THE UNKNOTTING NUMBER AND CLASSICAL INVARIANTS II

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Abstract. In [BF12] the authors defined a knot invariant $n_{\mathbb{R}}(K)$, for a given knot K, as the minimal size of a matrix over $\mathbb{R}[t^{\pm 1}]$, which represents the Blanchfield form. In this paper we show that $n_{\mathbb{R}}(K)$ is determined by the Levine–Tristram signatures and nullities of K. In the proof we show that the Blanchfield form for any knot K is diagonalizable over $\mathbb{R}$. We relate that result to the classification of isometric structures given by [Neu82].

1. Introduction

Let $K \subset S^3$ be a knot. We denote by $X(K) = S^3 \setminus \nu K$ its exterior. The Blanchfield form (see [Bl57]) is a linking form on $H_1(X(K), \mathbb{Z}[t^{\pm 1}])$, i.e. a non-singular hermitian pairing

$$\lambda(K) \colon H_1(X(K), \mathbb{Z}[t^{\pm 1}]) \times H_1(X(K), \mathbb{Z}[t^{\pm 1}]) \mapsto \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

In [BF12] we denoted by $n(K)$ the minimal size of a square matrix $A(t)$ over $\mathbb{Z}[t^{\pm 1}]$, which represents the Blanchfield form and such that $A(1)$ is diagonalizable over $\mathbb{Z}$. We furthermore showed that $n(K)$ is a lower bound on the unknotting number $u(K)$. (Here $u(K)$ denotes the unknotting number of a knot K, i.e. the minimal number of crossing changes needed to turn K into the unknot.) Unfortunately, $n(K)$ is, in general, hard to compute. The weaker invariant $n_{\mathbb{R}}(K)$ is the minimal size of a square matrix over $\mathbb{R}[t^{\pm 1}]$, which represents the Blanchfield form over $\mathbb{R}$. In this paper we shall study its properties.

Before we state the main result of the paper, let us recall that for a Seifert matrix $V = V_K$ of K and $z \in S^1$ we define

$$\begin{aligned} \sigma_K(z) &= \operatorname{sign}(V(1-z) + V^T(1-z^{-1})) \\ \eta_K(z) &= \operatorname{null}(V(1-z) + V^T(1-z^{-1})),\ z \neq 1 \\ \eta_K(1) &= 0. \end{aligned}$$

The invariant σ_K is called the Levine–Tristram signature and η_K is called the nullity of the knot K. We need also to introduce the following notation: given a knot K we

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set

$$\begin{array}{ll} \mu(K) & := \frac{1}{2}\left(\max\{\eta_K(z)+\sigma_K(z)\,|\,z\in S^1\}+\max\{\eta_K(z)-\sigma_K(z)\,|\,z\in S^1\}\right) \\ \eta(K) & := \max\{\eta_K(z)\,|\,z\in \mathbb{C}\setminus\{0\}\}. \end{array}$$

It is relatively straightforward to see that $\mu(K)$ and $\eta(K)$ are lower bounds on $n_{\mathbb{R}}(K)$. Our main theorem is now the following result announced in [BF12] which says that $n_{\mathbb{R}}(K)$ is in fact determined by $\mu(K)$ and $\eta(K)$.

Theorem 1.1. *For any knot K we have*

$$n_{\mathbb{R}}(K) = \max\{\mu(K), \eta(K)\}.$$

Remark.

1. Since $V(1-z)+V^t(1-z^{-1}) = (Vz-V^t)(z^{-1}-1)$ and $\Delta_K(z) = \det(Vz-V^t)$ it follows that $\eta(K)$ is determined by the values of η_K at the set of zeros of $\Delta_K(t)$. Similarly we will show (see Proposition 4.5) that $\mu(K)$ is determined by the values of σ_K and η_K at the zeros of $\Delta_K(t)$ on the unit circle.
2. We now denote by $W(\mathbb{Q}(t))$ the Witt group of hermitian non-singular forms $\mathbb{Q}(t)^r \times \mathbb{Q}(t)^r \to \mathbb{Q}(t)$. Livingston [Li11] introduced the knot invariant

$$\rho(K) := \begin{array}{c}\text{minimal size of a hermitian matrix } A(t) \\ \text{representing } (1-t)V_K + (1-t^{-1})V_K^T \text{ in } W(\mathbb{Q}(t))\end{array}$$

and showed that it is a lower bound on the 4-genus. Furthermore Livingston showed that $\rho(K)$ can be determined using the Levine-Tristram signature function. These results are related in spirit to our result that $n_{\mathbb{R}}(K)$ is a lower bound on the unknotting number and that $n_{\mathbb{R}}(K)$ can be determined using Levine-Tristram signatures and nullities.

There are two main ingredients of the proof. The first one is that the Blanchfield form over $\mathbb{R}$ can be represented by a diagonal matrix, see Section 4.1. The other one is the Decomposition Theorem in Section 3.3 which is used twice in the proof of Theorem 1.1. More precisely, we first show that the Blanchfield form can be represented by an elementary diagonal matrix E in Section 4.2. Then, we use the Decomposition Theorem to carefully rearrange terms on the diagonal of E so as to decrease its size to exactly $n_{\mathbb{R}}$. This is done in Section 4.3.

To conclude the introduction we point out, that passing from a matrix $A(t)$ representing the Blanchfield form to an elementary diagonal matrix $E(t)$ (see Section 4.2) is tightly related to the classification of isometric structures over $\mathbb{R}$ done in [Mi69] (see also [Neu82, Ne95]). For example, there is a one to one correspondence between indecomposable parts of isometric structures over $\mathbb{R}$ and polynomials occurring on the diagonal of $E(t)$. We refer to [BN12] for another applications of this classification in knot theory.

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2. Generalized Blanchfield forms

In this section we review the material from [BF12].

2.1. The Blanchfield form. Let $R \subset \mathbb{R}$ be a subring. We denote by $p(t) \mapsto \overline{p(t)} := p(t^{-1})$ the involution on $R[t^{\pm 1}]$ which is given by $t \mapsto t^{-1}$. Similarly we define an involution on $Q(t)$, where Q is a subfield of $\mathbb{R}$. We will henceforth always view $R[t^{\pm 1}]$ and $Q(t)$ as rings with involution.

Let $K \subset S^3$ be a knot. We consider the following sequence of maps:

$$\begin{array}{rcl} \Phi\colon H_1(X(K);\mathbb{Z}[t^{\pm1}]) & \to & H_1(X(K),\partial X(K);\mathbb{Z}[t^{\pm1}]) \\ & \to & \overline{H^2(X(K);\mathbb{Z}[t^{\pm1}])} \xleftarrow{\cong} \overline{H^1(X(K);\mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}])} \\ & \to & \overline{\mathrm{Hom}_{\mathbb{Z}[t^{\pm1}]}(H_1(X(K);\mathbb{Z}[t^{\pm1}]),\mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}])}. \end{array} \tag{1}$$

Here the first map is the inclusion induced map, the second map is Poincaré duality, the third map comes from the long exact sequence in cohomology corresponding to the coefficients $0 \to \mathbb{Z}[t^{\pm1}] \to \mathbb{Q}(t) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}] \to 0$, and the last map is the evaluation map. All these maps are isomorphisms and we thus obtain a non-singular hermitian pairing

$$\begin{array}{rcl} \lambda(K)\colon H_1(X(K);\mathbb{Z}[t^{\pm1}]) \times H_1(X(K);\mathbb{Z}[t^{\pm1}]) & \to & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}] \\ (a,b) & \mapsto & \Phi(a)(b), \end{array}$$

called the *Blanchfield pairing of K*.

For any subring $R \subset \mathbb{R}$ in an obvious way we can extend the definition to the pairing

$$\lambda_R(K)\colon H_1(X(K),R[t^{\pm1}]) \times H_1(X(K),R[t^{\pm1}]) \longrightarrow Q(t)/R[t^{\pm1}],$$

where Q is the field of fractions of R. We will be mostly interested in the case $R = \mathbb{R}$.

We now introduce some abstract definitions, which allow us to translate Theorem 1.1 into a purely algebraic language.

Definition. An abstract *Blanchfield form* over $R[t^{\pm1}]$ is a non-singular hermitian form

$$\lambda\colon H \times H \longrightarrow Q(t)/R[t^{\pm1}],$$

where H is a finitely generated torsion $R[t^{\pm1}]$-module such that multiplication by $t-1$ is an isomorphism.

Remark. Note that λ hermitian means that

$\lambda(p_1a_1 + p_2a_2, b) = \overline{p_1}\lambda(a_1,b) + \overline{p_2}\lambda(a_2,b)$ for any $a_1,a_2 \in H$ and any $p_1,p_2 \in R[t^{\pm1}]$,

and that

$$\lambda(a_1,a_2) = \overline{\lambda(a_2,a_1)} \text{ for any } a_1,a_2 \in H.$$

Also recall that a form is *non-singular* if the map

$$\begin{array}{rcl} H & \otimes & \overline{\mathrm{Hom}(H,Q(t)/R[t^{\pm1}])} \\ a & \mapsto & \lambda(a,b) \end{array}$$

is an isomorphism.

2.2. Definition of n_R.

Definition. (1) Given a hermitian $n \times n$ matrix A over $R[t^{\pm 1}]$ with $\det A(1) = \pm 1$ we denote by $\lambda(A)$ the pairing

$$\begin{array}{rcl} \Lambda^n / A\Lambda^n \times \Lambda^n / A\Lambda^n & \to & \Omega/\Lambda \\ (a,b) & \mapsto & \overline{a}^t A^{-1} b, \end{array}$$

where we view a, b as represented by column vectors in Λ^n. Note that $\lambda(A)$ is a hermitian, non-singular form.

(2) Let λ be an abstract Blanchfield form over $R[t^{\pm 1}]$. We say a hermitian matrix A over $R[t^{\pm 1}]$ is a *presentation matrix* for λ if
- $\lambda(A)$ is isometric to λ;
- $A(1)$ is diagonalizable over R with entries ± 1 on the diagonal.

If λ admits a presentation matrix, then we define the *Blanchfield Form dimension* $n_R(\lambda)$ to be the minimal size of a presentation matrix. If λ does not admit a presentation matrix, then we write $n_R(\lambda) := \infty$.

Remark. If $R = \mathbb{R}$, then the second condition of a presentation matrix is always satisfied.

Example. Given a knot K and a Seifert matrix V of size $2n \times 2n$ for it, let us choose a basis of $\mathbb{Z}^{2n}$ such that $V - V^t = \begin{pmatrix} 0 & \mathrm{id}_k \\ -\mathrm{id}_k & 0 \end{pmatrix}$. Then the hermitian matrix

$$A(t) = \begin{pmatrix} (1-t^{-1})^{-1}\,\mathrm{id}_k & 0 \\ 0 & \mathrm{id}_k \end{pmatrix} V \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & (1-t)\,\mathrm{id}_k \end{pmatrix} + \\ + \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & (1-t^{-1})\,\mathrm{id}_k \end{pmatrix} V^t \begin{pmatrix} (1-t)^{-1}\,\mathrm{id}_k & 0 \\ 0 & \mathrm{id}_k \end{pmatrix}. \tag{2}$$

has the property that $\lambda(A(t)) \cong \lambda(K)$. The diagonal sum of $A(t)$ with (± 1) is then a presentation matrix for the Blanchfield form for K. See [Ko89, Section 4] or [BF12, Section 2.2] for the details. It follows in particular that $n(K)$ is finite.

If $\lambda = \lambda_R(K)$ is the Blanchfield form for a knot, then $n_R(\lambda)$ shall be denoted by $n_R(K)$. Obviously $n_R(K) \leq n_{\mathbb{Z}}(K)$. The latter invariant, denoted $n(K)$ was studied in [BF12]. The authors showed, that $n(K) \leq u(K)$, where $u(K)$ is the unknotting number of K. In particular $n(K)$ is finite.

In the present paper we shall focus on $n_{\mathbb{R}}(K)$. Our approach will be purely algebraic.

2.3. Classification theorem for matrices representing Blanchfield forms.

We shall often appeal to the following result of Ranicki's [Ra81, Proposition 1.7.1]:

Proposition 2.1. *Let A and B be hermitian matrices over $R[t^{\pm 1}]$ with $\det(A(1)) = \det(B(1)) = \pm 1$. Then $\lambda(A) \cong \lambda(B)$ if and only if A and B are related by a sequence of the following three moves:*

(1) *replace C by $PC\overline{P}^t$ where P is a matrix over $R[t^{\pm 1}]$ with $\det(P)$ a unit in $R[t^{\pm 1}]$,*
(2) *replace C by the block sum $C \oplus D$ where D is a hermitian matrix over $R[t^{\pm 1}]$ with $\det(D) = \pm 1$,*
(3) *the inverse of (2).*

Given a hermitian matrix A over $R[t^{\pm 1}]$ and $z \in S^1$ we now define

$$\sigma_A(z) := \operatorname{sign}(A(z)) - \operatorname{sign}(A(1))$$

and given any $z \in \mathbb{C} \setminus \{0\}$ we define

$$\eta_A(z) := \operatorname{null}(A(z)).$$

We can now formulate the following corollary to Proposition 2.1.

Corollary 2.2. *Let A and B be hermitian matrices over $R[t^{\pm 1}]$ with $\det(A(1)) = \det(B(1)) = \pm 1$. If $\lambda(A)$ and $\lambda(B)$ are isometric, then for any $z \in S^1$ we have*

$$\sigma_A(z) = \sigma_B(z)$$

and for any $z \in \mathbb{C} \setminus \{0\}$ we have

$$\eta_A(z) = \eta_B(z).$$

Proof. The first claim concerning nullity is an immediate consequence of Proposition 2.1. We now turn to the proof of the claim regarding signatures. First suppose that $B = PA\overline{P}^t$ where P is a matrix over $R[t^{\pm 1}]$ with $\det(P) = \pm 1$. Note that $\det(P(z)) \neq 0$ for any z. We now calculate

$$\begin{aligned} \sigma_B(z) &= \operatorname{sign}(B(z)) - \operatorname{sign}(B(1)) \\ &= \operatorname{sign}(P(z)A(z)\overline{P(z)}^t) - \operatorname{sign}(P(1)A(1)P(1)^t) \\ &= \operatorname{sign}(A(z)) - \operatorname{sign}(A(1)) \\ &= \sigma_A(z). \end{aligned}$$

Now suppose that $B = A \oplus D$ where D is a hermitian matrix over $R[t^{\pm 1}]$ with $\det(D) = \pm 1$. It is well-known that for any hermitian matrix Q over $R[t^{\pm 1}]$ the map

$$\begin{aligned} S^1 &\to \mathbb{Z} \\ z &\mapsto \operatorname{sign}(Q(z)) \end{aligned}$$

is continuous on $\{z \in S^1 \mid \det(Q(z)) \neq 0\}$. Since $\det(D(z)) = \det(D)(z) = \pm 1$ for any z we see that $\operatorname{sign}(D(z)) = \operatorname{sign}(D(1))$ for any z. It now follows immediately that

$$\sigma_A(z) = \operatorname{sign}(A(z)) - \operatorname{sign}(A(1)) = \operatorname{sign}(B(z)) - \operatorname{sign}(B(1)) = \sigma_B(z).$$

The corollary now follows from Proposition 2.1. $\square$

From now on, we shall call $\sigma_\lambda(z)$ and $\eta_\lambda(z)$ the signature and the nullity of an abstract Blanchfield form λ. The following fact is proved in [BF12, Section 3.1].

Lemma 2.3. *If $\lambda = \lambda(K)$ is the Blanchfield form for a knot K, then $\sigma_\lambda(z)$ is the Levine–Tristram signature for K and $\eta_\lambda(z)$ is the nullity.*

Let $\lambda : H \times H \to \mathbb{R}(t)/\mathbb{R}[t^{\pm 1}]$ be an abstract Blanchfield form over $\mathbb{R}[t^{\pm 1}]$. Let B be a matrix over $R[t^{\pm 1}]$ which represents λ, i.e. such that $\lambda(B) \cong \lambda$. Given $z \in S^1$ we define $\sigma_\lambda(z) = \sigma_B(z)$ and given $z \in \mathbb{C} \setminus \{0\}$ we define $\eta_\lambda(z) = \eta_B(z)$. Note that by Corollary 2.2 these invariants do not depend on the choice of B. We now define

$$\begin{array}{ll} \mu(\lambda) & = \frac{1}{2}\big(\max\{\eta_\lambda(z) + \sigma_\lambda(z) \,|\, z \in S^1\} + \max\{\eta_\lambda(z) - \sigma_\lambda(z) \,|\, z \in S^1\}\big) \\ \eta(\lambda) & = \max\{\eta_\lambda(z) \,|\, z \in \mathbb{C} \setminus \{0\}\}. \end{array}$$

We can now rephrase Theorem 1.1 in a purely algebraic language. The following is now our main technical theorem.

Theorem 2.4. *Let $\lambda : H \times H \to \mathbb{R}(t)/\mathbb{R}[t^{\pm 1}]$ be a Blanchfield form over $\mathbb{R}[t^{\pm 1}]$ such that multiplication by $t + 1$ is an isomorphism of H. Then*

$$n_{\mathbb{R}}(\lambda) = \max\{\mu(\lambda), \eta(\lambda)\}.$$

The proof of this theorem will require all of Section 4. Assuming Theorem 2.4 we can now easily provide the proof of Theorem 1.1.

Proof of Theorem 1.1. It is well-known that the Alexander polynomial Δ_K of K satisfies $\Delta_K(1) = \pm 1$. Since $\Delta_K(-1) \equiv \Delta_K(1) = 1 \bmod 2$ we know that $\Delta_K(-1) \neq 0$. It now follows easily that multiplication by $t - 1$ and $t + 1$ are isomorphisms of the Alexander module $H_1(X(K); \mathbb{R}[t^{\pm 1}])$. By Lemma 2.3 we infer that $\mu(\lambda) = \mu(K)$ and $\eta(\lambda) = \eta(K)$. Hence $n_{\mathbb{R}}(\lambda) = n_{\mathbb{R}}(K)$. The theorem now follows immediately from Theorem 2.4. $\square$

3. Technical lemmas

From now on we write

$$\Lambda := \mathbb{R}[t, t^{-1}] \text{ and } \Omega := \mathbb{R}(t).$$

Moreover, we write S^1_+ for the set of all points on S^1 with non-negative imaginary part. Let $z_1, z_2 \in S^1_+$. We can write $z_i = e^{2\pi i t_i}$ for a unique $t_i \in [0, \pi]$. We write $z_1 > z_2$ if $t_1 > t_2$. Given $a, b \in S^1_+$ we use the usual interval notation to define subsets $[a, b), (a, b)$ etc. of S^1_+. We will frequently make use of the fact that for a hermitian matrix $A(t)$ over Λ we have $A(z) = A(z^{-1})^t$. In particular it suffices to consider $\sigma_z(A)$ and $\eta_z(A)$ on S^1_+ to determine $\mu(A)$ and $\eta(A)$.

3.1. **Palindromic polynomials and elementary palindromic polynomials.** Let us recall the well known definition

Definition. An element $p \in \Lambda$ is called *palindromic* if $p(t) = p(t^{-1})$ as polynomials.

In particular, if p is palindromic and $z \in S^1$, then $p(z) = p(\overline{z})$. Among other things it follows that p can not have a zero of an odd order at ± 1. The next result will be used in the proof of Lemma 3.5 below.

Lemma 3.1. *Palindromic polynomials form a dense subset in the space of all real-valued continuous functions on S^1 such that $f(z) = f(\overline{z})$ for all $z \in S^1$.*

In the lemma we mean 'dense' with respect to the supremum norm.

Proof. Each such function f is determined by its values on S^1_+. On S^1_+, the palindromic polynomials form a real algebra which separates points (note that they do not separate points on the whole S^1). By the Weierstrass theorem (see e.g. [Rud76, Theorem 7.32]), for any real-valued continuous function f, there exists a sequence p_n of palindromic polynomials converging to f uniformly on S^1_+. As $f(\overline{z}) = f(z)$ and $p_n(\overline{z}) = p_n(z)$ for all $z \in S^1$, this convergence extends to the convergence on S^1. $\square$

We shall use the following obvious definition.

Definition. Let g be a palindromic polynomial and let $z \in S^1$. We say g *changes sign at z* if in any neighborhood of z on S^1 the function g has both positive and negative values.

We will make use of the following terminology. We write

$$\Xi := \{\xi \in \mathbb{C} \mid \operatorname{Im} \xi \geq 0 \text{ and } |\xi| \leq 1\}.$$

Definition. Given $\xi \in \Xi$ we now define

$$B_\xi(t) = \begin{cases} t - (\xi + \overline{\xi}) + t^{-1} & \text{if } |\xi| = 1, \\ (t-\xi)(t-\overline{\xi})(1-t^{-1}\overline{\xi}^{-1})(1-t^{-1}\xi^{-1}) & \text{if } |\xi| < 1 \text{ and } \xi \notin \mathbb{R}, \\ (t-\xi)(1-\xi^{-1}t^{-1}) & \text{if } \xi \in \mathbb{R} \setminus \pm 1. \end{cases} \tag{3}$$

The polynomials are called the *elementary palindromic* polynomials.

We conclude with the following observations:

(1) For any $\xi \in \Xi$ the polynomial $B_\xi(t)$ is a real, palindromic and monic polynomial.
(2) For any ξ we have $B_\xi(1) > 0$, furthermore if $|\xi| < 1$, then B_ξ has no zeros on S^1, i.e. B_ξ is positive on S^1.
(3) Given any $z \in \mathbb{C} \setminus \{0\}$ there exists a unique $\xi \in \Xi$ such that z is a zero of $B_\xi(t)$. Furthermore $B_\xi(t)$ is the unique real, palindromic, monic polynomial of minimal degree which has a zero at z.
(4) If $\xi = \pm 1$, $B_\xi(t)$ is not indecomposable in Λ.

(5) Any palindromic polynomial in Λ factors uniquely as the product of elementary palindromic polynomials and a constant in $\mathbb{R}$.

3.2. **First results.**

Lemma 3.2. *Let $P \in \Lambda$ be palindromic. Then there exists $U \in \Lambda$ with $P = U\overline{U}$ if and only if $P(z) \geq 0$ for every $z \in S^1_+$.*

Proof. If $P = U\overline{U}$, then for each $z \in S^1_+$ we obviously have $P \geq 0$. So assume that $P(z) \geq 0$ on S^1_+. Note that by the above discussion this implies that $P(z) \geq 0$ on S^1. We proceed by induction on the number of zeros of P. If P has no zeros, then P is constant and there is nothing to prove.

Let θ be a zero of $P(t)$. Since $P(t) = P(t^{-1})$ we see that θ^{-1} is also a zero of P. Furthermore, since $P(t)$ is a real polynomial we see that if μ is a zero, then $\overline{\mu}$ is also a zero. Thus, if θ is a zero, then $\theta, \overline{\theta}, \theta^{-1}, \overline{\theta}^{-1}$ are all zeros.

Assume now that there exists $\theta \in \mathbb{C} \setminus S^1$, $\theta \neq 0$, such that $P(\theta) = 0$. Let $\xi \in \Xi$ be the unique element such that θ is a zero of the elementary palindromic polynomial $B_\xi(t)$. Note that $B_\xi(t)$ divides $P(t)$. Furthermore note that $P_2 = \frac{P(t)}{B_\xi(t)}$ has a smaller number of zeros and is positive on S^1. By induction we have $P_2 = U_2\overline{U_2}$. The polynomial $U = (t-\theta)(t-\overline{\theta})U_2$ then satisfies $P = U\overline{U}$.

Now let $\theta \in S^1 \setminus \{\pm 1\}$ be a zero of $P(t)$. As $P \geq 0$ on S^1, the order of the root of P at θ must be even. Let $\xi \in \Xi$ be the unique element such that θ is a zero of $B_\xi(t)$. As B_ξ has only simple roots, $B_\xi(t)^2$ divides $P(t)$. As above note that $P_2 = \frac{P(t)}{B_\xi(t)^2}$ has a smaller number of zeros and is positive on S^1. We can thus again appeal to the induction hypothesis.

Finally, of $\theta = \pm 1$, then P is divisible by $(t-\theta)^2$, for the same reason. We write $P_2 = \frac{P(t)}{B_\theta}$ and by induction we have $P_2 = U_2\overline{U_2}$. Then we put $U = (t-\theta)U_2$. $\square$

Lemma 3.3. *Let $A, B \in \Lambda$ be palindromic coprime polynomials. If for every $z \in S^1_+$, either $A(z)$ or $B(z)$ is positive, then there exist palindromic P and Q in Λ such that $PA + QB = 1$ and such that $P(z)$ and $Q(z)$ are positive for any $z \in S^1$.*

Proof of Lemma 3.3. Note that if for every $z \in S^1_+$, either $A(z)$ or $B(z)$ is positive, then the same conclusion holds for any point on S^1 since A and B are assumed to be palindromic.

The idea behind the proof is that if a, b are real numbers and at least one of them is positive, then we can obviously find real numbers $p, q > 0$ such that $pa + qb = 1$. The statement of the lemma is that this can be done for palindromic coprime polynomials A and B and any $z \in S^1$ by palindromic polynomials P and Q.

As A and B are coprime, there exist P' and Q' in Λ such that $P'A + Q'B = 1$ by Euclid's algorithm. We now define $\widetilde{P} := \frac{1}{2}(P' + \overline{P'})$ and $\widetilde{Q} := \frac{1}{2}(Q' + \overline{Q'})$. Note that $\widetilde{P}$ and $\widetilde{Q}$ are palindromic and satisfy the equality $\widetilde{P}A + \widetilde{Q}B = 1$ since A, B are palindromic.

The functions $\widetilde{P}$ and $\widetilde{Q}$ are not necessarily positive on S^1. Our goal is to find a palindromic Laurent polynomial $\gamma(t)$ such that $\widetilde{P}-\gamma B\geq 0$ and $\widetilde{Q}+\gamma A\geq 0$ on S^1. To this end, let us define two functions $\gamma_{\max},\gamma_{\min}\colon S^1\to\mathbb{R}\cup\{\infty,-\infty\}$ as follows:

$$\gamma_{\max}(z)=\begin{cases}\frac{\widetilde{P}(z)}{B(z)} & \text{if } B(z)>0\\ \infty & \text{if } B(z)\leq 0\end{cases}\quad\text{and } \gamma_{\min}(z)=\begin{cases}\frac{-\widetilde{Q}(z)}{A(z)} & \text{if } A(z)>0\\ -\infty & \text{if } A(z)\leq 0.\end{cases}\tag{4}$$

We also consider the usual ordering on the set $\mathbb{R}\cup\{-\infty,\infty\}$.

Lemma 3.4. *The functions $\gamma_{\min}$ and $\gamma_{\max}$ have the following properties:*

(a) $\gamma_{\min}(\overline{z})=\gamma_{\min}(z)$ *and* $\gamma_{\max}(\overline{z})=\gamma_{\max}(z)$ *for all* $z\in S^1$.

(b) *Let* $z\in S^1$. *If* $\gamma\in[\gamma_{\min}(z),\gamma_{\max}(z)]$, *then*

$$\widetilde{P}(z)-\gamma B(z)\geq 0 \textit{ and } \widetilde{Q}(z)+\gamma A(z)\geq 0;$$

(c) *for all* $z\in S^1$, $\gamma_{\min}(z)<\gamma_{\max}(z)$;

(d) *the functions*

$$\begin{array}{rcl} S^1 & \to & [-\pi/2,\pi/2]\\ z & \mapsto & \arctan(\gamma_{max}(z)) \textit{ and}\\ z & \mapsto & \arctan(\gamma_{min}(z))\end{array}$$

are continuous (here we define $\arctan(\infty)=\pi/2$ *and* $\arctan(-\infty)=-\pi/2$*).*

Proof. The point (a) is obvious since $\widetilde{P}(\overline{z})=\widetilde{P}(z)$, and the same holds for A, B and Q, as all these functions are palindromic polynomials.

Let $z\in S^1$ and let $\gamma\in[\gamma_{\min}(z),\gamma_{\max}(z)]$. First suppose that $A(z)>0$ and $B(z)>0$. Then it follows from the definitions that

$$\begin{array}{rcccl}\widetilde{P}(z)-\gamma B(z) & \geq & \widetilde{P}(z)-\gamma_{max}(z)B(z) & = & 0\\ \widetilde{Q}(z)+\gamma A(z) & \geq & \widetilde{Q}(z)+\gamma_{min}(z)A(z) & = & 0.\end{array}$$

Now suppose that $A(z)=0$. Note that this implies that $B(z)>0$ by our assumption on A and B. In this case we see that $\widetilde{P}(z)-\gamma B(z)\geq 0$ as above. Furthermore, we have

$$\widetilde{Q}(z)+\gamma A(z)=\widetilde{Q}(z)=\frac{1}{B(z)}>0.$$

Now suppose that $A(z)<0$. As above, $B(z)>0$ and $\widetilde{P}(z)-\gamma B(z)\geq 0$. Furthermore,

$$\widetilde{Q}(z)+\gamma A(z)\geq\widetilde{Q}(z)+\gamma_{max}(z)A(z)=$$
$$=\widetilde{Q}(z)+\frac{\widetilde{P}(z)}{B(z)}A(z)=\frac{\widetilde{Q}(z)B(z)+\widetilde{P}(z)A(z)}{B(z)}=\frac{1}{B(z)}>0.$$

Similarly we deal with the case that $B(z)\leq 0$ and $A(z)>0$. This proves (b).

We now turn to the proof of (c). Clearly we only have to consider the case that $A(z) > 0$ and $B(z) > 0$. In that case we have

$$\gamma_{max}(z) = \frac{\widetilde{P}(z)}{B(z)} = -\frac{\widetilde{Q}(z)}{A(z)} + \frac{1}{A(z)B(z)} > -\frac{\widetilde{Q}(z)}{A(z)} = \gamma_{min}(z).$$

Finally we turn to the proof of (d). We will first show that $z \mapsto \arctan(\gamma_{max}(z))$ is continuous. Clearly we only have to show continuity for $z \in S^1$ such that $B(z) = 0$. We will show the following: if z_i is a sequence of points on S^1 with $\lim_{i\to\infty} z_i = z$ such that $B(z_i) > 0$ for any i, then $\lim_{i\to\infty} \frac{\widetilde{P}(z_i)}{B(z_i)} = \infty$. Indeed, since $B(z) = 0$ we have $A(z) > 0$ by our assumption. Now $\widetilde{Q}$ is bounded on S^1, in particular from $\widetilde{P}A + \widetilde{Q}B = 1$ we deduce that

$$\lim_{i\to\infty} \widetilde{P}(z_i) = \lim_{i\to\infty} \frac{1 - \widetilde{Q}(z_i)B(z_i)}{A(z_i)} = \frac{1}{A(z)} > 0.$$

It now follows that $\lim_{i\to\infty} \frac{\widetilde{P}(z_i)}{B(z_i)} = \infty$ as desired.

This completes the proof that $z \mapsto \arctan(\gamma_{max}(z))$ is continuous. Similarly one can prove that $z \mapsto \arctan(\gamma_{min}(z))$ is continuous. □

Lemma 3.5. *There exists a palindromic polynomial γ such that $\gamma_{min}(z) < \gamma(z) < \gamma_{max}(z)$ for any $z \in S^1$.*

The proof of Lemma 3.5 might be shortened, but one would have to consider continuous functions with values in $\mathbb{R} \cup \{\pm\infty\}$. The trick with arctan function allows us to avoid such functions.

Proof. We write $f_1 = \arctan(\gamma_{min}(z))$ and $f_2 = \arctan(\gamma_{max}(z))$. By Lemma 3.4 we know that f_1 and f_2 are continuous functions on S^1 with $f_1(z) < f_2(z)$ for all $z \in S^1$. We can now pick continuous functions $g_1, g_2 : S^1 \to (-\pi/2, \pi/2)$ (note the open intervals), such that $f_1(z) < g_1(z) < g_2(z) < f_2(z)$ for any $z \in S^1$. We can assume that $g_1(\overline{z}) = g_1(z)$ and $g_2(\overline{z}) = g_2(z)$, as f_1 and f_2 have this property by Lemma 3.4(a). We have the inequality

$$\gamma_{min}(z) < \tan(g_1(z)) < \tan(g_2(z)) < \gamma_{max}(z).$$

Let now

$$c = \inf\left\{\tan(g_2(z)) - \tan(g_1(z)) \colon z \in S^1\right\}.$$

We have $c \geq 0$. But since S^1 is compact and the functions $\tan g_1$ and $\tan g_2$ are continuous, we have in fact $c > 0$. By Lemma 3.1 we can find a palindromic polynomial h which satisfies

$$\left|\gamma(z) - \frac{1}{2}\big(\tan(g_2(z)) + \tan(g_1(z))\big)\right| < \frac{c}{2}$$

for any $z \in S^1$. It clearly follows that for any $z \in S^1$ we have the desired inequalities

$$\gamma_{min}(z) < \gamma(z) < \gamma_{max}(z).$$

□

We can now conclude the proof of Lemma 3.3. By Lemma 3.5 we can find a palindromic polynomial γ such that $\gamma_{min}(z) < \gamma(z) < \gamma_{max}(z)$ for all $z \in S^1$. Then $P = \widetilde{P} - \gamma B$ and $Q = \widetilde{Q} + \gamma A$ satisfy $P > 0$ and $Q > 0$ on S^1 by Lemma 3.4 and they satisfy

$$P(z)A(z) + Q(z)B(z) = 1 \text{ for all } z \in S^1. \tag{5}$$

But both sides of (5) are Laurent polynomials on $\mathbb{C} \setminus \{0\}$ which agree on infinitely many points. Hence the equality (5) holds on $\mathbb{C} \setminus \{0\}$. So it must also hold in Λ. □

3.3. **The Decomposition Theorem.** In the following recall that for a palindromic $p(t) \in \Lambda$ and any $z \in S^1$ we have $p(z) \in \mathbb{R}$.

Theorem 3.6. *Assume that $A(t)$ and $B(t)$ are two coprime palindromic Laurent polynomials in Λ. Suppose there exists $\varepsilon \in \{-1, 1\}$ such that for all $z \in S^1_+$, at least one of the numbers $\varepsilon A(z) > 0$ or $\varepsilon B(z) > 0$ is strictly positive, then*

$$\lambda \begin{pmatrix} A(t) & 0 \\ 0 & B(t) \end{pmatrix} \cong \lambda(\varepsilon AB)$$

as Blanchfield forms over Λ.

We shall prove the theorem by combining Lemmas 3.2 and 3.3 with the following lemma.

Lemma 3.7. *For $\varepsilon = \pm 1$, we have*

$$\lambda \begin{pmatrix} A(t) & 0 \\ 0 & B(t) \end{pmatrix} \cong \lambda(\varepsilon AB)$$

if there exist $U, V \in \Lambda$ such that

$$U\overline{U} \cdot A + V\overline{V} \cdot B = \varepsilon. \tag{6}$$

Proof of Lemma 3.7. Suppose that there exist $U, V \in \Lambda$ which satisfy (6). Then write $X := \overline{V}B$ and $Y := -\overline{U}A$ and take $N = \begin{pmatrix} X & Y \\ U & V \end{pmatrix}$. Note that $\det(N) = \epsilon$. Then one calculates that

$$N \begin{pmatrix} A(t) & 0 \\ 0 & B(t) \end{pmatrix} \overline{N}^t = \begin{pmatrix} \varepsilon AB & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

The lemma now follows from Proposition 2.1. □

We can now prove Theorem 3.6.

Proof of Theorem 3.6. Suppose there exists $\varepsilon \in \{-1, 1\}$ such that for all $z \in S^1_+$, $\varepsilon A(z) > 0$ or $\varepsilon B(z) > 0$. By Lemma 3.3 there exist palindromic P and Q in Λ such that $PA + QB = \varepsilon$ and such that $P(z)$ and $Q(z)$ are positive for any $z \in S^1$. By Lemma 3.2 there exist $U \in \Lambda$ and $V \in \Lambda$ with $P = U\overline{U}$ and $Q = V\overline{V}$. The theorem now follows from Lemma 3.7. □

Remark. One easily sees from Theorem 3.6 that if $\xi \in \Xi$ and $|\xi| < 1$, then for any $n \geq 1$, $\lambda(B_\xi^n) \cong \lambda(-B_\xi^n)$. On the other hand, if $|\xi| = 1$ then $\lambda(B_\xi^n)$ and $\lambda(-B_\xi^n)$ are different. This is a counterpart of the known fact from the classification of isometric structures (see [Neu82, Proposition 3.1] or [Ne95, Section 2], compare also [Mi69]): for any $\lambda \in S^1 \setminus \{1\}$ and for any $n \geq 1$ there exist exactly two distinct isometric structures such that the corresponding monodromy operator is the single Jordan block of size n and eigenvalue λ. For any $\lambda \in \mathbb{C} \setminus \{S^1 \cup 0\}$, and any $n \geq 1$, there exists a unique isometric structure such that the corresponding monodromy operator is a sum of two Jordan blocks of size n: one with eigenvalue λ and the other one with eigenvalue $1/\lambda$.

4. The proof of Theorem 2.4

In this section we will prove Theorem 2.4. First let us prove that $\max\{\mu(\lambda), \eta(\lambda)\} \leq n_{\mathbb{R}}(\lambda)$. The argument is very easy and resembles [BF12, Theorem 4.1].

Let λ be a Blanchfield form over Λ and let B be a hermitian matrix over Λ of size $n := n_{\mathbb{R}}(\lambda)$ which represents λ. Of course for any $z \in \mathbb{C} \setminus \{0\}$ we have $\operatorname{null}(B(z)) \leq n_R(\lambda)$, in particular

$$\eta(\lambda) \leq n_{\mathbb{R}}(\lambda).$$

To show that $\mu(\lambda) \leq n_{\mathbb{R}}(\lambda)$ let us assume that $\operatorname{sign}(B(1)) = a \in [-n, n]$. As for any z, $\pm \operatorname{sign}(B(z)) + \operatorname{null}(B(z)) \in [-n, n]$, it follows that $\sigma_\lambda(z) + \eta_\lambda(z) \in [-n - a, n - a]$ and $\eta_\lambda(z) - s_\lambda(z) \in [-n + a, n + a]$, we infer that $\mu_\lambda(z) \leq \frac{1}{2}((n - a) + (n + a)) = n$.

In the remainder of Section 4 we will show that $n_{\mathbb{R}}(\lambda) \leq \max\{\mu(\lambda), \eta(\lambda)\}$.

4.1. **Diagonalizing Blanchfield forms.**

Recall that $\Lambda = \mathbb{R}[t^{\pm 1}]$ and $\Omega = \mathbb{R}(t)$. We say that a Blanchfield form λ over Λ is *diagonalizable* if λ can be represented by a diagonal matrix over Λ. The following is the main result of this section.

Proposition 4.1. *Let $\lambda : H \times H \to \Omega/\Lambda$ be a Blanchfield form over Λ such that multiplication by $t + 1$ is an isomorphism. Then λ is diagonalizable.*

In order to prove the proposition we will first consider the following special case.

Proposition 4.2. *Let $p \in \Lambda$ be a palindromic polynomial, irreducible over $\mathbb{R}$. Let $H = \Lambda/p^n\Lambda$ for some n and let $\lambda : H \times H \to \Omega/\Lambda$ be a Blanchfield form over H. Then λ is diagonalizable.*

Proof. Note that $\deg(p) = 2$ since p is an irreducible palindromic polynomial over $\mathbb{R}$. It follows that the zeros of p lie on $S^1 \setminus \{\pm 1\}$ (see Section 3.1). Throughout this proof let w be the (unique) zero of p which lies in S^1_+. Since $p(1) \neq 0$ we can multiply $p \in \Lambda$ by the sign of $p(1)$ and we can therefore, without loss of generality, assume that $p(1) > 0$.

Claim. Let q be a palindromic polynomial coprime to p. Then there exists $g \in \Lambda$ and $\epsilon \in \{-1, 1\}$ such that $q = \epsilon g\overline{g} \in \Lambda/p^n\Lambda$.

We first show that the claim implies the proposition. Note that λ takes values in $p^{-n}\Lambda/\Lambda$. We pick a representative $q' \in \Lambda$ of $p^n \cdot \lambda(1,1) \in \Lambda/p^n\Lambda$. Since λ is hermitian we have $q' \equiv \overline{q}' \bmod p^n\Lambda$. We now let $q = \frac{1}{2}(q' + \overline{q}')$. Note that q is palindromic and $q \in \Lambda$ is a representative of $\lambda(1,1)p^n \in \Lambda/p^n\Lambda$. Since λ is non-singular it follows that q is coprime to p. By the claim there exists $g \in \Lambda$ and $\epsilon \in \{-1,1\}$ such that $q = \epsilon g\overline{g} \in \Lambda/p^n\Lambda$. The map $\Lambda/p^n\Lambda \to \Lambda/p^n\Lambda$ which is given by multiplication by g is easily seen to define an isometry from $\lambda(\epsilon p^n)$ to λ. (Here recall that $\lambda(\epsilon p^n)$ is the Blanchfield form defined by the 1×1-matrix (ϵp^n).) In particular λ is represented by the 1×1-matrix ϵp^n.

We now turn to the proof of the claim. Given $g \in \Lambda$ we define

$$s(g) := \#\{z \in S^1_+ \,|\, g \text{ changes sign at } z\}.$$

We will prove the claim by induction on $s(q)$. If $s(q) = 0$, then we denote by ϵ the sign of $q(1)$. It follows that $q\epsilon$ is non-negative on S^1, hence by Lemma 3.2 there exists $g \in \Lambda$ with $q\epsilon = g\overline{g}$.

Now suppose the conclusion of the claim holds for any palindromic q with $s(q) < s$. Let q be a palindromic polynomial in Λ with $s(q) = s$. Let $v \in S^1_+$ be a point where q changes sign. Recall that we denote by w the unique zero of p which lies in S^1_+. Note that $v \neq w$ since we assumed that p and q are coprime.

First consider the case that $v < w$. Let $f \in \Lambda$ be an irreducible polynomial such that $f(v) = 0$. Note that f is palindromic and $f(1) \neq 0$. We can thus arrange that $f(1) < 0$. Note that p changes sign on S^1_+ precisely at w and f changes sign precisely at v. We thus see that for any $z \in S^1_+$ with $z < w$ we have $p(w) > 0$ and for any $z \in S^1_+$ with $z > v$ we have $f(z) > 0$. It follows that for any $z \in S^1_+$ either f or p is positive. Note that f and p are coprime, we can thus apply Lemma 3.3 to conclude that there exist palindromic x and y in Λ such that $p^n x + fy = 1$ and such that $x(z)$ and $y(z)$ are positive for any $z \in S^1$.

We now define $q' := qfy$. Note that

$$q = q(p^n x + fy) = qfy = q' \in \Lambda/p^n\Lambda.$$

Also note that q and f change sign at v. It follows that qf does not change sign at v. Since z is the only zero of f in S^1_+ and since y is positive for any $z \in S^1$ it follows that $s(q') = s(q) - 1$. By our induction hypothesis we can thus write

$$q = q' = \epsilon g \cdot \overline{g} \in \Lambda/p^n\Lambda$$

for some $g \in \Lambda$.

Now consider the case that $v > w$. Let $f \in \Lambda$ be an irreducible polynomial such that $f(w) = 0$ (note that v can not be equal to -1, because q changes sign at v and q is palindromic). Note that f is palindromic and $f(1) \neq 0$. We can thus arrange that $f(1) > 0$. As above we see that for any point on S^1_+ either f or p is negative. By Lemma 3.3 there exist palindromic x and y in Λ such that $p^n x + fy = 1$ and such that $x(z)$ and $y(z)$ are negative for any $z \in S^1$. The proof now proceeds as in the previous case. $\square$

Let $p \in \Lambda$ be an irreducible polynomial. In the following we say that a Λ-module H is *p-primary* if any $x \in H$ is annihilated by a sufficiently high power of p. If H is p-primary, then given $h \in H$ we write $l(h) := \min\{k \in \mathbb{N} \,|\, p^k h = 0\}$ and we write $l(H) := \max\{l(h) \,|\, h \in H\}$. We also denote by $s(H)$ the minimal number of generators of H. We will later need the following lemma.

Lemma 4.3. *Let $p \in \Lambda$ be an irreducible polynomial. Let H be a finitely generated p-primary module. Let $v \in H$ with $l(v) = l(H)$. Then there exists a direct sum decomposition*

$$H = H' \oplus v\Lambda/p^{l(v)}$$

with $s(H') = s(H) - 1$.

Proof. We write $l = l(H)$ and $s = s(H)$. Since Λ is a PID we can apply the classification theorem for finitely generated Λ-modules (see e.g. [La02, Theorems 7.3 and 7.5]) to find $e_1, \dots, e_k \in \Lambda$ and a submodule $H'' \subset H$ with the following properties:

(1) $l(e_i) = l$ for $i = 1, \dots, k$,
(2) $H = H'' \oplus \bigoplus_{i=1}^k e_i\Lambda/p^l\Lambda$,
(3) $s(H'') = s - k$,
(4) for any $w \in H''$ we have $l(w) < l$.

Now we can write $v = v'' + \sum_{i=1}^k a_i e_i$ for some $v'' \in H''$ and $a_i \in \Lambda/p^l\Lambda$. Note that $l(v) = l$ implies that there exists at least one a_j which is coprime to p. We now pick $x \in \Lambda$ with $xa_j = 1 \in \Lambda/p^l$. It is now clear that

$$H = H'' \oplus \bigoplus_{i \neq j} e_i\Lambda/p^l\Lambda \oplus xv\Lambda/p^l\Lambda.$$

But since $xv\Lambda/p^l\Lambda = v\Lambda/p^l\Lambda$ we get the desired decomposition. Furthermore, it is clear that

$$s\big(H'' \oplus \bigoplus_{i \neq j} e_i\Lambda/p^l\Lambda\big) = s - 1.$$

$\square$

Lemma 4.4. *Let $p \in \Lambda$ be a non-zero irreducible palindromic polynomial. Let H be a p-primary module and let $\lambda : H \times H \to \Omega/\Lambda$ be a Blanchfield form. Then λ is diagonalizable.*

Proof. We will prove the lemma by induction on the number s of generators of H. We will use an algorithm, which is a version of the Gram–Schmidt orthogonalization procedure from linear algebra. If $s(H) = 0$, then clearly there is nothing to prove. Now let $\lambda : H \times H \to \Omega/\Lambda$ be a Blanchfield form over a p-primary module H which is generated by $s > 0$ elements. We write $l = l(H)$. Given $f \in p^{-n}\Lambda/\Lambda$ we also write $l(f) := \min\{k \in \mathbb{N} \,|\, p^k f = 0 \in p^{-n}\Lambda/\Lambda\}$.

Claim. There exists $v \in H$ with $l(\lambda(v,v)) = l$.

To prove the claim, we pick $v \in H$ with $l(v) = l$. It follows from Lemma 4.3 that v generates a subsummand of H, in particular we can find a Λ-homomorphism $\varphi : H \to p^{-l}\Lambda/\Lambda$ such that $\varphi(v) = p^{-l} \in p^{-l}\Lambda/\Lambda$. Since λ is non-singular we can thus find $w \in H$ with $\lambda(w,v) = p^{-l} \in p^{-l}\Lambda/\Lambda$. If $l(\lambda(v,v)) = l$ or if $l(\lambda(w,w)) = l$, then we are done. Otherwise we consider $\lambda(v+w, v+w)$ which equals

$$\begin{array}{rcl} \lambda(v+w,v+w) & = & \lambda(v,v) + \lambda(w,w) + \lambda(v,w) + \overline{\lambda(v,w)} \\ & = & \lambda(v,v) + \lambda(w,w) + p^{-l} + \overline{p}^{-l} \\ & = & \lambda(v,v) + \lambda(w,w) + 2p^{-l} \in p^{-l}\Lambda/\Lambda. \end{array}$$

If $l(\lambda(v,v)) < l$ and if $l(\lambda(v,w)) < l$, then one can now easily see that $l(\lambda(v+w,v+w)) = l$, i.e. $v+w$ has the desired property. This concludes the proof of the claim.

Given the claim, let us pick $v \in H$ with $l(\lambda(v,v)) = l$, i.e. we can write $\lambda(v,v) = xp^{-l}\Lambda/\Lambda$ for some $x \in \Lambda$ coprime to p. We can in particular find $y \in \Lambda$ such that $yx \equiv 1 \bmod p^l$.

By Lemma 4.3 we can find $v_1, \dots, v_{s-1}$ and $l_1, \dots, l_{s-1}$ such that

$$H = v\Lambda/p^l\Lambda \oplus \bigoplus_{i=1}^{s-1} v_i\Lambda/p^{l_i}\Lambda.$$

For $i = 1, \dots, s-1$ we now define

$$w_i := v_i - y\lambda(v, v_i)v.$$

It follows immediately that $\lambda(v, w_i) = 0 \in \Omega/\Lambda$. It follows that H splits as the orthogonal sum of the submodule generated by v and the submodule generated by $w_1, \dots, w_{s-1}$.

By Proposition 4.2 the former is diagonalizable, and by our induction hypothesis the latter is also diagonalizable. It follows that λ is diagonalizable.

$\square$

In the following we write $p \doteq q \in R[t^{\pm 1}]$ if p and q differ by multiplication by a unit in $R[t^{\pm 1}]$. We say that p and q are *equivalent* if $q \doteq p$. We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1. We denote by $\mathcal{P}$ the set of equivalence classes of all non-constant irreducible elements in Λ which are not equivalent to $1+t$. Let $p \in \mathcal{P}$ with $p(t^{-1}) \doteq p(t)$. Since p is irreducible and since we excluded $1+t$ it follows easily that p is in fact represented by a palindromic polynomial.

Note that $\mathcal{P}$ inherits an involution $p \mapsto \overline{p}$ coming from the involution on Λ. We write $\mathcal{P}' = \{p \in \mathcal{P} \,|\, p \text{ palindromic}\}$ and we define $\mathcal{P}'' := \{\{p, \overline{p}\} \,|\, p \text{ not palindromic}\}$. In our notation we will for the most part ignore the distinction between an element $p \in \Lambda$ and the element it represents in $\mathcal{P}, \mathcal{P}'$ and $\mathcal{P}''$.

Given $p \in \mathcal{P}$ we denote by

$$H_p := \{v \in H \,|\, p^i v = 0 \text{ for some } i \in \mathbb{N}\}$$

the p-primary part of H. Note that $H_{1+t} = 0$ by our assumption on H.

Claim. Suppose that p and $\overline{q}$ are non-equivalent irreducible polynomials in Λ. Then $\lambda(a,b) = 0$ for any $a \in H_p$ and $b \in H_q$.

Suppose that p and $\overline{q}$ are not equivalent. Since p and $\overline{q}$ are irreducible this means that they are coprime, we can thus find x, y with $xp^l + y\overline{q} = 1$, where $l = l(H_p)$. Let $a \in H_p$ and $b \in H_q$. Note that multiplication by $\overline{q}$ is an automorphism of H_p with the inverse given by multiplication by y. We can thus write $a = \overline{q}a'$ for some $a' \in H_p$. We then conclude that

$$\lambda(a,b) = \lambda(\overline{q}a', b) = \lambda(a,b)q = \lambda(a, bq) = 0.$$

This concludes the proof of the claim.

Note that by the classification of modules over PIDs we get a unique direct sum decomposition

$$H = \bigoplus_{p \in \mathcal{P}'} H_p \oplus \bigoplus_{\{p,\overline{p}\} \in \mathcal{P}''} (H_p \oplus H_{\overline{p}})$$

and it follows from the claim that this is an orthogonal decomposition. In particular, (H, λ) is diagonalizable if the restrictions to H_p is diagonalizable for every $p \in \mathcal{P}'$ and if the restriction of λ to $H_p \oplus H_{\overline{p}}$ is diagonalizable for every $\{p, \overline{p}\} \in \mathcal{P}''$.

It follows from Lemma 4.4 that given $p \in \mathcal{P}'$ the restriction of λ to H_p is diagonalizable. The following claim thus concludes the proof of the proposition.

Claim. Let $p \in \Lambda$ be a non-palindromic irreducible polynomial. The restriction of λ to $H_p \oplus H_{\overline{p}}$ is diagonalizable.

First note that by the first claim of the proof we have $\lambda(H_p, H_p) = 0$ and $\lambda(H_{\overline{p}}, H_{\overline{p}}) = 0$. Since Λ is a PID we can write $H_p = \oplus_{i=1}^r V_i$ where the V_i are cyclic Λ-modules. We then define $\overline{V}_i$ to be the orthogonal complement in $H_{\overline{p}}$ to $\oplus_{i \neq j} V_j$, i.e.

$$\overline{V}_i := \{w \in H_{\overline{p}} \,|\, \lambda(w, v) = 0 \text{ for any } v \in \oplus_{i \neq j} V_j\}.$$

Since λ is non-singular it follows easily that $H_{\overline{p}} = \oplus_{i=1}^r \overline{V}_i$. In fact the decomposition

$$H_p \oplus H_{\overline{p}} \cong \bigoplus_{i=1}^r (V_i \oplus \overline{V}_i)$$

is an orthogonal decomposition into the r subsummands $V_i \oplus \overline{V}_i$.

It now suffices to show that the restriction of λ to any $V_i \oplus \overline{V}_i$ is diagonalizable. So let $i \in \{1, \ldots, r\}$. Note that $V_i \cong \Lambda/p^n\Lambda$ for some n. Let a be a generator of the cyclic Λ-module V_i. Since λ is non-singular there exists $b \in \overline{V}_i$ such that $\lambda(a,b) = \overline{p}^{-n} \in \overline{p}^{-n}\Lambda/\Lambda$. Note that b is necessarily a generator of the cyclic Λ-module $\overline{V}_i$.

Since p^n and $\overline{p}^n$ are coprime we can find $u, v \in \Lambda$ such that $up^n + v\overline{p}^n = 1$. We write $x := \frac{1}{2}(u + \overline{v})$. Then one can easily verify that $xp^n + \overline{x}\,\overline{p}^n = 1$. Note that it follows in particular that x is coprime to $\overline{p}$.

We now write $w := a \oplus xb$. It is straightforward to see that $\overline{p}^n w$ generates V_i and $p^n w$ generates $\overline{V}_i$, in particular w generates $V_i \oplus \overline{V}_i$. Furthermore,

$$\begin{aligned} \lambda(w, w) &= \lambda(a, xb) + \overline{\lambda(xb, a)} \\ &= \lambda(a, xb) + \overline{\lambda(a, xb)} \\ &= x\overline{p}^{-n} + \overline{x}p^{-n} \\ &= (xp^n + \overline{x}\overline{p}^n)p^{-n}\overline{p}^{-n} \\ &= p^{-n}\overline{p}^{-n}. \end{aligned}$$

This shows that sending 1 to w defines an isometry from $\lambda(p^n\overline{p}^n)$ (i.e. the Blanchfield form defined by the 1×1-matrix $(p^n\overline{p}^n)$) to the restriction of λ to $V_i \oplus \overline{V}_i$. □

Using Proposition 4.1 we can now also prove the following result.

Proposition 4.5. *Let λ be a Blanchfield form over Λ. Let $B = B(t)$ be a hermitian matrix over Λ representing λ. Denote by $Z \in S^1_+$ the set of zeros of $\det(B(t)) \in \Lambda$. Then*

$$\mu(\lambda) = \frac{1}{2}\left(\max\{\eta_B(z) + \sigma_B(z) \,|\, z \in Z\} + \max\{\eta_B(z) - \sigma_B(z) \,|\, z \in Z\}\right).$$

Proof. By Proposition 4.1 there exists a hermitian diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_r)$ over Λ with $\lambda(D) \cong \lambda(B)$. Recall that $\det(D) = \det(B)$ and $\eta_B(z) = \eta_D(z)$, $\sigma_B(z) = \sigma_D(z)$ for any $z \in S^1$. It thus suffices to prove the claim for D.

Given a hermitian matrix C we write

$$\Theta^{\pm}_C(z) := \eta_C(z) \pm \sigma_C(z).$$

Note that

$$\Theta^{\pm}_D(z) = \sum_{i=1}^{r} \Theta^{\pm}_{d_i}(z).$$

It is straightforward to see that for any i the function $\Theta^{\pm}_{d_i}(z)$ is constant away from the zeros of d_i and that the values at a zero are relative maxima. The proposition now follows immediately. □

4.2. **Elementary diagonal forms.** We say that a matrix is *elementary diagonal* if it is of the form

$$E = \operatorname{diag}(e_1, \dots, e_M),$$

where for $k = 1, \dots, M$ we have $e_k = \varepsilon_k B^{n_k}_{\xi_k}$ for some $\varepsilon_k \in \{-1, 1\}$, $n_k \in \mathbb{N}$ and $\xi_k \in \Xi$.

Lemma 4.6. *Let D be a hermitian matrix over Λ such that $D(\pm 1)$ is non-degenerate. Then there exists an elementary diagonal matrix E such that $\lambda(D) \cong \lambda(E)$.*

Proof. First, by Proposition 4.1 we can assume that D is a diagonal $n \times n$-matrix. We shall use an inductive argument. Let us consider the $k-$th element on the diagonal of D, $d_k(t)$. As d_k is a real polynomial and $\overline{D} = D$, we have a unique decomposition

$$d_k(t) = \varepsilon_k c_k \prod_{\xi} B_\xi(t)^{n_{k,\xi}},$$

with $\varepsilon_k \in \{-1, 1\}, c_k \in \mathbb{R}_{\geq 0}$ and where ξ runs over all elements Ξ, and where $n_{k,\xi}$ is zero for all but finitely many ξ.

We write $C = \operatorname{diag}(\sqrt{c_1}, \ldots, \sqrt{c_n})$. After replacing D by $C^{-1}D(C^{-1})^t$ we can assume that $c_i = 1$ for all i.

Assume now that there exists an ξ with $|\xi| < 1$ such that $n_{k,\xi} > 0$. Let us define

$$A(t) = B_\xi(t)^{n_{k,\xi}} \text{ and } \varepsilon_k B(t) = \frac{d_k(t)}{A(t)}.$$

Note that $A(1) > 0$ and since $A(t)$ has no zeros on S^1 we have in fact that $A(z) > 0$ for any $z \in S^1$. We can therefore use Theorem 3.6 to show that the matrix $\operatorname{diag}(d_1, d_2, \ldots, d_k, \ldots, d_n)$ is congruent to $\operatorname{diag}(d_1, \ldots, d_{k-1}, A, B, d_{k+1}, \ldots, d_n)$. In this way we can split out all terms with $|\xi| < 1$.

It remains to consider the case when $d_k(t) = \varepsilon_k \prod_{\xi \in S^1_+} B_\xi(t)^{n_{k,\xi}}$. Let $\xi \in S^1_+$ be the minimal number in S^1_+ with $n_{k,\xi} > 0$. We now define

$$A(t) = \varepsilon_k B_{\xi_1}(t)^{n_{k,\xi}} \text{ and } B(t) = \varepsilon_k (-1)^{n_{k,\xi}} \prod_{\xi' \neq \xi} B_{\xi'}(t)^{n_{k,\xi'}}.$$

Note that for $z \in S^1_+$ with $z > \xi$ we have $\operatorname{sign}(A(z)) = \varepsilon_k(-1)^{n_{k,\xi}}$ and for $z \in S^1_+$ with $z \leq \xi$ we have $\operatorname{sign}(B(z)) = \varepsilon_k(-1)^{n_{k,\xi}}$. It thus follows from Theorem 3.6 that the matrices (d_k) and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ give rise to the same Blanchfield form. The lemma now follows from a straightforward induction argument. $\square$

4.3. **Conclusion of the proof of Theorem 2.4.** Let B be a hermitian matrix over Λ such that $B(1)$ and $B(-1)$ are non-degenerate. Note that $B(z) = B(z^{-1})^t$ for $z \in S^1$. It thus follows that

$$\begin{aligned} \mu(B) &= \tfrac{1}{2}\left(\max\{\eta_B(z) + \sigma_B(z) \,|\, z \in S^1_+\} + \max\{\eta_B(z) - \sigma_B(z) \,|\, z \in S^1_+\}\right) \\ \eta(B) &= \max\{\eta_B(z) \,|\, z \in \mathbb{C} \setminus \{0\}\}. \end{aligned}$$

(Note that we now take the maximum over S^1_+.) For the reader's convenience we now recall the statement of Theorem 2.4.

Theorem 4.7. *Let B be a hermitian matrix over Λ such that $B(1)$ and $B(-1)$ are non-degenerate. Let $n_{\mathbb{R}}(B)$ be the minimal size of a hermitian matrix A defined over Λ such that $\lambda(A) \cong \lambda(B)$. Then* $\max\{\mu(B), \eta(B)\} = n_{\mathbb{R}}(B)$

We will first prove the following two important special cases of Theorem 4.7.

Proposition 4.8. *Let B be a hermitian matrix over Λ such that all zeros of* $\det(B) \in \Lambda$ *lie on $S^1 \setminus \{\pm 1\}$. Then $n_{\mathbb{R}}(B) = \mu(B)$.*

Note that if $\det(B) \in \Lambda$ has no zero outside of the unit circle then it can also be seen directly that $\eta(B) \leq \mu(B)$.

Proof. By Lemma 4.6 it suffices to prove the proposition for an elementary diagonal matrix of the form $E = \operatorname{diag}(e_1, \dots, e_M)$, where for $k = 1, \dots, M$ we have $e_k = \varepsilon_k B_{\xi_k}^{n_k}$ for some $\varepsilon_k \in \{-1, 1\}$, $n_k \in \mathbb{N}$ and $\xi_k \in \Xi \cap S^1 = S^1_+$. Also recall that by the discussion at the beginning of Section 4 it remains to show that $\mu(B) \geq n_{\mathbb{R}}(B)$. This will be achieved by proving the following claim.

Claim. Let $E = \operatorname{diag}(e_1, \dots, e_M)$ be such an elementary diagonal matrix. We write $s = \mu(E)$. Then there exists a positive integer a and a decomposition

$$\{1, \dots, M\} = \bigcup_{a=1}^{s} I_a$$

into pairwise disjoint sets, and for each $a = 1, \dots, s$ there exists $\kappa_a \in \{-1, 1\}$ such that

$$\lambda(E) \cong \lambda\Big(\operatorname{diag}\big(\kappa_1 \prod_{i \in I_1} e_i,\, \dots,\, \kappa_s \prod_{i \in I_s} e_i\big)\Big).$$

We will prove the claim by induction on the size M of the elementary diagonal matrix. The case $M = 0$ is trivial. So now suppose that the statement of the claim holds whenever the size of the elementary diagonal matrix is at most $M - 1$. Let $E = \operatorname{diag}(e_1, \dots, e_M)$ be an elementary diagonal matrix such that $\xi_k \in S^1 \cap \Xi \subset S^1_+$ for $k = 1, \dots, M$. Without loss of generality we can assume that $\xi_1 \leq \dots \leq \xi_M$ on S^1_+.

We now write $E' := \operatorname{diag}(e_1, \dots, e_{M-1})$. We write $s := \mu(E)$ and $s' := \mu(E')$. We now apply our induction hypothesis to E'. We obtain the corresponding decomposition $\{1, \dots, M-1\} = I'_1 \cup \dots \cup I'_{s'}$ and signs $\kappa'_1, \dots, \kappa'_{s'}$. For $a = 1, \dots, s'$, let

$$\rho_a = \kappa'_a \prod_{i \in I'_a} e_i.$$

In the following we write $\varepsilon = \varepsilon_M, n = n_M, e = e_M$ and $\xi_M = \xi$.

Case 1. First suppose there exists an $a \in \{1, \dots, s'\}$ such that $\rho_a(\xi) \neq 0$ and such that $\operatorname{sign}(\rho_a(\xi)) = \varepsilon$. Note that

$$\operatorname{sign}(B_\xi^n(z)) = \operatorname{sign}(B_\xi^n(1)) = \operatorname{sign}(\varepsilon)$$

for any $z \in [1, \xi) \subset S^1_+$ since B_ξ^n has no zeros on $z \in [1, \xi)$. Now recall that we assumed that $\xi_1 \leq \dots \leq \xi_M = \xi$ on S^1_+. It follows that ρ_a has no zeros on $[\xi, -1] \subset S^1_+$. It thus follows that

$$\operatorname{sign}(\rho_a(z)) = \operatorname{sign}(\rho_a(\xi)) = \varepsilon$$

for any $z \in [\xi, -1]$. We can thus apply Theorem 3.6 to conclude that

$$\lambda(\varepsilon\rho_a \cdot e) \cong \lambda(\mathrm{diag}(\rho_a, e)). \tag{7}$$

We will now prove the following claim.

Claim. $s = s'$.

Note that (7) implies that $\lambda(E)$ can be represented by an $s' \times s'$-matrix, in particular it follows that $s \le s'$. We will now show that $s \ge s'$. Given a hermitian matrix $C = C(t)$ over $\mathbb{C}[t^{\pm 1}]$ and $z \in S^1$ we write

$$\Theta_C^{\pm}(z) := \eta_C(z) \pm \sigma_C(z).$$

By Proposition 4.5 we have

$$\begin{aligned} \mu(E') &= \tfrac{1}{2}\left(\max\{\Theta_{E'}^{+}(z) \,|\, z \in S^1\} + \max\{\Theta_{E'}^{-}(z) \,|\, z \in S^1_+\}\right) \\ &= \tfrac{1}{2}\left(\max\{\Theta_{E'}^{+}(z) \,|\, z \in [1,\xi]\} + \max\{\Theta_{E'}^{-}(z) \,|\, z \in [1,\xi]\}\right) \\ &= \tfrac{1}{2}\left(\max\{\Theta_{E'}^{+}(z) + \varepsilon \,|\, z \in [1,\xi]\} + \max\{\Theta_{E'}^{-}(z) - \varepsilon \,|\, z \in [1,\xi]\}\right). \end{aligned}$$

Note that $\Theta_E^{\pm}(z) = \Theta_{E'}^{\pm}(z) + \Theta_e^{\pm}(z)$. It is straightforward to verify that $\Theta_e^{\pm}(z) \mp \varepsilon$ is greater or equal than zero for any $z \in [0, \xi]$. We thus conclude that

$$\Theta_{E'}^{\pm} \pm \varepsilon = \Theta_E^{\pm} - (\Theta_\varepsilon^{\pm} \mp \varepsilon) \le \Theta_E^{\pm}$$

on S^1_+. It follows that

$$\begin{aligned} \mu(E') &\le \tfrac{1}{2}\left(\max\{\Theta_{E}^{+}(z) \,|\, z \in [1,\xi]\} + \max\{\Theta_{E}^{-}(z) \,|\, z \in [1,\xi]\}\right) \\ &\le \tfrac{1}{2}\left(\max\{\Theta_{E}^{+}(z) \,|\, z \in S^1\} + \max\{\Theta_{E}^{-}(z) \,|\, z \in S^1_+\}\right) \\ &= \mu(E). \end{aligned}$$

This concludes the proof that $s = s'$. We now define $I_a = I_a' \cup \{M\}$ and $I_b = I_b'$ for $b \ne a$ and the induction step is proved in Case 1.

Case 2. Now suppose that for any $a \in \{1, \dots, s'\}$ we either have $\rho_a(\xi) = 0$ or $\mathrm{sign}(\rho_a(\xi)) = -\varepsilon$. We claim that $s = s' + 1$. We write $R := \mathrm{diag}(\rho_1, \dots, \rho_{s'}, e)$. We can thus represent E by the matrix R of size $s' + 1$. It follows that $s \le s' + 1$. We now write $k := \#\{a \in \{1, \dots, s'\} \,|\, \rho_a(\xi) = 0\}$. We have

$$\begin{aligned} \mu(E') &= \mu(R) \\ &\ge \tfrac{1}{2}\max\{\eta_R(z) + \varepsilon\sigma_R(z) \,|\, z \in S^1\} \\ &\ge \tfrac{1}{2}\left(\eta_R(\xi) + \varepsilon\sigma_R(\xi)\right) \\ &= (k+1) + (s' - k) = s' + 1. \end{aligned}$$

We now take $I_a := I_a'$ for $a \in \{1, \dots, s'\}$ and we define $I_{s'+1} = \{M\}$. □

We now consider the next special case of Theorem 2.4.

Proposition 4.9. *Let B be a hermitian matrix over Λ such that $\det(B) \in \Lambda$ has no zero on the unit circle. Then $n_{\mathbb{R}}(B) = \eta(B)$.*

Note that if $\det(B) \in \Lambda$ has no zero on the unit circle, then η_B and σ_B are constant functions on the unit circle, hence $\mu(B) = 0$.

Proof. As in the proof of Proposition 4.8 we only have to consider the case that B is an elementary diagonal matrix $B = \operatorname{diag}(e_1, \dots, e_M)$. Note that the zeros of $e_1, \dots, e_M$ do not lie on S^1. We write $s = \eta(B)$.

It is straightforward to see that one can decompose $\{1, \dots, M\}$ into subsets $I_1, \dots, I_s$ with the following property: given $k, l \in I_b$ with $k \neq l$ the polynomials e_k and e_l have different roots. It is clear that one can find such $I_1, \dots, I_s$, since for any $\xi \notin S^1$ there exist at most s indices $k \in \{1, \dots, M\}$ for which e_k has root at ξ.

Since the sign of any product of product of the e_i is constant on the unit circle we can now apply Theorem 3.6 repeatedly to show that there exist $\mu_b \in \{-1, 1\}$ such that

$$\lambda(B) \cong \lambda\big(\operatorname{diag}\big(\epsilon_1 \prod_{j \in I_1} e_j, \dots, \epsilon_s \prod_{j \in I_s} e_j\big)\big).$$

□

We are now ready to finally provide a proof of Theorem 4.7.

Proof of Theorem 4.7. Let B be a square matrix over Λ such that $B(1)$ and $B(-1)$ are non-degenerate. We write $s_i := s_i(B)$. It follows from Lemma 4.6 together with the proofs of Propositions 4.8 and 4.9 that there exist palindromic $f_1, \dots, f_\mu \in \Lambda$ with no zeros outside of S^1 and palindromic $g_1, \dots, g_\eta \in \Lambda$ with no zeros on S^1 such that

$$\lambda(B) \cong \lambda(\operatorname{diag}(f_1, \dots, f_\mu, g_1, \dots, g_\eta)).$$

Note that the sign of any g_i is constant on the unit circle. It follows from Theorem 3.6 that for any $k \in \{1, \dots, \min(\mu, \eta)\}$ we have $\lambda(\operatorname{diag}(f_k, g_k)) \cong \lambda((\varepsilon_k f_k g_k))$ for some $\varepsilon_k \in \{\pm 1\}$. This shows, that if $\mu \geq \eta$,

$$\lambda(B) \cong \lambda(\operatorname{diag}(\varepsilon_1 f_1 g_1, \dots, \varepsilon_\eta f_\eta g_\eta, f_{\eta+1}, \dots f_\mu)),$$

while, if $\eta > \mu$

$$\lambda(B) \cong \lambda(\operatorname{diag}(\varepsilon_1 f_1 g_1, \dots, \varepsilon_\mu f_\mu g_\mu, g_{\mu+1}, \dots g_\eta)).$$

□

We point out that the proof of Theorem 4.7 in fact provides a proof of the following slightly more precise statement.

Theorem 4.10. *Let B be a hermitian matrix over Λ such that $B(1)$ and $B(-1)$ are non-degenerate. Let $s = \max\{\mu(B), \eta(B)\}$. Then there exists a diagonal hermitian $s \times s$-matrix D over Λ such that $\lambda(D) \cong \lambda(B)$.*

5. Examples

It is easy to construct examples for different values of η and μ. Before we start, let us make some general and almost obvious remarks about η.

5.1. **Basic properties of η.** We have the following result.

Lemma 5.1. *For any knot K, the following numbers are equal.*

(a) *The maximum of nullities $\eta(K)$;*
(b) *The real Nakanishi index, i.e. the minimal number of generators of the $\mathbb{R}[t^{\pm 1}]$ module $H_1(X(K), \mathbb{R}[t^{\pm 1}])$;*
(c) *The rational Nakanishi index, i.e. the minimal number of generators of the $\mathbb{Q}[t^{\pm 1}]$ module $H_1(X(K), \mathbb{Q}[t^{\pm 1}])$;*
(d) *The maximal index k, for which the $k-$th Alexander polynomial Δ_k is not 1;*
(e) *The bigger of the two following numbers*

$$\max_{\lambda:\,0<|\lambda|<1} \sum_{k=1}^{\infty} q_\lambda^k \text{ and } \max_{|\lambda|=1} \sum_{k=1}^{\infty} \sum_{u=\pm 1} p_\lambda^k(u),$$

where the numbers $p_\lambda^k(u)$ and q_λ^k are the Hodge numbers defined in [BN12].

Proof. The fact that (a), (b), (c) and (d) are equal is well known to the experts, for a convenience of the reader we point out that (a)=(e) follows from [BN12, Lemma 4.4.6], (c)=(d)=(e) is [BN12, Proposition 4.3.4]. It is obvious that (b)≤(c) and (d)≤(b). □

From this lemma it follows that $\eta(nK) = n \cdot \eta(K)$ (here nK is a connected sum of n copies of a knot K), and

$$\eta(K_1 \# K_2) \in \{\max(\eta(K_1), \eta(K_2)), \dots, \eta(K_1) + \eta(K_2)\}.$$

To be more precise, we have the following corollary

Corollary 5.2. *Given two knots K_1 and K_2, if their Alexander polynomials are coprime, then $\eta(K_1 \# K_2) = \max(\eta(K_1), \eta(K_2))$.*

Now we show some examples for knots with up to 12 crossings, we refer to the authors' webpage [BF11] for more details.

5.2. **The knot $12a_{896}$.** Its Alexander polynomial is $2-11t+26t^2-40t^3+45t^4-40t^5+26t^6-11t^7+2t^8$. It has no multiple roots. The graph of the function $x \to \sigma(e^{2\pi i x})$ is presented on Figure 1. The maximum of the Levine-Tristram signature is 2, the minimum is -2. All the jumps of the Levine–Tristram signatures correspond to single roots of the Alexander polynomial, hence $\eta = 1$. But $\mu = 2$ is bigger than half the maximum of the Levine–Tristram signature.

5.3. **Some concrete examples.** We list now some examples, which are built from connected sums of different knots.

(1) For any knot K with non-trivial Alexander polynomial and $\eta = 1$ (for example $K = 3_1$), the knot $K' = K \# -K$ has $\mu = 0$ and $\eta = 2$. The connected sum of n copies of K' has $\mu = 0$ but $\eta = n$ can be arbitrarily large.

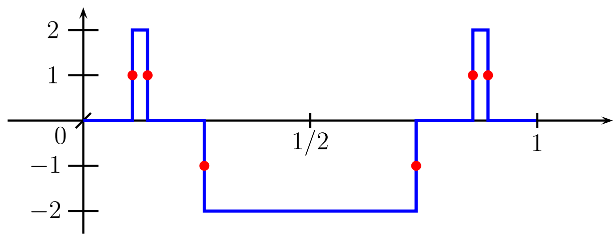

FIGURE 1. Graph of the signature function of knot $12a_{896}$, more precisely the function $x \to \sigma(e^{2\pi i x})$. The jumps of the signature function occur at the places, corresponding to roots of the Alexander polynomial $(2-3t+2t^2)(1-4t+6t^2-7t^3+6t^4-4t^5+1)$ on the unit circle. Numerically they are $x \sim 0.115$, $x \sim 0.12149$, $x \sim 0.2697$, $x \sim 0.7302$, $x \sim 0.8785$, $x \sim 0.8850$. The graph is taken from [CL11].

.

(2) The torus knots $T_{2,2n+1}$ have signature $2n$, the span of signatures is $\mu(T_{2,2n+1}) = n$ but $\eta = 1$. This example and the example above show that μ and η are, in general, completely independent.
(3) For any torus knot $T_{2,2n+1}$ we saw in (2) that $n_{\mathbb{R}} = n$, but for $T_{2,2n+1}\#-T_{2,2n+1}$ we have $\mu = 0$, $\eta = 2$, so $n_{\mathbb{R}} = 2$. The Blanchfield Form dimension $n_{\mathbb{R}}$ is therefore not additive.
(4) The torus knot $T_{3,4}$ has $\mu = 3$ and $n_{\mathbb{R}} = 3$ (this is the unknotting number for $T_{3,4}$). The torus knot $T_{2,5}$ has $\mu = 2$ and $n_{\mathbb{R}} = 2$. But the connected sum $T_{3,4}\# - T_{2,5}$ has $n_{\mathbb{R}} = 1$, because the maximal Levine–Tristram signature is 2 and minimal is 0. So we may even have $n_{\mathbb{R}}(K_1\#K_2) < \min(n_{\mathbb{R}}(K_1), n_{\mathbb{R}}(K_2))$.
(5) The knots 6_2 and 10_{32} have both $n_{\mathbb{R}} = 1$ (see [CL11] for graphs of their signature functions). But their sum $6_2\#10_{32}$ also has $n_{\mathbb{R}} = 1$. Therefore, $n_{\mathbb{R}}(K_1\#K_2)$ can be equal to 1 even if $n_{\mathbb{R}}(K_1) = n_{\mathbb{R}}(K_2) = 1$.

Finally note that in [Li11, Theorem 18] Livingston uses the Levine–Tristram signature function to define a new invariant $\rho(K)$ which gives a lower bound on the 4-genus and in particular on the unknotting number. Livingston furthermore shows that $\rho(-5_1\#10_{132}) = 3$, whereas $n_{\mathbb{R}}(K) = 2$. This shows that the Blanchfield Form dimension $n_{\mathbb{R}}(K)$ is not the optimal unknotting information, which can be obtained from Levine–Tristram signatures and nullities.

On the other hand there are many examples for which $\rho(K) = 0$, e.g. for all knots with vanishing Levine-Tristram signature function, but for which $\eta(K) > 0$. This shows that $\rho(K)$ and $n_{\mathbb{R}}(K)$ are independent lower bounds on the unknotting number.

We conclude this paper with the following question:

Question 5.3. *What is the optimal lower bound on the unknotting number that can be obtained using Levine–Tristram signatures and nullities.*

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