\Gamma_3\colon \mathbf{G}^{\mathrm{op}}\times\mathbf{K}\to\mathbf{Ab}^{\mathrm{op}}\times\mathbf{K}\xrightarrow{\Gamma_3}\mathbf{Ab}\end{aligned}\tag{12.7.6}$$

be the following structure on $\mathbf{K}$, see (12.5.1). The functor k is the projection. The functor Γ_4 carries $\eta\colon A\to B\in Ob(\mathbf{K})$ to the direct sum

(a) $$\Gamma_4(\eta)=\Gamma_2^2(\eta)\oplus\Gamma T(A).$$

Moreover Γ_4 carries a morphism $\chi=(\varphi_1,\varphi_0,\xi)\colon\eta\to\eta'$ to the induced map

$$\Gamma_4(\chi)\colon\Gamma_2^2(\eta)\oplus\Gamma T(A)\to\Gamma_2^2(\eta')\oplus\Gamma T(A')$$

given by the coordinates $\Gamma_2^2(\varphi_1,\varphi_0),\Gamma\mathrm{T}(\varphi_0)$ and

$$\Gamma T(A)\xrightarrow{H}A*A\xrightarrow{\xi_*}B'\otimes A\xrightarrow{1\otimes\varphi_0}B'\otimes A'\xrightarrow{q}\Gamma_2^2(\eta').$$

Compare (11.4.8). Next the functor Γ_3 in (12.7.6) carries the pair of objects (H,η) to the direct sum

(b) $$\Gamma_3(H,\eta)=\mathrm{Ext}(H,\Gamma_2^2(\eta))\oplus\Gamma T_*(H,A).$$

Now Γ_3 carries a morphism (ψ,χ) with $\psi\colon H'\to H\in\mathbf{Ab}$ to the induced map $\psi^*\chi_*$ where

$$\psi^*=\mathrm{Ext}(\psi,\Gamma_2^2(\eta))\oplus\Gamma T_*(\psi,A)$$

and where

$$\chi_*\colon\mathrm{Ext}(D,\Gamma_2^2(\eta))\oplus\Gamma T_*(D,A)\to\mathrm{Ext}(D,\Gamma_2^2(\eta'))\oplus\Gamma T_*(D,A')$$

has the coordinates $\mathrm{Ext}(D,\Gamma_2^2(\varphi_0,\varphi_1))$, $\Gamma T_*(D,\varphi_0)$, and

$$\Gamma T_*(D,A)\xrightarrow{h_*}[d_D,d_A\otimes A]\xrightarrow{\xi_*}\mathrm{Ext}(D,B'\otimes A)\xrightarrow{(\varphi_0)_*}\mathrm{Ext}(D,\Gamma_2^2\eta').$$

Here we set $(\varphi_0)_*=\mathrm{Ext}(D,q(1\otimes\varphi_0))$ where $q(1\otimes\varphi_0)$ is defined as in (12.7.6). Compare (11.4.9). There is an obvious (Δ,μ)-short exact sequence for Γ_3.

(12.7.7) Lemma *Using the equivalence* τ *in* (12.7.5) *the structure* (12.7.6) *is good in the sense of* (12.5.4).

Proof This is a consequence of Theorem 12.7.2 and Theorem 11.4.7. See (11.3.7) and (11.3.8). □

The lemma shows that we can apply the classification theorem 12.5.7:

(12.7.8) Theorem *Let* A_2^3*-systems be defined by the structure on* $\mathbf{K} = \Gamma\mathbf{Ab}[1/2] \times E$ *in* (12.7.5) *above. Then there is a detecting functor from the homotopy category of simply connected* 5*-dimensional CW-spaces* X *for which* H_2X *is uniquely* 2*-divisible to the category* A_2^3-**Systems**(**K**).

12.8 The case H_2X free abelian

We here consider simply connected 5-dimensional homotopy types X for which H_2X is a free abelian group. Let **Ab**(free) be the full subcategory of **Ab** consisting of free abelian groups and let $\mathbf{M}^2$(free) be the full homotopy category of Moore spaces $M(A,2)$ with $A \in \mathbf{Ab}$(free). Then one has the equivalence of categories

(12.8.1) $$\mathbf{Ab}(\text{free}) = \mathbf{M}^2(\text{free})$$

defined by the homology functor H_2. We now consider the functors π_4, π_3 on $\mathbf{M}^2$(free). Recall that we have the equivalence of categories $G: \mathbf{M}^3 = \mathbf{G}$ in Theorem 1.6.7; for abelian groups A, B the group $\mathbf{G}(A,B)$ is the set of morphisms $A \to B$ in **G**.

(12.8.2) Theorem *The composite functor*

$$\mathbf{Ab}(\text{free}) = \mathbf{M}^2(\text{free}) \xrightarrow{\pi_4} \mathbf{Ab}$$

is naturally isomorphic to the functor Γ_2^2; *see Lemma* 11.1.7. *Moreover the composite functor*

$$\mathbf{G}^{\text{op}} \times \mathbf{Ab}(\text{free}) = (\mathbf{M}^3)^{\text{op}} \times \mathbf{M}^2(\text{free}) \xrightarrow{\pi_3} \mathbf{Ab}$$

is naturally isomorphic to the functor which carries (B, A) *to the direct sum* $\text{Ext}(B, L_3(A,1)) \oplus \mathbf{G}(B, \Gamma(A))$ *and* $(\bar{\psi}, \varphi)$ *to* $\text{Ext}(\psi, L_3(\varphi,1)) \oplus \mathbf{G}(\bar{\psi}, \Gamma(\varphi))$. *The isomorphism is compatible with* Δ *and* μ.

Proof Since $\Gamma T(A) = 0$ we obtain $\pi_4 M(A,2) = \Gamma_2^2(A)$ by (11.1.9). Moreover the generalized Hopf map

$$\eta_A \in [M(\Gamma(A),3), M(A,2)]$$

is natural in A, therefore we obtain the result on π_3 by (11.2.3). □

Let $\Gamma\mathbf{Ab}$(free) be the full subcategory of $\Gamma\mathbf{Ab}$ consisting of all quadratic functions $\eta: A \to B$ with $A \in \mathbf{Ab}$(free). Let $\mathbf{types}_2^1$(free) be the full homotopy category of all $K(\eta, 2)$ with $\eta \in \Gamma\mathbf{Ab}$(free). Then the functor k_2 in (7.1.8) yields the equivalence of categories

$$\tau: \mathbf{types}_2^1(\text{free}) \xrightarrow{\sim} \Gamma\mathbf{Ab}(\text{free}). \tag{12.8.3}$$

Now let $\mathbf{K} = \Gamma\mathbf{Ab}$(free) and let

$$\begin{aligned} k&: \mathbf{K} \to \Gamma\mathbf{Ab} \\ \Gamma_4&: \mathbf{K} \to \mathbf{Ab} \\ \Gamma_3&: \mathbf{G}^{\mathrm{op}} \times \mathbf{K} \to \mathbf{Ab} \end{aligned} \tag{12.8.4}$$

be the following structure on $\mathbf{K}$; see (12.5.1). The functor k is the inclusion and Γ_4 is defined by Γ_2^2 in Definition 11.3.3, that is $\Gamma_4(\eta) = \Gamma_2^2(\eta)$. Moreover Γ_3 carries the pair of objects (H, η) with $\eta: A \to B$ to the abelian group $\Gamma_3(H, \eta)$ defined by the push-out diagram

$$\begin{tikzcd} \mathrm{Ext}(D, \Gamma(A) \otimes \mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, "\mathrm{Ext}(D,q)"'] \arrow[dr, phantom, "\text{push}"] & \mathbf{G}(D, \Gamma(A)) \arrow[d] \\ \mathrm{Ext}(D, \Gamma_2^2(\eta)) \arrow[r, tail] & \Gamma_3(D, \eta) \end{tikzcd}$$

A morphism $(\bar{\psi}, \varphi)$ with $\bar{\psi} \in \mathbf{G}$ and $\varphi = (\varphi_1, \varphi_2) \in \Gamma\mathbf{Ab}$ induces $\bar{\psi}^* \varphi_* = \mathrm{Ext}(\psi, \Gamma_2^2(\varphi)) \oplus \mathbf{G}(\bar{\psi}, \Gamma(\varphi_0))$.

(12.8.5) Lemma *Using the equivalence τ in* (12.8.3) *the structure* (12.8.4) *is good in the sense of* (12.5.4).

Proof We apply Theorem 11.3.4 and (11.3.8) and Theorem 12.8.2 above. □

The lemma yields the following application of the classification theorem 12.5.7.

(12.8.6) Theorem *Let A_2^3-systems be defined by the structure on $\mathbf{K} = \Gamma\mathbf{Ab}$(free) in* (12.8.4) *above. Then there is a detecting functor from the homotopy category of simply connected 5-dimensional CW-spaces X for which H_2X is free abelian to the category A_2^3-$\mathbf{Systems}(\mathbf{K})$.*

Appendix A

PRIMARY HOMOTOPY OPERATIONS AND HOMOTOPY GROUPS OF MAPPING CONES

A CW-complex is built by attaching cells. The attaching maps yield elements in homotopy groups, $\pi_n(X^n)$, where X^n is a CW-complex of dimension $\leq n$. Here X^n is also formed by attaching cells, say $X^n = Y \cup_g e^m$, so that we have to consider homotopy groups of the form $\pi_n(Y \cup_g e^m)$. The computation of such homotopy groups in terms of g is therefore one of the obstacles to analysing the interior structure of a homotopy type. Very little is known about such groups. For example groups of the form $\pi_n(S^k \cup_g e^m)$, given by elements $g \in \pi_{m-1}(S^k)$ in homotopy groups of spheres, are very hard to compute. The main results of this appendix yield a method to compute such groups via an $E_g H_g P_g$-sequence. This sequence generalizes the classical EHP-sequence. In fact, James introduced the EHP-sequence to study homotopy groups $\pi_n(\Sigma A)$ of a suspension. We introduce the $E_g H_g P_g$-sequence to study relative homotopy groups $\pi_n(C_g, B)$ of mapping cones C_g where g: $A \to B$. If $B = *$ is a point then $C_g = \Sigma A$ is the suspension, and, in this case, the $E_g H_g P_g$-sequence coincides with the EHP-sequence. We describe the operators E_g, P_g, H_g explicitly in terms of primary homotopy operations, in particular, the operator P_g is induced by a complicated sum of Whitehead products. This very explicit expression for the operator P_g is of great importance in applications.

The proof of the $E_g H_g P_g$-sequence uses a new combinatorial model N_g of the fibre of the inclusion $i_g: B \subset C_g$. The sophisticated proof relies on the geometric bar-construction and quasi-fibration techniques. We use the model N_g also for the proof of the surprising homotopy equivalence

$$\Sigma P_{i_g} \simeq (\Sigma A) \times \Omega C_g / \{*\} \times \Omega C_g.$$

Here ΩC_g is the loop space of the mapping cone C_g and P_{i_g} is the fibre of the inclusion $i_g: B \subset C_g$ with $g: A \to B, A = \Sigma A'$; see Theorems A.8.2 and A.8.13.

In the first three sections we introduce and study primary homotopy operations which satisfy intricate distributivity laws. With respect to addition

$$+: [\Sigma A, U] \times [\Sigma A, U] \to [\Sigma A, U]$$

we have for example the left distributivity law (obtained in Baues [CC])

$$(x+y) \circ f = x \circ f + y \circ f - \sum_{n \geq 2} c_n(x,y) \gamma_n f$$

where $f \in [\Sigma X, \Sigma A]$. More generally we deal in Section A.9 with the action

$$+ : [C_g, U] \times [\Sigma A, U] \to [C_g, U]$$

and we describe for $f \in [\Sigma X, C_g]$ an expansion for $(u + y) \circ f$ with $u \in [C_g, U]$.

The distributivity laws of primary homotopy operations have a long progression in the literature starting with the work of Whitehead, Hilton, Barratt, James, etc. In Baues [CC] we explored the connection of such distributivity laws with classical commutator calculus of group theory. Recently, in his thesis, Dreckman [DH] described for the first time the complete list of distributivity laws of primary homotopy operations.

A.1 Whitehead products

We recall some basic definitions and facts. Throughout let a *space* be a pointed space of the homotopy type of a CW-complex. Maps and homotopies are base-point preserving. The set of homotopy classes $X \to Y$ is denoted by $[X, Y]$; it contains the trivial class $0: X \to * \in Y$. The groups of homotopy classes

$$\pi_n^A(X) = [\Sigma^n A, X], \quad n \geq 1,$$

are equipped with various operations. We here study the Whitehead product, the cup products, the Hopf construction, and the partial suspension. For the product $A \times B$ of spaces we have the cofibre sequence

$$A \vee B \xrightarrow{i} A \times B \xrightarrow{\pi} A \wedge B.$$

Here $A \vee B = A \times \{*\} \cup \{*\} \times B$ is the one-point union and $A \wedge B = A \times B / A \vee B$ is the smash product. The n-fold products will be denoted by $A^n = A \times \cdots \times A$ and $A^{\wedge n} = A \wedge \cdots \wedge A$. The cofibre sequence of $A \vee B \subset A \times B$ yields a short exact sequence of groups

(A.1.1)

$$0 \to [\Sigma(A \wedge B), Z] \xrightarrow{(\Sigma\pi)^*} [\Sigma(A \times B), Z] \to [\Sigma A, Z] \times [\Sigma B, Z] \to 0.$$

Let p_1, p_2 be the projections of $\Sigma(A \times B)$ onto $\Sigma A, \Sigma B$. The *Whitehead product*

$$[\ \ ,\ \ ]: [\Sigma A, Z] \times [\Sigma B, Z] \to [\Sigma(A \wedge B), Z] \tag{1}$$

is defined by the commutator

$$(\Sigma\pi)^*([\alpha, B]) = -p_1^*\alpha - p_2^*\beta + p_1^*\alpha + p_2^*\beta$$

for $\alpha \in [\Sigma A, Z]$, $\beta \in [\Sigma B, Z]$. Compare Baues [CC]. Let i_1, i_2 be the inclusions of ΣA and ΣB into $\Sigma A \vee \Sigma B$ respectively. Then

$$w_{A,B} = [i_1, i_2] \in \pi_1^{A \wedge B}(\Sigma A \vee \Sigma B) \tag{2}$$

is the *Whitehead product map* for which we have $[\alpha, \beta] = w_{A,B}^*(\alpha, \beta)$.

We say that an element $\alpha \in \pi_n^X(A \vee B)$ is *trivial on B* if the retraction $r_2 = (0,1)\colon A \vee B \to B$ carries α to 0, that is $(r_2)_*(\alpha) = 0$. For example the Whitehead product map $w_{A,B}$ is trivial on ΣB. Let

$$\pi_n^X(A \vee B)_2 = \text{kernel } (r_2)_* : \pi_n^X(A \vee B) \to \pi_n^X B$$

be the subgroup of all elements trivial on B. We have for $n \geq 1$ the *partial suspension*

$$E\colon \pi_n^X(A \vee B)_2 \to \pi_{n+1}^X(\Sigma A \vee B)_2 \tag{3}$$

defined by $E = j^{-1}(\pi \vee 1)_* \partial^{-1}$:

$$\begin{array}{ccc} \pi_n^X(A \vee B)_2 \xleftarrow{\partial} & \pi_n^X(CA \vee B, A \vee B) & \\ & \downarrow{\scriptstyle (\pi \vee 1)_*} & \\ & \pi_n^X(\Sigma A \vee B, B) & \xleftarrow{j} \pi_n^X(\Sigma A \vee B)_2 \end{array}$$

Here $\pi\colon (CA, A) \to (\Sigma A, *)$ is the quotient map and the isomorphisms ∂ and j are obtained from the exact homotopy sequences of pairs of spaces; see (2.1.3). The Whitehead product map is compatible with the partial suspension, that is

(A.1.2) Proposition $\quad Ew_{A,B} = w_{\Sigma A, B}$.

This is proved in (3.1.11) of Baues [OT]. From the definition of the Whitehead product we obtain the following commutator rule in the group $[\Sigma(X_1 \times \cdots \times X_n), Y]$. For $a = \{a_1 < \cdots < a_r\} \subset \bar{n} = \{1, \ldots, n\}$ let

$$p_a\colon X_1 \times \cdots \times X_h \to \wedge X_a = X_{a_1} \wedge \cdots \wedge X_{a_r}$$

be the obvious projection. Then we have for $a, b \subset \bar{n}$ and $\alpha \in [\Sigma \wedge X_a, Y]$ and $\beta \in [\Sigma \wedge X_b, Y]$ the *commutator rule*

(A.1.3)

$$-\alpha(\Sigma p_a) - \beta(\Sigma p_b) + \alpha(\Sigma p_a) + \beta(\Sigma p_b) = [\alpha, \beta] T_{a,b}(\Sigma p_{a \cup b})$$

where

$$T_{a,b}\colon \Sigma \wedge X_{a \cup b} \to \Sigma(\wedge X_a) \wedge (\wedge X_b)$$

is defined by $T_{a,b}(t, x_{a\cup b}) = (t, x_a, x_b)$ with $x_a = (x_{a_1}, \ldots, x_{a_r})$. Clearly, if for $i \in a \cap b$, X_i is a co-H-space then $T_{a,b} \simeq 0$ since the reduced diagonal $\overline{\Delta}_A: A \to A \wedge A$ is null-homotopic if A is a co-H-space.

For any three elements a, b, c of a group G we have the *Witt–Hall identities*

$$(A.1.4)\qquad \begin{aligned} &(a, b\cdot c) = (a,c)\cdot(a,b)\cdot((a,b),c)\\ &((a,b),c^a)\cdot((c,a),b^c)\cdot((b,c),a^b) = 1\end{aligned}$$

where $(x,y) = x^{-1}y^{-1}xy$ and $z^x = x^{-1}zx = z\cdot(z,x)$. Thus with $a = (x,y), b = z, c = (z,x)$ the first equation in (A.1.4) yields the equation

$$(A.1.5)\quad ((x,y),z^x) = ((x,y),(z,x))\cdot((x,y),z)\cdot(((x,y),z),(x,z)).$$

For elements $\alpha_i \in [\Sigma X_i, Z]$ we define the element corresponding to (A.1.5) by

$$(A.1.6)\quad \begin{aligned} W(\alpha_1,\alpha_2,\alpha_3) = {}&[[\alpha_1,\alpha_2],[\alpha_3,\alpha_1]]T_{1231} + [[\alpha_1,\alpha_2],\alpha_3]\\ &+[[[\alpha_1,\alpha_2],\alpha_3],[\alpha_1,\alpha_3]]T_{12313}\end{aligned}$$

in the group $[\Sigma X_1 \wedge X_2 \wedge X_3, Z]$. Here the shuffle

$$T = T_{n_1,\ldots,n_r}: X_1 \wedge \cdots \wedge X_k \to X_{n_1} \wedge \cdots \wedge X_{n_r}$$

for $n_1, \ldots, n_r \in \{1, \ldots, k\}$ maps the tuple $(x_1, \ldots, x_k)$ to $(x_{n_1}, \ldots, x_{n_r})$. Clearly, $T_{1231} = 0$ if X_1 is a co-H-space and $T_{12313} = 0$ if X_1 or X_3 is a co-H-space. The suspension ΣT is also denoted by $T_{n_1,\ldots,n_r}$. We now derive from (A.1.4), (A.1.5), and (A.1.3) the following Jacobi identities for Whitehead products.

(A.1.7) Proposition *The general Jacobi identity for Whitehead products is*

$$W(\alpha_1,\alpha_2,\alpha_3) + W(\alpha_3,\alpha_1,\alpha_2)T_{312} + W(\alpha_2,\alpha_3,\alpha_1)T_{231} = 0.$$

(A.1.8) Corollary *If X_1, X_2, X_3 are* co-H-spaces *the Jacobi identity is*

$$[[\alpha_1,\alpha_2],\alpha_3] + [[\alpha_3,\alpha_1],\alpha_2]T_{312} + [[\alpha_2,\alpha_3],\alpha_1]T_{231} = 0.$$

(A.1.9) Corollary *If X_2, X_3 are* co-H-spaces *the Jacobi identity is*

$$\begin{aligned}&[[\alpha_1,\alpha_2],\alpha_3] + [[\alpha_3,\alpha_1],\alpha_2]T_{312} + [[\alpha_2,\alpha_3],\alpha_1]T_{231}\\ &\qquad = [[\alpha_1,\alpha_2],[\alpha_1,\alpha_3]]T_{1213}.\end{aligned}$$

Moreover, we need the following properties of the Whitehead product map $w_{A,B}$. For pairs of spaces $(A \subset X, B \subset Y)$ we define the product

$$(A.1.10)\qquad \begin{aligned} &(X,A)\times(Y,B) = (X\times Y, X\overline{\times} Y)\\ &X\overline{\times} Y = X\times B \cup A \times Y.\end{aligned}$$

We consider the mapping

(A.1.11) $$\bar{h}\colon \Sigma(A\times B)\to CA\,\overline{\times}\,CB = CA\times B\cup A\times CB$$

which is defined by adding the following homotopies:

$$H_1\colon I\times A\times B\to CA\times B,\quad (t,a,b)\mapsto((t,a),*)$$

$$H_2\colon I\times A\times B\to A\times CB,\quad (t,a,b)\mapsto(a,(t,b))$$

$$H_3\colon I\times A\times B\to CA\times B,\quad (t,a,b)\mapsto((t,a),b)$$

$$H_4\colon I\times A\times B\to A\times CB,\quad (t,a,b)\mapsto(*,(t,b)).$$

Clearly,

$$\bar{h} = -H_1 - H_2 + H_3 + H_4$$

is defined on $\Sigma(A\times B)$ and is null-homotopic on $\Sigma(A\vee B)$. Therefore $\bar{h}$ defines, up to homotopy, a unique map h such that

(A.1.12)
$$\begin{tikzcd} & \Sigma(A\times B) \arrow[dl, "\Sigma\pi"'] \arrow[dr, "\bar{h}"] & \\ \Sigma A\wedge B \arrow[rr, "\simeq", "h"'] & & CA\,\overline{\times}\,CB \end{tikzcd}$$

homotopy commutes. The map h is a homotopy equivalence which we call the *join construction*. One easily checks that the quotient map $j_0\colon(CA,A)\to(CA/A,*)=(\Sigma A,*)$ yields the composition

(A.1.13) $$w_{A,B} = \left(j_0\,\overline{\times}\,j_0\right)h\colon\ \Sigma A\wedge B\to\Sigma A\vee\Sigma B$$

which is the Whitehead product map. We say that $Y\subset X$ is a *principal cofibration* with attaching map $g\colon A\to Y$ if there exists a homotopy equivalence $X\simeq C_g$ under Y where $C_g = Y\cup_g CA$ is the mapping cone of g. Since there is a homotopy equivalence $C(CA\,\overline{\times}\,CB)\simeq CA\times CB$ under $CA\,\overline{\times}\,CB$ and since

$$\begin{tikzcd} CA\times CB \arrow[r, "j_0\times j_0"] & \Sigma A\times\Sigma B \\ CA\,\overline{\times}\,CB \arrow[u, hook] \arrow[r] & \Sigma A\vee\Sigma B \arrow[u, hook] \end{tikzcd}$$

is a push-out diagram one readily gets

(A.1.14) Lemma *$\Sigma A\vee\Sigma B\subset\Sigma A\times\Sigma B$ is a principal cofibration with attaching map $w_{A,B}$.*

We can iterate this process. Consider the product of cones

$$(CX_1, X_2) \times \cdots \times (CX_n, X_n) = (P_n, P_n^\circ).$$

Then there is a homotopy equivalence

(A.1.15) $$\Sigma^{n-1} X_1 \wedge \cdots \wedge X_n \overset{h}{\simeq} P_n^\circ$$

which we obtain inductively by

$$\begin{aligned} P_n^\circ &= (P_{n-1}, P_{n-1}^\circ) \overline{\times} (CX_n, X_n) \\ &\simeq (CP_{n-1}^\circ, P_{n-1}^\circ) \overline{\times} (CX_n, X_n) \\ &\underset{h}{\simeq} \Sigma(P_{n-1}^\circ \wedge X_n). \end{aligned}$$

We call (A.1.15) the iterated join construction; see Baues [IJ]. Now the product of suspensions

$$(\Sigma X_1, *) \times \cdots \times (\Sigma X_n, *) = (T_n, T_n^\circ)$$

yields the pair (T_n, T_n°) where T_n° is called the 'fat wedge'. As in Lemma A.1.14 one can show that the inclusion $T_n^\circ \subset T_n$ is a principal cofibration with the attaching map

(A.1.16) $$w_n \colon \Sigma^{n-1} X_1 \wedge \cdots \wedge X_n \xrightarrow[h]{\simeq} P_n^\circ \xrightarrow[j_0^n]{} T_n^\circ .$$

This map is called the *nth-order Whitehead product map.* Here j_0^n is the restriction of the n-fold product $j_0 \times \cdots \times j_0 \colon CX_1 \times \cdots \times CX_n \to \Sigma X_1 \times \cdots \times \Sigma X_n$. Thus the mapping cone of w_n is homotopy equivalent to T_n.

Further operations we need are the (geometric) cup products. The *exterior cup products* are pairings

(A.1.17) $$\#, \underline{\#} \colon [\Sigma X, \Sigma A] \times [\Sigma Y, \Sigma B] \to [\Sigma X \wedge Y, \Sigma A \wedge B]$$

defined by the composites

$$\alpha \# \beta \colon \Sigma X \wedge Y \xrightarrow{\alpha \wedge Y} \Sigma A \wedge Y \xrightarrow{A \wedge \beta} \Sigma A \wedge B$$

$$\alpha \underline{\#} \beta \colon \Sigma X \wedge Y \xrightarrow{X \wedge \beta} \Sigma X \wedge B \xrightarrow{\alpha \wedge B} \Sigma A \wedge B$$

where $\alpha \wedge Y = \alpha \wedge 1_Y$ and where $A \wedge \beta$ is the map $1_A \wedge \beta$, up to the shuffle of the suspension coordinate.

The interior *cup products* are defined by composing with the reduced diagonal $\bar{\Delta} \colon A \to X \wedge X$,

(A.1.18) $$\cup, \underline{\cup} \colon [\Sigma X, A] \times [\Sigma X, B] \to [\Sigma X, \Sigma A \wedge B]$$

where $\alpha \cup \beta = (\alpha \# \beta)(\tilde{\Delta})$ and similarly $\alpha \underline{\cup} \beta = (\alpha \underline{\#} \beta)(\Sigma\tilde{\Delta})$. If α, β in (A.1.17) are co-H-maps we have $\alpha \# \beta = \alpha \underline{\#} \beta$ and for $\alpha' \in [\Sigma A, Z]$ and $\beta' \in [\Sigma B, Z]$ we get in this case

(A.1.19) $$[\alpha'\alpha, \beta'\beta] = [\alpha', \beta'](\alpha \# \beta).$$

If α and β are not co-H-maps we have to use, instead of (A.1.19), the Barcus–Barratt formula. We now use the exact sequence (A.1.1) for the definition of the *Hopf construction Hf*. Let $f\colon A \times B \to Z$ be given and let

$$(\alpha, \beta)\colon A \vee B \subset A \times B \xrightarrow{f} Z$$

be the restriction of the map f. Then by (A.1.1) there is a unique homotopy class

(A.1.20) $$Hf \in [\Sigma A \wedge B, \Sigma Z]$$

for which $(\Sigma\pi)^*(Hf) = -p_1^*(\Sigma\alpha) - p_2^*(\Sigma\beta) + (\Sigma f)$. Let $\mu_L\colon \Omega L \times \Omega L \to \Omega L$ be the loop addition map for the loop space ΩL and let

(A.1.21) $$H\mu_L \in [\Sigma\Omega L \wedge \Omega L, \Sigma\Omega L]$$

be its Hopf construction. Moreover, let

(A.1.22) $$R = R_L\colon \Sigma\Omega L \to L$$

be the *evaluation map* with $R(t, \sigma) = \sigma(t)$. We have the following connection with the Whitehead product.

(A.1.23) Proposition *Let i_1, i_2 be the inclusions of ΣK and L into $(\Sigma K) \vee L$ respectively. Then*

$$[[i_1, i_2R_L], i_2R_L] = [i_1, i_2R_L] \circ (K \wedge H\mu_L)$$

in $[\Sigma K \wedge \Omega L \wedge \Omega L, \Sigma K \vee L]$.

This result can be proved in the same way as (3.1.22) in Baues [OT].

A.2 The James–Hopf invariants

For a connected space B let $J(B)$ be the *infinite reduced product* of James. The underlying set of $J(B)$ is the free monoid generated by $B - \{*\}$. The topology is obtained by the quotient map

$$\bigcup_{n \geq 0} B^n \xrightarrow{\pi} J(B)$$

mapping a tuple $(b_1,\dots,b_n)$ of the n-fold product $B^n = B\times\cdots\times B$ to the word $(\pi b_1)\cdots(\pi b_n)$ where $\pi(*)$ denotes the empty word and $\pi b = b$ for $b\in B-\{*\}$. The subspace $J_n(B)=\pi(B^n)$ is the *n-fold reduced product* of B and $B=J_1(B)$ generates the monoid $J(B)$.

Let $i\colon B\to\Omega\Sigma B$ be the adjoint of the identity of ΣB. James [RP] has shown that the map

(A.2.1) $$g=g_B\colon J(B)\xrightarrow{\simeq}\Omega\Sigma B$$

with $g(b)=i(b)$ and $g(x\circ b)=g(x)+i(b)$ for $x\in J(B), b\in B-\{*\}$ is a homotopy equivalence. The map g induces the isomorphism of groups

(A.2.2) $$[\Sigma A,\Sigma B]=[A,\Omega\Sigma B]=[A,J(B)],\ \alpha\mapsto\bar{\alpha}.$$

There are mappings

(A.2.3) $$\begin{cases} g_r\colon J(B)\to J(B^{\wedge r}) \\ g_r(b_1\cdots b_n)=\prod\limits_a b_{a_1}\wedge\cdots\wedge b_{a_r} \end{cases}$$

where the product is taken in the lexicographical order from the left over all subsets $a=\{a_1<\cdots<a_r\}$ of $\{1,\dots,n\}$. The *James–Hopf invariants* are the functions

(A.2.4) $$\gamma_r\colon[\Sigma A,\Sigma B]\to[\Sigma A,\Sigma B^{\wedge r}]$$

induced by g_r, that is $\overline{\gamma_r(\alpha)}=(g_r)_*\bar{\alpha}$ where we use the operator $\alpha\mapsto\bar{\alpha}$ in (A.2.2). Clearly γ_1 is the identity.

Remark The map g_r in (A.2.3) can be defined with respect to any 'admissible ordering' of the set of finite subsets of $\mathbb{N}=\{1,2,\cdots\}$, see (I.1.8) and (II.2.3) in Baues [CC]. The lexicographical ordering from the left (resp. from the right) is an example of an admissible ordering.

Let $\tilde{g}\colon\Sigma J(B)\to\Sigma B$ be the adjoint of g in (A.2.1). Then the composite

$$\Sigma(\Omega\Sigma B)\simeq\Sigma J(B)\xrightarrow{\tilde{g}}\Sigma B$$

is homotopic to the evaluation map $R_{\Sigma B}$ in (A.1.22). Moreover,

$$\tilde{g}_r=\gamma_r(\tilde{g})\colon\Sigma J(B)\to\Sigma B^{\wedge r}$$

is the adjoint of g_r in (A.2.3) and

(A.2.5) $$J_B\colon\Sigma\Omega\Sigma B\simeq\Sigma J(B)\underset{G}{\xrightarrow{\simeq}}\bigvee_{r\geq 1}\Sigma B^{\wedge r}$$

is a homotopy equivalence. Here $G = \Sigma_{r \geq 1} j_r \tilde{g}_r$ is the limit of the finite subsums and j_r is the inclusion of $\Sigma B^{\wedge r}$ into the wedge. We will use the following formulas.

(A.2.6) Proposition *For a composite*

$$fg\colon \Sigma X \to \Sigma A \to \Sigma L,$$

with A and L connected and X finite dimensional, we have in $[\Sigma^2 X, \Sigma^2 L^{\wedge n}]$ *the formula*

$$\Sigma(\gamma_n(fg)) = \Sigma\Big(\sum_{r \geq 1} \Gamma_n^r(f) \circ \gamma_r(g)\Big)$$

where

$$\Gamma_n^r(f) = \sum_{\substack{n = i_1 + \cdots + i_r \\ i_1, \ldots, i_r \geq 1}} \gamma_{i_1}(f) \# \cdots \# \gamma_{i_r}(f).$$

For the proof of this formula see Boardman and Steer [HI]. Proposition A.2.6 corresponds to:

(A.2.7) Proposition *Let K be a* co-H-space *and let B and L be finite dimensional and connected spaces. For a map* $\beta\colon \Sigma B \to \Sigma L$ *and for the homotopy equivalences in* (*A.2.5*) *the diagram*

$$\begin{tikzcd} \Sigma K \wedge \Omega\Sigma B \arrow[r, "\Sigma K \wedge \Omega\beta"] \arrow[d, "K \wedge J_B", "\simeq"'] & \Sigma K \wedge \Omega\Sigma L \arrow[d, "K \wedge J_L", "\simeq"'] \\ \bigvee_{r \geq 1} \Sigma K \wedge B^{\wedge r} \arrow[r, "\Gamma(\beta)"'] & \bigvee_{n \geq 1} \Sigma K \wedge L^{\wedge n} \end{tikzcd}$$

homotopy commutes where

$$\Gamma(\beta)_{|\Sigma K \wedge B^{\wedge r}} = \sum_{n=1}^{\infty} i_n \circ \Gamma_n^r(\beta).$$

Here i_n *denoted the inclusion of* $\Sigma K \wedge L^{\wedge n}$ *into the wedge.*

(A.2.8) Proposition *Let A be a* co-H-space *and let B be a connected space. Then for the reduced diagonal $\bar{\Delta}$ on $\Omega\Sigma B$ the diagram*

$$\begin{tikzcd} \bigvee_{n\geq 1} \Sigma A \wedge B^{\wedge n} \arrow[d, "\Delta"'] & & \Sigma A \wedge \Omega\Sigma B \arrow[ll, "\simeq"', "A\wedge J_B"] \arrow[d, "\Sigma A\wedge\bar{\Delta}"] \\ \bigvee_{n,m\geq 1} \Sigma A \wedge B^{\wedge m} \wedge B^{\wedge n} & & \Sigma A \wedge \Omega\Sigma B \wedge \Omega\Sigma B \arrow[ll, "A\wedge(J_b \# J_B)"] \end{tikzcd}$$

homotopy commutes. Here Δ is defined by

$$\Delta_{|\Sigma A \wedge B^{\wedge n}} = \sum_{a\cup b=\bar{n}} i_{\#a,\#b}(\Sigma A \wedge T_{a,b})$$

where we sum over all pairs (a,b) of non-empty subset $a, b \subset \bar{n} = \{1,\ldots,n\}$ with $a \cup b = \bar{n}$. The shuffle map $T_{a,b}$ is described in $(A.1.3)$ and $i_{m,n}$ is the inclusion of $\Sigma A \wedge B^{\wedge m} \wedge B^{\wedge n}$.

(A.2.9) Proposition *Let K be a* co-H-space *and let L be a connected finite dimensional space. Then the Hopf construction $H\mu_{\Sigma L}$ in $(A.1.21)$ for the loop addition map on $\Omega\Sigma L$ makes the following diagram homotopy commute*

$$\begin{tikzcd} \Sigma K \wedge \Omega\Sigma L \wedge \Omega\Sigma L \arrow[r, "K\wedge H\mu_{\Sigma L}"'] \arrow[d, "\simeq"', "K\wedge(J_L \# J_L)"] & \Sigma K \wedge \Omega\Sigma L \arrow[d, "\simeq"', "K\wedge J_L"] \\ \bigvee_{r,s\geq 1} \Sigma K \wedge L^{\wedge r} \wedge L^{\wedge s} \arrow[r, "\varphi"'] & \bigvee_{t\geq 1} \Sigma K \wedge L^{\wedge t} \end{tikzcd}$$

Here φ is the folding map given by the identity $L^{\wedge r} \wedge L^{\wedge s} = L^{\wedge t}$ for $r + s = t$.

We leave the proofs of Propositions A.2.8 and A.2.9 as exercises.

A.3 The fibre of the retraction $A \vee B \to B$ and the Hilton–Milnor theorem

Let $B^+ = \{*\} \cup B$ be the disjoint union of the base point $*$ with the space B. We have

(A.3.1) $$A \rtimes B = A \wedge B^+ = A \times B / * \times B.$$

If A is a co-H-space with comultiplication μ then also $A \rtimes B$ is a co-H-space with comultiplication

$$\mu \rtimes B \colon A \rtimes B \to (A \vee A) \rtimes B = A \rtimes B \vee A \rtimes B.$$

This yields the homotopy equivalence

(A.3.2) $$\bar{\mu} \colon A \rtimes B \xrightarrow{\simeq} A \vee A \wedge B$$

by $\bar{\mu} = (pr \vee \pi)(\mu \rtimes B)$. Here pr and π are the projections of $A \rtimes B$ onto A and onto $A \wedge B$ respectively.

Let $q: P_f \twoheadrightarrow A$ be the principal fibration with classifying map $f: A \to B$; the space P_f is also called the *homotopy theoretic fibre of f* or simply the *fibre of f*. We have

$$P_f = \{(a, \sigma) \in A \times B^I \mid f(a) = \sigma(0), * = \sigma(1)\}$$

where B^I is the function space of all maps $I = [0, 1] \to B$. For the retraction $r_2: A \vee B \to B$ we have the following result.

(A.3.3) Proposition *There is a canonical homotopy equivalence*

$$\pi: P_{r_2} \xrightarrow{\simeq} A \rtimes \Omega B.$$

Proof By definition of P_{r_2} we have

$$P_{r_2} = A \times \Omega B \cup * \times WB \subset (A \vee B) \times WB$$

where $WB = \{\sigma \in B^I \mid \sigma(1) = *\}$ is the contractible path object. Since $\Omega B \subset WB$ is a cofibration the quotient map $\pi: P_{r_2} \to P_{r_2}/WB = A \rtimes \Omega B$ is a homotopy equivalence. Compare also B. Gray's proof of the Hilton–Milnor theorem in Gray [NH]. □

Let

$$q_0: A \rtimes \Omega B \simeq P_{r_2} \twoheadrightarrow A \vee B$$

be given by a homotopy inverse of π in Proposition A.3.3. We thus have the fibre sequence

$$A \rtimes \Omega B \xrightarrow[q_0]{} A \vee B \xrightarrow[r_2]{} B$$

which yields a short exact sequence of homotopy groups

$$0 \to \pi_0^X(A \rtimes \Omega B) \xrightarrow{(q_0)_*} \pi_0^X(A \vee B) \xrightarrow{(r_2)_*} \pi_0^X(B) \to 0$$

where X is a suspension. This shows that q_0 induces the isomorphism

(a) $$(q_0)_*: \pi_0^X(A \rtimes \Omega B) \cong \pi_0^X(A \vee B)_2 = \mathrm{kernel}(r_2)_*.$$

We therefore get the following corollary of Proposition A.3.3.

(b) Corollary *Let A be $(a-1)$-connected and let Y be a subcomplex of the CW-complex X with $X^d \subset Y \subset X$ where X^d is the d-skeleton of X. Then the homomorphism*

$$\pi_k(A \vee Y)_2 \to \pi_k(A \vee X)_2$$

induced by $Y \subset X$ is surjective for $k \leq a + d - 1$ and is an isomorphism for $k \leq a + d - 2$.

Proof We have cell decompositions

$$\begin{aligned} A \rtimes \Omega X &= A \rtimes \Omega(Y \cup e^{d+1} \cup \cdots) \\ &\simeq A \rtimes (\Omega Y \cup e^{d} \cup \cdots) \\ &= (A \rtimes \Omega Y) \cup e^{a+d} \cup \cdots \end{aligned}$$

where we may assume that A is a CW-complex with $A^{a-1} = *$. Now the result follows from the cellular approximation theorem by use of (a) above. □

Using the isomorphism in (a) the partial suspension E, defined in (A.1.1)(3), can be expressed by the usual suspension Σ as follows.

(A.3.4) Proposition *Let X be a suspension, then*

$$\begin{tikzcd} \pi_0^X(A \rtimes \Omega B) \arrow[r, "\cong", "q_{0*}"'] \arrow[d, "\Sigma"] & \pi_0^X(A \vee B)_2 \arrow[d, "E"] \\ \pi_1^X(\Sigma A \rtimes \Omega B) \arrow[r, "\cong", "q_{0*}"'] & \pi_1^X(\Sigma A \vee B)_2 \end{tikzcd}$$

commutes.

Proof We prove the result only in case A is a suspension. In this case commutativity of the diagram is a consequence of Proposition A.3.5 below and of Proposition A.1.2. □

(a) Corollary *Let A be $(a - 1)$-connected. Then*

$$E\colon \pi_k(A \vee B)_2 \to \pi_{k+1}(\Sigma A \vee B)_2$$

is surjective for $k \leq 2a - 1$ and is an isomorphism for $k < 2a - 1$.

This follows from Proposition A.3.4 by using the Freudenthal suspension theorem. We remark that Corollary A.3.4(a) is also a consequence of Theorem A.3.10 below. For the fibre of the retraction $r_2\colon (\Sigma A) \vee B \to B$ we get:

(A.3.5) Proposition *The diagram*

$$\begin{tikzcd} P_{r_2} \arrow[r, phantom, "\overset{\pi}{\simeq}"] \arrow[dr, "q"'] & \Sigma A \rtimes \Omega B \arrow[r, phantom, "\overset{\bar{\mu}}{\simeq}"] \arrow[d, "q_0"] & \Sigma A \vee \Sigma A \wedge \Omega B \arrow[dl, "{(i_1, [i_1, i_2 R_B])}"] \\ & (\Sigma A) \vee B & \end{tikzcd}$$

homotopy commutes. Here i_1, i_2 are the inclusions of ΣA and B into $\Sigma A \vee B$, and R_B is the evaluation map in $(A.1.22)$, $[\ \ ,\ \ ]$ *is the Whitehead product and $\bar{\mu}$ is defined in* $(A.3.2)$.

Proof Consider the diagram

$$\begin{tikzcd}
C\Omega B \arrow[r, dashed, "r"] \arrow[d, "j"] & WB \arrow[d, "p"] \\
\Sigma\Omega B \arrow[r, "R"] & B
\end{tikzcd}$$

where j is the pinch map and $p(\sigma) = \sigma(0)$. There is a mapping r which extends the inclusion $\Omega B \subset WB$ over the cone and for which the diagram homotopy commutes *relative* to ΩB. With r we obtain the homotopy commutative diagram

$$\begin{tikzcd}
P_{r_2} = \Sigma A \times \Omega B \cup_{\Omega B} WB \arrow[dr, "\pi"] \arrow[d, "q"] & & \Sigma A \times \Omega B \cup C\Omega B \arrow[ll, "1 \cup r"', "\simeq"] \arrow[dl, "\simeq"'] \arrow[d, "i_1 pr \cup i_2 j"] \\
\Sigma A \vee B & \Sigma A \rtimes \Omega B \arrow[l, "q_0"] & \Sigma A \vee \Sigma\Omega B \arrow[ll, bend left, "\mathrm{id} \vee R_B"]
\end{tikzcd}$$

Proposition A.3.5 now follows from the general fact that

$$\begin{tikzcd}
 & (\Sigma A) \rtimes B \arrow[dr, "\bar{\mu}", "\simeq"'] & \\
\Sigma A \times B \cup_B CB \arrow[ur, "p", "\simeq"'] \arrow[dr, "i_1 pr \cup i_2 j"'] & & \Sigma A \vee \Sigma A \wedge B \arrow[dl, "{(i_1, [i_1, i_2])}"] \\
 & \Sigma A \vee \Sigma B &
\end{tikzcd}$$

homotopy commutes. Here $j: CB \to \Sigma B$ is the pinch map and p is the quotient map. □

In addition to Proposition A.3.5 we have the following result for the fibre of the retraction $r_2: \Sigma A \vee \Sigma B \to \Sigma B$.

(A.3.6) Proposition *Let A be a* co-H-space *and let B be connected. Then the diagram*

$$\begin{tikzcd}
P_{r_2} \arrow[r, "\pi", "\simeq"'] \arrow[dr, "q"'] & \Sigma A \vee \Sigma A \wedge \Omega\Sigma B \arrow[r, "J", "\simeq"'] \arrow[d, "{(i_1, [i_1, i_2 R_B])}"] & \bigvee_{r \geq 0} \Sigma A \wedge B^{\wedge r} \arrow[dl, "W"] \\
 & \Sigma A \vee \Sigma B &
\end{tikzcd}$$

homotopy commutes. Here $\pi = \bar{\mu}\pi$ *as in Proposition A.3.5 and* $\bar{J} = 1 \vee (A \wedge J_B)$ *as in* $(A.2.5)$. *The mapping* W *is given by the r-fold Whitehead products*

$$W|_{\Sigma A \wedge B^{\wedge r}} = [\cdots[i_1, i_2], \ldots, i_2].$$

For $r = 0$ *we set* $W|_{\Sigma A} = i_1$.

Proof We have to show

$$W \circ (A \wedge J_B) = [i_1, i_2 R_B],$$

but this is a consequence of II.3.4 in Baues [CC]. □

Proposition A.3.6 is the basic step in the proof of the Hilton–Milnor theorem. To state the Hilton–Milnor theorem precisely, we need a certain amount of formal algebra (we follow the excellent presentation of Boardman and Steer [HI]). Let $B = B_1 \vee \cdots \vee B_k$ be a one-point union of co-H-spaces. Take abstract symbols $z_1, z_2, \ldots, z_k$ and let L be the free Lie algebra (over $\mathbb{Z}$) generated by the letters $z_1, \ldots, z_k$. Let F be the free non-associative algebraic object generated by $z_1, \ldots, z_k$ with one binary operation [,]. F is the set of 'brackets' or of 'formal commutators' in the letters $z_1, \ldots, z_k$. There is an obvious map $F \to L$ which we suppress from the notation. The *weight* $wt(a)$ of an element $a \in F$ is the number of factors in it. By induction on weight we define for each $c \in F$ the space

$$\wedge^c B = \begin{cases} B_r & \text{if} \quad c = z_r \\ (\wedge^a B) \wedge (\wedge^b B) & \text{if} \quad c = [a, b] \end{cases}$$

and the iterated Whitehead product $w_c \in [\Sigma \wedge^c B, \Sigma B]$ by

$$w_c = \begin{cases} \text{the class of the inclusion } \Sigma B_r \subset \Sigma B & \text{if} \quad c = z_r, \\ [w_a, w_b] & \text{if} \quad c = [a, b]. \end{cases}$$

For a family of spaces (P_α) let $\prod_\alpha P_\alpha$ be the direct limit of the finite subproducts. It is well known that the free Lie algebra L is a free abelian group and that there exists a subset Q of F which yields a base of the free abelian group L; such a set Q is called a *set of basic commutators*.

(A.3.7) Theorem (*Hilton–Milnor*): *Let* Q *be a set of basic commutators and give* Q *any total ordering. Then the map*

$$\prod_{c \in Q} \Omega w_c : \prod_{c \in Q} \Omega \Sigma \wedge^c B \to \Omega \Sigma B,$$

defined by using the multiplication in $\Omega \Sigma B$ *in the order indicated by* Q, *is a homotopy equivalence.*

The following *recipe for the construction of a set of basic commutators, Q*, is available. We define and order Q inductively. The elements of weight 1 in Q are the elements $z_1, \dots, z_k$ with $z_1 < \cdots < z_k$. Now suppose that all elements of weight $< w$ in Q are defined and ordered. Then an element in Q of weight $w > 1$ is a bracket $[a, b]$ where $wt(a) + wt(b) = w, a < b$, and if $b = [c, d]$ then $c \leq a$. The elements of weight w are then ordered arbitrarily among themselves and are greater than any element of less weight.

Proof of Theorem A.3.7 We indicate the proof for the wedge $A \vee B$ of two connected co-H-spaces ($B_1 = A, B_2 = B$). Since $r_2 \colon A \vee B \to B$ is the retraction we obtain by Proposition A.3.6 the isomorphism

$$\begin{array}{c} \pi_n(\Sigma B) \oplus \pi_n\left(\bigvee\limits_{r \geq 0} \Sigma A \wedge B^{\wedge r} \right) \\ \cong \Big\downarrow (i_2)_* + W_* \\ \pi_n(\Sigma A \vee \Sigma B) \end{array} \tag{$*$}$$

Here the group $\pi_n(\Sigma A' \vee \Sigma B')$ with $A' = \vee\{A \wedge B^{\wedge r} \mid r \geq 1\}$ and $B' = A$ has again a splitting as in $(*)$. This way we obtain inductively the proposition of the Hilton–Milnor theorem. Since we assume A and B to be connected the connectivity of the fibres is raised by the inductive steps. Such considerations are also valid if A and B are not co-H-spaces. $\square$

Let $g \colon A \to B$ be a map. The fibre of the retraction $A \vee B \to B$ is the first *approximation* of the fibre P_{i_g} of the principal cofibration $i_g \colon B \rightarrowtail C_g$. To see this we consider the commutative diagram of unbroken arrows:

(A.3.8)
$$\begin{tikzcd}
A \rtimes \Omega B \arrow[rrr, dashed, bend left, "\tau"] \arrow[dr, dashed, "q_0"'] & P_{r_2} \arrow[l, "\simeq", "\pi"'] \arrow[d, "q"] & P_{i_0} \arrow[l, "\simeq", "\alpha"'] \arrow[r, "\tau_0"'] \arrow[d] & P_{i_g} \arrow[d] \\
& A \vee B \arrow[r, equal] \arrow[d, "r_2"] & A \vee B \arrow[r, "{(g,1)}"'] \arrow[d, "i_0"] \arrow[dr, phantom, "\circledast"] & B \arrow[d, "i_g"] \\
& B & CA \vee B \arrow[l, "\simeq", "r_2"'] \arrow[r, "{(\pi_g,1)}"'] & C_g
\end{tikzcd}$$

Here all columns are fibre sequences. Clearly, the map α, induced by r_2, is a homotopy equivalence. Thus for the map τ_0, induced by $(\pi_g, 1)$, we obtain $\tau = \tau_0 \alpha^{-1} \pi^{-1}$ by homotopy inverses of α and π. The subdiagram $\circledast$ is a push-out diagram which defines the mapping cone C_g; the map i_0 is given by the inclusion $A \subset CA$.

For the inclusion $V \subset W$ we have the natural isomorphism of relative homotopy groups

$$\pi_1^X(W, V) \cong \pi_0^X(P_i)$$

where P_i is the fibre of $i: V \subset W$. This isomorphism carries the homotopy class of a pair map $F:(CX, X) \to (W, V)$ to the homotopy class of the adjoint map $\bar{F}: X \to P_i$ with $\bar{F}(x) = (F(x), \sigma_x)$, $\sigma_x(t) = F(t, x)$. Now diagram (A.3.8) shows that the diagram of homotopy groups

(A.3.9)
$$\begin{array}{ccc}
\pi_0^X(A \rtimes \Omega B) & \xrightarrow{\tau_*} & \pi_0^X(P_{i_g}) \\
\cong \downarrow (q_0)_* & & \\
\pi_0^X(A \vee B)_2 & & \uparrow \cong \\
\cong \uparrow a & & \\
\pi_1^X(CA \vee B, A \vee B) & \xrightarrow[(\pi_g, 1)_*]{} & \pi_1^X(C_g, B)
\end{array}$$

commutes. Here we assume that X is a suspension and a CW-complex. A map $i: V \to W$ is *n-connected* if the fibre P_i is $(n-1)$-connected or equivalently if $i_*: \pi_i V \to \pi_i W$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. If $i: V \to W$ is n-connected we know that

$$i_*: \pi_0^X(V) \to \pi_0^X(W)$$

is an isomorphism if $\dim X < n$ and an epimorphism for $\dim X \leq n$. This is easily seen by the cellular approximation theorem since we may assume that V is a subcomplex of the CW-complex W and that V contains the n-skeleton of W.

(A.3.10) Theorem *Let $g: A \to B$ be a mapping where A is $(a$-$1)$-connected. Then*

$$\tau: A \rtimes \Omega B \to P_{i_g}$$

is $(2a-1)$-connected, or equivalently $(\pi_g, 1)_*$ in (A.3.9) is an isomorphism for $\dim X < 2a - 1$ *and an epimorphism for* $\dim X \leq 2a - 1$.

Using (A.3.9) this result is a special case of the general suspension theorem (V.7.6) Baues [AH], see also (3.4.7) Baues [OT]. A different proof of Theorem A.3.10 can be obtained by Corollary A.6.3 below.

We point out that the maps q and q_0 in (A.3.8) can be replaced by the corresponding maps in Propositions A.3.5 and A.3.6 provided A and B are suspensions. Moreover, the map τ in (A.3.8) has the following property.

(A.3.11) Lemma *The diagram*

$$\begin{array}{ccc}
A \rtimes \Omega B & \xrightarrow{pr} & A \\
\tau \downarrow & & \downarrow i \\
P_{i_g} & \xrightarrow[\lambda]{} & \Omega \Sigma A
\end{array}$$

homotopy commutes. Here i is the adjoint of the identity on ΣA and λ is induced by the pair map $j_g: (C_g, B) \to (\Sigma A, *)$.

A.4 The loop space of a mapping cone

The loop space ΩB is the subspace of B^I consisting of all paths $\sigma: I \to B$ with $\sigma(0) = \sigma(1) = *$. The loop space is an H-space by the addition of paths; this addition, however, is not associative. Therefore, we also use the following loop space of Moore which is a topological monoid.

(A.4.1) Definition Let $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$. The *Moore loop space* $\tilde{\Omega} B$ is the subspace of $B^{\mathbb{R}_+} \times \mathbb{R}_+$ consisting of all pairs (σ, k_σ) where $\sigma: \mathbb{R}_+ \to B$ is a map with $\sigma(0) = \sigma(t) = *$ for $t \geq k_\sigma$. The space $\tilde{\Omega} B$ is a topological monoid by addition

$$(\sigma, k_\sigma) + (\tau, k_\tau) = (\sigma + \tau, k_\sigma + k_\tau)$$

with $(\sigma + \tau)(t) = \sigma(t)$ for $t \leq k_\sigma$ and $(\sigma + \tau)(k_\sigma + t) = \tau(t)$ for $t \geq 0$. The inclusion $i: \Omega B \subset \tilde{\Omega} B, \sigma \mapsto (\sigma, 1)$ is an H-map and a homotopy equivalence.

We now consider the loop space $\tilde{\Omega} C_g$ of a mapping cone C_g where $g: \Sigma A \to B$ is defined on a suspension ΣA. In this case we have the following commutative diagram

(A.4.2)
$$\begin{tikzcd} A \arrow[r, "i_0"'] \arrow[d, "\bar{g}"] & CA \arrow[d, "\bar{\pi}_g"] \\ \Omega B \arrow[d, phantom, "\cap"] & \Omega C_g \arrow[d, phantom, "\cap"] \\ \tilde{\Omega} B \arrow[r, "\tilde{\Omega} i_g"] & \tilde{\Omega} C_g \end{tikzcd}$$

Here $\bar{g}$ is the adjoint of g and $\bar{\pi}_g$ is the adjoint of

$$\pi_g T: \Sigma CA \xrightarrow[T]{} C\Sigma A \xrightarrow[\pi_g]{} C_g$$

where T interchanges the C-coordinate and the Σ-coordinate. The map π_g is given by the definition of a mapping cone, see (A.3.8), and $i_g: B \subset C_g$ is the inclusion.

We derive from (A.4.2) the following push-out diagram in the category of topological monoids:

(A.4.3)
$$\begin{tikzcd} J(A) \arrow[r, "J(i_0)"] \arrow[d, "(\bar{g})_\infty"'] \arrow[dr, phantom, "\text{push}"] & J(CA) \arrow[d] \arrow[dd, bend left=60, "(\bar{\pi}_g)_\infty"] \\ \tilde{\Omega} B \arrow[r, "j"'] \arrow[dr, "\tilde{\Omega} i_g"'] & M_g \arrow[d, "m"] \\ & \tilde{\Omega} C_g \end{tikzcd}$$

Here $(\bar{g})_\infty$ and $(\bar{\pi}_g)_\infty$ are the extensions of $\bar{g}$ and $\bar{\pi}_g$ in (A.4.2) which are homomorphisms of topological monoids; $J(A)$ is the infinite reduced product of James, see Section A.2. For the push-out M_g we have the unique map m which is a homomorphism of monoids and for which diagram (A.4.3) commutes.

(A.4.4) Theorem *Assume that the space A is connected and that B is 1-connected. Then the map m in (A.4.3) is a homotopy equivalence of spaces. We call $M_g \xrightarrow{\simeq} \Omega C_g$ the model of ΩC_g.*

Proof Using the work of Adams and Hilton [CA] and Lemaire [AC] we see that m induces an isomorphism in homology and hence is a homotopy equivalence since M_g and ΩC_g are connected. The theorem can also be considered as a very special case of the model construction in Baues [GL]. Theorem A.4.4 is the basic step for the CW-models of loop spaces of Husseini [TC], [CR] and Toda [CS]. □

We obtain the topology of M_g by the surjective quotient map

(A.4.5)
$$\pi\colon \bigcup_{n\geq 0} \tilde{\Omega} B \times (CA \times \tilde{\Omega} B)^n \to M_g$$

with $\pi(b_0, a_1, b_1, \dots, a_n, b_n) = (ib_0)\cdot(\pi a_1)\cdot(ib_1)\cdot \cdots \cdot(\pi a_n)\cdot(ib_n)$ where $b_i \in \tilde{\Omega} B$ and $a_i \in CA$. The monoid multiplication on M_g is represented by

$$(b_0, a_1, b_1, \dots, a_n, b_n)(b'_0, a'_1, \dots, a'_m, b'_m)$$
$$= (b_0, a_1, b_1, \dots, a_n, b_n + b'_0, a'_1, \dots, a'_m, b'_m).$$

Clearly, the spaces

(A.4.6)
$$M_n = \pi\left(\tilde{\Omega} B \times (CA \times \tilde{\Omega} B)^n\right)$$

give us the filtration

$$\tilde{\Omega} B = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots \subset M_g$$

of M_g. We obtain M_n from M_{n-1} by the push-out diagram of spaces in (A.4.7) below. For the pairs of spaces $(X, A), (Y, B)$ we define the product as in (A.1.10). Moreover, let

$$((CA)^n, (CA)^{\circ n}) = (CA, A) \times \cdots \times (CA, A)$$

be the n-fold product as in (A.1.15) and let

$$T: (CA)^n \times (\tilde{\Omega} B)^{n+1} \to (\tilde{\Omega} B) \times (CA \times \tilde{\Omega} B)^n$$

be the canonical permutation of coordinates. The mapping π in (A.4.5) yields the commutative diagram

(A.4.7)
$$\begin{tikzcd} T((CA)^{\circ n} \times (\tilde{\Omega} B)^{n+1}) \arrow[d] \arrow[r, phantom, "\subset"] & (\tilde{\Omega} B) \times (CA \times \tilde{\Omega} B)^n \arrow[d, "\pi"] \\ M_{n-1} \arrow[r, phantom, "\subset"] & M_n \end{tikzcd}$$

This diagram is a push-out diagram of spaces. For example M_1 is given by the push-out diagram

(A.4.8)
$$\begin{tikzcd} \tilde{\Omega} B \times A \times \tilde{\Omega} B \arrow[r, hook] \arrow[d, "\pi"] & \tilde{\Omega} B \times CA \times \tilde{\Omega} B \arrow[d] \\ \tilde{\Omega} B \arrow[r, hook] & M_1 \end{tikzcd}$$

with $\pi(b_0, a, b_1) = b_0 + (\bar{g}a) + b_1$ for $a \in A$ and with $\bar{g}$ as in (A.4.2).

The pinch map $j: C_g \to C_g/B = \Sigma^2 A$ has the following property:

(A.4.9) Proposition *The diagram*

$$\begin{tikzcd} M_g \arrow[r, "\simeq", "m"'] \arrow[d, "j"] & \Omega C_g \arrow[d, "\Omega j"] \\ J(\Sigma A) \arrow[r, "\simeq", "g"'] & \Omega \Sigma^2 A \end{tikzcd}$$

homotopy commutes. Here g is the equivalence in (A.2.1) and j is defined on M_g by

$$j(a_0, b_1, \ldots, b_n, a_n) = (j_0 a_0) \cdot \ldots \cdot (j_0 a_n)$$

where $j_0: CA \to CA/A = \Sigma A$ is the quotient map.

A.5 The fibre of a principal cofibration

Let $g: \Sigma A \to B$ be a mapping as in Section A.4 where ΩB and A are connected. For the principal cofibration $i_g: B \rightarrowtail C_g$ we have the fibre sequence

$$\to \Omega B \xrightarrow[\Omega i_g]{} \Omega C_g \xrightarrow[i]{} P_{i_g} \xrightarrow[q]{} B \xrightarrow[i_g]{} C_g;$$

compare (A.3.8). The map i is the inclusion of the fibre $q^{-1}(*) = \Omega C_g$. In Section A.4 we constructed the model

$$M_g \xrightarrow{\simeq} \Omega C_g$$

for the loop space ΩC_g. We now consider the fibre P_{i_g} and we define a model N_g for the space P_{i_g} as follows.

(A.5.1) Definition Let $\sim$ be the equivalence relation on M_g which is generated by $x \sim i(b) \cdot x$ for $b \in \tilde{\Omega} B$, $x \in M_g$. We define N_g by the quotient space $N_g = M_g / \sim$. Let

$$\nu: M_g \to M_g / \sim \, = N_g$$

be the quotient map.

(A.5.2) Theorem *There is a canonical homotopy equivalence n such that the diagram*

$$\begin{array}{ccc} C_g & \xrightarrow{i} & P_{i_g} \\ \simeq \uparrow m & & \simeq \uparrow n \\ M_g & \xrightarrow{\nu} & N_g \end{array}$$

homotopy commutes.

We prove this theorem, which is a basic result of this chapter, in (A.5.10) below. The filtration of M_g in (A.4.6) induces the following filtration on N_g. Let $N_n = \nu(M_n)$ be the image of the subspace M_n of M_g in (A.4.6). Then clearly we have the filtration

$$N_0 = * \subset N_1 \subset N_2 \subset \cdots \subset N_n \subset \cdots \subset N_g$$

on N_g. From (A.4.8) and Definition A.5.1 we derive

(A.5.3) $$N_1 = CA \times \tilde{\Omega} B / A \times \tilde{\Omega} B = (\Sigma A) \rtimes \tilde{\Omega} B.$$

(A.5.4) Proposition *The mapping τ in (A.3.8) makes the diagram*

$$\begin{tikzcd} (\Sigma A) \rtimes \tilde{\Omega} B \arrow[r, phantom, "="] & N_1 \arrow[r, phantom, "\subset"] & N_g \arrow[d, "\simeq"', "n"] \\ (\Sigma A) \rtimes \Omega B \arrow[u, "\simeq", "(-1)\rtimes i"'] \arrow[rr, "\tau"'] & & P_{i_g} \end{tikzcd}$$

homotopy commutative.

We prove this result in (A.5.11) below.

(A.5.5) Theorem *The inclusion*

$$N_{n-1} \subset N_n$$

is a principal cofibration with attaching map

$$\omega_n \colon \Sigma^{n-1} A^{\wedge n} \rtimes (\Omega B^n) \to N_{n-1}$$

where $\Omega B^n = \Omega B \times \cdots \times \Omega B$ is the n-fold product.

Proof The mapping $\nu\pi$ with ν in Theorem A.5.2 and π in (A.4.7) gives us the push-out diagram of spaces (compare (A.4.7)):

$$\begin{tikzcd} (CA)^{\circ n} \rtimes (\tilde{\Omega} B^n) \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d, "\nu\pi"'] & (CA \times \tilde{\Omega} B)^n / (* \times \tilde{\Omega} B)^n \arrow[d] \\ N_{n-1} \arrow[r, phantom, "\subset"] & N_n \end{tikzcd} \tag{1}$$

Here we use the fact that $\nu\pi(* \times \tilde{\Omega} B)^n = *$ by Definition A.5.1. We omit the obvious permutation T in (1); see (A.4.7). Now i in (1) is a closed cofibration into a contractible space. Therefore i is equivalent to the inclusion into the cone. Moreover, we have the homotopy equivalence

$$(CA)^{\circ n} \xleftarrow[h_n]{\simeq} \Sigma^{n-1} A^{\wedge n} \tag{2}$$

which is the iterated join construction in (A.1.15). Now the mapping ω_n is given by

$$\omega_n = \nu\pi\big(h_n \rtimes (\tilde{\Omega} B^n)\big). \qquad \square$$

The results above are also available for the trivial map $g = 0 \colon \Sigma A \to * = B$. In this case N_g is essentially the reduced product space $J(\Sigma A)$ in (A.2.1). The reduced product filtration $J_n(\Sigma A)$ corresponds to the filtration of N_g above. The inclusion

(A.5.6) $$J_{n-1}(\Sigma A) \subset J_n(\Sigma A)$$

is a principal cofibration with attaching map

(A.5.7) $$W_n\colon \Sigma^{n-1}A^{\wedge n} \to J_{n-1}(\Sigma A).$$

Here W_n factors through the higher-order Whitehead product map

$$W_n\colon \Sigma^{n-1} A^{\wedge n} \to T_n^{\circ} \xrightarrow{\pi} J_{n-1}(\Sigma A)$$

where $(T_n, T_n^{\circ}) = (\Sigma A, *)^n$; see (A.1.16). The map π is the projection in (A.2.1). In particular

$$W_2 = [1_{\Sigma A}, 1_{\Sigma A}]\colon \Sigma A \wedge A \to \Sigma A$$

is the Whitehead product for the identity $1_{\Sigma A}$ of ΣA. Theorem A.5.5 shows that N_g has an iterated mapping cone structure which generalizes the one of $J(\Sigma A)$. We now compare N_g and $J(\Sigma A)$ by using the quotient map $j_g\colon (C_g, B) \to (\Sigma^2 A, *)$.

We consider the homotopy commutative diagram

(A.5.8)
$$\begin{tikzcd} N_g \arrow[r, "\simeq", "n"'] \arrow[d, "j"] & P_{i_g} \arrow[d, "\lambda"] \\ J(\Sigma A) \arrow[r, "\simeq", "n_0"'] & \Omega\Sigma\Sigma A \end{tikzcd}$$

where λ is induced by the pinch map j_g as in Lemma A.3.11 and where

$$n_0 = (\Omega T) g_{\Sigma A} \simeq (\Omega(-1)) g_{\Sigma A}$$

is given by the homotopy equivalence $g_{\Sigma A}$ in (A.2.1). The map $T\colon \Sigma^2 A \to \Sigma^2 A$ with $T(t_1, t_2, a) = (t_2, t_1, a)$ is homotopic to -1 where 1 is the identity on $\Sigma^2 A$. We define j in (A.5.8) by

$$j(b_1, a_1, \ldots, b_n, a_n) = (j_0 a_1) \cdot \cdots \cdot (j_0 a_n)$$

where $j_0\colon CA \to \Sigma a$ is the pinch map; see Proposition A.4.9. We derive from Proposition A.4.9 that (A.5.8) actually homotopy commutes. Moreover, by the proof of Theorem A.5.5 we obtain:

(A.5.9) Proposition *The map j in (A.5.8) is filtration preserving and the pair map*

$$j\colon (N_n, N_{n-1}) \to (J_n, J_{n-1})$$

with $J_n = J_n(\Sigma A)$ is a canonical map between principal cofibrations, that is: the attaching maps can be chosen such that the diagram

$$\begin{tikzcd}
\Sigma^{n-1}A^{\wedge n} \rtimes \Omega B^n \arrow[r, "pr"] \arrow[d, "\omega_n"] & \Sigma^{n-1}A^{\wedge n} \arrow[d, "W_n"] \\
N_{n-1} \arrow[r, "j"'] & J_{n-1}
\end{tikzcd}$$

commutes and there are homotopy equivalences for which the diagram

$$\begin{tikzcd}
C_{\omega_n} \arrow[r, "\bar{j}"'] \arrow[d, "\simeq"] & C_{W_n} \arrow[d, "\simeq"'] \\
N_n \arrow[r, "j"'] & J_n
\end{tikzcd}$$

homotopy commutes relative to N_{n-1}. Here $\bar{j}$ is the map $\bar{j} = j \cup C(pr)$.

Recall that the map $Cr: CU \to CV$ denotes the cone on $r: U \to V$ with $(Cr)(t, u) = (t, ru)$ for $t \in I, u \in U$.

(A.5.10) Proof of Theorem A.5.2 For a path-connected topological monoid M let

$$\underline{B}M: \Delta^{\mathrm{op}} \to \mathbf{Top} \tag{1}$$

be the geometric bar construction; see (I.1.5) in Baues [GL]. The realization of this simplicial space, $|\underline{B}M|$, is a classifying space for M. As in (I.1.6) in Baues [GL] we have the quasi-fibration

$$M \to |E_M| \to |\underline{B}M| \tag{2}$$

where $|E_M|$ is contractible. A space B gives us the homotopy equivalence

$$|\underline{B}\tilde{\Omega}B| \simeq B. \tag{3}$$

For the inclusion $i: \tilde{\Omega}B \subset M_g$ of monoids we consider the pull-back

$$\begin{tikzcd}
i^*|E_{M_g}| \arrow[r, phantom, "\subset"] \arrow[d, "q"] & {|E_{M_g}|} \arrow[d] \\
B \simeq |\underline{B}\tilde{\Omega}B| \arrow[r, phantom, "\subset"'] \arrow[r, phantom, "i"', yshift=-8pt] & {|\underline{B}M_g|}
\end{tikzcd} \tag{4}$$

Clearly, we have a homotopy equivalence n_1 for which

$$\begin{tikzcd}
M_g \arrow[r, "m", "\simeq"'] \arrow[d] & \Omega C_g \arrow[d] \\
i^*|E_{M_g}| \arrow[r, "n_1"'] & P_{i_g}
\end{tikzcd} \tag{5}$$

homotopy commutes. We now define a map n_2 for which

$$\begin{tikzcd} & M_g \arrow[dl, "\nu"'] \arrow[d, "j"] \\ M_g/\sim\, = N_g & i^*|E_{M_g}| \arrow[l, "n_2"'] \end{tikzcd} \tag{6}$$

is commutative and we show that n_2 is a homotopy equivalence. First we observe that $i^*|E_{M_g}|$ is the realization of the following simplicial space F,

$$F\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$$

$$\begin{cases} F(\Delta(n)) = (\tilde{\Omega}B)^n \times M_g \\ F(d_0) = pr_1\colon (\tilde{\Omega}B)^n \times M_g \to (\tilde{\Omega}B)^{n-1} \times M_g \\ F(d_i) = \mu_i\colon (\tilde{\Omega}B)^n \times M_g \to (\tilde{\Omega}B)^{n-1} \times M_g \\ \qquad \text{for} \quad i = 1,2,\dots,n \\ F(s_i) = j_{i+1}\colon (\tilde{\Omega}B)^{n-1} \times M_g \to (\tilde{\Omega}B)^n \times M_g \\ \qquad \text{for} \quad i = 0,1,\dots,n-1. \end{cases} \tag{7}$$

Here d_i are the face operators and s_i are the degeneracy operators in the simplicial category Δ^{op}; see §1 in Baues [GL]. Moreover

$$\mu_i(x_1,\dots,x_n) = (x_1,\dots,x_i \cdot x_{i+1},\dots,x_n)$$

$$pr_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

$$j_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},*,x_i,\dots,x_n).$$

We have by definition in the proof of I.1.6 in Baues [GL] the canonical equivalence

$$|F| = i^*|E_{M_g}| \tag{8}$$

of realizations. The projection q in (4) is the realization of the natural transformation

$$q\colon F \to \underline{B}\tilde{\Omega}B \tag{9}$$

with

$$q = pr_{n+1}\colon F(\Delta(n)) = (\tilde{\Omega}B)^n \times M_g \to (\tilde{\Omega}B)^n. \tag{10}$$

We now define n_2 in (6). Let

$$n_2 : |F| \to N_g \tag{11}$$

be induced by the projections

$$\Delta^n \times (\Omega B)^n \times M_g \to M_g \xrightarrow{\nu} N_g.$$

We have to check that n_2 is well defined. In fact, this is clear by definition of μ_n above and by Definition A.5.1. Moreover, it is clear that n_2 makes (6) commute since j in (6) is given by

$$M_g = \Delta^0 \times F(\Delta(0)) \subset |F|.$$

We now claim that

$$|E_{\tilde{\Omega} B}| \subset |F| \xrightarrow{n_2} N_g \tag{12}$$

is a quasi-fibration with fibre $|E_{\tilde{\Omega} B}|$. Since $|E_{\tilde{\Omega} B}|$ is contractible we conclude that n_2 is a homotopy equivalence. Thus by (5) and (6) we have the proposition in Theorem A.5.2.

For the proof of (12) we consider the restrictions:

$$\begin{array}{ccccc} |F|_{n-1} & \subset & |F|_n & \subset & |F| \\ \downarrow{\scriptstyle p_{n-1}} & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle n_2} \\ N_{n-1} & \subset & N_n & \subset & N_g \end{array} \tag{13}$$

with $|F|_n = n_2^{-1}(N_n)$. By definition of F in (7) and of n_2 in (11) we see that

$$|F|_0 = |E_{\tilde{\Omega} B}|. \tag{14}$$

Now we assume that p_{n-1} is a quasi-fibration with fibre $|E_{\Omega B}|$. In (1) of the proof for Theorem A.5.5 we obtained a push-out diagram for the inclusion $N_{n-1} \subset N_n$ which gives us the push-out diagram

$$\begin{array}{ccc} (CA)^{\circ n} \times (\tilde{\Omega} B)^n & \subset & (CA \times \tilde{\Omega} B)^n \\ {\scriptstyle \nu\pi}\downarrow & & \downarrow{\scriptstyle \nu\pi} \\ N_{n-1} & \subset & N_n \end{array} \tag{15}$$

It is clear from the definition of F, that if we pull back $|F|_n$ over $\nu\pi$, we have

$$(\nu\pi)^* |F|_n = |E_{\tilde{\Omega} B}| \times (CA \times \tilde{\Omega} B)^n. \tag{16}$$

Therefore we obtain a push-out diagram

$$\begin{array}{ccc} |E_{\tilde{\Omega}B}| \times (CA)^{\circ n} \times (\tilde{\Omega}B)^n & \subset & |E_{\tilde{\Omega}B}| \times (CA \times \tilde{\Omega}B)^n \\ \downarrow & & \downarrow \\ |F|_{n-1} & \subset & |F|_n \end{array}$$

lying over the push-out (15). The push-out property of quasi-fibrations (see Hardie [QA]) gives us the result that p_n in (13) is a quasi-fibration. Thus also n_2 is a quasi-fibration (see V. Puppe [RH]). □

(A.5.11) Proof of Proposition A.5.4 Since the mappings in Proposition A.5.4 are equivariant with respect to the action from the right of ΩB we obtain the result by proving that the following diagram homotopy commutes:

$$\begin{array}{ccccc} \Sigma A & \subset & N_1 & \subset & N_g \\ \uparrow{\scriptstyle -1} & & & & \simeq \downarrow{\scriptstyle n} \\ \Sigma A & & \xrightarrow[\tau|_{\Sigma A}]{} & & P_{i_g} \end{array}$$

Now $\tau|_{\Sigma A}$ corresponds to the map $\Sigma A \to WC\Sigma A$ which is adjoint to the identity of $C\Sigma A$. However, $n|_{\Sigma A}$ corresponds to the map $s: \Sigma A \to WC\Sigma A$ given by

$$s(\tau, a)(r) = (0, 1 - 2r + 2\tau r, a) \in C\Sigma A$$

for $0 \leq \tau \leq \frac{1}{2}$ and

$$s(\tau, a)(r) = (2\tau - 1, r, a) \in C\Sigma A$$

for $\frac{1}{2} \leq \tau \leq 1, r \in I, a \in A$. In $\pi_2^A(C\Sigma A, \Sigma A)$ the mapping s represents the element $-1_{C\Sigma A}$. □

(A.5.12) Remark For the special case that $g: \Sigma A \to B = \Sigma B'$ is a map between suspensions we gave a different proof of Theorem A.5.2 in Baues [RP]. This proof relies on a construction of Gray [HM] which is only available if B is a suspension. All results in Baues [RP] are special cases of the results in this chapter.

A.6 EHP sequences

For a map $g: \Sigma A \to B$ we have the isomorphism

(A.6.1) $$(\pi_g, 1)_*: \pi_n(C\Sigma A \vee B, \Sigma A \vee B) \to \pi_n(C_g, B)$$

between relative homotopy groups. In this section we embed $(\pi_g, 1)_*$ into a long exact sequence which generalizes the EHP sequences of James. This shall give us a fundamental tool for the computation of the groups $\pi_n(C_g, B)$. Ganea [GH] and Gray [HM] also studied the homotopy groups of a mapping cone. The improvement here is the fact that the exact sequences below are available in a considerably better range of dimensions. The map $-1: \Sigma A \to \Sigma A$ induces an automorphism of $\pi_n(\Sigma A)$ which we denote by $(-1)_*$.

(A.6.2) Theorem *Let A be connected and let B be 1-connected. Then we have a commutative diagram*

$$\begin{tikzcd}
\pi_{n+1}(C_g, B) \arrow[r, "\bar{\gamma}"] \arrow[dd, "(-1)_*(j_g)_*"] & \pi_n(N_g, N_1) \arrow[r, "\bar{\partial}"] \arrow[dd, "j_*"] & \pi_n(C\Sigma A \vee B, \Sigma A \vee B) \arrow[r, "(\pi_g, 1)"] \arrow[d, "\partial", "\cong"'] & \pi_n(C_g, B) \arrow[r, "\bar{\gamma}"] \arrow[dd, "(-1)_*(j_g)_*"] & {} \\
 & & \pi_{n-1}(\Sigma A \vee B) \arrow[d, "(-1)_*(r_1)_*"] & & \\
\pi_{n+1}(\Sigma^2 A) \arrow[r, "\gamma"'] & \pi_n(J\Sigma A, \Sigma A) \arrow[r, "\partial"'] & \pi_{n-1}(\Sigma A) \arrow[r, "\Sigma"'] & \pi_n(\Sigma^2 A) \arrow[r, "\gamma"'] & {}
\end{tikzcd}$$

in which the rows are long exact sequences. Here j_g is the pinch map and $j: N_g \to J\Sigma A$ is defined in (A.5.8); $r_1: \Sigma A \vee B \to \Sigma A$ is the retraction.

Clearly, if $B = *$ is a point all vertical arrows in the diagram of Theorem A.6.2 are isomorphisms. The bottom sequence in the diagram is the classical *suspension sequence of James* [ST] which is given by the use of the equivalence $g_{\Sigma A}: \Omega\Sigma\Sigma A \simeq J\Sigma A$ in (A.2.1). The operator γ is the composite

$$\gamma: \pi_{n+1}(\Sigma^2 A) = \pi_n(\Omega\Sigma^2 A) \xleftarrow[(g_{\Sigma A})_*]{\simeq} \pi_n(J\Sigma A) \xrightarrow[j]{} \pi_n(J\Sigma A, \Sigma A).$$

The operator ∂ is the usual boundary operator and Σ is the suspension. We now define the operator $\bar{\gamma}$ by

$$\bar{\gamma}: \pi_{n+1}(C_g, B) = \pi_n(P_{i_g}) \cong \pi_n(N_g) \xrightarrow[j]{} \pi_n(N_g, N_1).$$

Moreover, the operator $\bar{\partial}$ is the composite

$$\pi_{n+1}(N_g, N_1) \xrightarrow[\partial]{} \pi_n(N_1) \xrightarrow[(\pi\alpha)_*^{-1}(-1)_*]{\cong} \pi_{n-1}(P_{i_0}) = \pi_n(C\Sigma A \vee B, \Sigma A \vee B)$$

see (A.3.8). For $N_1 = \Sigma A \rtimes \Omega B$ the map $-1: N_1 \to N_1$ is $-1_{\Sigma A} \rtimes \Omega B$.

Proof of Theorem A.6.2 We consider the map of pairs

$$j\colon (N_g, N_1) \to (J\Sigma A, \Sigma A)$$

which induces a map of the corresponding long homotopy sequences. Now the proposition is a consequence of Theorem A.3.10, Proposition A.5.4, and Theorem A.5.2. □

This shows that a result as in Theorem A.6.2 is also available for the homotopy functor π_n^X instead of π_n; compare (II.7.8) in Baues [AH].

If ΣA is $(a-1)$-connected we derive easily from the iterated mapping cone structure of $J\Sigma A$ in (A.5.6) that $(J\Sigma A, \Sigma A)$ is $(2a-1)$-connected. Therefore the exactness of the James sequence gives us the classical suspension theorem of Freudenthal. More generally, (N_g, N_1) is also $(2a-1)$-connected, therefore we obtain from the exact row in Theorem A.6.2 the following special case of Theorem A.3.10.

(A.6.3) Corollary *Let ΣA be $(a-1)$-connected and let B be simply connected. Then $(\pi_g, 1)_*$ in (A.6.1) is epimorphic for $n \le 2a$ and an isomorphism for $n < 2a$.*

We can apply this corollary to the principal cofibration $N_1 \subset N_2$ in Theorem A.5.5. The attaching map for N_2 is

(A.6.4) $$\omega_2\colon (\Sigma A \wedge A) \rtimes (\Omega B)^2 \to (\Sigma A) \rtimes \Omega B = N_1$$

and thus we obtain (compare Theorem A.6.2):

(A.6.5) Corollary *Let ΣA be $(a-1)$-connected and let B be simply connected. Then*

$$(\pi_{\omega_2}, 1)_* \partial^{-1}\colon \pi_{n-1}(\Sigma A \wedge A \rtimes (\Omega B)^2 \vee \Sigma A \rtimes \Omega B)_2 \to \pi_n(N_2, N_1)$$

is epimorphic for $n \le 4a-2$ and is isomorphic for $n < 4a-2$.

Clearly from the iterated cone structure of N_g we see that

(A.6.6) $$\pi_n(N_2, N_1) \to \pi_n(N_g, N_1)$$

is epimorphic for $n \le 3a$-1 and is isomorphic for $n < 3a-1$ in case ΣA is $(a-1)$-connected. Thus in the appropriate range of dimensions we can replace the group $\pi_n(N_g, N_1)$ in Theorem A.6.2 by the groups in Corollary A.6.5. This leads to the following corollary of Theorem A.6.2.

(A.6.7) Theorem *Let ΣA be $(a-1)$-connected and let B be 1-connected and let $g\colon \Sigma A \to B$. For $n \le 3a-1$ we have the following commutative diagram with exact columns:*

$$\begin{tikzcd}
\pi_{n-1}(\Sigma A \rtimes \Omega B) \arrow[r, "(-1)_* pr_*"'] \arrow[d, "E_g"] & \pi_{n-1}(\Sigma A) \arrow[d, "\Sigma"] \\
\pi_n(C_g, B) \arrow[r, "(-1)_*(j_g)_*"'] \arrow[d, "H_g"] & \pi_n(\Sigma^2 A) \arrow[d, "H"] \\
\pi_{n-2}(\Sigma A \wedge A \rtimes \Omega B^2) \arrow[r, "pr_*"'] \arrow[d, "P_g"] & \pi_{n-2}(\Sigma A \wedge A) \arrow[d, "P"] \\
\pi_{n-2}(\Sigma A \rtimes \Omega B) \arrow[r, "(-1)_* pr_*"'] \arrow[d, "E_g"] & \pi_{n-2}(\Sigma A) \arrow[d, "\Sigma"] \\
\vdots & \vdots
\end{tikzcd}$$

The map pr is the projection and we set

$$P_g = (-1)_*(\omega_2)_*, \quad P = [1_{\Sigma A}, 1_{\Sigma A}]_*.$$

Moreover, for $n = 3a-1$ the following commutative diagram extends the diagram above such that the columns remain exact:

$$\begin{tikzcd}
\pi_{3a-2}(\Sigma A \wedge A \rtimes \Omega B^2 \vee \Sigma A)_2 \arrow[r, "(pr \vee 1)_*"'] \arrow[d, "(-1)_*(\omega_2, i_1)_*"] & \pi_{3a-2}(\Sigma A \wedge A \vee \Sigma A)_2 \arrow[d, "([1_{\Sigma A}, 1_{\Sigma A}], 1_{\Sigma A})_*"] \\
\pi_{3a-2}(\Sigma A \rtimes \Omega B) \arrow[r, "(-1)_* pr_*"'] & \pi_{3a-2}(\Sigma A)
\end{tikzcd}$$

Addendum *For the operator E_g the diagram*

$$\begin{tikzcd}
\pi_{n-1}(\Sigma A \rtimes \Omega B) \arrow[r, "="', "\bar{\mu}_*"'] \arrow[d, "E_g"] & \pi_{n-1}(\Sigma A \vee \Sigma A \wedge \Omega B) \arrow[d, "(g, [g, R_B])_*"] \\
\pi_n(C_g, B) \arrow[r, "\partial"'] & \pi_{n-1}(B)
\end{tikzcd}$$

commutes, where $\bar{\mu}$ is defined in $(A.3.2)$ and where $[g, R_B]$ is the Whitehead product of g and of the evaluation map $R_B\colon \Sigma\Omega B \to B$. If B is a suspension we can replace $[g, R_B]$ by use of Proposition $A.3.6$.

Proof The operator E_g is defined by

$$\begin{tikzcd}
\pi_{n-1}(\Sigma A \rtimes \Omega B) \arrow[dr, "E_g"'] & \pi_{n-1}(P_{i_0}) = \pi_n(C\Sigma A \vee B, \Sigma A \vee B) \arrow[l, "\cong", "(\pi\alpha)_*"'] \arrow[d, "(\pi_g, 1)_*"] \\
 & \pi_n(C_g, B)
\end{tikzcd}$$

or equivalently by

$$E_g \colon \pi_{n-1}(\Sigma A \rtimes \Omega B) \xrightarrow[\tau_*]{} \pi_{n-1}\left(P_{i_g}\right) = \pi_n(C_g, B)$$

where τ and $\pi\alpha$ are defined in (A.3.8). The operator H_g is defined for $n+1 \le 3a-1$ by

$$\begin{tikzcd}
\pi_{n+1}(C_g, B) = \pi_n\left(P_{i_g}\right) \arrow[d, "H_g"] & \pi_n(N_g) \arrow[l, "\cong", "n_*"'] \arrow[dr, "j"] & \\
 & & \pi_n(N_g, N_1) \\
\pi_{n-1}(\Sigma A \wedge A \rtimes \Omega B^2) \arrow[r, "\cong"] & \pi_n(N_2, N_1) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where we use Corollary A.6.5 and (A.6.6). The operator H is similarly given by

$$\begin{tikzcd}
\pi_{n+1}(\Sigma^2 A) = \pi_n(\Omega\Sigma\Sigma A) \arrow[r, "\cong", "(g_{\Sigma A})_*"'] \arrow[d, "H"] & \pi_n(J\Sigma A) \arrow[dr, "j"] & \\
 & & \pi_n(J\Sigma A, \Sigma A) \\
\pi_{n-1}(\Sigma A \wedge A) \arrow[r, "\cong"] & \pi_n(J_2\Sigma A, \Sigma A) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where we use the equivalence in (A.2.1). If B is a point all horizontal arrows of the diagram in Theorem A.6.7 are isomorphisms and one easily checks that in this case the diagram of Theorem A.6.7 commutes.

The proposition of Theorem A.6.7 is a consequence of Proposition A.5.9, Theorem A.6.2, and Proposition A.3.5. We have

$$\pi_{3a-2}(\Sigma A \wedge A \rtimes \Omega B^2 \vee \Sigma A)_2 = \pi_{3a-2}(\Sigma A \wedge A \rtimes \Omega B^2 \vee \Sigma A \rtimes \Omega B)_2$$

since $\pi_{3a-2}(\Sigma A \wedge A \wedge A \wedge \Omega B) = 0$. □

(A.6.8) Proposition *For the operator H in Theorem A.6.7 and for the James–Hopf invariant γ_2 the following diagram commutes:*

$$\begin{tikzcd}
\pi_n(\Sigma(\Sigma A)) \arrow[r, "\gamma_2"'] \arrow[d, "H"] & \pi_n(\Sigma(\Sigma A \wedge \Sigma A)) \\
\pi_{n-2}(\Sigma A \wedge A) \arrow[r, "\cong", "\Sigma^2"'] & \pi_n(\Sigma^3 A \wedge A) \arrow[u, "\cong"', "(-\iota)_*"]
\end{tikzcd}$$

Here $\nu(t_1,t_2,t_3,a,b)=(t_1,t_2,a,t_3,b)$ *for* $t_i \in I$ *and* $a,b \in A$. *Since* ΣA *is* $(a-1)$*-connected and since* $n \le 3a-1$ *the operator* $\Sigma\Sigma$ *is an isomorphism.*

By the isomorphism $(q_0)_*$ in Proposition A.3.4 we derive from Theorem A.6.7 the following exact sequence in which the operator P_g has a simple description.

(A.6.9) Theorem *Let* $g: \Sigma A \to B$ *and let* ΣA *be* $(a-1)$*-connected and* B *be* $(b-1)$*-connected,* $b \ge 2$. *Moreover, let* A *be a* co-H-space. *For* $n \le \min(3a-1, 2a+b-1)$ *we have the exact sequence*

$$\pi_{n-1}(\Sigma A \vee B)_2 \xrightarrow[E'_g]{} \pi_n(C_g, B) \xrightarrow[H_g]{} \pi_{n-2}(\Sigma A \wedge A) \xrightarrow[P'_g]{} \pi_{n-2}(\Sigma A \vee B)_2 \xrightarrow[E'_g]{} .$$

Here H_g *is defined in Theorem A.6.7.* E'_g *is given by*

$$E'_g = E_g(q_{0*})^{-1} = (\pi_g, 1)_* \partial^{-1},$$

and the operator P'_g *is induced by a Whitehead product:*

$$P'_h = (q_0)_* P_g = [i_1, i_1 - i_2 g]_*$$

where i_1 *and* i_2 *are the inclusions of* ΣA *and* B *respectively into* $\Sigma A \vee B$. *If* $n = 3a-1 < 2a+b-1$ *the exact sequence has a prolongation*

$$\pi_{3a-2}(\Sigma A \wedge A \vee \Sigma A)_2 \xrightarrow[P'_g]{} \pi_{3a-2}(\Sigma A \vee B)_2$$

with $P'_g = ([i_1, i_1 - i_2 g], -i_1)_*$.

For the operator E'_g *the diagram*

$$\begin{array}{ccc} \pi_{n-1}(\Sigma A \vee B)_2 & \subset & \pi_{n-1}(\Sigma A \vee B) \\ \downarrow{\scriptstyle E'_g} & & \downarrow{\scriptstyle (g,1)_*} \\ \pi_n(C_g, B) & \xrightarrow{\partial} & \pi_{n-1}(B) \end{array}$$

commutes. Here ∂ *is the boundary operator. As in Theorem A.6.7 the* (E'_g, P'_g, H_g)*-sequence forms a commutative diagram with the* (E, H, P)*-sequence.*

Proof By the assumption on n we know

$$\pi_{n-2}(\Sigma A \wedge A) = \pi_{n-2}(\Sigma A \wedge A \rtimes \Omega B^2).$$

Moreover, we use the isomorphism $(q_0)_*$ in Proposition A.3.4. We have to check that

$$(q_0)_* P_g = [i_1, i_1 - i_2 g]_*.$$

Here we use the formula for ω_g^{12} in Theorem A.7.4(a) in the next section. Now $(q_0)_* P_g$ is induced by

$$\begin{aligned}(q_0(-1)\omega_2)|_{\Sigma A \wedge A} &= (i_1, [i_1, i_2 R_B])\,\omega_g^{12} \\ &= (i_1, [i_1, i_2 R_B])([a, a] - b(\Sigma A \wedge \bar{g})) \\ &= [i_1, i_1] - [i_1, i_2 R_B](\Sigma A \wedge \bar{g}) \\ &= [i_1, i_1] - [i_1, i_2 g] = [i_1, i_1 - i_2 g].\end{aligned}$$

Here we use $R_B(\Sigma \bar{g}) = g$ in (A.1.19), and we use the bilinearity of the Whitehead product. □

A.7 The operator P_g

The operator P_g is induced by the mapping $(-1)\omega_2$. This mapping can be described as follows. By the homotopy equivalence $\bar{\mu}$ in (A.3.2) we obtain the homotopy commutative diagram

(A.7.1)

$$\begin{tikzcd}
\Sigma A \wedge A \rtimes (\Omega B \times \Omega B) \arrow[r, "\omega_2"] \arrow[d, "\bar{\mu}", "\simeq"'] & \Sigma A \rtimes \Omega B \arrow[r, "-1"] & \Sigma A \rtimes \Omega B \arrow[dd, "\bar{\mu}", "\simeq"'] \\
\Sigma A \wedge A \wedge \Sigma A \wedge A \wedge (\Omega B \times \Omega B) \arrow[d, "\bar{\bar{\mu}}", "\simeq"'] & & \\
\Sigma A \wedge A \wedge (S^0 \vee \Omega B \vee \Omega B \vee (\Omega B)^{\wedge 2}) \arrow[rr, "\omega_g"] & & \Sigma A \vee \Sigma A \wedge \Omega B
\end{tikzcd}$$

where ω_2 is defined in Theorem A.5.5. The homotopy equivalence $\bar{\bar{\mu}}$ is given by

(A.7.2)
$$\begin{aligned}&\mu_0 \colon \Sigma(X_1 \times X_2) \simeq \Sigma X_1 \vee \Sigma X_2 \vee \Sigma X_1 \wedge X_2 \\ &\mu_0 = i_1(\Sigma p_1) + i_2(\Sigma p_2) + i_3(\Sigma \pi)\end{aligned}$$

which we easily derive from the exact sequence (A.1.1). The homotopy commutative diagram (A.7.1) defines up to homotopy the map ω_g. This map has the components

(A.7.3)
$$\omega_g = \left(\omega_g^{12}, \omega_g^{123}, \omega_g^{124}, \omega_g^{1234}\right)$$

given by the restrictions of ω_g to the factors of the $\vee$-product. We denote by

$$a\colon \Sigma A \subset \Sigma A \vee \Sigma A \wedge \Omega B \quad \text{and} \quad b\colon \Sigma A \wedge \Omega B \subset \Sigma A \vee \Sigma A \wedge \Omega B$$

the inclusions and by

$$H_B = H\mu_B\colon \Sigma\Omega B \wedge \Omega B \to \Sigma\Omega B$$

the Hopf construction on the loop addition map; see (A.1.21). Moreover, for $n_1, \dots, n_r \in \{m_1 < \cdots < m_k\}$ let

$$T_{n_1,\dots,n_r}\colon X_{m_1} \wedge \cdots \wedge X_{m_k} \to X_{n_1} \wedge \cdots \wedge X_{n_r}$$

be the shuffle map mapping the tuple $(x_{m_1}, \dots, x_{m_k})$ to $(x_{n_1}, \dots, x_{n_r})$ with $x_{m_i} \in X_{m_i}$. In the following theorem we set

$$X_1 = X_2 = A, \qquad X_3 = X_4 = \Omega B.$$

(A.7.4) Theorem *Let A be a* co-H-space. *Then for a map $g\colon \Sigma A \to B$ and its adjoint $\bar{g}\colon A \to \Omega B$ the components of ω_g are given by the following formulas*:

(*a*) *for* $\omega_g^{12} \in [\Sigma X_1 \wedge X_2, \Sigma A \vee \Sigma A \wedge \Omega B]$ *we have*

$$\omega_g^{12} = [a,a] - b(\Sigma A \wedge \bar{g});$$

(*b*) *for* $\omega_g^{123} \in [\Sigma X_1 \wedge X_2 \wedge X_3, \Sigma A \vee \Sigma A \wedge \Omega B]$ *we have*

$$\omega_g^{123} = [b,a]T_{132} - b(A \wedge H_B)(\Sigma A \wedge \Omega B \wedge \bar{g})T_{132};$$

(*c*) *for* $\omega_g^{124} \in [\Sigma X_1 \wedge X_2 \wedge X_4, \Sigma A \vee \Sigma A \wedge \Omega B]$ *we have*

$$\omega_g^{124} = [b,a]T_{142} + [a,b] + [b,b]T_{1424} - b(A \wedge H_B)(\Sigma A \wedge \bar{g} \wedge \Omega B);$$

(*d*) *for* $\omega_g^{1234} \in [\Sigma X_1 \wedge X_2 \wedge X_3 \wedge X_4, \Sigma A \vee \Sigma A \wedge \Omega B]$ *we have*

$$\begin{aligned} \omega_g^{1234} &= [b(A \wedge H_B), a]T_{1342} + [b(A \wedge H_B), b]T_{13424} \\ &\quad + [b,b]T_{1324} \\ &\quad - b(A \wedge H_B)(A \wedge \Omega B \wedge H_B)(\Sigma A \wedge \Omega B \wedge \bar{g} \wedge \Omega B)T_{1324}. \end{aligned}$$

Thus, all components of ω_g are expressed solely in terms of the Whitehead product, the Hopf construction H_B, and the adjoint $\bar{g}$. Clearly, by (A.7.1), *we can replace the operator P_g* in Theorem A.6.7 *by the operator* $(\omega_g)_*$ with ω_g described in (A.7.1) and Theorem A.7.4.

Proof of Theorem A.7.4 For a product $X = X_1 \times \cdots \times X_n$ of spaces and for $i_1, \ldots, i_r \in \{1, \ldots, n\}$ we denote a map

$$T\colon (\Sigma X)/\sim \to \big(\Sigma(X_{i_1} \times \cdots \times X_{i_r})\big)/\approx \tag{1}$$

by $T = T_{i_1 \ldots i_r}$ if

$$T[t, x_1, \ldots, x_n]_\sim = [t, x_{i_1}, \ldots, x_{i_r}]_\approx$$

for $t \in I, x_i \in X_i$. Here $\sim$ and $\approx$ are equivalence relations and $[x]_\sim$ is the equivalence class of x in $(\Sigma X)/\sim$. Now let $X = A \times A \times \Omega B \times \Omega B$. We derive from the definition of ω_2 in Theorem A.5.5 and from (A.1.12) that the following diagram homotopy commutes

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\xi} & U = \Sigma X_1 \vee \Sigma X_1 \wedge (X_2 \times X_3 \times X_4) \vee \Sigma X_2 \vee \Sigma X_2 \wedge X_4 \\ \downarrow{\scriptstyle T_{1234}} & & \downarrow{\scriptstyle m} \\ \Sigma X_1 \wedge X_2 \rtimes (X_3 \times X_4) & \xrightarrow[\bar{\mu}(-1)\omega_2]{} & \Sigma A \vee \Sigma A \wedge \Omega B \end{array} \tag{2}$$

Here we have

$$m = \big(a, b(\Sigma A \wedge m_g), a, b\big)$$

with

$$\begin{cases} m_g\colon X_2 \times X_3 \times X_4 \to \Omega B \\ m_g(a, b_1, b_2) = b_1 + \bar{g}(a) + b_2. \end{cases} \tag{3}$$

The map ξ is the sum

$$\xi = p_1 + p_{134} + p_2 + p_{24} - p_{1234} - p_1 - p_{24} - p_2 \tag{4}$$

where $p_{i_1 \ldots i_r}$ is $T_{i_1 \ldots i_r}$ followed by the inclusion into U. In particular p_{134} is the composite

$$\Sigma X \xrightarrow[T_{134}]{} \Sigma X_1 \wedge (X_3 \times X_4) \subset \Sigma X_1 \wedge (X_2 \times X_3 \times X_4) \subset U.$$

Let $j_{12}\colon \Sigma(X_1 \times X_2) \to \Sigma X$ be the inclusion. Then we have

$$\xi j_{12} = p_1 + p_2 - p_{12} - p_1 - p_2 \tag{5}$$

with $p_{12} = p_{134} j_{12}$. The commutator rule (A.1.3) shows that

$$\xi j_{12} = (-p_1, -p_2) - p_{12} = (p_1, p_2) - p_{12}.$$

Therefore we have

$$T_{12}^*\big(\omega_g^{12}\big) = m_* \xi j_{12} = T_{12}^*([a, a] - b(\Sigma A \wedge \bar{g})) \tag{6}$$

and thus (a) is proven.

We now compute $\hat{P}_g = \omega_g \bar{\bar{\mu}}|_{\Sigma A \wedge A \wedge Y}$ with $Y = \Omega B \times \Omega B$; see (A.7.1). As in (A.7.2) we have the equivalence

$$\mu_0 : \Sigma(X_2 \times Y) \simeq \Sigma U_0$$
$$U_0 = X_2 \vee Y \vee X_2 \wedge Y.$$

This gives us

$$\mu_0 : U \xrightarrow{\simeq} U' = \Sigma X_1 \vee \Sigma X_1 \wedge U_0 \vee \Sigma X_2 \vee \Sigma X_2 \wedge X_4.$$

We consider

$$T^*_{1234}\hat{P}_g = -m\xi j_{12} + m\xi = m_*(-\xi j_{12} + \xi). \tag{7}$$

Let m_0: $U' \to \Sigma A \vee \Sigma A \wedge \Omega B$ be defined by

$$m_0 \mu_0 \simeq m. \tag{8}$$

Then we derive from (7), (5), and (4)

$$T^*_{1234}\hat{P}_g = (m_0)_* \xi_0 \tag{9}$$

where

$$\begin{aligned} \xi_0 &= -\mu_0 \xi j_{12} + \mu_0 \xi \\ &= -(p_1 + p_2 - p_{12} - p_1 - p_2) + p_1 + p_{134} + p_2 + p_{24} \\ &\quad -(p_{12} + p_{134} + p_{1234}) - p_1 - p_{24} - p_2. \end{aligned} \tag{10}$$

Now the commutator rule (A.1.3) shows

$$\xi_0 = (p_{134}, p_2) + (p_{134}, p_{24}) - p_{1234} + (p_1, p_{24}) \tag{11}$$

where (,) denotes the commutator. Clearly, all these commutators correspond to Whitehead products lying in the group $[\Sigma X_1 \wedge X_2 \wedge Y, U']$. Using the definition of the Hopf construction $H_B = H\mu_B$ in (A.1.20) we are able to compute from (11) the term $(m_0)_*(\xi_0)$. By (9) this yields the formulas (b), (c), and (d) in Theorem A.7.4. □

(A.7.5) Corollary *Let $g: \Sigma A \to B$ be a map where A is a connected* co-H-space *and where B is simply connected. For the mapping ω_g in (A.7.3) the composition $(g, [g, R_B])\omega_g$ is null-homotopic.*

Proof The following diagram homotopy commutes

$$\begin{array}{ccccc} \Sigma A \rtimes \Omega B & \xrightarrow[(-1)]{\simeq} & \Sigma A \rtimes \Omega B & \xrightarrow[\simeq]{\bar{\mu}} & \Sigma A \vee \Sigma A \wedge \Omega B \\ \downarrow i & & \downarrow \tau & & \downarrow (g,[g,R_B]) \\ N_g & \xrightarrow[\simeq]{n} & P_{i_g} & \xrightarrow{q} & B \end{array}$$

Compare Proposition A.3.5, (A.3.8), and Proposition A.5.4. Now the corollary follows from Theorems A.7.4 and A.5.5. □

The relation in Corollary A.7.5 is not a 'new' homotopy relation since

(A.7.6) Theorem *The formula*

$$(g,[g,R_B])_* \omega_g = 0$$

can be proved by use of the Jacobi identity (Corollary A.1.9), the relation in Proposition A.1.23, and by (A.1.19).

Proof Since $g = R_B(\Sigma\bar{g})$ we see from (A.1.19) that

$$(g,[g,R_B])_* \omega_g^{12} = [g,g] - [g,R_B](\Sigma A \wedge \bar{g})$$

is trivial. Moreover, we derive from Proposition A.1.23 and (A.1.19) the equations

$$\begin{aligned}(g,[g,R_B])_* \omega_g^{123} &= [[g,R_B],g]T_{132} \\ &\quad -[g,R_B](A \wedge H_B)(\Sigma A \wedge \Omega B \wedge \bar{g})T_{132} \\ &= [g,[g,R_B]] - [[g,R_B],R_B](\Sigma A \wedge \Omega B \wedge \bar{g})T_{132} \\ &= 0.\end{aligned}$$

Similarly, we derive from Corollary A.1.9, Proposition A.1.23, and A.1.19 the remaining two equations for (c) and (d) in Theorem A.7.4. For (d) we apply Corollary A.1.9 with $\alpha_3 = [g,R_B]$, $\alpha_1 = R_B$, $\alpha_2 = g$. □

We now consider the special case where g in (A.7.1) is a map between suspensions. For a map

$$g\colon \Sigma A \to \Sigma B \quad (B \text{ connected})$$

we derive (A.7.1) the mapping W_g for which

$$\text{(A.7.7)}\qquad \begin{tikzcd} \Sigma A \wedge A \wedge \Omega \wedge \Omega \arrow[r, "\omega_g"] \arrow[d, "T_{1324}", "\cong"'] & \Sigma A \wedge \Omega \arrow[dd, "A \wedge J_B", "\cong"'] \\ \Sigma A \wedge \Omega \wedge A \wedge \Omega \arrow[d, "(A \wedge J_B)\#(A \wedge J_B)", "\cong"'] & \\ \Sigma A \wedge B^* \wedge A \wedge B^* \arrow[r, "W_g"'] & \Sigma A \wedge B^* \end{tikzcd}$$

homotopy commutes. Here we set

$$\Omega = S^0 \vee \Omega\Sigma B, \qquad B^* = \bigvee_{n \geq 0} B^{\wedge n}, \qquad B^{\wedge 0} = S^0.$$

The homotopy equivalence $A \wedge J_B$ is described in Proposition A.3.6. Let

$$i_n\colon \Sigma A \wedge B^{\wedge n} \hookrightarrow \Sigma A \wedge B^*$$

for $n \geq 0$ be the inclusion.

(A.7.8) Theorem *Let A be a* co-H-space. *Then for $n, m \geq 0$ the map W_g in $(A.7.7)$ satisfies the formula*

$$W_g|_{\Sigma A \wedge B^{\wedge m} \wedge A \wedge B^{\wedge n}} = \sum_{a \cup b = \bar{n}} [i_{m+\#a}, i_{\#b}]\nu_{a,b}$$
$$- \sum_{r \geq 1} i_{r+m+n}(A \wedge B^{\wedge m} \wedge \gamma_r(g) \wedge B^{\wedge n}).$$

The first sum is taken over all pairs (a,b) of subsets $a, b \subset \bar{n} = \{1, \ldots, n\}$ with $a \cup b = \bar{n}$. The shuffle map

$$\nu_{a,b}\colon \Sigma A \wedge B^{\wedge m} \wedge A \wedge B^{\wedge n} \to \Sigma A \wedge B^{\wedge(m+\#a)} \wedge A \wedge B^{\wedge \#b}$$

is defined by

$$\nu_{a,b}(t, u, x, v, y) = (t, u, x \wedge y_a, v, y_b)$$

for $t \in I, u, v \in A, x \in B^{\wedge m}, y \in B^{\wedge n}$; compare $(A.1.3)$. *The James–Hopf invariants $\gamma_r(g)$ are defined in Section A.2.*

If B is a co-H-space $\nu_{a,b}$ is trivial if $a \cap b$ is non-empty. Theorem A.7.8 is a corollary of Theorem A.7.4 by use of (A.1.19) and Propositions A.2.8 and A.2.9. We proved Theorem A.7.8 in a slightly different way in Baues [RP]. Again *we can replace the operator P_g in* Theorem A.6.7 *by* (A.7.7) *and by the operator* $(W_g)_*$ where W_g is given by the formula in Theorem A.7.8. This is of great value for the study of the groups $\pi_n(C_f, B)$ for $n \leq 3a - 1$. Explicitly we obtain the following result which is a special case of Theorem A.6.7.

(A.7.9) Theorem *Let A be a* co-H-space. *Let ΣA be $(a-1)$-connected and let ΣB be 1-connected and let $g\colon \Sigma A \to \Sigma B$ be a map. For the relative homotopy* *groups of the mapping cone $(C_g, \Sigma B)$ we have the following commutative diagram with exact columns, $n \leq 3a-1$.*

$$\begin{array}{ccc}
\pi_{n-1}(\Sigma A \wedge B^*) & \xrightarrow[(-1)_* pr_*]{} & \pi_{n-1}(\Sigma A) \\
\downarrow{\scriptstyle \bar{E}_g} & & \downarrow{\scriptstyle \Sigma} \\
\pi_n(C_g, \Sigma B) & \xrightarrow[(-1)_*(j_g)_*]{} & \pi_n(\Sigma^2 A) \\
\downarrow{\scriptstyle \bar{H}_g} & & \downarrow{\scriptstyle H} \\
\pi_{n-2}(\Sigma A \wedge B^* \wedge A \wedge B^*) & \xrightarrow[pr_*]{} & \pi_{n-2}(\Sigma A \wedge A) \\
\downarrow{\scriptstyle (W_g)_*} & & \downarrow{\scriptstyle P} \\
\pi_{n-2}(\Sigma A \wedge B^*) & \xrightarrow[(-1)_* pr_*]{} & \pi_{n-2}(\Sigma A) \\
\downarrow{\scriptstyle \bar{E}_g} & & \downarrow{\scriptstyle \Sigma} \\
\pi_{n-1}(C_g, \Sigma B) & \xrightarrow[(-1)_*(j_g)_*]{} & \pi_{n-1}(\Sigma^2 A) \\
\vdots & & \vdots
\end{array}$$

The map pr is induced by the projection $B^* \to S^0$; recall that $B^* = S^0 \vee B \vee B^{\wedge 2} \vee \cdots$. The map W_g is defined in Theorem A.7.8. Moreover, for $n = 3a-1$ the following commutative diagram extends the diagram above such that the columns remain exact:

$$\begin{array}{ccc}
\pi_{3a-2}(\Sigma A \wedge B^* \wedge A \wedge B^* \vee \Sigma A)_2 & \xrightarrow[(pr \vee 1)_*]{} & \pi_{3a-2}(\Sigma A \wedge A \vee \Sigma A) \\
\downarrow{\scriptstyle (W_g, -i_0)_*} & & \downarrow{\scriptstyle ([1,1],1)_*} \\
\pi_{3a-2}(\Sigma A \wedge B^*) & \xrightarrow[(-1)_* pr_*]{} & \pi_{3a-2}(\Sigma A)
\end{array}$$

Here i_0 is induced by $S^0 \subset B^*$.

Addendum *For the operator $\bar{E}_g$ the diagram*

$$\begin{array}{ccc}
\pi_{n-1}(\Sigma A \wedge B^*) & & \\
\downarrow{\scriptstyle \bar{E}_g} & \searrow{\scriptstyle W_*} & \\
\pi_n(C_g, \Sigma B) & \xrightarrow[\partial]{} & \pi_{n-1}(\Sigma B)
\end{array}$$

is commutative where

$$W_{|\Sigma A \wedge B^{\wedge i}} = [\cdots[g, 1_{\Sigma B}], \ldots, 1_{\Sigma B}]$$

is the i-fold Whitehead product: clearly, for $i = 0$ we set $W_{|\Sigma A} = g$. Compare Proposition A.3.6 and the addendum of Theorem A.6.7.

Proof of Theorem A.7.9 We define $\bar{E}_g$ by the composition (see Theorem A.6.7):

$$\pi_{n-1}(\Sigma A \wedge B^*) \cong \pi_{n-1}(\Sigma A \rtimes \Omega\Sigma B) \xrightarrow[E_g]{} \pi_n(C_g, B)$$

where the isomorphism is induced by $\bar{\mu}$ in (A.7.1) and by $A \wedge J_B$ in (A.7.7). We define the operator $\bar{H}_g$ by the composition (see Theorem A.6.7):

$$\pi_n(C_g, \Sigma B) \xrightarrow[H_g]{} \pi_{n-2}(\Sigma A \wedge A \rtimes \Omega(\Sigma B)^2) \cong \pi_{n-2}(\Sigma A \wedge B^* \wedge A \wedge B^*).$$

Here the isomorphism is induced by $\bar{\bar{\mu}}\bar{\mu}$ in (A.7.1) and by

$$(A \wedge J_B)\#(A \wedge J_B)T_{1324}$$

in (A.7.7). □

A.8 The difference map ∇

The difference operator ∇ is of importance for the computation of the left distributivity law for maps between mapping cones; see also (II.2.8) Baues [AH]. Let $g: A \to B$ be a map between connected spaces. Then the difference operator ∇ is induced by a map ∇_0, namely, there is a commutative diagram

$$\begin{array}{ccc}
\pi_1^X(C_g, B) & \xrightarrow[\nabla]{} & \pi_1^X(\Sigma A \vee C_g)_2 \\
\| & & \cap \\
\pi_0^X(P_{i_g}) & \xrightarrow[(\nabla_0)_*]{} & \pi_o^X(\Omega(\Sigma A \vee C_g))
\end{array}$$

The map ∇_0 is constructed as follows. The inclusions $i_1: \Sigma A \subset \Sigma A \vee C_g$ and $i_2: C_g \subset \Sigma A \vee C_g$ yield the map

$$i_2 + i_1: C_g \to \Sigma A \vee C_g$$

which we defined by the cooperation on the mapping cone C_g. We consider the diagram

$$\begin{array}{ccc}
 & & \Omega(\Sigma A \rtimes \Omega C_g) \\
 & \nearrow^{\nabla} & \downarrow{\scriptstyle \Omega q_0} \\
P_{i_g} & \xrightarrow[\nabla_0]{} & \Omega(\Sigma A \vee C_g) \\
 & & \downarrow{\scriptstyle \Omega r_2} \\
 & & \Omega C_g
\end{array} \tag{A.8.1}$$

with $\nabla_0(b,\sigma) = -i_2\sigma + (i_2+i_1)\sigma$ for $(b,\sigma)\in P_{i_g} \subset B\times WC_g$, $\sigma(0) = i_g(b)$. The element $-i_2\sigma+(i_2+i_1)\sigma$ is given by addition of paths in $\Sigma A \vee C_g$. It is clear that this element is actually a loop in $\Sigma A\vee C_g$. We call ∇_0 the *difference map* for the mapping cone C_g. It is easy to verify that ∇_0 actually induces the difference operator ∇ as described above.

Since in diagram (A.8.1) the composition $(\Omega r_2)\nabla_0$ is null-homotopic there is by (A.3.1) up to homotopy a unique map ∇ which lifts ∇_0, that is $(\Omega q_0)\nabla \simeq \nabla_0$. Let

$$\bar{\nabla}\colon \Sigma P_{i_g} \to \Sigma A \rtimes \Omega C_g$$

be the adjoint of ∇. We shall prove:

(A.8.2) Theorem *If $A = \Sigma A'$ is a suspension and if A' and ΩB are connected then $\bar{\nabla}$ is a homotopy equivalence.*

This result is proved in Theorem A.8.13. We point out that $\bar{\bar{\nabla}}$ is a homotopy equivalence for any map $g\colon A\to B$ where A and B are connected.

(A.8.3) Remark If $B = *$ is a point we have $C_g = \Sigma A$ and $P_{i_g} = \Omega\Sigma A$. In this case it is well known that there is a homotopy equivalence as in Theorem A.8.2, namely

$$\begin{tikzcd}
P_{i_g} = \Sigma\Omega\Sigma A \arrow[r, "\simeq", "J_{\Sigma A}"'] & \bigvee_{r\geq 1} \Sigma A^{\wedge r} = \Sigma A \vee \Sigma A \wedge \left(\bigvee_{r\geq 1} A^{\wedge r}\right) \\
& \Sigma A \vee \Sigma A \wedge \Omega\Sigma A \arrow[u, "\simeq", "1\vee(A\wedge J_{\Sigma A})"'] \\
& \Sigma A \rtimes \Omega C_g = \Sigma A \rtimes \Omega\Sigma A \arrow[u, "\simeq"]
\end{tikzcd} \tag{1}$$

compare (A.2.5) and (A.3.2). The equivalence, however, does not coincide with the canonical map $\bar{\nabla}$. In fact, we have the homotopy commutative diagram

$$\begin{tikzcd}
\Sigma\Omega\Sigma A \arrow[r, "-\bar{\nabla}"] \arrow[d, "\simeq"', "J_{\Sigma A}"] & \Sigma A \rtimes \Omega\Sigma A \arrow[d, "\simeq"'] \\
\bigvee_{r\geq 1} \Sigma A^{\wedge r} \arrow[r, "\bar{c}", "\simeq"'] & \bigvee_{r\geq 1} \Sigma A^{\wedge r}
\end{tikzcd} \tag{2}$$

The vertical arrows are the homotopy equivalences which are defined by the row and the column of diagram (1) respectively and $\bar{c}$ is the map with $\bar{c}|_{\Sigma A^{\wedge r}} = \bar{c}_r(j_1,\ldots,j_r)$. Here $\bar{c}_r(j_1,\ldots,j_r)$ is the element defined in the proof of

Theorem A.9.5 below. By (6) in the proof of Theorem A.9.5 we see that $\bar{c}$ is a homotopy equivalence.

We define the map Λ by the homotopy commutative diagram

$$\text{(A.8.4)}\qquad \begin{array}{ccc} \Omega(\Sigma A \rtimes \Omega Y) \times \Omega Y & \xrightarrow{\Lambda} & \Omega(\Sigma A \rtimes \Omega Y) \\ \simeq \uparrow & & \simeq \uparrow \\ J(A \rtimes \Omega Y) \times \Omega Y & \xrightarrow{\lambda} & J(A \rtimes \Omega Y) \end{array}$$

where the homotopy equivalences are defined as in (A.2.1). The map λ is given by the formula

$$\lambda((a_1, \sigma_1) \cdots (a_n, \sigma_n), \sigma) = (a_1, \sigma_1 + \sigma) \cdots (a_n, \sigma_n + \sigma)$$

where $a_1 \in A, \sigma_i, \sigma \in \Omega Y$. It is easy to see that λ is a well-defined map. The map Λ is used in the following diagram

$$\begin{array}{ccc} P_{i_g} \times \Omega C_g & \xrightarrow{\nabla \times 1} & \Omega(\Sigma A \rtimes \Omega C_g) \times \Omega C_g \\ \downarrow \mu & & \downarrow \Lambda \oplus \nabla i \\ \Omega C_g \subset P_{i_g} & \xrightarrow[\nabla]{} & \Omega(\Sigma A \rtimes \Omega C_g) \end{array}$$

where μ is the operation and i is the inclusion. The map $\Lambda \oplus \nabla i$ is defined by $(x, y) \mapsto \Lambda(x, y) + (\nabla i)y$ where $+$ is the addition of loops in $\Omega(\Sigma A \rtimes \Omega C_g)$.

(A.8.5) Theorem *This diagram homotopy commutes.*

Proof We consider

$$\text{(1)}\qquad \begin{array}{ccc} P_{i_g} \times \Omega C_g & \xrightarrow{\nabla_0 \times 1} & \Omega(\Sigma A \vee C_g) \times \Omega C_g \\ \downarrow \mu & & \downarrow I \oplus \nabla_0 i \\ F_{i_g} & \xrightarrow[\nabla_0]{} & \Omega(\Sigma A \vee C_g) \end{array}$$

with $I(x, \sigma) = -i_2\sigma + x + i_2\sigma$. We set $(I \oplus \nabla_0 i)(x, \sigma) = I(x, \sigma) + \nabla_o i(\sigma)$. Now (1) homotopy commutes as we see by the homotopies

$$\begin{aligned} \nabla_o \mu((b, \sigma), \sigma') &= -i_2(\sigma + \sigma') + (i_2 + i_1)(\sigma + \sigma') \\ &= -i_2\sigma' - i_2\sigma + (i_2 + i_1)\sigma + (i_2 + i_1)\sigma' \\ &\simeq -i_2\sigma' + \nabla_0(b, \sigma) + i_2\sigma' + \nabla_0(*, \sigma'). \end{aligned}$$

Now Ωq_0 in (A.8.1) induces a monomorphism $(\Omega q_0)_*$ on homotopy sets. Therefore Theorem A.8.5 follows from the following lemma and (1). □

(A.8.6) Lemma *For a space Y the diagram*

$$\begin{tikzcd}
\Omega(\Sigma A \rtimes \Omega Y) \times \Omega Y \arrow[r, "(\Omega q_0)\times 1"] \arrow[d, "\Lambda"] & \Omega(\Sigma A \vee Y) \times \Omega Y \arrow[d, "I"] \\
\Omega(\Sigma A \rtimes \Omega Y) \arrow[r, "\Omega q_0"'] & \Omega(\Sigma A \vee Y)
\end{tikzcd}$$

homotopy commutes. Here $I(x,\sigma) = -i_2\sigma + x + i_2\sigma$ *and* q_0 *is defined as in Proposition* $A.3.3$.

Proof Let $i\colon A \rtimes \Omega Y \to \Omega\Sigma(A \rtimes \Omega Y)$ be the adjoint of the identity and let

$$i^n\colon (A \rtimes \Omega Y)^n \to \Omega\Sigma(A \rtimes \Omega Y)$$
$$i^n(x_1,\dots,x_n) = i^{n-1}(x_1,\dots,x_{n-1}) + (ix_n), \qquad i^1 = i.$$

It is enough to prove that for all $n \geq 1$ we have

$$I((\Omega q_0)\times 1)(i^n \times 1) \simeq (\Omega q_0)\Lambda(i^n \times 1). \tag{1}$$

But if (1) holds for $n = 1$, then by definition of Λ we see that (1) holds for all n. In fact, we derive this from the following homotopies with $x_i = (a_i, \sigma_i)$ for $i = 1,\dots,n$.

$$\begin{aligned}
&(\Omega q_0)\Lambda(i^n \times 1)(x_1,\dots,x_n,\sigma) \\
&\quad\simeq (\Omega q_0)i^n((a_1,\sigma_1+\sigma),\dots,(a_n,\sigma_n+\sigma)) && (2)\\
&\quad= (\Omega q_0)i(a_1,\sigma_1+\sigma) + \cdots + (\Omega q_0)i(a_n,\sigma_n+\sigma) && (3)\\
&\quad\simeq -i_2\sigma + (\Omega q_0)(a_1,\sigma_1) + i_2\sigma - i_2\sigma + (\Omega q_0)(a_2,\sigma_2) + \cdots && (4)\\
&\quad\simeq -i_2\sigma + (\Omega q_0)i^n(x_1,\dots,x_n) + i_2\sigma && (5)\\
&\quad= I(\Omega q_0 \times 1)(i^n \times 1)(x_1,\dots,x_n,\sigma).
\end{aligned}$$

Here (2) follows from the definition of Λ, (3) from the definition of i^n, and (4) is a consequence of (1) with $n = 1$. (5) follows from the standard homotopy $\sigma - \sigma \simeq 0$. For $n = 1$ (1) is a consequence of the following lemma. □

(A.8.7) Lemma *For a space Y the diagram*

$$\begin{tikzcd}
A \rtimes \Omega Y \arrow[d, "i"] & A \times \Omega Y \arrow[l, "\pi"'] \arrow[d, "I_0"] \\
\Omega(\Sigma A \rtimes \Omega Y) \arrow[r, "\Omega q_0"'] & \Omega(\Sigma A \vee Y)
\end{tikzcd}$$

homotopy commutes, where π is the quotient map and where $I_0(a,\sigma) = -i_2\sigma + i_1\hat{a} + i_2\sigma$ *with* $\hat{a} = ia$ *for* $i\colon A \subset \Omega\Sigma A$.

Proof It is easy to see that

$$\begin{tikzcd}
\Omega(\Sigma A \rtimes \Omega Y) \arrow[d, "\Omega\bar{\mu}"] & A \rtimes \Omega Y \arrow[l, "i"'] \\
\Omega(\Sigma A \vee \Sigma A \wedge \Omega Y) \arrow[d, "\Omega(i_1,[i_1,i_2])"] & \\
\Omega(\Sigma A \vee \Sigma\Omega Y) & A \times \Omega Y \arrow[l, "I_1"] \arrow[uu, "\pi"']
\end{tikzcd}$$

homotopy commutes with $I_1(a,\sigma) = -i_2\hat{\sigma} + i_1\hat{a} + i_2\hat{\sigma}$; compare Proposition A.1.2. Thus we conclude Lemma A.8.7 from Proposition A.3.5. Clearly, $R_Y\hat{\sigma} = \sigma$ and $(\Omega(1 \vee R_Y))I_1 = I_0$. □

For the mapping τ in (A.3.8) we obtain the diagram

(A.8.8)
$$\begin{tikzcd}
\Sigma A \rtimes \Omega B \arrow[r, phantom, "\subset"] \arrow[dr, "\Sigma\tau"'] & \Sigma A \rtimes \Omega C_g \arrow[d, "\nabla"] \\
& \Sigma P_{i_g}
\end{tikzcd}$$

where the inclusion is $1 \rtimes \Omega i_g$.

Proposition (A.8.8) *homotopy commutes.*

Proof This follows from Theorem A.8.5 since

$$\begin{tikzcd}
A \times \Omega B \arrow[r, "\tau|_{A\times\Omega i_g}"] \arrow[d, "\pi"] & P_{i_g} \times \Omega C_g \arrow[d, "\mu"] \\
A \rtimes \Omega B \arrow[r, "\tau"] & P_{i_g}
\end{tikzcd}$$

homotopy commutes and since $\nabla_0(\tau|_A)$ is the inclusion of A. Clearly, $\nabla_0 \mid \Omega B$ and thus $\nabla i\,|_{\Omega B}$ is null-homotopic. □

We now replace A above by the suspension ΣA so that $g\colon \Sigma A \to B$. We assume that A and ΩB are connected. We want to give a *combinatorial description* ∇_N of the mapping ∇, that is we want to describe explicitly a map ∇_N for which the diagram

(A.8.9)
$$\begin{tikzcd}
N_g \arrow[r, "\nabla_N"] \arrow[d, "n", "\simeq"'] & J(\Sigma A \rtimes M_g) \arrow[d, "m", "\simeq"'] \\
P_{i_g} \arrow[r, "\nabla"] & \Omega(\Sigma\Sigma A \rtimes \Omega C_g)
\end{tikzcd}$$

homotopy commutes. Here n is given in Theorem A.5.2 and m is defined by (A.2.1) and by m in Theorem A.5.2. The mapping ∇ on P_{i_g} is defined in (A.8.1). Let

$$\nu\colon M_g \to N_g$$

be a quotient map in Theorem A.5.2 and let

$$i\colon C_g \subset M_g$$

be the inclusion of the mapping cone of $\bar{g}\colon A \to \tilde{\Omega} B$ given by $i \cup \pi|_{CA}$ in (A.4.3). Each element in M_g is of the form

$$x_1 \dots x_n = i(x_1) \cdot \dots \cdot i(x_n)$$

where $x_i \in C_g, i = 1, \dots, n, n \geq 1$. Each element in N_g is of the form

$$[x_1 \dots x_n] = \nu(x_1 \dots x_n).$$

Let

$$j_0\colon C_g \to \Sigma A \xrightarrow{-1} \Sigma A$$

be given by the pinch map. We define ∇_N in (A.8.9) by the formula

(A.8.10) $$\nabla_N[x_1 \dots x_n] = (j_0 x_1, x_2 \dots x_n) \cdot (j_0 x_2, x_3 \dots x_n) \dots \cdot \dots (j_0 x_{n-1}, x_n) \cdot (j_0 x_n, *).$$

This is the product in the monoid $J(\Sigma A \rtimes M_g)$ of the elements

$$(j_0 x_i, x_{i+1} \cdot x_{i+2} \cdot \dots \cdot x_n) \in \Sigma A \rtimes M_g$$

with $i = 1, \dots, n$. One can check that ∇_N is a well-defined map.

(A.8.11) Theorem *For the mapping ∇_N in (A.8.10) diagram (A.8.9) homotopy commutes.*

It is easy to see that (A.8.9) is homotopy commutative if we restrict to $N_1 = \Sigma A \rtimes \tilde{\Omega} B$; compare (A.8.8). Moreover, ∇_N in (A.8.9) yields the following *commutative* diagram which corresponds to Theorem A.8.5:

(A.8.12)
$$\begin{tikzcd} & N_g \times M_g \arrow[r, "\nabla_N \times 1"] \arrow[d, "\mu"] & J(\Sigma A \rtimes M_g) \times M_g \arrow[d, "\lambda \oplus \nabla_N \nu"] \\ M_g \arrow[r, "\nu"] & N_g \arrow[r, "\nabla_N"'] & J(\Sigma A \rtimes M_g) \end{tikzcd}$$

Here λ is defined in the same way as λ in (A.8.4) and we set

$$(\lambda \oplus \nabla_N v)(x,y) = \lambda(x,y)\cdot \nabla_N \nu y$$

and

$$\mu([x_1 \dots x_n], x_{n+1} \dots x_m) = [x_1 \dots x_m].$$

We deduce directly from the definition that (A.8.12) commutes.

Proof of Theorem A.8.11 We have to show that

$$\begin{tikzcd}
N_g \arrow[r, "\nabla_N"] \arrow[d, "n", "\simeq"'] & J(\Sigma A \rtimes M_g) \arrow[d, "m", "\simeq"'] \\
P_{i_g} \arrow[dr, "\nabla_0"'] & \Omega(\Sigma\Sigma A \rtimes \Omega C_g) \arrow[d, "\Omega q_0"] \\
& \Omega\left(\Sigma^2 A \vee C_g\right)
\end{tikzcd} \tag{1}$$

homotopy commutes. Let $\hat{\Omega} X$ be the *loop group* of Kan which is the topological group equivalent to the loop space ΩX, X connected. (If SX is the reduced singular complex and if GSX is the semi-simplicial loop group of Kan [HT], then $\hat{\Omega} X = |GSX|$ is the realization.) In all considerations of this chapter we can replace Ω or $\tilde{\Omega}$ by $\hat{\Omega}$. In particular, we can replace Ω in (1) by $\hat{\Omega}$. Let

$$\hat{i}\colon \Sigma A \to \hat{\Omega}\Sigma(\Sigma A)$$

be the 'adjoint' of the identity on $\Sigma^2 A$ and let

$$\hat{m}\colon C_g \subset M_g \xrightarrow[\simeq]{m} \hat{\Omega} C_g, \bar{g}\colon A \to \hat{\Omega} B.$$

Then the composite in (1), namely

$$\hat{\nabla} = \left(\hat{\Omega} q_0\right) m \nabla_N \colon N_g \to \hat{\Omega}\left(\Sigma^2 A \vee C_g\right),$$

satisfies by Lemma A. 8.7 the formula

$$\begin{aligned}
\hat{\nabla}[x_1 \cdots x_n] = {} & \overline{x_2 \cdots x_n}^{-1} \cdot \overline{j_0 x_1} \cdot \overline{x_2 \cdots x_n} \\
& \cdot \overline{x_3 \cdots x_n}^{-1} \cdot \overline{j_0 x_2} \cdot \overline{x_3 \cdots x_n} \dots \\
& \cdot \overline{x_n}^{-1} \cdot \overline{j_0 x_{n-1}} \cdot \overline{x_n} \cdot \overline{j_o x_n}
\end{aligned} \tag{2}$$

where we set $\overline{x_1 \cdots x_n} = (\hat{\Omega} i_2)(\hat{m}(x_1) \cdot \cdots \cdot \hat{m}(x_n))$, $\overline{j_0 x_i} = (\hat{\Omega} i_1) \hat{i} \hat{j}_0 x_i$. Since $\hat{\Omega}$ is a group we see

$$\hat{\nabla}[x_1 \cdots x_n] = \overline{x_1 \cdots x_n}^{\,-1} \left(\overline{x_1} \cdot \overline{j_0 x_1}\right) \ldots \left(\overline{x_n} \cdot \overline{j_0 x_n}\right), \tag{3}$$

We have to show $\nabla_0 n \simeq \hat{\nabla}$. We first give a characterization of ∇_0 as follows. We consider the commutative diagram

$$\begin{tikzcd}
 & \Omega\left(\Sigma^2 A \vee C_g\right) \arrow[rr, "\Omega r_2"] \arrow[dr] \arrow[d, phantom, "\circledast"] & & \Omega(C_g) \\
P_{i_g} \arrow[ur, "\nabla_0"] \arrow[rr, "\bar{\nabla}"'] \arrow[d, "q"] & {} & P_{i_2} \arrow[d] & \\
B \arrow[rr, "i_g"'] \arrow[d, "i_g"] & & C_g \arrow[d, "i_2"] & \\
C_g \arrow[rr, "i_2 + i_1"'] & & \Sigma^2 A \vee C_g &
\end{tikzcd} \tag{4}$$

where $\bar{\nabla}$ is the map induced on fibres. Since $i_g q \simeq 0$ there exists a map ∇_0 which lifts $\bar{\nabla}$. Now ∇ in (1) is up to homotopy the *unique* lifting of $\bar{\nabla}$ with

$$(\Omega r_2)\nabla_0 \simeq 0. \tag{5}$$

Clearly, the inclusion $i_2: C_g \subset \Sigma^2 A \vee C_g$ is the principal cofibration with attaching map $g_0: \Sigma A \to \{*\} \subset C_g$. Therefore ⊛ in (4) corresponds to

$$\begin{tikzcd}
 & M_{g_0} \overset{\psi}{=} J(\Sigma A) * \hat{\Omega} C_g \arrow[d, "\nu"] \\
N_g \arrow[ur] \arrow[r, "\bar{\nabla}_N"'] & N_{g_0} = M_{g_0}/\hat{\Omega} C_g
\end{tikzcd} \tag{6}$$

where M_{g_0} is the free product of monoids and where N_{g_0} is given by the action from the left of $\hat{\Omega} C_g$ on M_{g_0}. We claim

$$\bar{\nabla}_N[x_1 \cdots x_n] = [\hat{x}_1 \cdot (j_0 x_1) \cdot \ldots \cdot \hat{x}_n \cdot (j_0 x_n)] \tag{7}$$

where $\hat{x}_i = \hat{m} x_i$. From (7) and (3) we derive that $\hat{\nabla}$ satisfies the characterization of ∇_0 in (5) and thus we have proven $\nabla_0 n \simeq \hat{\nabla}$.

For the proof of (6) we remark that ψ in (6) is given by

$$\psi[y_1 \cdots y_n] = (p_1 y_1) \cdot (p_2 y_1) \cdot \ldots \cdot (p_1 y_n) \cdot (p_2 y_n) \tag{8}$$

for $y_i \in \Sigma A \vee \hat{\Omega} C_g, i = 1, \ldots, n$. Moreover, $\tilde{\nabla}_N$ is, by the naturality of the construction N_g, induced by the map

$$\begin{aligned} \chi &: C_g \to \Sigma A \vee \hat{\Omega} C_g \\ \chi &= i_2 \hat{m} + i_1 j_0. \end{aligned} \tag{9}$$

Thus $\hat{\nabla}_N[x_1 \cdots x_n] = [(\chi x_1) \cdots (\chi x_n)]$. This and (8) proves (7). □

(A.8.13) Theorem *The adjoint of* ∇_N *in* $(A.8.9)$

$$\overline{\nabla}_N : \Sigma N_g \to \Sigma^2 A \rtimes M_g$$

is a homotopy equivalence.

Proof From the combinatorial definition of ∇_N in (A.8.9) we derive that

$$(\nabla_N)_\infty : J(N_g) \to J(\Sigma A \rtimes M_g)$$

induces an isomorphism in homology and thus is a homotopy equivalence. Since $(\nabla_N)_\infty$ corresponds to $\Omega\overline{\nabla}_N$ also $\overline{\nabla}_N$ is a homotopy equivalence. □

Similarly we get for the filtrations in (A.4.6) and Theorem A.5.5 the following result.

(A.8.14) Theorem $\overline{\nabla}_N$ *restricts to a homotopy equivalence*

$$\overline{\nabla}_N : \Sigma N_n \simeq \Sigma^2 A \rtimes M_{n-1}$$

for $n \geq 1$. *With* $M_0 = \Omega B$ *we have* $\overline{\nabla}_1 = -1$ *on* ΣN_1; *see* $(A.5.3)$.

A.9 The left distributivity law

In (II.2.8) of Baues [CC] we obtained the following left distributivity law. Let X be finite dimensional and let Y be a co-H-space. Given $f \in [\Sigma X, \Sigma Y]$ and $x, y \in [\Sigma Y, U]$ we have the formula

(A.9.1)
$$xf + yf = (x+y)f + \sum_{n \geq 2} c_n(x,y) \circ \gamma_n(f).$$

Here $\gamma_n(f)$ is the James–Hopf invariant defined with respect to the lexicographical ordering from the left, see (A.2.4), and the terms $c_n(x,y) \in [\Sigma Y^{\wedge n}, U]$ are given by

(A.9.2)
$$c_n(x,y) = \sum_{d \in D_n} [x,y]_{\phi(d)} T_{\tau(d)}.$$

Here $[x, y]_{\phi(d)}$ is an iterated Whitehead product of weight n. The set D_n is computed in (I.1.13), (I.1.16) of Baues [CC]. For the map $f: \Sigma X \to \Sigma Y$ we have the difference element (see Baues [AH] II.§12)

$$\nabla f = -i_2 f + (i_2 + i_1) f: \Sigma X \to \Sigma Y \vee \Sigma Y$$

which is trivial on the second $\Sigma Y, r_{2*}\nabla f = 0$. Therefore the fibre sequence in Proposition A.3.6 yields the diagram

(A.9.3)
$$\begin{tikzcd}
\Sigma Y^{\wedge n} & \bigvee_{n \geq 1} \Sigma Y^{\wedge n} \arrow[l, "r_n"'] \arrow[d, "W"] \\
 & \Sigma Y \vee \Sigma Y \arrow[d, "{(0,1)=r_2}"] \\
\Sigma X \arrow[uu, "H_n f"] \arrow[uur, dashed] \arrow[ur, "\nabla f"'] & \Sigma Y
\end{tikzcd}$$

Since $r_{2*}\nabla f = 0$ and since r_2 is a retraction there is a unique homotopy class $\overline{\nabla f}$ which lifts the class ∇f, that is $W_* \overline{\nabla f} = \nabla f$. We define the element

(A.9.4) $$H_n f = r_{n*} \overline{\nabla f} \in [\Sigma X, \Sigma Y^{\wedge n}]$$

by the retraction r_n in (A.9.3). The element $H_n f$ is one of the Hilton–Hopf invariants which, in particular, was considered by Barcus and Barratt [HC]. The advantage of definition (A.9.4) is the fact that $H_n f$ depends only on the definition of ∇f since the map W in Proposition A.3.6 is defined canonically. On the other hand, the James–Hopf invariants depend on the choice of the 'admissible ordering'. In (A.2.4) we have chosen the admissible ordering given by the lexicographical ordering from the left. In fact this is a good choice since we prove:

(A.9.5) Theorem *Let $f \in [\Sigma X, \Sigma Y]$ where X is finite dimensional and where Y is a* co-H-space. *If the James–Hopf invariant, $\gamma_n f$, is defined with respect to the lexicographical ordering from the left we have the formula*

$$-H_n f = (-1) \circ (\gamma_n f).$$

Here $-1 \in [\Sigma Y^{\wedge n}, \Sigma Y^{\wedge n}]$ is given by the identity 1 of $\Sigma Y^{\wedge n}$. The formula implies $\Sigma H_n f = \Sigma \gamma_n f$.

The result $\Sigma^{n-1} H_n f = \Sigma^{n-1} \gamma_n f$ is also proved in Theorem (4.18)(a) of Boardman and Steer [HI].

Proof of Theorem A.9.5 We deduce the theorem from formula (A.9.1). By (A.9.1) we know

$$\nabla f = i_1 f - \sum_{n\geq 2} c_n(i_2, i_1)\gamma_n(f) \tag{1}$$

where

$$c_n(i_2, i_1) \in [\Sigma Y^{\wedge n}, \Sigma Y \vee \Sigma Y]$$

is given by (A.9.2) with $r_{2*}c_n(i_2, i_1) = 0$. Therefore there is a unique

$$\bar{c}_n(j_1, \ldots, j_n) \in \left[\Sigma Y^{\wedge n}, \bigvee_{k\geq 1} \Sigma Y^{\wedge k}\right] \tag{2}$$

with $W_* \bar{c}_n(j_1, \ldots, j_n) = c_n(i_2, i_1)$. Here

$$j_m\colon \Sigma Y^{\wedge m} \subset \bigvee_{k\geq 1} \Sigma Y^{\wedge k}$$

is the inclusion and $\bar{c}_n(j_1, \ldots, j_n)$ is a sum of iterated Whitehead products of such inclusions. By the explicit formula for $c_n(i_2, i_1)$ in (A.9.2) it is possible to derive an explicit formula for $\bar{c}_n(j_1, \ldots, j_n)$ in (2). For example we have

$$-\bar{c}_2(j_1, j_2) = j_2, \tag{3}$$

$$-\bar{c}_3(j_1, j_2, j_3) = j_3 + [j_2, j_1] + [j_2, j_1]T_{132}. \tag{4}$$

By (1), (2) and by the definition of $\overline{\nabla} f$ in (A.9.3) we see that

$$\overline{\nabla} f = j_1 f - \sum_{n\geq 2} \bar{c}_n(j_1, \ldots, j_n)\gamma_n(f). \tag{5}$$

We claim that for the retraction r_n in (A.9.3) we have $r_{n*}\bar{c}_m = 0$ for m not equal to n and

$$r_{n*}\bar{c}_n(j_1, \ldots, j_n) = -j_n, \quad n \geq 2. \tag{6}$$

This follows from the fact that the only bracket $[x, y]_{\phi(d)}$ with $d \in D_n$ in which y appears only once is

$$[x_2, y_1, x_3, \ldots, x_n].$$

(For this compare the explicit description of D_n in (I.1.16) in Baues [CC].) Now (5) and (6) yield the result in Theorem A.9.5. □

We now consider the partial suspension of the element $\nabla f \in [\Sigma X, \Sigma Y \vee \Sigma Y]_2$; compare Proposition A.3.4 and (A.1.1)(3). By Theorem A.9.5 we get

(A.9.6) Corollary *Let $f \in [\Sigma X, \Sigma Y]$ where X is finite dimensional and where Y is a* co-H-space *(it is enough to assume that Y is connected). With the inclusions*

$$\Sigma^2 Y \xrightarrow[j_1]{} \Sigma^2 Y \vee \Sigma Y \xleftarrow[j_2]{} \Sigma Y$$

we have the formula in $[\Sigma^2 X, \Sigma^2 Y \vee \Sigma Y]_2$

$$E\nabla f = j_1(\Sigma f) + \sum_{n \geq 2} [j_1, j_2^{n-1}](\Sigma \gamma_n f)$$

where $[j_1, j_2^{n-1}] = [\cdots[j_1, j_2]\cdots, j_2]]$ is an $(n\text{-}1)$-fold Whitehead product.

Proof We have

$$E\nabla f = E(W\overline{\nabla f}) = (EW)(\Sigma\overline{\nabla f}) \tag{1}$$

where

$$E\overline{\nabla f} = \Sigma\Big(\sum_{m \geq 1} j_m H_m f\Big) \tag{2}$$

by (A.9.3). By Proposition A.1.2 and by definition of W in Proposition A.3.6 we obtain the proposition in Corollary A.9.6 where we use the fact that $\Sigma H_m f = \Sigma \gamma_m f$. Compare the proof of (3.3.19) in Baues [OT]. □

Corollary A.9.6 is of importance for the homotopy classification of maps. From (II.13.10) in Baues [AH] we derive:

(A.9.7) Theorem *Let $C = C_f$ be the mapping cone of $f: \Sigma X \to \Sigma Y$ where X is finite dimensional and where Y is connected. Moreover, let $w: C_f \to U$ be a map with restriction $u: \Sigma Y \to U$. Then we have the long exact sequence ($k \geq 2$):*

$$\to [\Sigma^{k+1} Y, U] \xrightarrow{\nabla^k(u,f)} [\Sigma^{k+1} X, U] \xrightarrow{w^+} \pi_k(U^{C|\bullet}, w) \xrightarrow{i} \cdots$$

$$\cdots \xrightarrow{i} [\Sigma^2 Y, U] \xrightarrow{\nabla(u,f)} [\Sigma^2 X, U] \xrightarrow{w^+} [C_f, U] \xrightarrow{i_f^*} [\Sigma Y, U].$$

Here $\nabla^k(u,f)$ is given by the formula

$$\begin{aligned}\nabla^k(u,f)(\alpha) &= (E^k \nabla f)^*(\alpha, u)\\ &= \alpha(\Sigma^k f) + \sum_{n \geq 2} [\alpha, u^{n-1}](\Sigma^k \gamma_n f)\end{aligned}$$

with $\alpha \in [\Sigma^{k+1} Y, U]$ and $k \geq 1$. We set $\nabla = \nabla^1$.

The element $[\alpha, u^{n-1}] = [\cdots[\alpha, u], \cdots u]]$ denotes an $(n-1)$-fold Whitehead product. Exactness in Theorem A.9.7 implies that the image of $\nabla(u, f)$ is the isotropy group in $w \in [C_f, U]$ of the action $[\Sigma^2 X, U]+$ on the set $[C_f, U]$, that is:

$$\text{(A.9.8)} \qquad \operatorname{image} \nabla(u, f) = \{\beta \in [\Sigma^2 X, U] \mid \{w\} + \beta = \{w\}\}.$$

Therefore we have the bijection

$$\text{(A.9.9)} \qquad [C_f, U] \approx \bigcup_{\substack{u \in [\Sigma Y, U] \\ f^* u = 0}} [\Sigma^2 X, U]/\operatorname{image} \nabla(u, f).$$

The bijection is defined by choosing elements w in each orbit of the action $[\Sigma^2 X, U]+$ on the set $[C_f, U]$. This shows that Theorem A.9.7 yields an explicit method for the enumeration of the set $[C_f, U]$. A first result of the type in Theorem A.9.7 was obtained by Barcus and Barratt [HC], see also Rutter [HC], however, in these papers only equation (A.9.8) and not the exact sequence in Theorem A.9.7 is discussed.

For a mapping cone $C_g, g: Y \to B$, and for a map $f: \Sigma X \to C_g$ we obtain a diagram similar to (A.9.3) as follows:

$$\text{(A.9.10)} \qquad \begin{tikzcd} \Sigma Y \wedge \Omega C_g & \Sigma Y \vee \Sigma Y \wedge \Omega C_g \arrow[l, "r_2"'] \arrow[d, "W = (i_1, [i_1, i_2 R_{C_g}])"] \\ \Sigma X \arrow[u, "H(f)"] \arrow[ur, dashed] \arrow[r, "\nabla f"'] & \Sigma Y \vee C_g \arrow[d, "r_2"] \\ & C_g \end{tikzcd}$$

Compare Propositions A.3.5 and A.3.6. Here ∇f is again defined by $f = -i_2 f + (i_2 + i_1) f$. Since $r_{2*} \nabla f = 0$ and since r_2 is a retraction there is a unique homotopy class $\overline{\nabla f}$ which lifts ∇f. We call

$$\text{(A.9.11)} \qquad H(f) = r_{2*} \overline{\nabla f} \in [\Sigma X, \Sigma Y \wedge \Omega C_g]$$

the *total Hopf invariant* of f. If $B = *$ is a point we can derive from $H(f)$ all invariants $H_n f$ in (A.9.3). Clearly, we have

$$\text{(A.9.12)} \qquad r_{1*} \overline{\nabla f} = j_0 f \in [\Sigma X, \Sigma Y]$$

where $j_0: C_g \to C_g/B = \Sigma Y$ is the pinch map. By the Hilton–Milnor theorem

$$\text{(A.9.13)} \qquad \Sigma \overline{\nabla f} = i_1(\Sigma j_0 g) + i_2(\Sigma H(f))$$

where i_1 and i_2 are the inclusions.

(A.9.14) Problem Is it possible to express $\overline{\nabla} f$ solely in terms of $j_0 f$ and $H(f)$? If $B = *$ is a point this is true by the left distributivity law described above; see (6) in the proof of Theorem A.9.5. Theorem A.8.11 might be helpful.

We now consider the partial suspension of the difference element $\nabla f \in [\Sigma X, \Sigma Y \vee C_g]_2$. This generalizes Corollary A.9.6. By (A.9.10) we get:

(A.9.15) Proposition *With the inclusions*

$$\Sigma^2 Y \xrightarrow[j_1]{} \Sigma^2 Y \vee C_g \xleftarrow[j_2]{} C_g$$

we have the formula

$$(E\nabla f) = j_1 \Sigma(j_0 f) + \left[j_1, j_2 R_{C_g}\right] \circ (\Sigma Hf).$$

Thus $E\nabla f$ depends only on $j_0 f$ and on the total Hopf invariant Hf.

Proof of Proposition A.9.15 We have

$$E(\nabla f) = E(W\nabla f) = (EW)(\Sigma \nabla f)$$

and therefore the result follows from (A.9.13) and Proposition A.1.2. □

Clearly, we can derive from Proposition A.9.15 an expression for the operators $\nabla^k(u, f)$ in the exact sequence (II.13.10) Baues [AH] in the same way as in Theorem A.9.7:

(A.9.16) Proposition $f \in [\Sigma X, C_g]$, $u \colon C_g \to U$, $\beta \in [\Sigma^n Y, U]$.

$$\begin{aligned}\nabla^n(u, f)(\beta) &= (E^n \nabla f)^*(\beta, u)\\ &= \beta(\Sigma^n j_0 f) + \left[\beta, u R_{C_g}\right](\Sigma^n Hf)\end{aligned}$$

where $R_{C_g} \colon \Sigma \Omega C_g \to C_g$ *is the evaluation map.*

A.10 Distributivity laws of order 3

We here describe all distributivity laws of order 3 in homotopy theory. They are needed in the proof of Chapter 9. We derive these distributivity laws from 2.51 in Dreckmann [DH]. Recall that a map $f \colon X \to Y$ is a *phantom map* if for all finite dimensional CW-complexes K and for all maps $g \colon K \to X$ the composite gf is null-homotopic.

(A.10.1) Notation Let $\mathfrak{X}$ be a class of pointed spaces and consider the homotopy group $[\Sigma A, B]$ given by pointed spaces A, B. We define the subgroup $U(\mathfrak{X}) \subset [\Sigma A, B]$ to be the normal subgroup generated by all composites

$$\Sigma A \to X \to B \quad \text{with} \quad X \in \mathfrak{X}$$

and all phantom maps $\Sigma A \to B$. For $f, g \in [\Sigma A, B]$ we write

$$f = g \quad \textit{modulo} \quad \mathfrak{X}$$

if $\{f\} = \{g\}$ in the quotient group $[\Sigma A, B]/U(\mathfrak{X})$.

In the following theorem we consider the *primary homotopy operations*

$+ : [\Sigma A, Z] \times [\Sigma A, Z] \to [\Sigma A, Z]$	(addition)
$\circ : [\Sigma A, \Sigma B] \times [\Sigma B, Z] \to [\Sigma A, Z]$	(composition)
$\#, \underline{\#} : [\Sigma A, \Sigma X] \times [\Sigma B, \Sigma Y] \to [\Sigma A \wedge B, \Sigma X \wedge Y]$	(exterior cup products)
$\cup, \underline{\cup} : [\Sigma A, \Sigma X] \times [\Sigma A, \Sigma Y] \to [\Sigma A, \Sigma X \wedge Y]$	(interior cup products)
$\gamma_n : [\Sigma A, \Sigma B] \to [\Sigma A, \Sigma B^{\wedge n}]$	(James–Hopf invariant)
$[\ ,\ ] : [\Sigma A, Z] \times [\Sigma B, Z] \to [\Sigma A \wedge B, Z]$	(Whitehead product).

Here $\gamma_n = \gamma_n^{<}$ is defined with respect to an admissible ordering $<$ (satisfying $\{1\} < \{2\}$, see (A.2.4)). Moreover, let $T_{n_1, \dots, n_r}$ be the shuffle map in (A.1.6).

(A.10.2) Theorem *Let A, A', B, B', C, Z be path-connected pointed CW-spaces. Then the primary homotopy operations above satisfy the following formulas (a)–(h).*

(a) Let $<$ and $\ll$ be two admissible orderings and $f \in [\Sigma A, \Sigma B]$. Then we have for $n = 2, 3$

$$\gamma_n^{<}(f) = \gamma_n^{\ll}(f) \quad \textit{modulo} \quad \Sigma B^{\wedge r}, r \geq 4.$$

(b) Let $u, v \in [\Sigma B, Z]$ and $f \in [\Sigma A, \Sigma B]$. Then

$$uf + vf = (u + v)f + ([u, v]T_{21} + [[u, v], v]T_{212})\gamma_2(f)$$
$$+ ([[u, v], v]T_{213} + [[u, v], v]T_{312} + [[u, v], u]T_{231})\gamma_3(t)$$

modulo $\Sigma B^{\wedge r}, r \geq 4$.

(c) Let $u \in [\Sigma B, Z], v \in [\Sigma B', Z], f \in [\Sigma A, \Sigma B]$, and $g \in [\Sigma A', \Sigma B']$. Then

$$[uf, vg] = [u, v](f \# g) + [[u, v], v](f \# \gamma_2(g)) + [[u, v], u]T_{132}(\gamma_2(f) \# g)$$

modulo $\Sigma B^{\wedge r} \wedge (B')^{\wedge s}, r + s \geq 4$.

(d) Let $f \in [\Sigma A, \Sigma B]$, $g \in [\Sigma A', \Sigma B']$. Then $f \# g = f \underline{\#} g$, and $f \cup g = f \underline{\cup} g$ for $A = A'$ where both equations hold modulo $\Sigma B^{\wedge r} \wedge (B')^{\wedge s}, r + s \geq 4$.

(e) Let $f \in [\Sigma A, \Sigma B]$. Then

$$f \cup f = T_{11} f + (T_{12} + T_{21}) \gamma_2(f)$$

$$f \cup \gamma_2(f) = (T_{112} + T_{212}) \gamma_2(f) + (T_{123} + T_{213} + T_{312}) \gamma_3(f)$$

$$\gamma_2(f) \cup f = (T_{121} + T_{122}) \gamma_2(f) + (T_{123} + T_{132} + T_{231}) \gamma_3(f)$$

where these equations hold modulo $\Sigma B^{\wedge r}, r \geq 4$.

(f) Let $f, g \in [\Sigma A, \Sigma B]$. Then

$$\gamma_2(f + g) = \gamma_2(f) + \gamma_2(g) + f \cup g$$

$$\gamma_3(f + g) = \gamma_3(f) + \gamma_3(g) + f \cup \gamma_2(g) + \gamma_2(f) \cup g.$$

These equations hold modulo $\Sigma B^{\wedge r}, r \geq 4$.

(g) Let $f \in [\Sigma B, \Sigma C]$, $g \in [\Sigma A, \Sigma B]$. Then

$$\gamma_2(fg) = \gamma_2(f) g + (f \# f) \gamma_2(g)$$

$$\gamma_3(fg) = \gamma_3(f) g + (f \# \gamma_2(f)) \gamma_2(g) + (\gamma_2(f) \# f) \gamma_2(g) + (f \# f \# f) \gamma_3(g).$$

These equations hold modulo $\Sigma B^{\wedge r}, r \geq 4$.

(h) Let $f \in [\Sigma A, \Delta B]$, $g \in [\Sigma A', \Sigma B']$. Then

$$\gamma_2([f, g]) = (T_{12} - T_{21})(f \# g)$$

$$\begin{aligned} \gamma_3([f, g]) = {} & (T_{121} - T_{112} - T_{212} + T_{221})(f \# g) \\ & + (T_{123} - T_{213} - T_{312} + T_{321})(f \# \gamma_2(g)) \\ & + (T_{132} - T_{312} - T_{213} + T_{231})(\gamma_2(f) \# g). \end{aligned}$$

These equations hold modulo $\Sigma B^{\wedge r}, r \geq 4$.

W. Dreckmann in 2.51 of [DH] described formulas as in Theorem A.10.2 which actually hold modulo the empty class of spaces; the reduction of these formulas yields the formulas above.

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NOTATION FOR CATEGORIES

Boldface sans serif letters like **C**, **A**, **Top** denote categories.

Ob(**C**) class of objects
C$(A,B) = \mathrm{Hom}_{\mathbf{C}}(A,B) = \mathrm{Hom}(A,B)$ set of morphisms
$\mathrm{Aut}_{\mathbf{C}}(A)$ group of automorphisms

$\mathbf{C}^{\mathrm{op}}$ opposite category
$F\mathbf{C}$ category of factorizations 10
Pair(**C**) category of pairs 224
$\mathbf{C} \times D$ split linear extension 11

Set* category of pointed sets 398
Gr category of groups
Ab category of abelian groups
Ab[1/2] 419 **Ab** (free) 423
FAb finitely generated abelian groups 25
^{f}**Ab** finite abelian groups 202
Cyc cyclic groups 22
^{f}**Cyc** finite cyclic groups 386
PCyc elementary cyclic groups 361 $\mathbf{PCyc}^0$ 402
Add($\mathbb{Z}$) finitely generated free abelian groups 215
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Top category of topological spaces
Top* pointed topological spaces 32
Top*/$\simeq$ homotopy category 32

CW category of CW-complexes with trivial 0-skeleton 32
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