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The certain exact sequence of Whitehead and the classification of homotopy types of CW-complexes

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ABSTRACT

This paper defines an invariant associated to Whitehead's certain exact sequence of a simply connected CW-complex which is much more elementary — and less powerful — than the boundary invariant of Baues. Nevertheless, in good cases, it classifies the homotopy types of CW-complexes.

1. Introduction

Classification of spaces (for our purpose we restrict ourselves to simply connected CW-complexes) is a major task of algebraic topology. From the first fundamental invariants (homotopy and homology groups) to today's developments such as operads, an extensive collection of algebraic objects is used to try to determine CW-complexes and their morphisms. Rational homotopy builds an equivalence of categories between simply connected spaces without torsion and algebraic categories which are easy to define and to work with (Therefore our problem tackled below is solved in rational homotopy theory, see for example [4]).

Such a nice situation is out of reach for CW-complexes with torsion. In this paper we limit ourselves to certain specific morphisms, with this restriction we obtain a very simple criterion to detect topological morphisms in terms of algebraic data.

The starting point is the Hurewicz morphism which connects homotopy to homology; if X is a CW-complex, we denote it as usual by: $h_*:\pi_*(X)\to H_*(X,\mathbb{Z})$. Whitehead [10] inserted it into a long exact sequence:

$$\cdots\longrightarrow H_{n+1}(X,\mathbb{Z})\xrightarrow{b_{n+1}}\Gamma_n^X\longrightarrow\pi_n(X)\xrightarrow{h_n}H_n(X,\mathbb{Z})\longrightarrow\cdots$$

From this he obtained a good invariant for 4-dimensional CW-complexes.

The program of Whitehead was to extend these results to higher dimensions. Many years later, Baues [3] took afresh the problem and developed an elaborated theory. He mimicked the Postnikov sequence in a categorical and homological setting via towers of categories. The fundamental step is a recursive construction of CW-complexes, starting from the Whitehead certain exact sequence. One defines a category from the $(n-1)$-skeleton and a sophisticated algebraic "boundary invariant" which overlaps information derived from Whitehead's sequence. This invariant is built using a homotopical construction, the "principal reduction". We notice that this construction is theoretically defined, but in general non-effective (hardly reachable by direct calculations on examples). Nevertheless the whole theory is very nice; it gives a theoretical recursive tool for determining a complete invariant for CW complexes and their maps from Whitehead's exact sequence. Baues was

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able to give a complete sets of invariants for CW-complexes with cells in a short range of dimensions; we can view this last case as the first significative generalization of Whitehead's results on 4-dimensional complexes.

Our program is a compromise between Whitehead's and the elaborated results of Baues. To begin with, we suppose given (simply-connected) CW-complexes X and Y, and a commutative ladder of maps between their respective Whitehead certain exact sequences.

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,\mathbb{Z}) \arrow[r, "b_{n+1}"] \arrow[d, "f_{n+1}"] & \Gamma_n^X \arrow[r] \arrow[d, "\gamma_n"] & \pi_n(X) \arrow[r] \arrow[d, "\Omega_n"] & H_n(X,\mathbb{Z}) \arrow[r, "b_n"] \arrow[d, "f_n"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Y,\mathbb{Z}) \arrow[r, "b'_{n+1}"] & \Gamma_n^Y \arrow[r] & \pi_n(Y) \arrow[r] & H_n(Y,\mathbb{Z}) \arrow[r, "b'_n"] & \cdots
\end{tikzcd}$$

We add a collection of extensions belonging to a homotopy invariant set (the set of *the characteristic n-extensions*) to these data and suppose the above ladder of maps is compatible with these extensions. In general these data are not sufficient to define a topological map $\alpha : X \to Y$ from the algebraic map $f_* : H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$ in the ladder. We need a recursive condition about the compatibility of α with Whitehead's Γ-groups, and call *strong morphism*, denoted by (f_*, γ_*), such a ladder of maps between Whitehead's exact sequences. The main theorem is as follows:

Main theorem. *Let X and Y be two simply connected CW-complexes. Any strong morphism (f_*, γ_*) from Whitehead's certain exact sequence of X to Y's gives rise to a map $\alpha : X \to Y$ such that $H_*(\alpha) = f_*$.*

Analogous discussions and theorems take place in algebraic categories such as differential graded Lie algebras or differential graded free chain algebras (notice that we can define a homotopy theory for these categories such that they work "similarly" as for CW-complexes). These are simpler cases and lead to more powerful theorems [5,7,8].

The paper is organized as follows.

In Section 2, we recall the basic definitions of Whitehead's certain exact sequence and his theorem about 4-dimensional simply-connected CW-complexes and in Section 3, we define the characteristic n-extensions. In Section 4, we formulate and prove the main theorem.

2. The certain exact sequence of Whitehead

2.1. The cellular complex and the Hurewicz morphism

Let X be a simply connected CW-complex defined by the collection of its skeleta $(X^n)_{n\geqslant 0}$, where we can suppose $X^0 = X^1 = \star$.

The long exact sequence of the pair (X^n, X^{n-1}) in homotopy and in homology are connected by the Hurewicz morphism h_*:

$$\begin{tikzcd}
\cdots \arrow[r, "i_{m,n}"] & \pi_m(X^n) \arrow[r, "j_{m,n}"] \arrow[d, "h_m"'] & \pi_m(X^n, X^{n-1}) \arrow[r, "\beta_{m,n}"] \arrow[d, "h_m"] & \pi_{m-1}(X^{n-1}) \arrow[r] \arrow[d, "h_{m-1}"] & \cdots \\
\cdots \arrow[r, "i^H_{m,n}"] & H_m(X^n,\mathbb{Z}) \arrow[r, "j^H_{m,n}"] & H_m((X^n, X^{n-1}),\mathbb{Z}) \arrow[r, "\beta^H_{m,n}"] & H_{m-1}(X^{n-1},\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd} \tag{1}$$

Remark 2.1. The following elementary facts are well known.

(1) The Hurewicz morphism $h_m : \pi_m(X^n, X^{n-1}) \to H_m((X^n, X^{n-1}),\mathbb{Z})$ is an isomorphism if $m \leqslant n$, non-trivial only if $m = n$.
(2) $\pi_n(X^n, X^{n-1})$ is the free $\mathbb{Z}$-module generated by the n-cells of X.
(3) $C_nX = \pi_n(X^n, X^{n-1})$ with the differential $d_n = j_n \circ \beta_n$, where $\beta_n = \beta_{n,n}$ and $j_n = j_{n,n}$, defines the cellular chain complex of X (its homology is of course the singular homology $H_*(X)$). From now on, we omit reference to $\mathbb{Z}$; it is understood that we deal only with integral homology. Moreover $\beta_n : C_nX \to \pi_{n-1}(X^{n-1})$ represents by adjunction the attaching map for the n-cells $\bigvee S^n \to X^{n-1}$.

2.2. The definition of Whitehead's certain exact sequence

Now Whitehead [10] inserted the Hurewicz morphism in a long exact sequence connecting homology and homotopy. First he defined the following group

$$\Gamma_n^X = \operatorname{Im}\bigl(i_n : \pi_n\bigl(X^{n-1}\bigr) \to \pi_n\bigl(X^n\bigr)\bigr) = \ker j_n, \quad \forall n \geqslant 2. \tag{1}$$

We notice that $\beta_{n+1} \circ d_{n+1} = 0$ and so $\beta_{n+1} : \pi_{n+1}(X^{n+1}, X^n) \to \pi_n(X^n)$ factors through the quotient: $b_{n+1} : H_{n+1}(X) \to \Gamma_n^X$.

With this map, Whitehead [10] defined the following sequence:

$$\cdots \longrightarrow H_{n+1}(X,\mathbb{Z}) \xrightarrow{b_{n+1}} \Gamma_n^X \longrightarrow \pi_n(X) \xrightarrow{h_n} H_n(X,\mathbb{Z}) \longrightarrow \cdots \tag{2}$$

and proved the following.

Theorem 2.2. *The above sequence is a natural exact sequence, called the certain exact sequence.*

Noration. We shall denote the sequence (2) by $\mathrm{WES}(X)$.

This sequence improves the information provided by both homology and homotopy groups. Whitehead was led to the very natural question: for which class of CW-complexes and maps does the certain exact sequence define a complete invariant? In other words, given the following commutative diagram of group maps:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,\mathbb{Z}) \arrow[r, "b_{n+1}"] \arrow[d, "f_{n+1}"] & \Gamma_n^X \arrow[r] \arrow[d, "\gamma_n"] & \pi_n(X) \arrow[r] \arrow[d, "\Omega_n"] & H_n(X,\mathbb{Z}) \arrow[r, "b_n"] \arrow[d, "f_n"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Y,\mathbb{Z}) \arrow[r, "b'_{n+1}"] & \Gamma_n^Y \arrow[r] & \pi_n(Y) \arrow[r] & H_n(Y,\mathbb{Z}) \arrow[r, "b'_n"] & \cdots
\end{tikzcd} \tag{2}$$

what can we say about the existence of a cellular map $\alpha : X \to Y$ with $H_*(\alpha) = f_*$?

The question has no answer in general. Whitehead [10] gave a complete answer in the case of 4-dimensional simply connected CW-complexes. We recall it right now. Baues gave a more general and sophisticated answer; it needs long definitions and new formulations (see [3] for details).

2.3. 4-*dimensional CW-complexes*

We need first to define an algebraic functor which represents quadratic maps.

A function $f : A \to B$ between Abelian groups is called a quadratic map if $f(-a) = a$ and if the function $A \times A \to B$, defined by $(a,b) \mapsto f(a+b) - f(a) - f(b)$ is a bilinear map. The following assertion is both a definition and a proposition justifying it:

Definition 2.3. For every Abelian group A there exists a universal quadratic map

$$\gamma : A \to \Gamma(A)$$

such that every quadratic map $f : A \to B$ uniquely factors

$$\begin{tikzcd}
A \arrow[d, "\gamma"'] \arrow[dr, "f"] & \\
\Gamma(A) \arrow[r] & B
\end{tikzcd}$$

For any morphism $\phi : A \to A'$, the morphism $\Gamma(\phi) : \Gamma(A) \to \Gamma(A')$ is defined and Γ is a well-defined functor, called *Whitehead's quadratic functor*.

This functor has the following properties:

1) If $\eta : S^3 \to S^2$ is the Hopf map, the induced map $\eta^* : \pi_2(X) \to \pi_3(X)$ is quadratic;
2) $\Gamma_3^X = \Gamma(\pi_2(X))$;
3) $\Gamma_{n+1}^X = \pi_n(X) \otimes \mathbb{Z}/2$, $n \geqslant 3$.

Then we can formulate Whitehead's theorem on 4-dimensional CW-complexes:

Theorem 2.4. *Let X and Y be two simply connected* 4*-dimensional CW-complexes. We suppose there exists a commutative ladder of group maps from* $\mathrm{WES}(X)$ *towards* $\mathrm{WES}(Y)$ (*notice that the Hurewicz map is an isomorphism in degree* 2*, so we can shorten the exact sequences*).

$$\begin{tikzcd}
H_4(X^4,\mathbb{Z}) \arrow[r, "b_4"] \arrow[d, "f_4"] & \Gamma(H_2(X^4,\mathbb{Z})) \arrow[r] \arrow[d, "\gamma_4"] & \pi_3(X^4) \arrow[r, two heads] \arrow[d, "\Omega_3"] & H_3(X^4,\mathbb{Z}) \arrow[d, "f_3"] \\
H_4(Y^4,\mathbb{Z}) \arrow[r, "b_4"] & \Gamma(H_2(Y^4,\mathbb{Z})) \arrow[r] & \pi_3(Y^4) \arrow[r, two heads] & H_3(Y^4,\mathbb{Z})
\end{tikzcd}$$

If $\gamma_4 = \Gamma(f_2)$, there exists a cellular map $\alpha : X \to Y$ with $H_n(\alpha) = f_n$, $n = 2, 3, 4$.

Such a simple theorem is not valid for CW-complexes of higher dimensions. Nevertheless, we remark that an invariant for the 4-skeleton (namely the Γ group) can be calculated with invariants of the 3-skeleton (namely homology in degree 2).

The following section presents an elementary approach to the problem, easy to calculate with but less powerful than the boundary invariant of Baues. Nevertheless, in good cases, it classifies the homotopy types of CW-complexes.

3. The characteristic *n*-extensions

In order to give a partial generalization to Theorem 2.4, we shall see that we add two ingredients to our receipts. First some analogue to the condition $\gamma_4 = \Gamma(f_2)$. Second a homotopy invariant set to Whitehead's certain exact sequence, called the set of the characteristic n-extensions, which expresses a compatibility condition for morphisms. The purpose of this section is to define this set.

3.1. Splitting homotopy groups

Our task is to recursively define maps between spaces from morphisms of their Whitehead certain exact sequences.

Consider now the morphism $j_n : \pi_n(X) \to C_n X$ extracted from diagram (1). It gives rise to the short exact sequence

$$\Gamma_n^X \rightarrowtail \pi_n(X^n) \twoheadrightarrow \ker\beta_n = \operatorname{Im} j_n. \tag{3}$$

As $C_n X$ is a free Abelian group, $\ker\beta_n \subset C_n X$ is also free and the later short exact sequence splits. So we can choose a splitting

$$\mu_n : \pi_n\big(X^n\big) \xrightarrow{\cong} \Gamma_n^X \oplus \ker\beta_n. \tag{4}$$

Remark 3.1. Let us denote by $\mu_n^1 : \pi_n(X^n) \to \Gamma_n^X$ the composition of μ_n with the projection onto the first factor while the composition with the second projection is j_n (writing this in a matrix setting: $\mu_n = \binom{\mu_n^1}{j_n}$). If μ_n^{-1} is the inverse of μ_n, it is clear that $\mu_n^{-1}|_{\Gamma_n^X}$ is the inclusion $\Gamma_n^X \subset \pi_n(X^n)$. If we denote by $\sigma_n = \mu_n^{-1}|_{\ker\beta_n}$ the section of j_n defined by μ_n, we have the identification

$$\mu_n^1 = \mathrm{id}_{\pi_n(X^n)} - \sigma_n \circ j_n. \tag{5}$$

The sequence (3) is natural and induces a commutative diagram for any map $\alpha : X \to Y$

$$\begin{tikzcd} \Gamma_n^X \arrow[r, tail] \arrow[d, "\gamma_n^{\alpha_n}"'] & \pi_n(X^n) \arrow[r, two heads] \arrow[d, "\pi_n(\alpha_n)"] & \ker\beta_n \arrow[d, "C_n\alpha|_{\ker\beta_n}"] \\ \Gamma_n^Y \arrow[r, tail] & \pi_n(Y^n) \arrow[r, two heads] & \ker\beta_n' \end{tikzcd} \tag{3}$$

where α_n is induced from α by restriction to the n-skeleton, $\gamma_n^{\alpha_n}$ is the restriction of $\pi_n(\alpha_n)$ to $\Gamma_n^X \subset \pi_n(X^n)$. Using the splitting μ_n, we can form the following (non-commutative!) diagram:

$$\begin{tikzcd} \pi_n(X^n) \arrow[r, "\mu_n"] \arrow[d, "\pi_n(\alpha_n)"'] & \Gamma_n^X \oplus \ker\beta_n \arrow[d, "\gamma_n^{\alpha_n} \oplus C_n\alpha|_{\ker\beta_n}"] \\ \pi_n(Y^n) \arrow[r, "\mu_n'"] & \Gamma_n^Y \oplus \ker\beta_n' \end{tikzcd} \tag{4}$$

Let us examine $(\gamma_n^{\alpha_n} \oplus C_n\alpha|_{\ker\beta_n}) \circ \mu_n - \mu_n' \circ \pi_n(\alpha_n) : \pi_n(X^n) \to \Gamma_n^Y \oplus \ker\ \beta_n'$. By the above remark, the second summand is $C_n\alpha|_{\ker\beta_n} \circ j_n - j_n' \circ \pi_n(\alpha_n)$. By diagram (3) it is zero. Therefore:

$$\operatorname{Im}\big[\big(\gamma^{\alpha_n} \oplus C_n\alpha|_{\ker\beta_n}\big) \circ \mu_n - \mu_n' \circ \pi_n(\alpha_n) : \pi_n\big(X^n\big) \to \Gamma_n^Y \oplus \ker\beta_n'\big] \subset \Gamma_n^Y.$$

Moreover, using the decomposition of μ_n, formula (5) and the commutative diagram (3), we have:

$$\big(\gamma_n^{\alpha_n} \oplus C_n\alpha|_{\ker\beta_n}\big) \circ \mu_n - \mu_n' \circ \pi_n(\alpha_n) = \pi_n(\alpha_n) \circ \sigma_n \circ j_n - \sigma_n' \circ j_n' \circ \pi_n(\alpha_n). \tag{6}$$

3.2. Splitting the cellular complex. The characteristic n-extensions

Consider the differential of the cellular complex: $d_{n+1} : C_{n+1}X \to C_n X$; the image $\operatorname{Im} d_{n+1} \subset C_n X$ is a free Abelian group. We can choose a splitting:

$$t_{n+1} : \operatorname{Im} d_{n+1} \oplus \ker d_{n+1} \xrightarrow{\cong} C_{n+1}X \tag{7}$$

whose restriction to $\ker d_{n+1}$ is the inclusion.

We can now rewrite the morphism $\beta_{n+1} : C_{n+1}X \to \pi_n(X^n)$ using the above respective splittings of source (7) and target (4) groups:

$$\operatorname{Im} d_{n+1} \oplus \ker d_{n+1} \xrightarrow{t_{n+1}} C_{n+1}X \xrightarrow{\beta_{n+1}} \pi_n(X^n) \xrightarrow{\mu_n} \Gamma_n^X \oplus \ker \beta_n \tag{8}$$

and we write down the composition as a matrix:

$$\begin{pmatrix} \phi_n & \theta_n \\ \psi_n & \eta_n \end{pmatrix}. \tag{9}$$

First, using formulas (5) and the argument (3) in Remark 2.1 we get:

$$\begin{aligned} \phi_n &= \mu_n^1 \circ \beta_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} = (\mathrm{id}_{\pi_n(X^n)} - \sigma_n \circ j_n) \circ \beta_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} \\ &= (\beta_{n+1} - \sigma_n \circ d_{n+1}) \circ t_{n+1}|_{\operatorname{Im} d_{n+1}}. \end{aligned} \tag{10}$$

Second, using the same argument and Remark 3.1 we get:

$$\psi_n = d_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}}.$$

Finally, by definition of b_{n+1} given in (1):

$$\theta_n = b_{n+1} \circ \mathrm{pr}_{n+1}$$

(where $\mathrm{pr}_{n+1} : \ker d_{n+1} \twoheadrightarrow H_{n+1}(X)$ is the projection) and $\eta_n = 0$.

Among the four components of the matrix $\left(\begin{smallmatrix} \phi_n & \theta_n \\ \psi_n & \eta_n \end{smallmatrix}\right)$, only ϕ_n reflects data non-directly readable in the Whitehead exacts sequence. ϕ_n depends on our two splittings t_{n+1} and μ_n. Define now $\tilde{\phi}_n$ by composing ϕ_n with the projection $\Gamma_n^X \to \operatorname{Coker} b_{n+1}$. Then notice that:

$$\operatorname{Im} d_{n+1} \overset{\kappa_n}{\rightarrowtail} \ker d_n \longrightarrow H_n(X) \tag{11}$$

is a free resolution of $H_n(X)$ so $\tilde{\phi}_n$ defines the extension class:

$$[\tilde{\phi}_n] \in \operatorname{Ext}^1_{\mathbb{Z}}\big(H_n(X), \operatorname{Coker} b_{n+1}\big). \tag{12}$$

Definition 3.2. The class $[\tilde{\phi}_n]$ is called a *characteristic n-extension* of the CW-complex X.

Remark 3.3. It is important to notice the following fact:

Let $\operatorname{Ext}(\ker b_n, \operatorname{Coker} b_{n+1})$ be the Abelian group of the extensions classes of $\operatorname{Coker} b_{n+1}$ by $\ker b_n$. It is well known that $\operatorname{Ext}(\ker b_n, \operatorname{Coker} b_{n+1})$ and $\operatorname{Ext}^1_{\mathbb{Z}}(\ker b_n, \operatorname{Coker} b_{n+1})$ are isomorphic. Now if we consider the surjection:

$$\theta_n : \operatorname{Ext}^1_{\mathbb{Z}}(H_n(X), \operatorname{Coker} b_{n+1}) \twoheadrightarrow \operatorname{Ext}(\ker b_n, \operatorname{Coker} b_{n+1}) \tag{13}$$

induced by the inclusion $\ker b_n \subset H_n(X)$ and the above isomorphism, we can say that any characteristic n-extension of X satisfies:

$$\theta_n\big([\tilde{\phi}_n]\big) = \big[\pi_n(X)\big] \tag{14}$$

where $[\pi_n(X)]$ is the class represented by the short exact sequence $\operatorname{Coker} b_{n+1} \rightarrowtail \pi_n(X) \twoheadrightarrow \ker b_n(X)$ extracted for the Whitehead exact sequence (1). Therefore we can say that $S_n(X) = (\theta_n)^{-1}([\pi_n(X)])$ is the set of all the characteristic n-extensions of the CW-complex X.

The set $S_n(X)$ is an invariant of homotopy. Namely if $\alpha : X \to Y$ is a homotopy equivalence, then there exists a bijection $S_n(\alpha) : S_n(X) \to S_n(Y)$ defined by setting:

$$S_n(\alpha)\big([\tilde{\phi}_n]\big) = \big(\big[\tilde{\gamma}_n^\alpha \circ \tilde{\phi}_n\big]\big)$$

where $\tilde{\gamma}_n^\alpha : \operatorname{Coker} b_{n+1} \to \operatorname{Coker} b'_{n+1}$ is the quotient homomorphism induced by $\gamma_n^\alpha : \Gamma_n^X \to \Gamma_n^Y$. Note that the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Ext}^1_{\mathbb{Z}}(H_n(X), \operatorname{Coker} b_{n+1}) & \overset{\theta_n}{\twoheadrightarrow} & \operatorname{Ext}(\ker b_n, \operatorname{Coker} b_{n+1}) \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{Ext}^1_{\mathbb{Z}}(H_n(Y), \operatorname{Coker} b'_{n+1}) & \overset{\theta'_n}{\twoheadrightarrow} & \operatorname{Ext}(\ker b'_n, \operatorname{Coker} b'_{n+1}) \end{array}$$

assures that $([\gamma_n^\alpha \circ \tilde{\phi}_n] \in S_n(Y)$.

We can now tackle our main theorem.

4. The classification of CW-complexes

4.1. Preliminary settings

We now go back to the problem mentioned in Section 2: suppose given two simply connected CW-complexes X and Y and a graded homomorphism $f_*: H_*(X) \to H_*(Y)$. What can we say about the existence of a cellular map $\alpha: X \to Y$ with $H_*(\alpha) = f_*$?

We actually need to know f_* already at the chain complex level, say a representative $\xi_*: C_*X \to C_*Y$, the existence of which is certified by the homotopy extension theorem (see [9]).

We shall proceed by induction. So we first define:

Definition 4.1. The map $\alpha^n: X^n \to Y^n$ is an *n-realization* of f_* if $H_{\leqslant n-1}(\alpha) = f_{\leqslant n-1}$ and $C_n\alpha^n|_{\ker d_n} = \xi_n|_{\ker d_n}$. We denote by $A_n = \{\alpha^n\}$ the set of all n-realization of f_*.

We need further some compatibility with the Whitehead exact sequences.

Definition 4.2. The pair (f_*, γ_*), where $f_*: H_*(X) \to H_*(Y)$ and $\gamma_*: \Gamma_*^X \to \Gamma_*^Y$ are graded group maps, is called a morphism from WES(X) towards WES(Y) if the following two properties are satisfied:

1) There exists a graded homomorphism $\Omega_*: \pi_*(X) \to \pi_*(Y)$ making the diagram (2) commute.
2) For every $n \geqslant 2$, if there exists an n-realization of f_*, then A_n contains some α^n with $\gamma_n = \gamma_n^{\alpha^n}$.

Example 4.3. If $\theta: X \to Y$ is a map of CW-complexes, it induces a morphism $(H_*(\theta), \gamma_*^\theta)$ of Whitehead's certain exact sequences. Obviously, this is a morphism in the meaning of Definition 4.2.

Example 4.4. Theorem 2.4 gives a non-trivial illustration of Definition 4.2. Moreover, it is an example of the forthcoming Definition 4.6; this fact justifies the presentation of our main theorem as a generalization of the result of Whitehead recalled in Section 2.3.

Example 4.5. In this example we are motivated by the following results due to Anick [1,2].

Let R be a subring of $\mathbb{Q}$ and let p the least prime number which is not a unit in R. A free differential graded Lie algebra is called n-mild if it generated by the elements with degree i, where $n \leqslant i \leqslant np-1$.

Let $CW_n^{np}(R)/_\simeq$ and $DGL_n^{np}/_\simeq$ denote the homotopy category of R-localized, n-connected, np-dimensional CW-complexes and of n-mild free dgl's, respectively. Anick has proved that the universal enveloping algebra functor $U: DGL_n^{np} \to HAH_n^{np}$ induces an isomorphism on the homotopy categories. Here HAH_n^{np} is the category of n-mild Hopf algebras up to homotopy. Thus we have an equivalence of categories $L: CW_n^D(R)/_\simeq \to DGL_n^D/_\simeq$, for $D = \min(n+2p-3, np-1)$, by composing the Adams–Hilton model $L: CW_n^D/_\simeq \to HAH_n^D/_\simeq$ with U^{-1}. Moreover if $L(X) = (L(V), \partial)$, then for every $i < D$ we have:

$$\pi_i(X) \otimes R \cong H_{i-1}\big(L(V), \partial\big) \quad \text{and} \quad H_i(X, R) \cong H_i\big(s^{-1}V, d\big)), \quad \forall i < D,$$

here s^{-1} denotes the desuspension graded homomorphism.

Define $\Gamma_i^{L(V)} = \mathrm{Im}(i_n: H_i(L(V_{\leqslant i-1})) \to H_i(L(V_{\leqslant i})))$, for all $i < D$. Because of the above equivalence, we can say that the Abelian groups Γ_i^X and $\Gamma_{i-1}^{L(V)}$ are isomorphic.

THIS MIGHT WELL BE LABELED AS AN EXAMPLE WHY THE CHOICE $p = 7$?

Now let $n = 1$, $p = 7$ and let X be an object in $CW_1^5(R)/_\simeq$. By putting $H_i = H_i(X)$, for $2 \leqslant i \leqslant 5$, and by using Proposition 3.4 in [6] we derive that:

$$\Gamma_3^X = [H_2, H_2], \qquad \Gamma_4^X = [H_3, H_2] \oplus [H_2, \mathrm{Coker}\, b_5] \tag{15}$$

where $[H_i, H_j]$ is the submodule of $H_i \otimes H_j$ generated by the elements on the form:

$$h_i \times h_j - (-1)^{(i-1)(j-1)} h_j \times h_i, \quad h_i \in H_i,\ h_j \in H_j.$$

Therefore WES(X) can be written as follows:

$$H_5 \xrightarrow{b_5} [H_3, H_2] \oplus [H_2, \mathrm{Coker}\, b_4] \longrightarrow \pi_4(X) \longrightarrow H_4 \xrightarrow{b_4} [H_2, H_2] \longrightarrow \pi_3(X) \twoheadrightarrow H_3$$

Likewise it is also shown that if $\alpha: X \to Y$ is a morphism in $CW_1^5(R)/_\simeq$, then in the following diagram:

$$\begin{tikzcd}
H_5 \arrow[r, "b_5"] \arrow[d, "H_5(\alpha)"] & {[H_3, H_2] \oplus [H_2, \mathrm{Coker}\, b_4]} \arrow[r] \arrow[d, "\gamma_4^\alpha"] & \pi_4(X) \arrow[r] \arrow[d, "\pi_4(\alpha)"] & H_4 \arrow[r, "b_4"] \arrow[d, "H_4(\alpha)"] & {[H_2, H_2]} \arrow[r] \arrow[d, "\gamma_3^\alpha"] & \pi_3(X) \arrow[r, two heads] \arrow[d, "\pi_3(\alpha)"] & H_3 \arrow[d, "H_3(\alpha)"] \\
H'_5 \arrow[r, "b'_4"] & {[H'_3, H'_2] \oplus [H'_2, \mathrm{Coker}\, b'_4]} \arrow[r] & \pi_4(Y) \arrow[r] & H'_4 \arrow[r, "b'_4"] & {[H'_2, H'_2]} \arrow[r] & \pi_3(Y) \arrow[r, two heads] & H'_3
\end{tikzcd}$$

where the bottom sequence is WES(Y), we have:

$$\gamma_3^\alpha = \big[H_2(\alpha), H_2(\alpha)\big], \qquad \gamma_4^\alpha = \big[H_3(\alpha), H_2(\alpha)\big] \oplus \big[H_2(\alpha), \overline{[H_2(\alpha), H_2(\alpha)]}\big].$$

Here $\overline{[H_2(\alpha), H_2(\alpha)]} : \operatorname{Coker} b_4 \to \operatorname{Coker} b'_4$ is the quotient homomorphism induced by $[H_2(\alpha), H_2(\alpha)]$.

Hence if X and Y are in $CW_1^5(R)/_{\simeq}$, then morphisms from WES(X) towards WES(Y) defined in Definition 4.2 can be characterized as follows: they are homomorphisms $f_i : H_i(X) \to H_i(Y), i = 2, 3, 4, 5$, for which there exist homomorphisms Ω_3, Ω_4 making the following diagram commutes:

$$\begin{tikzcd}
H_5 \arrow[r, tail, "b_5"] \arrow[d, "f_5"] & {[H_3, H_2] \oplus [H_2, \operatorname{Coker} b_4]} \arrow[r] \arrow[d, "\gamma_4"] & \pi_4(X) \arrow[r] \arrow[d, "\Omega_4"] & H_4 \arrow[r, tail, "b_4"] \arrow[d, "f_4"] & {[H_2, H_2]} \arrow[r, tail] \arrow[d, "\gamma_3"] & \pi_3(X) \arrow[r, two heads] \arrow[d, "\Omega_3"] & H_3 \arrow[d, "f_3"] \\
H'_5 \arrow[r, tail, "b'_4"] & {[H'_3, H'_2] \oplus [H'_2, \operatorname{Coker} b'_4]} \arrow[r] & \pi_4(Y) \arrow[r] & H'_4 \arrow[r, tail, "b'_4"] & {[H'_2, H'_2]} \arrow[r, tail] & \pi_3(Y) \arrow[r, two heads] & H'_3
\end{tikzcd} \tag{A}$$

where:

$$\gamma_3 = [f_2, f_2], \qquad \gamma_4 = [f_3, f_2)] \oplus \big[f_2, \overline{[f_2, f_2]}\big].$$

Here $\overline{[f_2, f_2]} : \operatorname{Coker} b_4 \to \operatorname{Coker} b'_4$ is the quotient homomorphism induced by $[f_2, f_2]$.

Recall that we are given two CW-complexes X and Y and a map (f_*, γ_*) between their Whitehead's certain exact sequences. Let us denote by:

$$(f_n)^* : \operatorname{Ext}^1_{\mathbb{Z}}\big(H_n(Y), \operatorname{Coker} b'_{n+1}\big) \to \operatorname{Ext}^1_{\mathbb{Z}}\big(H_n(X), \operatorname{Coker} b'_{n+1}\big)$$

the obvious map induced by f_n. If (f_*, γ_*) is a morphism, $\gamma_n : \Gamma_n^X \to \Gamma_n^Y$ defines a quotient morphism $\tilde{\gamma}_n : \operatorname{Coker} b_{n+1} \to \operatorname{Coker} b'_{n+1}$, and therefore a group morphism

$$(\tilde{\gamma}_n)_* : \operatorname{Ext}^1_{\mathbb{Z}}\big(H_n(X), \operatorname{Coker} b_{n+1}\big) \to \operatorname{Ext}^1_{\mathbb{Z}}\big(H_n(X), \operatorname{Coker} b'_{n+1}\big).$$

Definition 4.6. (f_*, γ_*) is a *strong* morphism if there exist $[\tilde{\phi}_n] \in S_n(X)$ and $[\tilde{\phi}'_n] \in S_n(Y)$ such that:

$$(f_n)^*\big([\tilde{\phi}'_n]\big) = (\tilde{\gamma}_n)_*\big([\tilde{\phi}_n]\big), \quad \forall n \geqslant 2. \tag{16}$$

Remark 4.7. If X and Y are two CW-complexes whose homology $H_*(X)$ and $H_*(Y)$ are $\mathbb{Z}$-free, any morphism between their respective Whitehead exact sequences is strong.

For proving the main theorem, we shall make explicit condition (16) on representatives. As a preliminary, we achieve that in the following subsection.

4.2. Choosing explicit classes in Ext*-groups*

First let us choose the free resolution of $H_n(X)$ (resp. $H_n(Y)$) given in (11): $\operatorname{Im} d_{n+1} \overset{\kappa_n}{\rightarrowtail} \ker d_n \twoheadrightarrow H_n(X)$ (resp. $\operatorname{Im} d'_{n+1} \overset{\kappa'_n}{\rightarrowtail} \ker d'_n \twoheadrightarrow H_n(Y)$). The cycle ϕ_n and its quotient $\tilde{\phi}_n$ which represents the class $[\tilde{\phi}_n]$ can be inserted in the following diagram (resp. for ϕ'_n)

$$\begin{tikzcd}
\operatorname{Im} d_{n+1} \arrow[r, tail, "\kappa_n"] \arrow[d, "\tilde{\phi}_n"'] \arrow[dr, "\phi_n"] & \ker d_n \arrow[r, two heads] & H_n(X) \\
\operatorname{Coker} b_{n+1} \arrow[d, "\tilde{\gamma}_n"'] & \Gamma_n^X \arrow[l, two heads, "\mathrm{pr}"] \arrow[d, "\gamma_n"] & \\
\operatorname{Coker} b'_{n+1} & \Gamma_n^Y \arrow[l, two heads, "\mathrm{pr}'"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Im} d_{n+1} \arrow[r, tail, "\kappa_n"] \arrow[d, "\xi_{n+1}"'] & \ker d_n \arrow[r, two heads] & H_n(X) \arrow[d, "f_n"] \\
\operatorname{Im} d'_{n+1} \arrow[r, tail, "\kappa'_n"] \arrow[d, "\tilde{\phi}'_n"'] \arrow[dr, "\phi'_n"] & \ker d'_n \arrow[r, two heads] & H_n(Y) \\
\operatorname{Coker} b'_{n+1} & \Gamma_n^Y \arrow[l, two heads, "\mathrm{pr}'"] &
\end{tikzcd} \tag{5}$$

Notice that these commutative diagrams hold for any lifting ϕ_n (resp. ϕ'_n) of $\tilde{\phi}_n$ (resp. $\tilde{\phi}'_n$) — not only for the representatives defined by formula (16).

Now we define $(f_n)^*$ and $(\tilde{\gamma}_n)_*$ on cycles by using the free resolution (21); taking classes again we may write down:

$$(\tilde{\gamma}_n)_*\big([\tilde{\phi}_n]\big) = [\tilde{\gamma}_n \circ \tilde{\phi}_n] \quad \text{and} \quad (f_n)^*\big([\tilde{\phi}'_n]\big) = \big[\tilde{\phi}'_n \circ \xi_{n+1}\big].$$

With these descriptions, condition (16) turns out to be:

$$\big[\tilde{\gamma}_n \circ \tilde{\phi}_n - \tilde{\phi}'_n \circ \xi_{n+1}\big] = 0 \quad \text{in } \operatorname{Ext}^1_{\mathbb{Z}}\big(H_n(X), \operatorname{Coker} b'_{n+1}\big).$$

Going back to cycles we deduce the existence of a group morphism $\tilde{h}_n : \ker d_n \to \operatorname{Coker} b'_{n+1}$ satisfying:

$$\tilde{\gamma}_n \circ \tilde{\phi}_n - \tilde{\phi}'_n \circ \xi_{n+1} = \tilde{h}_n \circ \kappa_n. \tag{17}$$

As $\ker d_n$ is free, we can find a morphism $h_n : \ker d_n \to \Gamma^Y_n$ which lifts $\tilde{h}_n$:

$$\begin{tikzcd} \ker d_n \arrow[d, "h_n"'] \arrow[dr, "\tilde{h}_n"] & \\ \operatorname{Coker} b'_{n+1} & \Gamma^Y_n \subseteq \pi_n(Y^n) \arrow[l, "\mathrm{pr}"'] \end{tikzcd}$$

Thus, lifting Eq. (17), see diagrams (5), we get:

$$\operatorname{Im}\big(\gamma_n \circ \phi_n - \phi'_n \circ \xi_{n+1} - h_n \circ \kappa_n\big) \subset \operatorname{Im} b'_{n+1} \tag{18}$$

Recalling the splitting map (7):

$$t_{n+1} : \operatorname{Im} d_{n+1} \oplus \ker d_{n+1} \xrightarrow{\cong} C_{n+1}X$$

and the definition (1) of Γ^Y_n, we get the following lifting g_n of h_n:

$$\begin{tikzcd} C_nX \arrow[d, "(t_n)^{-1}"'] \arrow[ddr, "g_n"] & \\ \operatorname{Im} d_n \oplus \ker d_n \arrow[d, "h_m"'] & \\ \pi_n(Y^n) \supseteq \Gamma^Y_n & \pi_n(Y^{n-1}) \arrow[l, "i_n"] \end{tikzcd}$$

Finally formula (18) becomes

$$\operatorname{Im}\big(\gamma_n \circ \phi_n - \phi'_n \circ \xi_{n+1} - i_n \circ g_n \circ d_{n+1}\big) \subset \operatorname{Im} b'_{n+1} \tag{19}$$

where we use the fact that $\kappa_n \circ (t_{n+1})^{-1} = d_{n+1}$. Once again we emphasize that ϕ_n (resp. ϕ'_n) is any lifting of $\tilde{\phi}_n$ (resp. $\tilde{\phi}'_n$).

4.3. The main theorem

We can now formulate and prove the following:

Theorem 4.8. *Let X and Y be two simply connected CW complexes and $(f_*, \gamma_*) : \mathrm{WES}(X) \to \mathrm{WES}(Y)$ a strong homomorphism. Then there exists a cellular map $\alpha : X \to Y$ such that $H_*(\alpha) = f_*$.*

As an immediate consequence of the Whitehead theorem, we get:

Corollary 4.9. *Let X and Y be two simply connected CW complexes. If $\mathrm{WES}(X)$ and $\mathrm{WES}(Y)$ are strongly isomorphic, then X and Y are homotopic.*

Remark 4.10. If $H_*(X)$ and $H_*(Y)$ are free Abelian groups, the assertion “strong” in the above corollary is automatically satisfied.

The last pages are now devoted to the proof of Theorem 4.8.

Remark 4.11. For the proof of Theorem 4.8 we need the following elementary fact:

Let X and Y be CW-complexes and $F : X^n \to Y^n$ a map between their n-skeleta. If $\rho_{n+1} : C_{n+1}(X) \to C_{n+1}(Y)$ is a homomorphism such that the following diagram commutes:

$$\begin{tikzcd} C_{n+1}X \arrow[r, "\rho_{n+1}"] \arrow[d, "\beta_{n+1}"] & C_{n+1}Y \arrow[d, "\beta'_{n+1}"] \\ \pi_n(X^n) \arrow[r, "\pi_n(F)"] & \pi_n(Y^n) \end{tikzcd}$$

Then F can be extended to a map $G : X^{n+1} \to Y^{n+1}$ with $C_{n+1}(G) = \rho_{n+1}$.

Proof of Theorem 4.1. The proof goes recursively. At each step, we wish to set up the data in such a way that we can apply Remark 4.11.

Let X and Y be two simply connected CW complexes. By hypothesis we are given the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,\mathbb{Z}) \arrow[r, "b_{n+1}"] \arrow[d, "f_{n+1}"] & \Gamma_n^X \arrow[r] \arrow[d, "\gamma_n"] & \pi_n(X) \arrow[r] \arrow[d, "\Omega_n"] & H_n(X,\mathbb{Z}) \arrow[r, "b_n"] \arrow[d, "f_n"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Y,\mathbb{Z}) \arrow[r, "b'_{n+1}"] & \Gamma_n^Y \arrow[r] & \pi_n(Y) \arrow[r] & H_n(Y,\mathbb{Z}) \arrow[r, "b'_n"] & \cdots
\end{tikzcd}$$

and a chain map $\xi_* : C_*X \to C_*Y$ whose homology is $H_*(\xi) = f_*$.

Suppose now we have already constructed an n-realization of (f_*, γ_*). By Definition 4.1 it means there exists a map $\alpha^n : X^n \to Y^n$ with the following properties:

$$H_{\leqslant n-1}(\alpha^n) = f_{\leqslant n-1}, \qquad C_n\alpha^n|_{\ker d_n} = \xi|_{\ker d_n} \quad \text{and} \quad \gamma_n = \gamma_n^{\alpha^n}. \tag{20}$$

Now let us consider the following diagram:

$$\begin{tikzcd}
 & C_{n+1}X \arrow[rr, "\xi_{n+1}"] \arrow[d, "\beta_{n+1}"] \arrow[dddl, "d_{n+1}"'] & & C_{n+1}Y \arrow[d, "\beta'_{n+1}"] \arrow[dddr, "d'_{n+1}"] & \\
 & \pi_n(X^n) \arrow[rr, "\pi_n(\alpha^n)"] \arrow[d, "\mu_n", "\cong"'] \arrow[ddl, "j_n"'] & & \pi_n(Y^n) \arrow[d, "\mu'_n", "\cong"'] \arrow[ddr, "j'_n"] & \\
 & \Gamma_n^X \oplus \ker\beta_n \arrow[rr, "\gamma_n^{\alpha^n} \oplus C_n\alpha^n|_{\ker\beta_n}"'] & & \Gamma_n^Y \oplus \ker\beta'_n & \\
C_nX \arrow[rrrr, "\xi_n"] & & & & C_nY
\end{tikzcd} \tag{6}$$

In this diagram, both squares do not commute, but both side triangles and both trapezoids do. (These various commutative subdiagrams translate the respective definitions of the differential d_* of the chain complex C_*X, of the chain map $\xi_* : C_*X \to C_*Y$ and of the cellular map $\alpha^n : X^n \to Y^n$. The lower (non-commutative) square is merely diagram (4).)

We shall show that we can disrupt both α^n and ξ_{n+1} so that we can apply Remark 4.11 and still satisfy the recursive hypothesis, proving the "n to $n+1$ step". Let us now explain the details.

To begin with let us examine the big square in diagram (6); more precisely, we calculate

$$\big(\gamma_n^{\alpha^n} \oplus C_n\alpha^n|_{\ker\beta_n}\big) \circ \mu_n \circ \beta_{n+1} - \mu'_n \circ \beta'_{n+1} \circ \xi_{n+1}.$$

Let us make use of the matrix setting given in (9). In this setting the first summand has the following expression:

$$\begin{aligned}
&\begin{pmatrix} \gamma_n^{\alpha^n} & 0 \\ 0 & C_n\alpha^n|_{\ker\beta_n} \end{pmatrix} \circ \begin{pmatrix} \phi_n & b_{n+1} \circ \mathrm{pr}_{n+1} \\ d_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} & 0 \end{pmatrix} \\
&\quad = \begin{pmatrix} \gamma_n^{\alpha^n} \circ \phi_n & \gamma_n^{\alpha^n} \circ b_{n+1} \circ \mathrm{pr}_{n+1} \\ C_n\alpha^n|_{\ker\beta_n} \circ d_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} & 0 \end{pmatrix}.
\end{aligned} \tag{21}$$

For the second summand:

$$\begin{aligned}
&\begin{pmatrix} \phi'_n & b'_{n+1} \circ \mathrm{pr}'_{n+1} \\ d'_{n+1} \circ t'_{n+1}|_{\operatorname{Im} d'_{n+1}} & 0 \end{pmatrix} \circ \begin{pmatrix} \mathrm{pr}_1 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} & 0 \\ \mathrm{pr}_2 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} & \xi_{n+1}|_{\ker d_{n+1}} \end{pmatrix} \\
&\quad = \begin{pmatrix} \phi'_n \circ \mathrm{pr}_1 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} + \Delta_{n+1} & b_{n+1} \circ \mathrm{pr}'_{n+1} \circ \xi_{n+1}|_{\ker d_{n+1}} \\ d'_{n+1} \circ t'_{n+1}|_{\operatorname{Im} d'_{n+1}} \circ \mathrm{pr}_1 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} & 0 \end{pmatrix}
\end{aligned} \tag{22}$$

where $\Delta_{n+1} = b_{n+1} \circ \mathrm{pr}'_{n+1} \circ \mathrm{pr}_2 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}}$.

Remark 4.12. As we noticed in Section 3.2 both maps $\phi'_n \circ \mathrm{pr}_1 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} + \Delta_{n+1}$and $\phi'_n \circ \mathrm{pr}_1 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} : \operatorname{Im} d_{n+1} \to \Gamma_n^Y$ project identically onto $\operatorname{Coker} b'_{n+1}$. So the explicit formula (19) holds — just choose an other adequate g_n:

$$\operatorname{Im}\big(\gamma_n \circ \phi_n - \big(\phi'_n \circ \mathrm{pr}_1 \circ \big(t'_{n+1}\big)^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\operatorname{Im} d_{n+1}} + \Delta_{n+1}\big) - i_n \circ g_n \circ d_{n+1}\big) \subset \operatorname{Im} b'_{n+1}. \tag{23}$$

Determining the lack of commutativity of the big square in diagram (6), we have to calculate the difference of matrix (21) and matrix (22); we obtain:

$$\begin{pmatrix} \gamma_n \circ \phi_n - (\phi'_n \circ \mathrm{pr}_1 \circ (t'_{n+1})^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\mathrm{Im}\, d_{n+1}} + \Delta_{n+1}) & 0 \\ 0 & 0 \end{pmatrix}. \tag{24}$$

We used here: For the 0 on the first line, the hypothesis that $\gamma_n^{\alpha^n} = \gamma_n$; for the 0 on the second line the hypothesis that $C_n\alpha^n|_{\ker \beta_n} = C_n\alpha^n|_{\ker d_n} = \xi_n|_{\ker d_n}$.

We resume our calculation:

$$\begin{aligned} &\big(\gamma_n^{\alpha^n} \oplus C_n\alpha^n|_{\ker \beta_n}\big) \circ \mu_n \circ \beta_{n+1} - \mu'_n \circ \beta'_{n+1} \circ \xi_{n+1} \\ &\quad = \gamma_n \circ \phi_n - \big(\phi'_n \circ \mathrm{pr}_1 \circ \big(t'_{n+1}\big)^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\mathrm{Im}\, d_{n+1}} + \Delta_{n+1}\big). \end{aligned} \tag{25}$$

Now we focus on the lack of commutativity of the lower square and compose Eq. (6) on the right by β_{n+1}:

$$\begin{aligned} &\big[\mu'_n \circ \pi_n\big(\alpha^n\big) - \big(\gamma_n^{\alpha^n} \oplus C_n\alpha^n|_{\ker \beta_n}\big) \circ \mu_n\big] \circ \beta_{n+1} \\ &\quad = \big[\sigma'_n \circ j'_n \circ \pi_n\big(\alpha^n\big) - \pi_n\big(\alpha^n\big) \circ \sigma_n \circ j_n\big] \circ \beta_{n+1} = \big(\sigma'_n \circ \xi_n - \pi_n\big(\alpha^n\big) \circ \sigma_n\big) \circ j_n \circ \beta_{n+1} \\ &\quad = \big(\sigma'_n \circ \xi_n - \pi_n\big(\alpha^n\big) \circ \sigma_n\big) \circ d_{n+1} \end{aligned} \tag{26}$$

where we used the commutations $j'_n \circ \pi_n(\alpha^n) = \xi_n \circ j_n$ and $j_n \circ \beta_{n+1} = d_{n+1}$ we pointed to in diagram (6).

As $C_{n+1}X$ is free, we can lift the morphism $\sigma'_n \circ \xi_n - \pi_n(\alpha^n) \circ \sigma_n : C_{n+1}X \to \Gamma_n^Y$ to $\pi_n(Y^{n-1})$ as pictured in the following diagram:

$$\begin{tikzcd} C_{n+1}X \arrow[d, "\sigma'_n\xi_n - \pi_n(\alpha^n)\circ\sigma_n"'] \arrow[dr, "k_n"] & \\ \pi_n(Y^n) \supseteq \Gamma_n^Y & \pi_n(Y^{n-1}) \arrow[l, "i_n"] \end{tikzcd}$$

Summing formulas (25) and (26) we obtain:

$$\begin{aligned} &\mu'_n \circ \pi_n\big(\alpha^n\big) \circ \beta_{n+1} - \mu'_n \circ \beta'_{n+1} \circ \xi_{n+1} \\ &\quad = \gamma_n \circ \phi_n - \big(\phi'_n \circ \mathrm{pr}_1 \circ \big(t'_{n+1}\big)^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\mathrm{Im}\, d_{n+1}} + \Delta_{n+1}\big) + i_n \circ k_n \circ d_{n+1}. \end{aligned} \tag{27}$$

As μ'_n is an isomorphism we can rewrite this equation

$$\begin{aligned} &\pi_n\big(\alpha^n\big) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+1} \\ &\quad = \big(\mu'_n\big)^{-1} \circ \big[\gamma_n \circ \phi_n - \big(\phi'_n \circ \mathrm{pr}_1 \circ \big(t'_{n+1}\big)^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\mathrm{Im}\, d_{n+1}} + \Delta_{n+1}\big) + i_n \circ k_n \circ d_{n+1}\big]. \end{aligned} \tag{28}$$

We know first that $\gamma_n \circ \phi_n - (\phi'_n \circ \xi_{n+1} + \Delta_{n+1}) + i_n \circ k_n \circ d_{n+1}$ maps into Γ_n^Y, second that $(\mu'_n)^{-1}$ is the identity on Γ_n^Y. We then have :

$$\pi_n\big(\alpha^n\big) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+1} = \gamma_n \circ \phi_n - \big(\phi'_n \circ \mathrm{pr}_1 \circ \big(t'_{n+1}\big)^{-1} \circ \xi_{n+1} \circ t_{n+1}|_{\mathrm{Im}\, d_{n+1}} + \Delta_{n+1}\big) + i_n \circ k_n \circ d_{n+1}. \tag{29}$$

Let us now substitute this last equation in formula (23); we get:

$$\mathrm{Im}\big(\pi_n\big(\alpha^n\big) \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+1} - i_n \circ k_n \circ d_{n+1} - i_n \circ g_n \circ d_{n+1}\big) \subset \mathrm{Im}\, b'_{n+1}. \tag{30}$$

Recalling again $j_n \circ \beta_{n+1} = d_{n+1}$ this becomes:

$$\mathrm{Im}\big(\big[\big(\pi_n\big(\alpha^n\big) - i_n \circ (k_n - g_n)\big) \circ j_n\big] \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+1}\big) \subset \mathrm{Im}\, b'_{n+1}. \tag{31}$$

So there is a well-defined morphism:

$$(\mathrm{Im}\, d_{n+1})' \xrightarrow{[(\pi_n(\alpha^n) - i_n \circ (k_n - g_n)) \circ j_n] \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+1}} \mathrm{Im}\, b'_{n+1} \subset \Gamma_n^Y .$$

It admits a lifting λ_{n+1} as pictured in the diagram:

$$\begin{tikzcd} & \ker d'_{n+1} \arrow[d, "\beta'_{n+1}"] \\ \mathrm{Im}\, d_{n+1} \arrow[ur, "\lambda_{n+1}"] \arrow[r, "{[(\pi_n(\alpha^n) - i_n \circ (k_n - g_n)) \circ j_n] \circ \beta_{n+1} - \beta'_{n+1} \circ \xi_{n+1}}"'] & \mathrm{Im}\, b'_{n+1} \subset \Gamma_n^Y \end{tikzcd}$$

We are now ready to define maps $\psi_n : X^n \to Y^n$ and $\rho_{n+1} : C_{n+1}X \to C_{n+1}Y$ fitting the conditions of Remark 4.11.

First we easily define $\rho_{n+1} = \xi_{n+1} + \lambda_{n+1}$. As to ψ_n, to begin with, we choose a map $\omega : \bigvee_{k\in K_n} S^n_k \to Y^{n-1}$ (where K_n indices the n-cells of X^n) whose homotopy class corresponds to $g_n - k_n$ via the isomorphism Hom $(C_nX, \pi_n(Y^{n-1})) \cong [\bigvee_{k\in K_n} S^n_k, Y^{n-1}]$.

Let us recall that attaching cells by a map such as $f : \bigvee_{k\in K_n} S^n_k \to X^{n-1}$ induces the action:

$$\Big[\bigvee_{k\in K_n} S^n_k, Y^n\Big] \times \big[X^n, Y^n\big] \xrightarrow{\vee} \big[X^n, Y^n\big].$$

So we thus define $\psi_n = (i_n \circ \omega) \vee \alpha^n$ and check that:

$$\pi_n(\psi_n) = \pi_n\big(\alpha^n\big) - i_n \circ (k_n - g_n) \circ j_n. \tag{32}$$

Finally we remark the following fact: as $\operatorname{Im}\omega \subset Y^{n-1}$ the chain morphisms induced by ψ_n and $\alpha^n : C_nX \to C_nY$ agree and therefore:

$$H_{\leqslant n}\big(\psi^n\big) = H_{\leqslant n}\big(\alpha^n\big). \tag{33}$$

By the definitions of ψ_n, λ_{n+1} and ρ_{n+1}, the following diagram commutes:

$$\begin{tikzcd} C_{n+1}X \arrow[r, "\rho_{n+1}"] \arrow[d, "\beta_{n+1}"] & C_{n+1}Y \arrow[d, "\beta'_{n+1}"] \\ \pi_n(X^n) \arrow[r, "\pi_n(\psi_n)"] & \pi_n(Y^n) \end{tikzcd}$$

So, by Remark 4.11, there exists a map $\alpha^{n+1} : X^{n+1} \to Y^{n+1}$ which extends ψ_n and satisfies $H_{\leqslant n}(\alpha^{n+1}) = H_{\leqslant n}(\psi_n)$, $H_{n+1}(\alpha^{n+1}) = \rho_{n+1}|_{\ker d_{n+1}}$.

Now these two equations become: First $H_{\leqslant n}(\alpha^{n+1}) = H_{\leqslant n}(\alpha^n)$ by Eq. (33). Second $H_{n+1}(\alpha^{n+1}) = C_{n+1}\alpha^{n+1}|_{\ker\ d_{n+1}} = \xi_{n+1}|_{\ker d_{n+1}}$, again by the definitions of ρ_{n+1} and λ_{n+1}.

So we can phrase: α_{n+1} is an $(n+1)$-realization of (f_*, γ_*), and we got the $(n+1)$th-step of our recursive proof. □

Example 4.13. Classification of simply connected, 5-dimensional and R-localized CW-complexes.

Let us consider again Example 4.5. Let $X, Y \in CW^5_1(R)/_{\simeq}$, due to Corollary 4.9 we can say that if $f_i : H_i(X) \to H_i(Y)$, $i = 2, 3, 4, 5$, are isomorphisms for which there exist Ω_3, Ω_4 making the diagram (A) commutes and if there exist two characteristic 4-extensions $[\tilde{\phi}_4] \in S_4(X) \subset \operatorname{Ext}^1_{\mathbb{Z}}(H_4(X), \operatorname{Coker} b_5)$ and $[\tilde{\phi}_n] \in S_4(Y) \subset \operatorname{Ext}^1_{\mathbb{Z}}(H_4(Y), \operatorname{Coker} b'_5)$ satisfying Eq. (16), for $n = 4$. Then X and Y are homotopy equivalent.

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