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THE VIETORIS MAPPING THEOREM FOR BICOMPACT SPACES

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As a means of proving the isomorphism of the homology groups of two spaces, Vietoris' well-known theorem [5][1] has many applications in topology. In this paper we show that this theorem can be extended from the case of compact metric spaces to that of bicompact Hausdorff spaces. Since Vietoris' original proof is somewhat condensed, and also contains one minor slip, we present our proof in some detail.

The statement of the theorem, for an arbitrary coefficient group, is found in Section 3 below, as well as a simpler form of the theorem which holds when the coefficient group is suitably restricted. Sections 1 and 2 are devoted to preliminary material, and Section 4 to three lemmas which are used, in the last two sections, in the proofs of the two forms of the theorem.

1. Terminology and Notation

We shall deal only with bicompact Hausdorff spaces. By a covering M of a space X we shall always mean a finite covering by open sets, and if N is a refinement of M, we write $N < M$. If W is a subset of X, we denote by $\mathrm{St}(W; M)$ the union of those sets of M which meet W. We denote by $\mathrm{St}(m; M)$ or M^* the covering whose elements are the sets $\mathrm{St}(M; M)$, where m runs through the elements of M. If $\mathrm{St}(N; N) = N^* < M$, we say that N is a star-refinement of M, and we write $N <^* M$. Every covering has a star-refinement [4, p. 47]. For each covering M, we choose one of its star-refinements and denote it by $^*M$.

An n-simplex σ^n of X is a set of $n + 1$ points of X, and these are the vertices of σ^n. If M is a covering and W a subset of X, we write diam $W < M$ if there is an element of M which contains W. $X(M)$ is the simplicial complex consisting of all simplexes σ such that diam $\sigma < M$. Clearly, if $N < M$, then $X(N)$ is a subcomplex of $X(M)$. If W is again a subset of X, then $X(M) \cap W$ is the closed subcomplex of $X(M)$ consisting of all simplexes of $X(M)$ all of whose vertices are in W.

We shall consider only finite chains on the complexes $X(M)$. The coefficients, unless otherwise indicated, are in an arbitrary abelian group. If C^n is such a chain, we denote by $| C^n |$ the finite simplicial complex consisting of all the simplexes on which C^n has non-zero coefficients together with all their faces.

In what follows we make frequent use of the Cartesian product of a simplicial complex and the unit interval, so we recall here the definition of this product [1, p. 307]. Let K be a simplicial complex and let the vertices of K be simply ordered in an arbitrary fashion. Let $\{A'\}$ be a copy of the collection $\{A\}$ of

[1] Numbers in square brackets refer to the bibliography at the end of the paper.

vertices of K, ordered in the same way. Eor each n-simplex $\sigma^n = (A_0, A_1, \cdots, A_n)$ of K, with the vertices arranged in the chosen order, consider the $n + 1$ simplexes of the form $(A_0, A_1, \cdots, A_i, A_i', \cdots, A_n')$. The collection of all such simplexes, together with all their faces, constitutes the product $K \times I$. K is called the base of $K \times I$, and the set of all simplexes of $K \times I$, all of whose vertices are primed, is called the top of $K \times I$.

For each simplex $\sigma^n = (A_0, A_1, \cdots, A_n)$ of K, let $D(\sigma^n) = \sum_{i=0}^{i=n} (-1)^i (A_0, A_1, \cdots, A_i, A_i', \cdots, A_n')$, and if $C^n = \sum g_j \sigma_j^n$, let $D(C^n) = \sum g_j D(\sigma_j^n)$. For any chain C^n of K, a direct calculation shows that

$$FD(C^n) + DF(C^n) = C'^n - C^n,$$

where C'^n is the chain in the top of $K \times I$ formed by replacing each vertex of each simplex of C^n by the corresponding primed vertex. Hence if Z^n is a cycle of K,

$$FD(Z^n) = Z'^n - Z^n,$$

i.e., $Z^n \sim Z'^n$ on $K \times I$.

In one place (Lemma 3) it will be convenient to consider $K \times I$ as a cell-complex rather than a simplicial complex. This time the elements of $K \times I$ are all the cells of the form $\sigma \times 0$, $\sigma \times 1$, or $\sigma \times I$, where σ runs through the simplexes of K. The boundary relations in $K \times I$ are: $F(\sigma \times 0) = (F\sigma) \times 0$, $F(\sigma \times 1) = (F\sigma) \times 1$, and $F(\sigma \times I) = (F\sigma) \times I + (\sigma \times 1) - (\sigma \times 0)$. Then for any cycle Z on K, we have $F(Z \times I) = (Z \times 1) - (Z \times 0)$, i.e., $Z \times 1 \sim Z \times 0$ in $K \times I$.

2. Generalized Vietoris Cycles

As is the case with many problems involving continuous mappings, it is more convenient here to use not the usual Čech cycles, but rather a generalized form of Vietoris cycles. Such cycles have been defined by Spanier [2]. In this section we give his definition in a notation more suited to our purposes.

A collection $\Gamma^n = \{\Gamma^n(M)\}$ of n-cycles of X, one for each covering M of X, is a generalized Vietoris n-cycle (n-V-cycle) if $\Gamma^n(M)$ is a cycle of $X(M)$ and if, whenever $N < M$, $\Gamma^n(N) \sim \Gamma^n(M)$ on $X(M)$. The cycles $\Gamma^n(M)$ are the coordinates of Γ^n. If Γ^n and Δ^n are two n-V-cycles, then $\Gamma^n + \Delta^n$ is the n-V-cycle whose coordinate on $X(M)$ is $\Gamma^n(M) + \Delta^n(M)$. $\Gamma^n \sim 0$ if $\Gamma^n(M) \sim 0$ on $X(M)$ for every M. The n-dimensional Vietoris homology group of X, $H_v^n(X)$, is the factor group of the group of n-V-cycles of X by the subgroup of those which bound.

Let X and Y be two spaces and f a mapping of X into Y. Let Γ^n be an n-V-cycle of X. For each covering N of Y, $f^{-1}(N) = M$ is a covering of X. Clearly f maps each simplex of $X(M)$ onto a simplex of $Y(N)$, and hence is a simplicial mapping of $X(M)$ into $Y(N)$. We define $f\Gamma^n$ to be the n-V-cycle of Y whose coordinate on $Y(N)$ is $f(\Gamma^n(M))$. The correspondence of $f\Gamma^n$ to Γ^n clearly induces a homomorphism of $H_v^n(X)$ into $H_v^n(Y)$.

The Vietoris homology groups defined above do not give any new homology properties of X. If X is compact metric, it is easy to see that $H_v^n(X)$ is isomorphic to the ordinary Vietoris homology group. In the general case, these groups are isomorphic to the corresponding Čech groups, as we now show.

Given a covering M, let $N = {}^*M$. For each vertex A of $X(N)$, choose an element n of N which contains it and then choose an element m of M which contains $\mathrm{St}(n, N)$. Denote this set m by $\zeta(A)$. The function ζ thus defined is a simplicial mapping of $X(N)$ into the nerve $\bar{M}$ of M.

Next, given a covering N, let $P = {}^*N$. For each element p of P, let $\varphi(p)$ be a point in p. Then φ is a simplicial mapping of P into $X(N)$.

Now let Γ^n be an n-V-cycle. For each covering M, let $N = {}^*M$ and define $Z^n(M)$ to be $\zeta(\Gamma^n(N))$. We assert that $Z^n = \{Z^n(M)\}$ is a Čech cycle and that the correspondence of Z^n to Γ^n induces an isomorphism of $H_v^n(X)$ onto $H_c^n(X)$, the n-dimensional Čech homology group of X.

To see that Z^n is a Čech cycle, let $M_2 < M_1$ be two coverings of X. Let $N_1 = {}^*M_1$ and $N_2 = {}^*M_2$, and choose N to be a common refinement of N_1 and N_2. By the definition of Z^n, we have

$$Z^n(M_1) = \zeta_1\Gamma^n(N_1),$$

$$Z^n(M_2) = \zeta_2\Gamma^n(N_2).$$

Since $N < N_1$,

$$\Gamma^n(N) \sim \Gamma^n(N_1) \text{ on } X(N_1).$$

Therefore

$$\zeta_1\Gamma^n(N) \sim \zeta_1\Gamma^n(N_1) \text{ on } \bar{M}_1 .$$

Similarly, since $N < N_2$

$$\zeta_2\Gamma^n(N) \sim \zeta_2\Gamma^n(N_2) \text{ on } \bar{M}_2 ,$$

and hence

$$\pi\,\zeta_2\Gamma^n(N) \sim \pi\,\zeta_2\Gamma^n(N_2) \text{ on } \bar{M}_1 ,$$

where π is a projection of $\bar{M}_2$ into $\bar{M}_1$. Thus it will be sufficient to show that

$$(\alpha) \qquad \pi\,\zeta_2\Gamma^n(N) \sim \zeta_1\Gamma^n(N) \text{ on } \bar{M}_1 .$$

In order to show this, let $K = |\,\Gamma^n(N)\,|$. We define now a simplicial mapping ψ of $K \times I$ into $\bar{M}_1$. For each vertex A of K, the base of $K \times I$, let $\psi(A) = \pi\,\zeta_2(A)$, and for each vertex A' of the top of $K \times I$, let $\psi(A') = \zeta_1(A)$.

To see that this is indeed a simplicial mapping, let $(A_0, A_1, \cdots, A_i, A_i', \cdots, A_n')$ be a simplex of $K \times I$. By the definition of ζ_2, there is, for $0 \leqq j \leqq i$, a set $n_{2,j}$ containing A_j, and a set $m_{2,j} = \zeta_2(A_j)$ containing $\mathrm{St}(n_{2,j}; N_2)$. By the definition of π, there is a set $m_{1,j} = \pi\,\zeta_2(A_j)$ containing $m_{2,j}$. Similarly, for $i \leqq k \leqq n$, there is a set $n_{1,k}'$ containing A_k and a set $m_{1,k}' = \psi(A_k')$ containing $\mathrm{St}(n_{1,k}' ; N_1)$.

Since $(A_0, \cdots, A_n)$ is a simplex of $X(N)$, there is a set n containing $A_0, \cdots, A_n$. Therefore, since $N < N_2$, n is in $\mathrm{St}(n_{2,j}; N_2)$ for $0 \leqq j \leqq i$, and consequently n is contained in $m_{1,j}$ for $0 \leqq j \leqq i$. Similarly, since $N < N_1$, n is contained in $\mathrm{St}(n'_{1,k}; N_1)$ and hence in $m'_{1,k}$ for $i \leqq k \leqq n$. Therefore $m_{1,0} \cap m_{1,1} \cap \cdots \cap m_{1,i} \cap m'_{1,i} \cap \cdots \cap m'_{1,n}$ is not vacuous. Thus ψ maps the vertices of $(A_0, A_1, \cdots, A_i, A'_i, \cdots, A'_n)$ into the vertices of a simplex of $\bar{M}_1$ and therefore is simplicial.

Now $\Gamma^n(N) \sim \Gamma'^n(N)$ in $K + I$. By the definition of ψ, $\psi(\Gamma^n(N)) = \pi\zeta_2(\Gamma^n(N))$ and $\psi(\Gamma'^n(N)) = \zeta_1(\Gamma^n(N))$, and this proves ($\alpha$).

If $\Gamma^n \sim 0$, then clearly $Z^n \sim 0$ also. Suppose now that $Z^n \sim 0$. We shall show that $\Gamma^n \sim 0$. Given any covering M, let $N = {}^*M$ and let $P = {}^*N$. Since $\Gamma^n(P) \sim \Gamma^n(M)$ on $X(M)$, it will be sufficient to show that $\Gamma^n(P) \sim 0$ on $X(M)$. Now $Z^n(N) = \zeta\Gamma^n(P) \sim 0$ on $\bar{N}$. Hence $\varphi\zeta\Gamma^n(P) \sim 0$ on $X(M)$, so we are reduced to proving

$$(\beta) \qquad \Gamma^n(P) \sim \varphi\zeta(\Gamma^n(P)) \text{ on } X(M).$$

Let $K = |\Gamma^n(P)|$. We define a simplicial mapping ω of $K \times I$ into $X(M)$ as follows: For each vertex A in the base of $K \times I$, let $\omega(A) = A$, and for each vertex A' in the top of $K \times I$, let $\omega(A') = \varphi\zeta(A)$.

To see that ω is simplicial, let $(A_0, A_1, \cdots, A_i, A'_i, \cdots, A'_n)$ be a simplex of $K \times I$. By the definition of ζ, there is a set p'_k containing A_k and a set n'_k containing $\mathrm{St}(p'_k; P)$. By the definition of φ, $\varphi(n'_k)$ is a point in n'_k.

Since $(A_0, A_1, \cdots, A_n)$ is a simplex of $X(P)$, there is a set p containing $(A_0, A_1, \cdots, A_n)$. Hence p is contained in $\mathrm{St}(p'_k; P)$ for $i \leqq k \leqq n$ and therefore p is in n'_k. Thus n'_n meets n'_k for $i \leqq k < n$, so $\mathrm{St}(n'_n; N)$ contains each n'_k. Since $N = {}^*M$, there is an element m of M which contains $\mathrm{St}(n'_n; N)$ and hence each n'_k. Consequently m contains $\varphi\zeta(A_k)$ for $i \leqq k \leqq n$. But p is in n'_n and hence in m, so m contains $(A_0, A_1, \cdots, A_i)$. Hence all the vertices of $(A_0, A_1, \cdots, A_i, A'_i, \cdots, A'_n)$ are carried by ω into vertices contained in one element of M and hence into the vertices of a simplex of $X(M)$, and therefore ω is a simplicial mapping.

Now $\Gamma^n(P) \sim \Gamma'^n(P)$ on $K \times I$. By the definition of ω, $\omega(\Gamma^n(P)) = \Gamma^n(P)$ and $\omega(\Gamma'^n(P)) = \varphi\zeta(\Gamma^n(P))$, so we have proved ($\beta$).

Thus far we have shown that the correspondence of Z^n to Γ^n induces an isomorphism of $H_v^n(X)$ into $H_c^n(X)$. To complete the proof we must show that this isomorphism is onto, i.e., that for every Čech cycle Z^n there is an n-V-cycle Γ^n such that $\zeta\Gamma^n \sim Z^n$. But, given Z^n and a covering M, let $N = {}^*M$. Define $\Gamma^n(M)$ to be $\varphi(Z^n(N))$. Then $\Gamma^n = \{\Gamma^n(M)\}$ is an n-V-cycle and $\zeta\Gamma^n \sim Z^n$. We omit the proofs of these last two statements since they are analogous to those above.

3. The Vietoris Mapping Theorem

Let X and Y be two spaces. A mapping f of X onto Y is a Vietoris mapping of order n if for each covering M of X and each point y of Y there is a covering $P = P(M, y)$ of X, with $P < M$, such that any k-cycle, $0 \leqq k \leqq n$, on $X(P) \cap f^{-1}(y)$ bounds on $X(M) \cap f^{-1}(y)$.

THEOREM 1. *If f is a Vietoris mapping of order n of X onto Y, then the homomorphism of $H_v^n(X)$ into $H_v^n(Y)$ induced by f is an isomorphism and is onto.*

The hypothesis of this theorem can be put in a more convenient form if the coefficient group is restricted to lie in either of two classes of groups, the class of fields and the class of elementary compact topological groups [3, p. 672]. The latter class consists of the character groups of discrete groups with finite bases, and hence contains all finite groups as well as the group of real numbers mod 1.

THEOREM 2. *If the coefficient group is an elementary compact topological group or is a field, and if f is a mapping of X onto Y such that for each point y of Y, and for each integer k, $0 \leqq k \leqq n$, $H_v^k(f^{-1}(y)) = 0$, then the homomorphism of $H_v^n(X)$ into $H_v^n(Y)$ induced by f is an isomorphism and is onto.*

4. Preliminary Lemmas

LEMMA 1. *If f is a Vietoris mapping of order n of X onto Y, then for each covering M of X and each covering N of Y there is a refinement $Q = Q(M, N)$ of N such that if B is a subset of Y with* diam $B < Q$, *then there is a point y in Y such that*

$$1)\ \mathrm{St}(y; N) \supset B$$

$$2)\ \mathrm{St}(f^{-1}(y); {}^*P) \supset f^{-1}(B),$$

where $P = P(M, y)$.

PROOF.[2] For each $y \in Y$, let $A_y = X - \mathrm{St}(f^{-1}(y); {}^*P)$, where $P = P(M, y)$. Then A_y is closed, hence compact, so $f(A_y)$ is closed and does not contain y. Since Y is a normal space, there is an open set B_y containing y which does not meet $f(A_y)$. We may choose B_y to be in a set of N which contains y. Now a finite number of the sets B_y cover Y, and these constitute the covering Q.

LEMMA 2. *If f is a Vietoris mapping of order n of X onto Y, then for each covering M of X and each covering Y of N there is a covering $R = R(M, N)$ of Y, with $R < N$, and a chain-mapping T of the $(n + 1)$-skeleton of $Y(R)$ into $X(M)$ such that for any k-simplex σ^k of $Y(R)$, $0 \leqq k \leqq n + 1$, $fT\sigma^k$ is a barycentric subdivision, $\delta\sigma^k$, of σ^k, with* diam $|\delta\sigma^k| < N$.

PROOF. Let $M_{n+1} = M$ and $N_{n+1} = N$. Let Q_n be $Q(M_{n+1}, {}^*N_{n+1})$ and let $N_n = {}^*Q_n$. For each element $q_{n,i}$ of Q_n, diam $q_{n,i} < Q_n$, so, by Lemma 1, there is an associated point, $y_{n,i}$. Let $P_{n,i} = P(M_{n+1}, y_{n,i})$ and let M_n be a common refinement of the coverings ${}^*P_{n,i}$. Next let $Q_{n-1} = Q(M_n, {}^*N_n)$ and let $N_{n-1} = {}^*Q_{n-1}$. Let $\{y_{n-1,i}\}$ be the points associated, by Lemma 1, with the elements of Q_{n-1}, and let $P_{n-1,i} = P(M_n, y_{n-1,i})$. Let M_{n-1} be a common refinement of the coverings ${}^*P_{n-1,i}$.

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[2] This proof was suggested by the referee and is somewhat shorter than our original proof.

Vietoris proves [5, p. 465], an analogous lemma, for the metric case, but weaker in that 2) is replaced by $\mathrm{St}(f^{-1}(y); M) \supset f^{-1}(B)$. That this is not sufficient will be seen after an inspection of the proof of the next lemma.

Proceeding in this fashion, we construct a sequence $\{M_k\}$ of coverings of X and a sequence $\{N_k\}$ of coverings of Y, together with the associated sets $\{y_{k,i}\}$, so that

$$1)\ N_{k-1} = {}^*Q_{k-1}\ ;\ Q_{k-1} = Q(M_k\,,\ {}^*N_k)$$

$$2)\ M_{k-1} < {}^*P(M_k\,,\ y_{k-1,i}).$$

We assert that the covering N_0 will serve for $R(M, N)$.

To prove this assertion, we must construct the chain mapping T. First let σ^0 be a 0-simplex of $Y(N_0)$. Let Σ^0 be an arbitrary point of $f^{-1}(\sigma^0)$, and define $T(\sigma^0)$ to be Σ^0. Then $T(\sigma^0)$ is a 0-chain of $X(M_0)$, and $fT\sigma^0 = \sigma^0$.

Now suppose that T has been defined for all simplexes σ^m in $Y(N_0)$ with $m < k$ in such a way that $T(\sigma^m)$ is a chain of $X(M_m)$ and $fT\sigma^m$ is a barycentric subdivision $\delta\sigma^m$ of σ^m, with diam $|\delta\sigma^m| < N_m$.

Let σ^k be a k-simplex of $Y(N_0)$. Then T is defined on $F\sigma^k$, and $TF\sigma^k$ is a chain of $X(M_{k-1})$. Now consider $f\,|\,TF\sigma^k\,|$. Since σ^k is in $Y(N_0)$, there is an element n_0 of N_0 which contains σ^k. If σ^{k-1} appears in $F\sigma^k$, then $fT\sigma^{k-1} = \delta\sigma^{k-1}$ contains a vertex of σ^k. But diam $|\delta\sigma^{k-1}| < N_{k-1}$, so $\mathrm{St}(n_0\,,\ N_{k-1})$ contains $f\,|TF\sigma^k\,|$. But $N_0 < N_{k-1} <^* Q_{k-1}$, so diam $f\,|\,TF\sigma^k\,| < Q_{k-1} = Q(M_k\,,\ {}^*N_k)$. Let $y_{k-1,1}$, say, be the corresponding point of Y, so that $\mathrm{St}(y_{k-1,1}\ ;\ {}^*N_k)$ contains $f\,|\,TF\sigma^k\,|$ and $\mathrm{St}(f^{-1}(y_{k-1,1});\ {}^*P)$ contains $f^{-1}f\,|\,TF\sigma^k\,|$, which in turn contains $|\,TF\sigma^k\,|$, where $P = P(M_k\,,\ y_{k-1,1})$.

Denote now the cycle $TF\sigma^k$ by Z^{k-1}, and let $K = |\,Z^{k-1}\,|$. We define a simplicial mapping μ of $K \times I$ into $X(P)$ by first setting $\mu(A) = A$ for each vertex A in the base of $K \times I$. Next let A' be a vertex in the top of $K \times I$, and let A be the corresponding point in the base, so that A is a vertex of $|\,TF\sigma^k\,|$. Since $\mathrm{St}(f^{-1}(y_{k-1,1});\ {}^*P)$ contains $|\,TF\sigma^k\,|$, there is a set ${}^*p$ of ${}^*P$ which meets $f^{-1}(y_{k-1,1})$ and also contains A. Let $\mu(A')$ be a point in this set ${}^*p$ and in $f^{-1}(y_{k-1,1})$. If now $(A_0\,,\ \cdots,\ A_i\,,\ A_i'\,,\ \cdots,\ A_{k-1}')$ is a simplex of $K \times I$, then $(A_0\,,\ \cdots,\ A_{k-1})$ is a simplex of $|\,TF\sigma^k\,|$ and hence is in some element m_{k-1} of M_{k-1}. For each j, $i \leqq j \leqq k-1$, $\mu(A_j')$ is a point of ${}^*p_j$, where ${}^*p_j$ contains A_j, and therefore $\mu(A_0\,,\ \cdots,\ A_i\,,\ A_i'\,,\ \cdots,\ A_{k-1}') = (A_0\,,\ \cdots,\ A_i\,,\ \mu(A_i),\ \cdots,\ \mu(A_{k-1}))$ is in

$$\mathrm{St}(m_{k-1}\ ;\ {}^*P)$$

and hence in some element of P, since $M_{k-1} < {}^*P$. Thus μ maps $K \times I$ simplicially into $X(P)$.

Now let $\Sigma_1^k = \mu(DZ^{k-1})$, so that $F\Sigma_1^k = \mu(Z'^{k-1}) - \mu(Z^{k-1}) = \mu(Z'^{k-1}) - Z^{k-1}$. The cycle $\mu(Z'^{k-1})$ is on $X(P) \cap f^{-1}(y_{k-1,1})$, and since $P = P(M_k\,,\ y_{k-1,1})$, there is a chain Σ_2^k on $X(M_k) \cap f^{-1}(y_{k-1,1})$ such that $F\Sigma_2^k = \mu(Z'^{k-1})$. Set $\Sigma^k = \Sigma_2^k - \Sigma_1^k$, and set $T\sigma^k = \Sigma^k$. Then $FT\sigma^k = TF\sigma^k$, so T is a chain-mapping.

Finally, observe that each vertex of $|\,\Sigma^k\,|$ is either a vertex of $|\,TF\sigma^k\,|$ or is a vertex in $f^{-1}(y_{k-1,1})$, and f maps all the latter on the single point $y_{k-1,1}$. Hence $f\Sigma^k$ is the join of $y_{k-1,1}$ with $fTF\sigma^k = \delta F\sigma^k$ and thus is a barycentric subdivision $\delta\sigma^k$ of σ^k. Since $\mathrm{St}(y_{k-1,1}\ ;\ {}^*N_k)$ contains $f\,|\,TF\sigma^k\,|$, diam $|\,\delta\sigma^k\,| < N_k$.

Thus we can continue extending the definition of T until it is finally defined on all of the $(n+1)$-skeleton of $Y(N_0)$, and we have therefore completed the proof of the lemma.

LEMMA 3. *Let M and $\check{M}$ be coverings of X, with $\check{M} < M$, and let N and $\check{N}$ be coverings of Y. Let $R = R(M, N)$ and $\check{R} = R(\check{M}, \check{N})$. Let T and $\check{T}$ be the corresponding chain-mappings. Then there is a common refinement S of R and $\check{R}$ such that for any cycle Z^n on $Y(S)$, $TZ^n \sim \check{T}Z^n$ on $X(M)$.*

PROOF. We first recall the sequences $\{M_k\}$ and $\{N_k\}$ of coverings which were constructed in the proof of Lemma 2. Suppose now that we construct new sequences $\{M'_k\}$ and $\{N'_k\}$ by first choosing M'_{n+1} to be any refinement of M and N'_{n+1} to be any refinement of N. Then, at each step, choose Q'_k to be a common refinement of Q_k and of $Q(M'_{k+1}, {}^*N'_{k+1})$, and N'_k to be a common refinement of ${}^*Q'_k$ and of N_k. Let $\{y'_{k,i}\}$ be the set of points of Y associated with Q'_k, and let M'_k be a common refinement of M_k and of the coverings ${}^*P'_{k,i}$, where $P'_{k,i} = P(M'_{k+1}, y'_{k,i})$.

Now we can repeat the argument of Lemma 2 to obtain a chain-mapping T' of $Y(N'_0)$ into $X(M'_{n+1})$ such that for σ^k in $Y(N'_0)$, $T'(\sigma^k)$ is a chain of $X(M'_k)$. We assert that for any cycle Z^n of $Y(N'_0)$, $T(Z^n) \sim T'(Z^n)$ on $X(M)$.

Before proving this assertion, we show that our lemma follows from it. For we can choose M'_k and $\check{M}'_k$ to be the same covering of X for each k, and similarly for N'_k and $\check{N}'_k$. Then $N'_0 = \check{N}'_0$, and we take this to be S. Now, if Z^n is a cycle on $Y(S)$, $T(Z^n) \sim T'(Z^n)$ on $X(M)$ by our assertion. Similarly, $\check{T}(Z^n) \sim \check{T}'(Z^n)$ on $X(\check{M})$. But T' and $\check{T}'$ are the same chain-mapping, and $X(\check{M})$ is a subcomplex of $X(M)$, so $T(Z^n) \sim \check{T}(Z^n)$ on $X(M)$.

Returning now to our assertion, let Z^n be a cycle of $Y(N'_0)$ and let $K = |Z^n|$. We shall define a chain-mapping θ of the cell-complex $K \times I$ into $X(M)$. For a cell of $K \times I$ of the form $\sigma \times 0$, let $\theta(\sigma \times 0) = T'(\sigma)$, and for a cell of the form $\sigma \times 1$, let $\theta(\sigma \times 1) = T(\sigma)$. Now consider a 0-complex, σ^0, of K. $T(\sigma^0) = \Sigma^0$ and $T'(\sigma^0) = \Sigma'^0$ are, by construction, vertices of $f^{-1}(\sigma^0)$ and $fT(\sigma^0) = fT'(\sigma^0) = \sigma^0$. There is a point, say $y_{0,2}$, such that $\mathrm{St}(y_{0,2}; N_0)$ contains σ^0 and $\mathrm{St}(f^{-1}(y_{0,2}); {}^*P)$ contains $f^{-1}(\sigma^0)$, where $P = P(M_1, y_{0,2})$. Let B^0 be the cycle $T\sigma^0 - T'\sigma^0$, and let $L_0 = |B^0|$. We map the simplicial complex $L_0 \times I$ into $X(P)$ by a mapping ω_0 such that $\omega_0(A) = A$ for any vertex A in the base of $L_0 \times I$ and $\omega_0(A')$ is a point of $f^{-1}(y_{0,2})$ such that $\mathrm{St}(\omega_0(A'); {}^*P)$ contains A. That there exists such a point follows from the fact that $\mathrm{St}(f^{-1}(y_{0,2}); {}^*P)$ contains L_0. It is clear that ω_0 is a simplicial mapping of $L_0 \times I$ into $X(P)$. Let $\Xi^1_1 = \omega_0(DB^0)$, so that Ξ^1_1 is a chain of $X(P)$ and $F(\Xi^1_1) = \omega_0(B'^0) - B^0$. Now $\omega_0(B'^0)$ is a 0-cycle of $X(P) \cap f^{-1}(y_{0,2})$, so there is a 1-chain Ξ^1_2 of $X(M_1) \cap f^{-1}(y_{0,2})$ such that $F\Xi^1_2 = \omega_0(B'^0)$. Then $\Xi^1 = \Xi^1_2 - \Xi^1_1$ is a chain of $X(M_1)$ and $F\Xi^1 = B^0$. Clearly $f|\Xi^0|$ is the join of σ^0 and $y_{0,2}$. We define $\theta(\sigma^0 \times I)$ to be Ξ^1. Then $F\theta(\sigma^0 \times I) = B^0 = T\sigma^0 - T'\sigma^0 = \theta(\sigma^0 \times 1) - \theta(\sigma^0 \times 0) = \theta F(\sigma^0 \times I)$.

Now suppose that θ has been defined on every cell of $K \times I$ of the form $\sigma^m \times I$, for all $m < k$, in such a way that $\theta(\sigma^m \times I)$ is a chain of $X(M_{m+1})$ and $\operatorname{diam} f|\theta(\sigma^m \times I)| < N_{m+1}$. Let σ^k be a simplex of $Y(N'_0)$. Then θ is defined on $F(\sigma^k \times I)$, and we wish to consider the set $f \mid \theta F(\sigma^k \times I) \mid$. But $F(\sigma^k \times I) = ((F\sigma^k) \times I) + (\sigma^k \times 1) - (\sigma^k \times 0)$, so $f \mid \theta F(\sigma^k \times I) \mid$ is contained in

$$f \mid \theta((F\sigma^k) \times I) \mid \cup f \mid T\sigma^k \mid \cup f \mid T'\sigma^k \mid.$$

Let n_0' be an element of N_0' which contains σ^k. Since diam $f \mid T\sigma^k \mid < N_k$, $\mathrm{St}(n_0' ; N_k)$ contains $f \mid T\sigma^k \mid$. Similarly, since $N_k' < N_k$, $\mathrm{St}(n_0' ; N_k)$ contains $f \mid T'\sigma^k \mid$. Also, for any simplex σ^{k-1} in $F\sigma^k$, diam $f \mid \theta(\sigma^{k-1} \times I) \mid < N_k$ and $f \mid \theta(\sigma^{k-1} \times I) \mid$ contains a vertex of σ^k, so $\mathrm{St}(n_0' ; N_k)$ also contains

$$f \mid \theta((F\sigma^k) \times I) \mid.$$

But $N_k < {}^*Q_k$, where $Q_k = Q(M_{k+1}, {}^*N_{k+1})$, so diam $f \mid \theta F(\sigma^k \times I) \mid < Q_k$.

Therefore there is a point, say $y_{k,2}$, such that $\mathrm{St}(y_{k,2} ; {}^*N_{k+1})$ contains $f \mid \theta F(\sigma^k \times I) \mid$ and $\mathrm{St}(f^{-1}(y_{k,2}); {}^*P)$ contains $f^{-1}f \mid \theta F(\sigma^k \times I) \mid$, which in turn contains $\mid \theta F(\sigma^k \times I) \mid$, where $P = P(M_{k+1}, y_{k,2})$.

Now denote the cycle $\theta F(\sigma^k \times I)$ by B^k, and let $L_k = \mid B^k \mid$. We can define a simplicial mapping ω_k of the simplicial complex $L_k \times I$ into $X(P)$ in the same way that we defined ω_0, so that $F\omega_k(DB^k) = \omega_k(B'^k) - B^k$, and $\omega_k(B'^k)$ is a cycle of $X(P) \cap f^{-1}(y_{k,2})$. Let $\Xi_1^{k+1} = \omega_k(DB^k)$ and let Ξ_2^{k+1} be a chain of $X(M_{k+1}) \cap f^{-1}(y_{k,2})$ such that $F\Xi_2^{k+1} = \omega_k(B'^k)$. Then set $\theta(\sigma^k \times I) = \Xi^{k+1} = \Xi_2^{k+1} - \Xi_1^{k+1}$. We have $F\theta(\sigma^k \times I) = F\Xi^{k+1} = B^k = \theta F(\sigma^k \times I)$, so θ commutes with F. Also, $f \mid \theta(\sigma^k \times I) \mid$ is the join of $f \mid \theta F(\sigma^k \times I) \mid$ and $y_{k,2}$. Since $\mathrm{St}(y_{k,2} ; {}^*N_{k+1})$ contains $f \mid \theta F(\sigma^k \times I) \mid$, diam $f \mid \theta(\sigma^k \times I) \mid < N_{k+1}$. By construction, $\theta(\sigma^k \times I)$ is on $X(M_{k+1})$.

We can therefore continue extending the definition of θ until it is defined on all the cells of $K \times I$. Now $F(Z^n \times I) = (Z^n \times 1) - (Z^n \times 0)$ in $K \times I$, so $\theta F(Z^n \times I) = F\theta(Z^n \times I) = \theta(Z^n \times 1) - \theta(Z^n \times 0) = TZ^n - T'Z^n$. Since $\theta(Z^n \times I)$ is a chain of $X(M_{n+1}) = X(M)$, $TZ^n \sim T'Z^n$ on $X(M)$, which completes the proof of the lemma.

5. Proof of Theorem 1

We show first that under the homomorphism induced by f, each element of $H_v^n(Y)$ is the image of an element of $H_v^n(X)$.

With each covering M of X we associate a covering N of Y such that M is a refinement of $f^{-1}(N)$, and if $M = f^{-1}(N)$ for some N, we associate this N with M. Now let $Z^n = \{Z^n(N)\}$ be an n-V-cycle of Y. For each covering M of X, we define $\Gamma^n(M)$ to be $TZ^n(R)$, where $R = R(M, N)$, N being the covering associated with M as above, and T being the chain-mapping of $Y(R)$ into $X(M)$ given by Lemma 2.

We assert that the collection $\{\Gamma^n(M)\}$ is an n-V-cycle. For let $\check{M}$ be a refinement of M, and let $\check{N}$ be the covering of Y associated with $\check{M}$. Then $\Gamma^n(M) = TZ^n(R)$ and $\Gamma^n(\check{M}) = \check{T}Z(\check{R})$, where $\check{R} = R(\check{M}, \check{N})$. Let S be the common refinement of R and $\check{R}$ given by Lemma 3. Then $TZ^n(S) \sim \check{T}Z^n(S)$ on $X(M)$. Since Z^n is an n-V-cycle, $Z^n(S) \sim Z^n(R)$ on $Y(R)$. Hence $TZ^n(S) \sim TZ^n(R)$ on $X(M)$. Similarly, $\check{T}Z^n(S) \sim \check{T}Z^n(\check{R})$ on $X(\check{M})$. But $X(\check{M})$ is a subcomplex of $X(M)$, so $\Gamma^n(\check{M}) = \check{T}Z^n(\check{R}) \sim TZ^n(R) = \Gamma^n(M)$ on $X(M)$, which proves that $\{\Gamma^n(M)\}$ is an n-V-cycle.

Next we assert that $f\Gamma^n \sim Z^n$. For a given covering N of Y, let $M = f^{-1}(N)$. Then $\Gamma^n(M) = TZ^n(R)$, where $R = R(M; N)$. Also, $f\Gamma^n(M) = fTZ^n(R) = \delta Z^n(R)$, a barycentric subdivision of $Z^n(R)$ such that for each simplex σ^n of $| Z^n(R) |$, diam $| \delta\sigma^n | < N$. The standard argument for showing that a cycle is homologous to its barycentric subdivision applies here to show that $Z^n(R) \sim \delta Z^n(R)$ on $Y(N)$. But Z^n is an n-V-cycle, so $Z^n(R) \sim Z^n(N)$ on $Y(N)$. Therefore $Z^n(N) \sim \delta Z^n(R) = fTZ^n(R) = f\Gamma^n(M)$ on $Y(N)$.

Thus we have shown that f induces a homomorphism of $H_v^n(X)$ onto $H_v^n(Y)$. To complete the proof, it is only necessary to show that if $f\Gamma^n \sim 0$, then $\Gamma^n \sim 0$.

Let then M be a covering of X, and let N be the associated covering of Y, so that $M < f^{-1}(N)$. Let $R = R(M, N)$ and let $U = f^{-1}(R)$. Now recall the sequence $\{M_k\}$ of coverings of X constructed in the proof of Lemma 2, and choose a common refinement, V, of U and of M_0.

Since Γ^n is an n-V-cycle, $\Gamma^n(V) \sim \Gamma^n(U)$ on $X(U)$. Hence $f\Gamma^n(V) \sim f\Gamma^n(U)$ on $Y(R)$. But if $Z^n = f\Gamma^n \sim 0$ in Y, then $Z^n(R) = f\Gamma^n(U) \sim 0$ on $Y(R)$. Therefore, $f\Gamma^n(V) \sim 0$ on $Y(R)$ and $Tf\Gamma^n(V) \sim 0$ on $X(M)$, since T is a chain-mapping. We wish now to show that $\Gamma^n(V) \sim Tf\Gamma^n(V)$ on $X(M)$.

Let $L = | \Gamma^n(V) |$, and let $L \times I$ be considered as a cell-complex. Define a chain-mapping θ on the base and the top of $L \times I$ by $\theta(\tau^k \times 0) = \tau^k$ and

$$\theta(\tau^k \times 1) = Tf\tau^k$$

for any simplex τ^k of L. If we now examine the proof of Lemma 3, we see that, after the substitution of $Tf\tau^k$ for $T\sigma^k$ and τ^k for $T'\sigma^k$, this proof applies without change to show that θ can be extended to be a chain-mapping of all of $L \times I$ into $X(M)$. Thus $\theta(\Gamma^n(V) \times I)$ is a chain of $X(M)$ such that $F\theta(\Gamma^n(V) \times I) = \theta(\Gamma^n(V) \times 1) - \theta(\Gamma^n(V) \times 0) = Tf\Gamma^n(V) - \Gamma^n(V)$, i.e. $Tf\Gamma^n(V) \sim \Gamma^n(V)$ on $X(M)$.

Now, since $Tf\Gamma^n(V) \sim 0$ on $X(M)$, we have $\Gamma^n(V) \sim 0$ on $X(M)$. But Γ^n is an n-V-cycle, so $\Gamma^n(V) \sim \Gamma^n(M)$ on $X(M)$. Thus $\Gamma^n(M) \sim 0$ on $X(M)$, so $\Gamma^n \sim 0$. This completes the proof of Theorem 1.

6. Proof of Theorem 2

Let M be a covering of X and y a point of Y. Let $N_1 = {}^*M$, and let φ be the simplicial mapping, defined in Section 2 above, of $\bar{N}_1$ into $X(M)$. We now consider N_1 as a covering of the compact set $f^{-1}(y)$. Since the coefficient group is an elementary compact group or a field, there is [3, p. 678] and [1, p. 216], a refinement N_2 of N_1 such that if Z^k is a cycle of $\bar{N}_2$ on $f^{-1}(y)$, then πZ^k is the coordinate on $\bar{N}_1$ of a Čech cycle of $f^{-1}(y)$. Let $P = {}^*N_2$. We assert that any cycle Γ^k, $0 \leqq k \leqq n$, on $X(P) \cap f^{-1}(y)$ bounds on $X(M) \cap f^{-1}(y)$.

Let ζ be the simplicial mapping of $X(P)$ into $\bar{N}_2$ defined in Section 2. Then $\zeta\Gamma^k$ is a cycle of $\bar{N}_2$ on $f^{-1}(y)$. Therefore, $\pi\zeta\Gamma^k$ is the coordinate on $\bar{N}_1$ of a Čech cycle of $f^{-1}(y)$. Since $H_c^k(f^{-1}(y)) = H_v^k(f^{-1}(y)) = 0$, this Čech cycle bounds and

$\pi\zeta\Gamma^k \sim 0$ on $\bar{N}_1$. Then $\varphi\pi\zeta\Gamma^k \sim 0$ on $X(M) \cap f^{-1}(y)$. But it is easy to see, as in Section 2, that $\varphi\pi\zeta\Gamma^k \sim \Gamma^k$ in $X(M) \cap f^{-1}(y)$. Now we can choose $P(M, y)$ to be P, and the hypothesis of Theorem 1 is satisfied. This proves Theorem 2.

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