Stably hyperbolic ε-hermitian forms and doubly sliced knots

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Introduction

Let A be a commutative $\mathbb{Q}$-algebra, finite dimensional over $\mathbb{Q}$. Let us consider a $\mathbb{Q}$-involution on A, which we shall denote with an overbar. Let Λ be an order of A which is stable under this involution. All modules in this paper will be finitely generated and reflexive — but not necessarily projective. To a reflexive Λ-module M we associate the hyperbolic ε-hermitian (where $\varepsilon = 1$ or -1) form $H(M)$, see Section 1 for the definition. We shall say that a unimodular ε-hermitian form $h: L \times L \to \Lambda$ is *stably hyperbolic* if the orthogonal sum of (L, h) with some hyperbolic ε-hermitian form is hyperbolic. The purpose of this paper is to study the following problem: are stably hyperbolic ε-hermitian forms hyperbolic? We prove that this is the case under suitable hypothesis, see Section 6.

Our main motivation for the study of this problem is a knot theoretical question, which involves the ring $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$, where $\lambda \in \mathbb{Z}[X]$ is a polynomial such that $\lambda(X) = X^{\deg(\lambda)}\lambda(X^{-1})$, and $\lambda(1) = 1$, $\lambda(0) \neq 0$. Let us consider the $\mathbb{Z}$-involution of $\mathbb{Z}[X, X^{-1}]$ which sends X to X^{-1}. This induces an involution of Λ.

Theorem. *Let $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$ as above. Then stably hyperbolic ε-hermitian forms are hyperbolic.*

This theorem has the following

Corollary. *Let $q \geqq 2$ be an integer. Then stably doubly sliced simple $(2q-1)$-knots are doubly sliced.*

We prove this in Section 8. This corollary gives a partial solution of Problem 23, [6].

Let $\Lambda = \mathbb{Z}G$, where G is a finite abelian group. Stably hyperbolic ε-hermitian forms over this order also arise in connection with topological problems (cf. M. Kreck [11]). We prove the following

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Theorem. *Let $\Lambda = \mathbb{Z}G$, where G is a finite abelian group of order p^n, where p is an odd prime. Then stably hyperbolic ε-hermitian forms are hyperbolic.*

The proofs une results of H.-G. Quebbemann, R. Scharlau, W. Scharlau and M. Schulte, as well as strong approximation theorems in appropriate algebraic groups.

Although we do not prove that stably hyperbolic ε-hermitian forms are hyperbolic for arbitrary orders, we do not have any counter-examples either. On the other hand, counter-examples exist for commutative rings of Krull dimension two. In Section 7 we give such a counter-example, which has been communicated to us by M. Ojanguren.

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1. Definitions

Let Λ be a ring with an involution, which we shall denote with an overbar. Let M be a finitely generated left Λ-module. Set $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$ with the left Λ-module structure given by $(\lambda f)(m) = f(m)\bar{\lambda}$. We shall say that M is *reflexive* if the evaluation homomorphism $i_M: M \to M^{**}$, defined by $i_M(m)(f) = \overline{f(m)}$, is an isomorphism. Let $h: M \times M \to \Lambda$ be an ε-hermitian form. We shall say that (M, h) is *unimodular* if the homomorphism $\mathrm{ad}(h): M \to M^*$ induced by h is an isomorphism. It is easy to check that if (M, h) is a unimodular ε-hermitian form, then M is reflexive. Let M be a reflexive Λ-module. We associate to M the *hyperbolic* ε-hermitian form $H(M)$, defined as follows:

$$M \oplus M^* \times M \oplus M^* \to \Lambda, \quad (m, f) \times (n, g) \to f(n) + \varepsilon \overline{g(m)}.$$

As M is reflexive, $H(M)$ is a unimodular ε-hermitian form. Let $h: L \times L \to \Lambda$ be a unimodular ε-hermitian form. We shall say that (L, h) is *stably hyperbolic* if there exist reflexive Λ-modules M and N such that $(L, h) \boxplus H(M) \cong H(N)$, where $\boxplus$ denotes orthogonal sum.

We shall say that an ε-hermitian form $h: L \times L \to \Lambda$ is *even* if there exists a sesquilinear form $g: L \times L \to \Lambda$ such that $h = g + \varepsilon g^*$, where $g^*: L \times L \to \Lambda$ is defined by $g^*(n, m) = \overline{g(m, n)}$.

2. Local cancellation

Let R be a commutative ring. We shall say that a ring Λ is an R-algebra if there exists a ring homomorphism $f: R \to Z(\Lambda)$, where $Z(\Lambda)$ is the center of Λ, such that $f(1) = 1$. Moreover, we shall assume that an R-algebra is finitely generated as R-module.

The following theorem is essentially a consequence of results of H.-G. Quebbemann, R. Scharlau, W. Scharlau and M. Schulte:

Theorem 1. *Suppose that Λ is either a left artinian ring or an R-algebra, where R is a complete local ring. Let M_1, M_2 and M be finitely generated left Λ-modules, and let (M_1, h_1), (M_2, h_2) and (M, h) be unimodular, even ε-hermitian forms. Assume that*

$$(M_1, h_1) \boxplus (M, h) \cong (M_2, h_2) \boxplus (M, h).$$

Then $(M_1, h_1) \cong (M_2, h_2)$.

Proof. Let $\mathcal{M}$ be the category of finitely generated reflexive left Λ-modules. Let us define a duality $*: \mathcal{M} \to \mathcal{M}$ by $M^* = \operatorname{Hom}_\Lambda(M, \Lambda)$. We shall apply the cancellation results of Quebbemann, Scharlau and Schulte (cf. [19], § 3, or [22], Chapter 7, § 10). In order to do this, we have to check that the conditions (i), (ii) and (iii) of [19], § 3 are satisfied. Condition (i) clearly holds as $\mathcal{M}$ is a category of modules. The endomorphism rings of the indecomposable objects of $\mathcal{M}$ are local by Reiner [20], § 2, (10). This implies that condition (ii) is satisfied. Finally, let us check condition (iii). We have to show that for every M in $\mathcal{M}$, the ring of endomorphisms $\operatorname{End}(M)$ of M is $J(M)$-adically complete, where $J(M) = \operatorname{rad}\operatorname{End}(M)$. If Λ is artinian, then $\operatorname{End}(M)$ is also artinian, so $J(M)^m = 0$ for some positive integer m. Let R be a complete local ring with maximal ideal p, and let Λ be an R-algebra. Then $J(M)^m \subset p \operatorname{End}(M)$ for some positive integer m, cf. [20], § 2, (9). Therefore condition (iii) is satisfied in both cases. The assertion of Theorem 1 follows from [19], 3. 4, (1) or [22], Chapter 7, Theorem 10. 9. (iv). □

3. Locally hyperbolic ε-hermitian forms are hyperbolic: the case of a semi-simple algebra with a strongly non-trivial involution

Let A be a commutative $\mathbb{Q}$-algebra, finite dimensional over $\mathbb{Q}$. Let us consider a $\mathbb{Q}$-involution on A which we shall denote with an overbar. Let Λ be an order of A such that $\bar{\Lambda} = \Lambda$. If p is a prime number, we shall denote by $\mathbb{Z}_p$ the ring of p-adic integers. Let $\Lambda_p = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For any Λ-module N, we shall denote $N_p = N \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Let L be a finitely generated Λ-module, and let $h: L \times L \to \Lambda$ be a unimodular ε-hermitian form. Then (L, h) induces a unimodular ε-hermitian form $h_p: L_p \times L_p \to \Lambda_p$. We shall say that (L, h) is *locally hyperbolic* if there exists a reflexive Λ-module M such that for every prime number p, the ε-hermitian forms (L_p, h_p) and $H(M_p)$ are isometric.

Let $\operatorname{rad}(A)$ be the radical of the $\mathbb{Q}$-algebra A. Then $\overline{\operatorname{rad}(A)} = \operatorname{rad}(A)$, because $\operatorname{rad}(A)$ is the maximal nilpotent ideal of A. Therefore we also obtain an involution on $A_0 = A/\operatorname{rad}(A)$. We have $A_0 = A_1 \times \cdots \times A_r$, where the A_i's are algebraic number fields. We shall say that the involution is *strongly non-trivial* if for all A_i such that $\bar{A}_i = A_i$, the restriction of the involution to A_i is non-trivial.

The following example is particularly important for topological applications (cf. Section 8).

Example 1. Let $A=\mathbb{Q}[X,X^{-1}]/(\lambda)$, where $\lambda\in\mathbb{Z}[X]$ is a polynomial which satisfies $\lambda(X)=X^{\deg(\lambda)}\lambda(X^{-1})$ and $\lambda(1)=1$, $\lambda(0)\neq 0$. Let us consider the $\mathbb{Q}$-involution of $\mathbb{Q}[X,X^{-1}]$ which sends X to X^{-1}. This induces a $\mathbb{Q}$-involution of A which is strongly non-trivial.

In Sections 3–5, we shall study the following problem: when are locally hyperbolic ε-hermitian forms hyperbolic? We shall begin by proving that this is the case when A is semi-simple and the involution is strongly non-trivial (see Theorem 2). Then in Section 4 we shall generalize this result to other semi-simple $\mathbb{Q}$-algebras. However, the proof of Theorem 2 is much simpler, and as Example 1 shows the case of a strongly non-trivial involution is of special interest.

Theorem 2. *Assume that A is semi-simple, and that the involution is strongly non-trivial. Then locally hyperbolic ε-hermitian forms are hyperbolic.*

Proof. Let (L,h) be a locally hyperbolic ε-hermitian form, and let M be a reflexive Λ-module such that (L_p,h_p) and $H(M_p)$ are isometric for every prime number p. Let $V=L\otimes\mathbb{Q}$, and $W=M\otimes\mathbb{Q}$. By Landherr's theorem (cf. [12]) we may assume that $(V,h)=H(W)$. For every prime number p, there exists an automorphism ϕ_p in $U(V_p,h_p)$ such that $L_p=\phi_p H(M_p)$. We may assume that $\phi_p=1$ for almost all p, because $L_p=H(M_p)$ for almost all p. Let $x_p=\det(\phi_p)$. Then $x_p\bar{x}_p=1$, and $x_p=1$ for almost all p. By Hilbert's theorem 90, there exists an element y_p of A_p such that $x_p=\frac{y_p}{\bar{y}_p}$. Let α_p be an endomorphism of W_p such that $\det(\alpha_p)=y_p$. This induces an element $\psi_p=H(\alpha_p)$ of $U(V_p,h_p)$ with $\det(\psi_p)=x_p$. Set

$$M'_{(p)}=\begin{cases}\alpha_p(M_p) & \text{if } x_p\neq 1,\\ M_p & \text{if } x_p=1.\end{cases}$$

There exists a Λ-lattice M' on W such that $M'_p=M'_{(p)}$ for every prime number p (cf. [20], § 3, (5)). Notice that $L_p=(\phi_p\psi_p^{-1})\,H(M'_p)$ and that $\det(\phi_p\psi_p^{-1})=1$. Strong approximation in the special unitary group (cf. M. Kneser [9] or G. Shimura [23]) then yields an element σ of $SU(V,h)$ with $L=\sigma H(M')$. □

4. Locally hyperbolic forms are hyperbolic: more about the semi-simple case

We shall use the same notations as in Section 3, but we shall not assume that the involution of the $\mathbb{Q}$-algebra A is strongly non-trivial. We can write A under the form $A=B\times C$, where B and C are $\mathbb{Q}$-algebras which are stable under the involution, and such that the restriction of the involution to B is strongly non-trivial, whereas the restriction of the involution to C is trivial. Moreover let $C=C_1\times\cdots\times C_m$, where the C_i's are algebraic number fields.

Let $h: L \times L \to \Lambda$ be a unimodular ε-hermitian form, and let $V = L \otimes_Z Q$. We have an orthogonal splitting

$$(V, h) = (V', h') \boxplus (V_1, h_1) \boxplus \cdots \boxplus (V_m, h_m),$$

where V' is a B-module, V_i is a C_i-vector space and $h': V' \times V' \to B$, $h_i: V_i \times V_i \to C_i$ are unimodular ε-hermitian forms for all $i = 1, \ldots, m$. Let $\mathrm{Aut}(V, h)$ be the set of A-linear automorphisms $\phi: V \to V$ such that $h(\phi u, \phi v) = h(u, v)$ for all u, v in V. We have

$$\mathrm{Aut}(V, h) = U(V', h') \times O(V_1, h_1) \times \cdots \times O(V_m, h_m).$$

Let $\phi = (\psi, \phi_1, \ldots, \phi_m)$ be an element of $\mathrm{Aut}(V, h)$, and let $\det(\phi) = (x, e_1, \ldots, e_m)$, where $x = \det(\psi)$, $e_i = \det(\phi_i)$. Notice that $\det(\phi)\, \overline{\det(\phi)} = 1$. Therefore $x\bar{x} = 1$, and $e_i = 1$ or -1 for all $i = 1, \ldots, m$.

Let p be a prime number, and let Q_p be the field of p-adic numbers. For all Q-algebras D and D-modules U we shall set $D_p = D \otimes_Q Q_p$, $U_p = U \otimes_Q Q_p$. If ϕ_p is an element of $\mathrm{Aut}(V_p, h_p)$, we clearly have a splitting

$$\det(\phi_p) = (x, e_1, \ldots, e_m)$$

with x in B_p, $e_i = 1$ or -1 as above.

If $\varepsilon = 1$, we shall make the following hypothesis:

(*) There exists a prime number q which has the following property: for every prime number $p \neq q$ and for every m-uple $(e_1, \ldots, e_m)$, there exists an element ϕ_p of $\mathrm{Aut}(L_p, h_p)$ such that $\det(\phi_p) = (x, e_1, \ldots, e_m)$ for some x in B_p.

Theorem 3. *Let $h: L \times L \to \Lambda$ be a locally hyperbolic ε-hermitian form. If $\varepsilon = 1$, assume that (*) holds. Then (L, h) is hyperbolic.*

Proof. As (L, h) is locally hyperbolic, there exists a reflexive Λ-module M such that for every prime number p, (L_p, h_p) and $H(M_p)$ are isometric. Let $W = M \otimes_Z Q$. Then by Hasse-Minkowski's theorem (cf. for instance [22]) and Landherr's theorem (cf. [12]), we may assume that $(V, h) = H(W)$. Let ϕ_p be an element of $\mathrm{Aut}(V_p, h_p)$ such that $L_p = \phi_p H(M_p)$. Let $\det(\phi_p) = (x, e_1, \ldots, e_m)$ where $x \in B_p$ and $e_i = 1$ or -1 depend on p. Notice that if $\varepsilon = -1$, then $e_i = 1$ for all $i = 1, \ldots, m$ and for all p.

There exists an element ϕ of $\mathrm{Aut}(V, h)$ such that $\det(\phi)$ and $\det(\phi_q)$ coincide in their last m components: in other words, such that

$$\det(\phi) \det(\phi_q) = (x', 1, \ldots, 1)$$

for some x' in B_q. We have $(\phi L)_p = (\phi \phi_p)\, H(M_p)$ for all p, and

$$\det(\phi \phi_q) = (x', 1, \ldots, 1).$$

Clearly L is hyperbolic if and only if ϕL is hyperbolic, therefore we may assume that $\det(\phi_q) = (x', 1, \ldots, 1)$. But using hypothesis (*) we may also assume that $\det(\phi_p) = (y, 1, \ldots, 1)$ for some $y \in B_p$ if $p \neq q$.

As in the proof of Theorem 2, we see that there exists a Λ-lattice M' together with an element ϕ_p' of $\mathrm{Aut}(V_p, h_p)$ such that $L_p = \phi_p H(M_p')$ and

$$\det(\phi_p') = (1, 1, \ldots, 1).$$

If $\varepsilon = 1$, let $\theta = (\theta_1, \ldots, \theta_m)$ be the spinor norm of ϕ_p. There exists an element ψ_p of $\mathrm{Aut}(V_p, h_p)$ such that $\det(\psi_p) = (1, \ldots, 1)$, $\psi_p(W_p) = W_p$ and that the spinor norm of ψ_p is θ_p (cf. for instance [17], Example 55:1). Let M'' be a lattice on W such that

$$M_p'' = \begin{cases} M_p & \text{if } \theta_p = (1, \ldots, 1), \\ \psi_p(M_p) & \text{otherwise.} \end{cases}$$

Such a lattice exists by [17], 81:14.

Now we have $L_p = \sigma_p H(M_p'')$, with $\sigma_p = \phi_p \psi_p = (\sigma_p', \sigma_{1p}, \ldots, \sigma_{mp})$, where $\sigma' \in SU(V_p', h_p')$ and $\sigma_{ip} \in SO(V_{ip}, h_{ip})$ have spinor norm 1. If $\varepsilon = -1$ or if $n_i \neq 2$ for all $i = 1, \ldots, m$, then we can apply strong approximation (cf. M. Kneser [9]) and conclude that (L, h) and $H(M'')$ are isometric. Let us assume that $\varepsilon = 1$ and that $n_1 = \cdots = n_k = 2$, $n_i \neq 2$ for $i > k$. Let us apply strong approximation to obtain $\sigma' \in SU(V', h')$ and $\sigma_i \in SO(V_i, h_i)$, $i = k+1, \ldots, m$ which approximate σ_p', σ_{ip} for the finitely many prime numbers p such that $\sigma_p' \neq 1$ or $\sigma_{ip} \neq 1$ for some $i = k+1, \ldots, m$. Let

$$\Sigma = (\sigma', 1, \ldots, 1, \sigma_{k+1}, \ldots, \sigma_m).$$

If p is a prime number, we shall denote by $\mathbb{Z}_{(p)}$ the localisation of $\mathbb{Z}$ at p. Let us choose p such that $\Lambda_{(p)} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is Dedekind. Set $L_{(p)} = L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Then $L_{(p)} = L_1 \boxplus L_2$, where L_1 is a $\Lambda_{(p)}$-lattice on $(V_1, h_1) \boxplus \cdots \boxplus (V_k, h_k)$ and L_2 is a $\Lambda_{(p)}$-lattice on $(V', h') \boxplus (V_{k+1}, h_{k+1}) \boxplus \cdots \boxplus (V_m, h_m)$. Let $\pi: L \to L/pL$ be the projection. Notice that $L/pL = L_{(p)}/pL_{(p)}$.

Let N be the intersection of L with $\Sigma(W)$, and let N' be the intersection of L with $\Sigma(W^*)$. We want to prove that L and $H(N)$ are isometric. Clearly $h(N, N) = h(N', N') = 0$. It suffices to prove that $N \oplus N' = L$: then an easy argument (cf. for instance Bass [2], Chapter V, Lemma 2.1) shows that $L \cong H(N)$.

It is enough to check that $\pi(N \oplus N') = L/pL$. We have $L_1 = H(N_1)$, $L_2 = H(N_2)$ where $N_1 = N \cap L_1$ and $N_2 = N \cap L_2$. Indeed, $L_2 = H(N_2)$ by construction. On the other hand, recall that if $i = 1, \ldots, k$ then $\dim_{C_i}(V_i) = 2$, and the restriction of the involution to C_i is trivial. This implies that (V_i, h_i) contains exactly two isotropic lines. Using this, it is easy to see that $L_1 = H(N_1)$. Notice that

$$\pi(N \oplus N') = \pi(N_1 \oplus N_1^* \oplus N_2 \oplus N_2^*).$$

Therefore $\pi(N \oplus N') = L/pL$. This concludes the proof of Theorem 3. □

Corollary 1. *Let G be an abelian group of order q^n, where q is a prime number. Let $\Lambda = \mathbb{Z}G$. Let $h: L \times L \to \Lambda$ be a locally hyperbolic ε-hermitian form. Then (L, h) is hyperbolic.*

Proof. If $p \neq q$, then Λ_p is the maximal order of A_q. Therefore condition (*) of Theorem 3 is satisfied. □

5. Locally hyperbolic ε-hermitian forms are hyperbolic: the case of a non-semi-simple algebra

Let $A = \mathbb{Q}[X, X^{-1}]/(\lambda)$, where $\lambda \in \mathbb{Z}[X]$ is a polynomial which satisfies

$$\lambda(X) = X^{\deg(\lambda)}\lambda(X^{-1})$$

and $\lambda(1) = 1$, $\lambda(0) \neq 0$. Let us consider the $\mathbb{Q}$-involution of $\mathbb{Q}[X, X^{-1}]$ which sends X to X^{-1}. This induces a $\mathbb{Q}$-involution of A which is strongly non-trivial. Let Λ be an order of A such that $\bar{\Lambda} = \Lambda$. We shall prove the following:

Theorem 4. *Locally hyperbolic ε-hermitian forms are hyperbolic.*

We shall use some results of H.-G. Quebbemann, R. Scharlau, W. Scharlau and M. Schulte (cf. [19] or [22], Chapter 7). We shall begin by recalling these results in the special case which we are interested in. Some of these results already follow from J. Milnor [14] or G. E. Wall [27].

Let $\mathcal{M}$ be the category of finitely generated, reflexive Λ-modules, and let $\mathcal{R}$ be the radical of the category $\mathcal{M}$. Let $H^\varepsilon(\mathcal{M})$ be the category of unimodular ε-hermitian forms over $\mathcal{M}$. Let (V, h) be an object of $H^\varepsilon(\mathcal{M})$. We shall denote by $U(V, h)$ the set of A-linear automorphisms of V which are isometries for h. Let us denote by $\operatorname{ad}(h)\colon V \to V^*$ the adjoint of the ε-hermitian form h. Let U and W be A-modules and let f be an element of $\operatorname{Hom}(U, W)$. Then we shall denote by f^* the element of $\operatorname{Hom}(U^*, W^*)$ induced by f. Let $(U/\mathcal{R})(V, h)$ be the set of $f \in \operatorname{End}(V)$ such that $f^* \operatorname{ad}(h) f \equiv \operatorname{ad}(h) \bmod \mathcal{R}(V, V^*)$. Then the natural homomorphism

$$U(V, h) \to (U/\mathcal{R})(V, h)$$

is surjective (cf. [19], Theorem 2.2 (2) or [22], Chapter 7, Theorem 4.4 (ii). Recall that we have already checked the validity of conditions (i), (ii) and (iii) of [19] for the category $\mathcal{M}$ in Section 2, in the proof of Theorem 1).

There exists an orthogonal decomposition $(V, h) \cong \boxplus_{i=1}^{m} (V_i, h_i)$, where V_i is of type $\{U_i, U_i^*\}$ with U_i indecomposable and $U_i \not\cong U_j$, U_j^* for $i \neq j$. Moreover, this decomposition is unique up to isometry (cf. [19], Theorems 3.2 and 3.3 or [22], Theorems 10.8 and 10.9). We have

$$(U/\mathcal{R})(V, h) = (U/\mathcal{R})(V_1, h_1) \times \cdots \times (U/\mathcal{R})(V_m, h_m).$$

In the special case which we are dealing with, the endomorphism rings of the indecomposable modules are very simple. Let $\lambda = \lambda_1^{e_1} \cdots \lambda_r^{e_r}$ with λ_j irreducible. Let $A_j = \mathbb{Q}[X, X^{-1}]/(\lambda_j)$. The structure theorem for finitely generated torsion $\mathbb{Q}[X, X^{-1}]$-modules shows that if U is an indecomposable A-module, then U is isomorphic to $\mathbb{Q}[X, X^{-1}]/(\lambda_j^n)$ for some $1 \leqq j \leqq r$, and $1 \leqq n \leqq e_j$. Therefore

$$\operatorname{End}(U) = \mathbb{Q}[X, X^{-1}]/(\lambda_j^n).$$

Let $J(U) = \operatorname{rad} \operatorname{End}(U)$. We have

$$\operatorname{End}(U)/J(U) = \mathbb{Q}[X, X^{-1}]/(\lambda_j) = A_j.$$

Assume that $\bar{A}_j = A_j$, or equivalently that $\lambda_j(X) = X^{\deg(\lambda_j)} \lambda_j(X^{-1})$. Then there exists a unimodular hermitian form $h_0: A_j \times A_j \to A$. Indeed, let $i: A_j \to A$ be the inclusion of A_j in A: $i(x) = (\lambda/\lambda_j)(x)$. Set $h_0(x, y) = i(x\bar{y})$. If $\bar{A}_j \neq A_j$, then we obtain a unimodular hermitian form $h_0: (A_j \oplus \bar{A}_j) \times (A_j \oplus \bar{A}_j) \to A$ in a similar way. We shall apply the principle of transfer (cf. [19] or [22], Chapter 7) using these hermitian forms. We obtain the following:

Lemma 1 ("Hasse-Minkowski's theorem"). *Let* $h: V \times V \to A$ and $h': V' \times V' \to A$ *be unimodular ε-hermitian forms. Assume that* (V_p, h_p) *and* (V'_p, h'_p) *are isomorphic for every prime number p. Then* (V, h) *and* (V', h') *are isomorphic.*

Proof. Let us apply reduction and transfer to (V, h), (V', h'), (V_p, h_p) and (V'_p, h'_p) as above. By hypothesis we have an isomorphism between the images of (V_p, h_p) and of (V'_p, h'_p). Notice that reduction and transfer commute with localisation. Therefore by Landherr's theorem (cf. [12]) we have an isomorphism between the images of (V, h) and of (V', h'). This implies that (V, h) and (V', h') are isomorphic, cf. [19], Theorem 2.2 (2) or [22], Theorem 4.4 (ii). □

Proof of Theorem 4. Let $h: L \times L \to \Lambda$ be a locally hyperbolic ε-hermitian form, and let M be a reflexive Λ-module such that (L_p, h_p) and $H(M_p)$ are isomorphic for every prime number p. Set $V = L \otimes_Z \mathbb{Q}$, $W = M \otimes_Z \mathbb{Q}$. By Lemma 1 we may assume that $(V, h) = H(W)$.

Let $B_i = A_i$ if $\bar{A}_i = A_i$, and $B_i = A_i \times \bar{A}_i$ otherwise. We shall use the notations of the beginning of the section. We have an orthogonal decomposition

$$(V, h) = \boxplus_{i=1}^{m} (V_i, h_i),$$

where the V_i's are of type $\{U_i, U_i^*\}$, U_i indecomposable, and $U_i \not\cong U_j$, U_j^* if $i \neq j$.

Let

$$P_i = \operatorname{Hom}_{(\mathcal{M}/\mathcal{R})}(U_i, V_i) = \operatorname{Hom}(U_i, V_i)/\mathcal{R}(U_i, V_i) \quad \text{if} \quad \bar{A}_i = A_i,$$

and let

$$P_i = \operatorname{Hom}_{(\mathcal{M}/\mathcal{R})}(U_i \oplus U_i^*, V_i) \quad \text{if} \quad \bar{A}_i \neq A_i.$$

Then P_i is a projective $B_i = \operatorname{End}(U_i)/J(U_i)$-module. Using the hermitian form $h_0: B_i \times B_i \to A$ which we have defined above, we obtain by transfer a unimodular ε-hermitian form $h'_i: P_i \times P_i \to B_i$. The transfer also induces a canonical isomorphism between $(U/\mathcal{R})(V_i, h_i)$ and $U(P_i, h'_i)$. Therefore we have a surjective homomorphism

$$U(V, h) \to U(P_1, h'_1) \times \cdots \times U(P_m, h'_m).$$

We shall denote by $SU(V, h)$ the inverse image of $SU(P_1, h'_1) \times \cdots \times SU(P_m, h'_m)$ by this homomorphism. Let N be the kernel of the surjection

$$SU(V, h) \to SU(P_1, h'_1) \times \cdots \times SU(P_m, h'_m).$$

Notice that N is a unipotent group. Therefore by M. Kneser [9], Hilfssatz 2.4 and Hilfssatz 2.3, strong approximation holds for $SU(V, h)$ if and only if it holds for $SU(P_i, h'_i)$ for all $i = 1, \dots, m$. But (P_i, h'_i) is hyperbolic, therefore by M. Kneser [9], Satz 2, strong approximation holds for $SU(P_i, h'_i)$.

For every prime number p, let ϕ_p be an element of $U(V_p, h_p)$ such that $L_p = \phi_p H(M_p)$. Using reduction and transfer, we obtain a surjective homomorphism

$$U(V_p, h_p) \to U(P_{1p}, h'_{1p}) \times \cdots \times U(P_{mp}, h'_{mp}).$$

We shall denote by $x_p = \det(\phi_p)$ the determinant of the image of ϕ_p under this homomorphism. Then $x_p \bar{x}_p = 1$, and $x_p = 1$ for almost all p. By Hilbert's theorem 90, there exists an element y_p of A_p such that $x_p = y_p / \bar{y}_p$. We can also apply transfer to A_p-modules, without ε-hermitian forms. In this way we obtain a surjective homomorphism

$$\mathrm{End}(W_p) \to \mathrm{End}(Q_{1p}) \times \cdots \times \mathrm{End}(Q_{mp})$$

for some B_{ip}-modules Q_{ip} which satisfy $(P_{ip}, h'_{ip}) = H(Q_{ip})$. Therefore there exists an endomorphism α_p of W_p such that $\det(\alpha_p) = y_p$. We finish the proof with the same argument as in the proof of Theorem 2. □

6. Stably hyperbolic ε-hermitian forms are locally hyperbolic

Let A be a finite dimensional $\mathbb{Q}$-algebra, together with a $\mathbb{Q}$-involution. Let Λ be an order of A such that $\bar{\Lambda} = \Lambda$.

Proposition 1. *Let $h: L \times L \to \Lambda$ be a stably hyperbolic ε-hermitian form. Then (L, h) is locally hyperbolic.*

Proof. There exist reflexive Λ-modules N and N' such that

$$(L, h) \boxplus H(N) \cong H(N').$$

Let $U = N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Unique decomposition holds for finitely generated A-modules (cf. for instance [20], § 2 (11)). Therefore there exists a reflexive A-module W such that

$$(V, h) \boxplus H(U) \cong H(W) \boxplus H(U).$$

By Theorem 1, this implies that $(V, h) \cong H(W)$. Let $M' = L \cap W$, and $M'' = L \cap W^*$. As L and $M' \otimes M''$ are modules of maximal rank on the same vector space, their localisations coincide for almost all p. Let $p_1, \dots, p_n$ be prime numbers such that $L_p = M'_p \oplus M''_p$ if $p \neq p_1, \dots, p_n$. Clearly $h(M'_p, M'_p) = 0$ for all p. It is easy to check that this implies that $(L_p, h_p) \cong H(M'_p)$ if $p \neq p_1, \dots, p_n$, see for instance Bass [2], Chapter V, Lemma 2.1.

Let p be one of the prime numbers $p_1, \dots, p_n$. Unique decomposition holds for finitely generated Λ_p-modules (cf. [20], § 2 (11)). Therefore there exists a reflexive Λ_p-module $M_{(p)}$ such that

$$(L_p, h_p) \boxplus H(N_p) \cong H(M_{(p)}) \boxplus H(N_p).$$

By Theorem 1 this implies that $(L_p, h_p) \cong H(M_{(p)})$. Moreover, by possibly exchanging some of the indecomposable factors of $M_{(p)}$ and of $M^*_{(p)}$, we may assume that $M_{(p)}$ is contained in W_p.

Let M be a Λ-lattice on W such that $M_p = M'_p$ if $p \neq p_1, \dots, p_n$ and that $M_{p_i} = M_{(p_i)}$ for $i = 1, \dots, n$. Such a lattice exists by [20], § 3 (5). Therefore (L, h) is locally hyperbolic. □

Corollary 2. *Let $h: L \times L \to \Lambda$ be a stably hyperbolic ε-hermitian form, and assume that the hypotheses of Theorem 2, 3 or 4 are satisfied. Then (L, h) is hyperbolic.*

The following corollary will be important for topological applications:

Corollary 3. *Let $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$ with $\lambda \in \mathbb{Z}[X]$ such that $\lambda(X) = X^{\deg(\lambda)} \lambda(X^{-1})$, $\lambda(1) = 1$ and $\lambda(0) \neq 0$. Let us consider the $\mathbb{Z}$-involution of $\mathbb{Z}[X, X^{-1}]$ which sends X to X^{-1}. This induces a $\mathbb{Z}$-involution of Λ. Let $h: L \times L \to \Lambda$ be a stably hyperbolic ε-hermitian form. Then (L, h) is hyperbolic.*

Proof. Let $A = \mathbb{Q}[X, X^{-1}]/(\lambda) = \mathbb{Q}(\tau)$, where τ is the image of X in A. Set $\alpha = \dfrac{1}{1-\tau}$ and let $\Gamma = \mathbb{Z}[\alpha]$. Then $\Lambda = \Gamma[(\alpha\bar{\alpha})^{-1}]$. Notice that Γ is an order of A, and that $\bar{\Gamma} = \Gamma$.

Proposition 1 implies that (L, h) is locally hyperbolic. Hence there exists a reflexive Λ-module M such that for every prime number p, (L_p, h_p) and $H(M_p)$ are isometric.

Let $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ and $W = M \otimes_{\mathbb{Z}} \mathbb{Q}$. If P is a prime Γ-ideal, we shall denote by Γ_P the localisation of Γ with respect to the multiplicative subset $\Gamma \setminus P$ of Γ. Let $P_1, \dots, P_k$ be the prime ideals of Γ which contain $(\alpha\bar{\alpha})\Gamma$. For all $i = 1, \dots, k$, let N_{P_i} be a free Γ_{P_i}-lattice on W. We shall denote by $H(N_{P_i})$ the hyperbolic Γ_{P_i}-lattice associated to N_{P_i}. Then $H(N_{P_i})$ is a lattice on (V, h). Let us denote by L' the intersection of L with the $H(N_{P_i})$'s for $i = 1, \dots, k$. Notice that $\Lambda_P = \Gamma_P$ if $P \neq P_1, \dots, P_k$.

Therefore (L', h) is a unimodular Γ-lattice on (V, h). Clearly (L', h) is also locally hyperbolic. But Γ is an order of A which is stable under the involution, and A satisfies the hypothesis of Theorem 4. Therefore (L', h) is hyperbolic. Recall that L is the localisation of L' with respect to the multiplicative subset of Γ consisting of the powers of $\alpha\bar{\alpha}$. Therefore (L, h) is also hyperbolic. □

Corollary 4. *Let G be an abelian group of order p^n, where p is a prime number. Let $\Lambda = \mathbb{Z}G$. Then stably hyperbolic forms are hyperbolic.*

In the special case where G is a cyclic group of odd prime order, Corollary 4 follows from a result of J. Alexander, P. Conner and G. Hamrick, cf. [1], Chapter VI, (4. 3).

7. An example

We thank M. Ojanguren for the following example. Let Λ be the ring of algebraic functions on the real 2-sphere. Then there exists a projective Λ-module L and a unimodular symmetric bilinear form $h: L \times L \to \Lambda$ which is stably hyperbolic but not hyperbolic. The form (L, h) is constructed as follows. Let P be a projective Λ-module of rank 2 which is not stably free. Let $L = \mathrm{End}(P)$, and let $q: L \to \Lambda$ be defined by $q(F) = \det(F)$. Let $h: L \times L \to \Lambda$ be the associated symmetric bilinear form. Let A be the field of fractions of Λ, and set $L_A = L \otimes_\Lambda A$. Then $L_A \cong M_2(A)$, and (L_A, h_A) is hyperbolic. Therefore by M. Ojanguren [16], Theorem 17, (L, h) is stably hyperbolic. But (L, h) is not hyperbolic. Indeed, notice that (L, h) represents 1. On the other hand, (L, h) does not represent -1 (cf. M. Ojanguren [15], § 4). But a hyperbolic form which represents 1 also represents -1, therefore (L, h) is not hyperbolic.

8. Doubly sliced knots

An n-knot will be a smooth, oriented submanifold K^n of the $(n+1)$-sphere S^{n+2} such that K^n is homeomorphic to S^n. We shall say that K^n is *doubly sliced* if there exists a trivial $(n+1)$-knot Σ^{n+1} in S^{n+3} such that $\Sigma^{n+1} \cap S^{n+2} = K^n$, where S^{n+2} is identified with the equator of S^{n+3}. This definition is due to D. Sumners, see [26]. If K and K' are two n-knots, we shall denote by $K \# K'$ their connected sum (cf. [21] for the definition). We shall say that an n-knot K is *stably doubly sliced* if there exists a doubly sliced n-knot K' such that $K \# K'$ is doubly sliced. It is not known in general whether stably doubly sliced knots are necessarily doubly sliced. In the present section we shall show that this is the case for a certain type of knots:

Let K be a $(2q-1)$-knot with $q \geqq 2$. Let X be the complement of the knot K: $X = \overline{S^{2q+1} \setminus U}$, where U is a tubular neighborhood of K. We shall say that K is *simple* if $\pi_i(X) \cong \pi_i(S^1)$ for all $i < q$.

Theorem 5. *Let K be a simple $(2q-1)$-knot, $q \geqq 2$. Assume that K is stably doubly sliced. Then K is doubly sliced.*

Proof. Let Σ be any $(2q-1)$-knot. Let Y be the complement of Σ, and let $\tilde{Y}$ be the infinite cyclic cover of Y. Set $M = H_q(\tilde{Y}, \mathbb{Z})$. Then M is a finitely generated, torsion $\mathbb{Z}[t, t^{-1}]$-module (cf. for instance [8], § 3). M is called the knot module of Σ. Let us consider the $\mathbb{Q}$-involution of $\mathbb{Q}(t)$ which sends t to t^{-1}. Blanchfield has associated to any $(2q-1)$-knot Σ an $\varepsilon = (-1)^{q+1}$-hermitian form

$$b: M \times M \to \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]$$

(cf. [4] or [8], § 4). For any torsion $\mathbb{Z}[t, t^{-1}]$-module N, set

$$N^0 = \mathrm{Hom}_{\mathbb{Z}[t, t^{-1}]}(N, \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]).$$

We shall say that (M, h) is hyperbolic if and only if there exists a $\mathbb{Z}[t, t^{-1}]$-module N such that (M, b) is isometric to the ε-hermitian form

$$N \oplus N^0 \times N \oplus N^0 \to \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}], \quad (x, f) \times (y, g) \to f(y) + \varepsilon \overline{g(x)}.$$

Let $\lambda \in \mathbb{Z}[t]$ be a polynomial which annihilates M. We may assume that $\lambda(0) \neq 0$, $\lambda(1) = 1$ and that $\lambda(t) = t^{\deg(\lambda)} \lambda(t^{-1})$, cf. [13], p. 19. Set $\Lambda = \mathbb{Z}[t, t^{-1}]/(\lambda)$. Then M is a Λ-module, and b takes values in

$$\left(\frac{1}{\lambda}\right) \mathbb{Z}[t, t^{-1}]/\mathbb{Z}[t, t^{-1}] \cong \Lambda .$$

We can identify the Blanchfield form (M, b) to an ε-hermitian form $h: M \times M \to \Lambda$. It is straight-forward to check that (M, b) is hyperbolic if and only if (M, h) is hyperbolic.

By hypothesis, there exist doubly sliced $(2q-1)$-knots K_1 and K_2 such that $K \# K_1$ is isotopic to K_2. Let L, L_1 and L_2 be the knot modules associated to K, K_1 and K_2. Let $\lambda \in \mathbb{Z}[t]$ be a polynomial which annihilates L_2. We may assume that $\lambda(0) \neq 0$, that $\lambda(1) = 1$ and that $\lambda(t) = t^{\deg(\lambda)} \lambda(t^{-1})$. Let $\Lambda = \mathbb{Z}[t, t^{-1}]/(\lambda)$, and let (L, h), (L_1, h_1) and (L_2, h_2) be the Λ-valued Blanchfield forms associated to K, K_1 and K_2. Then $(L, h) \boxplus (L_1, h_1) \cong (L_2, h_2)$. As K_1 and K_2 are doubly sliced, (L_1, h_1) and (L_2, h_2) are hyperbolic (cf. D. Sumners [26], C. Kearton [7]).

Therefore (L, h) is stably hyperbolic. By Corollary 3, this implies that (L, h) is hyperbolic. But D. Sumners (cf. [26]) and C. Kearton (cf. [7]) have proved that this implies that K is doubly sliced. □

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