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Algebraic K-theory of Parameterized Endomorphisms

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Abstract. Assume that M is an R-bimodule. Let End(R, M) denotes the category whose objects are pairs (P, f), where P is a finitely generated projective right R-module and $f: P \to P \otimes M$. It has an exact structure obtained from the category of projectives over R by forgetting fs. We prove that, when R is a field, we have $\tilde{K}(\text{End}(R, M)) = \omega\tilde{K}\sigma^{-1}TM$ denotes certain localization of the tensor algebra spanned by M. This result should be viewed as a special case of the noncommutative extension of the results of [4].

Key words: algebraic K-theory, parameterized endomorphism, localization sequence.

1. Introduction

Let R be a ring and M an R-bimodule. Let TM denote the tensor algebra spanned on M. Denote by End(R,M) the category whose objects are pairs (P, f), where P is a finitely generated projective right R-module and $f: P \to P \otimes M$. Let Nil denote the full subcategory of End(R,M) consisting of nilpotent objects. Our ultimate goal (which is still far ahead) is to understand the inclusion functor Nil $\hookrightarrow$ End(R,M) on K-theoretical level in terms of K-theories of rings. The difference between $K(\text{End}(R,M))$ and $K(\text{Nil})$ should be described in terms of the K-theory of a suitable, noncommutative localization of the ring TM.

It is worth underlining here that the category End(R,M) and its K-theory appears naturally in K-theoretical investigations. For example, it played a crucial role in comparing stable K-theory and topological Hochschild homology in [3]. Recently, it was used by McCarthy in his studies on the de Rham–Witt complex. It looks like the meaning of this theory will grow in the K-theoretical investigations in the nearest future.

Let us give here some historical motivation for our investigations. When $M = R$ it is known that reduced $K(\text{Nil})$ is the same as reduced $K(R[x])$ with a shift of gradation while $\tilde{K}(\text{End}(R,M))$ is equal to $\Omega\tilde{K}(A)$ where A is equal to $R[X]$ localized in $(1 + xR[x])$ (see [4]) and $\Omega\tilde{K}(A)$ denotes the loop space of the reduced $K(A)$. Hence the effect on K-theory of our inclusion functor can be viewed as a part of the localization sequence for localizing polynomial algebra. The observation comparing $K(\mathrm{Nil})$ and $K(R[x])$ has its generalization to "larger" Ms: Waldhausen [9, see Theorem 3, p. 137] proved that for a projective M, the reduced $K(\mathrm{Nil})$ is the same as the reduced $\Omega K(TM)$. We are looking for the appropriate generalization of the second observation. Our investigations were stimulated by McCarthy, who after [3] conjectured that $K(\mathrm{End}(R,M))$ should be described via appropriate localization of TM.

Our final results only partially fulfill our expectations. There are two reasons for that. First of all our model of the cofiber of the map $K(\mathrm{Nil})\to K(\mathrm{End}(R,M))$, which we construct in Sections 2 and 3 following the work of Schlichting [7], is very special. To make it work we have to assume that our ground ring is of the semi-simple type. The other problem comes from the fact that as a main tool to work with the localized tensor algebra we use the localization sequence of Neeman and Ranicki [5]. To use it we have to assume that the localized tensor algebra is stably-flat over TM. While writing this note we discovered that algebraic properties of localizations of TM are largely unknown when only one assumes that the ground ring is not a field. Hence to get our final result (Theorem 3.3) we have to assume that R is a field. We suggest [8] as a good place to learn something about noncommutative localizations and its properties and also as a reference book on this subject.

2. Category of Parameterized Endomorphisms

Let R be a commutative ring with unit and M a finitely generated R-bimodule. We will assume that M is (bi)-projective of rank bigger than 1 and R satisfies the condition that every submodule of a finitely generated projective module P is itself finitely generated projective and splits as a direct summand in P. In other words, we assume that our ring is semi-simple. In such case the category of finitely generated projective R-modules is abelian. As one sees, we eventually assume that our ground ring R is commutative. The assumption that R is commutative can be obviously removed, but having it we do not have to write about right and left structures over R, which play no role in our investigations. The real "noncommutativity" here comes from the tensor algebra.

Let TM denote the full tensor algebra on M:

$$TM=\bigoplus_{0\leqslant i} M^{\otimes i}.$$

Denote by $\mathrm{End}(R,M)$ the category whose objects are pairs (P,f), where P is a finitely generated projective right R-module and $f: P\to P\otimes M$. Morphisms $\Phi:(P,f)\to(Q,g)$ are given by maps $\phi: P\to Q$ which satisfy

$$g\circ\phi=(\phi\otimes\mathrm{id})\circ f.$$

We will address End(R,M) as a category of parameterized endomorphisms. It has an obvious structure of an exact category coming from the exact category of projective modules over R (we forget about f). The following definition is taken from [9]:

DEFINITION 2.1. An object (P, f) of End(R,M) is called nilpotent if $P=\bigcup_i P^i$ where P^i is defined inductively by the formula $P^i=f^{-1}(P^{i-1}\otimes M)$ with $P^0=0$.

LEMMA 2.2. *An object* (P, f) *is nilpotent if and only if the map* $P\otimes TM\to P\otimes TM$ *induced by* $\mathrm{id}-f$ *is an isomorphism.*

Proof. This lemma is fully proved in [9, p. 160]. Shortly speaking the formula

$$\mathrm{id}+f+f^2+f^3+\cdots: P\otimes TM\to P\otimes TM$$

makes sense for nilpotent objects, where

$$f^i=(f\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id})\circ\cdots\circ(f\otimes\mathrm{id})\circ f: P\to P\otimes M^{\otimes i}$$

and describes well the inverse to the map induced by $\mathrm{id}-f$. For the implication in the opposite direction one can use 2.3 below. □

LEMMA 2.3. *Let* (P, f) *be an object of* End(R,M). *Then there is a unique submodule* P' *of* P, *such that* $(P', f|_{P'})$ *is nilpotent and* $f: P/P'\to P/P'\otimes M$ *is a monomorphism. Moreover any* $\phi: (P, f)\to(Q, g)$ *induces* $\phi': (P', f|_{P'})\to (Q', g|_{Q'})$

Proof of Lemma 2.3. Define

$$P'=\bigcup P^i,$$

where P^is are defined in 2.1. Then $f|_{P'}$ has its image in $P'\otimes M$ by definition and hence f defines the map $P/P'\to P/P'\otimes M$. Then, it is straightforward to check that all required properties are satisfied. The map $\phi'=\phi|_{P'}$. It is also easy to check that ϕ' is well defined.

In the future the quotient P/P' as above will be denoted P'' and the induced map $P''=P/P'\to P/P'\otimes M=P''\otimes M$ will be called f''.

Observe that the subcategory Nil of nilpotent objects inherits the structure of an exact category from End(R,M). Recall from [7, 1.3] the definition of a filtering cubcategory of an exact category. WARNING: in an exact category we will follow notation from [7] and will call an admissible epimorphism as deflation, an admissible monomorphism as inflation and an exact sequence as conflation.

DEFINITION 2.4. Let $\mathcal{U}$ be an exact category and let $\mathcal{A} \subset \mathcal{U}$ be an extension closed full subcategory. Then the inclusion $\mathcal{A} \subset \mathcal{U}$ is called right filtering if

(1) $\mathcal{A}$ is closed under taking admissible subobjects and admissible quotients in $\mathcal{U}$ and
(2) every map $U \to A$ from an object U of $\mathcal{U}$ to an object A of $\mathcal{A}$ factors through an object B of $\mathcal{A}$ such that the arrow $U \to B$ is a deflation:

$$\begin{tikzcd} U \arrow[r, "\forall"] \arrow[dr, two heads] & A \\ & \exists B \arrow[u] \end{tikzcd} .$$

The inclusion $\mathcal{A} \subset \mathcal{U}$ is called left filtering if $\mathcal{A}^{\mathrm{op}}$ is right filtering in $\mathcal{U}^{\mathrm{op}}$.

PROPOSITION 2.5. *Under our assumptions on R and M the category* Nil *is a full, extension closed subcategory of* $\mathrm{End}(R,M)$ *which is right and left filtering.*

Proof. By the definition Nil is a full subcategory of $\mathrm{End}(R,M)$. Every subobject and every quotient of a nilpotent object is again nilpotent. Moreover, 2.2 implies that when (P, f) is an object of $\mathrm{End}(R,M)$ and it has a nilpotent subobject with nilpotent quotient then (P, f) is nilpotent. Every arrow $\phi\colon (P, f) \to (Q, g)$ has its image $(\mathrm{im}\phi, g|_{\mathrm{im}\phi})$ in $\mathrm{End}(R,M)$ which is an admissible subobject of (Q, g) and an admissible quotient of (P, f). This implies both filtering properties. $\square$

We will follow the path described in [7]. We will call a map $\phi\colon (P, f) \to (Q, g)$ in $\mathrm{End}(R,M)$ a weak isomorphism when it is a finite composition of inflations with cokernels in Nil and deflations with kernels in Nil. Following [7, 1.16] we have a well defined quotient category $\mathcal{H} = \mathrm{End}(R,M)/\mathrm{Nil}$ obtained from $\mathrm{End}(R,M)$ by formally inverting the weak isomorphisms. Moreover, $\mathcal{H}$ has a natural exact structure in which a sequence $X \to Y \to Z$ is a conflation if it is isomorphic to the image of a conflation in $\mathrm{End}(R,M)$ under localization functor $\mathrm{End}(R,M) \to \mathcal{H}$. Obviously this localization functor is an exact functor of exact categories and we have [7, 2.1]:

THEOREM 2.6. *The sequence of exact categories* $\mathrm{Nil} \to \mathrm{End}(R, M) \to \mathcal{H}$ *induces a homotopy fibration of K-theory spaces*

$$K(\mathrm{Nil}) \to K(\mathrm{End}(R, M)) \to K(\mathcal{H}).$$

Remark 2.7. As Schlichting noticed in 1.13 the set of weak isomorphisms admits a calculus of fractions. Hence in $\mathcal{H}$ every morphism can be written as a map in End(R,M) followed by an inverse of a weak isomorphism.

Remark 2.8. Theorem 2.6 can also be obtained from Quillen's localization theorem [6, Theorem 5], by observing that Nil is a Serre subcategory of End(R,M) and $\mathcal{H}$ is equivalent to the associated quotient.

3. The Category of *TM*-Modules

Let A_P denote the right TM-module which fits into an exact sequence

$$0 \to P \otimes TM \to P \otimes TM \to A_P \to 0,$$

where (P, f) is an object of End(R,M) and the map $P \otimes TM \to P \otimes TM$ is induced by id $- f$. Obviously A_P is generated over TM by the image of the 0-grade of $P \otimes TM$ which is isomorphic to P as an R-module. Hence A_P is always finitely generated. Warning: for simplicity we do not include f into the notation for A_P assuming that it will be always clear (or not necessary to know) which map we have to take into account. Moreover, for a given TM-module A_P as above the R-module P is obviously not uniquely determined. Nevertheless we will use this notation to indicate that our module fits into the exact sequence as above. We will write $A_P = A_Q$ when P is a submodule of Q and the natural embedding $A_P \hookrightarrow A_Q$ is an isomorphism. We will use the same convention for the quotient map $P \twoheadrightarrow Q$ with the property that the natural quotient map $A_P \twoheadrightarrow A_Q$ is an isomorphism.

Denote $H(TM, E)$ the full subcategory of the category of right TM-modules consisting of objects isomorphic to A_Ps as above. We can endow the category $H(TM, E)$ with an exact structure by saying that a short sequence in it is a conflation when it comes from a conflation in End(R,M). In order to be sure that this way we do get an exact category structure we show that $H(TM, E)$ is equivalent as an exact category to the category of right TM-modules of projective dimension 1, which have resolutions of the type described above. Later we will show that $H(TM, E)$ is equivalent to $\mathcal{H}$ as an exact category. But before proving all these results we need first to show some technical lemmas.

LEMMA 3.1. *Assume that (P, f) and (Q, g) are objects of* End(R,M) *and $F\colon A_P \to A_Q$ is a morphism of TM-modules. Assume that there is homomorphism $\phi\colon P \otimes TM \to Q \otimes TM$ which covers F and is induced by a homomorphism $\Phi\colon P \to Q$. Moreover assume that g is a monomorphism. Then Φ is unique.*

Proof. Assume that $\Phi' : P \to Q$ is another homomorphism satisfying the same conditions as Φ. Then for every $p \in P$ the element $\Phi(p) - \Phi'(p)$ goes to 0 in A_Q. Assume that there is $p \in P$ such that $\Phi(p) - \Phi'(p) \neq 0$. But then $\Phi(p) - \Phi'(p) \in \mathrm{im}(\mathrm{id} - g)$. On the other hand this is possible only if g has nontrivial kernel.

LEMMA 3.2. *Let* (P, f) *and* (Q, g) *be objects of* End(R,M). *Assume that we have a commutative diagram of* TM*-modules*

$$\begin{array}{ccccc} P\otimes TM & \xrightarrow[\mathrm{id}-f]{} & P\otimes TM & \longrightarrow & A_P \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle F,} \\ Q\otimes TM & \xrightarrow[\mathrm{id}-g]{} & Q\otimes TM & \longrightarrow & A_Q \end{array}$$

where ϕ *is induced by an* R*-homomorphism* $\Phi: P \to Q$. *Moreover assume that* g *is a monomorphism. Then* $\phi = \phi'$.

Proof. The homomorphism ϕ' is uniquely determined by its values on P or in other words by its values on the 0-grade part of $P \otimes TM$. Write ϕ' restricted to P as a sum $\phi' = \phi'_0 + \phi'_1 + \cdots + \phi'_k$ where indices correspond to the gradation in $Q \otimes TM$. Then from the commutativity of the left-hand square in the diagram above we easily check that $\phi = \phi'_0$ because two elements in a graded object are the same when they are the same in every gradation. Let for a given $p \in P, s$ be the largest index such that $\phi'_s(p) \neq 0$. Then $(\mathrm{id} - g)(\phi'(p))$ has a nontrivial part in the grading $s+1$ because g is a monomorphism. On the other hand $\phi((\mathrm{id} - f)(p))$ is trivial above gradation 1. Hence $s = 0$ and the lemma is proved.

LEMMA 3.3. *Let* (P, f) *and* (Q, g) *be objects of* End(R,M). *Assume that we have given a commutative square in the category of* TM*-modules:*

$$\begin{array}{ccc} P\otimes TM & \longrightarrow & A_P \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle F,} \\ Q\otimes TM & \longrightarrow & A_Q \end{array}$$

where horizontally we have our standard projection maps. Then there exists an object (S, h) *of* End(R,M) *such that* $A_S = A_Q, h$ *is a monomorphism and we have a commutative diagram*

$$\begin{array}{ccc} P\otimes TM & \longrightarrow & A_P \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle F,} \\ S\otimes TM & \longrightarrow & A_S \end{array}$$

where ϕ' *is induced by an* R*-homomorphism* $\Phi' : P \to S$.

Proof. Let us start from some simple technical observation. Assume that (T, j) is an object in End(R,M). Let $T' = T \oplus T \otimes M \oplus \cdots \oplus T \otimes M^{\otimes k}$ for a certain k. Observe that T' is a finitely generated projective R-module. Let $H: T' \to T' \otimes M$ be a map defined in the described above decomposition of T' by the matrix

$$\begin{pmatrix} j & \mathrm{id} & 0 & \cdots & 0 \\ 0 & 0 & \mathrm{id} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \mathrm{id} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Then it is easy to check that A_T is the same as $A_{T'}$ as right TM-modules. The identification comes from the embedding $T \hookrightarrow T'$ on the first summand.

The image of the zero grade of $P \otimes TM$ is contained in $Q \oplus Q \otimes M \oplus \cdots \oplus Q \otimes M^{\otimes k}$ for a certain k. Put $\bar{S} = Q \oplus Q \otimes M \oplus \cdots \oplus Q \otimes M^{\otimes k}$ and $\bar{h} = H$ as above with g instead of j. Then we can easily define $\phi': P \to \bar{S}$ which induces a TM-homomorphism covering F. Observe that ϕ restricted to P treated as the 0-grade of $P \otimes TM$ induces an R homomorphism $\bar{\phi}: P \to \bar{S}$. Take $\phi' = \bar{\phi}$.

The obvious question which arises here is why ϕ' covers F. Obviously ϕ composed with the embedding i of Q into $\bar{S}$ at the first summand is not equal to ϕ'. But for any $p \in P$ the classes in $A_{\bar{S}}$ of $i \circ \phi(p)$ and $\phi'(p)$ are equal. This is easily seen from the way we identify A_Q and $A_{\bar{S}}$. The main point in the construction of $\bar{S}$ is to allow us to see elements from the first k-grades of $Q \otimes TM$ as elements of the first grade of $\bar{S} \otimes TM$.

At this stage we cannot guarantee that $\bar{h}$ is a monomorphism (usually it is not !). So in order to get our object (S, h) we have to follow the lines of 2.3 and put $S = \bar{S}''$ with the map h induced from $\bar{h}$.

LEMMA 3.4. *Assume that A and B are objects of $H(TM, E)$ and $F: A \to B$ is a TM-homomorphism. Then there exist objects (P, f) and (S, h) in* End(R,M) *and a map $\Phi: P \to S$ in* End(R,M) *such that A_P is isomorphic to A, A_S is isomorphic to B and under this identification the map of TM-modules induced by Φ covers F. Moreover when F is a monomorphism* (*epimorphism*) *we can get Φ of the same type.*

Proof. Both A and B are objects of $H(TM, E)$ hence the existence of (P, f) and (Q, g) such that $A_P = A$ and $A_Q = B$ is obvious from the definition. TM-modules $P \otimes TM$ and $Q \otimes TM$ are projective over TM so by general properties of projective objects we have a commutative diagram

$$\begin{array}{ccccc} P \otimes TM & \xrightarrow[\mathrm{id}-f]{} & P \otimes TM & \longrightarrow & A_P \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle F} \\ Q \otimes TM & \xrightarrow[\mathrm{id}-g]{} & Q \otimes TM & \longrightarrow & A_Q \end{array}$$

with exact rows. Now we can apply 3.3 to the right square of this diagram and get the required (S, h) for the first part of the lemma. In order to get mono- and epi- properties we have to work a little more.

Assume that F is a monomorphism. By 2.3 we can assume that f is a monomorphism either. When we know that f is mono then the quotient map $P \otimes TM \to A_P$ is mono after restriction to the 0-grade. This forces Φ to be a monomorphism.

Now assume that F is an epimorphism. If obtained Φ is not an epimorphism then call $(\bar{S}, \bar{h})$ the object of End(R,M) given by (im $\Phi, h|_\Phi$). Then one checks easily that $A_S = A_{\bar{S}}$ and $\Phi\colon P \to \bar{S}$ is an epimorphism.

Notation. In the notation of 3.4, instead of saying that the map of TM-modules induced by Φ covers F we will say in the future that "Φ covers F".

LEMMA 3.5. *Assume that $f\colon S \to P \otimes M \oplus P \otimes M^{\otimes 2} \oplus \cdots \oplus P \otimes M^{\otimes k}$ is an R-homomorphism for some natural number k. Let $\alpha\colon S \to P$ be an isomorphism. Then* coker$(\alpha - f)$ *belongs to $H(TM, E)$, when we treat here $(\alpha - f)$ as a map $S \otimes TM \to P \otimes TM$ (the obvious extension via tensoring with Id_{TM}).*

Proof of Lemma 3.5. We will proceed similarly to the proofs of previous lemmas. Assume first that $P = S$. Let then $f_i\colon P \to P \otimes M^{\otimes i}$ denote the composition of f with the projection on $P \otimes M^{\otimes i}$. It is easy to observe that the cokernel of $1 - f$ is isomorphic as a TM-module to the cokernel of $1 - F$: $Q \otimes TM \to Q \otimes TM$ where Q is an R-vector space isomorphic to $P \otimes M \oplus P \otimes M^{\otimes 2} \oplus \cdots \oplus P \otimes M^{\otimes k}$ and $F\colon Q \to Q \otimes M$ in the sum decomposition of Q as above is given by the matrix:

$$\begin{pmatrix} f_1 & \mathrm{id} & 0 & \cdots & 0 \\ f_2 & 0 & \mathrm{id} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{k-1} & 0 & \cdots & 0 & \mathrm{id} \\ f_k & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Hence coker$(1 - f)$ belongs to $H(TM, E)$. Because f was here arbitrary we can write $\alpha - f = (1 - f \circ \alpha^{-1}) \circ \alpha$ and get the general statement of 3.5.

THEOREM 3.6. *The category $H(TM, E)$ with conflations coming from* End(R,M) *(or equivalently from TM-modules) is an exact category.*

Proof. Because $H(TM, E)$ is a full subcategory of the category of TM-modules it is enough to show that the former is extension closed in the latter. Let (P, h) and (Q, g) be two objects of End(R,M). Assume that a TM-module X fits into an exact sequence

$$0 \to A_P \to X \to A_Q \to 0.$$

To get our statement we have only to show that X is in $H(TM, E)$. When we apply standard method for constructing a projective resolution of a module from projective resolutions a submodule and a quotient we immediately get that X fits into an exact sequence of TM-modules

$$0 \to Y \xrightarrow{\psi} Y \to X \to 0.$$

Moreover we know that Y is a projective TM-module and hence, under our assumptions on a ground ring, $Y = S \otimes TM$ for a certain S abstractly isomorphic to $P \oplus Q$ as R-modules. Easy diagram chase tells us that $\psi = \alpha - f$ where α and f are as in the previous lemma.

There is an exact functor Θ: End(R, M) $\to H(TM, E)$ taking (P, f) to A_P. It obviously factors through the localization functor End(R,M) $\to \mathcal{H}$. We will denote by θ the induced functor $\mathcal{H} \to H(TM, E)$. Our main result in this section is

THEOREM 3.7. *The functor θ is an equivalence of exact categories.*

Proof. We will construct an exact functor $\xi: H(TM, E) \to \mathcal{H}$. On objects we put $\xi(A_P) = (P'', f'')$, where the image was described in 2.3. In other words, we choose (P, f) which maps to A_P and kill its nilpotent part.

The morphisms part of ξ is a little more tricky, because here is the point where we really have to use $\mathcal{H}$, and not End(R,M). Let $F: A_P \to A_Q$ be a TM-homomorphism. Using 3.3 we can rise it to a map $\Phi: P'' \to \bar{Q}$ such that there is a map $\alpha: Q'' \to \bar{Q}$ with nilpotent cokernel covering identity on A_Q. Hence $\alpha^{-1} \circ \Phi$ is a well defined map in $\mathcal{H}$. This map defines $\xi(F)$. But, defining $\xi(F)$ we have made several choices so we have to show that our map $\xi(F)$ does not depend on them.

Assume that $\psi: P'' \to Q''$ is map in $\mathcal{H}$ which covers F. By the calculus of fractions we can assume that $\psi = \beta^{-1} \circ \Phi'$ where $\Phi': P'' \to S$ and $\beta: Q'' \to S$ is a composition of weak isomorphisms. We have to show that

$$\alpha^{-1} \circ \Phi = \beta^{-1} \circ \Phi'.$$

Proceeding as previously we can rise id: $A_Q \to A_Q$ to a map $\gamma^{-1} \circ \delta$, where $\delta: S \to \bar{S}$ and $\gamma: \bar{Q} \to \bar{S}$ and moreover both δ and γ are weak isomorphisms. We can, of course assume that $\bar{S}$ contains no nilpotent part. If that was not the case then we could quotient nilpotents out, as in 2.3. Notice that, accordingly to 3.1, we have equalities

$$\gamma \circ \alpha = \delta \circ \beta$$

and

$$\delta \circ \Phi' = \gamma \circ \Phi.$$

From this we easily calculate

$$\alpha^{-1} = \beta^{-1} \circ \delta^{-1} \circ \gamma$$

and eventually

$$\alpha^{-1} \circ \Phi = \beta^{-1} \circ \delta^{-1} \circ \gamma \circ \gamma^{-1} \circ \delta \circ \Phi' = \beta^{-1} \circ \Phi'$$

as we wanted. It is obvious from its definition that ξ maps identities to identities and compositions of morphisms to compositions. Similarly, it is obvious that ξ is exact because all conflations in $H(TM, E)$ are coming from conflations in $\mathcal{H}$. Hence we have proved that $H(TM, E)$ is equivalent via an exact functor to the category obtained from $\mathcal{H}$ by choosing at least one object from every isomorphism class in $\mathcal{H}$. This finishes the proof of 3.7. □

As an immediate corollary of 2.6 and 3.7 we get

COROLLARY 3.8. We have the following exact sequence of algebraic K-theory groups:

$$\begin{aligned}\cdots \to K_{i+1}(H(TM, E)) \to K_i(\mathrm{Nil}) \to K_i(\mathrm{End}(R,\ M))\\ \to K_i(H(TM, E)) \to \cdots\end{aligned}$$

4. Endomorphisms Against Localization

Now, we are in a position to access the noncommutative localizations of rings. We are going to use the theorem of Neeman and Ranicki on the K-theory of noncommutative localizations. But before stating it we need some more notation. Let A be ring and σ be a collection of maps between finitely generated projective right modules over A. In such a case there is a general construction of a ring $\sigma^{-1}A$ which is called a noncommutative localization of A with respect to σ. Let $H(A, \sigma)$ denote the exact category of σ-torsion A-modules of projective dimension one, i.e. the A-modules with a finitely generated projective A-module resolution

$$0 \to P \xrightarrow{s} Q \to T \to 0$$

where $\sigma^{-1}s : \sigma^{-1}P \longrightarrow \sigma^{-1}Q$ is an isomorphism. We have the following theorem ([5]):

THEOREM 4.1. *Let $\sigma^{-1}A$ be stably flat over A and assume that each $s \in \sigma$ is a monomorphism. Then, we have the long exact sequence of K-theory groups* (*localization sequence*)*:*

$$\cdots \to K_n(A) \to K_n(\sigma^{-1}A) \to K_{n-1}(H(A,\sigma)) \to K_{n-1}(A) \to \cdots$$

Now, we can come back to our considerations. Let σ denote the collection of TM-maps $1 - f : P \otimes TM \to P \otimes TM$ as in Section 2, where (P, f) is an object of $\mathrm{End}(R, M)$. We have:

LEMMA 4.2. *Assume that R is a field. Then:*

$$H(TM, E) \simeq H(TM, \sigma)$$

as exact categories.

We postpone the proof of 4.2 for a while. Observe that all theories K(Nil), K(End(R,M)), and $K(\sigma^{-1}TM)$ have obvious split surjective maps to $K(R)$. Moreover the middle map in the exact sequence of 3.8 is compatible with these splitings. Let $\tilde{K}$(Nil), $\tilde{K}(\mathrm{End}(R, M))$, and $\tilde{K}_{n+1}(\sigma^{-1}TM)$ denote the corresponding reduced theories. Observe that 3.8 and 4.2 yield the following theorem:

THEOREM 4.3. *Assume that R is a field. Then we have*

$$\tilde{K}_n(\mathrm{End}(R,\ M)) = \tilde{K}_{n+1}(\sigma^{-1}TM)$$

Proof. Our ring R is a field so obviously it is regular coherent in the sense of Waldhausen's Theorem 4 [9, p.138] and hence $K(TM) = K(R)$ and $\tilde{K}$(Nil) is trivial. Thus 3.8 tells us that $\tilde{K}_n(\mathrm{End}(R,\ M)) = K_n(H(TM, E))$. This latter group is the same as $K_n(H(TM,\sigma))$ by 4.2. Again, assumptions on R easily imply that $\sigma^{-1}TM$ is stably flat over TM because this latter ring is hereditary (see [1] and the introduction to [NR]). Then, we get our statement by using localization sequence 4.1.

Proof of 4.2. We have to show that categories $H(TM, E)$ and $H(TM,\sigma)$ are equivalent. There is an obvious exact embedding functor $H(TM, E) \to H(TM,\sigma)$. We have only to show that every object of $H(TM,\sigma)$ is isomorphic to some object of $H(TM, E)$. Using Lemma 3.5, we know that if $f : S \to P \otimes M \oplus P \otimes M^{\otimes 2} \oplus \cdots \oplus P \otimes M^{\otimes k}$ is an R-homomorphism for some natural number k and $\alpha\colon S \to P$ be an isomorphism then $\alpha - f$ treated as a map $S \otimes TM \to P \otimes TM$ gets inverted after localization with respect to σ.

Knowing this, while talking about $H(TM,\sigma)$ we can enlarge σ to Σ which consists of all maps $\alpha - f$ where $\alpha\colon S \to P$ is an isomorphism of a finitely generated projective R-modules and $f\colon S \to P \otimes TM_+$. The notation TM_+ stands here for the tensor algebra without the 0-grade. We will finish the proof of 4.2 if we show that any map between finitely generated TM-modules, which is invertible after localization, belongs to Σ.

With our assumption that R is a field we know that all projective objects over TM are free with the well defined rank (see for example [1, 2]). Let $f\colon X \to Y$ be a map invertible after localization with respect to σ, where $X = R^n \otimes TM$ and $Y = R^m \otimes TM$. Let X_i (Y_i) denote the i-th grade of X (Y). Let f_0 be equal to $f|_{X_0}$ composed with the projection on the 0-grade of Y. To finish the proof we have only to show that f_0 is an isomorphism.

First of all observe that $f_0 = f \otimes_{TM} \mathrm{id}_R\colon X \otimes_{TM} R = X_0 \to Y_0 = Y \otimes_{TM} R$. Moreover, the natural ring map $TM \to R$ factors through the localization map $l\colon TM \to \sigma^{-1}TM$. This follows from the universal property satisfied by l. But knowing this we can finish the proof by observing that we have an equality

$$f_0 = f \otimes_{TM} \mathrm{id}_{\sigma^{-1}TM} \otimes_{\sigma^{-1}TM} \mathrm{id}_R$$

as maps

$$X_0 = X \otimes_{TM} \sigma^{-1}TM \otimes_{\sigma^{-1}TM} R \to Y \otimes_{TM} \sigma^{-1}TM \otimes_{\sigma^{-1}TM} R = Y_0$$

and $f \otimes_{TM} \mathrm{id}_{\sigma^{-1}TM}$ is an isomorphism.

Remark 4.5. We can give a better description of $\sigma^{-1}TM$, more in the spirit of the commutative case. Let σ' be the set of all elements of TM of the form $1 - m_1 \otimes \cdots \otimes m_n$ for an arbitrary n. Then $\sigma^{-1}TM$ is isomorphic as a ring to $\sigma'^{-1}TM$. To see this it is enough to observe that any saturated class of morphisms (in the sense of [8, page 58]) between projectives, which contains σ' has to contain Σ. This is obvious because the multiplicative closure of σ' has this property.

Remark 4.6. Observe that our proof of 4.2 works well in the case when we can assume that resolutions describing elements of $H(TM,\sigma)$ consist of finitely generated free TM-modules. For example, this is the case if every finitely generated projective TM-module is stably free. But here our poor understanding of the ring TM comes into play and prevents us from getting stronger results.

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