CHARACTERISTIC CLASSES AND HOMOGENEOUS SPACES, II.*

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Chapter VI. Applications to Todd Genera.[1]

20. Integration over the fibre in $(B_T, B_G, G/T, \rho(T,G))$.

Throughout § 20, the coefficients for cohomology are the real numbers and will not be mentioned explicitly.

20.1. Let G be a compact connected Lie group, T a maximal torus, $2m$ the dimension of G/T, $\mathcal{J}$ an invariant almost complex structure on G/T, and $a_1, \cdots, a_m$ the roots of $\mathcal{J}$ (see 12.3, 13.4). $\mathcal{J}$ defines an *orientation* of G/T, and hence also an identification of $H^{2m}(G/T)$ with $\boldsymbol{R}$, which will always be used in this §. In the fibering $\xi = (B_T, B_G, G/T, \rho(T,G))$ (see [2, § 20] for its definition), the integration over the fibre is a linear map of $H^*(B_T)$ in $H^*(B_G)$ or of $H^{**}(B_T)$ into $H^{**}(B_G)$ which lowers degrees by $2m$, (see § 8).

The order q of $W(G)$ is equal to the Euler number $E(G/T)$ of G/T and the latter is equal to the value of the m-th Chern class on the fundamental cycle. Therefore, considering the a_i's as elements of $H^2(B_T)$, we have by 10.8

$$(1) \qquad (a_1 \cdots a_m)[G/T] = q = \text{order } W(G),$$

where the left side denotes the value of $a_1 \cdots a_m$ on an oriented fibre.

G/T is totally non-homologous to zero in ξ for real coefficients and q is also the dimension of $H^*(G/T)$, [2, § 26]. Therefore $\rho^*(T,G)$, which will be abbreviated by π^*, is injective, and we can choose homogeneous elements $h_1, \cdots, h_q \in H^*(B_T)$ with $h_q = a_1 \cdots a_m$, whose restrictions to a fibre form a basis of $H^*(G/T)$; an element $x \in H^*(B_T)$ can then be written in one and only one way in the form

$$(2) \qquad x = \pi^*(b_1)h_1 + \cdots + \pi^*(b_q)h_q \qquad (b_i \in H^*(B_G))$$

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[1] Part I of this paper appeared earlier in this Journal, Vol. 80 (1958), pp. 458-538. We refer to it for the notation and a general introduction.

and we have by 8.4(1) and (1) above that

$$x^{\natural} = q \cdot b_q. \tag{3}$$

For any $w \in W(G)$, the elements $w(h_1), \cdots, w(h_{q-1})$, $\operatorname{sgn} w \cdot h_q$ (see 2.6 for $\operatorname{sgn} w$), when restricted to a fibre, also form a base of $H^*(G/T)$. Therefore, if we apply w to (2) and use 8.4(1) again, we see that

$$(w(x))^{\natural} = \operatorname{sgn} w \cdot x^{\natural}. \tag{4}$$

20.2. LEMMA. *Let $x \in H^*(B_T)$ be such that $w(x) = \operatorname{sgn} w \cdot x$ for all $w \in W(G)$. Then*

$$q \cdot x = \pi^*(x^{\natural}) \cdot a_1 \cdot \cdots \cdot a_m.$$

We may consider $H^*(B_T)$ as the ring of polynomials with real coefficients on the universal covering V_T of T. Let S_i be the symmetry to the hyperplane $a_i = 0$. Then, $S_i(x) = -x$ implies that x is zero on $a_i = 0$, and hence that x is divisible by a_i. It follows that $x = y \cdot a_1 \cdots a_m$ with $y \in H^*(B_T)$ and, in view of our assumption, invariant under $W(G)$. Therefore [2, § 26], $y = \pi^*(b)$, $b \in H^*(B_G)$, and the lemma follows from (3).

20.3. THEOREM. *Let $a_1, \cdots, a_m$ be the roots of an invariant almost complex structure $\mathfrak{C}$ on G/T, and $\natural$ be the integration over the fibre in $(B_T, B_G, G/T, \pi)$ with respect to the orientation defined by $\mathfrak{C}$. Then for $x \in H^*(B_T)$, we have*

$$\sum_{w \in W(G)} \operatorname{sgn} w \cdot w(x) = \pi^*(x^{\natural}) \cdot a_1 \cdot \cdots \cdot a_m. \tag{5}$$

Let y be the left-hand side of (5). Then $y^{\natural} = q \cdot x^{\natural}$ by (4). On the other hand, we have $w(y) = \operatorname{sgn} w \cdot y$ for any $w \in W(G)$; therefore 20.2 shows

$$q \cdot y = \pi^*(y^{\natural}) \cdot a_1 \cdot \cdots \cdot a_m$$

which proves the theorem.

It follows from 20.3 that, if $x \in H^{**}(B_T)$, then

$$\sum_{w \in W(G)} \operatorname{sgn} w \cdot w(x) = \pi^{**}(x^{\natural}) \cdot a_1 \cdots a_m. \tag{6}$$

21. Multiplicative sequences.

21.1. In this paragraph, ξ is a bundle in which F_ξ is a compact connected n-dimensional oriented manifold, G_ξ is a group of diffeomorphisms of F_ξ, and $\hat{\xi}$ is the bundle along the fibres (7.4). Γ is a commutative ring with unit and the cohomology groups of the fibres of ξ with respect to Γ are assumed to form a constant sheaf on B_ξ. Then $\natural = \natural_\xi$ is a map of $H^{**}(E_\xi, \Gamma)$ into $H^{**}(B_\xi, \Gamma)$ which lowers degrees by n. In particular, we have

$$a^\natural = a[F_\xi] \cdot 1 \qquad (a \in H^n(E_\xi, \Gamma)), \tag{1}$$

where 1 is the unit of $H^{**}(B_\xi, \Gamma)$.

21.2. Let $\{K_j(p_1, \cdots, p_j)\}$ be a multiplicative sequence of polynomials in indeterminates p_i, with coefficients in Γ [19, § 1]. If η is a real vector bundle, we put

$$\mathcal{K}_\eta = \sum_{j \geqq 0} K_j(p_1(\eta), \cdots, p_j(\eta)). \tag{2}$$

We have $K_j(p_1(\eta), \cdots, p_j(\eta)) \in H^{4j}(B_\eta, \Gamma)$ and $\mathcal{K}_\eta \in H^{**}(B_\eta, \Gamma)$. If η is the tangent bundle to a compact oriented differentiable manifold X, then the genus $K(X)$ of X with respect to the sequence $\{K_j\}$ is defined by $K(X) = \mathcal{K}_\eta[X]$, i.e.

$$K(X) = K_r(p_1(\eta), \cdots, p_r(\eta))[X] \in \Gamma$$

if $4r = \dim X$, and $K(X) = 0$ if $\dim X \not\equiv 0 \bmod 4$.

21.3. Definition. *Let $\hat{\xi}$ be the bundle along the fibres of ξ. The multiplicative sequence $\{K_j\}$ is said to be strictly multiplicative in ξ if and only if*

$$(\mathcal{K}_{\hat{\xi}})^\natural \in H^0(B_\xi, \Gamma). \tag{i}$$

Let $\hat{p}_i$ be the Pontrjagin classes of $\hat{\xi}$. The condition (i) is equivalent to

$$K_j(\hat{p}_1, \cdots, \hat{p}_j)^\natural = 0 \qquad (4j > n). \tag{ii}$$

The restriction of $\hat{\xi}$ to a fibre of ξ is the tangent bundle to the fibre. Therefore we have by (1) for any multiplicative sequence

$$(K_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = K(F_\xi) \cdot 1 \qquad (4j = \dim F_\xi), \tag{3}$$

and therefore $\{K_j\}$ is strictly multiplicative in ξ if and only if

$$(\mathcal{K}_{\hat{\xi}})^\natural = K(F_\xi) \cdot 1. \tag{4}$$

A multiplicative sequence is always strictly multiplicative in the product bundle, because in this case, $\hat{\xi}$ may be identified with the bundle induced from the tangent bundle to F_ξ by the projection of $E_\xi = B_\xi \times F_\xi$ onto F_ξ, and then (ii) is obviously true.

21.4. In addition to 21.1, we assume that ξ is a differentiable bundle (7.4), and that B_ξ, E_ξ are compact connected oriented manifolds, the orientation of E_ξ being induced by those of B_ξ, F_ξ taken in this order. It follows then from the definition of the integration over the fibre (8.1) or from its equivalence with the Gysin homomorphism (8.3, remark) that

$$a[E_\xi] = a^\natural[B_\xi] \qquad (a \in H^*(E_\xi, \Gamma)).$$

Let η and η' be the tangent bundles to E_ξ and B_ξ respectively. We have an exact sequence (7.6):

$$(5) \qquad 0 \to \hat{\xi} \to \eta \to \pi^*\eta' \to 0, \qquad (\pi = \pi_\xi),$$

and the multiplication theorem (9.7) implies

$$p(\eta) = p(\hat{\xi}) \cdot \pi^* p(\eta') \qquad \operatorname{mod} \operatorname{Tors} H^*(E_\xi, \mathbf{Z}),$$

where $\operatorname{Tors} H^*(E_\xi, \mathbf{Z})$ is the torsion subgroup of $H^*(E_\xi, \mathbf{Z})$. By the fundamental property of multiplicative sequences [19, § 1.2], this yields

$$(6) \qquad \mathcal{K}_\eta = \mathcal{K}_{\hat{\xi}} \cdot \mathcal{K}_{\pi^*\eta'} = \mathcal{K}_{\hat{\xi}} \cdot \pi^*(\mathcal{K}_{\eta'}),$$

modulo the image of $\operatorname{Tors} H^*(E_\xi, \mathbf{Z}) \otimes \Gamma$ in $H^*(E_\xi, \Gamma)$.

If $E_\xi = B_\xi \times F_\xi$, then [19, § 5.2]

$$(7) \qquad K(E_\xi) = K(B_\xi) \cdot K(F_\xi).$$

More generally, we have

21.5. PROPOSITION. *Let ξ be a differentiable bundle satisfying the assumption 21.4 and let $\{K_j\}$ be a multiplicative sequence of polynomials with coefficients in Γ. If $\{K_j\}$ is strictly multiplicative in ξ, then $K(E_\xi) = K(B_\xi) \cdot K(F_\xi)$.*

Since $H^j(E_\xi, \mathbf{Z})$ has no torsion for $j = \dim E_\xi$, we get from 21.4 and 8.2

$$K(E_\xi) = \mathcal{K}_\eta[E_\xi] = (\mathcal{K}_{\hat{\xi}} \cdot \pi^* \mathcal{K}_{\eta'})^\natural [B_\xi] = \mathcal{K}_{\eta'} (\mathcal{K}_{\hat{\xi}})^\natural [B_\xi],$$

and 21.5 follows from 21.2, 21.3(4).

21.6. We repeat briefly this discussion for the case of Chern classes. Let $\{K_j(c_1, \cdots, c_j)\}$ be a multiplicative sequence of polynomials with coefficients in Γ, in indeterminates c_i. Given a complex vector bundle η, we introduce the elements $K_j(c_1(\eta), \cdots, c_j(\eta)) \in H^{2j}(X, \Gamma)$ and put

$$\mathcal{K}_\eta = \sum_{j \geqq 0} K_j(c_1(\eta), \cdots, c_j(\eta)).$$

It is an element of $H^{**}(X, \Gamma)$. If η is the complex tangent bundle to a compact connected almost complex manifold (7.3), canonically oriented, the genus of X with respect to $\{K_j\}$ or its "K-genus" is $K(X)=\mathcal{K}_\eta[X]$. It is equal to $K_m(c_1(\eta),\cdots,c_m(\eta))[X]$ if m is the complex dimension of X.

21.7. Let ξ be as in 21.1. Assume moreover that $\hat{\xi}$ has been endowed with a complex structure $\hat{\xi}_C$ of the type considered in 7.4, that is, defined by means of an almost complex structure of F_ξ, invariant under G_ξ, and let $\hat{c}_i$ be its Chern classes. The multiplicative sequence $\{K_j\}$ is then said to be strictly multiplicative in ξ with respect to $\hat{\xi}_C$ if one of the three following equivalent conditions is fulfilled

(i) $$(\mathcal{K}_{\hat{\xi}_C})^\natural\in H^0(B_\xi,\Gamma)$$

(ii) $$(K_j(\hat{c}_1,\cdots,\hat{c}_j))^\natural=0, \qquad (2j>\dim F_\xi)$$

(iii) $$(\mathcal{K}_{\hat{\xi}_C})^\natural=K(F_\xi)\cdot 1.$$

21.8. Let ξ and $\hat{\xi}_C$ be as before. Assume in addition that ξ is differentiable and that B_ξ carries an almost complex structure η'_C. Then an almost complex structure η_C of E_ξ is said to be *compatible with* η'_C and $\hat{\xi}_C$ if there is an exact sequence

(8) $$0\to\hat{\xi}_C\to\eta_C\to\pi^*(\eta'_C)\to 0, \qquad (\pi=\pi_\xi).$$

Since exact sequences of vector bundles of the type (5), (8) always split, η_C always exists and is determined up to isomorphism by η'_C and $\hat{\xi}_C$. The proof of the following propostition is exactly the same as in the case of Pontrjagin classes:

21.9. PROPOSITION. *We keep the assumptions of* 21.8 *and assume moreover that the multiplicative sequence* $\{K_j\}$ *is strictly multiplicative in* ξ *with respect to* $\hat{\xi}_C$. *Then* $K(E_\xi)=K(B_\xi)\cdot K(F_\xi)$.

22. The Todd genus of certain almost complex homogeneous spaces. Throughout this paragraph, all cohomology groups will be taken with real coefficients, and all characteristic classes which occur will be regarded as real classes unless otherwise mentioned. The symbol $\boldsymbol{R}$ will be omitted in real cohomology groups.

22.1. *Notation.* Let ξ and η be complex vector bundles with the same base space: $B=B_\xi=B_\eta$. We recall that the Todd multiplicative sequence $\{T_j(c_1,\cdots,c_j)\}$ has $x(1-e^{-x})^{-1}$ as its characteristic power series [19, § 1] and define the cohomology class $\mathcal{T}(\xi,\eta)\in H^{**}(B)$ by the equation

$$\mathcal{T}(\xi,\eta)=\mathrm{ch}(\eta)\sum_{j=0}^{\infty}T_j(c_1(\xi),\cdots,c_j(\xi)),$$

where $c_i(\xi) \in H^{2i}(B)$ is the i-th Chern class of ξ and $\mathrm{ch}(\eta)$ is the Chern character of η as defined in 9.1. For $d \in H^2(B)$, set

$$\mathcal{T}(\xi, d) = e^d \cdot \sum_{j=0}^{\infty} T_j(c_1(\xi), \cdots, c_j(\xi))$$

and for $d = 0$,

$$\mathcal{T}(\xi, 0) = \mathcal{T}(\xi).$$

It is clear that $\mathcal{T}(\xi, d) = \mathcal{T}(\xi, \eta)$ if η is the complex line bundle with d as its first Chern class.

More generally, as in [19, § 1], let $\{T_j(y; c_1, \cdots, c_j)\}$ be the generalized Todd sequence. This multiplicative sequence has

$$x(1+y)/(1-e^{-x(1+y)}) - yx$$

as its characteristic power series. We define $\mathcal{T}_y(\xi) \in H^{**}(B) \otimes \mathbf{R}[y]$ by the equation

$$\mathcal{T}_y(\xi) = \sum_{j=0}^{\infty} T_j(y; c_1(\xi), \cdots, c_j(\xi)).$$

Obviously, $\mathcal{T}_0(\xi) = \mathcal{T}(\xi)$.

If B is a compact almost complex manifold and if now ξ stands for the tangential complex vector bundle of B, then we set

$$\mathcal{T}(B, \eta) = \mathcal{T}(\xi, \eta), \qquad \mathcal{T}(B, d) = \mathcal{T}(\xi, d), \qquad \mathcal{T}_y(B) = \mathcal{T}_y(\xi).$$

Moreover, in agreement with the notations of [19, §§ 10, 11, 12], the following real numbers (respectively polynomials with real coefficients) are defined

$$T(B, \eta) = \mathcal{T}(B, \eta)[B],$$

$$T(B, d) = \mathcal{T}(B, d)[B],$$

$$T_y(B) = \mathcal{T}_y(B)[B] = \sum_{p=0}^{n} T^p(B) y^p, \text{ where } n = \dim_{\mathbf{C}} B.$$

$T(B)$ denotes the Todd genus ($T(B) = T_0(B) = T(B, 0)$).

Finally, let us recall the following formal fact: If c_i is the i-th elementary symmetric function in $\gamma_1, \cdots, \gamma_n$ ($c_i = 0$ for $i > n$), then

$$\sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j) = e^{c_1/2} \prod_{i=1}^{n} \gamma_i / (2 \sinh(\gamma_i/2)).$$

22.2. Let G be a compact connected Lie group and T a maximal torus of G. Let V be the universal covering of T and $a_1, \cdots, a_m$ the *positive* roots of G with respect to an ordering $\mathcal{S}$ on V^* (2.4). Let ξ be a principal G-bundle and $\xi_{\mathbf{C}}$ the complex vector bundle along the fibres (7.4) of

$(E_\xi/T, B_\xi, G/T, \pi)$ which belongs to the invariant almost complex structure on G/T having $\epsilon_1 a_1, \epsilon_2 a_2, \cdots, \epsilon_m a_m$, $(\epsilon_i = \pm 1)$, as its roots (12.3 and 13.4). We orient G/T according to this almost complex structure and denote by $\natural$ the integration over the fibre with respect to this orientation (8.1). The total Chern class of $\hat{\xi}_C$ is given by the formula (10.8):

$$(1) \qquad c(\hat{\xi}_C) = (1 + \epsilon_1 a_1)(1 + \epsilon_2 a_2) \cdots (1 + \epsilon_m a_m).$$

Now let d be an arbitrary element of $H^1(T) = V^*$. We may regard d as an element of $H^2(E_\xi/T)$ under the negative transgression. Then, using (1) for the cohomology class $\mathcal{T}(\hat{\xi}_C, d)$ introduced in 22.1, we have

$$\mathcal{T}(\hat{\xi}_C, d) = e^d \cdot \prod \epsilon_j a_j / (1 - \exp(-\epsilon_j a_j)) = e^{(c_1/2)+d} \prod_{j=1}^{m} a_j / (2 \sinh(a_j/2)),$$

where $c_1 = c_1(\hat{\xi}_C) = \sum_{i=1}^{m} \epsilon_i a_i$. We are going to calculate $\mathcal{T}(\hat{\xi}_C, d)^\natural$ by 20.3. In view of 8.3, this is possible since ξ is induced from the universal bundle. First observe that $\prod a_j/(2\sinh(a_j/2))$ is invariant under the operations of the Weyl group $W(G)$ since the roots are permuted up to sign and since $x/\sinh x$ is an even function in x. Thus we obtain, after setting $b = d + \frac{1}{2} c_1(\hat{\xi}_C)$ and $a = \sum_{j=1}^{m} a_j$,

$$\epsilon_1 \epsilon_2 \cdots \epsilon_m \pi^{**}(\mathcal{T}(\hat{\xi}_C, d)^\natural) = \sum_{w \in W(G)} \operatorname{Sgn}(w) e^{w(b)} / \prod_{i=1}^{m} 2 \sinh a_i/2$$

or, in the notations of 3.2:

$$(2) \qquad \epsilon_1 \epsilon_2 \cdots \epsilon_m \pi^{**}(\mathcal{T}(\hat{\xi}_C, d)^\natural) = E(b/2\pi(-1)^{\frac{1}{2}}) / E(a/4\pi(-1)^{\frac{1}{2}}).$$

It follows that $\mathcal{T}(\hat{\xi}_C, d)^\natural$ vanishes if b is singular. The right side of the preceding equation is a formal power series in $d, a_1, \cdots, a_m$ (regarded as elements of $H^2(E_\xi/T)$) and, as such, is an element of $H^{**}(E_\xi/T)$. On the other hand, $d, a_1, \cdots, a_m$ are originally elements of V^*, i.e. functions on V. If b is a non-singular weight, then $E(b)/E(a/2)$ is also a function on V, namely, up to a sign, the character of a certain irreducible representation of $\tilde{G}$ (for $\tilde{G}$, see 3.3). In fact, if b is a non-singular weight, then there is a unique element $w' \in W(G)$ such that $w'(b)$ is in the positive Weyl chamber (2.7) with respect to the ordering $\mathcal{S}$; i.e., $(w'(b), a_j) > 0$ for $1 \leqq j \leqq m$, and hence $w'(b) - a/2$ is the highest weight of an irreducible representation λ of $\tilde{G}$ which is uniquely determined up to equivalence. According to 3.4, the function $E(b)/E(a/2)$ on V equals the character of λ as a function on V multiplied by $\operatorname{Sgn}(w')$. Thus we have seen that $\mathcal{T}(\xi_C, d)^\natural$ is essentially given by a character.

It is clear that $w'(b) - a/2$ is integral on the unit lattice of G if and only if d has this property. Assume now that d has the property just mentioned (in other words, that $d \in H^1(T, \mathbf{Z}) \subset H^1(T) = V^*$) and that $b = d + \frac{1}{2}c_1(\hat{\xi}_C)$ is non-singular. Then λ also defines a representation of G. The λ extension (6.5) $\lambda(\xi)$ of the principal G-bundle ξ is then defined. It is a $\boldsymbol{U}(n)$-bundle and we have

$$\mathcal{T}(\hat{\xi}_C, d)^{\natural} = \operatorname{Sgn}(w')\epsilon_1\epsilon_2 \cdots \epsilon_m \cdot \operatorname{ch}(\lambda(\xi)). \tag{3}$$

Here ch is the Chern character as defined in 9.1. See also 10.2, 10.3.

22.3. In Sections 22.3 and 22.4, we shall apply the results of 22.2 to the very special case where B_ξ is a point; then $E_\xi = G$ and $E_\xi/T = G/T$. Every element of V^* may be regarded as an element of $H^2(G/T)$. Otherwise, we keep the notations of 22.2.

The homogeneous space G/T has 2^m invariant almost complex structures belonging to the 2^m possible choices of the signs $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$. Now endow G/T with the invariant almost complex structure having the roots $\epsilon_1 a_1, \epsilon_2 a_2, \cdots, \epsilon_m a_m$. Then the first Chern class of G/T is

$$c_1 = c_1(G/T) = \epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_m a_m.$$

Since $a/2$ is a weight, $c_1/2$ is also a weight, which implies by (10.1) that the integral first Chern class $c_1(G/T) \in H^2(G/T, \mathbf{Z})$ equals 0 when reduced to coefficients mod 2. *Thus the second Stiefel-Whitney class $w_2(G/T) \in H^2(G/T, \mathbf{Z}_2)$ vanishes.*

The Pontrjagin class p_i of G/T is the i-th elementary symmetric function in the a_j^2, and vanishes for $i > 0$ (10.9). Thus for $d \in H^2(G/T)$ (see end of 22.1),

$$\begin{aligned} &\mathcal{T}(G/T, d) = \exp(d + \tfrac{1}{2}c_1) \in H^*(G/T), \\ &m! \cdot T(G/T, d) = ((c_1/2) + d)^m[G/T], \\ &2^m m! \cdot T(G/T) = c_1^m[G/T]. \end{aligned} \tag{4}$$

On the other hand, in our special case, $T(G/T, d) \cdot 1 = \mathcal{T}(G/T, d)^{\flat}$ and we have, by 22.2, for an element $d \in V^*$, that

$$T(G/T, d) = 0 \text{ if } d + (c_1/2) \text{ is singular,} \tag{5}$$

$$T(G/T, d) = \pm \deg(\lambda), \tag{6}$$

if d is a non-singular weight, and if λ is a suitable representation.

By Theorem 4.3, the sum $c_1 = \epsilon_1 a_1 + \cdots + \epsilon_m a_m$ is non-singular if and only if $\epsilon_1 a_1, \cdots, \epsilon_m a_m$ is a positive system of roots of G. By 4.9 and 12.4, we get that $\epsilon_1 a_1 + \cdots + \epsilon_m a_m$ is non-singular if and only if the invariant almost complex structure on G/T with roots $\epsilon_1 a_1, \cdots, \epsilon_m a_m$ is integrable.

Putting the value 0 for d in (5), we see that the Todd genus of G/T endowed with a non-integrable invariant almost complex structure vanishes. If, however, the structure is integrable, i.e., c_1 is non-singular, then (for $d = 0$) we have in the notation of 22.2 that $w'(\frac{1}{2}c_1) = w'(b) = a/2$ and $\operatorname{Sgn}(w') = \epsilon_1\epsilon_2 \cdots \epsilon_m$. Thus λ is the trivial representation of degree 1 and the Todd genus of G/T equals $\deg(\lambda) = 1$.

22.4. Let $a_1, a_2, \cdots, a_m$ be as before the positive system of roots of G with respect to some ordering $\mathcal{S}$ and let $a = \sum_{j=1}^{m} a_j$. Choose the integrable invariant almost complex (i.e. complex) structure on G/T which has $a_1, a_2, \cdots, a_m$ as its roots ($\epsilon_i = 1$) and let G/T be oriented accordingly. An arbitrary element $b \in V^*$ can be regarded as element of $H^2(G/T)$ and then the number $\delta(b) = b^m[G/T]/m!$ is defined. δ defines a homogeneous polynomial of degree m on V^*, which vanishes if $(b, a_j) = 0$, see (4) and (5). Since a_i and a_j are not proportional for $i \neq j$ and since $\delta(a/2) = 1$ by (4), we get

$$(7) \qquad b^m[G/T] = m! \prod_{j=1}^{m} (b, a_j)/(a/2, a_j), \qquad (b \in V^*).$$

Formula (7) shows that b *is singular if and only if* $b^m[G/T] = 0$.

Theorem 20.3 implies immediately that

$$(8) \qquad b^m[G/T] = (a_1 a_2 \cdots a_m)^{-1} \sum_{w \in W(G)} \operatorname{Sgn}(w) w(b)^m,$$

where the right side of this equation has to be regarded as a quotient of two homogeneous polynomials of degree m on V. Assume now that b is a weight. Then b is in the positive Weyl chamber ($(b, a_j) > 0$ for $1 \leqq j \leqq m$) if and only if $b - a/2$ is in the closure of the positive Weyl chamber ($(b, a_j) \geqq 0$ for $1 \leqq j \leqq m$). By (3) and (6), we get:

If b is a weight contained in the positive Weyl chamber, then

$$(9) \qquad b^m[G/T]/m! = T(G/T, b - a/2) = \deg(\lambda),$$

where λ is the irreducible representation of $\tilde{G}$ with main weight $b - a/2$, and a is the sum of the roots a_j of the invariant complex structure on G/T; i.e., a is the first Chern class of G/T endowed with this complex structure.

By (7) and (8), we get well known formulas for the degree of λ, see 3.4.

22.5. In this and the following Section, we shall use 22.2 to prove the strictly multiplicative behavior of the Todd sequence in certain fibre bundles. For this purpose, we take the value 0 for d in 22.2. Then $2b$ equals the first Chern class of the complex vector bundle $\hat{\xi}_C$ along the fibres of $(E_\xi/T, B_\xi, G/T, \pi)$. By 22.2 and 22.3, we see that the integration over the fibre (with respect to the orientation of G/T induced by $\hat{\xi}_C$) gives, when applied to $\mathfrak{T}(\hat{\xi}_C)$, either 0 or $\mathrm{ch}(\lambda(\xi))$, where λ is the trivial 1-dimensional representation, and thus $\mathrm{ch}(\lambda(\xi)) = 1$. In either case, the integration over the fibre gives only a zero-dimensional term. According to the definition in 21.7, we get:

THEOREM. *Let ξ be a principle G-bundle. Choose an invariant almost complex structure on G/T and let $\hat{\xi}_C$ be the corresponding complex vector bundle along the fibres of $(E_\xi/T, B_\xi, G/T)$. Then the Todd sequence $\{T_j\}$ is strictly multiplicative in $(E_\xi/T, B_\xi, G/T)$ with respect to $\hat{\xi}_C$.*

For later use, we reformulate our result as follows: Let Ψ be a set of roots of G which contains for each root α exactly one of the roots α, $-\alpha$. Then Ψ is the set of roots of an invariant almost complex structure on G/T. Orient G/T by this structure and let $\natural(\Psi)$ be the integration over the fibre in $(E_\xi/T, B_\xi, G/T)$ with respect to that orientation. Then we have

$$(10)\qquad \Big(\prod_{\alpha\in\Psi} \alpha/(1-e^{-\alpha})\Big)^{\natural(\Psi)} = \begin{cases} 1, & \text{if } \Psi \text{ is a positive system} \\ 0, & \text{otherwise.}\end{cases}$$

Let ϵ be a map of Ψ into $\{1, -1\}$ and $s(\epsilon)$ the number of elements in Ψ which are mapped by ϵ on -1, and let $\mathrm{sgn}(\epsilon) = (-1)^{s(\epsilon)}$. We have

$$\prod_{\alpha\in\Psi} \epsilon(\alpha)\cdot\alpha/(1-e^{-\epsilon(\alpha)\cdot\alpha}) = \exp\big(\tfrac{1}{2}\sum_{\alpha\in\Psi}\epsilon(\alpha)\cdot\alpha\big)\cdot\prod_{\alpha\in\Psi}\alpha/(2\sinh(\alpha/2)).$$

Thus, by (10),

$$(10^*)\qquad \Big(\exp\big(\tfrac{1}{2}\sum_{\alpha\in\Psi}\epsilon(\alpha)\cdot\alpha\big)\cdot\prod_{\alpha\in\Psi}\alpha/(2\sinh(\alpha/2))\Big)^{\natural(\Psi)}$$

is equal to $\mathrm{sgn}(\epsilon)\cdot 1$, if $\{\epsilon(\alpha)\cdot\alpha \mid \alpha\in\Psi\}$ is a positive system, and to 0, otherwise.

We give two applications of formula (10): Let ξ be a principal G-bundle for which B_ξ is a compact *oriented* manifold. Let Ψ be a *positive* system of roots of G. Orient G/T accordingly and choose for E_ξ/T that orientation which is induced by those of B_ξ and G/T. Then, for $y \in H^*(E_\xi/T)$, we have (see 21.4):

$$y[E_\xi/T] = y^{\natural(\Psi)}[B_\xi].$$

This fact, together with (10) and 8.2, yields for every element x of $H^*(B_\xi)$ that

$$x[B_\xi] = (\pi^*(x) \cdot \prod_{\alpha \in \Psi} \alpha/(1 - \exp(-\alpha)))[E_\xi/T], \tag{11}$$

where π is the projection of E_ξ/T on B_ξ.

22.6. Now let ξ be again an arbitrary principal G-bundle. For a closed connected subgroup U of G which contains a maximal torus T of G, consider the diagram

$$E_\xi/T \xrightarrow{\pi_\nu} E_\xi/U \xrightarrow{\pi_\mu} B_\xi. \tag{12}$$

Orient G/U. Let Θ be a system of positive roots of U. Orient U/T by the invariant complex structure having Θ as its set of roots and give G/T the orientation induced by those of G/U and U/T.

According to the above diagram, we have the fibre bundles

$$\mu = (E_\xi/U, B_\xi, G/U, \pi_\mu), \qquad \nu = (E_\xi/T, E_\xi/U, U/T, \pi_\nu),$$

$$\tilde{\xi} = (E_\xi/T, B_\xi, G/T, \pi_{\tilde{\xi}}), \qquad \eta = (G/T, G/U, U/T, \pi_\eta),$$

which satisfy the assumptions of 8.4. (The ξ of 8.4 corresponds to the $\tilde{\xi}$ here and the θ of 8.4 to η.) If we define the integrations over the fibres with respect to the orientations already chosen, then formula (2) of 8.4 holds. Thus we can infer from (10) and 8.2 the

PROPOSITION. *Let x be an arbitrary element of $H^{**}(E_\xi/U)$. Then in the foregoing notation*

$$x^{\natural\mu} = (\pi_\nu^{**}(x) \cdot \prod_{\alpha \in \Theta} \alpha/(1 - e^{-\alpha}))^{\natural\tilde{\xi}}.$$

22.7. Let U be a closed connected subgroup of G containing a maximal torus T of G and assume that G/U has been endowed with an invariant almost complex structure $\mathcal{J}$. Thus G/U is oriented. Let Ψ be the set of the roots of $\mathcal{J}$ and Θ a system of positive roots of U according to which we orient U/T. Then G/T is oriented. Let ξ be again a principal G-bundle for which we consider the diagram (12) and the four fibre bundles μ, ν, $\tilde{\xi}$, η. We wish to prove that the generalized Todd sequence (22.1) is strictly multiplicative (21.7) in μ with respect to the complex vector bundle $\hat{\mu}_C$ along the fibres arising from the given invariant almost complex structure on G/U. Let n

be the complex dimension of G/U, i.e., the number of roots in Ψ. First we observe that $\pi_\nu{}^{**}(\mathcal{J}_y(\hat{\mu}_C))$ is, in virtue of 10.8, equal to the element

$$(13)\qquad \prod_{\alpha\in\Psi}((1+y)\alpha/(1-e^{-(1+y)\alpha})-y\alpha)\in H^{**}(E_\xi/T)\otimes\mathbf{R}[y]$$

which goes over into

$$(13^*)\qquad \prod_{\alpha\in\Psi}(1+ye^{-\alpha})\alpha/(1-\exp(-\alpha))\in H^{**}(E_\xi/T)\otimes\mathbf{R}[y]$$

if one multiplies the component of complex dimension i in (13) by $(1+y)^{n-i}$. We have to prove that $\mathcal{J}_y(\hat{\mu}_C)^{\natural\mu}$ is a zero-dimensional element of $H^{**}(B_\xi)$. In view of the proposition in 22.6 and the passage from (13) to (13*), we must prove that the element

$$(\prod_{\alpha\in\Psi}(1+ye^{-\alpha})\cdot\prod_{\beta\in\Theta\cup\Psi}\beta/(1-\exp(-\beta)))^{\natural\tilde{\xi}}$$

which is equal to

$$(14)\qquad (\exp(\sum_{\beta\in\Theta\cup\Psi}\beta/2)\cdot\prod_{\alpha\in\Psi}(1+ye^{-\alpha})\cdot\prod_{\beta\in\Theta\cup\Psi}\beta/(2\sinh(\beta/2)))^{\natural\tilde{\xi}}$$

is zero-dimensional. But this is a consequence of (10*). The set $\Theta\cup\Psi$ plays here the role of Ψ in (10*). Thus the element given in (14) equals the unit of $H^{**}(B_\xi)$ multiplied by $T_y(G/U)$, (G/U has the given almost complex structure). Here we have to use that in passing from (13) to (13*), the component of complex dimension n is not changed. Using (10*), we can obtain the value of $T_y(G/U)$. In order to formulate the final result more easily, we introduce the following definition.

DEFINITION. *Let U be a closed subgroup of G containing a maximal torus T of G. Let Ψ be a set of complementary roots of G with respect to U which contains for each complementary root α exactly one of the roots α, $-\alpha$. Let Θ be a system of positive roots of U. Then $k^p(G/U,\Psi,\Theta)$ is defined as the number of those positive systems of roots of G which contain Θ and exactly $n-p$ roots of Ψ and thus p roots of $-\Psi$ $(0\leqq 2p\leqq 2n=\dim_{\mathbf{R}} G/U)$.*

Using this definition, we have, in virtue of (10*) and the fact that the element given in (14) equals $T_y(G/U)\cdot 1$, that

$$(15)\qquad T_y(G/U)=\sum_{p=0}^{n}T^p(G/U)y^p=\sum_{p=0}^{n}(-y)^pk^p(G/U,\Psi,\Theta).$$

We notice that $k^p(G/U,\Psi,\Theta)$ depends only on Ψ and not on the choice of the positive system Θ of U if Ψ is the set of roots of an invariant almost complex structure on G/U. The number $k^p(G/U,\Psi,\Theta)$ is also well defined if G/U does not admit an invariant almost complex structure.

Formula (15) states in particular that the Todd genus $T(G/U)$ equals $k^0(G/U, \Psi, \Theta)$. Thus $T(G/U)$ equals 1 if $\Psi \cup \Theta$ is a positive system of roots of G and is 0 otherwise. If the invariant almost complex structure on G/U with Ψ as the set of its roots is integrable, then $\Psi \cup \Theta$ is a positive system (13.7) and thus $T(G/U) = 1$. If $T(G/U) = 1$, then $\Psi \cup \Theta$ is a positive system, but $\Psi \cup -\Theta$ is also positive, since $-\Theta$ is a system of positive roots of U and $k^0(G/U, \Psi, -\Theta) = T(G/U) = 1$. Therefore, $T(G/U) = 1$ implies that $\Psi \cup \Theta$, $\Psi \cup -\Theta$ are positive and thus closed systems. $\Theta \cup -\Theta$ is closed, since it is the set of roots of a subgroup. Thus $T(G/U) = 1$ implies that $\Psi \cup \Theta \cup -\Theta$ is closed and that the given invariant almost complex structure is integrable (12.4).

We express the results of this section in the following theorem.

22.8. THEOREM. *Let G be a compact Lie group and U a closed connected subgroup of maximal rank of G. The Todd genus of an invariant almost complex structure on G/U equals* 1 (*respectively* 0) *if the structure is integrable* (*respectively not integrable*). *With respect to a maximal torus T $(T \subset U \subset G)$, let Θ be a system of positive roots of U. Assume that G/U has been given an invariant almost complex structure $\mathfrak{J}$ and that Ψ is the set of roots of $\mathfrak{J}$. Then, letting n be the complex dimension of G/U and using the definition in* 22.7, *we have*

$$T_y(G/U) = \sum_{p=0}^{n} T^p(G/U) y^p = \sum_{p=0}^{n} (-y)^p k^p(G/U, \Psi, \Theta).$$

Let ξ be a principal G-bundle. The generalized Todd sequence

$$\{T_j(y; c_1, \cdots, c_j)\}$$

is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$ with respect to the complex vector bundle along the fibres $\hat{\xi}_C$ arising from $\mathfrak{J}$. In particular, if B_ξ is a compact almost complex differentiable manifold, if ξ is differentiable and if the differentiable manifold E_ξ/U has been endowed with an almost complex structure compatible (21.8) *with the almost complex structure of B_ξ and $\hat{\xi}_C$, then*

$$T_y(E_\xi/U) = T_y(B_\xi) \cdot T_y(G/U).$$

22.9. For $y = 1$, the preceding theorem gives results on the index $\tau(G/U)$, see [19, §§ 8 and 10]. These results remain correct for an arbitrary G/U not necessarily almost complex:

Let G be a compact connected Lie group and U a closed connected subgroup of G of maximal rank. Let T be a maximal torus of U. Then, with respect to T, let Ψ be a set of complementary roots containing for each complementary root α exactly one of the roots $\alpha, -\alpha$. Let ξ be a principal G-bundle. The element $\prod_{\alpha\in\Psi} \alpha/\operatorname{tgh}\alpha \in H^{**}(E_\xi/T)$ is symmetric in the α^2 $(\alpha\in\Psi)$, and thus belongs to $\pi_\nu^{**}(H^{**}(E_\xi/U))$. According to 10.7,

$$\prod_{\alpha\in\Psi} \alpha/\operatorname{tgh}\alpha = \pi_\nu^{**}\left(\sum_{j=0}^{\infty} L_j(p'_1, p'_2, \cdots, p'_j)\right),$$

where the p'_i are the Pontrjagin classes of the real vector bundle $\hat{\mu}$ along the fibres of $(E_\xi/U, B_\xi, G/U)$. (For the L_j, see [19, §1].) We orient the bundle along the fibres and thus also G/U by requiring that

$$\prod_{\alpha\in\Psi} \alpha = \pi_\nu^{**}(W_{2n}),$$

where W_{2n} denotes the Euler class of $\hat{\mu}$ $(2n = \dim_R G/U)$. Then the same calculations as in 22.7 show that $\{L_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$ and that

$$\tau(G/U) = \sum_{p=0}^{n} (-1)^p k^p(G/U, \Psi, \Theta),$$

Θ being an arbitrary system of positive roots of U. As a consequence of the strictly multiplicative behavior, we have

$$\tau(E_\xi/U) = \tau(B_\xi)\cdot\tau(G/U),$$

in case B_ξ is a compact oriented differentiable manifold and ξ a differentiable bundle and after introducing convenient orientations.

In a similar way, under the assumptions of this section, we get by setting $y = -1$ for the Euler number $E(G/U)$ that

$$E(G/U) = \sum_{p=0}^{n} k^p(G/U, \Psi, \Theta).$$

22.10. The strictly multiplicative behavior (22.8) of the Todd sequence has certain formal consequences. We follow the notations of 22.7. Let ξ be a differentiable principal G-bundle over the compact almost complex differentiable manifold B_ξ and η a complex vector bundle over B_ξ. Consider the bundle $\mu = (E_\xi/U, B_\xi, G/U, \pi_\mu)$ and the complex vector bundle $\hat{\mu}_C$ along the fibres arising from a given invariant almost complex structure on G/U. Then endow E_ξ/U with an almost complex structure compatible with that of B_ξ and with $\hat{\mu}_C$. We have

$$(16)\qquad \mathfrak{T}(E_\xi/U, \pi_\mu^*\eta)^{\natural\mu} = T(G/U)\cdot\mathfrak{T}(B_\xi, \eta).$$

Proof. Since the complex tangent bundle of E_ξ/U is the Whitney sum of $\hat{\mu}_C$ and the complex tangent bundle of B_ξ lifted under π_μ, we obtain from the Whitney multiplication theorem (9.7) that

$$\mathfrak{T}(E_\xi/U) = \mathfrak{T}(\hat{\mu}_C)\cdot \pi_\mu{}^*\mathfrak{T}(B_\xi).$$

Thus

$$\begin{aligned}\mathfrak{T}(E_\xi/U,\pi^*{}_\mu\eta)^{\natural\mu} &= (\pi_\mu{}^*(\mathrm{ch}(\eta))\cdot\pi_\mu{}^*(\mathfrak{T}(B_\xi))\cdot\mathfrak{T}(\hat{\mu}_C))^{\natural\mu}\\ &= \mathfrak{T}(\hat{\mu}_C)^{\natural\mu}\cdot\mathfrak{T}(B_\xi,\eta) = T(G/U)\cdot\mathfrak{T}(B_\xi,\eta),\end{aligned}$$

which completes the proof.

As a consequence of (16), we get (see 21.4):

$$(17)\qquad T(E_\xi/U,\pi_\mu{}^*\eta) = T(G/U)\cdot T(B_\xi,\eta).$$

There is a formula for the generalized Todd sequence which is analogous to (16) and which follows from the strictly multiplicative behavior of the generalized Todd sequence. In order to write it down, we introduce the element $\mathrm{ch}_y(\eta)$ as the element obtained from $\mathrm{ch}(\eta)$ by multiplying its component of complex dimension j with $(1+y)^j$. The element $\mathrm{ch}_y(\eta)$ was denoted by $\mathrm{t}_y(\eta)$ in [19, § 12.2]. We have

$$(18)\qquad (\mathrm{ch}_y(\pi_\mu{}^*\eta)\cdot\mathfrak{T}_y(E_\xi/U))^{\natural\mu} = T_y(G/U)\cdot\mathrm{ch}_y(\eta)\mathfrak{T}_y(B_\xi),$$

which implies

$$(19)\qquad T_y(E_\xi/U,\pi_\mu{}^*\eta) = T_y(G/U)\cdot T_y(B_\xi,\eta).$$

For the definition of $T_y(B_\xi,\eta)$ and $T_y(E_\xi/U,\pi_\mu{}^*\eta)$, see [19, § 12]. Compare also [19, § 14.4].

22.11. Let G be a compact connected Lie group, U a closed connected subgroup of maximal rank and T a maximal torus of U. Let d be an element of $H^1(T,\boldsymbol{Z})$ which is orthogonal to all roots of U; i.e., d is invariant under all operations of the Weyl group of U. By the canonical isomorphism of $H^1(T,\boldsymbol{Z})$ with $\mathrm{Hom}(T,\boldsymbol{U}(1))$, the element d gives rise to an homomorphism of T in $\boldsymbol{U}(1)$ which has a unique extension to an homomorphism of U in $\boldsymbol{U}(1)$, also denoted by d. Now let ξ be a principal G-bundle. We extend the principal U-bundle $(E_\xi,E_\xi/U,U)$ by the homomorphism d of U in $\boldsymbol{U}(1)$. We get a principal $\boldsymbol{U}(1)$-bundle over E_ξ/U and the associated line bundle whose first Chern class we also denote by d. Following the notations of 22.6, it is clear that $\pi_\nu{}^*(d)$ is that element of $H^2(E_\xi/T,\boldsymbol{Z})$ which is obtained from the original element $d\in H^1(T,\boldsymbol{Z})$ by the negative transgression in $(E_\xi,E_\xi/T,T)$. Therefore, we may also denote $\pi_\nu{}^*d$ by d.

Now assume moreover that G/U carries an invariant complex structure. Let Ψ be the set of roots of this structure and let Θ be a system of positive roots of U. Then, by (13.7), $\Theta \cup \Psi$ is a positive system of G, to which belongs a positive Weyl chamber. Assume furthermore that d belongs to the closure of this Weyl chamber; i.e., $(d,\alpha) \geqq 0$ for $\alpha \in \Psi$. Let λ be the representation of G with main weight d and let d_ξ be the complex vector bundle over B_ξ associated with the λ-extension of ξ. Now, as in 22.10, we make the hypothesis that B_ξ is a compact almost complex manifold and that E_ξ/U has been given an almost complex structure compatible with the almost complex structure of B_ξ and the complex vector bundle $\hat{\mu}_C$ along the fibres of E_ξ/U arising from the given invariant complex structures on G/U. We have then, using the notations of 22.6, that

$$\mathcal{T}(E_\xi/U, d)^{\natural\mu} = \mathcal{T}(B_\xi, d_\xi). \tag{20}$$

Proof.

$$\mathcal{T}(E_\xi/U, d)^{\natural\mu} = (\pi_\nu{}^*(e^d \cdot \pi_\mu{}^*(\mathcal{T}(B_\xi)) \cdot \mathcal{T}(\hat{\mu}_C)) \cdot \prod_{\alpha\in\Theta} \alpha/(1-e^{-\alpha}))^{\natural\tilde{\xi}}$$
$$= \mathcal{T}(B_\xi)(e^d \prod_{\alpha\in\Theta\cup\Psi} \alpha/(1-e^{-\alpha}))^{\natural\tilde{\xi}}.$$

Formula (20) follows then by applying 22.2.

Remarks. (1) Formulas (16) and (20) are closely related to Grothendieck's generalized Riemann-Roch formula (not yet published); compare also with [7b, p. 241].

(2) The representation λ induces a holomorphic map β of G/U into a complex projective space $\boldsymbol{P}_q(\boldsymbol{C})$ such that $d \in H^2(E_\xi/U)$ restricted to G/U equals $\beta^*(e^*)$, where $e^* \in H^2(\boldsymbol{P}_q(\boldsymbol{C}), \boldsymbol{Z})$ is the cohomology class dual to a hyperplane (see 14.4).

23. The A-genus of certain homogeneous spaces. Throughout this paragraph, all cohomology groups are taken with real coefficients and all characteristic classes which occur are regarded as real classes.

23.1. Let $\{\hat{A}_j(p_1, \cdots, p_j)\}$ be the multiplicative sequence of polynomials [19, § 1] with $\frac{1}{2}z^{\frac{1}{2}}/\sinh \frac{1}{2}z^{\frac{1}{2}}$ as characteristic power series. The polynomials $\hat{A}_j$ are related to the A_j introduced in [19, § 1.6] by the equation

$$A_j = 2^{4j}\hat{A}_j.$$

For a real vector bundle ξ, we define the cohomology class $\hat{\mathfrak{A}}(\xi) \in H^{**}(B_\xi)$ as follows:

$$\hat{\mathfrak{a}}(\xi) = \sum_{j=0}^{\infty} \hat{A}_j(p_1(\xi), \cdots, p_j(\xi)).$$

If ξ is the tangent bundle of a differentiable manifold X, then we set $\hat{\mathfrak{a}}(\xi) = \mathfrak{a}(X)$. The genus $\hat{A}(X)$ of a compact oriented differentiable manifold X is given by

$$\hat{A}(X) = \hat{\mathfrak{a}}(X)[X].$$

$\hat{A}(X)$ vanishes if the dimension of X is not divisible by 4. For $\dim X = 4k$, we have

$$\hat{A}(X) = \hat{A}_k(p_1(X), \cdots, p_k(X))[X],$$

and, obviously, $A(X) = 2^{4k}\hat{A}(X)$, where $A(X)$ is the genus corresponding to the power series $2z^{\frac{1}{2}}/\sinh 2z^{\frac{1}{2}}$, and which is called the A-genus of X. If X is almost complex with vanishing first Chern class, then its Todd genus equals its $\hat{A}$-genus; see 22.1 and [19, p. 15].

23.2. Let G be a compact connected Lie group, T a maximal torus of G and U a closed connected subgroup of G containing T. Choose an ordering $\mathscr{S}$ (2.4), let Θ be the set of those roots which are positive with respect to $\mathscr{S}$ and belong to U, and let Ψ be the set of positive complementary roots. Orient U/T and G/T by the invariant complex structures with root systems Θ and $\Theta \cup \Psi$ respectively. We orient G/U by its Euler class using Ψ (see 22.9). Then, in the fibre bundle $(G/T, G/U, U/T)$, the orientations of G/U and and U/T induce that of G/T. Let ξ be a principal G-bundle. We adhere now strictly to the notations given in 22.6. Consider the fibre bundle

$$\mu = (E_\xi/U, B_\xi, G/U, \pi_\mu).$$

We wish to calculate the value of $\hat{A}(G/U)$ and to investigate under which conditions the sequence $\{\hat{A}_j\}$ behaves strictly multiplicatively (21.3) in μ. Let $\hat{\mu}$ be the real vector bundle along the fibres of μ. Then, by 10.7,

$$\pi_\nu^{**}\hat{\mathfrak{a}}(\hat{\mu}) = \prod_{\alpha \in \Psi} \alpha/(2\sinh(\alpha/2))$$

and we get, by the proposition in 22.6, that

$$\hat{\mathfrak{a}}(\hat{\mu})^{\natural\mu} = (\prod_{\beta \in \Theta} \beta/(1-e^{-\beta}) \prod_{\alpha \in \Psi} \alpha/(2\sinh(\alpha/2)))^{\natural\xi}.$$

If we write s for the sum of all $\alpha \in \Theta$, then

$$\hat{\mathfrak{a}}(\hat{\mu})^{\natural\mu} = (e^{\frac{1}{2}s} \prod_{\alpha \in \Theta \cup \Psi} \alpha/(2\sinh(\alpha/2)))^{\natural\xi}.$$

Let a be the sum of all roots in $\Theta \cup \Psi$. Since $\Theta \cup \Psi$ is a positive system of roots for G and since G/T is oriented by the invariant complex structure having $\Theta \cup \Psi$ as its set of roots, we get, in virtue of 22.2, that

$$\pi_\xi^{**}(\hat{\mathfrak{A}}(\hat{\mu})^{\natural\mu}) = E(s/4\pi(-1)^{\frac{1}{2}})/E(a/4\pi(-1)^{\frac{1}{2}}). \tag{1}$$

The genus $\hat{A}(G/U)$ is given by the constant term on the right side of the preceding equation which, by 22.3 (4), is also equal to $(s^m/2^m m!)[G/T]$, where $m = \dim_{\mathbf{C}}(G/T)$. Thus, by 22.4 (7),

$$\hat{A}(G/U) = \prod_{\beta \in \Theta \cup \Psi} (s,\beta)/(a,\beta). \tag{2}$$

Now assume that U is the centralizer of a toral subgroup of G and that Ψ is the set of roots of an invariant complex structure on G/U. Then $a - s$ represents the first Chern class of G/U. The element $a - s$ is orthogonal to all roots of Θ (see 14.2 and 14.8). Therefore, $(s,\beta) = (a,\beta)$ for all $\beta \in \Theta$ and thus, by (2),

$$\hat{A}(G/U) = \prod_{\beta \in \Psi} (s,\beta)/(a,\beta). \tag{3}$$

23.3. THEOREM. *Let G be a compact connected Lie group, T a maximal torus of G and U a closed connected subgroup of G containing T. Choose an ordering $\mathcal{S}$ and let Θ be the set of those roots which are positive with respect to $\mathcal{S}$ and belong to U. Let s denote the sum of all $\alpha \in \Theta$. Then the following holds:*

i) *The genus $\hat{A}(G/U)$ vanishes if and only if s is a singular element.*

ii) *If ξ is a principal G-bundle and $\hat{A}(G/U) = 0$, then the sequence $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$. In particular, if B_ξ is a compact orientable differentiable manifold and ξ a differentiable bundle, then $\hat{A}(G/U) = 0$ implies that $\hat{A}(E_\xi/U) = 0$ also.*

iii) *If $\hat{A}(G/U)$ is not zero, then U is the centralizer of a toral subgroup of G;* i.e., *G/U is homogeneous algebraic* (§ 14). *In particular, $\hat{A}(G/U)$ vanishes if the second Betti number of G/U is zero.*

iv) *If ξ is the universal principal G-bundle and $U \neq G$, then $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$ if and only if $\hat{A}(G/U) = 0$.*

Proof of i) *and* ii). The statement i) follows from formula (2). If s is singular, then $E(s/4\pi(-1)^{\frac{1}{2}}) = 0$ (see 3.2). Thus, ii) follows from i) and formula (1).

The proof of iii) will be preceded by the following lemma.

23.4. LEMMA. *If G is compact, connected, and semi-simple, if T is a maximal torus of G, and if U $(U \neq G)$ is a closed connected semi-simple* *subgroup of G which contains T, then the sum s of all roots of U which are positive with respect to a given ordering $\mathcal{S}$ on $V_T{}^*$ is singular.*

Proof. It is enough to prove the lemma for the case that U is a maximal connected subgroup of G, i.e., U is not contained in a closed connected subgroup of G different from U and G. In this case, we have ([7, p. 205], compare also 10.1)

$$G/U \cong (G_1/U_1) \times (G_2/U_2) \times \cdots \times (G_k/U_k),$$

where G_i is simple, rank $U_i = \operatorname{rank} G_i$, and U_i is a maximal connected subgroup of G_i. The lemma holds for G/U if it is true for at least one of the factors. Thus it suffices to prove the lemma for the case G simple and U a maximal connected subgroup of G. These spaces G/U were listed in [7, p. 219], see also 13.3. Because of i), it suffices to prove the lemma for one ordering on $V_T{}^*$, for then it is proved for all orderings on $V_T{}^*$. The proof will proceed by checking the various cases with G simple and U maximal.

If $G = \mathbf{B}_l$ and $U = \mathbf{B}_i \times \mathbf{D}_{l-i}$ $(0 \leqq i \leqq l-2)$, the positive roots of U with respect to a suitable ordering and a suitable maximal torus are

$$\pm x_r + x_t \ (1 \leqq r < t \leqq i), \ \pm x_r + x_t \ (i+1 \leqq r < t \leqq l), \ x_r \ (1 \leqq r \leqq l).$$

The sum s of these roots is

$$\sum_{j=1}^{i} (2j-1)x_j + 2\sum_{j=1}^{l-i} (j-1)x_{i+j}$$

which is orthogonal to the root x_{i+1}, and thus s is singular.

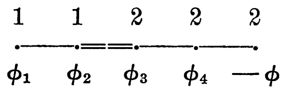
If $G = \mathbf{C}_l$ and $U = \mathbf{C}_i \times \mathbf{C}_{l-i}$, the positive roots of U with respect to a suitable ordering are (16.4)

$$\pm x_r + x_t \ (1 \leqq r < t \leqq i), \ \pm x_r + x_t \ (i+1 \leqq r < t \leqq l), \ 2x_r \ (1 \leqq r \leqq l).$$

The sum s of these roots is $2\sum_{j=1}^{i} jx_j + 2\sum_{j=1}^{l-i} jx_{i+j}$ which is orthogonal to the root $-x_1 + x_{i+1}$, and thus s is singular. Next we check $\mathbf{F}_4/\mathbf{B}_4$ and $\mathbf{G}_2/\mathbf{A}_1 \times \mathbf{A}_1$. In these cases, one can choose a set of complementary roots containing for each complementary root α exactly one of the roots α, $-\alpha$ and such that the sum of all roots of this set is 0 (see 18.3 and 19.2). But then it is an immediate consequence of 4.4 that s is singular. The space $\mathbf{G}_2/\mathbf{A}_2$ has dimension 6 (in fact, it is the 6-sphere). Thus $\hat{A}(\mathbf{G}_2/\mathbf{A}_2) = 0$, and the lemma is correct in this case by i).

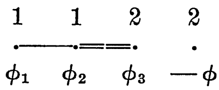
If G is simple and U maximal, one can choose orderings $\mathcal{S}$ and $\mathcal{S}'$ on $V_T{}^*$ such that each root of U simple with respect to $\mathcal{S}$ is a root of G which is either simple with respect to $\mathfrak{S}'$ or equals the negative dominant root of G with respect to $\mathfrak{S}'$ (see [7]). If $G=\boldsymbol{D}_l$, $\boldsymbol{E}_6$, $\boldsymbol{E}_7$, $\boldsymbol{E}_8$, then all $\mathfrak{S}'$-simple roots of G and the corresponding dominant root of G have equal lengths and thus all simple roots of U with respect to the ordering $\mathfrak{S}$ have all the same length ρ in the Killing metric of G. Then the sum s of all roots of U which are positive with respect to $\mathfrak{S}$ has a representative contravariant vector lying in the principal diagonal of the $\mathfrak{S}$-positive Weyl chamber of U [25a, p. 221], since $(s,\beta)=(\beta,\beta)$ for each $\mathfrak{S}$-simple root β of U, by 3.1. According to [25a, Théorème 7], the principal diagonal contains only singular vectors, and thus s is singular.

It remains to check our lemma for the spaces $\boldsymbol{F}_4/\boldsymbol{A}_1\times\boldsymbol{C}_3$ and $\boldsymbol{F}_4/\boldsymbol{A}_2\times\boldsymbol{A}_2$. The Schläfli diagram of $\boldsymbol{F}_4$ (including the dominant root) is



where the ϕ_i are the $\mathfrak{S}'$-simple roots of $\boldsymbol{F}_4$ and where $\phi=2\phi_1+4\phi_2+3\phi_3+2\phi_4$ is the $\mathfrak{S}'$-dominant root [25a]. The integers 1, 2 indicate the values of (ϕ_i,ϕ_i) and (ϕ,ϕ).

The subgroup $U=\boldsymbol{A}_1\times\boldsymbol{C}_3$ is represented by the following diagram which indicates the $\mathfrak{S}$-simple roots of U.



By an easy calculation, we get for the sum s of all $\mathfrak{S}$-positive roots of U

$$s=4\phi_1+6\phi_2+3\phi_3-2\phi_4$$

which is orthogonal to the root $\phi_1+2\phi_2+2\phi_3+\phi_4$. Thus s is singular.

Finally, we consider $\boldsymbol{F}_4/\boldsymbol{A}_2\times\boldsymbol{A}_2$. This space is almost complex (13.3) with vanishing Todd genus (22.8). Since the second Betti number of $\boldsymbol{F}_4/\boldsymbol{A}_2\times\boldsymbol{A}_2$ is zero, also the $\hat{A}$-genus vanishes (23.1) and thus the lemma is true in this case by (i).

The proof of the lemma is completed.[2]

23.5. *Proof of* iii). The proof proceeds by induction on the dimension of G/U. Assume iii) is proved for U', G' with $\dim G'/U'<n$. We prove it now for U, G with $\dim G/U=n$. If Q is the center of G, then G/U

[2] (Added in proof). Another proof of this lemma will be given in a forthcoming paper of the authors.

$=(G/Q)/(U/Q)$ and if U/Q is the centralizer of a toral subgroup in G/Q, then the same holds for U in G, and conversely. Thus, in proving iii) for U, G with $\dim G/U=n$, we may assume that G is semi-simple. Suppose $\hat{A}(G/U)\neq 0$. Then by i) and 23.4, the subgroup U is not semi-simple. Let T' be the identity component of its center and V the centralizer of T'. Then V is connected. $V\neq G$, since G is semi-simple. Thus $U\subset V\subsetneq G$. We have the fibre bundle $(G/U,G/V,V/U)$ and $\hat{A}(V/U)\neq 0$ by ii). By the induction hypothesis, U is the centralizer of T' in V; but then $U=V$.

23.6. *Proof of* iv). Let ξ be the universal principal G-bundle. It follows immediately from formula (1) in 23.2 that the sequence $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi, B_\xi, G/U)$ if and only if the function $E(s)\cdot E(a)^{-1}$, which is defined on V_T, is a constant (see 3.2). This happens in the following two cases and only then.

a) $E(s)$ is identically 0. b) $E(s)=\pm E(a)$.

We shall show that b) is impossible, if $U\neq G$. If b) holds, then s is not singular and U is the centralizer of a toral subgroup of G, according to i) and iii). From b), we infer more precisely that s is a transform of a under the Weyl group of G; i.e., $s=w(a)$ for some $w\in W(G)$. Now we can define a new ordering by letting the element $x\in V_T{}^*$ be positive if $(s,x)>0$. Since $s=w(a)$, we conclude that s equals the sum of all roots of G which are positive in this ordering. Since $(s,\beta)>0$ for all $\beta\in\Theta$ (notations of 23.3), all $\beta\in\Theta$ are positive in the new ordering. Since s is the sum of all roots in Θ, we infer that the sum of all complementary roots positive in the new ordering is zero which is impossible if $U\neq G$. Therefore $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$ if and only if a) holds, but this is the case if and only if s is singular (3.2). In virtue of 23.3, i), the element s is singular if and only if $\hat{A}(G/U)$ vanishes. This completes the proof.

23.7. *Remarks.*

1) As a corollary of 23.3, i), we mention that s is singular if the real dimension of G/U is not divisible by 4. Thus s can only be non-singular if G/U is homogeneous algebraic of even complex dimension, see 23.3, iii).

2) In view of 23.3, ii), one might formulate the following *conjecture*: Let ξ be a bundle for which F_ξ is a compact oriented differentiable manifold and G_ξ is a group of differentiable homeomorphisms of F_ξ. If $\hat{A}(F_\xi)=0$, then $\{\hat{A}_j\}$ is strictly multiplicative in ξ. (For the notations see 21.1-21.3.)

3) If the first Chern class of a compact almost complex manifold X vanishes, then $\hat{A}(X)$ is equal to the Todd genus of X. Taking this into account, 23.3, iii) is in agreement with 13.2 and 22.8.

4) In the proof of Lemma 23.4, we used the theorem of de Siebenthal on the principal diagonal. de Siebenthal proved his theorem by "checking all cases." There exists a general proof of it (A. Borel, unpublished).

24. Applications to simply connected algebraic homogeneous spaces.

24.1. Let X be a non-singular n-dimensional projective manifold, whose cohomology classes with respect to complex coefficients of type (p,q) vanish if $p \neq q$. Then the $h^{p,q}$ of X satisfy (see for instance [19]):

$$\text{(1)} \qquad \begin{aligned} \chi^p(X) &= \sum_{q=0}^{n} (-1)^q h^{p,q} = (-1)^p h^{p,p}, \\ \chi^p(X) &= (-1)^p \sum_{r+s=2p} h^{r,s} = (-1)^p b_{2p}, \end{aligned}$$

where b_i is the Betti number of X in the real dimension i; it vanishes if i is odd. From (1) and [19, § 21.3], we conclude

$$\text{(2)} \qquad T_y(X) = \sum_{p=0}^{n} (-y)^p b_{2p}.$$

For $y = 1$ (respectively $y = -1$), $T_y(X)$ is equal to the index $\tau(X)$ (respectively the Euler number $E(X)$) of X ([19], pp. 84, 122); hence

$$\text{(3)} \qquad \tau(X) = \sum_{p=0}^{n} (-1)^p b_{2p}, \qquad E(X) = \sum_{p=0}^{n} b_{2p}.$$

24.2. Let G be a compact connected Lie group, T a maximal torus, and U the centralizer of a toral subgroup of T. Let $\mathcal{J}$ be an invariant complex structure on G/U, Ψ its root system and Θ a system of positive roots of U. Then (13.7), $\Theta \cup \Psi$ is a positive system of roots of G. The complex manifold G/U (with the structure $\mathcal{J}$) is projective and satisfies the assumptions of 24.1 (see 14.4, 14.10). It follows then from (2) and 22.7, in the notations of 22.7, that

$$\text{(4)} \qquad b_{2p}(G/U) = k^p(G/U, \Psi, \Theta).$$

As was recalled in 2.7, the map $w \to w(\Theta \cup \Psi)$ is a 1-1 correspondence between the Weyl group $W(G)$ of G and the systems of positive roots. Let $W(G/U, \Psi, \Theta)$ be the set of those elements in $W(G)$ for which $\Theta \subset w(\Theta \cup \Psi)$. Each *right* coset $w(U) \cdot w$ of $W(G)$ modulo $W(U)$ contains at most one element of $W(G/U, \Psi, \Theta)$, since only the identity of $W(U)$ transforms Θ onto Θ. Moreover, given $w \in W(G)$, the system $w(\Theta \cup \Psi)$ contains a system Θ' of positive roots of U; hence, if u is the element of $W(U)$, carrying Θ' onto Θ, we have $u \cdot w \in W(G/U, \Psi, \Theta)$. Thus $W(G/U, \Psi, \Theta)$ is a system of representatives for the right cosets of $W(G)$ modulo $W(U)$.

Given $w \in W(G)$, let $\mu(w)$ be the number of elements in $w(\Theta \cup \Psi) \cap (-\Psi)$. Then, clearly, $k^p(G/U, \Psi, \Theta)$ is the number of elements in $W(G/U, \Psi, \Theta)$ for which $\mu(w) = p$. Since Ψ is invariant under $W(U)$, (13.4, remark), we have $\mu(w) = \mu(w')$ if w and w' belong to the same right coset of $W(G)$ modulo $W(U)$. By (4) and 13.7, we have the:

24.3. THEOREM. *Let U be the centralizer of a torus in the compact connected Lie group G. Let $\mathcal{S}$ be an ordering of the roots of G for which the set Ψ of positive complementary roots is closed. For $w \in W(G)$, let $\mu(w)$ be the number of positive roots whose image under w is a negative complementary root. Then we have, with $2n = \dim G/U$:*

$$(5)\qquad \begin{aligned} \sum_{p=0}^{n} b_{2p} t^{2p} &= (\operatorname{ord} W(U))^{-1} \sum_{w \in W(G)} t^{2\mu(w)}, \\ \tau(G/U) = \sum_{p=0}^{n} (-1)^p b_{2p} &= (\operatorname{ord} W(U))^{-1} \sum_{w \in W(G)} (-1)^{\mu(w)}, \end{aligned}$$

where $\tau(G/U)$ is the index of G/U, and b_{2p} its $2p$-th Betti number.

24.4. It follows in particular that $b_{2p}(G/T)$ (T maximal torus of G) equals the number of elements of $W(G)$ for which $w(\Psi)$ contains exactly p negative roots. Therefore, in the notations of 2.6, we have

$$\sum_{p=0}^{n} b_{2p}(G/T) t^{2p} = \sum_{w \in W(G)} t^{2s(w)}.$$

Theorem 24.3 was proved independently by R. Bott (Bull. Soc. Math. France 84 (1956), 251-281) in a slightly different formulation. 24.4 was also proved by C. Chevalley by means of a cellular decomposition (Tohoku Math. Journal 7 (1955), 14-66). The general case could also be read off from the cellular decomposition mentioned in [5].

24.5. *Kodaira's vanishing theorem.* Let X be a compact connected Kählerian manifold, n its complex dimension, and F a holomorphic complex line bundle over X. The bundle F is said to be negative of order $\geqq k$ if its first Chern class $c_1(F)$ can be represented by a closed real $(1,1)$-form ω of class C^∞ which, around every point $x \in X$, can be written in the form

$$\omega = i \sum g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

where $(g_{\alpha\beta})$ is a hermitian matrix with at least k negative eigenvalues. In particular, F is negative of order $\geqq n$ if and only if it is negative in the sense of Kodaira, that is, if and only if F^{-1} is positive in the sense of Kodaira. In [22], Kodaira has shown that *if F is negative, then the cohomology groups $H^q(X,F)$ of X with respect to the sheaf of germs of holomorphic sections of F vanish for $q < \dim_C X$.* By Serre's duality theorem, this is equivalent to the following statement: *Let K be the canonical bundle of X. If $F \otimes K^{-1}$ is positive, then $H^q(X,F) = 0$ for $q > 0$.*

Bott [7b, p. 231] has given a generalization of the first theorem in the case $q = 0$: *if F is negative of order $\geqq 1$, then $H^0(X,F) = 0$, that is, F does not admit a not identically zero holomorphic cross section.*

Remark. Bott formulates his theorem in a slightly different fashion; but, if one takes into account the lemma in [22, p. 1271], one gets Bott's theorem in the above form.

24.6. We keep the notations of 24.1 and 24.2. Since $H^{0,1}(G/U) = H^{0,2}(G/U) = 0$, by 14.10, the map assigning to a holomorphic line bundle over G/U its first Chern class defines an isomorphism between the group of isomorphism classes of line bundles and $H^2(G/U,\mathbf{Z})$. The negative transgression defines a homomorphism of the group A of weights which are orthogonal to the roots of U onto $H^2(G/U,\mathbf{Z})$ by 14.2. For any weight, we define $H^i(G/U,d)$ as the i-th cohomology group of G/U with respect to the sheaf of germs of holomorphic sections of a complex bundle with first Chern class d. $\chi(G/U,d)$ will denote the alternating sum of the dimensions of the $H^i(G/U,d)$.

24.7. THEOREM. *Let U be the centralizer of a torus in G, Ψ be the set of roots of an invariant complex structure on G/U, and Θ a set of positive roots for U. Let d be a weight orthogonal to the roots of U. If $(d,b) \geqq 0$ for all $b \in \Psi$, then $H^i(G/U,d) = 0$ $(i > 0)$ and $\dim_C H^0(G/U,d)$ equals $T(G/U,d)$, which is the degree of the irreducible representation of $\bar{G}$* (see 3.3) *with main weight d (in the ordering which has $\Theta \cup \Psi$ as positive roots). If $(d,b) < 0$ for at least one $b \in \Psi$, then $H^0(G/U,d) = 0$.*

The last assertion follows from Bott's theorem (24.5) and 14.6. Let c_1 be the first Chern class of G/U. Then $(c_1,b) > 0$ for $b \in \Psi$ by 14.8, and $(d,b) \geqq 0$ for $b \in \Psi$ implies $(d + c_1,b) > 0$. Since $c_1 = -c_1(K)$, the vanishing of $H^i(G/U,d)$, $(i > 0)$, follows then from 14.6 and 24.5.

Assume G/T and U/T to be endowed with the invariant complex structures having as root systems $\Theta \cup \Psi$ and Θ respectively. Then (14.3),

$(G/T, G/U, U/T, \nu)$ is a complex analytic fibering; we have by 22.8 and 22.10

(6) $$T(G/U, d) = T(G/T, d),$$

and, therefore, by Riemann-Roch

(7) $$\chi(G/U, d) = \chi(G/T, d).$$

Since $H^i(G/U, d) = H^i(G/T, d) = 0$ for $i > 0$, we get

$$\dim_C H^0(G/U, d) = T(G/T, d),$$

and the remaining assertion of the theorem follows from 22.4.

Remarks. (1) Assume that $U = T$. Let $d \in H^2(G/T, \boldsymbol{Z})$ be such that $d + (c_1/2)$ is in the closure of the positive Weyl chamber, but not inside; therefore it is singular, $(d, b) < 0$ for at least one positive root b, and $(d + c_1, b) > 0$ for all positive roots b. By 14.6 and 24.5, it follows that *all* cohomology groups $H^i(G/T, d)$ vanish, in agreement with the fact (22.3(5)) that $T(G/T, d) = \chi(G/T, d) = 0$ if $d + (c_1/2)$ is singular.

(2) If the weight d is orthogonal to the roots of U, the element $d \in H^2(G/T, \boldsymbol{Z})$ is the first Chern class of a line bundle which is the image under ν^* of a line bundle on G/U with first Chern class $d \in H^2(G/U, \boldsymbol{Z})$. Since $H^{p,q}(U/T) = 0$ for $p \neq q$ (14.10), one can deduce by a spectral argument applied to the fibering $(G/T, G/U, U/T, \nu)$ that, more generally than in the proof of 24.7, ν^* induces an isomorphism of $H^i(G/U, d)$ onto $H^i(G/T, d)$ for all i and all d.

24.8. We assume here that G is semi-simple. Then $H^2(G/T, \boldsymbol{Z})$ is isomorphic to the group of weights of G.

The projective space associated to the vector space $H^0(G/T, d)$ can be identified with the complete linear system of all positive divisors whose homology class is dual to d. Thus the preceding results on $\dim H^0(G/T, d)$ are also consequences of the results of [7a] quoted in 14.4.

Bott [7b] has proved the following theorem, which had been conjectured by the authors in view of 22.2 and 24.7:

THEOREM (Bott). *Let d be a weight. Then all groups $H^i(G/T, d)$ vanish if and only if $d + (c_1/2)$ is singular. If $d + (c_1/2)$ is regular and if w is the unique element of $W(G)$ which brings $d + (c_1/2)$ into the positive Weyl chamber, then $H^i(G/T, d)$ is zero if $i \neq s(w)$, and is equal to the degree of the irreducible representation $\bar{G}$ with main weight $w(d + (c_1/2)) - (c_1/2)$ if $i = s(w)$ (see 2.6 for $s(w)$).*

24.9. *Degrees of embeddings.* We follow the preceding notations. Let d be a weight orthogonal to all roots of U for which moreover $(d,b) > 0$ for all $b \in \Psi$. Let Γ be the representation with main weight d and $\check{\Gamma}$ the contragredient representation. G/U is *strictly* associated (14.4) to Γ, and $\check{\Gamma}$ induces an embedding j of G/U in the complex projective space $\boldsymbol{P}_q(\boldsymbol{C})$, where $q+1$ is the degree of the representation Γ. If $e^* = H^2(\boldsymbol{P}_q(\boldsymbol{C}),\boldsymbol{Z})$ is dual to a hyperplane of $\boldsymbol{P}_q(\boldsymbol{C})$, then $j^*(e^*) = d$ (d regarded now as element of $H^2(G/U,\boldsymbol{Z})$) and the value of the cohomology class d^n $(n = \dim_{\boldsymbol{C}} G/U)$ on the fundamental cycle of G/U is the degree of the embedding in the sense of algebraic geometry. The following formula is clear for an arbitrary $d \in H^2(G/U,\boldsymbol{R})$

$$d^n[G/U] = n!\lim_{r\to\infty} r^{-n}T(G/U,rd).$$

Let a be the sum of all roots in $\Theta \cup \Psi$, then (6) and 22.3(4) and 22.4(7) give

$$T(G/U,rd) = \prod_{c\in\Theta}(rd+a/2,c)/(a/2,c)\cdot\prod_{b\in\Psi}(rd+a/2,b)/(a/2,b).$$

Since $(d,c) = 0$ for $c \in \Theta$, the first product equals 1. Passing to the limit yields

$$d^n[G/U] = n!\prod_{b\in\Psi}(d,b)/(a/2,b) \qquad \text{for } d\in H^2(G/U,\boldsymbol{R}). \tag{8}$$

24.10. THEOREM. *Let G be a compact connected Lie group, T a maximal torus of G and U the centralizer of a toral subgroup of T. Endow G/U with an invariant complex structure, and let Ψ be the set of its roots. Choose an ordering $\mathcal{S}$ on V_T for which Ψ is the set of all positive complementary roots. Let a be the sum of all positive roots. Let d be a weight orthogonal to the roots of U and for which $(d,b) > 0$ for all $b \in \Psi$. The contragredient representation of the irreducible representation of $\tilde{G}$ (3.3) with main weight d induces an embedding of G/U in a complex projective space (14.4). The degree of this embedding in the sense of algebraic geometry is*

$$d^n[G/U] = n!\prod_{b\in\Psi}(d,b)/(a/2,b), \qquad (n=\dim_C G/U). \tag{9}$$

24.11. As an example, we take $G = \boldsymbol{U}(4)$ and $U = \boldsymbol{U}(2)\times\boldsymbol{U}(1)\times\boldsymbol{U}(1)$. In 13.9, two invariant complex structures $\mathcal{B}_1$, $\mathcal{B}_2$ on G/U were defined. We shall calculate the number $c_1{}^5[G/U]$ with respect to these two structures. Let $a^{(1)}$ (respectively $a^{(2)}$) be the sum of the positive roots with respect to the ordering $\mathcal{S}_1$ (respectively $\mathcal{S}_2$) defined in 13.9. We have

$$a^{(1)} = 3x_4 + x_1 - x_2 - 3x_3, \qquad a^{(2)} = 3x_1 + x_2 - x_3 - 3x_4.$$

The first Chern classes and the roots of these two structures have been given in 13.9. With respect to the coordinates x_i, the metric in the universal covering V_T of the maximal torus of $\boldsymbol{U}(4)$ is the usual euclidean metric. Thus, in the formulas (8), (9), the scalar product is the ordinary one, and, by a straightforward computation, the Chern number $c_1{}^5[G/U]$ of G/U with respect to $\mathscr{C}_1$ (respectively $\mathscr{C}_2$) is 4860 (respectively 4500). Therefore we get an example of two 5-dimensional algebraic varieties which are C^∞-differentiably homeomorphic, but have different Chern numbers.

Chapter VII. Genera Defined by Pontrjagin Classes.

In this chapter, a real number s is said to be an integer exc 2, or integral exc 2, if there exists an integer k such that $2^k \cdot s$ is an integer. Analogously, a real cohomology class x is integral exc 2 if x, multiplied by a suitable power of 2, is the image of an integral cohomology class under the coefficient homomorphism induced by $\boldsymbol{Z} \to \boldsymbol{R}$.

25. The integrality of the A-genus.

25.1. Let $\{L_j(p_1, \cdots, p_j)\}$ and $\{A_j(p_1, \cdots, p_j)\}$ be the multiplicative sequences [19, § 1] with $z^{\frac{1}{2}}/\mathrm{tgh}\, z^{\frac{1}{2}}$ and $2z^{\frac{1}{2}}/\sinh 2z^{\frac{1}{2}}$ respectively as characteristic power series. The polynomials A_k have rational coefficients which, when written as quotients of relatively prime integers, do not contain the factor 2 in their denominators. It suffices to prove this for the coefficients a_k of the power series $2z^{\frac{1}{2}}/\sinh 2z^{\frac{1}{2}}$. The coefficient of z^k $(k \geqq 1)$ in this series is

$$a_k = (-1)^k 2^{2k+1}(2^{2k-1}-1)B_k/(2k)\,!$$

and, by elementary number theory, $(2k)\,!$ is not divisible by 2^{2k}, whereas by von Staudt's theorem, the Bernoulli number B_k contains 2 exactly to the first power in its denominator, which proves the desired result.

25.2. If X is a compact oriented differentiable manifold, then the genera $L(X)$, $A(X)$, $\hat{A}(X)$ are defined (21.2, 23.1). They are rational numbers which vanish if the dimension of X is not divisible by 4. We have

$$A(X) = 2^{4k}\hat{A}(X) \qquad \text{for } \dim X = 4k.$$

$A(X)$ may be written with an odd denominator; by [19, Haupsatz 8.2.2], the rational number $L(X)$ equals the index $\tau(X)$ and thus is an integer. In this paragraph, we wish to prove in particular that the A-genus $A(X)$ is also an integer or, equivalently, that $\hat{A}(X)$ is integral exc 2.

For a compact almost complex manifold X with Chern classes $c_i \in H^{2i}(X, \boldsymbol{Z})$ and for elements $d_1, \cdots, d_s \in H^2(X, \boldsymbol{R})$, we define the virtual Todd genus as in [19, § 11] by the formula

$$(1) \qquad T(d_1, \cdots, d_s)_X = ((1-e^{-d_1}) \cdots (1-e^{-d_s}) \sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j))[X].$$

This virtual Todd genus is a real number.

For $d \in H^2(X, \boldsymbol{R})$ the number $T(X, d)$ is defined by the formula

$$(2) \qquad T(X, d) = (e^d \sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j))[X], \qquad \text{(see 22.1)}.$$

By [19, Satz 14.3.2], the Todd genus $T(X) = T(X, 0)$ is an integer exc 2 and the virtual Todd genus $T(d_1, \cdots, d_s)_X$ is integral exc 2 if $d_1, \cdots, d_s$ are images of integral cohomology classes. We have

$$T(X, d) = T(X) - T(-d)_X$$

and thus $T(X, d)$ is also integral exc 2 if d is the image of an integral class. We give now a slight generalization of these results.

25.3. Proposition. *If the elements $d, d_1, \cdots, d_s$ of $H^2(X, \boldsymbol{R})$, (X compact almost complex), are integral* exc 2, *then $T(X, d)$ and the virtual Todd genus $T(d_1, \cdots, d_s)_X$ are integral* exc 2.

By (1) and (2), it is sufficient to prove that $T(X, d)$ is integral exc 2 if $2^k d$ is the image of an integral class for some positive integer k. This statement will be proved by induction on k. It is proved already for $k = 0$, and we assume it to be true for $k - 1$. We have

$$e^d = (1 - (1 - e^{-2d}))^{-\frac{1}{2}},$$

and therefore

$$T(X, d) = \left(\sum_{r=0}^{\infty} (-1)^r \binom{-\frac{1}{2}}{r} (1 - e^{-2d})^r \cdot \sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j) \right) [X].$$

The coefficients $(-1)^r \binom{-\frac{1}{2}}{r} = 2^{-2r} \binom{2r}{r}$ are integers exc 2. If $2^k d$ is the image of an integral class, we see that $T(X, d)$ is a finite linear combination, with integers exc 2 as coefficients, of numbers $T(X, f)$, where f runs through certain elements of $H^2(X, \boldsymbol{R})$ for which $2^{k-1} f$ is the image of an integral class. By the induction assumption, it is therefore an integer exc 2.

The following theorem will include the integrality of the A-genus (25.2).

25.4. Theorem. *Let X be a compact oriented differentiable manifold with the Pontrjagin classes $p_j \in H^{4j}(X, \boldsymbol{Z})$. Let the element d of $H^2(X, \boldsymbol{R})$*

be integral exc 2. *Then the number* $\hat{A}(X,d)$ *defined by*

$$\hat{A}(X,d) = (e^d \sum_{j=0}^{\infty} \hat{A}_j(p_1,\cdots,p_j))[X]$$

is integral exc 2.

The theorem is trivial if the dimension of X is odd. Therefore we may put $\dim X$ equal to $2q$. Let $\xi = (E, X, \boldsymbol{SO}(2q))$ be the principal tangent bundle of X. Let T be a maximal torus of $\boldsymbol{SO}(2q)$ and $(x_1,\cdots,x_q)$ a base of $H^1(T,\boldsymbol{Z})$, see 10.1. We consider the fibre bundle

$$\zeta = (E/T, X, \boldsymbol{SO}(2q)/T, \pi).$$

Then $\pi^*(\xi)$ is the Whitney sum of q principal $\boldsymbol{U}(1)$-bundles $\xi_1,\cdots,\xi_q$, where ξ_j is the extension of $(E, E/T, T)$ with respect to $t \to \exp 2\pi i x_j(t)$. The first Chern class of ξ_i is x_i if we regard x_i under the negative transgression of $(E, E/T, T)$ as an element of $H^2(E/T,\boldsymbol{Z})$. Let $a_1,\cdots,a_m$ $(m = q(q-1))$, be the positive roots of $\boldsymbol{SO}(2q)$ with respect to T and an ordering. The a_i are the roots of an invariant integrable almost complex structure (§ 12) on $\boldsymbol{SO}(2q)/T$, to which belongs a complex structure of the vector bundle along the fibres of ζ. Thus the principal bundle η along the fibres of ζ is restricted to $\boldsymbol{U}(m)$ and the corresponding principal $\boldsymbol{U}(m)$-bundle η' is the Whitney sum of m principal $\boldsymbol{U}(1)$-bundles $\eta_1,\eta_2,\cdots,\eta_m$ whose first Chern classes are $a_1,\cdots,a_m$ regarded as elements of $H^2(E/T,\boldsymbol{Z})$.

The principal tangent bundle of E/T is the Whitney sum of $\pi^*\xi$ and η; thus E/T admits an almost complex structure whose principal tangent $\boldsymbol{U}(m+q)$-bundle is the Whitney sum of $\eta_1,\cdots,\eta_m,\xi_1,\cdots,\xi_q$. Hence E/T is an almost complex split manifold [19, § 13.5] with total Chern class

$$c(E/T) = (1+a_1)(1+a_2)\cdots(1+a_m)(1+x_1)(1+x_2)\cdots(1+x_q).$$

By 22.5(11), we get for an arbitrary element $d \in H^2(X,\boldsymbol{R})$,

$$\hat{A}(X,d) = (\pi^*(e^d)\prod_{j=1}^{m} a_j/(1-e^{-a_j}) \cdot \pi^*(\sum_{k=0}^{\infty}\hat{A}_k(p_1,\cdots,p_k)))[E/T].$$

Observing that

$$\pi^*(p(\xi)) = p(\pi^*\xi) = (1+x_1{}^2)(1+x_2{}^2)\cdots(1+x_q{}^2)$$

and using the identity $x/(1-e^{-x}) = (\frac{1}{2}x/\sinh\frac{1}{2}x)\cdot\exp(x/2)$, we obtain

$$\pi^*(\sum_{k=0}^{\infty}\hat{A}_k(p_1,\cdots,p_k))$$
$$= \prod_{i=1}^{q}(x_i/2)/\sinh(x_i/2) = e^{-\frac{1}{2}(x_1+\cdots+x_q)}\prod_{i=1}^{q} x_i/(1-e^{-x_i}).$$

Thus we see that

$$\hat{A}(X, d) = T(E/T, \pi^*(d) - \tfrac{1}{2}(x_1 + \cdots + x_q)),$$

and this, together with 25.3, proves 25.4.

25.5. THEOREM. *Let* $\eta = (E, X, \boldsymbol{U}(k))$ *be a principal bundle over a compact oriented differentiable manifold* X $(\dim X = 2q)$ *and let* p_j *denote the Pontrjagin classes of* X. *Let* $d \in H^2(X, \boldsymbol{R})$ *be integral* exc 2. *Then the number* $\hat{A}(X, d, \eta)$ *defined by*

$$\hat{A}(X, d, \eta) = (e^d \operatorname{ch}(\eta) \cdot \sum_{j=0}^{\infty} \hat{A}_j(p_1, \cdots, p_j))[X]$$

is integral exc 2. (As in 9.1, $\operatorname{ch}(\eta)$ denotes the Chern character of η).

We consider the associated bundle $\zeta = (E/T, X, \boldsymbol{U}(k)/T, \pi)$ where T is the standard maximal torus of $\boldsymbol{U}(k)$. Let $(x_1, \cdots, x_k)$ be the standard base of $H^1(T, \boldsymbol{Z})$. Then $\pi^*(\eta)$ is the Whitney sum of k principal $\boldsymbol{U}(1)$-bundles whose first Chern classes are $x_1, \cdots, x_k$ if we consider $x_1, \cdots, x_k$ via negative transgression as elements of $H^2(E/T, \boldsymbol{Z})$. We note that

$$\pi^* \operatorname{ch}(\eta) = \operatorname{ch}(\pi^*\eta) = e^{x_1} + e^{x_2} + \cdots + e^{x_k}. \tag{3}$$

Let $a_1, \cdots, a_m$ $(m = k(k-1)/2)$ be the positive roots of $\boldsymbol{U}(k)$ with respect to T and an ordering. By 10.7, the Pontrjagin class $\hat{p}$ of the bundle ζ along the fibres of ζ is given by

$$\hat{p} = \prod_1^m (1 + a_i^2) \tag{4}$$

and therefore

$$\sum \hat{A}_j(\hat{p}_1, \cdots, \hat{p}_j) = \prod (a_i/2)/\sinh(a_i/2) = e^{-(a_1 + \cdots + a_m)/2} \cdot \prod a_i/(1 - e^{-a_i}). \tag{5}$$

The tangent bundle to E/T is the direct sum of $\hat{\zeta}$ and of $\pi^*\sigma$ where σ is the tangent bundle to X (7.6). Thus if p'_i denotes the i-th Pontrjagin class of E/T, we have in view of (5)

$$\pi^*(\sum \hat{A}_j(p_1, \cdots, p_j)) \cdot \prod a_i/(1 - e^{-a_i}) = e^{(a_1 + \cdots + a_m)/2} \sum \hat{A}_j(p'_1, \cdots, p'_j). \tag{6}$$

On the other hand, it follows from 22.5(11) that

$$\hat{A}(X, d, \eta) = (\pi^*(e^d \cdot \operatorname{ch}(\eta) \cdot \sum \hat{A}_j(p_1, \cdots, p_j)) \cdot \prod a_i/(1 - e^{-a_i}))[E/T].$$

Together with (3) and (6), this gives

$$\hat{A}(X, d, \eta) = \sum_i \hat{A}(E/T, \pi^*(d) + x_i + (a_1 + \cdots + a_m)/2),$$

and the right hand side is an integer exc 2 by Theorem 25.4.

25.6. *Remarks.* The preceding theorem is the most general integrality theorem we give in this paper. All the theorems of integrality for the Todd genus, etc., [19, § 14.4, 2)] are formal consequences of it: Let X be a compact almost complex manifold of complex dimension q and η a principal $\boldsymbol{U}(k)$-bundle over X. We have $T(X,\eta) = \hat{A}(X, c_1/2, \eta)$. Thus $T(X)$ and $T(X,\eta)$ are integers exc 2. As a consequence, $T_y(X)$ and $T_y(X,\eta)$ are polynomials in y with integers exc 2 as coefficients [19, p. 93, (7), (8)]. The virtual T_y-characteristic $T_y(v_1,\cdots,v_r \,|, \eta)_X$ as defined in [19, p. 95], ($v_1,\cdots,v_r$ are elements of $H^2(X,\boldsymbol{Z})$), is a polynomial of degree $q-r$ in y which can be written as a formal power series in y, the coefficients being finite linear combinations with integral coefficients of polynomials $T_y(X,\xi)$, where ξ runs through certain unitary bundles depending on η, $v_1,\cdots,v_r$. This is purely formal (see also the analogous statement for the χ_y-theory, [19, p. 132]). Thus, $T_y(v_1,\cdots,v_r \,|, \eta)_X$ is also a polynomial with integers exc 2 as coefficients.

The proofs of 25.4 and 25.5 depend mainly on the strictly multiplicative behaviour of $x(1-e^{-x})^{-1}$, and on Proposition 25.3 which we actually would need only for almost complex split manifolds. The theory of Thom enters implicitly in the proof of 25.3 (integrality of virtual indices, see [19, § 9 and end of § 13]).

We do not know how far in 25.5 "integral exc 2" could be replaced by "integral." We can only dare the following *conjectures* which are motivated by the theorem of Riemann-Roch (see [18]). Let X be a compact oriented differentiable manifold and η a principal $\boldsymbol{U}(k)$-bundle over X.

1) *Let* w_2 *denote the second Stiefel-Whitney class of* X, ($w_2 \in H^2(X,\boldsymbol{Z}_2)$). *If* $d \in H^2(X,\boldsymbol{Z})$ *reduced* mod 2 *is* w_2, *then* $\hat{A}(X, d/2, \eta)$ *is an integer.*

2) *If* $w_2 = 0$ *and* $\dim X \equiv 4 \pmod 8$, *then* $\hat{A}(X)$ *is an even integer.*

2*) *If* $w_2 = 0$, $\dim X \equiv 4 \pmod 8$ *and if the structural group of* η *can be reduced to* $\boldsymbol{SO}(k)$, *then* $\hat{A}(X, 0, \eta)$ *is an even integer.*

These conjectures would be generalizations of Rohlin's theorem [24] that the Pontrjagin number $p_1[X]$ is divisible by 48 if $\dim X = 4$ and $w_2 = 0$. Rohlin's theorem goes over into conjecture 2 fro $\dim X = 4$.[3]

25.7. *Examples.* Putting the value 0 for d in 25.4 yields that $\hat{A}(X)$

[3] (Added in proof). A proof of (1), using the integrality of the Todd genus recently proved by Milnor (yet unpublished) will be given in the paper mentioned in footnote 2). For a different approach which proves (1) and (2*), see F. Hirzebruch, Séminaire Bourbaki, Exposé 177, Febr. 1959.

is integral exc 2 or, equivalently, that the A-genus of X (see 25.2) is an integer. This is non-trivial only if the dimension of X is divisible by 4. For $\dim X = 8$, we get [19, p. 14]

$$(-4p_2 + 7p_1^2)[X] \equiv 0 \pmod{45}. \tag{5}$$

The integrality of the L-genus (index) gives

$$(7p_2 - p_1^2)[X] \equiv 0 \pmod{45}. \tag{6}$$

The two congruences (5) and (6) are not independent of each other; (5) results if one multiplies (6) by -7. For $\dim X = 12$, the integrality of $A(X)$ and $L(X)$ respectively means

$$(16p_3 - 44p_2p_1 + 31p_1^3)[X] \equiv 0 \pmod{945}, \tag{7}$$

$$(62p_3 - 13p_2p_1 + 2p_1^3)[X] \equiv \pmod{945}. \tag{8}$$

In this case, neither of the two congruences is a formal consequence of the other, since one can derive from (7) and (8) that

$$(p_1p_2)[X] \equiv 0 \pmod{3}. \tag{9}$$

(8) is a formal consequence of (7) and (9), and (7) of (8) and (9). The congruence (9) can also be obtained by the use of Steenrod's reduced powers. In fact, by [15, Theorem 2.1],

$$p_1^3 \equiv -p_1(7p_2 - p_1^2) \pmod{3}.$$

Assume now that X is a compact connected oriented differentiable manifold of dimension $2q$, whose real Pontrjagin classes p_j vanish for $j \neq 0, \dim X$. Taking into account that $\hat{A}(X)$ is integral exc 2, we get

(10) *$d^q[X]/q!$ is integral* exc 2 *for all $d \in H^2(X, \mathbf{Z})$*

and also, by 25.5,

(11) $\mathrm{ch}(\eta)[X]$ *is integral* exc 2 *for every $\boldsymbol{U}(k)$-bundle over X.*

Let c_j be the Chern classes of η. We infer from the definition of the Chern character (9.1) that for $q \neq 0$, the $2q$-dimensional component $\mathrm{ch}(\eta)_q$ of $\mathrm{ch}\,\eta$ is of the form

$$q!\,\mathrm{ch}(\eta)_q = (-1)^{q+1} q \cdot c_q + P(c_1, \cdots, c_{q-1}), \tag{12}$$

where P is a polynomial in $q-1$ indeterminates with integral coefficients. Therefore (11) and (12) prove in particular the following:

25.8. THEOREM. *Let* ξ *be a* $\boldsymbol{U}(k)$*-bundle over* $\boldsymbol{S}_{2q}$, *and let* c_q *be its q-th Chern class. Then* $c_q[\boldsymbol{S}_{2q}]/(q-1)!$ *is an integer* exc 2.

The theorem is non trivial only for $k \geqq q$. For $q=k$, it implies that the spheres $\boldsymbol{S}_{2q}$ are not almost complex for $q \geqq 4$. (See also [18, § 2.1].) This was proved by Borel-Serre by showing that $c_q[\boldsymbol{S}_{2q}]$ is divisible by every prime p less than q and not dividing q, see [6, Propositions 12.4 and 15.1].

25.9. COROLLARY. *Let* η *be a principal* $\boldsymbol{O}(k)$*-bundle* ($\boldsymbol{Sp}(k)$*-bundle*) *over the sphere* $\boldsymbol{S}_{4q}$. *Let* p_q (*respectively* e_q) *be the q-th Pontrjagin class* (*q-th symplectic Pontrjagin class*) *of* η. *Then*

$$p_q[\boldsymbol{S}_{4q}]/(2q-1)! \text{ or } e_q[\boldsymbol{S}_{4q}]/(2q-1)! \text{ respectively}$$

is an integer exc 2.

For the proof, it is enough to observe that p_q (respectively e_q) is by definition up to sign the Chern class c_{2q} of the complex extension η' of η, with respect to the inclusion $\boldsymbol{O}(k) \subset \boldsymbol{U}(k)$ (respectively $\boldsymbol{Sp}(k) \subset \boldsymbol{U}(2k)$), and then to apply Theorem 25.8 to η'.

26. Applications to homotopy groups of Lie groups. In this paragraph, C_2 will be the class of finite commutative 2-groups.

26.1. The boundary homomorphism in the homotopy sequence of a bundle ξ will be denoted by ∂_ξ. We recall that there is a commutative diagram

$$\begin{array}{ccccc} \pi_i(B_\xi) & \longleftrightarrow & \pi_i(E_\xi \bmod F) & \xrightarrow{\partial_\xi} & \pi_{i-1}(F) \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\ H_i(B_\xi, \boldsymbol{Z}) & \longleftarrow & H_i(E_\xi \bmod F, \boldsymbol{Z}) & \xrightarrow{\partial_*} & H_{i-1}(F, \boldsymbol{Z}), \end{array} \tag{1}$$

where F is some fibre, ∂_* the boundary homomorphism of the relative homology sequence and α the Hurewicz homomorphism. Using the bottom line of (1) to define transgression in homology and the corresponding maps

$$H^i(B_\xi, \boldsymbol{Z}) \longrightarrow H^i(E_\xi \bmod F, \boldsymbol{Z}) \xleftarrow{\partial^*} H^{i-1}(F, \boldsymbol{Z})$$

to define the transgression τ_ξ in cohomology, we obtain readily the:

PROPOSITION. *Let* $x \in \pi_i(B_\xi)$ *and let* $y \in H^{i-1}(F_\xi, \boldsymbol{Z})$ *be transgressive. Then for any image* $\tau_\xi(y) \in H^i(B_\xi, \boldsymbol{Z})$ *of* y *by transgression, we have*

$$KI(\tau_\xi(y), \alpha(x)) = KI(y, \alpha\partial_\xi x),$$

where KI (for Kronecker index) denotes the standard pairing of homology and cohomology.

26.2. ι_n will denote a generator of $\pi_n(\boldsymbol{S}_n)$ or its image in $H_n(\boldsymbol{S}_n, \boldsymbol{Z})$, and $\iota_n{}^*$ the dual generator of $H^n(\boldsymbol{S}_n, \boldsymbol{Z})$. We recall [26, § 18.5] that if we associate to a principal G-bundle ξ over $\boldsymbol{S}_n$ the element $\partial_\xi(\iota_n) \in \pi_{n-1}(G)$, we define a 1-1 correspondence between the set of equivalence classes of principal G-bundles over $\boldsymbol{S}_n$ and $\pi_{n-1}(G)$. Also, since for any finite dimension, we may take a differentiable manifold as classifying space for G, each equivalence class may be represented by a differentiable bundle, and we shall assume our bundles to be differentiable whenever convenient. Clearly, if $\lambda\colon G \to G'$ is a homomorphism and if the G-bundle ξ is represented by α, then its λ-extension is represented by $\lambda_0(\alpha)$, where $\lambda_0\colon \pi_{n-1}(G) \to \pi_{n-1}(G')$ is induced by λ.

We shall be interested in the cases $n = 2q$, $G = \boldsymbol{U}(m)$ $(m \geqq q)$, $n = 4q$, $G = \boldsymbol{Sp}(r)$, $(r \geqq q)$, $n = 4q$, $G = \boldsymbol{SO}(s)$ $(s \geqq 2q+1)$, and shall denote by $c^*{}_k$ or $c^*{}_k(\xi)$ (respectively $e^*{}_k$ or $e^*{}_k(\xi)$, respectively $p^*{}_k$ or $p^*{}_k(\xi)$) the value of the k-th Chern (respectively symplectic Pontrjagin, respectively Pontrjagin) class on ι_n, where $n = 2k$ (respectively $n = 4k$, respectively $n = 4k$). It follows directly from the definition of the characteristic classes by means of classifying spaces that $\xi \to c^*{}_k(\xi)$ (respectively $\xi \to e^*{}_k(\xi)$, respectively $\xi \to p^*{}_k(\xi)$) is a homomorphism of the $(n-1)$-th homotopy group $\pi_{n-1}(G)$ of the structural group into $\boldsymbol{Z}$; hence this homomorphism depends only on $\pi_{n-1}(G)$ modulo torsion. Finally, we recall that the maps

$$\pi_{2q-1}(\boldsymbol{U}(r)) \to \pi_{2q-1}(\boldsymbol{U}(s)), \pi_{4q-1}(\boldsymbol{Sp}(r)) \to \pi_{4q-1}(\boldsymbol{Sp}(s)) \quad (s \geqq r \geqq q)$$
$$\pi_{4q-1}(\boldsymbol{SO}(r)) \to \pi_{4q-1}(\boldsymbol{SO}(s)) \quad (s \geqq r \geqq 4q+1)$$

induced by the standard inclusions are isomorphisms [26, § 22.8, 25.2, 25.5] and that

$$\pi_{4q-1}(\boldsymbol{SO}(2r+1)) \to \pi_{4q-1}(\boldsymbol{SO}(2s+1)), \qquad (s \geqq r \geqq q),$$

is an isomorphism mod C_2; this last fact follows from the homotopy sequence of the fibering $\boldsymbol{SO}(2r+1)/\boldsymbol{SO}(2r-1) = \boldsymbol{W}_{4r-1}$, where $\mathbf{W}_{4r-1}$ is the manifold of unit tangent vectors to $\boldsymbol{S}_{2r}$, and from the existence of a map $\boldsymbol{S}_{4r-1} \to \boldsymbol{W}_{4r-1}$ which induces a C_2-isomorphism of $\pi_i(\boldsymbol{S}_{4r-1})$ onto $\pi_i(\boldsymbol{W}_{4r-1})$ for all $i \geqq 0$ (see [25], Chapitre IV, Prop. 2).

26.3. Lemma. (a) *Let ξ be a principal $\boldsymbol{U}(q)$-bundle over $\boldsymbol{S}_{2q}$, and let η be the associated bundle with fibre $\boldsymbol{S}_{2q-1} = \boldsymbol{U}(q)/\boldsymbol{U}(q-1)$. Then*

$$\partial_\eta(\iota_{2q}) = \pm c^*{}_q(\xi) \cdot \iota_{2q-1}$$

(b) *Let ξ be a principal $\boldsymbol{Sp}(q)$-bundle over $\boldsymbol{S}_{4q}$, and η be the associated bundle with fibre $\boldsymbol{S}_{4q-1}=\boldsymbol{Sp}(q)/\boldsymbol{Sp}(q-1)$. Then $\partial_\eta\iota_{4q}=\pm e^*_q(\xi)\iota_{4q-1}$.*

The assertion (a) follows from the fact that $c_q(\xi)$ is the image by transgression of $\pm\iota_{2q-1}$ in η (see § 29) and from 26.1.

By definition, $e_q(\xi)=(-1)^q c_{2q}(\xi')$, where ξ' is the λ-extension of ξ under the inclusion $\lambda\colon \boldsymbol{Sp}(q)\to\boldsymbol{U}(2q)$. It is immediately seen that the pair inclusion $(\boldsymbol{Sp}(q-1),\boldsymbol{Sp}(q))\to(\boldsymbol{U}(2q-1),\boldsymbol{U}(2q))$ induces a homeomorphism of $\boldsymbol{Sp}(q)/\boldsymbol{Sp}(q-1)$ onto $\boldsymbol{U}(2q)/\boldsymbol{U}(2q-1)$. As a consequence, the bundle $(\xi',\boldsymbol{S}_{4q-1})$ associated to ξ' is the λ-extension of η, and then (b) is implied by (a).

26.4. The result quoted at the end of 26.2 implies in particular that $\pi_{4q-1}(\boldsymbol{W}_{4q-1})$ is the direct sum of $\boldsymbol{Z}$ and of a finite 2-group, and that there exists an integer $2^{a(q)}$ such that the image of the Hurewicz homomorphism

$$\alpha\colon \pi_{4q-1}(\boldsymbol{W}_{4q-1})\to H_{4q-1}(\boldsymbol{W}_{4q-1},\boldsymbol{Z})$$

is generated by $2^{a(q)}\cdot j_q$, where j_q is a generator of $H_{4q-1}(\mathbf{W}_{4q-1},\boldsymbol{Z})$.

LEMMA. *Let ξ be a principal $\boldsymbol{SO}(2q+1)$-bundle over $\boldsymbol{S}_{4q}$, η be the associated bundle with fibre $\boldsymbol{W}_{4q-1}=\boldsymbol{SO}(2q+1)/\boldsymbol{SO}(2q-1)$, and let γ_q be a generator of $\pi_{4q-1}(\boldsymbol{W}_{4q-1})$*mod 2*-torsion. Then we have in the previous notation*

$$\partial_\eta\iota_{4q}=\pm 2^{-a(q)-1}p^*_q(\xi)\gamma_q \text{ modulo 2-}\textit{torsion}.$$

Modulo 2-torsion, we have $\partial_\eta\iota_{4q}=c\cdot\gamma_q$, for some integer c, and therefore

$$\alpha\partial_\eta\iota_{4q}=2^{a(q)}\cdot c\cdot j_q.$$

By § 30, $p_q(\xi)$ is the image by transgression in η of $\pm 2j_q{}^*$, where $j_q{}^*$ is the generator of $H^{4q-1}(\boldsymbol{W}_{4q-1},\boldsymbol{Z})$ dual to j_q. Hence we have by 26.1

$$\pm p^*_q(\xi)=KI(2j_q{}^*,\ \alpha\partial_\eta\iota_{4q})=2^{a(q)+1}\cdot c$$

which proves the lemma.

26.5. THEOREM. There exists:

(a) *over $\boldsymbol{S}_{2q}$ a $\boldsymbol{U}(m)$-bundle with $c^*_q=(q-1)!$ for $m\geqq q$.*

(a*) *over $\boldsymbol{S}_{4q}$ a $\boldsymbol{SO}(n)$-bundle with $p^*_q=(2q-1)!\cdot 2$ for $n\geqq 4q$ and a $\boldsymbol{Sp}(m)$-bundle with $e^*_q=(2q-1)!\cdot 2$ for $m\geqq q$.*

(b) *over $\boldsymbol{S}_{4q}$, q even, a $\boldsymbol{SO}(n)$-bundle with $p^*_q=(2q-1)!$ for $n\geqq 4q+1$ (for $n\geqq 8$ if $q=2$).*

(c) *over* S_{4q}, *q odd, a* $Sp(m)$*-bundle with* $e^*_q = (2q-1)\,!$ *for* $m \geqq q$.

(d) *over* S_{4q} *a* $SO(n)$*-bundle with* p^*_q *equal, up to a power of* 2, *to the greatest odd factor of* $(2q-1)\,!$ *for* $n \geqq 2q+1$.

By 26.2 and the end remark in 9.7, it is enough to prove (a), (b), (c) for one particular value of m or n; (d) follows from (a*). The case $q=2$ in (b) will be dealt with in 26.6.

Let η be the principal $SO(2q)$-bundle of the tangential bundle to S_{2q}, and let $\lambda\colon Spin(2q) \to SO(2q)$ be the covering map. Since $w_2(\eta)=0$, the bundle η may be λ-restricted to a principal $Spin(2q)$-bundle. In fact, $\rho(\lambda)\colon B_{Spin(n)} \to B_{SO(n)}$ is (for any $n \geqq 2$) a fibre map with fibre B_{Z_2} (see [2] § 22 or [6] § 1), i.e. an Eilenberg-MacLane space $K(Z_2, 1)$; its spectral sequence shows readily that the obstruction to a cross section is the universal second Stiefel-Whitney class w_2; then, by a standard argument, every map $\sigma\colon B \to B_{SO(n)}$ with $\sigma^*(w_2)=0$ can be factorized through $\rho(\lambda)$, and this shows in our case the existence of a λ-restriction η' of η.

Let x_i, $(1 \leqq i \leqq q)$, be the standard basis of the usual maximal torus T of $SO(2q)$, and let T' be the inverse image of T in $Spin(2q)$; it is connected (see 10.1) and is a maximal torus of $Spin(2q)$; we shall also denote by x_i the image of x_i in $H^1(T', Z)$ under the covering map; these generate a subgroup of index 2 of $H^1(T', Z)$. Let $\beta\colon Spin(2q) \to U(2^{q-1})$ be one of the half-spinor representations, say the one with the highest weight $\frac{1}{2}(x_1 + \cdots + x_q)$, and let θ be the β-extension of η'). We want to prove

$$(2) \qquad c_q(\theta) = (-1)^{q-1}(q-1)\,! \cdot \iota_{2q}$$

which,[4] in view of our initial remark, will prove (a). Let ω_j $(1 \leqq j \leqq 2^{q-1})$, be the weights of β. It is known that these are just the linear forms

$$\tfrac{1}{2}(\epsilon_1 x_1 + \cdots + \epsilon_q x_q), \quad (\epsilon_i = \pm 1, (i=1, \cdots, q), \textstyle\prod \epsilon_i = 1)$$

(in fact, these are all transforms under the Weyl group $W(SO(2q))$ of the highest weight, hence they must be weights; moreover, since there are 2^{q-1} of them, they represent all weights). Let ρ be the projection of $E_{\eta'}/T'$ onto S_{2q}. By (9.5), we have $\rho^*(W_{2q}(\eta)) = x_1 \cdots x_q$, and hence

$$(3) \qquad x_1 \cdots x_q = 2 \cdot \rho^*(\iota_{2q}).$$

By (10.3), $\rho^*(c_q(\theta))$ is the q-th elementary symmetric function in the ω_j. Since the lower symmetric functions are zero here (because $H^i(S_{2q}, Z) = 0$

[4] The other half-spinor representation yields a bundle whose q-th Chern class is $-c_q(\theta)$.

for $0 < i < 2q$), we get

$$(4)\qquad (-1)^{q-1} q \cdot \rho^*(c_q(\theta)) = \omega_1^q + \cdots + \omega_s^q, \qquad (s = 2^{q-1}).$$

Let $\omega_j = \frac{1}{2}(\epsilon_1 x_1 + \cdots + \epsilon_q x_q)$ be one particular weight. We have

$$\omega_j^q = q! \cdot 2^{-q} \cdot x_1 \cdots x_q + b_j,$$

where b_j is a sum of monomials in the x_i's, none of which contains all variables x_i, and therefore

$$\sum \omega_j^q = q! \cdot 2^{-1} \cdot x_1 \cdots x_q + \sum b_j$$

or, taking (3) into account,

$$\sum \omega_j^q = q! \cdot \rho^*(\iota_{2q}) + \sum b_j$$

so that (2) will follow from (4) if we show that

$$(5)\qquad \sum b_j = 0.$$

$W(\boldsymbol{SO}(2q))$ is the group of permutations of the x_i combined with an even number of changes of signs. Thus, the ring I_W of invariants of $W(\boldsymbol{SO}(2q))$ is generated by $x_1 \cdots x_q$ and by the symmetric functions in the x_i^2. The Weyl group permutes the ω_j, and therefore $\sum b_j \in I_W$; since no monomial in this sum contains all variables x_i's, b_j must then be a symmetric function in the x_i^2; but, by (9.3), it is then the image under ρ^* of a polynomial in the Pontrjagin classes of η. Since the Pontrjagin classes of $\boldsymbol{S}_{2q}$ are all zero, this proves (5).

Let now $q = 2s$ be even; let ξ be the $\boldsymbol{U}(2s)$-bundle over $\boldsymbol{S}_{4s}$ with Chern class $(2s-1)!$, and ξ^* be its extension under the contragredient representation (10.6). We have $c_{2s}(\xi) = c_{2s}(\xi^*)$, hence

$$c_{2s}(\xi \oplus \xi^*) = (2s-1)! \cdot 2 \cdot \iota_{4s},$$

but in $\boldsymbol{U}(4s)$, the matrices of the form $A \dot{+} \bar{A}$ $(A \in \boldsymbol{U}(2s))$ form a subgroup conjugate to a subgroup of $\boldsymbol{Sp}(2s)$ or of $\boldsymbol{SO}(4s)$, and $\xi \oplus \xi^*$ can be considered as the complexification of a $\boldsymbol{Sp}(2s)$- or of a $\boldsymbol{SO}(4s)$-bundle. This proves (a*).

It is known that the image group of a half-spinor representation of $\boldsymbol{SO}(4q)$ is conjugate to a subgroup of $\boldsymbol{SO}(2^{2q-1})$ (respectively $\boldsymbol{Sp}(2^{2q-2})$), if q is even (respectively odd). (See E. Cartan, Jour. Math. Pur. Appl. 10 (1914), 149-186, § XV, p. 173, or A. I. Malcev, Isv. Ak. Nauk. SSSR Ser. Math. 8 (1944), 143-174, A. M. S. Translation 33, pp. 29-30.) This implies that θ is the complexification of a $\boldsymbol{SO}(2^{2q-1})$- (respectively $\boldsymbol{Sp}(2^{2q-2})$-) bundle, for q even (respectively odd), and (b) and (c) follow from (2).

26.6. *Remark.* The image group of $\mathbf{Spin}(8)$ under a half-spinor representation is conjugate to $\mathbf{SO}(8)$, as is well known and follows also from the result just quoted. The standard representation of $\mathbf{SO}(8)$ and the two half-spinor representations provide three homomorphisms of $\mathbf{Spin}(8)$ onto $\mathbf{SO}(8)$ which are, up to equivalence, all the representations of degree 8 of $\mathbf{Spin}(8)$. They may be distinguished by the element or order 2 of the center of $\mathbf{Spin}(8)$, (isomorphic to $\mathbf{Z}_2+\mathbf{Z}_2$), which they map onto the identity. They may be obtained from one another by performing on $\mathbf{Spin}(8)$ the automorphisms of the triality principle, (which are transitive on the elements of the center of $\mathbf{Spin}(8)$ different from the identity).

The last step of the proof of 26.5 shows therefore that θ is a $\mathbf{SO}(8)$-bundle over $\mathbf{S}_8$ with $p^*_2=-6$, which ends the proof of (b). On the other hand, the projective lines on the Cayley plane $\mathbf{W}$ are homeomorphic to $\mathbf{S}_8$ and a generator u of $H^8(\mathbf{W},\mathbf{Z})$ restricts on them to a fundamental cocycle; therefore 9.7 and 19.4 show that the normal bundle θ' to a projective line in $\mathbf{W}$ has also $p^*_2=-6$. In fact, it can be shown directly that θ and θ' are isomorphic; we sketch the proof, using 19.1 and some information on $\mathbf{W}$ to be found for instance in [1]: Let $\mathbf{U}$ be a subgroup of $\mathbf{F}_4$ isomorphic to $\mathbf{Spin}(9)$, $\mathbf{V}$ a subgroup of $\mathbf{U}$ isomorphic to $\mathbf{Spin}(8)$, and P the point of $\mathbf{W}$ fixed under $\mathbf{U}$. Then $\mathbf{V}$ leaves exactly two other points Q, R fixed, and the projective line M joining Q, R is operated upon transitively by $\mathbf{U}$. Thus the fibering $\xi=(\mathbf{U},\mathbf{U}/\mathbf{V},\mathbf{V})$ may be identified with $(\mathbf{Spin}(9),\ \mathbf{S}_8,\ \mathbf{Spin}(8))$. The natural representation of $\mathbf{V}$ into the tangent space $\mathbf{W}_Q$ of $\mathbf{W}$ at Q decomposes into the representations ρ_1, ρ_2 into M_Q and into the subspace N_Q of $\mathbf{W}_Q$ orthogonal to M_Q. Thus the tangent (respectively normal) bundle to M is the ρ_1- (respectively ρ_2-) extension of ξ. The representation of $\mathbf{V}$ in $\mathbf{W}_Q$ is faithful, because its kernel belongs to the center of $\mathbf{F}_4$, which is reduced to $\{e\}$. Hence (see beginning of this section) ρ_1 and ρ_2 are not equivalent, the normal bundle to M and the bundle θ of 26.5 arise from the tangent bundle to $\mathbf{S}_8$ by the same construction.

26.7. THEOREM. *The fibrations*

$$\mathbf{U}(q)/\mathbf{U}(q-1)=\mathbf{S}_{2q-1},\qquad \mathbf{Sp}(q)/\mathbf{Sp}(q-1)=\mathbf{S}_{4q-1},$$
$$\mathbf{SO}(2q+1)/\mathbf{SO}(2q-1)=\mathbf{W}_{4q-1},\qquad \mathbf{G}_2/\mathbf{Sp}(1)=\mathbf{W}_{11}$$

give rise to the following sequences, which are exact modulo the class C_2 *of finite commutative* 2-*groups:*

(a) $\quad 0\rightarrow \mathbf{Z}_{(q-1)!}\rightarrow \pi_{2q-2}(\mathbf{U}(q-1))\rightarrow \pi_{2q-2}(\mathbf{U}(q))\rightarrow 0 \qquad (q\geqq 2)$

(b) $0 \to \boldsymbol{Z}_{(2q-1)!} \to \pi_{4q-2}(\boldsymbol{Sp}(q-1)) \to \pi_{4q-2}(\boldsymbol{Sp}(q)) \to 0 \qquad (q \geqq 2)$

(c) $0 \to \boldsymbol{Z}_{(2q-1)!} \to \pi_{4q-2}(\boldsymbol{SO}(2q-1)) \to \pi_{4q-2}(\boldsymbol{SO}(2q+1)) \to 0 \qquad (q \geqq 2)$

(d) $0 \to \boldsymbol{Z}_{k15} \to \pi_{10}(\boldsymbol{S}_3) \to \pi_{10}(\boldsymbol{G}_2) \to 0$, *for some* $k \geqq 1$.[5]

(a) Let $\alpha \in \pi_{2q-1}(\boldsymbol{U}(q))$. Let ξ be a principal $\boldsymbol{U}(q)$-bundle over $\boldsymbol{S}_{2q}$ representing α, and η be the associated bundle with fibre $\boldsymbol{S}_{2q-1}$. Then $E\eta = E\xi/\boldsymbol{U}(q-1)$ and the restriction of the natural map $E\xi \to E\eta$ to a fibre is the projection map in the fibering $(\boldsymbol{U}(q), \boldsymbol{S}_{2q-1}, \boldsymbol{U}(q-1))$ hence we get a commutative diagram

$$\begin{array}{ccc} \pi_{2q}(\boldsymbol{S}_{2q}) & \xrightarrow{\partial\eta} & \pi_{2q-1}(\boldsymbol{S}_{2q-1}) \\ \updownarrow & & \uparrow \psi \\ \pi_{2q}(\boldsymbol{S}_{2q}) & \xrightarrow{\partial\xi} & \pi_{2q-1}(\boldsymbol{U}(q)), \end{array}$$

where ψ is part of the homotopy sequence of $(\boldsymbol{U}(q), \boldsymbol{S}_{2q-1}, \boldsymbol{U}(q-1))$. By definition, $\alpha = \partial\xi(\iota_{2q})$, hence we have by 26.3a.

$$\psi(\alpha) = \pm c^*{}_q(\xi)\iota_{2q-1}$$

which is divisible by the greatest odd factor of $(q-1)!$ in virtue of 25.8. Since α is arbitrary, this shows that $\psi(\pi_{2q-1}(\boldsymbol{U}(q))$ is contained in the subgroup generated by $b(q-1)\cdot\iota_{2q-1}$, where $b(q-1)$ is the greatest odd factor of $(q-1)!$; on the other hand, the same argument together with (26.5), shows that $\psi(\pi_{2q-1}(\boldsymbol{U}(q)))$ contains $(q-1)!\cdot\iota_{2q-1}$, and the mod C_2 exactness of (a) follows.

The proofs for (b) and (c) are quite analogous, the sole difference being that one has to invoke 26.3b and 26.4 instead of 26.3a.

For the fibration $\boldsymbol{G}_2/\boldsymbol{Sp}(1) = \boldsymbol{W}_{11}$, we refer to [4, § 17]. Let $\alpha \in \pi_{11}(\boldsymbol{G}_2)$, let ξ be a principal $\boldsymbol{G}_2$-bundle representing α, and let η be the associated bundle with fibre $\boldsymbol{W}_{11}$. We have the commutative diagram

$$(6) \qquad \begin{array}{ccc} \pi_{12}(\boldsymbol{S}_{12}) & \xrightarrow{\partial\eta} & \pi_{11}(\boldsymbol{W}_{11}) \\ \updownarrow & & \uparrow \psi \\ \pi_{12}(\boldsymbol{S}_{12}) & \xrightarrow{\partial\xi} & \pi_{11}(\boldsymbol{G}_2). \end{array}$$

Now $\boldsymbol{G}_2$ is embedded in $\boldsymbol{SO}(7)$ and its action on $\boldsymbol{W}_{11}$ extends to that of $\boldsymbol{SO}(7)$; in other words, $\boldsymbol{G}_2$, as a subgroup of $\boldsymbol{SO}(7)$, acts transitively on $\boldsymbol{W}_{11} = \boldsymbol{SO}(7)/\boldsymbol{SO}(5)$ and $\boldsymbol{G}_2 \cap \boldsymbol{SO}(5) = \boldsymbol{Sp}(1)$. This means that if we

[5] It will be shown later that $k = 1$.

extend the structural group of η to $\boldsymbol{SO}(7)$, we get the associated bundle to the extension ξ' of ξ. Therefore we have by 26.4:

$$\partial_\eta(\iota_{12}) = \pm 2^{-a(3)-1} p^*_3(\xi')\gamma_3 \bmod 2\text{-torsion}$$

and (6) gives then: $\psi(\alpha) = \pm p^*_3(\xi') \cdot \gamma_3$, up to a power of two. Since (25.9) the number $p^*_3(\xi')$ is divisible by 15, and since this argument is valid for any $\alpha \in \pi_{11}(\boldsymbol{G}_2)$, the mod C_2 exactness of (d) is established.

26.8. PROPOSITION. *If we have*

$$(7) \qquad \pi_9(\boldsymbol{SO}(7)) \equiv 0, \qquad \pi_{10}(\boldsymbol{U}(5)) \equiv \boldsymbol{Z}_{15} \bmod C_2,$$

then the following congruences mod C_2 *are valid:* $\pi_{10}(\boldsymbol{G}_2) \equiv 0$, $\pi_{10}(\boldsymbol{SO}(5)) \equiv \boldsymbol{Z}_{15}$, $\pi_{10}(\boldsymbol{SO}(n)) \equiv 0$ $(n \geqq 7, n \neq 8)$, $\pi_{10}(\boldsymbol{Sp}(n)) \equiv 0$ $(n \geqq 3)$, $\pi_{10}(\boldsymbol{U}(n)) \equiv 0$ $(n \geqq 6)$.

In this proof, all congruences are mod C_2. We use the following results:

$$(8) \qquad \pi_{n+1}(\boldsymbol{S}_n) \equiv \pi_{n+2}(\boldsymbol{S}_n) \equiv 0 \quad (n \geqq 3), \qquad \pi_{n+3}(\boldsymbol{S}_n) \equiv \boldsymbol{Z}_3 \quad (n \geqq 5),$$

$$(9) \qquad \pi_{10}(\boldsymbol{S}_3) \equiv \boldsymbol{Z}_{15}, \qquad \pi_9(\boldsymbol{S}_3) \equiv \boldsymbol{Z}_3.$$

(For the last equality of (8), see J-P. Serre, Comm. Math. Helv. 27 (1953), 198-232, for the other ones, see [25]; as to (7), see 26.9 and 26.10 below.)

The first equality in (9) shows that in 26.7d, we have $k = 1$ and $\pi_{10}(\boldsymbol{G}_2) \equiv 0$. The fibering $\boldsymbol{G}_2/\boldsymbol{Sp}(1) = \boldsymbol{W}_{11}$, discussed in [4, § 17], together with (9) and the congruence $\pi_i(\boldsymbol{W}_{11}) \equiv \pi_i(\boldsymbol{S}_{11})$ shows that $\pi_9(\boldsymbol{G}_2) \equiv \boldsymbol{Z}_3$. Applying this and (7) to the exact homotopy sequence of the fibering $\boldsymbol{Spin}(7)/\boldsymbol{G}_2 = \boldsymbol{S}_7$ (see [1]), we get the mod C_2 exact sequence

$$0 \rightarrow \pi_{10}(\boldsymbol{SO}(7)) \rightarrow \boldsymbol{Z}_3 \rightarrow \boldsymbol{Z}_3 \rightarrow 0;$$

hence $\pi_{10}(\boldsymbol{SO}(7)) \equiv 0$. The mod C_2 exact sequence 26.7c yields then, for $q = 3$, that $\pi_{10}(\boldsymbol{SO}(5)) \equiv \boldsymbol{Z}_{15}$. Since, as is well known, the universal covering $\boldsymbol{Spin}(5)$ of $\boldsymbol{SO}(5)$ is isomorphic to $\boldsymbol{Sp}(2)$, we deduce from 26.7b that $\pi_{10}(\boldsymbol{Sp}(3)) \equiv 0$, and hence also $\pi_{10}(\boldsymbol{Sp}(n)) \equiv 0$ for $n \geqq 3$.

Since $\boldsymbol{SO}(9)/\boldsymbol{SO}(7) \equiv \boldsymbol{W}_{15}$ has, mod C_2, the homotopy groups of $\boldsymbol{S}_{15}$, we have $\pi_{10}(\boldsymbol{SO}(9)) \equiv \pi_{10}(\boldsymbol{SO}(7)) \equiv 0$ and then $\pi_{10}(\boldsymbol{SO}(n)) \equiv 0$ $(n \geqq 9)$ follows from (8), the finiteness of $\pi_{10}(\boldsymbol{SO}(11))$ (see [25]) and the homotopy sequence of $\boldsymbol{SO}(n)/\boldsymbol{SO}(n-1) = \boldsymbol{S}_{n-1}$.

Finally, (7) and 26.7a give $\pi_{10}(\boldsymbol{U}(6)) \equiv 0$, and therefore $\pi_{10}(\boldsymbol{U}(n)) \equiv 0$ for $n \geqq 6$.

26.9. The preceding results (found in Spring 1957) contradict several of those of [30]. Since then, Toda has made new computations whose outcome (yet unpublished) agrees with the above. They have also been confirmed by Bott (Proc. Nat. Ac. Sci. USA 43 (1957), pp. 933-935) who in particular determines all stable homotopy groups of the classical groups.

26.10. Bott has also shown (to be published) that the sequence 26.7(a) is exact also for the 2-primary components (since $\pi_{2q-2}(\boldsymbol{U}(q))=0$, by Bott, loc. cit., 26.9, this gives $\pi_{2q-2}(\boldsymbol{U}(q-1))=\boldsymbol{Z}_{(q-1)!}$). This implies (see 26.3 and the proof of 26.7) the following generalization of 25.8, 25.9:

THEOREM (Bott). *Let ξ be a $\boldsymbol{U}(k)$-bundle over $\boldsymbol{S}_{2q}$. Then $c^*_q(\xi)$ is divisible by $(q-1)!$. Let η be a $\boldsymbol{SO}(k)$- (respectively $\boldsymbol{Sp}(k)$-), bundle over over $\boldsymbol{S}_{4q}$. Then $p^*_q(\eta)$ (respectively $e^*_q(\eta)$), is divisible by $(2q-1)!$.*

Using this theorem, Kervaire has proved more generally that $p^*_q(\eta)$ (respectively $e^*_q(\eta)$) is divisible by $(2q-1)!\cdot 2$ if q is odd (respectively even) (Amer. Jour. Math., vol. 80 (1958), pp. 632-638).

The stable homotopy groups $\pi_{2q-1}(\boldsymbol{U}(k))$, $(k\geqq q)$, and $\pi_{4q-1}(\boldsymbol{SO}(k))$ $(k\geqq 4q+1)$ are infinite cyclic according to Bott (loc. cit. in 26.9). The generator of the first group has the Chern number $c^*_q=\pm(q-1)!$, the generator of the second group has Pontrjagin number p^*_q equal to $\pm(2q-1)!$ if q is even and $\pm(2q-1)!\cdot 2$ if q is odd; similarly for the symplectic groups (with odd and even interchanged). This follows from the preceding theorem, the result of Kervaire and 26.5. The "spinor-method" in 26.5 gives an explicit construction for these generators.

The above theorem would follow by the same argument as 25.8, 25.9 if one could prove that the virtual Todd genus with respect to an *integral* class is an integer. In this respect, compare the conjectures in 25.6. The last one would contain Kervaire's result for the orthogonal groups.

Finally, we remark that the proof of 25.8, 25.9 also applies if the sphere is replaced by a compact connected oriented manifold X whose real Pontrjagin classes p_j vanish for $4j\neq 0, \dim X$, and if ξ (respectively η) is a $\boldsymbol{U}(k)$-bundle (respectively $\boldsymbol{O}(k)$- or $\boldsymbol{Sp}(k)$-bundle) whose real Chern (respectively Pontrjagin or symplectic Pontrjagin) classes vanish in all positive dimensions less than $\dim X$.

26.11. Milnor and Kervaire, independently, have deduced from the result of Bott quoted in 26.10 that $\boldsymbol{S}_n$, endowed with its usual differentiable structure, is not parallelizable if $n\neq 1,3,7$. An easy argument similar to 26.3 shows that if $\boldsymbol{S}_{2n-1}$ is parallelizable, that is if the fibering $\boldsymbol{SO}(2n)/\boldsymbol{SO}(2n-1)$ $=\boldsymbol{S}_{2n-1}$ has a cross section, then there exists a $\boldsymbol{SO}(2n)$-bundle η over $\boldsymbol{S}_{2n}$

whose Euler-Poincaré class W_{2n} is equal to $1 \cdot \iota_{2n}$. Therefore the theorem of Milnor-Kervaire is a consequence of the

THEOREM (Milnor). *Let ξ be a $\boldsymbol{SO}(2q)$-bundle over $\boldsymbol{S}_{2q}$, where $q \neq 1, 2, 4$. Then $W_{2q}(\xi)$ is divisible by 2.*

We want to give here a proof for this, different from Milnor's but also using 26.10.

Let η be a $\boldsymbol{SO}(2q)$-bundle whose second Stiefel-Whitney class w_2 vanishes. Then (see beginning of the proof of 26.5), η has a λ-restriction η', where λ is the projection of $\boldsymbol{Spin}(2q)$ onto $\boldsymbol{SO}(2q)$. We denote again by θ the extension of η' by means of the half-spinor representation β. It is a $\boldsymbol{U}(2^{q-1})$-bundle.

LEMMA. *In the previous notations, we have*

(i) $c_q(\theta)/(q-1)! = W_{2q}(\eta)/2$ *if q is odd,*

(ii) $$c_{2k}(\theta)/(2k-1)! = -(tg^{(2k-1)}(0)/((2k-1)!4))p_k(\eta) - W_{4k}(\eta)/2 + R_{2k}(\eta) \quad \text{if } q = 2k \geqq 2,$$

where $R_{2k}(\eta)$ is a polynomial with rational coefficients in $p_1(\eta), \cdots p_{k-1}(\eta)$, and $tg^{(2k-1)}(0)$ denotes the $(2k-1)$-th derivative of $tg\, x$ at $x=0$.

We keep the notations of 26.5. Since ρ^* is injective, we allow ourselves to omit the symbol ρ^*. The computations of 26.5 show first that

$$\sum \omega_j{}^q = q! \cdot x_1 \cdots x_q/2 + q!\, r'_q(x_1{}^2, \cdots, x_q{}^2), \tag{10}$$

where r'_q has rational coefficients; this can be written

$$\sum \omega_j{}^q = q!\, W_{2q}(\eta)/2 + q! \cdot r_q, \tag{11}$$

r_q *being* a polynomial in the $p_i(\eta)$ with rational coefficients. On the other hand, $c_q(\theta)$ is the q-th elementary symmetric function in the ω_j's, hence

$$\sum \omega_j{}^q = (-1)^{q-1} q \cdot c_q(\theta) + q!\, s_q,$$

where s_q is a polynomial in the $c_i(\theta)$ $(i < q)$ with rational coefficients. Now, θ is extension of a $\boldsymbol{Spin}(2q)$-bundle η', hence its characteristic ring is contained in the characteristic ring of η'. Since we consider real cohomology, $\rho^*(\lambda): H^*(B_{\boldsymbol{SO}(2q)}) \to H^*(B_{\boldsymbol{Spin}(2q)})$ is an isomorphism, and therefore the $c_i(\theta)$ belong to the characteristic ring of η, which is generated by the $p_i(\eta)$ $(i < q)$ and by $W_{2q}(\eta)$. Thus we get

$$(-1)^{q-1}c_q(\theta)/(q-1)! = W_{2q}(\eta)/2 + t_q,$$

where $t_q = r_q - s_q$ is a polynomial in the $p_i(\eta)$ $(2i < q)$ with rational coefficients. It is necessarily zero if q is odd, and this proves (i). In order to prove (ii), we have to compute the coefficient d_k of $p_k(\eta)$ in t_{2k}. By the above, d_k is equal to the coefficient of $p_k(\eta)$ in r_{2k}. Let S be defined by $S(x_1) = -x_1$ and $S(x_i) = x_i$ $(i \geqq 2)$ and put $\sigma_j = S(\omega_j)$. The set $\{\sigma_j\}$ is then the set of forms

$$(\epsilon_1 x_1 + \cdots + \epsilon_{2k} x_{2k})/2, \quad \prod \epsilon_i = -1,$$

(which are the weights of the second half spinor representation), and (10) yields

$$(12) \qquad \sum \sigma_j^{2k} = -(2k)!\, x_1 \cdots x_{2k}/2 + (2k)!\, r'_{2k}(x_1^2, \cdots, x_{2k}^2),$$

hence

$$(13) \qquad \sum (\omega_j^{2k} + \sigma_j^{2k}) = (2k)!\, 2 \cdot r_{2k},$$

so that $2d_k$ is the coefficient of p_k in

$$((2k)!)^{-1} \sum (\omega_j^{2k} + \sigma_j^{2k}).$$

We have clearly

$$\sum_{\epsilon_i = \pm 1} \exp(\epsilon_1 x_1 + \cdots + \epsilon_{2k} x_{2k})/2 = 2^{2k} \prod_{j=1}^{j=2k} \cosh(x_j/2).$$

Let $\{D_j(p_1, \cdots, p_j)\}$ be the multiplicative sequence with $\cosh z^{\frac{1}{2}}/2$ as characteristic power series (this is well defined since $\cosh x$ is an even function in x). Then $2^{-2k+1} d_k$ is for $k \geqq 1$ the coefficient of p_k in D_k. The formula 1.4(10) of [19] yields therefore, (with $2d_0 = 1$),

$$2 \sum_{j \geqq 0} d_j(-z/4)^j = \cosh(\tfrac{1}{2} z^{\frac{1}{2}}) \cdot d(z/\cosh(\tfrac{1}{2} z^{\frac{1}{2}}))/dz,$$

$$2 \sum_{j \geqq 0} d_j(-z/4)^j = 1 - (z^{\frac{1}{2}}/4)\operatorname{tgh}(\tfrac{1}{2} z^{\frac{1}{2}}).$$

Putting $z = -4x^2$, we get then

$$\sum_{j \geqq 1} d_j x^{2j-1} = (1/4)\operatorname{tg} x,$$

and this ends the proof of (ii).

Proof of the theorem. Since the base space of ξ is $\boldsymbol{S}_{2q}$ $(q \neq 1)$, we have $w_2(\xi) = 0$, hence we may apply the lemma to ξ. For q odd, the theorem follows then from (i) and from the divisibility theorem of Bott (26.10). Let now $q = 2k$ be even. We have $tg'(0) = 1$, $tg^{(3)}(0) = 2$, and it is well known, and easily checked, that $tg^{(2k-1)}(0)$ is an integer divisible by 4 for

$k \geqq 3$. Moreover, in (ii) of the lemma we have $R_{2k} = 0$ since $B_\xi = S_{4k}$. Thus, for $q = 2k$, the theorem follows from the lemma and 26.10.

27. Multiplicative properties of the index and consequences.

27.1. *Notation.* Throughout this and the following paragraph, all cohomology groups will be taken with real coefficients and all characteristic classes which occur will be regarded as real classes unless otherwise mentioned.

$\mathfrak{D}$ denotes the class of differentiable bundles ξ, where E_ξ, B_ξ, F_ξ are compact connected oriented differentiable manifolds, the orientation of E_ξ being induced by those of B_ξ, F_ξ taken in that order, and where the fundamental group of B_ξ operates trivially on $H^*(F_\xi)$.

We recall (21.5) that if $\{K_j\}$ is a multiplicative sequence with real coefficients which is strictly multiplicative in ξ, then

$$K(E_\xi) = K(B_\xi) \cdot K(F_\xi) \qquad (\xi \in \mathfrak{D}). \tag{1}$$

We wish to prove a theorem which is a sort of converse to 21.5.

27.2. THEOREM. *Let F be a compact connected oriented differentiable manifold. If $\{K_j\}$ is a multiplicative sequence of polynomials with real coefficients, for which* (1) *holds in every bundle $\xi \in \mathfrak{D}$ such that $F_\xi = F$, then $\{K_j\}$ is strictly multiplicative in each of these bundles.*

The proof uses essentially the theorem of Thom [29, Corollaire II 30] that every real cohomology class of B_ξ is a finite linear combination of real cohomology classes representable by submanifolds. By definition, a real cohomology class is representable by a submanifold if and only if it corresponds by Poincaré duality to a real homology class containing the fundamental cycle of a compact oriented differentiable manifold differentiably imbedded in B_ξ.

The strictly multiplicative behavior of $\{K_j\}$ is obviously true if $\dim B_\xi = 0$. Let us make the induction hypothesis that it is proved for $\dim B_\xi < n$. Then we will prove it, using (1), for an n-dimensional manifold B_ξ. Let $\hat{p}_i$ be the Pontrjagin classes of the bundle along the fibres of ξ. We have to show that

$$\Big(\sum_{j=0}^{\infty} K_j(\hat{p}_1, \cdots, \hat{p}_j)\Big)^{\natural} = K(F_\xi) \cdot 1,$$

where $\natural : H^*(E_\xi) \to H^*(B_\xi)$ is the integration over the fibre (8.1). We can restrict the bundle ξ to every submanifold Y of B_ξ. Since integration over the fibre and restriction commute (8.3), we obtain by our induction hypothesis that for $\dim Y < \dim B_\xi$ the restriction $(\sum_{j=0}^{\infty} K_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural$ to Y equals $K(F_\xi) \cdot 1$, where 1 denotes now the unit of the cohomology ring of Y. This, together with the above mentioned theorem of Thom, implies

$$(\sum_{j=0}^{\infty} K_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = K(F_\xi) \cdot 1 + c,$$

where $c \in H^n(B_\xi)$.

We denote the Pontrjagin classes of B_ξ by p'_i and those of E_ξ by p_i. Then (compare the proof of 21.5)

$$\begin{aligned} K(E_\xi) &= (\sum_{j=0}^{\infty} K_j(p_1, \cdots, p_j))[E_\xi] \\ &= ((K(F_\xi) \cdot 1 + c) \cdot \sum_{t=0}^{\infty} K_t(p'_1, \cdots, p'_t))[B_\xi] \\ &= K(F_\xi)(\sum_{t=0}^{\infty} K_t(p'_1, \cdots, p'_t))[B_\xi] + c[B_\xi] \\ &= K(F_\xi) \cdot K(B_\xi) + c[B_\xi]. \end{aligned}$$

According to (1) we have $K(E_\xi) = K(B_\xi) K(F_\xi)$ and thus obtain $c = 0$, which completes the proof.

It was proved recently [12] for the index τ and a bundle $\xi \in \mathfrak{D}$ that

(2) $$\tau(E_\xi) = \tau(B_\xi) \cdot \tau(F_\xi).$$

Since the index τ equals the genus L defined by the multiplicative sequence $\{L_j\}$, see [19, § 8], we get in virtue of the preceding theorem the following result:

27.3. THEOREM. *The sequence $\{L_j\}$ is strictly multiplicative in every $\xi \in \mathfrak{D}$.*

If $\xi \in \mathfrak{D}$ and $\natural$ is the integration over the fibre in ξ, we have therefore

$$(L_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = 0 \qquad (4j \neq \dim F_\xi),$$

$$(L_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = \tau(F_\xi) \cdot 1 \qquad (4j = \dim F_\xi).$$

27.4. *Examples.*

1) $\dim F_\xi = 2$. *In this case*, $\tau(F_\xi)$ vanishes. $L_j(\hat{p}_1, 0, \cdots, 0)$ is a non-zero multiple of $\hat{p}_1{}^j$, see [19, § 1]. Since $\hat{p}_1 = \hat{W}_2{}^2$, where $\hat{W}_2$ is the Euler class of the bundle along the fibres, we get (in real cohomology)

$$(\hat{W}_2{}^{2j})^\natural = 0, \qquad j = 1, 2, 3, \cdots.$$

If B_ξ is of dimension $4k-2$ and hence E_ξ of dimension $4k$, then $\hat{W}_2^{2k}=0$.

2) $\dim F_\xi = 3$. We get for the Pontrjagin class $\hat{p}_1$ of the bundle along the fibres that $(\hat{p}_1{}^j)^\natural$ $(j=1,2,3,\cdots)$, vanishes in real cohomology.

3) $\dim F_\xi = 4$. We have $45\cdot L_2(\hat{p}_1,\hat{p}_2)=7\hat{p}_2-\hat{p}_1{}^2$. Thus

$$(7\hat{p}_2-\hat{p}_1{}^2)^\natural=0.$$

This equation is proved in real cohomology and not known in integral cohomology. Again $\hat{p}_2=\hat{W}_4{}^2$, where $\hat{W}_4$ is the Euler class of the bundle along the fibres. If $\dim B_\xi=4$ (and hence $\dim E_\xi=8$), then

$$7\cdot\hat{W}_4{}^2=\hat{p}_1{}^2.$$

27.5. *Remark.* The strict multiplicativity of $\{L_j\}$ was proved in 22.9 for bundles $\xi\in\mathfrak{D}$ with $F_\xi=G/U$ $(\operatorname{rank}G=\operatorname{rank}U)$ and G as structural group. It was also shown (§ 23) in this special case that the sequence $\{A_j\}$ is strictly multiplicative provided $A(F_\xi)=0$. It might be conjectured that in every fibre bundle $\xi\in\mathfrak{D}$ the vanishing of $A(F_\xi)$ implies that of $A(E_\xi)$. As a consequence, we would have (27.2) that $\{A_j\}$ is strictly multiplicative in every fibre bundle $\xi\in\mathfrak{D}$ with $A(F_\xi)=0$.

28. A uniqueness theorem on the index.

28.1. In this paragraph, we shall prove that the sequence $\{L_j\}$ is essentially the only multiplicative sequence with coefficients in a field of characteristic 0 giving rise to a genus which is multiplicative in fibre bundles. Throughout this paragraph, we keep the notations of 27.1.

Let M be a $4k$-dimensional compact oriented differentiable manifold and p_i its Pontrjagin classes, which may be written formally as elementary symmetric functions:

$$1+p_1+\cdots+p_k=(1+\beta_1)\cdots(1+\beta_k).$$

Then the number $s(M)$ is defined by

$$s(M)=(\beta_1{}^k+\cdots+\beta_k{}^k)[M], \qquad \text{(see [19, § 6.3])}.$$

28.2. LEMMA. *Let $\xi\in\mathfrak{D}$ be a principal $\boldsymbol{U}(q)$-bundle over a 4-dimensional base space and c_i its Chern classes. If $2q+2=4k\geqq 8$, then*

$$s(E_\xi/\boldsymbol{U}(1)\times\boldsymbol{U}(q-1))=\left(-q(q+2)c_2+\left(\binom{q+1}{2}-1\right)c_1{}^2\right)[B_\xi].$$

For the proof, we use the notations of 15.1. We always take into account that B_ξ is 4-dimensional, and hence, for example, $c_i = 0$ for $i > 2$ and

$$(1) \qquad \pi^*(1 + c_1 + c_2) = (1 + x_1)(1 + x_2) \cdots (1 + x_q).$$

Furthermore

$$(2) \qquad x_1{}^q - x_1{}^{q-1}\pi^*(c_1) + x_1{}^{q-2}\pi^*(c_2) = 0.$$

Letting $a = \sum_{j=1}^{q} (x_j - x_1)^{q+1}$, we get by (1) that

$$(3) \qquad (-1)^q a = -q \cdot x_1{}^{q+1} + (q+1) x_1{}^q \cdot \pi^*(c_1) - \binom{q+1}{2} x_1{}^{q-1} \cdot \pi^*(c_1{}^2 - 2c_2).$$

We infer readily from (2) and (3) that

$$(-1)^q a = (q+2) q \cdot x_1{}^{q-1} \pi^*(c_2) + (1 - \binom{q+1}{2}) \cdot x_1{}^{q-1} \cdot \pi^*(c_1{}^2).$$

We may put $a = \rho^*(b)$ and $x_1 = \rho^*(\gamma_1)$. Then the preceding formula yields

$$(4) \qquad b = -q(q+2)(-\gamma_1)^{q-1}\sigma^*(c_2) + (\binom{q+1}{2} - 1) \cdot (-\gamma_1)^{q-1}\sigma^*(c_1{}^2)$$

M will denote the total space of the bundle $\theta = (E_\xi/\boldsymbol{U}(1) \times \boldsymbol{U}(q-1), B_\xi, \boldsymbol{P}_{q-1}(\boldsymbol{C}))$. The tangent bundle of M is the Whitney sum of $\hat{\theta}$, the bundle along the fibres of θ, and of the tangent vector bundle of B_ξ lifted by σ. We have for the total Pontrjagin classes

$$(5) \qquad p(M) = p(\hat{\theta}) \cdot \sigma^* p(B_\xi) = p(\hat{\theta}) \cdot (1 + \sigma^* p_1(B_\xi)).$$

In 15.1, one finds a formula for $\rho^* c(\eta')$ which yields

$$(6) \qquad \rho^* p(\hat{\theta}) = \prod_{j=1}^{q} (1 + (x_j - x_1)^2).$$

By (5) and (6), we get for $2q + 2 = 4k$

$$s(M) = (b + (\sigma^*(p_1(B_\xi))^k))[M].$$

Since $(p_1(B_\xi))^k = 0$ for $k \geqq 2$, we have $s(M) = b[M]$ which, by (4), completes the proof because the value of $(-\gamma_1)^{q-1}$ on the oriented fibres of θ equals 1.

28.3. We are going to construct a special base sequence for the algebra $\Omega \otimes \boldsymbol{Q}$ of Thom [29]. Consider over $X = \boldsymbol{P}_2(\boldsymbol{C})$ the differentiable principal $\boldsymbol{U}(q)$-bundle $\xi(q)$ which is the Whitney sum of $q-2$ trivial $\boldsymbol{U}(1)$-bundles and of the two principal $\boldsymbol{U}(1)$-bundles with g and $-g$ respectively as first

Chern classes, where g is the cohomology class dual to a complex projective line imbedded in X. The Chern classes of $\xi(q)$ are

$$c_1 = 0, \qquad c_2 = -g^2.$$

For $2q+2=4k \geqq 8$, let E^{4k} be the $4k$-dimensional manifold fibred with $\boldsymbol{P}_{q-1}(\boldsymbol{C})$ as fibre and associated to $\xi(q)$; i.e.

$$E^{4k} = E_{\xi(q)}/\boldsymbol{U}(1) \times \boldsymbol{U}(q-1), \qquad (2q+2=4k \geqq 8).$$

According to the preceding lemma, we have

$$(7) \qquad s(E^{4k}) = (2k+1)(2k-1) = 4k^2 - 1 \neq 0.$$

For $k=1$, we put $E^4 = \boldsymbol{P}_2(\boldsymbol{C})$ and then (7) holds also in this case.

By [19, § 6], the sequence $\{E^{4k}\}$, $(k=1,2,3,\cdots)$, of $4k$-dimensional manifolds is a base sequence of the algebra $\Omega \otimes \boldsymbol{Q}$ of Thom; and we have in terms of the usual base sequence $\boldsymbol{P}_{2k}(\boldsymbol{C})$

$$(8) \quad E^{4k} = (2k-1)\boldsymbol{P}_{2k}(\boldsymbol{C}) + \text{composite terms in the } \boldsymbol{P}_{2j}(\boldsymbol{C}), j < k.$$

28.4. THEOREM. *Let $\{K_r(p_1,\cdots,p_r)\}$ be a multiplicative sequence of polynomials with coefficients in a field of characteristic* 0 *and suppose that the corresponding genus K satisfies the equation $K(E) = K(B)\cdot K(F)$ for all differentiable fibre bundles in $\mathfrak{D}$, or equivalently* (27.2), *that $\{K_r(p_1,\cdots,p_r)\}$ is strictly multiplicative in $\mathfrak{D}$. Put $a = K(\boldsymbol{P}_2(\boldsymbol{C}))$. Then*

$$(9) \qquad K(Y) = a^r \cdot L(Y), \qquad (4r = \dim Y)$$

for all $4r$-dimensional compact oriented differentiable manifolds Y, and moreover

$$(10) \qquad K_r(p_1,\cdots,p_r) = a^r L_r(p_1,\cdots,p_r), \qquad (r=1,2,3\cdots).$$

We prove (9) by induction over r. It is true for $r=1$ since $\boldsymbol{P}_2(\boldsymbol{C})$ generates Ω^4. Suppose it is proved for all Y with $\dim Y < 4r$. The vector space $\Omega^{4r} \otimes \boldsymbol{Q}$ over the rationals is generated by E^{4r} and "composite" manifolds M^{4r} which are cartesian products of lower dimensional manifolds. Since K and L are both multiplicative in cartesian products, (9) is true on the composite manifolds M^{4r} by induction hypothesis, and since K and L are also both multiplicative in differentiable fibre bundles, (9) is true on E^{4r} too (again we have used the induction hypothesis). Thus (9) holds on $\Omega^{4r} \otimes \boldsymbol{Q}$.

This proves (9) in full generality which implies (10), [19, Satz 6.5.1].

Appendix I.

29. The different definitions of the Chern classes.

29.1. *Orientation conventions.* In the n-dimensional complex vector space V with coordinates $z_j = x_j + iy_j$, we take as usual the orientation defined by the order $x_1, y_1, \cdots, x_n, y_n$. This determines also an orientation for complex analytic manifolds as well as for the $2n - 1$ dimensional sphere S_{2n-1} of unit vectors with respect to some hermitian metric on V. The image of the fundamental cycle of S_{2n-1} thus defined in $\pi_{2n-1}(S_{2n-1})$, or $H_{2n-1}(S_{2n-1}, Z)$, or $H_{2n-1}(V - 0, Z)$, (0 being the origin in V), and the element of $H^{2n-1}(S_{2n-1}, Z)$ or $H^{2n-1}(V - 0, Z)$ taking the value 1 on it will be called the canonical generator of the corresponding group.

Let $W_{n,n-q+1}$ be the complex Stiefel manifold of ordered systems of $n - q + 1$ orthonormal vectors in C^n $(1 \leqq q \leqq n)$. We know that its first non-vanishing homotopy group is in dimension $2q - 1$ and is infinite cyclic. Now $W_{n,n-q+1}$ is fibered by $W_{q,1} = S_{2q-1}$, with base $W_{n,n-q}$; the projection assigning to each $(n - q + 1)$-frame the $(n - q)$-frame formed by its first $n - q$ elements. The fibre may thus be identified with the set of unit vectors in C^q and its injection in $W_{n,n-q+1}$ induces isomorphisms for the $(2q - 1)$-st homotopy or homology or cohomology groups. The element corresponding to the canonical generator previously defined will also be called the canonical generator.

Similarly, let $W^*_{n,n-q+1}$ be the manifold of ordered systems of $n - q + 1$ linearly independent vectors in C^n; it has $W_{n,n-q+1}$ as a deformation retract; let $e_1, \cdots, e_{n-q}$ be independent vectors, and let V be a q-dimensional subspace supplementary to the space spanned by the e_i's. The subspace U of $W^*_{n,n-q+1}$ made of the systems (f_j), $(j = 1, \cdots, n - q + 1)$, for which $f_j = e_j$ $(j \leqq n - q)$ and f_{n-q+1} is in V may be identified with $V - 0$. Its injection in the Stiefel manifold is an isomorphism for homotopy or homology in dimension $2q - 1$ and we define as before the canonical generator of $H_{2q-1}(W^*_{n,n-q+1}, Z)$ and $\pi_{2q-1}(W^*_{n,n-q+1})$.

29.2. *The Hopf fibering.* (x_i), $(1 \leqq i \leqq n + 1)$, are the coordinates of C^{n+1} and the homogeneous coordinates in the complex projective space P_n. By the Hopf fibering over P_n, we mean here $C^{n+1} - 0$ endowed with the usual $C^* = GL(n, 1)$ bundle structure. e will denote a hyperplane with the positive orientation or the corresponding homology class and $e^* \in H^2(P_n, Z)$ will be the dual cohomology class. Let U_i be the set of points in P_n for which $x_i \neq 0$, $(1 \leqq i \leqq n+1)$; using the usual conventions for the transition functions of a bundle [19, § 3.2.a] and of a line bundle associated to a divisor D [19, § 15.2] we see that in $U_i \cap U_j$ the Hopf fibering is given by $f_{ij} = x_i/x_j$, whereas the bundle $\{e\}$ associated to e is given by $g_{ij} = x_j/x_i$; thus the Hopf fibering is the inverse of $\{e\}$. We recall that the Hopf fibering over $\boldsymbol{P}_n$ is a $2n$-universal bundle for $\boldsymbol{C}^*$ or $\boldsymbol{U}(1)$.

29.3. *The definitions of Chern classes.* Including the definition (9.1), there are apparently seven definitions of Chern classes, which we proceed to list now; ${}^ic_j$ will be the j-th Chern class according to the i-th definition, and ${}^ic$ the sum of the ${}^ic_j$.

(1) *The definition* (9.1) *of this paper.* It may also be formulated in the following way: in the Hopf fibering, we put ${}^1c_1 = -\tau(x)$, where x is the canonical generator of $H^1(\boldsymbol{C}^*, \boldsymbol{Z})$; for a general $\boldsymbol{C}^*$-bundle, we use the characteristic map; for a general $\boldsymbol{GL}(n, C)$ bundle, we go over to the bundle of flags and take the elementary symmetric functions in the Chern classes of the different line bundles in which the lifted bundle decomposes.

(2) *The definition of* [19]: it is quite similar to (1), except that we put ${}^2c_1 = -e^*$ in the Hopf fibering.

(3) *The obstruction definition.* Given a complex vector bundle $(E, B, \boldsymbol{C}^n)$, we consider the associated bundle $(E', B, \boldsymbol{W}_{n,n-q+1})$. The first obstruction to the construction of a cross section (B is supposed to be a complex here) is an element of $H^{2q}(B, \pi_{2q-1}(\boldsymbol{W}_{n,n-q+1}))$. We identify the coefficient group with $\boldsymbol{Z}$ by sending the canonical generator onto 1, and thus get a class ${}^4c_q \in H^{2q}(B, \boldsymbol{Z})$.

This convention was introduced in [20], and was recalled at the beginning of [11] but was not made in [9], where consequently the obstruction classes are defined up to sign only.

(4) *The definition of* [6]: in the Hopf fibering, considered as universal bundle, we put ${}^4c_1 = \tau(x)$, and then proceed as in (1).

(5) *Schubert systems.* Let $\boldsymbol{H}(n, N)$ be the complex Grassmann manifold of n-dimensional subspaces of $\boldsymbol{C}^{n+N}$. It is the base space of the $2N$-universal bundle $(\boldsymbol{U}(n+N)/\boldsymbol{U}(N), \boldsymbol{H}(n, N), \boldsymbol{U}(n))$ for $\boldsymbol{U}(n)$, where $\boldsymbol{U}(N)$, (respectively $\boldsymbol{U}(n)$), is the subgroup of $\boldsymbol{U}(n+N)$, leaving the n first (respectively N last) coordinate vectors fixed. For $n=1$, we have the Hopf fibering. We take as universal ${}^5c_q$ the dual class to the Schubert cycle of dimension $2(nN-q)$ represented by the symbol $(N-1, \cdots, N-1, N, \cdots, N)$,

(q times $N-1$, $(n-q)$ times N). By the intersection properties of Schubert varieties, ${}^5c_q$ is also the cohomology class taking the value 1 on the Schubert cycle $(0,\cdots 0,1,\cdots,1)$ ($(n-q)$ times 0, q times 1), and zero on all other Schubert cycles of Ehresmann's cell decomposition of $\boldsymbol{H}(n,N)$; (Schubert cycles are defined for instance in [9, 10, 11].) This defines ${}^5c$ in the universal bundle; for a general bundle, we take its image by the characteristic map. For $n=1$, the Schubert symbol $(N-1)$ represents the hyperplane of $\boldsymbol{P}_N(\boldsymbol{C})=\boldsymbol{H}(1,N)$. Thus, in the Hopf fibering, we have ${}^5c_1=e^*$.

(6) *Definition by means of differential forms.* This leads to real cohomology classes, defined for differentiable bundles. Let ξ be a differentiable principal $\boldsymbol{U}(n)$-bundle and let Ω_{ij} $(1\leqq i,j\leqq n)$ be the curvature forms of a connexion on E_ξ. We then consider the (mixed) differential form:

$$\Psi=\sum\Psi_q=\det|\,\mathrm{Id}+(2\pi i)^{-1}\Omega_{ij}\,|$$

(Ψ_q of degree q; the product in the determinant is of course the exterior product). It defines a form on B_ξ, which is closed. The image of Ψ_q in $H^{2q}(B_\xi,\boldsymbol{R})$ is by definition ${}^6c_q$. This definition is introduced in [9] (our Ψ_q is the Ψ_{n-q+1} of Chern), although in an apparently slightly more restrictive way. Chern considers only bundles of (tangential) orthonormal frames to a hermitian manifold and a special connexion characterized by a property of its torsion tensor [9, p. 111]. However, by a theorem of Weil, whose proof is reproduced in [10, pp. 58-59], the cohomology class of Ψ_q is independent of the particular connexion chosen in ξ.

(7) *Definition by transgression.* ${}^7c_q$ is the image by transgression of the canonical generator of $H^{2q-1}(\boldsymbol{W}_{n,n-q+1},\boldsymbol{Z})$ in the bundle with fiber $\boldsymbol{W}_{n,n-q+1}$ associated to a given complex vector bundle.

The purpose of this Appendix is to prove the

29.4. THEOREM. *Let ${}^ic_j$ be the j-th Chern class of a bundle $(E,B,\boldsymbol{U}(n))$ with respect to the i-th definition $(j=1,\cdots,n;\,i=1,\cdots,7)$. Then* ${}^1c_j={}^2c_j={}^3c_j=(-1)^j\cdot{}^4c_j=(-1)^j\cdot{}^5c_j=(-1)^j\cdot{}^6c_j=-{}^7c_j$.

29.5. *Remarks.* (a) All these definitions have the naturality property: if $f\colon\xi\to\eta$ is a homomorphism, then $\bar{f}^*({}^ic(\eta))={}^ic(\xi)$, where $\bar{f}\colon B_\xi\to B_\eta$ is induced by f. This is obvious for $i=1,2,4,5,7$, and standard for $i=3$; for $i=6$, it follows by the theorem of Weil quoted above, because if Ω_{ij} are the curvature forms of a connexion $\mathcal{C}$ on E_η, then their images on E_ξ under f will be the curvature forms of the connexion induced from $\mathcal{C}$ by f.

(b) Let a, b $(1 \leqq a, b \leqq 7)$ be given, and assume that ${}^ac_1 = \epsilon \cdot {}^bc_1$ $(\epsilon = \pm 1)$ and that ${}^ac$ obeys duality.[6] Then ${}^bc$ obeys duality if and only if ${}^ac_j = \epsilon^j \cdot {}^bc_j$, $(1 \leqq j \leqq n)$. This is readily seen by using the bundle of flags. Thus ${}^ic$ has the duality property for $i \leqq 6$, but not for $i = 7$. For $i = 1, 2, 4$, the duality property follows immediately from an identity between elementary symmetric functions (see e. g. [6]). In the course of the proof of the theorem we shall *use* the fact that ${}^5c$ obeys duality, which is proved in [11], and therefore we do not provide a new proof for it. We note in passing that in the introduction of [11], Chern classes are defined as obstructions (with signs) but the proof for duality is carried out for Schubert cocycles. However, our 29. 4 shows that the obstruction classes obey duality, a fact for which there is, to our knowledge, no direct proof in the literature.

29. 6. LEMMA. *In the Hopf fibering, we have* ${}^7c_1 = + e^* = - {}^3c_1$.

In view of a general fact about transgression and obstructions, recalled below (29. 7), it is in fact enough to prove one equality, but both may be easily checked directly: As to the first one, we put $a = \sum x_i \cdot \bar{x}_i$ and consider

$$\Omega = (i/2\pi)(a^{-1} \cdot \sum dx_i \wedge d\bar{x}_i - a^{-2} \cdot \sum x_i \cdot \bar{x}_j \, dx_j \wedge d\bar{x}_i) ;$$

it is the imaginary part, multiplied by $1/\pi$, of the Fubini metric

$$(1) \qquad ds^2 = a^{-1} \cdot \sum dx_i \cdot d\bar{x}_i - a^{-2} \cdot \sum x_i \bar{x}_j \, dx_j \cdot d\bar{x}_i.$$

Ω is a closed real form of type $(1, 1)$ on $\boldsymbol{P}_n$ and we integrate it on the projective line $\boldsymbol{P}_1$: $x_3 = \cdots = x_{n+1} = 0$ with homogeneous coordinates (x_1, x_2); leaving out $(0, 1)$, we replace $\boldsymbol{P}_1$ by the cross section $x_1 = 1$ and get for the integral

$$(i/2\pi) \int_{\boldsymbol{P}_1} (1 + x_2 \cdot \bar{x}_2)^{-2} \cdot dx_2 \wedge d\bar{x}_2 = 1.$$

Hence Ω represents e^*. On the other hand, we have

$$\Omega = (i/2\pi) \, d\bar{\partial} \log a = d((i/2\pi) a^{-1} \cdot \sum x_i \cdot d\bar{x}_i),$$

and the restriction of $(i/2\pi)\bar{\partial} \log a$ to the fibre $x_2 = \cdots = x_{n+1} = 0$ is $i(2\pi\bar{x}_1)^{-1} d\bar{x}_1$, whose integral on the positively oriented unit circle is again 1; this proves the first equality.

${}^3c_1$ is the first obstruction to the construction of a cross section in the Hopf fibering and we only need to know its value on $\boldsymbol{P}_1$. Over this line, we consider the cross section defined (except at $(0, 1)$) by $(x_1, x_2) \rightarrow (1, x_2/x_1)$.

[6] By duality we mean the multiplication theorem 9. 7 (6).

Around $(0,1)$, we take as product representation of the bundle the one which identifies 1 on the typical fibre with $(x_1/x_2, 1)$; then we see easily that around $(0,1)$, the map $\boldsymbol{P}_1 \to \boldsymbol{C}^*$ which defines the value of the obstruction cochain on a 2 simplex having $(0,1)$ in its interior, is of the form $z \to 1/z$; this value will therefore be -1, and so will be that of the obstruction cochain on $\boldsymbol{P}_1$, hence the second equality.

29.7. *Proof of the theorem.* By the definitions, the lemma and naturality, we have

$$^1c_1 = {}^2c_2 = {}^3c_1 = -{}^4c_1 = -{}^5c_1 = -{}^7c_1$$

since all these classes are equal to $-e^*$ in the Hopf fibering. For $i = 1, 2, 4, 5$, ^ic satisfies duality, and we get therefore (see 29.5b)

$$(2) \qquad {}^1c_i = {}^2c_i = (-1)^i \cdot {}^4c_i = (-1)^i \cdot {}^5c_i \qquad (1 \leqq i \leqq n).$$

The equality

$$(3) \qquad {}^3c_j = -{}^7c_j, \qquad (1 \leqq j \leqq n),$$

follows from the more general fact that in a fibre bundle, "transgression is minus obstruction" for a proof of which we refer to [26, § 37.16].

The $2N$-universal bundle for $\boldsymbol{U}(n)$ over $\boldsymbol{H}(n,N)$ is $(\boldsymbol{W}_{n+N,n}, \boldsymbol{H}(n,N), \boldsymbol{U}(n))$ which may also be written as $(\boldsymbol{U}(n+N)/\boldsymbol{U}(N), \boldsymbol{H}(n,N), \boldsymbol{U}(n))$, where $\boldsymbol{U}(N)$, (respectively $\boldsymbol{U}(n)$), is the subgroup of $\boldsymbol{U}(n+N)$ leaving the n first (respectively N last) coordinate vectors fixed. Let (u_{ij}), $(1 \leqq i, j \leqq n+N)$, be the standard coordinates in the matrix space. Following Chern [9], we denote by θ_{ab} the left invariant Maurer-Cartan form on $\boldsymbol{U}(n+N)$ which is equal to du_{ba} (and not du_{ab}) at the neutral element. In these notations, we have

$$(4) \qquad d\theta_{ab} = \sum_{1}^{n+N} \theta_{ai} \wedge \theta_{ib}; \qquad \theta_{ba} = -\bar{\theta}_{ba}.$$

It is easily seen that for $1 \leqq a, b \leqq n$, the forms are right invariant under $\boldsymbol{U}(N)$ and satisfy

$$\theta_{ab} \cdot u = \sum_{ij} u_{ia} \theta_{ij} (u^{-1})_{bj} \qquad (u \in \boldsymbol{U}(n), u = (u_{ij})),$$

hence they induce forms on $\boldsymbol{U}(n+N)/\boldsymbol{U}(N)$ which define there a connexion for the $\boldsymbol{U}(n)$-bundle structure. (4) shows that its curvature forms are

$$\Omega_{ij} = \sum_{k=n+1}^{k=N} \theta_{ik} \wedge \theta_{kj}, \qquad (1 \leqq i, j \leqq n).$$

Thus, by definition

$$\Psi = \sum \Psi_q = \det | \mathrm{Id} + (2\pi i)^{-1}\Omega_{ij} |$$

represents ${}^6c$ in the universal bundle. By a computation which we shall not reproduce, Chern ([9], Chap. II, § 2, [10], p. 77) has shown that Ψ_q is equal to ${}^5c_q$; hence

$$(5) \qquad {}^5c_j = {}^6c_j, \qquad (j = 1, \cdots, n),$$

first in the universal bundle, and then by naturality in all differentiable $\boldsymbol{U}(n)$-bundles.

To conclude the proof, we have to compare the obstruction classes with one of the six other types, and it will be enough to show that

$$(6) \qquad {}^3c_j = (-1)^j \cdot {}^6c_j, \qquad (1 \leqq j \leqq n).$$

We have first

$${}^3c_j = \epsilon_j \cdot {}^6c_j \qquad (\epsilon_j = \pm 1, j = 1, \cdots, n),$$

with ϵ_j depending only on j; by naturality, we have only to prove this in the universal bundle; there it follows from (2), (3), (5), and from the fact that ${}^7c_j$ and ${}^4c_j$ both generate the kernel of $\rho^*(\boldsymbol{U}(j-1), \boldsymbol{U}(n))$ in dimension $2j$, which is infinite cyclic. The proof of this is identical with that of a similar statement on Stiefel-Whitney classes [3, Lemme 5.1] and will be left to the reader.

To determine ϵ_j, it is then enough to compute the 2 classes ${}^3c$ and ${}^6c$ for one bundle with Chern classes not zero or of order 2. To this end, we take the tangent bundle of $\boldsymbol{P}_n$. By [9, p. 118], we have

$$\Psi_j = (2\pi i)^{-j} \binom{n+1}{j} \Lambda^j$$

for the torsionless connexion associated to the Fubini metric, where Λ is the exterior form obtained from the metric (1) by replacing the symmetric products by exterior products; in the notation of 29.6, we have, therefore, $\Omega = i(2\pi)^{-1} \cdot \Lambda$ and

$$\Psi_j = \binom{n+1}{j} (-1)^j \Omega^j.$$

We have seen in the proof of 29.6 that Ω represents e^* and, consequently,

$$(7) \qquad {}^6c_j = \binom{n+1}{j} (-1)^j (e^*)^j, \qquad (1 \leqq j \leqq n),$$

as also follows from 15.1 and (2), (5). On the other hand, a direct construction of vector fields in [9, Theorem 13] shows that

$$(8) \qquad {}^3c_j = \binom{n+1}{j}(e^*)^j, \qquad (j=1,\cdots,n),$$

whence $\epsilon_j = (-1)^j$ and (6). Of course we must check that in the proof of (8), the indices of singularities are counted with the proper sign conventions; this presents no difficulty, but for the sake of completeness, we outline Chern's construction. Let

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{11} & a_{12} & & a_{1,n+1} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{n,n+1} \end{pmatrix}$$

be a matrix with constant coefficients and all minors of degree j $(1 \leqq j \leqq n+1)$ different from zero. Let H_j be the j-dimensional projective subspace defined by

$$\sum_k a_{ik}x_k = 0 \qquad (1 \leqq i \leqq n-j).$$

We denote by $\mathfrak{v}_j$ the vector field on $C^{n+1} - 0$ defined by

$$\mathfrak{v}_j(x) = \sum_k a_{jk}x_k\mathfrak{e}_k, \qquad ((\mathfrak{e}_k) \text{ canonical basis of } C^{n+1}).$$

It is invariant under $x \to a\cdot x$ $(a \in C^*)$ and defines a vector field $\mathfrak{w}_j$ on P_n. The vectors $\mathfrak{w}_j$ $(j \leqq r)$ are dependent at a point P with homogeneous coordinates (x_i) if and only if the $r+1$ vectors $\mathfrak{x} = \sum x_i\mathfrak{e}_i$ and $\mathfrak{v}_1,\cdots,\mathfrak{v}_r$ are dependent at $x = (x_i)$, which is equivalent with the vanishing of $n+1-r$ homogeneous coordinates of P.

Let now j be fixed and put $r = n-j+1$. On H_j, $\mathfrak{w}_1,\cdots,\mathfrak{w}_{r-1}$ are independent everywhere whereas $\mathfrak{w}_1,\cdots,\mathfrak{w}_r$ are dependent at $\binom{n+1}{j}$ points. We use these vector fields to compute the value of ${}^3c_j$ on H_j; since we already know that it is equal to $\pm\binom{n+1}{j}$, we have only to show that the singular points have non-negative indices. Let Q be a singular point, and W a neighborhood of Q on H_j. Using the fields $\mathfrak{w}_1,\cdots,\mathfrak{w}_{r-1}$ which are also independent at Q we see immediately that the map $W - Q \to W^*_{n,r}$ leading to the index is homotopic to a complex analytic map of $W - Q$ into a subspace of $W^*_{n,r}$ of the type of the space U introduced in 29.1, and that the resulting map of $W - Q$ in $C^j - 0$ extends analytically to Q and maps it onto the origin; hence it has a positive degree, and the index is $\geqq 0$ according to the conventions of 29.1 and 29.3(3).

29.8. *Remarks.* (a) The obstruction classes ${}^3c_j$ are the obstructions to the construction of *contravariant* vector fields. Hodge [20] uses covariant vector fields and, by 29.4 and 10.6a, or directly, the resulting classes are our ${}^4c_j$.

(b) According to Hodge [20] and Nakano (Mem. Coll. Sci. Kyoto 29 (1955), 145-149), the canonical classes of Eger-Todd are to be identified with the Schubert classes ${}^5c_j$.

(c) The lack of sign conventions for obstruction classes in [9] leads to a slight inaccuracy for the Chern classes of $\boldsymbol{P}_n$; Theorem 13 gives $c_j = \binom{n+1}{j}(e^*)^j$ and Theorem 12 gives the class of Ψ_j; as we have seen, these differ by $(-1)^j$.

29.9. Finally, to be complete, we list some properties or alternate definitions of the first Chern class. For the notations, and concepts used here, see [19].

(a) Let $\boldsymbol{C}^*{}_c$ be the sheaf of germs of continuous $\boldsymbol{C}^*$-valued functions on the space B. A complex line bundle over B is represented by an element $\xi \in H^1(B, \boldsymbol{C}^*{}_c)$ and we have

$$ {}^1c_1 = \delta\xi, $$

where δ is the coboundary operator $H^1(B, \boldsymbol{C}^*{}_c) \to H^2(B, \boldsymbol{Z})$ associated to the exact sequence $0 \to \boldsymbol{Z} \to \boldsymbol{C}_c \xrightarrow{e} \boldsymbol{C}^*{}_c \to 0$, where e is the exponential map [19, § 4.3.1].

(b) Let V be an oriented m-dimensional manifold, B an oriented $(m-2)$-dimensional submanifold, η the normal bundle oriented in such a way that orientation of B plus orientation of η gives the orientation of V. It has then a complex structure compatible with its orientation, determined up to isomorphism; its class ${}^1c_1$ is dual to the homology class defined by B [19, § 4.8.1].

(c) Kodaira has introduced the following definition for the Chern class c_1 of a holomorphic principal $\boldsymbol{C}^*$-bundle $\xi = (E, B, \boldsymbol{C}^*, \pi)$. Let (U_j) be a covering by coordinate neighborhoods, z_j the coordinate of the standard fibre over U_j, (f_{jk}) the transition functions, (a_j) a cross section of the bundle with fibre $\boldsymbol{R}^+$ defined by the transition functions $f_{jk} \cdot \bar{f}_{jk}$. Then $c_1(\xi)$ is the class of the form $\gamma = (i/2\pi)\partial\bar{\partial}\log a_j$. *This class is equal to* ${}^1c_1$.

Proof. We have $\pi^*(\gamma) = d\psi$, where ψ is a 1-form over E with the local representation $(-i/2\pi)\bar{\partial}\log(\bar{z}_j/a_j)$ over U_j; the restriction of ψ to a fibre is $(-i/2\pi)\bar{z}_j^{-1}d\bar{z}_j$ and its integral over the positively oriented unit circle is -1. Thus the Kodaira class is equal to $-\tau(x)$, where x is the canonical generator of $H^1(\boldsymbol{C}^*, \boldsymbol{Z})$, and is equal to ${}^1c_1$ by 29.4.

(d) On a complex manifold B of complex dimension n, the *canonical bundle* K is the line bundle of exterior forms of type $(n, 0)$, i.e. the bundle of n-forms on the tangent bundle. By (10.6a, b), its Chern class is $-{}^1c_1(\theta)$, where θ is the tangential bundle. In particular, the determinant g of a positive non-degenerate hermitian metric provides a section of the bundle with fibre $\boldsymbol{R}^+$ and transition functions $|f_{jk}|^2$, (f_{jk} being the transition functions of K). Thus the Ricci form

$$(-i/2\pi)\partial\bar{\partial}\log g,$$

where $R_{\alpha\bar{\beta}} = -\partial^2\log g/\partial z_\alpha\partial\bar{z}_\beta$ is the Ricci tensor, represents ${}^1c_1(\theta)$ (see Kodaira, Annals of Math. 60 (1954), 28-48).

Appendix II.

30. Pontrjagin classes.

30.1. *Notation.* Tors A is the torsion subgroup of the commutative group A, and $\text{Tors}_p A$ its p-primary component.

Let V be a vector space graded by finite dimensional subspaces V^i $(i \geqq 0)$. By $P(V, t)$, we denote its Poincaré polynomial

$$P(V, t) = \sum \dim V^i \cdot t^i,$$

and for a topological space X, we write $P_p(X, t)$ for $P(H^*(X, \boldsymbol{Z}_p), t)$.

f^*_p and $f^*_{\boldsymbol{Z}}$ denote the homomorphism of cohomology rings over $\boldsymbol{Z}_p$ and $\boldsymbol{Z}$ induced by a continuous map f.

Let ξ be a bundle with connected fibres, and A a commutative group. Then $T^i(F_\xi, A)$ or simply $T^i{}_\xi$ denotes the subgroup of transgressive elements in $H^i(F_\xi, A)$. We recall that the transgression τ_ξ is a homomorphism of $T^i{}_\xi$ into the quotient of $H^{i+1}(B_\xi, A)$ by a subgroup which will be denoted by $L^{i+1}(B_\xi, A)$ or $L^{i+1}{}_\xi$; we have $L^2{}_\xi = 0$.

30.2. *Cohomology* mod p *of* $B_{\boldsymbol{O}(n)}$ *and* $B_{\boldsymbol{SO}(n)}$. Let G be a compact Lie group, T a maximal torus. The ring of invariants of $W(G)$ operating in the usual way in $H^*(B_T, \boldsymbol{Z})$ is denoted by I_G. If $G = \boldsymbol{SO}(2n+1)$, $\boldsymbol{O}(2n+1)$, $\mathbf{O}(2n)$, (respectively $G = \boldsymbol{SO}(2n)$), and if (y_j) is the base induced by transgression in (E_G, B_T, T) from the basis of $H^1(T, \boldsymbol{Z})$ discussed in § 9, then $W(G)$ is the group of permutations of the y_j's combined with an arbitrary (respectively even) number of changes of signs, and consequently $I_G = S(y_1^2, \cdots, y_n^2)$, (respectively is generated by $S(y_1^2, \cdots, y_n^2)$ and by the product of the y_i's).

PROPOSITION. *Let $G = \boldsymbol{SO}(n)$ or $\boldsymbol{O}(n)$, let $\boldsymbol{T}$ be its standard maximal torus, and let $\boldsymbol{Q}$ be the subgroup of diagonal matrices. Then*

(a) *For $p \neq 2$, $\rho^*_p(\boldsymbol{T}, G)$ maps $H^*(B_G, \boldsymbol{Z}_p)$ isomorphically onto $I_G \otimes \boldsymbol{Z}_p$.*

(b) *$\rho^*_2(\boldsymbol{Q}, \boldsymbol{O}(n))$ maps $H^*(B_{\boldsymbol{O}(n)}, \boldsymbol{Z}_2)$ isomorphically onto $S(u_1, \cdots, u_n)$, where (u_i) is a suitable basis of $H^1(B_{\boldsymbol{Q}}, \boldsymbol{Z}_2)$.*

For (b), see [3, Théorème 5], where a similar statement is also proved for $\boldsymbol{SO}(n)$. For $G = \boldsymbol{SO}(n)$, the assertion (a) is proved in [2, § 29]. For $G = \boldsymbol{O}(n)$, it follows from the more general

30.3. PROPOSITION. *Let G be a compact Lie group, G_0 its largest connected subgroup, T a maximal torus. If $H^*(G_0, \boldsymbol{Z})$ has no p-torsion and if the order of G/G_0 is not divisible by p, then $\rho^*_p(T, G)$ is an isomorphism of $H^*(B_G, \boldsymbol{Z}_p)$ onto $I_G \otimes \boldsymbol{Z}_p$.*

For $p = 0$, see [2, Prop. 27.1]. For p prime, the proof is practically identical, in view of the absence of torsion on G_0/T, (14.2), and is left to the reader.

30.4. *A remark on the Bockstein homomorphism.* Let X be a space with finitely generated integral cohomology groups. Let r_i be the number of cyclic direct summands of the p-primary component of $H^i(X, \boldsymbol{Z})$. Then by the universal coefficient formula

$$P_p(X, t) - P_0(X, t) = (1 + 1/t) \sum r_i \cdot t^i.$$

Let β_p be the Bockstein homomorphism attached to the exact sequence

$$0 \to \boldsymbol{Z} \xrightarrow{\alpha} \boldsymbol{Z} \to \boldsymbol{Z}_p \to 0 \qquad (\alpha(x) = px, x \in \boldsymbol{Z})$$

followed by reduction $\bmod p$. Clearly $\beta_p \circ \beta_p = 0$, and

$$s_i = \dim \beta_p(H^{i-1}(X, \boldsymbol{Z}_p))$$

is the number of torsion coefficients of $\mathrm{Tors}_p H^i(X, \boldsymbol{Z})$ which are equal to p. Thus we see:

LEMMA. $\mathrm{Tors}_p H^*(X, \boldsymbol{Z})$ *consists of elements of order* p *if and only if*

$$P_p(X,t) - P_0(X,t) = (1 + 1/t)P(\beta_p(H^*(X, \boldsymbol{Z}_p)), t).$$

When this is the case, the kernel of β_p *is the reduction* $\bmod p$ *of* $H^*(X, \boldsymbol{Z})$, *and its image is the reduction* $\bmod p$ *of* $\mathrm{Tors}_p H^*(X, \boldsymbol{Z})$.

We recall that β_p is an antiderivation and that $\beta_2 = Sq^1$.

30.5. *Integral cohomology of* $B_{\boldsymbol{O}(n)}$ *and* $B_{\boldsymbol{SO}(n)}$. In this section, we shall prove that the torsion elements of the cohomology of these classifying spaces are all of order 2; first we insert a remark to be used in the proof.

Let H be an anticommutative graded algebra with unit over a field K, with $H^0 = K$, and let D be an antiderivation on H, raising degrees by one, of square zero. Let A be a graded subspace stable under D. We denote by N_A, M_A, I_A, J_A, respectively, the kernel of D in A, a graded supplementary subspace to N_A in A, the image of A under D, a graded supplementary subspace to I_A in N_A. Since D increases degrees by one and is an isomorphism of M_A onto I_A, we have

$$P(A,t) = (1 + 1/t)P(I_A, t) + P(J_A, t). \tag{1}$$

Let now B be a second graded subspace stable under D, linearly disjoint from A over K; i.e., such that the subspace $A \cdot B$ spanned by the products $a \cdot b$ $(a \in A, b \in B)$ is isomorphic to $A \otimes B$ under the natural map $a \otimes b \to a \cdot b$. Using the previous notations, we have, as an elementary special case of the Künneth formula

$$P(J_{A \cdot B}, t) = P(J_A, t) \cdot P(J_B, t). \tag{2}$$

PROPOSITION. *The torsion elements of* $H^*(B_{\boldsymbol{O}(n)}, \boldsymbol{Z})$ *and* $H^*(B_{\boldsymbol{SO}(n)}, \boldsymbol{Z})$ *are of order* 2.

It follows from 30.2 that these spaces have only 2-torsion. By 30.4, there remains to prove that for $G = \boldsymbol{SO}(n), \boldsymbol{O}(n)$, we have

$$P_2(B_G, t) - P_0(B_G, t) = (1 + 1/t)P(Sq^1(H^*(B_G, \boldsymbol{Z}_2)), t). \tag{3}$$

We have

$$\begin{aligned} &H^*(B_{\boldsymbol{SO}(n)}, \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_2, \cdots, w_n], Sq^1 w_i = (i+1)w_{i+1} \\ &H^*(B_{\boldsymbol{O}(n)}, \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_1, \cdots, w_n], Sq^1 w_i = w_1 w_i + (i+1)w_{i+1} \end{aligned} \tag{4}$$

($w_i = i$-th Stiefel-Whitney class, $i \leqq n, w_{n+1} = 0$), (see e.g. [3]). We first consider $\boldsymbol{SO}(n)$. By the foregoing, we may write

$$H^*(B_{SO(2m+1)}, \mathbf{Z}_2) \cong A_1 \otimes \cdots \otimes A_m,$$

$$H^*(B_{SO(2m)}, \mathbf{Z}_2) \cong A_1 \otimes \cdots \otimes A_{m-1} \otimes A'_m,$$

where

$$A_i = \mathbf{Z}_2[w_{2i}, w_{2i+1}] \quad (1 \leqq i \leqq m), \quad A'_m = \mathbf{Z}_2[w_{2m}]$$

are stable under the cup product and Sq^1, and A'_m is annihilated by Sq^1. By (4), the kernel of Sq^1 in A_i is spanned by the elements

$$w_{2i}{}^s \cdot w_{2i+1}{}^t \qquad (s \geqq 0, s \text{ even}, t \geqq 0),$$

and its image by the elements

$$w_{2i}{}^s \cdot w_{2i+1}{}^t \qquad (s \text{ even}, t > 0).$$

Thus, in the notations above, we may take as base for M_{A_i} (respectively J_{A_i}), the monomials

$$w_{2i}{}^s \cdot w_{2i+1}{}^t, \quad (s \text{ odd}, t \geqq 0), \quad (\text{respectively } w_{2i}{}^s, s = 0, 2, 4, 6, \cdots)$$

and we obtain

$$P(J_{A_i}, t) = (1 - t^{4i})^{-1}, \qquad (1 \leqq i \leqq m)$$

and, since $Sq^1(A'_m) = 0$

$$P(J_{A'_m}, t) = P(A'_m, t) = (1 - t^{2m})^{-1}.$$

By iterated application of (2) and by (1), we get

$$P_2(B_{SO(2m+1)}, t) = (1 + 1/t)P(Sq^1(H^*(B_{SO(2m+1)}, \mathbf{Z}_2)), t) + \prod_{i=1}^{m} (1 - t^{4i})^{-1},$$

$$P_2(B_{SO(2m)}, t) = (1 + 1/t)P(Sq^1(H^*(B_{SO(2m)}, \mathbf{Z}_2)), t) + (1 - t^{2m})^{-1} \prod_{i=1}^{m-1} (1 - t^{4i})^{-1},$$

which proves (3) for the unimodular case, since in both formulas, the last term on the right is the rational Poincaré series in view of 30.2.

We now pass to $\mathbf{O}(n)$. Choose a new basis of $H^*(B_{O(n)}, \mathbf{Z}_2)$ by

$$w^*_1 = w_1, w^*_{2i} = w_{2i}, \qquad (i \leqq [n/2]),$$

$$w^*_{2i+1} = w_{2i+1} + w_i \cdot w_{2i}, \qquad (i < [n/2]);$$

we have

$$H^*(B_{O(n)}, Z_2) = \mathbf{Z}_2[w^*_1, \cdots, w^*_n],$$

$$Sq^1 w^*_1 = (w^*_1)^2; Sq^1 w^*_{2i} = w^*_{2i+1} \qquad (i < [n/2]),$$

$$Sq^1 w^*_{2m} = w^*_1 \cdot w^*_{2m}, \qquad (n = 2m),$$

$$Sq^1 w^*_{2i+1} = Sq^1 Sq^1 \cdot w^*_{2i} = 0, \qquad (i \geqq 1).$$

We put

$$H^*(B_{\boldsymbol{O}(2m+1)}, \boldsymbol{Z}_2) = A_0 \otimes A_1 \otimes \cdots \otimes A_m,$$
$$H^*(B_{\boldsymbol{O}(2m)}, \boldsymbol{Z}_2) = A'_0 \otimes A_1 \otimes \cdots \otimes A_{m-1},$$

where

$$A_i = \boldsymbol{Z}_2[w^*_{2i}, w^*_{2i+1}] \qquad (1 \leqq i \leqq m),$$
$$A_0 = \boldsymbol{Z}_2[w^*_1] = \boldsymbol{Z}_2[w_1],$$
$$A'_0 = \boldsymbol{Z}_2[w^*_1, w^*_{2m}] = \boldsymbol{Z}_2[w_1, w_{2m}].$$

These are stable under cup-product and Sq^1, and we have as above

$$P(J_{A_i}, t) = (1 - t^{4i})^{-1}, \qquad (1 \leqq i \leqq m).$$

In A_0, the kernel of Sq^1 is spanned by w_1^{2s} $(s \geqq 0)$; and the image by w_1^{2s} $(s \geqq 1)$; hence $P(J_{A_0}, t) = 1$. Let us now prove that

(5) $$P(J_{A'_0}, t) = (1 - t^{2m})^{-1}.$$

We have

$$Sq^1(w_1^s \cdot w_{2m}^t) = (s + t) w_1^{s+1} w_{2m}^t$$

which shows that the monomials $w_1^s w_{2m}^t$ with $s + t$ even (respectively $s + t$ even and $s > 0$) span $N_{A'_0}$ (respectively $I_{A'_0}$). Thus we may take the elements w_{2m}^t (t even) as a base for $J_{A'_0}$, and this proves (5). The remainder of the proof of (3) for $\boldsymbol{O}(n)$ is then the same as for $\boldsymbol{SO}(n)$.

30.6. Corollary. *Let $G = \boldsymbol{O}(n)$ or $\boldsymbol{SO}(n)$. Then the kernel of Sq^1 on $H^*(B_G, \boldsymbol{Z})$ is the reduction* mod 2 *of $H^*(B_G, \boldsymbol{Z})$ and its image is the reduction* mod 2 *of* Tors $H^*(B_G, \boldsymbol{Z})$. *An element of $H^*(B_G, \boldsymbol{Z})$ is completely determined by its canonical images in $H^*(B_G, \boldsymbol{R})$ and $H^*(B_G, \boldsymbol{Z}_2)$.*

The first assertion follows from 30.4 and 30.5. The second one is an elementary fact about spaces with torsion elements of order 2 only.

In connection with the integral cohomology of $B_{\boldsymbol{O}(n)}$ and $B_{\boldsymbol{SO}(n)}$, let us also mention the following

30.7. Proposition. *Let $G = \boldsymbol{O}(n)$, $\boldsymbol{SO}(n)$ and let T be a maximal torus of G. Then $\rho^*_Z(T, G)$ maps $H^*(B_G, \boldsymbol{Z})$ onto I_G; its kernel is the torsion subgroup of $H^*(B_G, \boldsymbol{Z})$.*

We have seen in 9.3 that $S(y_1^2, \cdots, y_m^2)$, $(m = [n/2])$, is contained in $\rho^*_Z(T, G)$, which proves our statement for $G = \boldsymbol{SO}(2m+1)$, $\boldsymbol{O}(2m+1)$, $\boldsymbol{O}(2m)$. For $G = \boldsymbol{SO}(2m)$, we have to know moreover that $\rho^*_Z(T, \boldsymbol{SO}(2m))$ contains the product of the y_j's, but this follows from 9.5.

30.8. *Pontrjagin classes.* We follow the notations of 9.2, 9.3. By 30.6, the equalities

(6) $$\rho^*_0(T,G)(\tilde{p}) = \prod(1+y_i^2), \qquad (G = \boldsymbol{O}(n), \boldsymbol{SO}(n)),$$

(7) p_i (respectively $p_{i+\frac{1}{2}}$) reduced mod 2, is equal to w_{2i}^2 (respectively w_{2i+1}^2),

completely characterize the integral Pontrjagin classes. (6), over the integers, follows from 9.1, 9.3, 9.4. It implies that $p_{i+\frac{1}{2}}$ is a torsion element. (7) needs only to be proved for $G = \boldsymbol{O}(n)$ and, in view of 30.2(b), will follow from

(8) $$\rho^*_2(\boldsymbol{Q}(n), \boldsymbol{O}(n))(\tilde{p}) = \prod(1+u_j^2).$$

We have

$$\rho^*_2(\boldsymbol{Q}(n), \boldsymbol{O}(n))(\tilde{p}) = \rho^*_2(\boldsymbol{Q}(n), \boldsymbol{U}(n))(c) = \rho^*_2(\boldsymbol{Q}(n), \boldsymbol{T}')(\prod(1+x_i))$$

where $\boldsymbol{T}'$ is the subgroup of diagonal matrices of $\boldsymbol{U}(n)$ and (x_i) is the standard basis of $H^2(B_{T'}, \boldsymbol{Z})$, and therefore (8) follows from [4, § 11]:

$$\rho^*_2(\boldsymbol{Q}(n), \boldsymbol{T}')(x_i) = u_i^2 \qquad (1 \leqq i \leqq n).$$

30.9. *Remark on integral Stiefel-Whitney classes.* It follows also from 9.5, 30.6, (6), (7) that for an $\boldsymbol{SO}(2m)$-bundle, we have

(9) $$p_m = W_{2m}^2,$$

both sides being considered as integral classes. The relations $w_{2i+1} = Sq^1 w_{2i}$ for $\boldsymbol{SO}(n)$-bundles and (30.6), show that the universal w_{2i+1} is the reduction mod 2 of a uniquely determined element

$$W_{2i+1} \in H^{2i+1}(B_{SO(n)}, \boldsymbol{Z})$$

of order 2, the *integral* $2i+1$*-th Stiefel-Whitney class,* and that we also have

$$p_{i+\frac{1}{2}} = (W_{2i+1})^2$$

over the integers.

30.10. *Pontrjagin classes and transgression.* As usual, $V_{n,k}$ denotes the Stiefel-manifold of orthonormal k-frames in euclidean n-space. We recall [2, § 10] that for n odd, $H^j(V_{n,n-2i+1}, \boldsymbol{Z})$ is equal to $\boldsymbol{Z}$ for $j = 0, 4i-1$, to $\boldsymbol{Z}_2$ for j even running from $2i$ to $4i-2$, and is zero for the other values of j which are $\leqq 4i-1$. We denote by v_i a generator of $H^{4i-1}(\boldsymbol{V}_{n,n-2i+1}, \boldsymbol{Z})$.

The first non-vanishing integral cohomology group of strictly positive dimension of the complex Stiefel manifold $\boldsymbol{W}_{n,n-2i+1}$ is of dimension $4i-1$ and is infinite cyclic. We denote its canonical generator (see 29.1) by t_i.

The inclusion of $\boldsymbol{O}(n)$ in $\boldsymbol{U}(n)$ induces a natural injective map of $\boldsymbol{V}_{n,k} = \boldsymbol{O}(n)/\boldsymbol{O}(n-k)$ into $\boldsymbol{W}_{n,k} = \boldsymbol{U}(n)/\boldsymbol{U}(n-k)$.

LEMMA. *Let n be odd, and $\lambda_{n,i}$ be the natural inclusion of $\boldsymbol{V}_{n,n-2i+1}$ in $\boldsymbol{W}_{n,n-2i+1}$. Let v_i and t_i be defined as above. Then $\lambda^*_{n,i}(t_i) = \pm 2v_i$.*

This lemma will be proved in the next section. Let ξ be a principal $\boldsymbol{O}(n)$-bundle, and ξ' be its complex extension. The given homomorphism of ξ into ξ' induces in a natural way a homomorphism $\alpha\colon \eta \to \eta'$ of the associated bundles with respective typical fibres $\boldsymbol{V}_{n,n-2i+1}$ and $\boldsymbol{W}_{n,n-2i+1}$. By the transgression definition (29.3(7)), we have $c_{2i} = -\tau_{\eta'}(t_i)$. Hence 29.4 and the lemma imply the

THEOREM. *Let n be odd and t_i be the canonical generator of $H^{4i-1}(\boldsymbol{W}_{n,n-2i+1}, \boldsymbol{Z})$ and choose $v_i \in H^{4i-1}(\boldsymbol{V}_{n,n-2i+1}, \boldsymbol{Z})$ such that $2v_i = \lambda^*_{n,i}(t_i)$. Let ξ be a principal $\boldsymbol{O}(n)$-bundle, η the associated bundle with fibre $\boldsymbol{V}_{n,n-2i+1}$. Then $p_i(\xi) = (-1)^{i+1}\tau_\eta(2v_i)$ modulo $L^{4i}{}_\eta$.*

For the notation $L^{4i}{}_\eta$, see 30.1. Since the cohomology groups of $\boldsymbol{V}_{n,n-2i+1}$ in positive dimensions $< 4i-1$ are 2-groups, the spectral sequence definition of $L^{4i}{}_\eta$ [2, § 5] shows that $L^{4i}{}_\eta$ is a 2-group, so that the theorem characterizes p_i up to 2-torsion. We remark also that v_i itself is not universally transgressive (see following proof).

Let n be odd and β be the natural projection of E_ξ onto $E_\eta = E_\xi/\boldsymbol{O}(2i-1)$. Then we have of course

$$p_i(\xi) = (-1)^{i+1}\tau_\xi(\beta^*(2v_i)) \quad \mathrm{mod}\ L^{4i}{}_\xi.$$

By [2, § 10], $\beta^*(v_i)$ generates a direct summand of $H^{4i-1}(\boldsymbol{O}(n), \boldsymbol{Z})$ or of $H^{4i-1}(\boldsymbol{SO}(n), \boldsymbol{Z})$.

30.11. *Proof of the lemma.* We first consider the case where $n = 2i+1$ and denote by ξ the universal bundle for $\boldsymbol{O}(2i+1)$, by ξ' its complex extension, by η and η' the associated bundles with fibres $\boldsymbol{V}_{2i+1,2}$ and $\boldsymbol{W}_{2i+1,2}$ and by α the natural map of η in η'. We have $H^*(\boldsymbol{V}_{2i+1,2}, \boldsymbol{Z}_2) = \wedge(h_{2i-1}, h_{2i})$, with $Sq^1 h_{2i-1} = h_{2i}$ and $\tau_\eta(h_{2i-1}) = w_{2i}$, $\tau_\eta(h_{2i}) = w_{2i+1}$ (see for instance [3]); this implies easily that $h_{2i-1} \cdot h_{2i}$ is not universally transgressive. Since $h_{2i-1} \cdot h_{2i}$ is the reduction mod 2 of v_i, the latter is not universally transgressive in integral cohomology. For $\boldsymbol{V}_{2i+1,2}$, we have $H^0 = H^{4i-1} = \boldsymbol{Z}$, $H^{2i} = \boldsymbol{Z}_2$ (see e.g. [2], § 10), therefore, the non-zero terms in the spectral sequence of η over the integers have fibre degrees 0, $2i$, $4i-1$ and those with fibre degree $2i$ have order 2. Therefore $2t_i$ is universally transgressive and $L^{4i}{}_\eta$ consists of elements of order 2.

We know now that $2t_i$ generates $T^{4i-1}(\boldsymbol{V}_{2i+1,2}, \boldsymbol{Z})$; this group contains necessarily $\lambda^*{}_{2i+1,i}(v_i)$ since v_i is transgressive in η'. Thus we have $\lambda^*{}_{2i+1,i}(v_i) = 2k \cdot t_i$ for some integer k, hence also

$$p_i = \pm k\tau_\eta(2t_i) \quad \operatorname{mod} L^{4i}{}_\eta.$$

Let $\boldsymbol{T}$ be the standard maximal torus of $\boldsymbol{O}(2i+1)$. Then, in the notation of 9.3, we get from (1) in 9.3 and from the fact that $L^{4i}{}_\eta$ is a 2-group:

$$y_1{}^2 \cdot \cdots \cdot y_i{}^2 = \pm k\rho^*{}_Z(\boldsymbol{T}, \boldsymbol{O}(2i+1))(\tau_\eta(2t_i)).$$

Since $H^*(B_{\boldsymbol{T}}, \boldsymbol{Z})$ is the ring of polynomials in the y_j's, this yields $k = \pm 1$, and the lemma for $n = 2i+1$.

In the general case, we consider the commutative diagram

$$\begin{array}{ccc} \boldsymbol{V}_{n,n-2i+1} & \xrightarrow{\lambda_{n,i}} & \boldsymbol{W}_{n,n-2i+1} \\ \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \nu} \\ \boldsymbol{V}_{2i+1,2} & \xrightarrow{\lambda_{2i+1,i}} & \boldsymbol{W}_{2i+1,2}, \end{array}$$

where μ, ν are the injection of a fibre in the standard fiberings. It follows from §§ 9, 10 in [2] that μ^*, ν^* are isomorphisms in dimension $4i-1$; therefore the general case of the lemma follows by commutativity of the above diagram from the particular case already proved.

30.12. *A property of* $\rho^*{}_Z(\boldsymbol{O}(r), \boldsymbol{O}(n))$. We end this appendix by proving that $\rho^*{}_Z(\boldsymbol{O}(r), \boldsymbol{O}(n))$ is surjective $(r \leqq n)$, a fact which is useful in the discussion of relative Pontrjagin classes (see M. Kervaire, Amer. J. Math. 79 (1957)).

LEMMA. *Let* X, Y *be two spaces with finitely generated integral cohomology groups and* $f\colon X \to Y$ *be a continuous map. We assume*:

(a) *The orders of the torsion elements of* $H^*(X, \boldsymbol{Z})$ *and* $H^*(Y, \boldsymbol{Z})$ *are square free.*

(b) $f^*{}_0$ *is surjective.*

(c) *For all primes* p, $f^*{}_p$ *is a surjective map for the kernels of the Bockstein maps* β_p (see 30.4).

Then $f^*{}_Z$ *is surjective.*

Let M_i be the image of $H^i(Y,\boldsymbol{Z})$ under f^*. By (b), it has a finite index, say g_i, in $H^i(X,\boldsymbol{Z})$. In view of (30.4), the assumptions (a), (c) imply that

$$f^*\colon H^i(Y,\boldsymbol{Z})\otimes\boldsymbol{Z}_p\to H^i(X,\boldsymbol{Z})\otimes\boldsymbol{Z}_p \qquad (i\geqq 0, p \text{ prime})$$

is surjective, or in other words, that

$$H^i(X,\boldsymbol{Z})=p\cdot H^i(X,\boldsymbol{Z})+M_i,$$

hence, by iteration,

$$H^i(X,\boldsymbol{Z})=g_i\cdot H^i(X,\boldsymbol{Z})+M_i=M_i.$$

PROPOSITION. *The homomorphism* $\rho^*_Z(\boldsymbol{O}(r),\boldsymbol{O}(n))$, $(r\leqq n)$, *is surjective.*

By (30.5), $B_{\boldsymbol{O}(r)}$ and $B_{\boldsymbol{O}(n)}$ satisfy (a) of the lemma.

It follows from (9.3) and (30.2) that $H^*(B_{\boldsymbol{O}(n)},\boldsymbol{Z}_p)$ is the ring of polynomials in the universal Pontrjagin classes $(i\geqq 1)$ for $p\neq 2$. Since these are preserved under the natural inclusion $\boldsymbol{O}(r)\subset\boldsymbol{O}(n)$ (see 9.7), it follows that $\rho^*_p(\boldsymbol{O}(r),\boldsymbol{O}(n))$ is surjective for $p\neq 2$. This shows that (b) is fulfilled and also, in view of (30.2), (30.5), that (c) is true for $p\neq 2$. Thus, in order to derive the proposition from the lemma, there remains to show that

(10) $\rho^*_2(\boldsymbol{O}(r),\boldsymbol{O}(n))$ *is surjective for the kernels of* Sq^1.

We write ρ^* for $\rho^*_2(\boldsymbol{O}(r),\boldsymbol{O}(n))$, denote by w_i (respectively $\bar{w}_i$) the universal Stiefel-Whitney classes for $\boldsymbol{O}(r)$- (respectively $\boldsymbol{O}(n)$-) bundles and define w^*_i, $\bar{w}^*_i$ as in the proof of (3).

The assertion (10) is clearly true on any subalgebra $A\subset H^*(B_{\boldsymbol{O}(n)},\boldsymbol{Z}_2)$ which is stable under Sq^1 and mapped injectively by ρ^*; the latter is given by

$$\rho^*(\bar{w}_i)=w_i,\quad \rho^*(\bar{w}^*_i)=w^*_i,\quad (i\leqq r),\quad \rho^*(\bar{w}_j)=0\quad (j>r).$$

Therefore, by (4), this applies to

$$A=\boldsymbol{Z}_2[\bar{w}_1,\cdots,\bar{w}_i]\quad (i\leqq r, i \text{ odd}),\quad \boldsymbol{Z}_2[\bar{w}_1,\cdots,\bar{w}_{r-1},\bar{w}_r{}^2]\qquad (r \text{ even}),$$

$$\boldsymbol{Z}_2[\bar{w}_1{}^2, \bar{w}^*_2,\cdots,\bar{w}^*_{r-1},\bar{w}_r{}^2]\quad (r \text{ even}).$$

This establishes (10) for r odd; for $r=2m$ even, it reduces its proof to that of the following statement: given

$$x=w_{2m}\cdot P+Q, Sq^1x=0, (P,Q\in\boldsymbol{Z}_2[w_1,\cdots,w_{2m-1},w_{2m}{}^2]),$$

there exists $y\in H^*(B_{\boldsymbol{O}(n)},\boldsymbol{Z}_2)$ with the properties

$$Sq^1 y = 0, \qquad \rho^*(y) = x.$$

$Sq^1 x = 0$ gives

$$w_{2m}(w_1 P + Sq^1 P) + Sq^1 Q = 0$$

and, since $\mathbf{Z}_2[w_1, \cdots, w_{2m-1}, w_{2m}^2]$ is stable under Sq^1,

$$w_1 P + Sq^1 P = Sq^1 Q = 0.$$

We may write

$$P = w_1 R + S, \qquad (R, S \in \mathbf{Z}_2[w_1^2, w^*_2, \cdots, w^*_{2m-1}, w_{2m}^2].$$

Hence

$$Sq^1 P = w_1^2 R + w_1 \cdot Sq^1 R + Sq^1 S,$$

$$0 = w_1 P + Sq^1 P = w_1(S + Sq^1 R) + Sq^1 S,$$

and, as before,

$$S + Sq^1 R = Sq^1 S = 0.$$

Now let $\bar{P}, \bar{Q}, \bar{R} \in H^*(B_{O(n)}, \mathbf{Z}_2)$ be the elements obtained from P, Q, R by barring the w_i's. Then a trivial computation shows that

$$y = \bar{w}_{2m} \cdot \bar{P} + \bar{Q} + \bar{w}_{2m+1} \bar{R}$$

has the desired properties.

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