QUEEN MARY COLLEGE MATHEMATICS NOTES

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Homological Dimension of Discrete Groups

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HOMOLOGICAL DIMENSION OF DISCRETE GROUPS

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Introduction

These Notes grew out of a course of advanced lectures given at Queen Mary College, London, in the Spring of 1975.

The general theme is the classification of discrete groups using information on (mostly high dimensional) homology and cohomology groups. More specifically we discuss the following three topics:

In Chapter I we investigate groups of type $(FP)_n$, i.e.,roughly speaking, groups G whose cohomology functor $H^k(G; -)$ commutes with direct limits (or, equivalently, whose homology functor $H_k(G; -)$ commutes with direct products) for all $k \leq n$. All finitely presented groups are of type $(FP)_2$ and it is conceivable that the converse holds, also. Thus type $(FP)_n$, for $n \geq 3$, provides a useful classification of finitely presented groups which takes into consideration the whole homological iceberg below the group theory, the top of which is just finite presentation.

Chapter II is devoted to the homological dimensions $cdG = \max\{ n \mid H^n(G; -) \not\equiv 0 \}$, $hdG = \max\{ n \mid H_n(G; -) \not\equiv 0 \}$. In many ways this chapter is just an improved and largely extended version of K.W. Gruenberg's Chapter 8 in [30]. I have tried to give a reasonably complete survey of the present status of knowledge on cd, hd, but I am painfully aware that there are still many gaps. In particular, the reader will fruitlessly look for Serre's important result that all finitely generated torsion-free subgroups of $GL(n,\mathbb{Q})$ are of finite cohomological dimension [52]. In

In Section 8 we apply the results of all preceding sections to obtain some purely group theoretic results, notably on groups of cohomology dimension 2.

Finally, in Chapter III, we present the theory of duality groups, i.e., groups satisfying a homological duality $H^k(G; -) \cong H_{n-k}(G; C \otimes -)$, $k \in \mathbb{Z}$. The most important examples of duality groups occur as discrete subgroups of Lie-groups. They have a particularly smooth homological behaviour, being, in particular, of type $(FP)_\infty$ and of finite cohomological dimension, so that all results of Chapters I and II apply.

The audience was (and the reader is assumed to be) familiar with the basic techniques of homological algebra, including the use of spectral sequences, as well as with some basic group theoretic constructions. A few topological aspects are mentioned, but otherwise the presentation is purely algebraic.

I am greatly indebted to Karl Gruenberg for reading the manuscript, for his most valuable criticism and for supervising the typing after I had left London. My first and very best thanks are due to him. Also, I express my thanks to all participants of the course for their interest which manifested itself in a large number of stimulating questions and discussions. My very special and best thanks finally to Mrs. Lola Buer, who typed the manuscript, for her excellent work as well as for her enormous patience with me.

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Our knowledge on the three topics of these notes has grown substantially since 1976. This is particularly the case for the notion of groups of type $(FP)_m$ and for Poincaré duality groups: what was really just a couple of interesting and perhaps intriguing observations, five years ago, seems now to have become serious mathematics. As a consequence any attempt to incorporate the new aspects fully into the text would mean changing its style and level completely. Therefore I have preferred to add a short appendix ("some recent developments" on p.184) where some of the new results are sketched without proofs, but with full reference to the literature.

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Preliminary remarks and notations

1. Ext and Tor. Let Λ be an arbitrary ring with unit element $1 \neq 0$. We shall assume that the reader is familiar with the functors Ext^k_Λ and Tor^Λ_k from the category of Λ-modules into the category of Abelian groups, but to fix the notation we make a few preliminary remarks.

$\mathrm{Ext}^k_\Lambda(-,-)$ is contravariant in the first and covariant in the second argument. $\mathrm{Ext}^k_\Lambda(A,A')$ is defined whenever both A and A' are left modules or both are right modules, and we use the convention that $\mathrm{Ext}^k_\Lambda \equiv 0$ for $k < 0$. Of course, one has $\mathrm{Ext}^0_\Lambda(A,A') \cong \mathrm{Hom}_\Lambda(A,A')$.

$\mathrm{Tor}^\Lambda_k(-,-)$ is covariant in both arguments. $\mathrm{Tor}^\Lambda_k(B,A)$ is defined whenever A is a left and B a right Λ-module, and we use the convention that $\mathrm{Tor}^\Lambda_k \equiv 0$ for $k < 0$. Of course, one has $\mathrm{Tor}^\Lambda_0(B,A) \cong B \otimes_\Lambda A$.

Remark. Usually, (but not always) we shall stick to the rule that the letters A, A',... denote left modules and the letters B,B',... right modules.

The Ext-groups can be computed using either a projective resolution of the first argument or an injective resolution of the second argument. The Tor-groups are usually computed using projective resolutions of either the first or the second argument, but one may, and this is often most convenient, also use flat resolutions for this purpose.

2. Change of ring. For later reference we recall the four change-of-ring isomorphisms. Let $\gamma: \Gamma \to \Lambda$ be a unitary ring homomorphism. Then every Λ-module can be regarded as a Γ-module via γ and the following holds:-

(a) Let B be a right Λ-module and C a left Γ-module. If Λ is <u>flat</u> as a Γ-module via γ, then one has natural isomorphisms

$$\mathrm{Tor}_k^{\Gamma}(B,C) \cong \mathrm{Tor}_k^{\Lambda}(B,\Lambda\otimes_\Gamma C),\ k \in \mathbb{Z}.$$

(b) Let B be a right Γ-module and C a left Λ-module. If Λ is <u>flat</u> as a Γ-module via γ, then one has natural isomorphisms

$$\mathrm{Tor}_k^{\Gamma}(B,C) \cong \mathrm{Tor}_k^{\Lambda}(B\otimes_\Gamma\Lambda, C),\ k \in \mathbb{Z}.$$

(c) Let A be a left Λ-module and C a left Γ-module. If Λ is <u>flat</u> as a Γ-module via γ, then one has natural isomorphisms

$$\mathrm{Ext}_\Gamma^k(C,A) \cong \mathrm{Ext}_\Lambda^k(\Lambda\otimes_\Gamma C,A),\ k \in \mathbb{Z}.$$

(d) Let A be a left Γ-module and C a left Λ-module. If Λ is <u>projective</u> as a Γ-module via γ, then one has natural isomorphisms

$$\mathrm{Ext}_\Gamma^k(C,A) \cong \mathrm{Ext}_\Lambda^k(C,\mathrm{Hom}_\Gamma(\Lambda,A)),\ k \in \mathbb{Z}.$$

<u>Proof</u>. For (a) and (c) choose a Γ-projective resolution $\underline{P} \twoheadrightarrow C$. Then $\Lambda\otimes_\Gamma P$ is a Λ-projective resolution of $\Lambda\otimes_\Gamma C$ and the result follows from the obvious (co)chain isomorphisms

$$B\otimes_\Lambda(\Lambda\otimes_\Gamma\underline{P}) \cong B\otimes_\Gamma\underline{P},\quad \mathrm{Hom}_\Lambda(\Lambda\otimes_\Gamma\underline{P},A) \cong \mathrm{Hom}_\Gamma(\underline{P},A).$$

(b) is analagous to (a). For (d) choose a Γ-injective resolution $A \rightarrowtail \underline{I}$. Then $\mathrm{Hom}_\Gamma(\Lambda,\underline{I})$ is an injective resolution of $\mathrm{Hom}_\Gamma(\Lambda,A)$ and the result follows from the obvious cochain isomorphism

$$\mathrm{Hom}_\Lambda(C,\mathrm{Hom}_\Gamma(\Lambda,\underline{I})) \cong \mathrm{Hom}_\Gamma(C,\underline{I}).\square$$

<u>Remark</u>. Of course there is also a right-module version of (c) and (d).

3. The group ring case. We are mainly interested in the group ring case. Throughout these Notes R will denote a commutative ring with unit element $1 \neq 0$, and we shall consider the (co)homology theory of groups over R. If G is a group and A a left RG-module then the cohomology groups of G over R with coefficients in A are defined as

$$H^k(G; A) = \mathrm{Ext}^k_{RG}(R,A), \quad k \in \mathbb{Z},$$

where R is regarded as an RG-module with trivial G-action. Analogously, for a right RG-module B, one has the homology groups of G over R with coefficients in B defined as

$$H_k(G; B) = \mathrm{Tor}^{RG}_k(B,R), \quad k \in \mathbb{Z},$$

where again, R is the RG-module with trivial G-action.

Remark. We shall use the convention that coefficient modules for cohomology groups are left RG-modules and for homology groups right RG-modules. The reason for doing so is that we are going to use one and the same projective resolution $\underline{P} \twoheadrightarrow R$ of the trivial left RG-module R in order to compute both homology and cohomology groups

$$H_k(G; B) = H_k(B \otimes_{RG} \underline{P}), \quad H^k(G; A) = H^k(\mathrm{Hom}_{RG}(\underline{P},A)).$$

Notice, however, that every left RG-module A can be converted in a canonical way into a right module by putting. $a \cdot x = x^{-1}a$, $a \in A$, $x \in G$ (and vice versa).

Thus the cohomology functor $H^k(G; -)$, $k \in \mathbb{Z}$, as defined above, is a functor from the category of left RG-modules into the category of R-modules. $H_k(G; -)$ is a functor from the category of right RG-modules into the category of R-modules.

It is sufficient, in a sense, to consider homology and cohomology groups over $\mathbb{Z}$ instead of over R. Indeed one has natural isomorphisms

$$\mathrm{Ext}^k_{RG}(R,A) \cong \mathrm{Ext}^k_{\mathbb{Z}G}(\mathbb{Z},A)$$

$$\mathrm{Tor}_k^{RG}(B,R) \cong \mathrm{Tor}_k^{\mathbb{Z}G}(B,\mathbb{Z})$$

for every left RG-module A and right RG-module B and all $k \in \mathbb{Z}$.

<u>Proof</u>. Let $\underline{P} \twoheadrightarrow \mathbb{Z}$ be a G-projective resolution. This resolution is $\mathbb{Z}$-split, hence $R\otimes_{\mathbb{Z}}\underline{P} \twoheadrightarrow R\otimes_{\mathbb{Z}}\mathbb{Z} \cong R$ is an RG-projective resolution of R. Now one has the natural isomorphisms

$$\phi : \quad \mathrm{Hom}_{RG}(R\otimes_{\mathbb{Z}}\underline{P},A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}G}(\underline{P},A)$$

$$\psi : \quad B\otimes_{RG}(R\otimes_{\mathbb{Z}}\underline{P}) \xrightarrow{\sim} B\otimes_{\mathbb{Z}G}\underline{P}$$

given by $\phi(f)(p)=f(1\otimes p)$, $\psi(b\otimes r\otimes p)=br\otimes p$, $f \in \mathrm{Hom}_{RG}(R\otimes_{\mathbb{Z}}\underline{P},A)$, $p \in P$, $b \in B$, $r \in R$. This yields the result.□

Considering (co)homology groups over R is thus equivalent with restricting the coefficient category of all $\mathbb{Z}G$-modules to the category of RG-modules.

If S is a subgroup in G then RS is embedded in RG and the change-of-ring isomorphisms (b) and (d) yield

$$H_k(S;\, B) \approx H_k(G;\, B \otimes_{RS} RG), \quad H^k(S;\, A) \approx H^k(G;\, \mathrm{Hom}_{RS}(RG, A)),$$

for every right RS-module B and left RS-module A. This result is sometimes called the "Shapiro Lemma".

4. Special notations. For "$\mathbb{Z}G$-module", "$\mathbb{Z}G$-projective", etc. we shall write "G-module", "G-projective", etc. Also, tensor products and Hom-functors over $\mathbb{Z}G$ will be denoted by $- \otimes_G -$ and $\mathrm{Hom}_G(-,-)$.

Hom(−,−) and $-\otimes-$ denotes Hom-functor and tensor product over the ring $\mathbb{Z}$.

If G, F, H, S,... are groups, their augmentation ideals over R are denoted by the corresponding small German letters $\mathfrak{g}, \mathfrak{f}, \mathfrak{h}, \mathfrak{s}$ (e.g. $\mathfrak{g} = \ker\,(RG \twoheadrightarrow R)$).

CHAPTER I

FINITELY GENERATED RESOLUTIONS

1. Homological finiteness criteria

1.1 Type $(FP)_n$. Let A be an RG-module. A projective resolution $\underline{P} \twoheadrightarrow A$ is said to be finitely generated if the RG-modules P_i are finitely generated in each dimension $i \geq 0$. Every module A has projective resolutions, but not necessarily finitely generated projective resolutions. In this section we deduce homological conditions on A which are equivalent with the existence of finitely generated free resolutions. As no additional difficulties are involved it is natural (and in fact easier) to consider the more general situation where the group-ring RG is replaced by an arbitrary ring Λ with unit.

Definition. The Λ-module A is said to be of type $(FP)_n$ if there is a projective resolution $\underline{P} \twoheadrightarrow A$ with P_i finitely generated for all $i \leq n$. If the modules P_i are finitely generated for all i then we say that A is of type $(FP)_\infty$.

Remarks. 1) Notice that A is of type $(FP)_0$ if and only if A is finitely generated and that A is of type $(FP)_1$ if and only if A is finitely presented.

2) If A is of type $(FP)_n$, $0 \leq n \leq \infty$, then one can even construct a free resolution which is finitely generated

in dimensions $\leq n$. For, let ... $P_2 \to P_1 \xrightarrow{d_1} P_0 \twoheadrightarrow A$ be a projective resolution with P_0 finitely generated. Then there is a finitely generated projective module Q such that $P_0 \oplus Q$ is a free module. Thus replacing P_0 by $P_0 \oplus Q$ and P_1 by $P_1 \oplus Q$ and extending d_1 by Id_Q yields a new resolution which is finitely generated and free in dimension 0. Iterating this process yields the result. Notice that in the case $(FP)_\infty$ this construction pushes the difficulties to infinity where they vanish!

1.2 Exact limits and colimits. Let $\mathcal{G}$ be a directed graph without loops. A $\mathcal{G}$-diagram in the category of Λ-modules is given by (1) a Λ-module M_v for every vertex $v \in \mathcal{G}$ and (2) a Λ-homomorphism α_e: $M_v \to M_w$ for every edge e from v to w. For every $\mathcal{G}$-diagram M_* one has limit $\lim M_*$ and colimit $\operatorname{colim} M_*$ defined by the usual universal properties.

Let F be a covariant functor from the category of Λ-modules to the category of Abelian groups. The canonical maps $M_v \to \operatorname{colim} M_*$ and $\lim M_* \to M_v$ induce a compatible system of maps $F(M_v) \to F(\operatorname{colim} M_*)$ and $F(\lim M_*) \to F(M_v)$ respectively and hence limiting homomorphisms,

$$\operatorname{colim} F(M_*) \to F(\operatorname{colim} M_*), \quad F(\lim M_*) \to \lim F(M_*) ,$$

respectively. We say that F commutes with colimits or limits if the corresponding limiting homomorphism is an isomorphism.

For a fixed graph $\mathcal{G}$, lim and colim are functors from the category of ($\mathcal{G}$-diagrams in the category of Λ-modules) into the category of Λ-modules. Neither of these functors is exact in general, but there are interesting special cases, i.e., special graphs $\mathcal{G}$ with the property that lim or colim are exact functors. In this case we shall speak of exact limits or exact colimits respectively.

Examples. 1) If $\mathcal{Y}$ is the graph consisting of a set of vertices I with no edges, then the limit lim is the (direct) product $\prod_I$. This is easily seen to be an exact functor. Thus the direct product is an exact limit.

2) If $\mathcal{Y}$ has the property that for any two vertices u,v there is a vertex w and (directed) paths from u to w and from v to w, then colim is the direct limit $\varinjlim$. This is an exact functor. Thus the direct limit is an exact colimit.

Proposition 1.1. For every (left) Λ-module A and all $k \geq 0$ one has:

(a) the functor $\mathrm{Tor}_k(-,A)$ commutes with exact colimits,

(b) the functor $\mathrm{Ext}^k(A,-)$ commutes with exact limits. *

Proof. Colimits commute with $-\otimes_\Lambda A$ and limits with $\mathrm{Hom}_\Lambda(A,-)$. If these are exact, they commute with the functors $\mathrm{Tor}_k(-,A)$ and $\mathrm{Ext}^k(A,-)$ respectively. □

Proposition 1.2. Let A be a (left) Λ-module of type $(\mathrm{FP})_n$, $0 \leq n \leq \infty$. Then one has

(a) For every exact limit the natural homomorphism $\mathrm{Tor}_k(\lim M_*,A) \to \lim \mathrm{Tor}_k(M_*,A)$ is an isomorphism for $k \leq n-1$ and an epimorphism for $k=n$.

(b) For every exact colimit the natural homomorphism

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* Throughout Section I we write $\mathrm{Ext}^k(-,-)$ and $\mathrm{Tor}_k(-,-)$ for $\mathrm{Ext}^k_\Lambda(-,-)$ and $\mathrm{Tor}^\Lambda_k(-,-)$ respectively.

colim $\mathrm{Ext}^k(A,M_*) \to \mathrm{Ext}^k(A, \mathrm{colim}\, M_*)$ is an isomorphism for $k \leq n-1$ and a monomorphism for $k=n$.

Proof. By the above remark we can choose a free resolution $\underline{F} \twoheadrightarrow A$ such that the modules F_k are finitely generated for all $k \leq n$. Since lim is an additive functor it commutes with finite direct sums and hence the natural homomorphism

$$(\lim M_*) \otimes_\Lambda F_k \to \lim (M_* \otimes_\Lambda F_k)$$

is an isomorphism for all $k \leq n$. Since lim is assumed to be exact it commutes with the homology functor, $H_*(\lim(M_* \otimes_\Lambda \underline{F})) \cong \lim H_*(M_* \otimes_\Lambda \underline{F})$ and (a) follows by easy diagram chasing. Analogously, $\mathrm{Hom}_\Lambda(A,-)$ commutes with finite direct sums, hence the natural homomorphism

$$\mathrm{colim}\, \mathrm{Hom}_\Lambda(F_k, M_*) \to \mathrm{Hom}_\Lambda(F_k, \mathrm{colim}\, M_*)$$

is an isomorphism for all $k \leq n$. Since colim is assumed to be exact one has $H^*(\mathrm{colim}\, \mathrm{Hom}_\Lambda(\underline{F}, M_*)) \cong \mathrm{colim}\, H^*(\mathrm{Hom}_\Lambda(\underline{F}, M_*))$, whence (b) □

1.3 The main result. The main result of the present section asserts that the converse of Proposition 1.2 holds.

Theorem 1.3. The following conditions are equivalent for a (left) Λ-module A :

(i) A is of type $(FP)_n$.

(iia) For any exact limit the natural map $\mathrm{Tor}_k(\lim M_*, A) \to \lim \mathrm{Tor}_k(M_*, A)$ is an isomorphism for $k < n$ and an epimorphism for $k=n$.

(iib) For any exact colimit the natural map $\operatorname{colim} \operatorname{Ext}^k(A,M_*) \to \operatorname{Ext}^k(A, \operatorname{colim} M_*)$ is an isomorphism for $k < n$ and a monomorphism for $k = n$.

(iiia) For a direct product $\Pi\Lambda$ of arbitrary many copies of Λ the natural map $\operatorname{Tor}_k(\Pi\Lambda, A) \to \Pi \operatorname{Tor}_k(\Lambda, A)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.

(iiib) For the direct limit of any directed system of Λ-modules $\{M_*\}$ with $\varinjlim M_* = 0$, one has $\varinjlim \operatorname{Ext}^k(A,M_*) = 0$ for all $k \le n$.

<u>Proof</u>. The implication (i) $\Rightarrow$ (iia) and (iib) are contained in Proposition 1.2. (iia) $\Rightarrow$ (iiia) and (iib) $\Rightarrow$ (iiib) are trivial. The remaining implications (iiia) $\Rightarrow$ (i) and (iiib) $\Rightarrow$ (i) shall be proved by induction on n.

(iiia)$\Rightarrow$(i): Let $n=0$. We take A itself as an index set and consider $\prod_A \Lambda$. By assumption the natural map $\mu: (\prod_A \Lambda)\otimes_\Lambda A \to \prod_A A$ is an epimorphism. In particular there is an element $c \in (\prod_A \Lambda)\otimes_\Lambda A$ which is mapped onto the diagonal $\prod_{a\in A} a$. c is of the form $c = \sum_{i=1}^m \prod_a \lambda_i^a \otimes a_i$, $\lambda_i^a \in \Lambda, a_i \in A$, hence

$$\mu(c) = \sum_{i=1}^m \prod_a \lambda_i^a a_i = \prod_a \sum_{i=1}^m \lambda_i^a a_i = \prod_a a .$$

It follows that $a = \sum_{i=1}^m \lambda_i^a a_i$ for all $a \in A$, i.e., A is generated by the finite set $a_1, a_2, \ldots, a_m$.

Now let $n \ge 1$. As in the case $n=0$ we first conclude that A is finitely generated. Then take a short exact sequence of Λ-modules $K \rightarrowtail F \twoheadrightarrow A$ with F finitely generated free. By naturality we have the following commutative diagram

$$\begin{array}{ccccccccc}
\to \mathrm{Tor}_n(\Pi\Lambda,F) & \to & \mathrm{Tor}_n(\Pi\Lambda,A) & \to & \mathrm{Tor}_{n-1}(\Pi\Lambda,K) & \to & \mathrm{Tor}_{n-1}(\Pi\Lambda,F) & \to & \mathrm{Tor}_{n-1}(\Pi\Lambda,A) \\
\downarrow\wr & & \downarrow & & \downarrow & & \downarrow\wr & & \downarrow\wr \\
\to \Pi\,\mathrm{Tor}_n(\Lambda,F) & \to & \Pi\,\mathrm{Tor}_n(\Lambda,A) & \to & \Pi\,\mathrm{Tor}_{n-1}(\Lambda,K) & \to & \Pi\,\mathrm{Tor}_{n-1}(\Lambda,F) & \to & \Pi\,\mathrm{Tor}_{n-1}(\Lambda,A)
\end{array}$$

It follows by the 5-lemma that $\mathrm{Tor}_k(\Pi\Lambda,K) \to \Pi\,\mathrm{Tor}_k(\Lambda,K)$ is an isomorphism for $k < n-1$ and an epimorphism for $k = n-1$. By the induction hypothesis we know that K is of type $(FP)_{n-1}$, hence A is of type $(FP)_n$.

(iiib) $\Rightarrow$ (i) Let $n=0$. We consider the direct system $\{A/A'\}$ where A' ranges over all finitely generated submodules of A. Then $\varinjlim A/A' = 0$, so we have $\varinjlim \mathrm{Hom}_\Lambda(A,A/A') = 0$; in particular $\varinjlim(A \overset{\pi}{\to} A/A') = 0$. But this means that $\pi: A \to A/A'$ is zero for some A', i.e. $A=A'$, hence A is finitely generated.

Now let $n \geq 1$. As above we first conclude that A is finitely generated. Then take a short exact sequence of Λ-modules $K \rightarrowtail F \twoheadrightarrow A$ with F finitely generated free. Let M_* be a direct system of Λ-modules with $\varinjlim M_* = 0$. Then, by the long exact Ext-sequence we see that $\varinjlim \mathrm{Ext}^k(K,M_*) = 0$ for all $k \leq n-1$. By induction hypothesis this implies that K is of type $(FP)_{n-1}$, hence A is of type $(FP)_n$.□

<u>Remarks</u> (concerning condition (iiia)). 1) Notice that $\mathrm{Tor}_k(\Lambda,A) = 0$ for $k \neq 0$. Thus for $n \geq 1$, the assertion of (iiia) is simply:

(iiia)' $\mu: (\Pi\Lambda) \otimes_\Lambda A \to \Pi A$ is an isomorphism and $\mathrm{Tor}_k(\Pi\Lambda,A)=0$ for $1 \leq k \leq n-1$.

2) The condition $\mu: (\Pi\,\Lambda)\otimes_\Lambda A \xrightarrow{\approx} \Pi A$ for all direct products is equivalent with "A is of type $(FP)_1$". Thus (iiia)' is furthermore equivalent to

(iiia)" A is finitely presented and $\mathrm{Tor}_k(\Pi\,\Lambda, A) = 0$ for $1 \le k \le n-1$.

3) The proof of (iiia) $\Rightarrow$ (i) yields a slightly stronger result: It is sufficient, in condition (iiia), to consider direct products $\prod_{\varkappa} \Lambda$ over an index set of cardinality $\varkappa \le \max(|\Lambda|, |A|)$. Hence if A is known to be finitely generated (e.g. in condition (iiia)") we only need to consider direct products $\prod_{\varkappa} \Lambda$ with $\varkappa \le |\Lambda|$.

As an application we prove

<u>Proposition 1.4</u>. Let $A' \rightarrowtail A \twoheadrightarrow A''$ be a short exact sequence of Λ-modules. Then the following statements hold.

(a) If A' is of type $(FP)_{n-1}$ and A of type $(FP)_n$, then A'' is of type $(FP)_n$,

(b) If A is of type $(FP)_{n-1}$ and A'' is of type $(FP)_n$ then A' is of type $(FP)_{n-1}$.

(c) If A' and A'' are of type $(FP)_n$ then so is A.

<u>Proof</u>. Apply either of (iiia) or (iiib) of Theorem 1.3, and the long exact Tor(Ext)-sequences. □

Let A be a finitely generated ($\Leftrightarrow (FP)_0$) Λ-module.

Choose a presentation

$$K_0 \rightarrowtail P_0 \twoheadrightarrow A$$

with P_0 a finitely generated projective Λ-module. If A is finitely presented ($\Leftrightarrow (FP)_1$), i.e. there is some short exact sequence $K \rightarrowtail F \twoheadrightarrow A$ with F a free Λ-module and both F and K finitely generated, then it follows by Proposition 1.4(b) that K_0 is of type $(FP)_0$, i.e. finitely generated. Next choose a presentation

$$K_1 \rightarrowtail P_1 \twoheadrightarrow K_0$$

with P_1 finitely generated projective. Now, if A is of type $(FP)_2$ then K_0 is of type $(FP)_1$ and hence K_1 is of type $(FP)_0$. Iterating this argument yields

<u>Proposition 1.5</u>. Let A be an Λ-module of type $(FP)_n$ and let $P_{n-1} \to \dots \to P_1 \to P_0 \twoheadrightarrow A$ be the first n terms of a projective resolution of A. If $P_0, P_1, \dots P_{n-1}$ are finitely generated, then the kernel of $P_{n-1} \to P_{n-2}$ is finitely generated (so that one can extend the resolution by a finitely generated projective module P_n one step further to the left $P_n \to P_{n-1} \to P_{n-2} \to \dots \to P_0 \twoheadrightarrow A$.).

Corollary 1.6. For a (left) Λ-module A the following conditions are equivalent

(i) A is of type $(FP)_\infty$,

(iia) The functor $\mathrm{Tor}_k(-,A)$ commutes with exact limits for all $k \geq 0$,

(iib) The functor $\mathrm{Ext}^k(A,-)$ commutes with exact colimits for all $k \geq 0$,

(iiia) $\mathrm{Tor}_k(\prod_{\varkappa} \Lambda, A) = 0$ for all $k \geq 1$ and all cardinalities $\varkappa \leq |\Lambda|$, and the natural map $\mu:(\prod_{\varkappa} \Lambda) \otimes_\Lambda A \to \prod A$ is an isomorphism for all $\varkappa \leq \max(|\Lambda|, |A|)$.

(iiib) $\varinjlim \mathrm{Ext}^k(A, M_*) = 0$ for all $k \geq 0$ and all direct systems $\{M_*\}$ of Λ-modules with $\varinjlim M_* = 0$.

Proof. The implications (i) $\Rightarrow$ (iia) $\Rightarrow$ (iiia) and (i) $\Rightarrow$ (iib) $\Rightarrow$ (iiib) are obvious. Theorem 1.3 shows that either of (iiia) or (iiib) implies that A is of type $(FP)_n$ for all $n \geq 0$. By Proposition 1.5 this enables us to construct a finitely generated projective resolution, i.e. A is of type $(FP)_\infty$. □

1.4 Topological remarks. The main results of this section hold in a more general situation, namely for projective chain complexes rather than for resolutions only. The proof in this more general circumstance is a little bit more complicated but not any more difficult than for resolutions (cf. Kenneth Brown "Homological criteria for finiteness", to appear in Comment.Math.Helv.). We are not going to use the result later, but we state it for completeness.

Let Λ be a ring with unit and let $\underline{C}$ be a positive chain complex of projective left Λ-modules. For left Λ-modules A and right Λ-modules B the (co)homology groups of $\underline{C}$ with coefficients in B(A) are defined to be

$$H^n(\underline{C};A) = H^n(\mathrm{Hom}_\Lambda(\underline{C},A)), \quad H_n(\underline{C};B) = H_n(B\otimes_\Lambda \underline{C})$$

Theorem 1.7. (K.S. Brown [15]) The following conditions on $\underline{C}$ are equivalent.

(i) $\underline{C}$ is chain homotopy equivalent to a complex of finitely generated projective modules.

(iia) $H_n(\underline{C};-)$ commutes with exact limits for all $n \geq 0$.

(iib) $H^n(\underline{C};-)$ commutes with exact colimits for all $n \geq 0$.

(iiia) The natural map $\mu: H_n(\underline{C};\Pi\,\Lambda) \to \Pi\, H_n(\underline{C})$ is an isomorphism for all $n \geq 0$ and all direct products Π.

(iiib) $\varinjlim H^n(\underline{C};M_*) = 0$ for all $n \geq 0$ and all direct systems of Λ-modules $\{M_*\}$ with $\varinjlim M_* = 0$.

Theorem 1.7 has a topological translation. Recall that a CW-complex X is said to be of finite type if X has only finitely many cells in each dimension.

Theorem 1.8. (K.S. Brown [15]) Let X be a connected CW-complex with finitely presented fundamental group $\pi_1(X) = G$. Then the following conditions on X are equivalent.

(i) X is homotopy equivalent to a complex of finite type,

(iia) the homology functor $H_n(X;-)$ - regarded as a functor from the category of local coefficient systems on X to the category of Abelian groups - commutes with exact limits for all $n \geq 0$,

(iib) the cohomology functor $H^n(X;-)$ - regarded as a functor from the category of local coefficient systems on X to the category of Abelian groups - commutes with exact colimits for all $n \geq 0$.

(iiia) $\mu: H_n(X; \prod_{\aleph_0} \mathbb{Z}G) \cong \prod_{\aleph_0} H_n(X; \mathbb{Z}G)$ for all $n \geq 0$,

(iiib) $\varinjlim H_n(X; M_*) = 0$ for every direct system of local coefficient systems $\{ M_* \}$ with $\varinjlim M_* = 0$ and all $n \geq 0$.

Proof. It is well known from the work of C.T.C. Wall [61] that X is homotopy equivalent to a complex of finite type if and only if the singular chain complex of its universal cover, $\underline{\underline{C}}(\tilde{X})$, is chain homotopy equivalent to a complex of finitely generated projective $\mathbb{Z}G$-modules. Thus Theorem 1.8 follows immediately from Theorem 1.7. □

Local coefficient systems are not so easy to deal with; therefore the following sufficient condition for finite type is somewhat more down-to earth.

Theorem 1.9. Let X be a connected CW-complex with finitely presented fundamental group $\pi_1(X) = G$ and let $\tilde{X}$ denote its universal cover. If the homology groups $H_k(\tilde{X})$ are of type $(FP)_\infty$ as $\mathbb{Z}G$-modules for all $k \geq 0$, then X is homotopy equivalent to a complex of finite type.

Proof. By Theorem 1.8 one has to show that $\mu: H_k(X;\prod_{\aleph_0} \mathbb{Z}G) \simeq \prod_{\aleph_0} H_k(X; \mathbb{Z}G)$ for all $k \geq 0$. For this we use the covering spectral sequence (cf.Cartan-Eilenberg "Homological Algebra" p.335).

$$E^{(2)}_{p,q} = H_p(G; H_q(\tilde{X}; \Pi\mathbb{Z}G)) \Rightarrow H_{p+q}(X; \Pi\mathbb{Z}G).$$

By the Universal Coefficients Theorem, $H_q(\tilde{X}; \Pi\mathbb{Z}G) \simeq H_q(\tilde{X})\otimes\Pi\mathbb{Z}G$, and hence by Lemma 1.10 below $E^{(2)}_{p,q} = \mathrm{Tor}_p(H_q(\tilde{X}); \Pi\mathbb{Z}G)$. Since $H_p(\tilde{X})$ is of type $(FP)_\infty$ we conclude $E^{(2)}_{p,q} = 0$ for $p \neq 0$, i.e., the spectral sequence collapses and yields the isomorphisms

$$\mu: H_q(X; \Pi\mathbb{Z}G) \simeq H_q(\tilde{X})\otimes_G \Pi\mathbb{Z}G \simeq \Pi H_q(\tilde{X}) = \Pi H_q(X; \mathbb{Z}G).\square$$

Lemma 1.10. Let A be a left and B a right G-module. If either A or B is torsion-free as an Abelian group then one has natural isomorphisms $H_k(G; B\otimes A) \simeq \mathrm{Tor}_k^{\mathbb{Z}G}(B,A)$ for all $k \in \mathbb{Z}$ with diagonal G-action on $B \otimes A$.

Proof. Without loss of generality assume that A is torsion-free over $\mathbb{Z}$. Let $\underline{P} \twoheadrightarrow \mathbb{Z}$ be a G-projective resolution. Then $A\otimes\underline{P} \twoheadrightarrow A$ is a G-flat resolution and can be used to compute $\mathrm{Tor}_k^{\mathbb{Z}G}(B,A)$. Moreover, one has the obvious natural isomorphism $B\otimes_G(A\otimes\underline{P}) \cong (B\otimes A)\otimes_G\underline{P}$, whence the result. □

Proposition 1.11. Let G be a group. If the trivial G-module $\mathbb{Z}$ is of type $(FP)_\infty$ then so is every G-module A whose underlying Abelian group is finitely generated.

Proof. We make repeated use of Corollary 1.6 (iiia). For all $k \in \mathbb{Z}$ one has

$$\begin{aligned}
\mathrm{Tor}_k^{\mathbb{Z}G}(A,\Pi\mathbb{Z}G) &\cong \mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},A\otimes\Pi\mathbb{Z}G) && \text{by Lemma 1.10}\\
&\cong \mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},\Pi(A\otimes\mathbb{Z}G)) && \text{by Cor.1.6 for } \Lambda=\mathbb{Z}\\
&\cong \Pi\mathrm{Tor}_k^{\mathbb{Z}G}(\mathbb{Z},A\otimes\mathbb{Z}G) && \text{by Cor.1.6 for } \Lambda=\mathbb{Z}G.\\
&\cong \Pi\mathrm{Tor}_k^{\mathbb{Z}G}(A,\mathbb{Z}G) && \text{by Lemma 1.10.}\ \square
\end{aligned}$$

If the trivial G-module $\mathbb{Z}$ is of type $(FP)_\infty$ then we say that the group G is of type $(FP)_\infty$. Theorem 1.9 together with Proposition 1.11 yields

Corollary 1.12. Let X be a connected CW-complex and $\tilde{X}$ its universal cover. If the fundamental group $\pi_1(X)$ is finitely presented and of type $(FP)_\infty$, and all homology groups $H_k(\tilde{X})$ are finitely generated Abelian groups, $k \in \mathbb{Z}$, then X is homotopy equivalent to a complex of finite type.

2. Groups of type $(FP)_n$

2.1. Definition and basic facts. We recall that R will always denote a commutative ring with unit $1 \neq 0$.

Definition. A group G is said to be of type $(FP)_n$ over R, $n = \infty$ or an integer ≥ 0, if the trivial G-module R is of type $(FP)_n$ as an RG-module.

If G is of type $(FP)_n$ over $\mathbb{Z}$ then we merely say that G is of type $(FP)_n$. If G is of type $(FP)_n$, then clearly G is of type $(FP)_n$ over any ring R.

Remarks. 1) R is finitely generated as an RG-module, hence every group is of type $(FP)_0$ over R.

2) Proposition 2.1. G is of type $(FP)_1$ over R if and only if G is finitely generated.

Proof. If G is finitely generated, then the augmentation ideal I_G is finitely generated as a left $\mathbb{Z}G$-module and hence one can construct a free resolution $\to \underset{\text{gens.}}{\oplus \mathbb{Z}G} \to \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$, i.e. G is of type $(FP)_1$ over any ring R. Conversely, assume that G is of type $(FP)_1$ over R. This means that the kernel $\mathfrak{g}$ of $RG \twoheadrightarrow R$ is finitely generated over RG. It follows that $\mathfrak{g}$ can be generated, as an RG-module, by a finite number of elements of the form $(1-x_i)$, $x_i \in G$. Let S be the subgroup generated by $x_1,\ldots,x_n$, and γ its augmentation ideal. Then $RG \cdot \gamma = \mathfrak{g}$. Now, consider the short exact sequence

$\mathfrak{T} \rightarrowtail RS \twoheadrightarrow R$. Tensoring with $RG\otimes_{RS}-$ yields

$$RG\otimes_{RS}\mathfrak{T} \rightarrowtail RG \twoheadrightarrow RG\otimes_{RS}R$$

But (1) $RG\otimes_{RS}R \cong R(G/S)$ $\qquad x\otimes r \mapsto r\cdot xS$

(2) $RG\otimes_{RS}\mathfrak{T} \cong RG.\mathfrak{T}$ $\qquad x\otimes(s-1) \mapsto x(s-1)$

It follows that $\mathfrak{g} \rightarrowtail RG \twoheadrightarrow R(G/S)$ is a short exact sequence, i.e. $G = S$.□

3) A group G is said to be <u>almost finitely presented over</u> R, if there is a short exact sequence of groups $K \rightarrowtail F \twoheadrightarrow G$ with F a finitely generated free group and $R\otimes K/[K,K]$ finitely generated as an RG-module. Finitely presented groups are clearly almost finitely presented over any ring R. Whether or not the converse holds is still an open question.

<u>Proposition 2.2</u>. G is of type $(FP)_2$ over R if and only if G is almost finitely presented over R.

<u>Proof</u>. $\mathfrak{f} \rightarrowtail RF \twoheadrightarrow R$ is an RF-free resolution, hence one has the exact sequence

$$0 \to H_1(F;RG) \to RG\otimes_{RF}\mathfrak{f} \to RG\otimes_{RF}RF \to RG\otimes_{RF}R \to 0,$$

But $H_1(F;RG) \cong H_1(F;R\otimes_{RK}RF) \cong H_1(K;R) \cong R\otimes K/[K,K]$, hence we get an exact sequence of RG-modules

$$0 \to R\otimes K/[K,K] \to RG\otimes_{RF}\mathfrak{f} \to RG \to R \to 0;$$

$RG\otimes_{RF}\mathfrak{f}$ is RG-free on the set $\{1\otimes(x_i-1)\}$, where $x_1, x_2,\ldots,x_n$ are the free generators of F. Thus G is of type $(FP)_2$ over R if and only if $R\otimes K/[K,K]$ is a finitely generated RG-module.□

Theorem 1.3 yields necessary and sufficient conditions for a group G to be of type $(FP)_n$ over R. In particular, we get

Proposition 2.3. A group G is of type $(FP)_n$ over R $(1 \le n \le \infty)$, if and only if G is finitely generated and $H_k(G;\prod_{\varkappa} RG) = 0$ for all $1 \le k < n$ and all direct products of $\varkappa = \max(\aleph_0, |R|)$ copies of RG.

Proof. Theorem 1.3 together with Proposition 2.1. One has to prove that certain kernels of maps between finitely generated free RG-modules are finitely generated, and these kernels are always of cardinality $\le \varkappa = \max(\varkappa_0, |R|)$, so that it is sufficient in the proof of Theorem 1.3 to consider direct products of $\varkappa$ copies of RG.□

Proposition 2.4. For a group G the following conditions are equivalent:

(i) G is of type $(FP)_\infty$ over R.

(ii) $H_k(G; -)$ commutes with direct products for all $k \ge 0$.

(iii) $H^k(G; -)$ commutes with direct limits for all $k \ge 0$.

2.2 Extension properties. In the remainder of section 2 we shall construct examples of type $(FP)_\infty$.

Proposition 2.5. Let G be a group, $S \le G$ a subgroup of finite index. Then G is of type $(FP)_n$, $0 \le n \le \infty$, if and only if S is.

For our "resolution-free" proof of this, we need the following Lemma. Let G be an arbitrary group, $S \leq G$ a subgroup, and A an RS-module. Then $RG \otimes_{RS} A$ and $\mathrm{Hom}_{RS}(RG,A)$ are RG-modules by the (so-called single) action given by $x(g \otimes a) = xg \otimes a$, $(x \circ f)(g) = f(gx)$, $x,g \in G$, $a \in A$, $f \in \mathrm{Hom}_{RS}(RG,A)$.

Lemma 2.6. If $|G:S| < \infty$ then there is a natural isomorphism of RG-modules

$$\theta\colon \mathrm{Hom}_{RS}(RG,A) \xrightarrow{\sim} RG \otimes_{RS} A.$$

Proof. Let $r_1, r_2, \ldots r_m$ be a right transversal for G mod S, and define $\theta(f) = \sum_{i=1}^{m} r_i^{-1} \otimes f(r_i)$. It is easy to see that θ does not depend upon the choice of the transversal. Moreover

$$\begin{aligned}\theta(x \circ f) &= \sum r_i^{-1} \otimes (x \circ f)(r_i) = \sum r_i^{-1} \otimes f(r_i x) \\ &= \sum r_i^{-1} \otimes f(r_i x \cdot \overline{r_i x}^{-1} \overline{r_i x}) \quad (\overline{g} = \text{representative of } g \in G) \\ &= \sum x \cdot \overline{r_i x}^{-1} \otimes f(\overline{r_i x}) \quad = x \cdot \theta(f),\end{aligned}$$

thus θ is an RG-homomorphism. Finally, θ is obviously an isomorphism of RS-modules, since $\mathrm{Hom}_{RS}(RG,A) \cong \mathrm{Hom}_{RS}(\oplus RS,A) \cong \oplus A$ and $RG \otimes_{RS} A \cong (\oplus RS) \otimes_{RS} A \cong \oplus(RS \otimes_{RS} A) \cong \oplus A$; this proves the Lemma. □

Proof (of Proposition 2.5). For all $p \geq 0$ one has

$$\begin{aligned}H_p(S; \Pi\, RS) &\cong H_p(G; RG \otimes_{RS} (\Pi\, RS)), \text{ by the Shapiro Lemma,} \\ &\cong H_p(G; \mathrm{Hom}_{RS}(RG, \Pi\, RS)), \text{ by Lemma 2.6,} \\ &\cong H_p(G; \Pi\, \mathrm{Hom}_{RS}(RG, RS)), \\ &\cong H_p(G; \Pi\, (RG \otimes_{RS} RS)), \text{ by Lemma 2.6,} \\ &\cong H_p(G; \Pi\, RG),\end{aligned}$$

and the result follows by Proposition 2.3.□

<u>Proposition 2.7</u>. Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups and assume that N is of type $(FP)_\infty$ over R. Then G is of type $(FP)_n$, $0 \leq n \leq \infty$, if and only if Q is.

<u>Proof</u>. Consider the LHS-spectral sequence $H_p(Q; H_q(N; \Pi\, RG) \Rightarrow H_{p+q}(G; \Pi\, RG)$. Since N is of type $(FP)_\infty$ one has $H_q(N; \Pi\, RG) \simeq \Pi\, H_q(N; RG) = 0$ for $q \geq 1$ and $\Pi(RG)_N \simeq \Pi RQ$ for $q = 0$. Thus the spectral sequence collapses and yields isomorphisms $H_p(Q; \Pi\, RG) \cong H_p(G; \Pi\, RG)$ for all $p \geq 0$. By Proposition 2.3 this implies the result.

<u>Exercise</u>. Similar results assuming type $(FP)_n$ for N rather than $(FP)_\infty$.

In order to construct further examples of groups of type $(FP)_\infty$ we are now going to consider amalgamated free products and HNN-extensions of groups. We shall deduce long exact Mayer-Vietoris sequences for these constructions and use them to show that certain amalgamated products and HNN-extensions of groups of type $(FP)_\infty$ are again of type $(FP)_\infty$.

2.3. Free differential calculus. Let G be a group. A derivation of G is a map $d: G \to A$, A an RG-module, with the property that $d(xy) = d(x) + x\cdot d(y)$, for all $x,y \in G$. Every derivation d extends uniquely to an R-homomorphism $d: RG \to A$ with the property $d(\lambda\mu) = d(\lambda)\cdot\varepsilon(\mu) + \lambda\cdot d(\mu)$. The set of all derivations, $Der(G,A)$, is an R-module, and it is easy to check that the map

$$\rho: Der(G,A) \to Hom_{RG}(\mathfrak{g}, A),$$

given by $\rho(d)(x-1) = d(x)$, $x \in G$, $d \in Der(G,A)$, is a natural isomorphism.

Now, let $G = F$ be a free group generated by free generators $\{x_i\}$. Then $\mathfrak{f}$ is RF-free on $\{x_i - 1\}$. It follows that an arbitrary choice of values $\alpha_i \in A$ determines a unique derivation $d: F \to A$ with $d(x_i) = \alpha_i$.

Definition. We denote by $\frac{\partial}{\partial x_i}$ the derivation $F \to RF$ which is defined by the values $\frac{\partial}{\partial x_i}(x_j) = \frac{\partial x_j}{\partial x_i} = \delta_{ij}$. Those derivations are called the partial derivatives with respect to x_i.

Let $d: F \to A$ be a derivation. Define $\tilde{d}(w) = \sum_i \frac{\partial w}{\partial x_i} d(x_i)$, $w \in F$. (This is well defined since $\frac{\partial w}{\partial x_i} = 0$ for all but a finite number of x_i's). $\tilde{d}: F \to A$ is again a derivation since

$$\tilde{d}(u\cdot v) = \sum \frac{\partial (uv)}{\partial x_i} d(x_i) = \sum \frac{\partial u}{\partial x_i} d(x_i) + \sum u \frac{\partial v}{\partial x_i} d(x_i) = \tilde{d}(u) + u\tilde{d}(v)$$

$u, v \in F$.

But $\tilde{d}(x_i) = \sum_j \delta_{ij} d(x_j) = d(x_i)$, whence $d = \tilde{d}$.

Thus one has the formula

$$d(w) = \sum_i \frac{\partial w}{\partial x_i} d(x_i), \quad w \in F,$$

for all derivations $d: F \to A$. If we apply this to the special inner derivation $d: F \to RF$, $d(w) = w - 1$, we get, in particular,

$$\boxed{w - 1 = \sum_i \frac{\partial w}{\partial x_i}(x_i - 1), \; w \in F} \quad \text{"fundamental formula".}$$

There are two ways to generalize this formula slightly. Firstly, extended to RF by linearity it reads

$$\lambda - \varepsilon(\lambda) = \sum \frac{\partial \lambda}{\partial x_i} (x_i - 1), \qquad \lambda \in RF.$$

Secondly, let G be a group with generators $\{g_i\}$ and let $\pi: F \twoheadrightarrow G$ be a free presentation, F freely generated by $\{x_i\}$ and $\pi(x_i) = g_i$. Then the fundamental formula extended to RF and mapped to RG by the induced homomorphism $\pi: RF \twoheadrightarrow RG$, reads

$$\lambda - \varepsilon(\lambda) = \sum \pi\left(\frac{\partial \Lambda}{\partial x_i}\right)(g_i - 1)$$

where $\lambda \in RG$, and $\Lambda \in RF$ with $\pi(\Lambda) = \lambda$.

2.4. Amalgamated products. Let G_1, G_2 be groups with subgroups $S_1 \le G_1$, $S_2 \le G_2$ and assume that S_1 and S_2 are isomorphic via an isomorphism $\theta: S_1 \xrightarrow{\sim} S_2$. Then the amalgamated free product of G_1 and G_2 with amalgamated subgroups S_i is defined to be

$$G = G_1 *_{S_1 = S_2} G_2 = \langle G_1, G_2 : \mathrm{rel}G_1, \mathrm{rel}G_2, s = \theta(s), \forall s \in S_1 \rangle.$$

There are obvious homomorphisms $j_\alpha : G_\alpha \to G$ induced by the identity on G_α, $\alpha = 1,2$; and one can prove

(i) the maps $j_\alpha : G_\alpha \to G$ are injections

(ii) $j_1(G_1) \cap j_2(G_2) = j_1(S_1) = j_2(S_2)$.

We shall use j_α as identifications, i.e. we consider G_1 and G_2 as being subgroups in G, with $S = G_1 \cap G_2 = S_1 = S_2$.

Normal Form Theorem. For $\alpha = 1,2$ let T_α denote left transversals for G_α mod S containing 1. Let T be the set of all products $g_1 g_2 \cdots g_k$, $1 \le k < \infty$, $1 \ne g_i \in T_1 \cup T_2$, with the property that consecutive factors g_i, g_{i+1} are not both in T_α, $\alpha = 1,2$. Then T together with 1 is a left transversal for $G = G_1 *_S G_2$ mod S.

Proof. cf [42].

Remark. Every coset $xS \ne S$ contains a unique representative $g_1 g_2 \cdots g_k \in T$ and we define the integer k to be the length of xS, $\ell(xS) = k$. $\ell(S)$ is defined to be 0. It is not hard to see that $\ell(xS)$ in fact does not depend upon the choice of T_1 and T_2. Notice that $\ell(xS) \le 1 \Leftrightarrow x \in G_1 \cup G_2$.

We shall also say that an element $x \in G$ is of length $\ell(x) = K$ if k is the length of its coset mod S.

Proposition 2.8. (Swan [60]). Let $G = G_1 *_S G_2$. Then there is a short exact sequence of (left) RG-modules:

$$R(G/S) \overset{\alpha}{\rightarrowtail} R(G/G_1) \oplus R(G/G_2) \overset{\varepsilon}{\twoheadrightarrow} R$$

where α and ε are defined by $\alpha(xS) = (xG_1, -xG_2)$ and $\varepsilon(xG_1, 0) = \varepsilon(0, xG_2) = 1$, $x \in G$.

Proof. Clearly ε is an epimorphism and $\varepsilon \circ \alpha = 0$. Next let $\{\bar{x}_i\}$ and $\{\bar{y}_j\}$ be generators for G_1 and G_2 respectively, construct the free group F on symbols $\{x_i, y_j\}$ and the presentation $F \overset{\pi}{\twoheadrightarrow} G_1 *_S G_2 = G$ with $\pi(x_i) = \bar{x}_i$, $\pi(y_j) = \bar{y}_j$. By the fundamental formula one has for all $\lambda \in RG$

$$\lambda - \varepsilon(\lambda) = \sum \pi(\frac{\partial \Lambda}{\partial x_i})(\bar{x}_i - 1) + \sum \pi(\frac{\partial \Lambda}{\partial y_j})(\bar{y}_j - 1), \quad \pi(\Lambda) = \lambda$$

and hence

$$\lambda G_1 - \varepsilon(\lambda G_1) = \sum \pi(\frac{\partial \Lambda}{\partial y_j})(\bar{y}_j - 1)G_1$$

$$\lambda G_2 - \varepsilon(\lambda G_2) = \sum \pi(\frac{\partial \Lambda}{\partial x_i})(\bar{x}_i - 1)G_2.$$

Let $(\lambda G_1, \mu G_2) \in R(G/G_1) \oplus R(G/G_2)$ with $\varepsilon(\lambda G_1, \mu G_2) = \varepsilon(\lambda G_1) + \varepsilon(\mu G_2) = 0$, and choose $\Lambda, M \in RF$ with $\pi(\Lambda) = \lambda$, $\pi(M) = \mu$. Then

$$\alpha(\sum \pi(\frac{\partial \Lambda}{\partial y_j})(\bar{y}_j - 1)S - \sum \pi(\frac{\partial M}{\partial x_i})(\bar{x}_i - 1)S + \varepsilon(\lambda)S) =$$

$$= \left(\sum \pi\left(\frac{\partial \Lambda}{\partial y_j}\right)(\overline{y}_j-1)G_1 + \varepsilon(\lambda)G_1, \sum \pi\left(\frac{\partial M}{\partial x_i}\right)(\overline{x}_i-1)G_2+\varepsilon(\mu)G_2\right)$$

$$= (\lambda G_1, \mu G_2).$$

It remains to be seen that α is a monomorphism. Let $\lambda \in RG$ with $\alpha(\lambda S) = (\lambda G_1, -\lambda G_2) = 0$, i.e., $\lambda G_1 = \lambda G_2 = 0$. Let wS be an element of maximum length $\ell(wS)$ in the support of λS; w is a word in the x_i's and y_j's $\neq 1$. Assume that w ends in x_i. Then wG_2 is of length $\ell(wS)$ and hence cannot cancel in λG_2 unless there was already cancellation mod S. Thus w cannot end in x_i; but the same argument shows that w cannot end in y_j either. We conclude that there are no elements of maximum length in the support of λ, i.e., $\lambda = 0$. This proves Proposition 2.8.□

Let A be a left RG-module and B a right RG-module. As the sequence of Proposition 2.8 is R-free, we obtain the short exact sequence of RG-modules

$$B\otimes_R R(G/S) \rightarrowtail B\otimes_R R(G/G_1) \oplus B\otimes_R R(G/G_2) \twoheadrightarrow B$$

(*)

$$A \rightarrowtail \mathrm{Hom}_R(R(G/G_1),A) \oplus \mathrm{Hom}_R(R(G/G_2),A) \twoheadrightarrow \mathrm{Hom}_R(R(G/S),A).$$

Hereby the modules $B\otimes_R R(G/S)$, $\mathrm{Hom}_R(R(G/S),A)$, etc. are considered to be endowed with the diagonal G-module structure, i.e. an element $x \in G$ acts as $(b\otimes gS)\cdot x = bx\otimes x^{-1}gS$, $x\cdot f(gS) = xf(x^{-1}gS)$, $b \in B$, $g \in G$, $f \in \mathrm{Hom}_R(R(G/S),A)$.

Lemma 2.9. Let G be a group, $H \leq G$ a subgroup, A a left and B a right RG-module. Then one has natural RG-module isomorphisms

$$u: \overset{\swarrow}{B} \otimes_R \overset{\searrow}{R}(G/H) \xrightarrow{\sim} B \otimes_{RH} RG^{\swarrow} ,$$

$$v: \mathrm{Hom}_R(\overset{\searrow}{R(G/H)}, \overset{\searrow}{A}) \xrightarrow{\sim} \mathrm{Hom}_{RH}(R\overset{\swarrow}{G}, A) ,$$

where the G-action is understood as indicated by the arrows (diagonal action on the left hand side and single action (on RG) on the right hand side).

Proof. u is defined by $u(b \otimes xH) = bx \otimes x^{-1}$, $b \in B$, $x \in G$. It is easily seen that this is well defined and a G-homomorphism. The inverse of u is given by $b \otimes x \mapsto bx \otimes x^{-1}H$. Analogously, v is defined by $v(f)(x) = xf(x^{-1}H)$, $x \in G$, $f \in \mathrm{Hom}_R(R(G/H),A)$, and its inverse v' by $v'(g)(xH) = xg(x^{-1})$, $x \in G$, $g \in \mathrm{Hom}_{RH}(RG,A)$. □

By Lemma 2.9 we can write the short exact sequences (*) in the form

$$B \otimes_{RS} RG \rightarrowtail B \otimes_{RG_1} RG \oplus B \otimes_{RG_2} RG \twoheadrightarrow B ,$$

(**)

$$A \rightarrowtail \mathrm{Hom}_{RG_1}(RG,A) \oplus \mathrm{Hom}_{RG_2}(RG,A) \twoheadrightarrow \mathrm{Hom}_{RS}(RG,A) .$$

These sequences give rise to long exact coefficient sequences in homology and cohomology of G. Notice that, by the Shapiro Lemma, one has natural isomorphisms

$$H_k(G; B \otimes_{RH} RG) \cong H_k(H;B), \quad H^k(G; \mathrm{Hom}_{RH}(RG,A)) \cong H^k(H;A)$$

for all $k \in \mathbb{Z}$ and all subgroups $H \leq G$. Thus we get

Theorem 2.10. Let $G = G_1 *_S G_2$, A a left RG-module and B a right RG-module. Then one has natural long exact sequences (=Mayer-Vietoris sequences)

$$\ldots \to H_k(S;B) \xrightarrow{(cor,-cor)} H_k(G_1;B) \oplus H_k(G_2;B) \xrightarrow{(cor,cor)} H_k(G;B) \xrightarrow{\partial} H_{k-1}(S;B) \to \ldots$$

$$\ldots \to H^k(G;A) \xrightarrow[(res,res)]{} H^k(G_1;A) \oplus H^k(G_2;A) \xrightarrow[(res,-res)]{} H^k(S;A) \xrightarrow[\delta]{} H^{k+1}(G;A) \to \ldots$$

Remark. The following is an even shorter way to get Theorem 2.10: Apply the short exact sequence of Proposition 2.8 to $Tor_k^{RG}(B,-)$ and $Ext_{RG}^k(-,A)$ respectively and use the change-of-ring isomorphisms (a), (c) in Section 2 of the introduction.

2.5 HNN-groups. Let G be a group with isomorphic subgroups S,T and let $\sigma: S \xrightarrow{\approx} T$ be a given isomorphism. The HNN-group $G^* = G*_{S,\sigma}$ over the base group G with associated subgroups S,T and stable letter p is defined to be

$$G^* = \langle G,p;\ \text{rel } G,\ psp^{-1} = \sigma(s)\ \text{ all } s \in S \rangle.$$

One can show that the obvious homomorphism $j: G \to G^*$ is a monomorphism. We shall use j as identification, i.e. we consider G as a subgroup of G^*. Then G^* is, in a sense, the universal group containing G such that σ is given by an inner automorphism.

Normal Form Theorem. Let X and Y be left transversals for G mod S and G mod T respectively, both X and Y containing $1 \in G$. Let Z denote the set of all products $x_1p^{n_1}x_2p^{n_2}\ldots x_kp^{n_k}$ (and 1) such that $0 \neq n_i \in \mathbb{Z}$, $x_i \in X$ if $n_i < 0$ and $x_i \in Y$ if $n_i > 0$, and $x_i \neq 1$ except perhaps for $i = 1$. Then Z is a left transversal for G^* mod G.

Sketch of a proof. We take the free product $(G * \langle u\rangle) * (G * \langle v\rangle)$ and amalgamate the subgroups $\langle G, uSu^{-1}\rangle = \langle G, vTv^{-1}\rangle$ in the obvious way. The result is a group $\overline{G}$ with presentation

$$\overline{G} = \langle G, u, v;\ \text{rel}G,\ usu^{-1} = v\sigma(s)v^{-1},\ \text{all } s \in S\rangle.$$

By the Tietze transformation $p = v^{-1}u$, we get

$$\overline{G} = \langle G, u, p;\ \text{rel}G,\ psp^{-1} = \sigma(s),\ \text{all } s \in S\rangle \cong G^* * \langle u\rangle.$$

So $\overline{G}$ differs from the HNN-group G^* only by an infinite cyclic free factor, and one deduces the Normal Form Theorem for HNN-groups from the corresponding theorem for amalgamated products. For details cf. [43]. □

Remark. Every coset xG contains an element $x_1p^{n_1}x_2p^{n_2}\ldots x_kp^{n_k} \in Z$, and we define the length $\ell(xG)$ of the coset xG to be the integer $\sum_{i=1}^{k} |n_i|$, $(\ell(G) = 0)$. One can show that this definition does not depend upon the choice of the transversals X and Y. We shall also say that an element $x \in G^*$ has length $\ell(x) = m$ if m is the length of its coset mod G.

Proposition 2.11. Let $G^* = G*_{S,\sigma}$ be an HNN-group. Then one has a short exact sequence of (left) RG^*-modules

$$R(G^*/S) \xrightarrow{\beta} R(G^*/G) \xrightarrow{\varepsilon} R$$

where ε is induced by the augmentation and β is given by $\beta(xS) = xG - xp^{-1}G$, $x \in G^*$, p being the stable letter of G^*. (This is well defined since $sp^{-1}G = p^{-1}\sigma(s)G = p^{-1}G$ for $s \in S$).

Proof. Clearly ε is an epimorphism and $\varepsilon \circ \alpha = 0$. Let $\{\bar{x}_1, \bar{x}_2, \ldots\}$ be a set of generators for G. Consider the free group F on letters $\{x_1, x_2, \ldots, q\}$ and the presentation $\pi: F \twoheadrightarrow G^*$, $\pi(x_i) = \bar{x}_i$, $\pi(q) = p$. By the fundamental formula one has for all $\lambda \in RG^*$

$$\lambda - \varepsilon(\lambda) = \sum \pi\left(\frac{\partial \Lambda}{\partial x_i}\right) (\bar{x}_i - 1) + \pi\left(\frac{\partial \Lambda}{\partial q}\right) (p-1)$$

where $\Lambda \in RF$ with $\pi(\Lambda) = \lambda$. Thus

$$\lambda G - \varepsilon(\lambda G) = \pi\left(\frac{\partial \Lambda}{\partial q}\right)(p-1)G$$

$$= \pi\left(\frac{\partial \Lambda}{\partial q}\right) p(1-p^{-1})G = \beta\left(\pi\left(\frac{\partial \Lambda}{\partial q}\right)pS\right),$$

whence $\ker\varepsilon = \operatorname{im}\beta$. It remains to prove that β is a monomorphism. For this we choose (left) transversals X and Y of G mod S and G mod $T = \sigma(S)$ respectively, both X and Y containing $1 \in G$, and the corresponding transversal Z of G^* mod G. Clearly the set of element of the form zx_o $z \in Z$, $x_o \in X$ is a left transversal for G^* mod S. Let $\lambda S \in R(G^*/S)$ with $\beta(\lambda S) = \lambda G - \lambda p^{-1}G = 0$ and let $wS = zx_oS$ be an element of maximum length $\ell(wS) = m$ in the support of λS. If either $x_o \neq 1$ or $z = x_1p^{n_1}x_2p^{n_2}\ldots x_kp^{n_k}$ with

$n_k < 0$, then wS gives rise to a term $wp^{-1}G$ of length $m+1$ in the support of $\lambda p^{-1}G$. All terms in λG have length $\leq m$, hence $wp^{-1}G$ must cancel within $\lambda p^{-1}G$ against an element of the same type $\tilde{w}p^{-1}G$. But since our coset representatives are unique we have $w = \tilde{w}$ and then there was already cancellation mod S.

It follows that wS is of the form $x_1p^{n_1}x_2p^{n_2}\ldots x_kp^{n_k}S$ with $n_k > 0$ and that all elements of maximum length m in λS are of this type. Now, we look at the term wG in the support of λG. All elements in the support of $\lambda p^{-1}G$ now have length $\leq m$, and the maximal ones are of the form $x_1p^{n_1}x_2p^{n_2}\ldots x_kp^{-1}G$ and hence cannot cancel against wG; thus wG must cancel within λG against an element of the same type $\tilde{w}G$. Again, our coset representatives are unique, so this implies that there was already cancellation mod S.

We conclude that there are no terms of maximal length in the support of λS, i.e. $\lambda S = 0$. □

Let A be a left and B a right RG-module. As before we get the short exact sequences of RG-modules (diagonal G-action)

$$B \otimes_R R(G^*/S) \rightarrowtail B \otimes_R R(G^*/G) \twoheadrightarrow B$$

$$A \rightarrowtail \mathrm{Hom}_R(R(G^*/G), A) \twoheadrightarrow \mathrm{Hom}_R(R(G^*/S), A),$$

and therefore, by Lemma 2.9, short exact sequences of RG-modules (single G-action)

$$B \otimes_{RS} RG^* \rightarrowtail B \otimes_{RG} RG^* \twoheadrightarrow B$$

$$A \rightarrowtail \mathrm{Hom}_{RG}(RG^*, A) \twoheadrightarrow \mathrm{Hom}_{RS}(RG^*, A).$$

The corresponding long exact coefficient sequences yield

Theorem 2.12. Let $G^* = G *_{S,\sigma}$ be an HNN-group over the base group G with associated subgroups S,T and stable letter p. Then one has long exact sequences (= Mayer-Vietoris sequences)

$$\ldots \to H_k(S;B) \xrightarrow{\mathrm{cor}_S - \mathrm{cor}_T \circ c_{p^*}} H_k(G;B) \xrightarrow{\mathrm{cor}} H_k(G^*;B) \xrightarrow{\partial} H_{k-1}(S;B) \to \ldots$$

$$\ldots \to H^{k-1}(S;A) \xrightarrow{\delta} H^k(G^*;A) \xrightarrow{\mathrm{res}} H^k(G;A) \xrightarrow{\mathrm{res}_S - c_p^* \circ \mathrm{res}_T} H^k(S;A) \to \ldots$$

for every left RG^*-module A and right RG^*-module B. The maps res and cor are induced by inclusion of the indicated subgroups, $c_{p^*} : H_k(S;B) \to H_k(T;B)$ and $c_p^* : H^k(T;A) \to H^k(S;A)$ are the isomorphisms induced by conjugation in G^*.

Remark. The slightly more general situation of an HNN-group of rank > 1 can be dealt with in the same way. If $G^* = \langle G, p_1, p_2, \ldots ;\ \mathrm{rel}\ G,\ p_i s p_i^{-1} = \sigma_i(s)\ \text{all}\ s \in S_i,\ i = 1,2,\ldots \rangle$ then one has a short exact sequence

$$\oplus R(G^*/S_i) \overset{\beta}{\rightarrowtail} R(G^*/G) \overset{\varepsilon}{\twoheadrightarrow} R,$$

$\beta(xS_i) = xG - xp_i^{-1}G$, $x \in G^*$, and long exact sequences

$$\ldots \to \bigoplus_i H_k(S_i;B) \to H_k(G;B) \to H_k(G^*;B) \to \bigoplus_i H_{k-1}(S_i;B) \to \ldots$$

$$\ldots \to \prod_i H^{k-1}(S_i;A) \to H^k(G^*;A) \to H^k(G;A) \to \prod_i H^k(S_i;A) \to \ldots$$

for every left RG-module A and right RG-module B.

Proposition 2.13(a) Let $G = G_1 *_S G_2$. If G_1, G_2 are of type $(FP)_n$ and S is of type $(FP)_{n-1}$ over R then G is of type $(FP)_n$ over R. If G and S are of type $(FP)_n$ over R then so are G_1 and G_2. If G_1 and G_2 are of type $(FP)_{n-1}$ and G of type $(FP)_n$ over R, then S is of type $(FP)_{n-1}$ over R.

(b) Let $G = G_1 *_{S,\sigma}$. If G_1 is of type $(FP)_n$ and S of type $(FP)_{n-1}$ over R then G is of type $(FP)_n$ over R. If G,S are $(FP)_n$, so is G_1. If G_1 is $(FP)_{n-1}$ and G is $(FP)_n$, then S is $(FP)_{n-1}$.

Proof. We use the criterion of Proposition 2.3. If G_1 and G_2 are of type $(FP)_n$ and S is of type $(FP)_{n-1}$ then the homology Mayer-Vietoris sequences yield $H_k(G; \Pi RG) = 0$ for $n > k \geq 1$ and a natural isomorphism $H_o(G; \Pi RG) = \Pi R$ for all direct products Π, i.e. G is of type $(FP)_n$ over R. □

If G and S are of type $(FP)_n$ then the same method yields $H_k(G_\alpha; \Pi RG) = 0$, $n > k \geq 1$ and $H_o(G_\alpha; \Pi RG) \simeq R(G/G_\alpha)$ only. But as RG_α-modules one has the direct sum splittings $RG \simeq RG_\alpha \oplus R[G-G_\alpha]$ $RG/G_\alpha \simeq R \oplus R[G/G_\alpha - G_\alpha]$, whence $H_k(G_\alpha; \Pi RG_\alpha) = 0$, $n > k \geq 1$, and $H_o(G_\alpha; \Pi RG_\alpha) \simeq \Pi R$, i.e. G_α is of type $(FP)_n$. The remaining case can be proved by the same argument. □

Remark. Ian M. Chiswell has recently obtained Mayer-Vietoris sequences in the theory of groups acting on a tree which generalize both Theorem 2.10 and 2.12. Cf. Chiswell [17].

2.6. Examples. The following is a list of groups of type $(FP)_\infty$.

(1) The trivial group 1.

(2) All finite groups (by Proposition 2.5)

(3) All finitely generated free groups (by Proposition 2.13)

(4) All poly (f.g. free or finite) groups (by Proposition 2.7), in particular all polycyclic groups.

(5) All f.g. one-relator groups.

The last statement follows from Lyndon's Identity Theorem [40], but we sketch also a direct proof using the "Freiheitssatz" only. Let G be a (finitely generated) group with one defining relator r of length $\ell(r)$. If the number of generators involved in the relator r is ≥ 2, then G can be embedded as an "amalgamated factor" into a group $G_1 = G *_{\langle u^m \rangle} \langle u \rangle$ which is a one relator group with the property that one of its generators has exponent sum 0 in the relator of G_1. Then G_1 is an HNN-group over a base group G_2 with finitely generated free associated subgroups, $G_1 = G_2 *_{F,\sigma}$, and G_2 is a group with a single defining relator of length $< \ell(r)$. The proof now goes with induction on the length of r. If $\ell(r) = 1$ then G is free and hence of type $(FP)_\infty$. Let $\ell(r) \geq 2$. If G is cyclic, it is, of course, of type $(FP)_\infty$. If G is not cyclic, then decompose it as sketched above: G_2 is of type $(FP)_\infty$ by induction. By Proposition 2.13(b) G_1 is of type $(FP)_\infty$ and by Proposition 2.13(a) this implies that G is of type $(FP)_\infty$.

Problem. Let C be the smallest class of groups with the following properties

(i) C contains all finite groups.

(ii) C is extension closed and closed with respect to taking subgroups of finite index.

(iii) Proposition 2.13 holds with "of type $(FP)_*$" replaced by "in C".

Construct a group of type $(FP)_\infty$ outside C. Is every torsion-group of type $(FP)_\infty$ finite? Are all arithmetic groups $SL(n, \mathbb{Z})$ in C? (They are known to be of type $(FP)_\infty$,cf [52])*.

In the remainder of Section 2 we construct groups which are "almost of type $(FP)_\infty$". Let $D_n = \langle x_1,y_1\rangle \times \langle x_2,y_2\rangle \times \ldots \times \langle x_n,y_n\rangle$ the direct product of n free groups of rank 2. We define D_n-action on two different infinitely generated groups: Firstly let F_∞ be the free group on generators $\{a_k\}$, $k \in \mathbb{Z}$, and put $x_i \cdot a_k = y_i \cdot a_k = a_{k+1}$ for all i,k. Secondly let Q_d be the additive group of all rational numbers q with denominator a power of d, d an integer ≥ 2, and put $x_i \cdot q = y_i \cdot q = dq$, all i, q. Then define

$$A_n = F_\infty \rtimes D_n \quad , \quad B_n = Q_d \rtimes D_n .$$

Proposition 2.14. Both A_n and B_n are of type $(FP)_n$ but not of type $(FP)_{n+1}$.

Proof. First we show that A_n is of type $(FP)_n$. The case n=0 is trivial and $A_1 = F_\infty \rtimes \langle x_1,y_1\rangle$ is generated by a_o,x_1,y_1 and hence of type $(FP)_1$; thus assume $n \geq 2$. Clearly A_n is finitely generated and hence all we have to show is that $H_k(A_n; \Pi \mathbb{Z} A_n) = 0$ for $1 \leq k \leq n-1$.

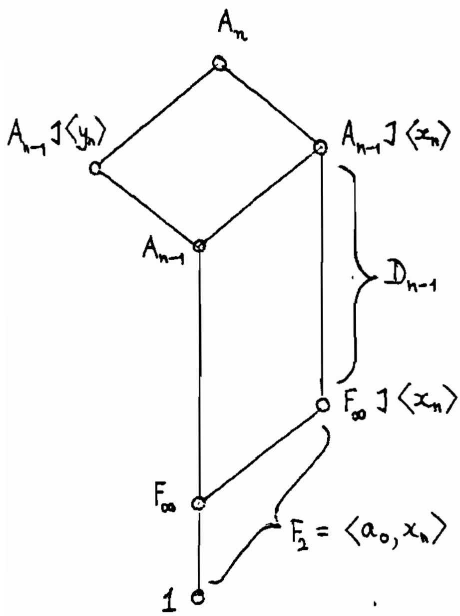
Now, $A_n = A_{n-1} \rtimes \langle x_n,y_n\rangle$ can be considered as an HNN-extension

* See Appendix 4.

with stable letters x_n, y_n. Thus one has the Mayer-Vietoris sequence

$$\ldots \to \bigoplus_1^2 H_k(A_{n-1}; \Pi\, \mathbb{Z}A_n) \xrightarrow{(1-x_n, 1-y_n)} H_k(A_{n-1}; \Pi\, \mathbb{Z}A_n) \xrightarrow{\text{cor}} H_k(A_n; \Pi\, \mathbb{Z}A_n) \to \ldots$$

By induction hypothesis $H_k(A_{n-1}; \Pi\, \mathbb{Z}A_n) = 0$, $1 \le k \le n-2$, hence $H_k(A_n; \Pi\, \mathbb{Z}A_n) = 0$, $1 \le k \le n-2$. It remains to show that $H_{n-1}(A_n; \Pi\, \mathbb{Z}A_n) = 0$, and this will follow if we can prove that the map $(1-x_n): H_{n-1}(A_{n-1}; \Pi\, \mathbb{Z}A_n) \to H_{n-1}(A_{n-1}; \Pi\, \mathbb{Z}A_n)$ is an isomorphism.



For this we consider the situation in the diagram on the left. As $F_\infty \rtimes \langle x_n \rangle$ is the free group freely generated by a_0 and x_n it follows that $A_{n-1} \rtimes \langle x_n \rangle$ is of type $(FP)_\infty$ by Prop. 2.7. Thus $H_k(A_{n-1} \rtimes \langle x_n \rangle; \Pi\, \mathbb{Z}A_n) = 0$ for $k \neq 0$, whence considering the Mayer-Vietoris for the HNN-extension $A_{n-1} \rtimes \langle x_n \rangle$ we get $(1-x_n): H_k(A_{n-1}; \Pi\, \mathbb{Z}A_n) \xrightarrow{\sim} H_k(A_{n-1}; \Pi\, \mathbb{Z}A_n)$ for all $k > 0$.

The proof that B_n is of type $(FP)_n$ is strictly analogous. In order to prove that neither A_n nor B_n is of type $(FP)_{n+1}$ we prove

(i) $H_{n+1}(A_n;\mathbb{Z})$ is free-Abelian of rank $\aleph_0$

(ii) $H_{n+1}(B_n;\mathbb{Z}) \simeq Q_d$.

Again we use induction on n to show $H_{n+2}(A_n;\mathbb{Z}) = 0$ and $H_{n+1}(A_n;\mathbb{Z}) = \underset{\aleph_0}{\oplus}\mathbb{Z}$. This is clear for $A_0 = F_\infty$. If $n \geq 1$ consider the Mayer-Vietoris sequence for $A_{n-1} \rtimes \langle x_n, y_n\rangle = A_n$

$$\ldots \to \underbrace{H_{n+1}(A_{n-1};\mathbb{Z})}_{0} \to H_{n+1}(A_n;\mathbb{Z}) \to \underbrace{\overset{2}{\underset{1}{\oplus}} H_n(A_{n-1};\mathbb{Z})}_{\underset{\aleph_0}{\oplus}\mathbb{Z}} \xrightarrow{(1-x_n,1-y_n)} \underbrace{H_n(A_{n-1};\mathbb{Z})}_{\underset{\aleph_0}{\oplus}\mathbb{Z}} \to \ldots$$

As the action of $(1-x_n)$ and $(1-y_n)$ on $H_n(A_{n-1};\mathbb{Z})$ coincide, the kernel of $(1-x_n, 1-y_n)$ must contain a copy of $H_n(A_{n-1};\mathbb{Z})$, hence $H_{n+1}(A_n;\mathbb{Z}) \simeq \underset{\aleph_0}{\oplus}\mathbb{Z}$; $H_{n+2}(A_n;\mathbb{Z}) = 0$ is obvious. This proves the assertion (i) - the proof of (ii) is again similar. Now, (i) and (ii) imply the assertion of Prop.2.14. by the following remark.□

<u>Proposition 2.15</u>. If G is a group of type $(FP)_n$ then $H_k(G;\mathbb{Z})$ and $H^k(G;\mathbb{Z})$ are finitely generated Abelian groups for all $0 \leq k \leq n$.

<u>Proof</u>. Let $\underline{F} \twoheadrightarrow \mathbb{Z}$ be a finitely generated G-free resolution. Then $\mathbb{Z} \otimes_G \underline{F}$ and $\mathrm{Hom}_G(\underline{F}, \mathbb{Z})$ are complexes of finitely generated Abelian groups, whence the assertion. □

Remarks. 1) Notice that the groups A_n, B_n are finitely presented for $n \geq 2$. In particular one has

$$A_2 = \langle a, x_1, x_2, y_1, y_2;\ a^{x_1} = a^{x_2} = a^{y_1} = a^{y_2}, [x_1,x_2]=[y_1,x_2]=[x_1,y_2]=[y_1,y_2]=1 \rangle$$

$$B_2 = \langle a, x_1, x_2, y_1, y_2;\ a^{x_1} = a^{x_2} = a^{y_1} = a^{y_2} = a^2,$$
$$[x_1,x_2]=[y_1,x_2]=[x_1,y_2]=[y_1,y_2] = 1 \rangle.$$

A_2 is Stallings' group [54].

2) Notice that $H_4(B_2; \mathbb{Z}) = 0$; this follows from the Lyndon-Hochschild-Serre spectral sequence together with the fact that $H_k(Q_d; \mathbb{Z}) = 0$ for $k \geq 2$ (see Section II, Proposition 1.8). By the Universal Coefficients Theorem it follows that $H^4(B_2; \mathbb{Z})$ is isomorphic to Ext $(Q_d, \mathbb{Z})$ and hence uncountable, despite the fact that B_2 is finitely presented.

3) Exercise. Show that the converse of Proposition 2.15 is false.

3. Universal Coefficients for groups of type $(FP)_\infty$

In this section we show that groups G of type $(FP)_\infty$ have a special feature: there are spectral sequences which approximate the (co)homology of G with coefficients in arbitrary RG-modules of finite projective dimension (cf. II Section 1) in terms of the cohomology groups with group ring coefficients $H^k(G; RG)$.

3.1. Dual modules. First we recall some facts about dual modules. Let Λ be an arbitrary ring with 1 and M a left (right) Λ-module. The dual of M is defined to be $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$ where Λ is regarded as a left (right) module by left (right) multiplication. M^* is a right (left) Λ-module with Λ-action given by $(f\circ\lambda)(m) = f(m)\lambda, \quad \lambda \in \Lambda, m \in M, f \in M^*$.

For left (right) modules M and N one has natural isomorphisms $(M \oplus N)^* \simeq M^* \oplus N^*$ defined in the obvious way. If Λ is regarded as a left Λ-module by left multiplication then Λ^* is naturally isomorphic to the right module Λ with Λ-action given by right multiplication. It follows readily that if P is a finitely generated projective left Λ-module then P^* is a finitely generated projective right Λ-module.

Let M and A be left Λ-modules and B a right Λ-module. Then there are natural homomorphisms

$$\phi : \quad M^* \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(M,A)$$
$$\psi : \quad B \otimes_\Lambda M \to \mathrm{Hom}_\Lambda(M^*, B)$$

given by $\phi(f \otimes a)(m) = f(m)a$ and $\psi(b \otimes m)(f) = bf(m)$ for $m \in M$, $f \in M^*$, $a \in A$, $b \in B$.

Proposition 3.1. The following statements are equivalent for a Λ-module M.

(i) M is finitely generated and projective.

(ii) ϕ is an isomorphism for every Λ-module A.

(iii) ψ is an isomorphism for every Λ-module B.

Proof. (i) $\Rightarrow$ (ii), (iii): If $M = \Lambda$ then ϕ is the identity on A and also ψ is the identity on B (provided the natural isomorphism of left Λ-modules $\Lambda \simeq \Lambda^{**}$ is used as an identification). It follows that ϕ and ψ are isomorphisms for all finitely generated free modules M. By naturality this is still true of direct summands i.e. for finitely generated projectives.

(ii) $\Rightarrow$ (i): If ϕ is an isomorphism for $A = M$, there is an element $\sum_i f_i \otimes m_i \in M^* \otimes_\Lambda M$ with $\phi(\sum f_i \otimes m_i) = \mathrm{Id}_M$, hence $\sum f_i(m)m_i = m$ for all $m \in M$. This shows that M is generated by the finite set $\{m_i\}$. Let F be the free Λ-module on free generators x_i and define a map $\pi: F \twoheadrightarrow M$ by $\pi(x_i) = m_i$. Then π has a splitting $\sigma: M \to F$ given by $\sigma(m) = \sum_i f_i(m)x_i$, $m \in M$, hence M is projective.

(iii) $\Rightarrow$ (i): If ψ is an isomorphism for $B = M^*$, there is an element $\sum f_i \otimes m_i \in M^* \otimes_\Lambda M$ with $\psi(\sum f_i \otimes m_i) = \mathrm{Id}_{M^*}$, hence $\sum f_i \cdot h(m_i) = h$ for all $h \in M^*$. This shows that M^* is generated by the finite set $\{f_i\}$. Moreover, let F be the free right module on free generators y_i and define a map $\pi: F \twoheadrightarrow M^*$ by $\pi(y_i) = f_i$. Then π has a splitting $\sigma: M^* \to F$ given by $\sigma(h) = \sum_i y_i h(m_i)$, $h \in M^*$, hence M^* is projective. This now implies that its dual M^{**} is also finitely generated and projective - but (iii) for $B = \Lambda$ yields $M \cong M^{**}$, whence the result. □

3.2. Universal Coefficients for cohomology. Let G be a group of type $(FP)_\infty$ over R and consider a resolution

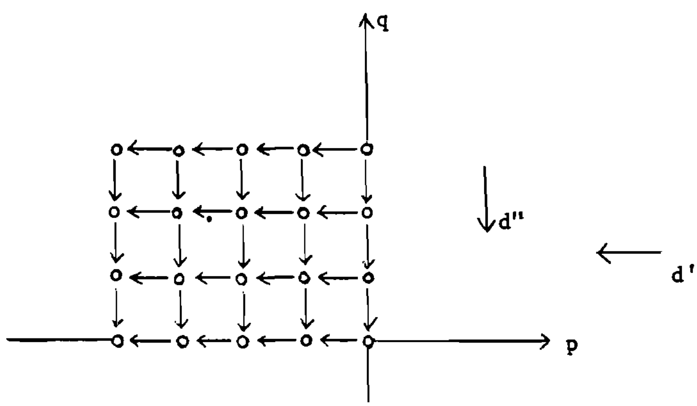
$$\ldots \to P_k \to P_{k-1} \to \ldots \to P_o \twoheadrightarrow R$$

of the trivial G-module R by finitely generated projective left RG-modules. Let A be a left RG-module and consider a flat resolution

$$\ldots \to Q_k \to Q_{k-1} \to \ldots \to Q_o \twoheadrightarrow A.$$

Now we construct the double complex $K_{p,q} = P^*_{-p} \otimes_{RG} Q_q$. Notice that the non-trivial terms of $K_{p,q}$ lie in the second quadrant. K_{**} may be visualized by the diagram

$K_{p,q}$



where d' and d'' denote the partial differential. As usual one has two natural filtrations on the total complex $X = \text{Tot } K_{**}$, $X_n = \bigoplus_{p+q=n} K_{p,q}$, and the associated spectral sequences.

The first filtration is defined by $F'_p X_n = \sum_{h \leq p} K_{h,n-h}$. If either $Q_k = 0$ for all sufficiently large k or $P_i = 0$ for all sufficiently large i, i.e., if one of the resolutions has finite length, then F' is a finite filtration. In this case we know that the associated spectral sequence converges to the homology of the total complex X.

$$E^{(2)}_{p,q} = H_p(H_q(K_{**},d''), d') \underset{p}{\Rightarrow} H_{p+q}(X) .$$

As P_{-p} is finitely generated projective, so is P^*_{-p} and one has

$$H_q(K_{p,*},d'') = \text{Tor}_q^{RG}(P^*_{-p},A) = \begin{cases} 0 \text{ if } q \neq 0 \\ P^*_{-p} \otimes_{RG} A \text{ if } q = 0 . \end{cases}$$

Moreover, Proposition 3.1 yields a natural isomorphism $\phi : P^*_{-p} \otimes_{RG} A \cong \text{Hom}_{RG}(P_{-p},A)$, whence

$$E^{(2)}_{p,q} = \begin{cases} 0 \text{ if } q \neq 0 \\ H^{-p}(G;A), \text{ if } q = 0, \end{cases}$$

i.e. the first spectral sequence collapses and yields natural isomorphisms $H_p(\text{Tot } K_{**}) \cong H^{-p}(G; A)$, $p \in \mathbf{Z}$.

Now we consider the second filtration of $X = \mathrm{Tot}\, K_{**}$, defined by $(F_q''X)_n = \sum_{h \le q} K_{n-h,h}$. If either of the resolutions $\underline{P}$ or $\underline{Q}$ is of finite length, then this second filtration is again finite, and the corresponding spectral sequence converges.

$$E^{(2)}_{p,q} = H_q(H_p(K_{**},d'),d'') \underset{q}{\Rightarrow} H_{p+q}(X).$$

Since tensoring with the flat module Q_q is an exact functor one has

$$H_p(K_{*,q},d') \cong H_p(\underline{P}^*) \otimes_{RG} Q_q = H^{-p}(G;\, RG) \otimes_{RG} Q_q$$

and hence

$$E^{(2)}_{p,q} \cong \mathrm{Tor}^{RG}_q(H^{-p}(G;RG),A) \underset{q}{\Rightarrow} H_{p+q}(X) \cong H^{-(p+q)}(G;A).$$

Replacing $-p$ by p yields the required "Universal coefficient Theorem".

We say that the group G has finite cohomology dimension over R, in symbols $\mathrm{cd}_R G < \infty$, if the trivial G-module R has an RG-projective resolution $\underline{P} \twoheadrightarrow R$ of finite length (i.e. $P_k = 0$ for large k). Also we say that the RG-module A has finite flat dimension, $\mathrm{fl.dim}\ A < \infty$, if A admits a flat resolution of finite length. These homological dimensions shall be discussed in Section 4.1, but we preintroduce them here in order to summarize our result:

<u>Theorem 3.2</u>. Let G be a group of type $(FP)_\infty$ over R, and let A be a (left) RG-module. If either $\mathrm{cd}_R G < \infty$ or $\mathrm{fl.dim}\ A < \infty$

then there is a convergent 2nd quadrant spectral sequence

$$E^{(2)}_{-p,q} = \mathrm{Tor}_q^{RG}(H^p(G;\ RG), A) \underset{q}{\Rightarrow} H^{p-q}(G;\ A).$$

3.3. Universal Coefficients for homology. Now we deduce the dual of Theorem 3.2. Recall that an injective resolution of a Λ-module M is an exact sequence of Λ-modules

$$0 \to M \to I^0 \to I^1 \to \dots \to I^k \to I^{k+1} \to \dots$$

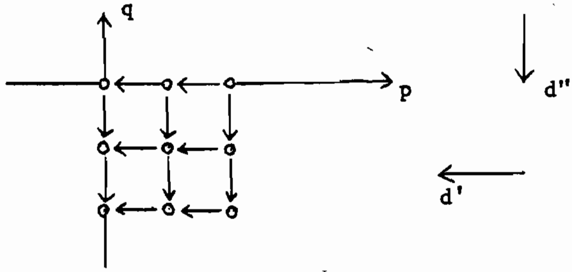
such that I^k is injective for all $k \geq 0$. If there is an injective resolution of finite length, we say that M has finite injective dimension and write $\mathrm{inj.dim}\ M < \infty$.

Theorem 3.3. Let G be a group of type $(FP)_\infty$ over R, and let B be a (right) RG-module. If either $\mathrm{cd}_R G < \infty$ or $\mathrm{inj.dim}\ B < \infty$ then there is a convergent 4th quadrant spectral sequence

$$E^{(2)}_{p,-q} = \mathrm{Ext}^q_{RG}(H^p(G;\ RG), B) \underset{q}{\Rightarrow} H_{p-q}(G;B)$$

Proof. Let $\underline{P} \twoheadrightarrow R$ be an *RG-projective resolution which* is finitely generated in each dimension, and let $B \rightarrowtail \underline{I}$ be an RG-injective resolution for B. We consider the double complex $K_{p,q} = \mathrm{Hom}_{RG}(P_p^*, I^{-q})$. Notice that its non-trivial terms lie in the fourth quadrant; K_{**} can be visualized by the diagram

$K_{p,q}$



where d', and d" denote the partial differentials.

Let $X_n = (\text{Tot } K_{**})_n = \bigoplus_{p+q=n} K_{p,q}$. The first filtration of X, $(F'_p X)_n = \sum_{h\leq p} K_{h,n-h}$, gives rise to a spectral sequence

$$E^{(2)}_{p,q} = H_p(H_q(K_{**}, d''), d') \underset{p}{\Rightarrow} H_{p+q}(X)$$

which converges if either $\underline{\underline{P}}$ or $\underline{\underline{I}}$ is of finite length. Now, as P_p is finitely generated projective, so is its dual P^*_p and hence one has

$$H_q(K_{p,*}, d'') = \text{Ext}^{-q}_{RG}(P^*_p, B) = \begin{cases} 0 \text{ if } q \neq 0, \\ \text{Hom}_{RG}(P^*_p, B) \text{ if } q \neq 0. \end{cases}$$

Moreover, Proposition 3.1 yields a natural isomorphism ψ: $B \otimes_{RG} P_p \simeq \text{Hom}_{RG}(P^*_p, B)$ whence

$$E^{(2)}_{p,q} = \begin{cases} 0, \text{ if } q \neq 0, \\ H_p(G; B), \text{ if } q = 0, \end{cases}$$

i.e. the spectral sequence collapses and yields a natural isomorphism $H_p(\text{Tot}_{**}) \simeq H_p(G;B)$.

The second filtration of X, $(F''_q X)_n = \sum_{h\leq q} K_{n-h,h}$, yields a spectral sequence

$$E^{(2)}_{p,q} = H_q(H_p(K_{**},d'),d'') \underset{q}{\Rightarrow} H_{p+q}(X)$$

which again converges if either $\underline{P}$ or $\underline{I}$ is of finite length. Since $Hom_{RG}(-, I^{-q})$ is an exact functor one has

$$H_p(K_{*,q}, d') = Hom_{RG}(H_p(\underline{P}^*), I^{-q}) = Hom_{RG}(H^p(G;RG), I^{-q})$$

and hence

$$E^{(2)}_{p,q} \cong Ext^{-q}_{RG} (H^p(G;RG), B) \underset{q}{\Rightarrow} H_{p+q}(X) \cong H_{p+q}(G; B).$$

Replacing $-q$ by q now yields the result.□

Remarks (1) The G-action on $H^p(G; RG)$ is given by the (right) G-module structure of $\underline{P}^*$, and hence is induced by right multiplication in the coefficient bi module RG.

(2) Without the assumption either $cd_R G < \infty$ or fl.dim $A < \infty$ and inj.dim $B < \infty$ in Theorems 3.2 and 3.3 one still gets spectral sequences, but they might not converge to the homology of the total complex. (E.g. for G a finite group).

3.4. Application. Let G be a group. Recall that RG-modules of the form $L \otimes_R RG$ and $Hom_R(RG, L)$, where L is any R-module, are said to be induced and coinduced respectively. If A is an induced and B a coinduced RG-module then $H_k(G;A) = 0$ and $H^k(G; B) = 0$ for all $k \neq 0$. If G is finite then induced = coinduced and hence one has also $H^k(G; A) = 0$ and $H_k(G; B) = 0$

for $k \neq 0$. This is not true in general, but one can say something if G is of type $(FP)_\infty$.

Let K, L be R-modules of finite flat and injective dimension, respectively. If $\underline{P} \twoheadrightarrow K$ is an R-flat resolution then $\underline{P} \otimes_R RG$ is an RG-flat resolution for $K \otimes_R RG$ and hence $\mathrm{fl.dim}\,(K \otimes_R RG) < \infty$. If $L \rightarrowtail \underline{Q}$ is an R-injective resolution then $\mathrm{Hom}_R(RG,L) \rightarrowtail \mathrm{Hom}_R(RG, \underline{Q})$ is an RG-injective resolution and hence $\mathrm{inj.dim}\,(\mathrm{Hom}_R(RG,L)) < \infty$. Thus, if G is of type $(FP)_\infty$ over R, we may apply Theorems 3.2 and 3.3. Moreover one can simplify the corresponding $E^{(2)}$ terms by the following Lemma

<u>Lemma 3.4.</u> For every (right) RG-module M one has natural isomorphisms.

$$\mathrm{Tor}_q^{RG}(M, K \otimes_R RG) \cong \mathrm{Tor}_q^R(M, K)$$

$$\mathrm{Ext}_{RG}^q(M, \mathrm{Hom}_R(RG,L)) \cong \mathrm{Ext}_R^q(M,L).$$

<u>Proof.</u> use the resolutions $\underline{P} \otimes_R RG$, $\mathrm{Hom}_R(RG,\underline{Q})$ and notice that $M \otimes_{RG}(\underline{P} \otimes_R RG) \cong M \otimes_R \underline{P}$, and $\mathrm{Hom}_{RG}(M, \mathrm{Hom}_R(RG, \underline{Q})) \cong \mathrm{Hom}_R(M, \underline{Q})$.□

<u>Corollary 3.5.</u> Let G be a group of type $(FP)_\infty$ over R. Then one has convergent spectral sequences

$$E^{(2)}_{-p,q} = \mathrm{Tor}_q^R(H^p(G; RG), K) \underset{q}{\Rightarrow} H^{p-q}(G; K \otimes_R RG)$$

$$E^{(2)}_{p,-q} = \mathrm{Ext}_R^q(H^p(G; RG), L) \underset{q}{\Rightarrow} H_{p-q}(G; \mathrm{Hom}_R(RG, L))$$

for all R-modules K and L with fl.dim $K < \infty$ and inj.dim $L < \infty$, respectively (or for arbitary K,L when $cd_R G < \infty$). □

If R is a hereditary ring (i.e. R has the property that submodules of projectives are projective) then the spectral sequences collapse to short exact sequences.

Corollary 3.6. Let R be a hereditary ring and let G be a group of type $(FP)_\infty$ over R. Then one has natural short exact sequences

$$H^q(G; RG) \otimes_R L \rightarrowtail H^q(G; L \otimes_R RG) \twoheadrightarrow Tor_1^R(H^{q+1}(G; RG), L)$$

$$Ext_R^1(H^{q+1}(G; RG), L) \rightarrowtail H_q(G; Hom_R(RG,L)) \twoheadrightarrow Hom_R(H^q(G; RG),L)$$

for every R-module L and all $q \in \mathbb{Z}$. Moreover these sequences split (but the splitting is not natural).

Proof. We give direct proof, also. Let $\underline{P} \twoheadrightarrow R$ be an RG-projective resolution. Then, by Proposition 3.1, one has natural isomorphisms

$$Hom_{RG}(\underline{P}, L \otimes_R RG) \cong \underline{P}^* \otimes_{RG} (L \otimes_R RG) \cong \underline{P}^* \otimes_R L,$$

$$Hom_R(RG,L) \otimes_{RG} \underline{P} \cong Hom_{RG}(\underline{P}^*, Hom_R(RG,L)) \cong Hom_R(\underline{P}^*,L),$$

and the assertion follows by the usual Universal Coefficient Theorem for complexes. □

<u>Remark</u>. It is conceivable that the cohomology groups $H^p(G; RG)$ are always R-projective (or even R-free). This is, of course, trivially the case if $p = 0$ and it has been proved by Swan [60] for $p = 1$. If it were true in general, then the Corollaries 3.5 and 3.6 would simply read

$$H^p(G; L \otimes_R RG) \approx H^p(G; RG) \otimes_R L$$

$$H_p(G; \mathrm{Hom}_R(RG,L)) \approx \mathrm{Hom}_R(H^p(G; RG), L).$$

Of course this holds when G is of type $(FP)_\infty$ and R is a field.

CHAPTER II

HOMOLOGICAL DIMENSIONS

Chapter II splits into three parts of different length. The first part consists of Sections 4-6 where we define the homological dimensions cdG and hdG of a group G and deduce general theorems. In the second part, Section 7, we compute cd and hd for special classes of groups. Finally, in Section 8, we shall apply the theorems of Sections 4-6 together with the information of Section 7 to get purely group theoretic results.

4. Homology and cohomology dimension

4.1 Flat and projective dimensions of modules. Let Λ be an arbitrary ring with non-trivial unit and let C be a left Λ-module. Recall that a resolution $\ldots \to K_i \to K_{i-1} \to \ldots \to K_0 \twoheadrightarrow C$ is said to be of length n, if $K_r = 0$ for all $r > n$.

Proposition 4.1 a) The following statements are equivalent for a left module C and an integer $n \geq 0$:

(i) If $Q_{n-1} \to Q_{n-2} \to \ldots \to Q_0 \twoheadrightarrow C$ is the beginning of a flat resolution, then $K = \ker (Q_{n-1} \to Q_{n-2})$ is flat (interpret $Q_{-1} = C$ and $Q_{-2} = 0$).

(ii) C admits a flat resolution of length n,

(iii) $\mathrm{Tor}_k^{\Lambda}(B,C) = 0$ for all (right) Λ-modules B and all $k > n$,

(iv) $\mathrm{Tor}^{\Lambda}_{n+1}(B,C) = 0$ for all (right) Λ-modules B,

(v) $\mathrm{Tor}^{\Lambda}_{n}(-,C)$ is a left exact functor.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (v) follows from the long exact Tor-sequence in the first argument.

(v) $\Rightarrow$ (i): Let $B' \rightarrowtail B$ be a monomorphism of Λ-modules. Then one has the following commutative diagram with exact rows $(n \geq 1)$

$$\begin{array}{ccccccc} 0 \to & \mathrm{Tor}^{\Lambda}_{n}(B', C) & \to & B' \otimes_{\Lambda} K & \to & B' \otimes_{\Lambda} Q_{n-1} \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \mathrm{Tor}^{\Lambda}_{n}(B, C) & \to & B \otimes_{\Lambda} K & \to & B' \otimes_{\Lambda} Q_{n-1} \end{array},$$

the left vertical map is a monomorphism by (v), the right vertical map as Q_{n-1} is flat. It follows that the middle vertical map is monomorphic, hence K is flat. The case $n = 0$ is obvious.□

Definition. For every Λ-module C, the minimum integer $n \geq 0$ with the property that C and n satisfy either of the equivalent conditions (i) - (v) of Prop. 1.1 a) is said to be the flat dimension of C and written $\mathrm{fl.dim}_{\Lambda} C = n$. If no such integer exists we write $\mathrm{fl.dim}_{\Lambda} C = \infty$.

Proposition 4.1 b). The following are equivalent for an integer $n \geq 0$:

(i) If $P_{n-1} \to P_{n-2} \to \ldots \to P_0 \twoheadrightarrow C \ (= P_{-1})$ is part of a projective resolution, then $K = \ker(P_{n-1} \to P_{n-2})$ is projective,

(ii) There is a projective resolution of length n,
$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \twoheadrightarrow C$,

(iii) $\mathrm{Ext}_\Lambda^k(C,A) = 0$ for all (left) Λ-modules A and all $k > n$,

(iv) $\mathrm{Ext}_\Lambda^{n+1}(C,A) = 0$ for all (left) Λ-modules A,

(v) $\mathrm{Ext}_\Lambda^n(C,-)$ is a right exact functor.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious. (iv) $\Rightarrow$ (v) follows from the long exact Ext-sequence in the second argument

(v) $\Rightarrow$ (i): Let $A \twoheadrightarrow A''$ be an epimorphism of Λ-modules. Then one has the commutative diagram with exact rows ($n \geq 1$)

$$\begin{array}{ccccccc} \mathrm{Hom}_\Lambda(P_{n-1}, A) & \to & \mathrm{Hom}_\Lambda(K,A) & \to & \mathrm{Ext}_\Lambda^n(C, A) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}_\Lambda(P_{n-1}, A'') & \to & \mathrm{Hom}_\Lambda(K,A'') & \to & \mathrm{Ext}_\Lambda^n(C,A'') & \to & 0 . \end{array}$$

The left hand side vertical map is an epimorphism since P_{n-1} is projective, the right hand side map by (v). It follows that the middle vertical map is epimorphic, hence K is projective. The case $n = 0$ is obvious. □

Definition. For every Λ-module C, the minimum integer $n \geq 0$ with the property that C and n satisfy either of the equivalent conditions (i) - (v) of Prop 1.1 b) is said to be the projective dimension of C and written $\mathrm{pr.dim}_\Lambda C = n$. If no such integer exists we write $\mathrm{pr.dim}_\Lambda C = \infty$

Remarks. 1) Proposition 4.1 b) together with Proposition 1.5 shows that if C is of type $(FP)_n$ and $\mathrm{pr.dim}_\Lambda C = n$, $n < \infty$, then there is a projective resolution

$$0 \to P_n \to P_{n-1} \to \dots \to P_0 \twoheadrightarrow C$$

which is both finitely generated and of finite length. Such a resolution is said to be finite. If a module C has a finite projective resolution we say that C is of type (FP). Thus C is of type (FP) if and only if C is of type $(FP)_\infty$ and of finite projective dimension.

Warning. If C is of type (FP) it is in general not possible to find a finite free resolution. This contrasts the fact that every module of type $(FP)_\infty$ admits a finitely generated free resolution.

4.2. Direct limits Direct limits of flat modules are flat. The situation is somewhat more difficult but not too bad for the direct limit of projective modules.

Lemma 4.2. Let $\{P_\alpha, \alpha \in I\}$ be a countable direct system of projective Λ-modules. Then the direct limit $P = \varinjlim P_\alpha$ is of projective dimension ≤ 1.

Proof. Since I is countable we can pick a cofinal sequence $\{S_k\} \subset I$ with $\varinjlim_k P_{S_k} = P$, i.e., one can assume that P is the direct limit of a diagram of the form

$$P_1 \xrightarrow{\lambda_1} P_2 \xrightarrow{\lambda_2} P_3 \to P_4 \to \cdots \to P_k \xrightarrow{\lambda_k} P_{k+1} \to \cdots$$

with P_i projective for all $i \geq 1$. Then one has an exact sequence

$$(*) \qquad \bigoplus_i P_i \xrightarrow{\oplus_i \{\lambda_i - Id_{P_i}\}} \bigoplus_i P_i \twoheadrightarrow P$$

The map $\bigoplus_i \{\lambda_i - \mathrm{Id}_{P_i}\}$ is obviously a monomorphism since the component of lowest degree of an element $a \in \bigoplus_i P_i$ is preserved. Thus $(*)$ is a projective resolution of P. $\square$

Remark. See B. Osofsky [45] for the following generalization of Lemma 4.2. If the directed set I has cardinality $|I| = \aleph_n$, then P is of projective dimension $\le n+1$. Moreover this bound is best possible.

Theorem 4.3. Let $\{C_i, i \in I\}$ be a direct system of Λ-modules. Then one has

(a) $\mathrm{fl.dim.}_\Lambda (\varinjlim C_i) \le \sup \{\mathrm{fl.dim}_\Lambda C_i\}$.

Moreover, if I is countable, then

(b) $\mathrm{pr.dim}_\Lambda (\varinjlim C_i) \le \sup \{\mathrm{pr.dim}_\Lambda C_i\} + 1$.

Proof. For every $i \in I$ we construct a free Λ-resolution $\underline{F}^{(i)} \twoheadrightarrow C_i$ as follows: $F_0^{(i)}$ is the free Λ-module over the set C_i and $\varepsilon_i : F_0^{(i)} \twoheadrightarrow C_i$ the obvious epimorphism; then $F_1^{(i)}$ is the free Λ-module on the set $K_1^{(i)} = \ker \varepsilon_i$ and $d_1^{(i)} : F_1^{(i)} \to F_0^{(i)}$ the homomorphism $F_1^{(i)} \twoheadrightarrow K_1^{(i)} \rightarrowtail F_0^{(i)}$, and so on. This choice of the resolutions enables one to lift homomorphisms $f: C_i \to C_j$ in a canonical way to maps of resolutions $f: \underline{F}^{(i)} \to \underline{F}^{(j)}$: if $f_{n-1}: F^{(i)}_{n-1} \to F^{(j)}_{n-1}$ is already constructed then restrict it to the kernels $K^{(i)}_n \to K^{(j)}_n$ and this, regarded as a map on the basis, induces $f_n: F^{(i)}_n \to F^{(j)}_n$. Thus we get a direct system of free resolutions

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F_n^{(i)} & & \longrightarrow & F_{n-1}^{(i)} & \longrightarrow \cdots \longrightarrow & F_o^{(i)} & \twoheadrightarrow & C_i \\ & & \searrow\!\!\!\searrow & & \nearrow & & & & \\ & & & K_n^{(i)} & & & & & \end{array}$$

Taking the direct limit preserves exactness and moreover, $F_n = \varinjlim_{i} F_n^{(i)}$ is the free Λ-module on $K_n = \varinjlim_{i} K_n^{(i)}$. So we obtain a free resolution

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F_n & & \longrightarrow & F_{n-1} & \longrightarrow \cdots \longrightarrow & F_o & \twoheadrightarrow & \varinjlim C_i \\ & & \searrow\!\!\!\searrow & & \nearrow & & & & \\ & & & K_n & & & & & \end{array}$$

Now, suppose $\sup\{\text{fl.dim}_\Lambda C_i\} = n$. Then, by Proposition 1.1 a), $K_n^{(i)}$ is flat for all i and hence so is $K_n = \varinjlim_{i} K_n^{(i)}$, i.e., $\text{fl.dim}_\Lambda (\varinjlim C_i) \le n$. On the other hand, if $\sup\{\text{pr.dim.}_\Lambda C_i\} = m$ then $K_n^{(i)}$ is projective for all i and hence $K_n = \varinjlim_{i} K_n^{(i)}$ is of projective dimension ≤ 1 by Lemma 4.2, provided I is countable. This implies that $\text{pr.dim}_\Lambda (\varinjlim C_i) \le m+1$. □

4.3. <u>Connections between the flat and the projective dimensions.</u> A Λ-module C is said to be countably presented if there is a short exact sequence of Λ-modules $K \rightarrowtail F \twoheadrightarrow C$ where F is free and both F and K generated by a countable set of elements.

The connection between the flat and projective dimension relies on the trivial fact that projective modules are flat and the following partial converse

Lemma 4.4 (a) Every finitely presented flat module is projective.

(b) Every countably presented flat module is of projective dimension ≤ 1.

Proof. (b) Consider a countably presented flat module C given by a countable set of generators $e_1, e_2, \ldots$ and a countable set of defining linear relations among those, $f_1, f_2, f_3, \ldots$. By a result of Daniel Lazard [39] every flat module is the direct limit of finitely generated free modules F_k; thus $C = \varinjlim_K F_k$ where K is a directed set; $\mu_k: F_k \to C$.

For every positive integer j let $m_j = \max\{ i \mid f_{j'}$ involves e_i for some $j' \leq j \}$. Now, for each $j > 1$ there is an element $k(j) \in K$ such that firstly there are elements $\overline{e_i} \in F_{k(j)}$ with $\mu_{k(j)}(\overline{e}_i) = e_i$ for all $i \leq m_j$, and secondly $f_{j'}(\overline{e}_{i_1}, \ldots, \overline{e}_{i_n}) = 0$ for all $j' \leq j$ (i.e. the relations $f_{j'}$ are satisfied already in $F_{k(j)}$. It follows that C is a direct summand of $\varinjlim F_{k(j)}$, which is a direct limit over a countable system of free modules, whence (b) by Lemma 4.2.

(a) Now, in addition, let C be finitely presented. Then we can find a finite projective resolution $P_1 \overset{i}{\rightarrowtail} P_0 \twoheadrightarrow C$. Let P_i^* denote the dual module of P_i (cf. Section 3.1), $i = 1,2$. These are right Λ-modules and hence we get an induced right Λ-module structure on the cokernel

$$P_0^* \to P_1^* \twoheadrightarrow D, \quad D = \operatorname{Ext}^1_\Lambda(C,\Lambda).$$

The natural homomorphisms $\phi\colon P_i^* \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(P_i, A)$ and $\psi\colon B \otimes_\Lambda P_i^* \to \mathrm{Hom}_\Lambda(P_i^*, B)$ yield commutative diagrams

$$\begin{array}{ccccccc}
P_0^* \otimes_\Lambda A & \longrightarrow & P_1^* \otimes_\Lambda A & \longrightarrow & D \otimes_\Lambda A & \to & 0 \\
\phi \downarrow & & \phi \downarrow & & \downarrow & & \\
\mathrm{Hom}_\Lambda(P_0, A) & \to & \mathrm{Hom}_\Lambda(P_1, A) & \to & \mathrm{Ext}_\Lambda^1(C, A) & \to & 0,
\end{array}$$

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Tor}_1^\Lambda(B, C) & \to & B \otimes_\Lambda P_1 & \longrightarrow & B \otimes_\Lambda P_0 \\
 & & \downarrow & & \downarrow \psi & & \downarrow \psi \\
0 & \to & \mathrm{Hom}_\Lambda(D, B) & \to & \mathrm{Hom}_\Lambda(P_1^*, B) & \to & \mathrm{Hom}_\Lambda(P_0^*, B),
\end{array}$$

for every left Λ-module A and right Λ-module B. Since P_0 and P_1 are finitely generated projective modules ϕ and ψ are isomorphisms, whence

$$D \otimes_\Lambda A \cong \mathrm{Ext}_\Lambda^1(C, A), \quad \mathrm{Tor}_1^\Lambda(B, C) \cong \mathrm{Hom}_\Lambda(D, B).$$

Now, as C is flat, $\mathrm{Hom}_\Lambda(D,B) = 0$ for all B. In particular $\mathrm{Hom}_\Lambda(D,D) = 0$ and hence $D = 0$. This in turn implies $\mathrm{Ext}_\Lambda^1(C,A) = 0$ for all A and hence C is projective. □

<u>Corollary 4.5</u>. Let Λ be an arbitrary ring with unit and let C be a Λ-module. Then the following holds:

(a) $\mathrm{fl.dim}_\Lambda C \le \mathrm{pr.dim}_\Lambda C$.

(b) If C has a resolution by countably generated free Λ-modules, then $\mathrm{pr.dim}_\Lambda C \le \mathrm{fl.dim}_\Lambda C + 1$.

(c) If C is of type $(FP)_\infty$ then $\mathrm{pr.dim}_\Lambda C = \mathrm{fl.dim}_\Lambda C$.

Proof (a) is trivial. For (b) notice that the kernels in a countably generated free resolution are countably presented. Thus if one of those kernels is flat, then the next higher one is projective. And for (c) notice that the kernels in a finitely generated free resolution are finitely presented.

4.4. The group ring case. We now come back to the group ring case. Let R be a commutative ring with unit and let G be a group. The flat dimension of R as an RG module with trivial G-action is called the homology dimension of G over R and denoted by $hd_R G$. The projective dimension of R, again as an RG-module with trivial G-action, is called the cohomology dimension of G over R and denoted by $cd_R G$.

The above results on flat and projective dimensions yield immediately

Theorem 4.6. Let G be a group; then the following holds:

(a) $hd_R G \le cd_R G$.

(b) If G is countable then $cd_R G \le hd_R G + 1$.

(c) If G is of type $(FP)_\infty$ over R, then $cd_R G = hd_R G$.

Remark. R has a resolution by countably generated free RG-modules if and only if G is a countable group. For if $|G| = \aleph_0$, then the bar-resolution is countably generated. Conversely, the existence of a countably generated RG-resolution for R implies that the augmentation ideal $\mathfrak{g} = \ker(RG \twoheadrightarrow R)$ is countably generated. Let H be the subgroup generated by all elements of G involved in a countable set of generators for $\mathfrak{g}$; then H is countable and $RG.\mathfrak{f} = \mathfrak{g}$ ($\mathfrak{f}$ the augmentation ideal of H). But $R(G/H) \cong RG/RG\mathfrak{f} = RG/\mathfrak{g} \cong R$, whence $G = H$.

In particular the existence of a countably generated RG-free resolution for R is independent of the ring R. I do not know whether the same holds for type $(FP)_\infty$ over R.

<u>Theorem 4.7</u>. Let $\{G_\alpha, \alpha \in I\}$ be a direct system of groups, $G = \varinjlim G_\alpha$. Then the following holds:

(a) $hd_R G \leq \sup \{hd_R G_\alpha\}$,

(b) if I is countable then $cd_R G \leq \sup \{cd_R G_\alpha\} + 1$.

<u>Proof</u>. Let $\underline{\underline{B}}(G)$ be the bar-resolution of G and $K_n(G)$ its n-th kernel. Then $\underline{\underline{B}}(G) \cong \varinjlim \underline{\underline{B}}(G_\alpha) \cong \varinjlim (RG \otimes_{RG_\alpha} \underline{\underline{B}}(G_\alpha))$, and $K_n(G) = \varinjlim K_n(G_\alpha) \cong \varinjlim (RG \otimes_{RG_\alpha} K_n(G_\alpha))$. Hence the assertion follows from Theorem 4.3.

<u>Proposition 4.8</u>. Let $\{G_\alpha, \alpha \in I\}$ be a direct system of groups, $G = \varinjlim G_\alpha$, and let B be a right RG-module. Then the limiting map yields a natural isomorphism

$$\varinjlim H_n(G_\alpha, B) \xrightarrow{\sim} H_n(G; B)$$

for all $n \in \mathbb{Z}$, where B is an RG_α-module via the canonical map $\pi_\alpha : G_\alpha \to G$.

Proof. Using the notation above one has

$$H_n(G;B) \simeq H_n(B \otimes_{RG} \underline{B}(G)) \simeq \varinjlim H_n(B \otimes_{RG} (RG \otimes_{RG_\alpha} \underline{B}(G_\alpha))$$

$$\simeq \varinjlim H_n(B \otimes_{RG_\alpha} \underline{B}(G_\alpha)) \simeq \varinjlim H_n(G_\alpha;B).$$

Proposition 4.9. If S is a subgroup in G then $hd_R\, S \le hd_R\, G$ and $cd_R\, S \le cd_R\, G$.

This is an immediate consequence of the Shapiro Lemma and shall be used without further reference. Notice that it implies the following Corollary of Theorem 4.7.

Corollary 4.10(a) Every group G of finite homology dimension over R contains a finitely generated subgroup S with $hd_R\, S = hd_R\, G$.

(b) Every countable group G of finite cohomology dimension over R contains a finitely generated subgroup T with $cd_R\, T \le cd_R\, G \le cd_R\, T + 1$.

It is an almost untouched question to what extent the homological dimensions of a group G depend upon the ring R.

The only relevant result in this direction is the following easy observation.

Let G be a group and let R be a commutative ring with 1. We say that G has no R-torsion, if the order of every element in G is either infinite or a unit in R. Thus G has no $\mathbb{Z}$-torsion if and only if G is torsion-free.

Proposition 4.11. If $\mathrm{hd}_R G$ (or $\mathrm{cd}_R G$) is finite, then G has no R-torsion.

Proof. Let S be a finite cyclic subgroup in G. Then $\mathrm{hd}_R S$ is finite and hence $H_{2n+1}(S; R) = R/|S|R = 0$ for sufficiently large n, whence $|S|$ is a unit in R.□

Proposition 4.12 (a) $\mathrm{cd}_R G = 0$ if and only if G is a finite group with no R-torsion (i.e., $|G|$ is a unit in R).

(b) $\mathrm{hd}_R G = 0$ if and only if G is a locally finite group with no R-torsion.

Proof. (a) If $|G|$ is invertible, then the augmentation map $RG \overset{\varepsilon}{\twoheadrightarrow} R$ has a splitting $\sigma: R \to RG$, $\sigma(r) = \frac{r}{|G|}\left(\sum_{x \in G} x\right)$. Conversely if ε splits then RG contains G-invariant elements, hence G is finite and (a) follows by Proposition 4.11.

(b) Clearly if G is locally of cohomology dimension 0 over R then $hd_R G = 0$. Conversely, assume $hd_R G = 0$ and let S be a finitely generated subgroup. Since S is countable, we know $cd_R S \leq 1$ by Theorem 4.6 (b). Since S is finitely generated, this implies that S is of type $(FP)_\infty$ and hence $cd_R S = hd_R S = 0$ by Theorem 4.6 (c). □

5. Normal subgroups and extensions

5.1 Projective and injective coefficients. The following easy observation shows that knowledge of the homology of a group G with injective coefficient modules or of the cohomology of G with projective coefficient modules is mostly sufficient to compute $hd_R G$ or $cd_R G$ respectively.

Proposition 5.1 (a) If $cd_R G = n < \infty$ then there is a free RG-module F with $H^n(G; F) \neq 0$ (so F is $= L \otimes_R RG$ with L a free R-module).

(b) If $hd_R G = m < \infty$ then there is an injective module I of the form $I = \mathrm{Hom}_R(RG, L)$, with L an injective R-module, such that $H_m(G; I) \neq 0$.

Proof. Since $H^n(G, -)$ is right exact $H^n(G; F) = 0$ for all free modules F would imply that $H^n(G,-) \equiv 0$, hence (a). As to (b), notice first that every RG-module B embeds in an RG-module of the required form and apply the dual argument. □

With regard to this remark we shall now concentrate to give more precise information on the cohomology with projective and the homology with injective coefficients.

Let Λ be an arbitrary ring with unit $1 \neq 0$, let A and K be left Λ-modules and B a right Λ-module, and recall from Section 3.1 that one has natural homomorphisms:

(*) $\phi: K^* \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(K, A)$ $\psi: B \otimes_\Lambda K \to \mathrm{Hom}_\Lambda(K^*, B)$ given by $\phi(f \otimes a)(k) = f(k)a$, $\psi(b \otimes k)(f) = bf(k)$, $f \in K^*$, $a \in A$, $b \in B$, $k \in K$.

Lemma 5.2 (a) If K is finitely presented and A flat then ϕ is an isomorphism.

(b) If K is finitely presented and B injective then ψ is an isomorphism.

Proof. Let $P_1 \to P_0 \to K \to 0$ be an exact sequence of Λ-modules with P_0, P_1 finitely generated projective. By naturality of ϕ and since A is flat, we get the following commuting diagram with exact rows:

$$\begin{array}{ccccc} 0 \to K^* \otimes_\Lambda A & \longrightarrow & P_0^* \otimes_\Lambda A & \longrightarrow & P_1^* \otimes_\Lambda A \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ 0 \to \mathrm{Hom}_\Lambda(K, A) & \longrightarrow & \mathrm{Hom}_\Lambda(P_0, A) & \longrightarrow & \mathrm{Hom}_\Lambda(P_1, A), \end{array}$$

and the assertion (a) follows by the 5-lemma. The proof of (b) is dual.

Remark. There is a very useful variant of Lemma 5.2 (a): It is straightforward that if A is projective then ϕ is always monomorphic - and this is all we need to apply the 5-lemma. Thus we have

(c) If K is finitely generated and A projective, then ϕ is an isomorphism.

Now we come back to the group ring case $\Lambda = RG$. Replacing the module K in $(*)$ by an RG-projective resolution $\underline{P} \twoheadrightarrow R$ yields complex homomorphisms and hence induced maps in homology

$$H^k(\underline{P}^* \otimes_{RG} A) \to H^k(G;A), \quad H_k(G;B) \to H_k(\mathrm{Hom}_{RG}(\underline{P}^*, B)).$$

These can be combined with the functorial homomorphisms

$$\alpha: H^k(\underline{P}^*) \otimes_{RG} A \to H^k(\underline{P}^* \otimes_{RG} A), \quad \alpha': H_k(\mathrm{Hom}_{RG}(\underline{P}^*, B)) \to \mathrm{Hom}_{RG}(H^k(\underline{P}^*), B)$$

and so we get natural homomorphisms

$$\left.\begin{array}{l} \phi^k: H^k(G;RG) \otimes_{RG} A \to H^k(G;A) \\ \\ \psi_k: H_k(G;B) \to \mathrm{Hom}_{RG}(H^k(G;RG), B) \end{array}\right\} \quad (**)$$

for every left RG-module A and right RG-module B and all $k \in \mathbf{Z}$.

Proposition 5.3. Let G be a group of type $(FP)_n$ over R. Then the following holds:

(a) If A is flat then ϕ^k is an isomorphism for all $k \le n-1$.

(b) If A is projective then ϕ^k is an isomorphism for all $k \le n$.

(c) If B is injective then ψ_k is an isomorphism for all $k \le n-1$.

Proof. Let $\underline{P} \twoheadrightarrow R$ be a projective resolution which is finitely generated in dimensions $\le n$, and let K_s be the kernel of $P_{s-1} \to P_{s-2}$. Then one has a commuting diagram with exact bottom row

$$\begin{array}{ccccc}
P^*_{s-1} \otimes_{RG} A & \longrightarrow & K_s^* \otimes_{RG} A & \longrightarrow & H^s(G; RG) \otimes_{RG} A \to 0 \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi^s \\
\mathrm{Hom}_{RG}(P_{s-1}, A) & \longrightarrow & \mathrm{Hom}_{RG}(K_s, A) & \longrightarrow & H^s(G; A) \to 0
\end{array} .$$

If A is flat, the top-row is exact as well and Lemma 5.2 (with Remark) yields the assertions (a) and (b). The proof of (c) is dual. □

5.2. Extensions. Let N be a normal subgroup in a group G and let A and B be RG-modules. Recall that in this situation the (co)homology groups $H^k(N; A)$ $H_k(N; B)$ have a natural RG-module structure, which is imposed by the fact that the (co)homology functor is "natural in the group variable". For an easy explicit description on the (co) chain level, take an RG-projective resolution $\underline{P} \twoheadrightarrow R$; then the G-action is given by

$$(xf)(p) = xf(x^{-1}p) \quad , \quad (b \otimes p)x = bx \otimes x^{-1}p,$$

$f \in \mathrm{Hom}_{RN}(\underline{P}, A)$, $p \in \underline{P}$, $b \in B$, $x \in G$. (According to our convention we think of $H^k(N; A)$ as left G-modules and of $H_k(N; B)$ as right G-modules).

Next we consider $H^k(N; RN)$. This is a right RN-module by right multiplication in RN. Explicitly $(f\cdot n)(p) = f(p)n$ for all $f \in \mathrm{Hom}_{RN}(\underline{P}, RN)$, $p \in \underline{P}$, $n \in N$. This can also be interpreted as $(f\cdot n)(p) = n^{-1}f(np)n$, and in this form, the N action can be extended to a G-action, provided $\underline{P} \twoheadrightarrow R$ is actually an RG-projective resolution:

$$(f\cdot x)(p) = x^{-1}f(xp)x,$$

$f \in \mathrm{Hom}_{RN}(\underline{P}, RN)$, $p \in \underline{P}$, $x \in G$. Of course one has to check that $f\cdot x$ is again an RN-homomorphism: $(f\cdot x)(np) = x^{-1}f(xnp)x$ $= x^{-1}f(xnx^{-1}\cdot xp)x = nx^{-1}f(xp)x = n(fx)(p)$.

Proposition 5.4. With respect to the G-actions above the homomorphisms

$$\phi^k: H^k(N; RN) \otimes_{RN} A \to H^k(N; A)$$

$$\psi_k: H_k(N; B) \to \mathrm{Hom}_{RN}(H^k(N; RN), B)$$

are G-module homomorphisms (diagonal action on $- \otimes_{RN} -$ and on $\mathrm{Hom}_{RN}(-,-)$).

Proof. Let $\underline{P} \twoheadrightarrow R$ be an RG-projective resolution, and let $x \in G$, $f \in \mathrm{Hom}_{RN}(\underline{P}, RN)$, $a \in A$. Then one has for every $p \in \underline{P}$,

$$\begin{aligned}\phi(x(f \otimes a))(p) &= \phi(fx^{-1} \otimes xa)(p) = (fx^{-1})(p)xa \\ &= xf(x^{-1}p)x^{-1}\cdot xa = xf(x^{-1}p)a \\ &= x[\phi(f \otimes a)(x^{-1}p)] = [x\cdot\phi(f \otimes a)](p).\end{aligned}$$

hence $\phi(x(f \otimes a)) = x\phi(f \otimes a)$. As to ψ_k, let $p \in \underline{P}$, $x \in G$ and $b \in B$; then one has for every $f \in \mathrm{Hom}_{RN}(\underline{P}, RN)$

$$\psi((b \otimes p)x)(f) = \psi(bx \otimes x^{-1}p)(f) = bx\ f(x^{-1}p)$$
$$= bx\ f(x^{-1}p)x^{-1}.\ x = b\ (fx^{-1}))p).x$$
$$= \psi(b \otimes p)(fx^{-1}).x = [\psi(b \otimes p)x](f)\ ,$$

hence $\psi((b \otimes p)x) = \psi(b \otimes p)x$ and the proposition is proved.□

Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups. The (co)homology of N, G and Q is linked together by the Lyndon-Hochschild-Serre (LHS) spectral sequences

$$H_p(Q;\ H_q(N;\ B)) \Rightarrow H_{p+q}(G;\ B), \quad H^p(Q;\ H^q(N;\ A)) \Rightarrow H^{p+q}(G;\ A),$$

for arbitrary RG-modules A and B. It follows immediately that one has always

$$hd_R G \le hd_R N + hd_R Q \quad \text{and} \quad cd_R G \le cd_R N + cd_R Q.$$

We shall find a large number of examples where these inequalities are strict - but in many interesting cases they are actually equalities.

<u>Theorem 5.5</u>. (Feldman [27]). Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups. Assume that N is of type (FP) over R and that $H^n(N;\ RN)$ is R-free for $n = cd_R N$ (= $hd_R N$). Then

(i) if $cd_R Q < \infty$ then $cd_R G = cd_R N + cd_R Q$

(ii) if $hd_R Q < \infty$ then $hd_R G = hd_R N + hd_R Q$.

Proof. Let $q = cd_R Q$, $n = cd_R N$. By Proposition 5.1 there is a free RQ-module $F \cong L \otimes_R RQ$ (L a free R-module) with $H^q(Q; F) \neq 0$. The LHS spectral sequence yields $cd_R G \leq n + q$ and an isomorphism $H^{n+q}(G; L \otimes_R RG) \cong H^q(Q; H^n(N; L \otimes_R RG)$. By Proposition 5.3 $H^n(N; L \otimes_R RG) \cong H^n(N; RN) \otimes_{RN} (L \otimes_R RG)$, and by Proposition 5.4 this is a G-isomorphism if we take diagonal action on the right hand side. By Lemma 5.6 below the right hand side, in turn, is isomorphic to $(H^n(N; RN) \otimes_R L) \otimes_R RQ$ with single G-action. Since $H^n(N; RN)$ is R-free the latter contains F as a direct summand. It follows $H^{n+q}(G; L \otimes_R RG) \neq 0$, whence $cd_R G = n + q$. The proof of (ii) is precisely the dual and urgently recommended as an exercise. □

It remains to prove

Lemma 5.6. Let G be a group N be a normal subgroup and C a right RG-module. Then one has natural RG-module isomorphisms

$$u: \; C \otimes_R \overset{\swarrow}{R(G/N)} \xrightarrow{\sim} \overset{\swarrow}{C} \otimes_{RN} \overset{\searrow}{RG},$$

$$v: \; \mathrm{Hom}_R(\overset{\swarrow}{R(G/N)}, C) \xrightarrow{\sim} \mathrm{Hom}_{RN}(\overset{\swarrow}{RG}, \overset{\swarrow}{C}),$$

where the G-action is understood as indicated by the arrows (single action on the left and diagonal action on the right hand side).

Proof. u is defined by $u(c \otimes xN) = cx \otimes x^{-1}$, $c \in C$, $x \in G$. It is easy to see that this is well defined and a

G-homomorphism. The inverse of u is given by $u^{-1}(c \otimes x) = cx \otimes x^{-1}N$. Analogously v is defined by $v(f)(x) = f(xN)x$, $x \in G$, $f \in \mathrm{Hom}_R(R(G/N), C)$ and its inverse $v^{-1}(h)(xN) = h(x)x^{-1}$, $x \in G$, $h \in \mathrm{Hom}_{RN}(RG,C)$.□

Lemma 5.6 should be compared with Lemma 2.9. The isomorphisms u, v in both statements coincide (the fact that one of the modules in Lemma 2.9 is a left module is irrelevant). The action of G described in Lemma 5.6, however, is only defined for <u>normal</u> subgroups N, whereas the action in Lemma 2.9 is available for arbitrary subgroups $H \le G$.

<u>Remarks</u>. 1) The statements of Theorem 5.5 are definitely false without the assumption that N be of type (FP) (e.g. G free of rank 2 and $N = [G,G]$). However, one can show that "type (FP)" can be replaced by the much weaker condition that there is a projective resolution $\underline{P} \twoheadrightarrow R$ which is merely finitely generated in the top dimension $n = \mathrm{cd}_R N$. For details cf.[6] and [59].

2) By a result of Swan's [60], $H^1(N; RN)$ is R-free for every finitely generated group N. Thus all assumptions are fulfilled if N is a finitely generated group with $\mathrm{cd}_R N \le 1$. <u>Exercise</u>: Prove that the statements (i) and (ii) of Theorem 5.5 hold if N is of type (FP) over R and Q is finitely generated with $\mathrm{cd}_R Q \le 1$ (without assuming that $H^n(N; RN)$ is R-free).

5.3 Subgroups of finite index Let G be an arbitrary group and $S \leq G$ a subgroup of finite index $|G:S| = d$. For every (left) RG-module K one has a natural isomorphism

$$v: \operatorname{Hom}_{RS}(K, RS) \to \operatorname{Hom}_{RG}(K, RG)$$

given by $v(f)(k) = \sum_{i=1}^{d} r_i^{-1} f(r_i k)$, where $1 = r_1, r_2, r_3, \ldots, r_d$ is a right transversal for G mod S. Obviously v does not depend upon the choice of this transversal, and v is an S-module homomorphism. Moreover, if S is normal in G, then $\operatorname{Hom}_{RS}(K, RS)$ is a G-module (diagonal action on K and RS (conjugation), and v is a G-homomorphism. Indeed, we have for $x \in G$ and $f \in \operatorname{Hom}_{RS}(K, RS)$

$$v(fx)(k) = \sum r_i^{-1}(fx)(r_i k) = \sum r_i^{-1} x^{-1} f(x r_i k) x$$

$$= v(f)(k)x \quad = \quad (v(f)x)(k)$$

for all $k \in K$, i.e., $v(fx) = v(f)x$.

We claim that v is actually an isomorphism. To see this notice that the group ring, considered as an RS-module, has a canonical direct sum decomposition $RG = RS \oplus R[G-S]$, where $R[G-S]$ is the RS-submodule freely generated, as an R-module, by all elements $x \in G$ $x \in S$. Combing the restriction map with the projection onto the direct summand RS yields a map

$$\sigma: \operatorname{Hom}_{RG}(K, RG) \to \operatorname{Hom}_{RS}(K, RG) \to \operatorname{Hom}_{RS}(K, RS),$$

$\sigma(f) = f_1$, where $f(k) = \sum_i r_i^{-1} f_i(k)$, $k \in K$. Now, notice that $f(r_j k) = r_j f(k) = \sum r_j r_i^{-1} f_i(k)$ implies $f_j(k) = f_1(r_j k)$, hence

$$f(k) = \sum_i r_i^{-1} f_1(r_i k) \;,$$

i.e. $v \circ \sigma = \text{Id}$. $\sigma \circ v$ is straightforward, hence $\sigma = v^{-1}$.

Replacing the module K by an RG-projective resolution $\underline{P} \twoheadrightarrow R$ yields

Proposition 5.7. Let G be a group and S a subgroup of finite index in G. Then there is a canonical isomorphism of right RS-modules $v: H^k(S; RS) \tilde{\rightarrow} H^k(G; RG)$, for all $k \in \mathbb{Z}$. If S is normal in G then v is an RG-module isomorphism.

Next we recall some notation. Let G be a group $S \leq G$ a subgroup, M and A left RG-modules and B a right RG-module. The usual restriction maps are denoted by

$$\text{res}: \text{Hom}_{RG}(M, A) \to \text{Hom}_{RS}(M, A) \qquad \text{cor}: B \otimes_{RS} M \to B \otimes_{RG} M.$$

Replacing M by an RG-projective resolution yields the restriction homomorphisms in (co)homology

$$\text{res}^*: H^k(G; A) \to H^k(S; A) \qquad \text{cor}_*: H_k(S; B) \to H_k(G; B).$$

Furthermore, if S is of finite index in G one has the transfer maps

$$\text{cor}: \text{Hom}_{RS}(M, A) \to \text{Hom}_{RG}(M, A) \qquad \text{res}: B \otimes_{RG} M \to B \otimes_{RS} M$$

given by $\text{cor}(f)(m) = \sum r_i^{-1} f(r_i m)$, $\text{res}(b \otimes m) = \sum b r_i^{-1} \otimes r_i m$, $f \in \text{Hom}_{RS}(M, A)$, $m \in M$, $b \in B$, where $1 = r_1, r_2, \ldots, r_d$ is a right transversal for G mod S. Replacing M by an RG-projective resolution $\underline{\underline{P}} \twoheadrightarrow R$ yields the <u>transfer</u> homomorphisms in (co)homology

$$cor^*: H^k(S; A) \to H^k(G; A) \qquad res_*: H_k(G; B) \to H_k(S; B).$$

<u>Theorem 5.8</u>. Let G be a group and S a subgroup of finite index. For fixed $k \in \mathbb{Z}$ let C denote the right RG-module $H^k(G; RG)$ and identify $H^k(S; RS)$ with C via v. Then one has the following four commutative squares

$$\begin{array}{ccc} C \otimes_{RG} A & \xrightarrow{res} & C \otimes_{RS} A \\ \phi \downarrow & & \downarrow \phi \\ H^k(G; A) & \xrightarrow[res^*]{} & H^k(S; A) \end{array} \qquad \begin{array}{ccc} H_k(S; B) & \xrightarrow{cor_*} & H_k(G; B) \\ \psi \downarrow & & \downarrow \psi \\ Hom_{RS}(C, B) & \xrightarrow[cor]{} & Hom_{RG}(C, B) \end{array}$$

$$\begin{array}{ccc} C \otimes_{RS} A & \xrightarrow{cor} & C \otimes_{RG} A \\ \phi \downarrow & & \downarrow \phi \\ H^k(S; A) & \xrightarrow[cor^*]{} & H^k(G; A) \end{array} \qquad \begin{array}{ccc} H_k(G; B) & \xrightarrow{res_*} & H_k(S; B) \\ \psi \downarrow & & \downarrow \psi \\ Hom_{RG}(C, B) & \xrightarrow[res]{} & Hom_{RS}(C, B) \end{array}$$

<u>Proof.</u> a) To prove commutativity of the top left square, we have to show that $\phi\,(v^{-1} \otimes_{RS} A) \circ res = res^* \circ \phi$. Let $c \otimes a \in C \otimes_{RG} A$ and let $f \in Hom_{RG}(\underline{\underline{P}}, RG)$ be a cocycle representing c. Then $v^{-1}(f) = f_1$, where $f(p) = \sum f_i(p) r_i$, $p \in \underline{\underline{P}}$. Now,

$$\operatorname{res}\,(f \otimes a)(p) = \sum_i f(p)r_i^{-1} \otimes r_i a$$

$$= \sum_{i,j} f_j(p) r_j r_i^{-1} \otimes r_i a,$$

and hence

$$(v^{-1} \otimes_{RS} A)\ \operatorname{res}\ (f \otimes a) = \sum_j f_j \otimes r_j a.$$

It follows that $\phi(v^{-1} \otimes_{RS} A)\ \operatorname{res}\ (f \otimes a)(p) = \sum f_j(p) r_j a$ $= f(p)a$ for all $p \in \underline{P}$, whence the assertion.

b) To prove commutativity of the top right square, one has to show that $\operatorname{cor}\circ \operatorname{Hom}_{RS}(v^{-1}, B)\circ\psi = \psi\circ\operatorname{cor}_*$. For every $f \in \operatorname{Hom}_{RG}(\underline{P}, RG)$ we put $f(q) = \sum f_i(q) r_i$, $q \in \underline{P}$. Then $f(q)r_j^{-1} = \sum f_i(q) r_i r_j^{-1}$, hence $v^{-1}(fr_j^{-1}) = f_j$. It follows for $b \otimes p \in \underline{B} \otimes_{RS} \underline{P}$.

$$\operatorname{cor}(\operatorname{Hom}(v^{-1},B)\psi(b \otimes p))(f) = \sum_i \operatorname{Hom}(v^{-1},B)\psi(b \otimes p)(fr_i^{-1})r_i$$

$$= \sum \psi(b \otimes p)(f_i) r_i$$

$$= \sum b\ f_i(p) r_i = bf(p),$$

whence the assertion.

c) To prove commutativity of the bottom left square, let $f \in \operatorname{Hom}_{RS}(\underline{P}, RS)$, $a \in A$. Then one has for all $p \in \underline{P}$

$$\operatorname{cor}^* \phi(f \otimes a)(p) = \sum r_i \phi(f \otimes a)(r_i^{-1} p)$$

$$= \sum r_i f(r_i^{-1} p) a$$

$$= v(f)(p)a = \phi(\operatorname{cor}(v(f) \otimes a))$$

whence the assertion

d) Finally, to prove commutativity of the bottom right square, let $b \otimes p \in B \otimes_{RG} \underline{P}$. Then one has for all $f \in \mathrm{Hom}_{RS}(\underline{P}, RS)$

$$\psi(\mathrm{res}_*(b \otimes p))(f) = \psi(\sum br_i^{-1} \otimes r_i p)(f)$$

$$= \sum br_i^{-1} f(r_i p) = bv(f)(p)$$

$$= \mathrm{res}\ \psi(b \otimes p)(v(f)),$$

whence the assertion. This completes the proof of Theorem 5.8. □

5.4 Serre's Theorem. The following preliminary remark is a slight generalization of Proposition 5.7.

Proposition 5.9. Let S be a subgroup of finite index in a group G. Then one has for all $k \in \mathbb{Z}$ and all R-modules L

(i) $H^k(G; L \otimes_R RG) \simeq H^k(S; L \otimes_R RS)$,

(ii) $H_k(G; \mathrm{Hom}_R(RG, L)) \simeq H_k(S; \mathrm{Hom}_R(RS, L))$.

Proof. $H^k(S, L \otimes_R RS) \simeq H^k(G; \mathrm{Hom}_{RS}(RG, L \otimes_R RS))$

$\simeq H^k(G; (L \otimes_R RS) \otimes_{RS} RG)$, by Lemma 2.6,

$\simeq H^k(G; L \otimes_R RG)$.

This proves (i); the proof of (ii) is dual. □

Propositions 5.1 and 5.9 imply immediately.

Corollary 5.10. Let S be a subgroup of finite index in a group G. Then one has

(i) if $cd_R G < \infty$ then $cd_R S = cd_R G$

(ii) if $hd_R G < \infty$ then $hd_R S = hd_R G$

Theorem 5.11. (Serre [52]). Let S be a subgroup of finite index in a group G. If G has no R-torsion then $cd_R S = cd_R G$.

By Corollary 5.10 all that is left to show is that $cd_R S < \infty$ implies $cd_R G < \infty$. Moreover, since every subgroup of finite index in G contains a normal subgroup of G with finite index, it suffices to prove Theorem 5.11. for a normal subgroup S. Finally, the following Lemma allows a further technical simplification.

Lemma 5.12. If a module A has a projective resolution of length $n \geq 1$, then A has also a free resolution of length n.

Proof. By induction on n it is clear that it suffices to prove the Lemma for $n = 1$. Let $0 \to P_1 \xrightarrow{\partial} P_0 \overset{\varepsilon}{\twoheadrightarrow} A$ be a projective resolution for A. Let Q_1 be a projective module such that $P_1 \oplus Q_1$ is free; then $0 \to P_1 \oplus Q_1 \xrightarrow{\partial \oplus Id} P_0 \oplus Q_1 \overset{\varepsilon \oplus 0}{\twoheadrightarrow} A$ is a projective resolution which is free in dimension 1. If we manage to find a free module Q_0 such that $(P_0 \oplus Q_1) \oplus Q_0$ is free then $0 \to P_1 \oplus Q_1 \oplus Q_0 \to P_0 \oplus Q_1 \oplus Q_0 \twoheadrightarrow A$ is the required free resolution, and we are done. To find such a free complement Q_0 we use Eilenberg's "projective module swindle". Let P be a projective module then there is a module P' such that $P \oplus P' \cong P' \oplus P$ is free. Now

$$(P \oplus P') \oplus (P \oplus P') \oplus \ldots \cong P \oplus (P' \oplus P) \oplus (P' \oplus P) \oplus \ldots ,$$

ie, the infinite sum $(P' \oplus P) \oplus (P' \oplus P) \oplus \ldots$ is a free complement for P. □

Proof (of Theorem 5.11). Assume $S \lhd G$, $|G/S| = d < \infty$ and let $\underline{F} \twoheadrightarrow R$ be an RS-free resolution. Let $\underline{E}$ be the d-fold tensor product of $\underline{F}$

$$\underline{E} = \underline{F} \otimes_R \underline{F} \otimes_R \cdots \otimes_R \underline{F}.$$

$\underline{E} \twoheadrightarrow R$ is an $R[S\times S\times\ldots\times S]$-free resolution. We shall now define a G-action on $\underline{E}$ which is compatible with the differential. Choose coset representatives x_i so that $G = \cup x_i S$. If $g \in G$, let $g^{-1}x_i = x_{v_i} h_{v_i}^{-1}$, $h_{v_i} \in S$ and define

$$g(p_1 \otimes p_2 \otimes \ldots \otimes p_d) = (-1)^{\alpha}\, h_{v_1} p_{v_1} \otimes \ldots \otimes h_{v_d} p_{v_d}$$

$p_k \in P_{i_k}$, $k = 1,2,\ldots,d$, where $\alpha = \sum i_r i_s$ with summation over all pairs $r < s$ with $v_r > v_s$. This extends uniquely to an R-automorphism of $\underline{E}$. We leave it as an exercise to verify that we have defined a genuine action of G on $\underline{\underline{E}}$ which is compatible with the differentials (for details see [18]).

We shall now show that $\underline{\underline{E}}$ is RG-projective. For this we can forget about the differential in $\underline{\underline{E}}$ and regard it as a free RS-module. Let $\{ b_\sigma \}_{\sigma\in J}$ be an RS-basis for $\underline{\underline{E}}$. Then $\{ hb_\sigma \}, h\in S\ \sigma\in J$ is an R-basis, and so $\underline{\underline{E}}$ has an R-basis consisting of all elements

$$w = h_1 b_{\sigma_1} \otimes h_2 b_{\sigma_2} \otimes \ldots \otimes h_\alpha b_{\sigma_\alpha} .$$

G permutes the R-modules Rw and hence $\underline{\underline{E}}$ is isomorphic, as an RG-module, to the direct sum $\oplus RGw_i$ for some basis elements w_i. Thus it is sufficient to show that all cyclic modules of the form RGw are RG-projective.

Let $K_w = \{ x \in G \mid xw = \varepsilon_x w,\ \varepsilon_x = \pm 1\}$. Then K_w is a subgroup of G and we claim that $K_w \cap S = 1$. Indeed for $h \in S$ one has $h^{-1} x_i = x_i (x_i^{-1} h x_i)^{-1}$ with $x_i^{-1} h x_i \in S$ since S is normal in G. So

$$hw = \pm h(h_1 b_{\sigma_1} \otimes h_2 b_{\sigma_2} \otimes \ldots \otimes h_d b_{\sigma_\alpha}) = \pm (x_1^{-1} h x_1 h_1 b_{\sigma_1} \otimes \ldots \otimes x_d^{-1} h x_d h_d b_{\sigma_\alpha})$$

$$\neq \pm w \text{ unless } h = 1 .$$

Therefore $K_w \leq G/S$, i.e., K_w is a finite subgroup of G. Let $m = | K_w |$; then m is a unit in R and hence one can define an RG-homomorphism $\rho\colon RGw \to RG$ by

$$\rho(\lambda w) = \frac{1}{m} \lambda(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \ldots + \varepsilon_m k_m).$$

where $k_1, k_2, \ldots, k_m$ are all elements in K_w with $k_i w = \varepsilon_i w$. It is easily checked that ρ splits the projection $RG \twoheadrightarrow RGw$. Thus RGw is a direct summand of RG and hence projective.

It follows that if $\underline{F} \twoheadrightarrow R$ was an RS-free resolution of finite length $\leq n$ then $\underline{E} \twoheadrightarrow R$ is an RG-projective resolution of length $\leq n \cdot d$.□

Theorem 5.13. Let S be a subgroup of finite index in a group G. If G has no R-torsion then $hd_R G = hd_R S$.

Proof. By Corollary 5.10, all that is left to show is that $hd_R S < \infty$ implies $hd_R G < \infty$. Notice that if G happens to be countable, this follows from Theorem 4.6 together with Theorem 5.11. In the general case we can argue as follows: we take an RS-resolution of the form

$$\underline{F}: \quad 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \twoheadrightarrow R$$

where the RS-modules F_i are free for $i = 0,1,\ldots,n-1$ and F_n is flat. Then we take the $d = |G:S|$ -fold tensor-product $\underline{E} = \underline{F} \otimes_R \underline{F} \otimes_R \cdots \otimes_R \underline{F}$ and give it the same RG-structure as in the proof of Theorem 5.11. It is easy to see that $\underline{F}$ splits over R, so that $\underline{E}$ is exact. We have to show that $\underline{E}$ is RG-flat.

By D. Lazard's result[39] F_n is the direct limit of (finitely generated) free modules F_n^α. Thus for each α we get a (in general not exact) complex of free RS-modules

$$\underline{\underline{F}}^{\alpha} : \; 0 \to F_n^{\alpha} \to F_{n-1} \to F_{n-2} \to \ldots \to F_0 \twoheadrightarrow R.$$

We construct the d-fold tensor-product $\underline{\underline{E}}^{\alpha} = \underline{\underline{F}}^{\alpha} \otimes_R \underline{\underline{F}}^{\alpha} \otimes \ldots \otimes_R \underline{\underline{F}}^{\alpha}$ with the same RG-structure as above. Notice that the canonical maps $\underline{\underline{E}}^{\alpha} \to \underline{\underline{E}}^{\alpha}$ are RG-homomorphisms, so that we find

$$\begin{aligned}
\varinjlim_{\alpha} \underline{\underline{E}}^{\alpha} &= \varinjlim_{\alpha} (\underline{\underline{F}}^{\alpha} \otimes_R \underline{\underline{F}}^{\alpha} \otimes_R \ldots \otimes_R \underline{\underline{F}}^{\alpha}) \\
&\simeq \varinjlim_{(\alpha,\beta,\ldots,\omega)} (\underline{\underline{F}}^{\alpha} \otimes \underline{\underline{F}}^{\beta} \otimes \ldots \otimes \underline{\underline{F}}^{\omega}) \\
&\simeq \underline{\underline{F}} \otimes_R \underline{\underline{F}} \otimes_R \ldots \otimes_R \underline{\underline{F}} = \underline{\underline{E}}
\end{aligned}$$

Now, the proof of Theorem 5.11 shows that $\underline{\underline{E}}^{\alpha}$ is RG-projective, therefore $\underline{\underline{E}} = \varinjlim \underline{\underline{E}}^{\alpha}$ is RG-flat.□

6. Amalgamated products and HNN-extensions

6.1 General results The first result follows readily from the Mayer-Vietoris sequences (Theorem 2.10).

Proposition 6.1 Let $G = G_1 *_S G_2$ be the free product of two groups G_1, G_2 with amalgamated subgroup S, and let $n = \max(cd_R G_1, cd_R G_2)$ and $m = \max(hd_R G_1, hd_R G_2)$. Then one has

$$n \leq cd_R G \leq n+1, \quad m \leq hd_R \leq m+1.$$

Moreover $cd_R G = n+1$ implies $cd_R G_1 = cd_R G_2 = cd_R S = n$ and $hd_R G = m+1$ implies $hd_R G_1 = hd_R G_2 = hd_R G = m$.

Notice that the converse of the last statement is false. It is easy to construct a non-trivial amalgamated product $G = G_1 *_S G_2$ where $cd_R G = cd_R G_1 = cd_R G_2 = cd_R S = n$ or $hd_R G = hd_R G_1 = hd_R G_2 = hd_R S = m$ (e.g. G_1 and G_2 free of rank 2 and S an infinite cyclic free factor). Analogously, the Mayer-Vietoris sequences for HNN-groups (Theorem 2.12) yields

Proposition 6.12 Let $G = G_1 *_{S,\sigma}$ be the HNN-group over the base group G_1 and with associated subgroups $\{S, \sigma(S)\}$. If $cd_R G_1 = n$ and $hd_R G_1 = m$, then one has

$$n \leq cd_R G \leq n+1, \qquad m \leq hd_R G \leq m+1.$$

Moreover $cd_R G = n+1$ implies $cd_R G_1 = cd_R S = n$ and $hd_R G = m+1$ implies $hd_R G_1 = hd_R S = m$.

Again, the converse of the last statement is false: e.g. $G_1 = \langle x,y \rangle$, $S = \langle x \rangle$, $\sigma(x) = y$.

6.2 The finite index case I: Amalgamated products. Here we examine the situation of Proposition 6.1 when S has finite index in G_1 and G_2.

Theorem 6.3 Let $G = G_1 *_S G_2$ be the free product of G_1 and G_2 with amalgamated subgroup S of finite index $\neq 1$ in both G_1 and G_2. Assume that S (and hence G_1, G_2, and G) is of type $(FP)_n$. Then, for every $k \leq n$, the map

$$(\text{res}^*, -\text{res}^*) : H^k(G_1; RG) \oplus H^k(G_2; RG) \to H^k(S; RG)$$

is an R-split monomorphism. Its cokernel is trivial if and only if $H^k(G_1; RG) = H^k(G_2; RG) = H^k(S; RG) = 0$.

Proof. Let $C = H^k(S; RS)$ and identify this with $H^k(G_1; RG_1)$ and $H^k(G_2; RG_2)$ via the canonical map of Proposition 5.7. By Theorem 5.8 one then has a commutative diagram

$$\begin{array}{ccc} (C \otimes_{RG_1} RG) \oplus (C \otimes_{RG_2} RG) & \xrightarrow{(\text{res}, -\text{res})} & C \otimes_{RS} RG \\ \downarrow \phi \oplus \phi & & \downarrow \phi \\ H^k(G_1; RG) \oplus H^k(G_2; RG) & \xrightarrow[(\text{res}^*, -\text{res}^*)]{} & H^k(S; RG) \end{array} ;$$

by Proposition 5.3 the vertical maps are isomorphisms.

Let Γ_j denote a right transversal of G_j mod S, with $1 \in \Gamma_j$. There is a natural isomorphism $u: C \otimes_{RH} RG \to C \otimes_R R(G/H)$ for every subgroup $H \leq G$, given by $u(c \otimes x) = cx \otimes Hx$, $c \in C$, $x \in G$.

As a map $C \otimes_R R(G/G_j) \to C \otimes_R R(G/S)$ the transfer is now given by

$$\begin{aligned} u \text{ res } u^{-1} (c \otimes G_j x) &= u \text{ res } (cx^{-1} \otimes x) \\ &= u(\sum_{r \in \Gamma_j} cx^{-1} r^{-1} \otimes rx) \\ &= \sum_{r \in \Gamma_j} c \otimes Srx, \end{aligned}$$

$c \in C$, $x \in G$. So we have to consider the map

$$\tau: \ R(G/G_j) \to R(G/S),$$

given by $\tau(G_j x) = \sigma_j x$, $x \in G$, where $\sigma_j = \sum_{r \in \Gamma_j} Sr$. Notice that this is obviously a monomorphism. Now, Proposition 6.3 follows readily from

<u>Lemma 6.4.</u> Let $G = G_1 *_S G_2$ be an amalgamated product with amalgamated subgroup of finite index $\neq 1$ in both factors G_1 and G_2. Then the map

$$(\tau, -\tau): \ R(G/G_1) \oplus R(G/G_2) \to R(G/S)$$

is an R-split monomorphism, but not an epimorphism.

<u>Proof</u> We use the following notation: letters $a, a', a'', \ldots$ shall always denote elements $\neq 1$ in Γ_1, $b, b', b'', \ldots$ elements $\neq 1$ in Γ_2. Recall that the words of the form $w = ba'b'a'' \ldots$ represent the right cosets $\neq G_1$ of G mod G_1. Let $\ell(w)$ be the length of w. An element $\alpha \in R(G/G_1)$ is a finite sum $\alpha = \sum m\, G_1 ba'b'a'' \ldots$ with coefficients in R. Its image $\tau(\alpha) \in R(G/S)$ is of the form

$$\tau(\alpha) = \sum m\, S\, ba'b' \ldots + \sum_a \sum m\, S\, aba'b' \ldots \quad .$$

We have divided the sum into two parts according to whether the first letter to the right of mS is in Γ_1 or in Γ_2. Let $\ell(\alpha)$ denote the maximum length of words occurring in α, and let $G_1 w$ be a term in α with $\ell(w) = \ell(\alpha)$. Then there is a term Saw in the second part of $\tau(\alpha)$ with $\ell(aw) = \ell(\alpha) + 1$.

If we now assume that $\tau(\alpha) = \tau(\beta)$ for some $\beta \in R(G/G_2)$, the term Saw must occur in the "first part" of $\tau(\beta)$, i.e. $G_2 aw$ must occur in β and thus $\ell(\beta) \geq \ell(\alpha) + 1$. But the situation is entirely symmetric in α and β, so that $\ell(\alpha) \geq \ell(\beta) + 1 \geq \ell(\alpha) + 2$, a contradiction. It follows that there are no words of maximum length in α and β, i.e., $\alpha = 0 = \beta$. Thus $\tau R(G/G_1) \cap \tau R(G/G_2) = 0$, hence $(\tau, -\tau)$ is a monomorphism.

It remains to prove that $I = \tau R(G/G_1) + \tau R(G/G_2)$ has a non-trivial R-complement in $R(G/S)$. As $G_1 \neq S \neq G_2$ we can choose fixed representatives $1 \neq \tilde{a} \in \Gamma_1$ and $1 \neq \tilde{b} \in \Gamma_2$.

Let M denote the R-submodule of $R(G/S)$ spanned by S and all cosets of the form $Saba'b'\ldots$, $a \neq \tilde{a}$, or $Sbab'a'\ldots$, $b \neq \tilde{b}$. The claim is that $R(G/S) = I \oplus M$. Now, every element $\lambda \in I$ is of the form $\lambda = \sigma_1\alpha + \sigma_2\beta$, $\alpha,\beta \in RG$, where the support of α consists of words $bab'a'\ldots$, and the support of β consists of words $aba'b'\ldots$. Considering an element of maximum length in the union of the supports of α and β shows that the support of $\lambda = \sigma_1\alpha + \sigma_2\beta$ contains either an element of the form $S\tilde{a}ba'b'\ldots$ or an element of the form $S\tilde{b}aba'\ldots$. In particular $\lambda \notin M$ unless $\lambda = 0$, i.e. $I \cap M = 0$.

It remains to show that $I + M = R(G/S)$. By induction on the length of $Sw \in G/S$ we prove that $Sw \in I + M$ for all $w \in G$. By definition $S \in M$. Now, let $\ell(Sw) \geq 1$, w of the form $aba'b'\ldots$ or $bab'a'\ldots$. If the initial letter of w is neither $\tilde{a}$ nor $\tilde{b}$ then $Sw \in M$; otherwise

$$Sw = S\tilde{a}w' = \sigma_1 w' - Sw' - \sum_{1\neq a\neq\tilde{a}} Saw',$$

say. By induction $Sw' \in I + M$, and clearly $\sigma_1 w' \in I$, $Saw' \in M$ for $1 \neq a \neq \tilde{a}$. Thus $Sw \in I + M$. □

<u>Remark.</u> Notice that the cokernel M is a free R-module of infinite rank unless $|G_1;S| = |G_2: S| = 2$, in which case $M \cong R$.

<u>Corollary 6.5.</u> Let $G = G_1 *_S G_2$ be an amalgamated product of groups of type $(FP)_\infty$ over R, with amalgamated subgroup S of finite index $\neq 1$ in both factors G_1 and G_2. Then the

Mayer-Vietoris sequences (cf Thm.2.10) for RG-flat coefficient modules A or RG-injective coefficient modules B, respectively, decompose into short exact sequences

$$0 \to H^k(G_1; A) \oplus H^k(G_2; A) \to H^k(S; A) \to H^{k+1}(G; A) \to 0$$

$$0 \to H_{k+1}(G; B) \to H_k(S; B) \to H_k(G_1; B) \oplus H_k(G_2; B) \to 0$$

for all $k \in \mathbf{Z}$. Moreover,

one has $cd_R G = cd_R G_i + 1$ (and $hd_R G = hd_R G_i + 1$).

Proof. Theorem 6.3 asserts the existence of short exact sequences

$$0 \to H^k(G_1; RG) \oplus H^k(G_2; RG) \to H^k(S; RG) \to H^{k+1}(G ; RG) \to 0$$

for all $k \in \mathbf{Z}$. Applying the exact functors $(- \otimes_{RG} A)$ and $Hom_{RG}(-, B)$ respectively, and noticing that one has natural isomorphisms such as e.g. $H^k(S; RG) \otimes_{RG} A \simeq H^k(S; RS) \otimes_{RS} A \simeq H^k(S; A)$ and its dual $Hom_{RG}(H^k(S, RG), B) \simeq H_k(S; B)$ (cf.Prop.5.3), yields the required short exact sequences. If $cd_R S = n < \infty$, then, by Theorem 5.11, $cd_R G_1 = cd_R G_2 = n$, and we have the short exact sequence

$$0 \to H^n(G_1; RG) \oplus H^n(G_2; RG) \to H^n(S; RG) \to H^{n+1}(G; RG) \to 0.$$

Identifying $C = H^n(S; RS)$ with $H^n(G_i; RG_i)$ via the canonical map v of Proposition 5.7 yields the short exact sequence

$$0 \to C \otimes_R (R(G/G_1) \oplus R(G/G_2)) \to C \otimes_R R(G/S) \to H^{n+1}(G; RG) \to 0.$$

Since S is of type (FP), $C \neq 0$; and since $R(G/G_1) \oplus R(G/G_2) \to R(G/S)$ is a <u>split</u> monomorphism (Lemma 6.4) with non-trivial cokernel, this implies $H^{n+1}(G; RG) \neq 0$. □

6.3. The finite index case II: HNN-groups

Theorem 6.6. Let $G = G_1 *_{S,\sigma}$ be the HNN-group over the base group G_1 with associated subgroups S and $T = \sigma(S)$ both of finite index in G. Assume that G_1 (and hence S, T, and G) is of type $(FP)_n$ over R. Then, for every $k \leq n$, the map

$$(res_S^* - c_p^* \; res_T^*) \; : \; H^k(G_1; RG) \to H^k(S; RG)$$

is an R-split monomorphism. Its cokernel is trivial if and only if $H^k(G_1; RG) = H^k(S; RG) = 0$. Hereby c_p^* denotes the isomorphism $H^k(T; RG) \to H^k(S; RG)$ induced by conjugation with the stable letter p.

Proof. Let C be the right RG_1-module $H^k(G_1; RG_1)$ and identify it with $H^k(S; RS)$ and $H^k(T; RT)$ via the canonical map Ψ of Proposition 5.7. By Proposition 5.3 one has isomorphisms

$$H^k(G_1; RG) \simeq C \otimes_{RG_1} RG \qquad H^k(S; RG) \simeq C \otimes_{RS} RG \; ,$$

and by Theorem 5.9 we know that res_S^* and res_T^* can be replaced by the corresponding transfer homomorphism in the tensor product. The homomorphism $c_p^* : H^k(T; RG) \to H^k(S; RG)$ must be replaced by the composite map

$$c_p: \quad C \otimes_{RT} RG \xrightarrow{v^{-1} \otimes RG} H^k(T;\, RT) \otimes_{RT} RG$$

$$\Big\downarrow \gamma_p \otimes p$$

$$H^k(S;\, RS) \otimes_{RS} RG \xrightarrow{v \otimes RG} C \otimes_{RS} RG \,,$$

γ_p being given by $\gamma_p(f)(d) = p^{-1} f(pd)p$, $f \in \mathrm{Hom}_{RT}(\underline{P}, RT)$, $d \in \underline{P}$, where $\underline{P} \twoheadrightarrow R$ is an RG-projective resolution.

Let Γ_1 and Γ_2 be right transversals (both including 1) of G_1 mod S and G_1 mod T, respectively. Then the transfer res_S: $C \otimes_{RG_1} RG \to C \otimes_{RS} RG$ is given by

$$\mathrm{res}_S(c \otimes x) = \sum_{a \in \Gamma_1} ca^{-1} \otimes ax, \quad c \in C,\ x \in G.$$

I have no nice description of $(c_p \circ \mathrm{res}_T)$: $C \otimes_{RG_1} RG \to C \otimes_{RS} RG$, but one has obviously

$$c_p \circ \mathrm{res}_T(c \otimes x) = \sum_{b \in \Gamma_2} c_b \otimes p^{-1} bx, \quad c \in C, \quad x \in G,$$

where $c \to c_b$ defines an R-automorphism for all $b \in \Gamma_2$.

First we prove that the map $\Delta = \mathrm{res}_S - c_p \circ \mathrm{res}_T$ is a monomorphism. By the (right version of the) Normal Form Theorem for HNN-groups (§ 2.5), an element $t \in C \otimes_{RG_1} RG$ is a finite sum of the form $t = \sum d_w \otimes w$, where w runs through all elements in G of the form $p^{n_1} x_1 p^{n_2} x_2 \ldots p^{n_r} x_r$ with $x_i \in \Gamma_1$ if $n_i > 0$ and $x_i \in \Gamma_2$ if $n_i < 0$ and with $x_i \neq 1$ except possibly for $i = r$. Now, let $\mathrm{res}_S(t) = c_p \mathrm{res}_T(t)$, i.e.,

$$(*) \qquad \sum_w \sum_{a \in \Gamma_1} d_w\, a^{-1} \otimes aw = \sum_w \sum_{b \in \Gamma_2} d_{w,b} \otimes p^{-1} bw \;.$$

Let $\bar{w} = p^{n_1}x_1p^{n_2}x_2\dots p^{n_r}x_r$ be a word of maximum length $\ell(\bar{w}) = \sum|n_i|$ in the support of t. If $n_1 < 0$ then the word $p^{-1}\bar{w}$ which occurs in the right hand side of $(*)$ has length $\ell(\bar{w})+1$ and hence cannot cancel against anything else. It follows that all elements of maximum length have $n_1 > 0$. But in this case $\bar{w}$ occurring on the left hand side cannot cancel there and cannot cancel against an element of length $\ell(\bar{w})$ in the right hand side either, since all of those have $n_1 < 0$. It follows that there are no elements of maximum length in the support of t, i.e., $t = 0$.

Now, let I be the image of $\Delta = \mathrm{res}_S - c_p \circ \mathrm{res}_T$ in $C \otimes_{RS} RG$. We claim that I is a direct summand as an R-module. To see this we distinguish two cases. First case: either $S \neq G_1$, or $T \neq G_1$. As the situation is symmetric we may assume that $S \neq G_1$ and pick a fixed representative $1 \neq \tilde{a} \in \Gamma_1$. Let M denote the R-submodule of $C \otimes_{RS} RG$ generated by all elements of the form $c \otimes aw$, $c \in C$, $a \neq \tilde{a} \in \Gamma_1$, $w = p^{n_1}x_1p^{n_2}x_2\dots p^{n_r}x_r$ $(0 \neq n_i \in \mathbb{Z}$, $1 \neq x_i \in \Gamma_1$ if $n_i > 0$, $1 \neq x_i \in \Gamma_2$ if $n_i < 0)$. Considering elements of maximal length $\ell(w)$ in the support of $t = \sum_w c_w \otimes w \in C \otimes_{RG_1} RG$ shows that $\Delta(t)$ involves always a summand of the form $c' \otimes \tilde{a}w'$ which does not cancel, whence $I \cap M = 0$. Next we use induction on $\ell(w)$ to prove that $c \otimes aw \in I + M$ for all $c \in C$, $a \in \Gamma$, w as above. By definition $c \otimes aw \in M$ for $a \neq \tilde{a}$, and one has

$$c \otimes \tilde{a}w = \Delta(c\tilde{a} \otimes w) - \sum_{\substack{a \neq \tilde{a} \\ a \in \Gamma_1}} c\tilde{a}a^{-1} \otimes aw + \sum_{b \in \Gamma_2} c_b \otimes p^{-1}bw.$$

Notice that if $b \neq 1$ or if $n_1 < 0$ then $c_b \otimes p^{-1}b\, w \in M$; If $b = 1$ and $n_1 > 0$ then $\ell(p^{-1}bw) < \ell(w)$, hence $c_b \otimes p^{-1}bw \in I + M$ by induction (the case $\ell(w) = 0$ follows with the same argument from $c \otimes 1 \in M$). Thus $C \otimes_{RS} RG = I \oplus M$; and clearly $M \neq 0$, unless $C = 0$. Second case: $S = T = G_1$. Then G is the split extension of G_1 by an infinite cycle generated by p. The map $\Delta: C \otimes_{RG_1} RG \to C \otimes_{RG_1} RG$ is given by $\Delta(c \otimes p^n) = c \otimes (1-p)p^n = c \otimes p^n - c \otimes p^{n+1}$, $c \in C$, $n \in \mathbf{Z}$. Let M be the R-submodule of $C \otimes_{RG_1} RG$ generated by $c \otimes 1$. Clearly $I \cap M = 0$ and induction on n shows that $c \otimes p^n \in I + M$ for all $c \in C$ and $n \in \mathbf{Z}$, whence $C \otimes_{RS} RG = I \oplus M$. Notice that $M \cong C$. This completes the proof of Theorem 6.6. □

<u>Remark</u> Notice that the cokernel M is isomorphic to the direct sum of $\aleph_0$ copies of C, unless $S = T = G$, in which case $M \cong C$.

<u>Corollary 6.7</u>. Let $G = G_1 *_{S,\sigma}$ be an HNN-group with base group G_1 of type $(FP)_\infty$ over R, and with associated subgroups S and $T = \sigma(S)$ of finite index in G_1. Then the Mayer-Vietoris sequences (cf.Thm.2.12) for RG-flat coefficient modules A or RG-injective coefficient modules B, respectively, decompose into short exact sequences

$$0 \to H^k(G_1; A) \to H^k(S; A) \to H^{k+1}(G; A) \to 0$$

$$0 \to H_{k+1}(G; B) \to H_k(S; B) \to H_k(G_1; B) \to 0$$

for all $k \in \mathbb{Z}$. Moreover, one has

$cd_R G = cd_R G_1 + 1$ (and $hd_R G = hd_R G_1 + 1$).

Proof. Strictly analogous to the proof of Corollary 6.5. □

Exercise. Generalize Corollaries 6.5 and 6.7 to the fundamental group of a finite graph of groups of type $(FP)_\infty$ over R, (cf. Section 7.1)

7. Low dimensions and solvable groups

7.1. Cohomology dimension 1. We have seen that the groups G with $cd_R G = 0$ are just all finite groups without R-torsion. The problem of classifying all groups G with $cd_R G \leq 1$ is still open, but it is solved in the torsion-free case by Stallings and Swan and much is known in the general case.

Let G be a group. If G can be written as $G = G_1 *_S G_2$, $G_1 \neq S \neq G_2$, with $|S| < \infty$, then we say that G has an α-decomposition. If G can be written as $G = G_1 *_{S,\sigma}$, again with $|S| < \infty$, then we say that G has a β-decomposition. The fundamental result which made the breakthrough possible was proved by Stallings [55] in 1968. A slightly more general version of it is

Theorem 7.1 Let G be a finitely generated group with $H^1(G; RG) \neq 0$. Then G has an α-decomposition or a β-decomposition.

For a proof see e.g. [60]. A group G is called 0-accessible (or $\alpha\beta$-indecomposable) if it has no α-decomposition and no β-decomposition. G is called n-accessible (n a positive integer) if G has an α-decomposition $G = G_1 *_S G_2$ or a β-decomposition $G = G_1 *_{S,\sigma}$ with G_1, G_2 $(n-1)$-accessible. The factors and base groups occurring in an iterated $\alpha\beta$-decomposition of G are called the subfactors of G. Accessible means n-accessible for some n.

Lemma 7.2. Every finitely generated torsion-free group is accessible.

Proof. Since G is torsion-free both α- and β-decompositions are ordinary free product decompositions of G. Let d(G) be the minimal number of generators of G. If $G = G_1 * G_2$ one has by Gruško's Theorem (see e.g. [18]) $d(G) = d(G_1) + d(G_2)$, and therefore d(G) is a bound for the number of free factors of G. □

It is an open question whether Lemma 7.2 holds without the assumption that G is torsion-free. A very nice criterion for accessibility of almost finitely presented groups has recently been obtained by Bamford and Dunwoody [1]:

Theorem 7.3. Let G be an almost finitely presented group. Then G is accessible if and only if $H^1(G; \mathbb{Z}G)$ is finitely generated as a G-module.

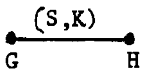
Remark. Bamford-Dunwoody's proof of Theorem 7.3 does not apply for arbitrary rings R. It would be interesting, in particular, to know whether the result holds e.g. for $R = \mathbb{Q}$. (See Appendix 7.)

For the main result it is convenient to introduce Serre's concept of the fundamental group of a graph of groups. A graph of groups is a graph $\mathcal{G}$ of the following kind. The set of vertices of $\mathcal{G}$, $V(\mathcal{G})$, is a non-empty set of groups; and an edge between two vertices

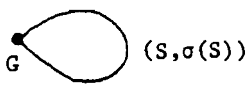
$G, G' \in V(\mathfrak{G})$ is a pair of isomorphic subgroups (S, S'), $G \geq S \cong S' \leq G'$.

If $\mathfrak{G}$ is such a graph of groups, we choose a maximal tree $\mathfrak{T}$ in $\mathfrak{G}$ and denote by $G(\mathfrak{T})$ the tree product with respect to $\mathfrak{T}$ (= successive amalgamated products along $\mathfrak{T}$). For every edge of $\mathfrak{G} - \mathfrak{T}$, $G(\mathfrak{T})$ contains a pair of isomorphic subgroups, so that we can extend $G(\mathfrak{T})$ by HNN-extensions for each e of $\mathfrak{G} - \mathfrak{T}$. One can show that the group $G(\mathfrak{G})$ we obtain by doing so does not depend upon the choice of the maximal tree $\mathfrak{T}$. $G(\mathfrak{G})$ is called the fundamental group of the graph of groups $\mathfrak{G}$.

<u>Examples</u>

1. $\mathfrak{G}$: 

, $G(\mathfrak{G}) = G *_{S=K} H$.

2. $\mathfrak{G}$: 

, $G(\mathfrak{G}) = G *_{S,\sigma}$

The following Lemma is left as an exercise (use subgroup theorems for α- and β-decompositions). Notice that by a finite graph we mean a graph with a finite number of edges and vertices.

<u>Lemma 7.4</u> The following statements are equivalent for a group G:

(i) G is accessible,

(ii) G is the fundamental group of a finite graph of groups $\mathfrak{G}$ with αβ-indecomposable vertices and finite edges.

Moreover, if (i) and (ii) hold for G, then the vertices of $\mathcal{G}$ are precisely the $\alpha\beta$-indecomposable subfactors of G which are unique up to isomorphism.

Corollary 7.5. Let G be a finitely generated accessible group. Then $cd_R G \leq 1$ if and only if G is the fundamental group of a finite graph of groups whose vertices are finite groups without R-torsion.

Proof. If G is the fundamental group of a finite graph of finite groups without R-torsion, then it follows readily from the Mayer-Vietoris sequences that $cd_R G \leq 1$. Conversely, assume that G is finitely generated with $cd_R G \leq 1$, and let K be an $\alpha\beta$-indecomposable subfactor of G. Clearly $cd_R K \leq 1$, and by (iterated application of) Proposition 2.13 K is again finitely generated. Now, assume $cd_R K = 1$; as K is of type (FP) over R this implies $H^1(K; RK) \neq 0$ which is a contradiction by Theorem 7.1. Therefore $cd_R K = 0$, i.e., K is finite without R-torsion. Thus the corollary follows from Lemma 7.4. □

It follows, in particular, that if G is a finitely generated torsion-free group with $cd_R G \leq 1$ then G is the fundamental group of a finite graph of groups with trivial vertices and hence G is free. Swan [60] has extended this result to infinitely generated groups, so that one has quite generally

Theorem 7.6 (Stallings [55] - Swan [60]). Let G be a torsion-free group. Then $cd_R G \leq 1$ if and only if G is free.

Remark One can show that a group G is the fundamental group of a finite graph of finite groups if and only if G contains a finitely generated free subgroup of finite index.

7.2. Cohomology dimension 2. The problem of classifying all groups of cohomology dimension ≤ 2 is still wide open. The best known examples are perhaps the groups with only one defining relation.

Theorem 7.7. Let G be a subgroup without R-torsion of a 1-relator group. Then $cd_R G \leq 2$.

Proof. (Sketch) Let $G \leq G_1$ where G_1 is a group with one defining relator of length $\ell(r)$. As usual we make an induction on $\ell(r)$. If $\ell(r) = 1$ then G_1 is free and the result is trivial. So let $\ell(r) \geq 2$. Now, if r involves only one generator of G_1, then G_1 is (finite cyclic)*free and the result is again obvious; otherwise G_1 can be embedded in a group of the form $G_2 = G_1 *_{\langle u^m \rangle} \langle u \rangle$ which is a 1-relator group with the property that one of its generators (namely u) has zero exponent sum in the relation of G_2 (see [42]p.265). It follows that G_2 is an HNN-group over a base group G_3 with free associated subgroups F, $\sigma(F)$, whereby G_3 is a 1-relator group with relator of length $< \ell(r)$. By the subgroup theorem for HNN-groups (cf. [19a] or [38]) it follows that G is the fundamental group of a graph of groups with vertices of the form $G \cap G_3^x$ and edges of the form $G \cap F^y$. By induction $cd_R(G \cap G_3^x) \leq 2$, and since the groups $G \cap F^y$ are free one can conclude from Chiswell's Mayer-Vietoris sequence [17] that $cd_R G \leq 2$. □

Notice that Theorem 7.7 contains Lyndon's result [40] that torsion-free 1-relator groups are of cohomology dimension ≤ 2. A further class of groups with cohomology dimension ≤ 2 (over $\mathbb{Z}$) arises in Topology. Let k be a tame knot (= diffeomorphic image of S^1 in S^3); the fundamental group $G = \pi_1(S^3 - k)$ is called "<u>the group of the knot</u> k". One can show that $G \simeq \mathbb{Z}$ if and only if the knot is trivial (i.e. unknotable). Now, Papakyriakopulos [48] has obtained the very deep result that the space $X = S^3 - k$ is aspherical (i.e., $\pi_i(X) = 0$ for all $i > 1$) for every non-trivial knot k. Thus X is an Eilenberg-MacLane space; since X has obviously the homotopy type of a compact 3-dimensional manifold with non-empty boundary this implies:

<u>Theorem 7.8</u>. If G is the fundamental group of a non-trivial knot then $cdG = 2$.

<u>Remarks</u>. 1) It is conceivable but unknown that all knot groups have actually a one relator presentation.

2) It should be mentioned that the class of all groups G with $cdG \leq 2$ is very much larger than the class of all (subgroups of) knot groups or one relator groups, due to the fact that it is closed with respect to free products with <u>free</u> amalgamations. In fact, one has simple examples such as the direct product of two non-Abelian free groups which are of cohomology dimension 2 but are neither subgroups of one relator groups nor of knot groups. Finally, notice that by Theorem 4.6(b) every countable locally free group has cohomology dimension ≤ 2.

7.3. Solvable and nilpotent groups. A solvable group G has a finite series of subgroups

$$G = G_o \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_{n-1} \trianglerighteq G_n = 1$$

with Abelian quotient groups $A_k = G_{k-1}/G_k$, $1 \leq k \leq n$. Let h_k be the torsion-free rank of A_k, $h_k = \dim_{\mathbb{Q}} (A_k \otimes \mathbb{Q})$ and define the Hirsch number hG to be the sum $hG = h_1 + h_2 + \ldots + h_n$. From Schreier's refinement Theorem it follows readily that hG does not depend upon the choice of the series, i.e., hG which is either a non-negative integer or ∞ is an invariant of G.

Lemma 7.9. Every torsion-free solvable group G with finite Hirsch number is countable.

Proof. In order to make an induction on hG we prove the somewhat stronger result

(*) every solvable group G of finite Hirsch number without non-trivial periodic normal subgroups is countable.

If $hG = 0$, then G is periodic hence $G = 1$. So assume $0 < hG < \infty$. Let A be a maximal Abelian normal subgroup in G. The torsion-subgroup of A is characteristic in A and hence normal in G and therefore trivial, i.e., A is torsion-free. As the automorphism group of A embeds into $GL(n,\mathbb{Q})$ with $n = hA$, we have a homomorphism $G \to GL(n,\mathbb{Q})$ whose image is certainly countable and whose kernel is the centralizer $C = C_G(A)$. C contains A which is countable, hence it remains to prove that C/A is countable. Clearly $h(C/A) \leq h(G/A) < hG$; thus by induction hypothesis it is sufficient to prove that C/A contains no non-trivial periodic normal subgroups.

Let K/A be the maximal periodic normal subgroup of C/A, and let S/A be the last non-trivial term in the derived series of K/A. Then $A \rightarrowtail S \twoheadrightarrow S/A$ is a central extension with S/A locally finite and, therefore, by Schur's Theorem, $[S,S]$ is locally finite. But $[S,S] \leq A$, hence $[S,S] = 1$ and S is Abelian. Since S is characteristic in K and K is characteristic in C and C is normal in G, it follows that $S \trianglelefteq G$, whence $S = A$ by maximality of A. This completes the proof.□

<u>Theorem 7.10</u>. Let G be a torsion-free solvable group. Then one has

a) $hdG = hG$

b) $hG \leq cdG \leq hG + 1$.

We split the proof into different steps; the first one being

<u>Proposition 7.11</u>. If G is a solvable group without R-torsion then $hd_R G \leq hG$.

<u>Proof</u>. Induction on $hG = n$. If $n = 0$, then G is locally finite and hence $hd_R G = 0$. So assume $n \geq 1$. By Corollary 4.10 there is a finitely generated subgroup $S \leq G$ with $hd_R S = hd_R G = n$.

Consider the derived series

$$S = S^{(0)} > S^{(1)} > \ldots > S^{(d)} = 1,$$

and let k be the least integer with the property that $S^{(k)}/S^{(k+1)}$ is infinite. Then $S^{(k)}$ is of finite index in S and hence still finitely generated, so that one can find a subgroup $K < S$, $S^{(k)} \triangleright K \trianglerighteq S^{(k+1)}$, with $S^{(k)}/K \simeq \mathbb{Z}$. By induction one has $hd_R S^{(k)} \leq hd_R K + 1 \leq hK + 1 = hS^{(k)}$. Since $hd_R S = hd_R S^{(k)}$ by Theorem 5.13 it follows $hd_R G = hd_R S \leq hS \leq hG$. □

<u>Proposition 7.12</u>. Let G be a torsion-free nilpotent group of finite Hirsh number $hG = n < \infty$. Then $H_n(G; \mathbb{Z})$ is isomorphic to a subgroup of the additive group of all rational numbers $\mathbb{Q}$. Moreover, $H_n(G; \mathbb{Z})$ is cyclic if and only if G is finitely generated (and hence polycyclic).

<u>Proof</u>. The upper central series of G has torsion-free quotients, hence there is a refined central series

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = 1$$

with all quotients G_k/G_{k+1} torsion-free Abelian of rank 1. We claim one has

$$(*) \quad H_n(G; \mathbb{Z}) = G/G_1 \otimes G_1/G_2 \otimes \ldots \otimes G_{n-1}.$$

The proof goes by induction on n. If $n=1$, then G itself is a subgroup of $\mathbb{Q}$ and $H_1(G; \mathbb{Z}) = G$. For $n \geq 2$ we consider the Lyndon-Hochschild-Serre (LHS) spectral sequence for $G_{n-1} \triangleleft G \twoheadrightarrow G/G_{n-1}$.

The usual corner argument yields an isomorphism

$$H_n(G;\mathbb{Z}) \cong H_{n-1}(G/G_{n-1};\ H_1(G_{n-1};\mathbb{Z}))$$

$$\cong H_{n-1}(G/G_{n-1};\ G_{n-1}).$$

As G_{n-1} is central, its G-module structure is trivial, hence we may apply the Universal Coefficients Theorem

$$H_n(G;\mathbb{Z}) \cong H_{n-1}(G/G_{n-1};\mathbb{Z}) \otimes G_{n-1},$$

and the induction hypothesis on G/G_{n-1} yields $(*)$. It follows that $H_n(G;\mathbb{Z})$ is cyclic if and only if G is polycyclic. □

<u>Proposition 7.13</u>. (U.Stammbach [57]) Let G be a solvable group of finite Hirsch number $hG = n$. Then there is a G-module A whose underlying Abelian group is the additive group $\mathbb{Q}$, with $H_n(G;A) \cong \mathbb{Q}$.

<u>Proof</u>. Consider the derived series $G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \ldots \triangleright G^{(d)} = 1$, and let $S_i = G^{(i-1)}/G^{(i)}$, $hS_i = n_i$. Let $L_i = H_{n_i}(S_i;\mathbb{Q})$ with G-module structure induced by conjugation. If T_i denotes the torsion subgroup of S_i then clearly $H_k(S_i;\mathbb{Q}) \cong H_k(S_i/T_i;\mathbb{Q}) \cong H_k(S_i/T_i;\mathbb{Z}) \otimes \mathbb{Q}$ for all $k \in \mathbb{Z}$, so that it follows from Proposition 7.12 that the underlying Abelian group of L_i is $\cong \mathbb{Q}$. Define <u>inverse</u> action on L_i by $x \circ k = x^{-1}k$, $x \in G$, $k \in L_i$ and let L_i^{op} be the additive group of L_i with this inverse G-action. Now we put $A = L_1^{op} \otimes L_2^{op} \otimes \ldots \otimes L_d^{op}$, with diagonal G-action and claim that $H_n(G;A) \cong \mathbb{Q}$.

We prove this by induction on d. If $d = 1$ then A is the trivial G-module $\mathbb{Q}$ and $H_n(G; A) = \mathbb{Q}$ as above. So let $d \geq 2$. The usual corner argument in the LHS-spectral sequence for $G^{(d-1)} \rightarrowtail G \twoheadrightarrow G/G^{(d-1)}$ yields

$$H_n(G; A) \cong H_{n-n_d}(G/G^{(d-1)}; H_{n_d}(G^{(d-1)}; A)).$$

$G^{(d-1)}$ acts trivially on A so that we can apply the universal-coefficients Theorem:

$$H_{n_d}(G^{(d-1)}; A) \cong H_{n_d}(G^{(d-1)}; \mathbb{Q}) \otimes A \cong L_d \otimes L_1^{op} \otimes \ldots \otimes L_d^{op}.$$

But $L_d \otimes L_d^{op} \cong \mathbb{Q}$ with trivial G-action, so that

$$H_{n_d}(G^{(d-1)}; A) \cong L_1^{op} \otimes L_2^{op} \otimes \ldots \otimes L_{d-1}^{op},$$

and the assertion follows by the inductive hypothesis. □

Proof (of Theorem 7.10). Proposition 7.11 together with Proposition 7.13 yields $hG \leq hd_{\mathbb{Q}} G \leq hdG \leq hG$ (provided G is torsion-free), hence $hdG = hG$. The cohomology statement follows from this by Theorem 4.6, because we know, by Lemma 7.9, that $hG < \infty$ implies G countable. □

Remark. By a result of Merzljakov [42a] a torsion-free locally solvable group with bounded Abelian subgroup rank is, in fact, solvable. This answers the question how to extend Theorem 7.10 to the locally solvable case: Locally solvable groups of finite (co)homology dimension are solvable.

7.4 Solvable and nilpotent groups (continuation). In the remainder of Section 7 we shall try to get more precise information on the cohomology part of Theorem 7.10. Let $\mathfrak{C}$ denote the class of all torsion-free solvable groups with $cdG = hG < \infty$.* By Theorem 4.6 (c) $\mathfrak{C}$ contains all solvable groups of type (FP), and hence in particular all torsion-free polycyclic groups. The question whether $cdG = hG < \infty$ implies that G is of type (FP) is still open in general, but it is known in the nilpotent case.

Theorem 7.14. (K.W.Gruenberg [30; § 8.8]) Let G be a torsion-free nilpotent group with finite Hirsch number. Then $cdG = hG$ if and only if G is finitely generated (and hence polycyclic).

Proof. It remains to prove that $n = cdG = hG < \infty$ implies G finitely generated. Now, by the Universal-Coefficients-Theorem one has for all Abelian groups L

$$0 = H^{n+1}(G; L) \cong \mathrm{Ext}(H_n(G;\mathbb{Z}), L),$$

hence $H_n(G; \mathbb{Z})$ must be free-Abelian. By Proposition 7.12 we conclude that $H_n(G; \mathbb{Z})$ is infinite cyclic and hence G finitely generated.□

One can try to extend the idea in the proof of Theorem 7.14 to the solvable case by using the following structure theorem: Let G be a torsion-free soluble group whose Abelian subgroups are all of finite rank (e.g. $hG < \infty$). Then G has a unique maximal nilpotent normal subgroup $N \trianglelefteq G$ (The Hirsch-Plotkin-radical) and the quotient

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* See Appendix 6.

G/N contains a finitely generated free Abelian group of finite index. (cf. [15a] or [O]). With regard to Theorem 5.11, we thus can restrict ourselves to extensions $N \rightarrowtail G \twoheadrightarrow Q$, where N is torsion-free nilpotent with $hN = n < \infty$ and Q is free-Abelian of rank $r < \infty$.

We need a few elementary remarks on the subgroups of the additive group $\mathbb{Q}$ of all rational numbers. Consider all formal products Πp^{α_p}, $0 \le \alpha_p \le \infty$, where p runs through all primes > 0. Call two such formal products equivalent if they coincide up to a finite number of finite exponents α_p, and let $[\Pi p^{\alpha_p}]$ denote the equivalence class of Πp^{α_p}. Let $S \le \mathbb{Q}$ and without loss of generality assume $1 \in S$; then $L^* = \Pi p^{\alpha_p}$, with $\alpha_p = \sup\{k \mid p^{-k} \in S\}$, is called "<u>the type of</u> S". The mapping $L \mapsto L^*$ defines a bijective correspondence between the isomorphism classes of subgroups of $\mathbb{Q}$ and the equivalence classes of formal products. Note furthermore that every automorphism $\phi: L \to L$ is given by $\phi(1) = \frac{b}{a}$, a and b coprime integers. Let $\pi(\phi)$ denote the set of prime divisors of ab; if $A \le \mathrm{Aut}\,(S)$ write $\pi(A) = \bigcup_{\phi \in A} \pi(\phi)$. Notice that $\pi(\mathrm{Aut}(L))$ is the set of all primes p with exponent $\alpha_p = \infty$ in $L^* = [\Pi p^{\alpha_p}]$.

<u>Theorem 7.15</u>. Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups. Assume that N is a torsion-free nilpotent group of finite Hirsch number $hN = n$ and that Q is a free Abelian group of finite rank r. Then the following statements are equivalent:

(i) $H^{r+n+1}(G; A) = 0$ for all Q- modules A

(ii) $H_n(N;\mathbb{Z})$ is a cyclic Q- module

(iii) $H_n(N;\mathbb{Z}) = L \leq \mathbb{Q}$ is of type $[p_1^{\infty} p_2^{\infty} \cdots p_s^{\infty}]$,

$0 \leq s < \infty$ and $\pi(\mathrm{im}(G \to \mathrm{Aut}\, L)) = \{p_1, p_2, \ldots, p_s\}$.

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) is elementary and left as an exercise.

(i) $\Rightarrow$ (iii) We assume that (iii) is false and consider 2 cases.

a) L is of type $[\Pi p^{\alpha_p}]$, where $0 < \alpha_p < \infty$ for an infinite set P of primes p. We claim that in this case $\mathrm{Ext}\,(L, L) \neq 0$. To see this consider the short exact sequence $L \rightarrowtail \mathbb{Q} \overset{\sigma}{\twoheadrightarrow} \mathbb{Q}/L$ and its induced sequence

$$\mathrm{Hom}(L,\mathbb{Q}) \to \mathrm{Hom}(L,\mathbb{Q}/L) \to \mathrm{Ext}\,(L,L) \to 0.$$

Let $\phi\colon L \to \mathbb{Q}/L$ be the homomorphism given by $\phi(x) = \sum_{p\in P} xp^{-1} + L$, $x \in L$. Notice that this is well defined since only finitely many $xp^{-1} \notin L$. For all $p \in P$ one has $\phi(p^{-\alpha_p}) = p^{-1-\alpha_p} + L \neq L$. If there were a map $\psi\colon L \to \mathbb{Q}$ with $\sigma\psi = \phi$ then one would have $p^{-\alpha_p}\psi(1) - p^{-1-\alpha_p} \in L$ for all $p \in P$. Let $\psi(1) = \frac{a}{b}$, $(a,b) = 1$. It follows that the denominator of $\frac{a}{b} - p^{-1} = \frac{ap-b}{bp}$ is prime to p, hence p/b; but this is impossible for infinitely many primes!

Now, the Universal Coefficients Theorem for N together with the usual corner argument in the Lyndon-Hochschild spectral sequence yields

$$H^{n+r+1}(G; L) \cong H^r(Q; H^{n+1}(N; L))$$

$$\cong H^r(Q; \mathrm{Ext}\,(L,L)).$$

An element $x \in Q$ acts on L by multiplication with $r_x \in \mathbb{Q}$. The induced action on $\mathrm{Ext}(L, L)$ is given by $\mathrm{Ext}(r_x^{-1}, r_x) = r_x^{-1} r_x \mathrm{Ext}(\mathrm{Id},\mathrm{Id}) = \mathrm{Ext}(\mathrm{Id},\mathrm{Id})$, i.e. $\mathrm{Ext}(L, L)$ is a trivial Q-module; thus by Proposition 6.11 we get

$$H^{n+r+1}(G; L) \cong H^r(Q;\mathbb{Z}) \otimes \mathrm{Ext}(L, L) \cong \mathrm{Ext}(L\ L) \neq 0.$$

b) L is of type $[p_1^\infty p_2^\infty \dots p_s^\infty]$, $0 < s \leq \infty$, and one of the primes $p_i = q$ does not lie in $\pi(\mathrm{im}(Q \to \mathrm{Aut}\, L))$. Let $\mathbb{Q}_q$ be the additive group of all rational numbers with denominator prime to q ($= \mathbb{Z}$ localized at q). We claim that $\mathrm{Ext}(L,\mathbb{Q}_q) \neq 0$. To see this consider the short exact sequences of Abelian groups

$$\mathbb{Z} \rightarrowtail L \twoheadrightarrow \bigoplus_i \mathbb{Z}(p_i^\infty) \qquad \mathbb{Q}_q \rightarrowtail \mathbb{Q} \twoheadrightarrow \mathbb{Z}(q^\infty)$$

where $\mathbb{Z}(p^\infty)$ denotes the quasicyclic p-group. These give rise to exact sequences

$$\mathbb{Q}_q \to \prod_i \mathrm{Ext}(\mathbb{Z}(p_i^\infty),\mathbb{Q}_q) \to \mathrm{Ext}(L,\mathbb{Q}_q) \to 0$$

$$0 \to \mathrm{Hom}(\mathbb{Z}(q^\infty), \mathbb{Z}(q^\infty)) \to \mathrm{Ext}(\mathbb{Z}(q^\infty),\mathbb{Q}_q) \to 0.$$

$\mathrm{End}(\mathbb{Z}(q^\infty))$ is isomorphic to the ring of q-adic integers and hence uncountable. As $p_i = q$ for some i and $\mathbb{Q}_q$ is countable this implies that $\mathrm{Ext}(L,\mathbb{Q}_q)$ is countable.

As $q \notin \pi(\mathrm{im}(Q \to \mathrm{Aut}\, L))$ one has an inclusion $\mathrm{im}(Q \to \mathrm{Aut}\, L) \subseteq \mathrm{Aut}(\mathbb{Q}_q)$, which defines a Q-module structure on $\mathbb{Q}_q$. With respect to this Q-action one proves in exactly the same way as in case a) that $H^{n+r+1}(G; \mathbb{Q}_q) \cong \mathrm{Ext}(L, \mathbb{Q}_q) \neq 0$.

(iii) $\Rightarrow$ (i). First we notice that one can restrict oneself to the case $Q \cong \mathbb{Z}$. For if (iii) holds one can find an element $x \in Q$ such that $Q/\langle x\rangle$ is free Abelian and $\pi(\mathrm{im}(x)) = \{p_1, p_2, \ldots, p_s\}$. So if we can prove that $H^{n+2}(\langle N, x\rangle; A) = 0$ for all $\langle x\rangle$-modules A then the spectral sequence argument yields also $H^{n+r+1}(G; A)$ for all Q-modules A.

So assume $Q = \langle x\rangle$ and let A be an arbitrary Q-module. The spectral sequence argument together with the universal coefficients theorem for N yields

$$H^{n+2}(G; A) \cong H^1(Q; H^{n+1}(N; A))$$

$$\cong \mathrm{Ext}(L,A)_Q$$

($H^1(Q; M) \cong M_Q$ for $Q \cong \mathbb{Z}$ is easily checked; cf. also chapter 7). Let x act on L by multiplication with $\frac{a}{b}$ where a and b are coprime integers. By assumption one has $\pi(ab) = \{p_1, p_2, \ldots, p_s\}$, hence L can be given in the form $L = \{\frac{m}{(ab)^i} \mid m \in \mathbb{Z},\ 0 \le i \in \mathbb{Z}\}$. We thus have the presentation

$$Y \overset{\alpha}{\rightarrowtail} X \overset{\beta}{\twoheadrightarrow} L$$

where X and Y are free-Abelian groups over free generators $\{x_i\}$ and $\{y_j\}$ respectively, $i,j = 1,2,3,\ldots,$ and α, β are given by $\beta(x_i) = (ab)^{-i}$, $\alpha(y_j) = abx_{j+1} - x_j$. We have to compute

$$\mathrm{Ext}(L,A)_x = \mathrm{Hom}(Y,A) \,/\, \alpha^*\mathrm{Hom}(X,A) + (x-1)\mathrm{Hom}(Y,A),$$

In order to compute the action of x on $\mathrm{Hom}(Y,A)$ we notice that one has the following commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \overset{\alpha}{\to} & X & \overset{\beta}{\to} & L & \to & 0 \\ & & \eta\downarrow & & \downarrow\xi & & \downarrow x & & \\ 0 & \to & Y & \underset{\alpha}{\to} & X & \underset{\beta}{\to} & L & \to & 0 \end{array},$$

$x(\ell) = \frac{a}{b}\,\ell,\quad (\ell \in L),\quad \xi(x_i) = a^2 x_{i+1},\quad \eta(y_j) = a^2 y_{j+1}$.

Hence one has $(xf)(y_j) = a^2 x\, f(y_{j+1})$ for all $f \in \mathrm{Hom}(Y,A)$. For arbitrary given elements $a_i \in A$, $i = 1,2,3,\ldots$. We now consider two homomorphisms $f \in \mathrm{Hom}(Y,A)$, $g \in \mathrm{Hom}(X,A)$ defined by

$$f(y_j) = \lambda^2 x^{-1} a_j + \lambda\mu\, a_j - \mu b a_{j+1}$$

$$g(x_i) = \mu^2 x\, a_i + \lambda\mu\, a_i + \mu a\, x\, a_{i+1},$$

where λ and μ are integers with the property that $\lambda a + \mu b = 1$.

A little computation shows that

$$((x-1)f + \alpha^* g)(y_j) = a_{j+1} - (\lambda + \mu x)^2\, x^{-1} a_j.$$

Now, if h is an arbitrary given homomorphism in $\mathrm{Hom}\,(Y,A)$ we choose the elements $a_j \in A$ to be

$$a_1 = 0,\quad a_{j+1} = h(y_j) + (\lambda + \mu x)^2\, x^{-1} a_j \qquad j \geq 1\,.$$

Then obviously $(x-1)f + \alpha^* g = h$. Thus we have proved $\mathrm{Hom}(Y,A) = (x-1)\mathrm{Hom}(Y,A) + \alpha^*\mathrm{Hom}(X,A)$, whence the result. This completes the proof of Theorem 7.15. □

A group G is called minimax if it admits a finite series of subgroups $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_r = 1$ whose quotients G_{i-1}/G_i, $1 \leq i \leq r$ satisfy either the minimal condition or the maximal condition on subgroups. Subgroups and quotient groups of minimax groups are again minimax and so are extensions of minimax groups by minimax groups. Notice that a subgroup of $\mathbb{Q}$ is minimax if and only if it is of type $[p_1^\infty p_2^\infty \dots p_s^\infty]$ with $0 \leq s < \infty$.

Corollary 7.16. Every torsion-free soluble group G with $cdG = hG < \infty$ is minimax.

Proof. Let N be the Hirsch-Plotkin radical of G and take a subgroup G_1, $N \trianglelefteq G_1 \trianglelefteq G$ such that G/G_1 is finite and G_1/N free-Abelian. Theorem 7.15 applies for G_1 and shows that $H_n(N;\mathbb{Z})$ is of type $[p_1^\infty p_2^\infty \dots p_s^\infty]$, $s < \infty$, $n = hN$. Now, formula (*) in the proof of Proposition 7.12 shows that N has a central series whose factors are subgroups of $H_n(N;\mathbb{Z})$. Thus N is minimax. As G/N is polycyclic we conclude that G is minimax. □

The problem of classifying all solvable groups G with $cdG \leq 2$ is still open.* The Abelian ones clearly are 1, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, and all non-cyclic subgroups of $\mathbb{Q}$. If G is non-Abelian then it is not too hard to see that G must be a semi direct product of a torsion-free Abelian group N of rank 1 by an infinite cycle generated by x, say; $G = N \rtimes \langle x \rangle$. It follows by Theorem 7.15 that N is of type $[p_1^\infty p_2^\infty \dots p_s^\infty]$ and that x acts on N by

* See Appendix 6.

multiplication with $\pm p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, $\mathbf{Z} \ni \alpha_i \neq 0 \quad i = 1,2,\dots,s$ (this includes the fundamental group of the Klein-bottle for $s = 0$). In particular it follows that G is finitely generated. Conversely let G be such a semi direct product. If all exponents α_i have the same sign then it is easily seen that G is in fact a 1-relator group and hence $cdG \leq 2$. If not all α_i's have the same sign, then, as shown by Baumslag-Strebel [2a], G is not of type $(FP)_2$ (in particular not finitely presented) and in this case it is not known whether $cdG = 2$ or $cdG = 3$.

Simplest explicit example: let N be the additive group of all rational numbers with denominator a power of 6, and let x act on N by multiplication with 2/3; $G = N \rtimes \langle x \rangle$. Is $cdG = 2$ or $= 3$?

8. Applications

Our aim in this Section is to use our knowledge on homological dimensions in order to obtain purely group theoretic results.

8.1 Normal subgroups of type (FP). Here the main idea is to combine Theorem 5.5. with Stalling's Theorem 7.1. The following simple observation will be crucial.

Lemma 8.1. Let G be a group and let A be a non-trivial induced RG-module. Then $H^0(G; A) \neq 0$ if and only if G is finite.

Proof. Let $A = L \otimes_R RG$, L an R-module. If $0 \neq \sum_{i=1}^{m} k_i \otimes x_i$ is an element in the G-invariant part of A ($k_i \in L$, $x_i \in G$) then xx_i must occur in $\{x_1, x_2, \ldots, x_m\}$ for all i and all x, hence $\{x_1, x_2, \ldots, x_m\} = G$. The converse is equally obvious. □

Now comes our first main result. The rather complicated looking assumptions on the cohomology of N shall be satisfied automatically in interesting special cases.

Theorem 8.2 Let G be a finitely generated group with $cd_R G = n < \infty$ and let N be a normal subgroup in G. Assume that N meets the following conditions

(i) N is of type (FP) over R,

(ii) $H^k(N; RN) = 0$ for $0 \leq k \leq n-2$,

(iii) $H^{n-1}(N; RN)$ is R-projective.

Then either $cd_R N = n-1$ or N is of finite index in G.

Proof. Since N is of type (FP), $H^k(N; RN) \neq 0$ for $k = cd_R N$, therefore we have either $cd_R N = n$ or $cd_R N = n-1$. Assuming that $cd_R N = n$ we have to show that $Q = G/N$ is finite. The first step is to observe that Q is periodic. Indeed, if x were an element of G which maps to an element of infinite order in Q one would find a short exact sequence $N \rightarrowtail S \twoheadrightarrow \mathbb{Z}$, $S = \langle N, x \rangle$. But Theorem 5.5 (with Remark 2) shows $cd_R S = n+1$, contradicting $S \leq G$.

By Proposition 5.1 there is a free RG-module $F = L \otimes_R RG$ (L a free R-module), with $H^n(G; F) \neq 0$. So we consider the LHS-spectral sequence

$$E_2^{p,q} = H^p(Q; H^q(N; F)) \Rightarrow H^{p+q}(G; F).$$

The iterated cohomology can be simplified: using Propositions 5.3, 5.4 and Lemma 5.6 we find RQ isomorphisms

$$\begin{aligned} H^q(N; F) &\cong H^q(N; RN)^{\swarrow} \otimes_{RN} (L \otimes_R \overset{\searrow}{RG}) \\ &\cong H^q(N; RN) \otimes_R L \otimes_R RQ^{\swarrow}, \end{aligned}$$

for all $q \in \mathbb{Z}$ (the arrows indicate the Q-action). Thus condition

(ii) implies $E_2^{p,q} = 0$ unless $q = n$ or $q = n-1$. Moreover, assumption (iii) together with the fact that Q is finitely generated allows to apply Proposition 5.3 again, whence

$$E_2^{1,n-1} \simeq H^1(Q; H^{n-1}(N;RN) \otimes_R L \otimes RQ)$$
$$\simeq H^1(Q; RQ) \otimes_R H^{n-1}(N; RN) \otimes_R L .$$

But we know that Q is periodic and hence, in particular, $\alpha\beta$-indecomposable. Thus Theorem 7.1 yields $H^1(Q; RQ) = 0$, whence $E_2^{1,n-1} = 0$, and hence the spectral sequence yields a monomorphism

$$0 \neq H^n(G; F) \simeq E_\infty^{0,n} \rightarrowtail E_2^{0,n} \simeq H^0(Q; H^n(N; RN) \otimes_R L \otimes_R RQ).$$

It follows by Lemma 8.1 that Q is finite. □

<u>Proposition 8.3</u>. Let G and N be as in Theorem 8.2. Then G is of type (FP) over R. (Strebel [59]).

<u>Proof</u>. If $cd_R N = n$ then, by Theorem 8.2, N is of finite index in G, whence G is of type (FP) over R. So let $cd_R N = n-1$. In this case we need an extremely useful criterion due to Ralph Strebel [59] (cf. 8.6 Appendix), saying that a group G is of type (FP) over R if and only if (i) $cd_R < \infty$ and (ii) the canonical map $\oplus H^k(G; RG) \to H^k(G; \oplus RG)$ is an isomorphism for arbitrary direct sums of copies of RG and all $k \in \mathbb{Z}$. In our case we have $H^k(N; \oplus RG) = 0$ for $k \neq n-1$ and $cd_R G = n$, hence $H^k(G; \oplus RG) = 0 = \oplus H^k(G; RG)$ unless $k = n-1$ or $k = n$. Moreover, the LHS-spectral

sequence yields natural isomorphisms

$$H^{n-1}(G; \oplus RG) \approx H^0(Q; H^{n-1}(N; \oplus RG)),$$

$$H^n(G; \oplus RG) \approx H^1(Q; H^{n-1}(N; \oplus RG)),$$

and the required property follows from the fact that $Q = G/N$ is finitely generated. □

As Bamford and Dunwoody's accessibility criterion is not available in the general case, we have to restrict ourselves to $R = \mathbb{Z}$ for the next result. But see Appendix 7.

<u>Theorem 8.4</u>. Let G be a finitely generated group with $\operatorname{cd} G = n < \infty$ and let N be a normal subgroup in G. Assume that N meets the following conditions

(i) N is of type (FP)

(ii) $H^k(N; \mathbb{Z}N) = 0$ for $k \neq n-1$

(iii) $H^{n-1}(N; \mathbb{Z}N)$ is free-Abelian. **

Then $Q = G/N$ contains a non-trivial finitely generated free subgroup of finite index.

<u>Proof</u>. Notice first that G is of type (FP) by Proposition 8.3, whence $H^n(G; \mathbb{Z}G) \neq 0$. We consider the natural isomorphism given by the LHS-spectral sequence

$$H^n(G; \mathbb{Z}G) \approx H^1(Q; H^{n-1}(N; \mathbb{Z}G)).$$

** In the terminology of Chapter III, N is just an "inverse duality group" or a "duality group with free-Abelian dualizing module".

By Propositions 5.3, 5.4 and Lemma 5.6 one has the Q-isomorphism.

$$H^{n-1}(N;\mathbb{Z}G) \cong H^{n-1}(N;\mathbb{Z}N)^{\swarrow} \otimes_N {}^{\searrow}\mathbb{Z}G \cong H^{n-1}(N;\mathbb{Z}N) \otimes \mathbb{Z}Q^{\swarrow}$$

and because of (iii) and the fact that Q is finitely generated we conclude, again using Proposition 5.3, that

$$(*) \qquad 0 \neq H^n(G;\mathbb{Z}G) \cong H^1(Q;\mathbb{Z}Q) \otimes H^{n-1}(N;\mathbb{Z}N).$$

Notice that this is an isomorphism of right G-modules, where G acts diagonally on the right hand side.

The first thing we notice from (*) is that $H^1(Q;\mathbb{Z}Q) \neq 0$ hence, by Theorem 7.1, Q has an α- or a β-decomposition. Secondly we may use (*) to prove that Q is accessible. Indeed, $H^n(G;\mathbb{Z}G)$ is a quotient of the dual of a finitely generated projective module and hence clearly a finitely generated G-module. By Lemma 8.5 below this implies that $H^1(Q;\mathbb{Z}Q)$ is finitely generated, hence Q is accessible by Theorem 7.2. Let R be an $\alpha\beta$-indecomposable subfactor of Q. Iterated application of Proposition 2.13 shows that R is of type (FP), and so is its preimage $S \leq G$. Now, (*) holds for G replaced by S, whence $H^n(S;\mathbb{Z}S) \cong H^1(R;\mathbb{Z}R) \otimes H^{n-1}(N;\mathbb{Z}N) = 0$. This tells that $cdS = n-1 = cdN$, hence $R = S/N$ is finite by Theorem 8.2.

Thus Q is the fundamental group of a finite graph of groups with finite vertices and hence contains a finitely generated free subgroup of finite index (cf. Remark at the end of Section 7.1). □

Lemma 8.5. Let G be a group, C and D right G-modules, and assume that the underlying Abelian group of C is free. If the diagonal G-module $C \otimes D$ is finitely generated then so is D.

Proof. $C \otimes D$ is generated by a finite number of elements of the form $e_{ij} = c_i \otimes d_j$, $c_i \in C$, $d_j \in D$, $1 \leq i \leq r$, $1 \leq j \leq s$. Let D_o be the submodule of D generated by $d_1, d_2, \ldots, d_s$. The embedding $i: D_o \rightarrowtail D$ induces an epimorphism $(C \otimes i): C \otimes D_o \twoheadrightarrow C \otimes D$, hence $C \otimes (D/D_o) = 0$. Since C is $\mathbb{Z}$-free this implies $D = D_o$.

Exercise Let G be a group with $cd_R G = n < \infty$ and $N \lhd G$ a normal subgroup satisfying (i) N is of type (FP) over R and (ii) $H^k(N; RN) = 0$ for $k \neq n$. Prove that G/N is finite.

8.2 Low dimensions. Let G be a finitely generated group with $cd_R G \leq 2$ and N an infinite almost finitely presented (over R) normal subgroup in G. Then clearly N is of type (FP) over R and $H^0(N; RN) = 0$, and $H^1(N; RN)$ is R-free by [60], Corollary 3.7. Thus the assumptions of Theorem 8.2 are fulfilled, hence either $cd_R N \leq 1$ or $|G/N| < \infty$. In the torsion-free case one can apply Theorem 7.6, whence

Corollary 8.6. Let G be a finitely generated torsion-free group with $cd_R G \leq 2$, and $N \lhd G$ a normal subgroup which is almost finitely presented over R. Then either N is free or of finite index in G.

Examples 1) Let G be the direct product of two free groups of rank 2, $G = \langle x,y\rangle \times \langle u,v\rangle$, and N the subgroup generated by x, yu, v. Then $N \lhd G$, with $G/N \cong \mathbf{Z}$ and one has $cdG = 2$. Moreover, N contains a free-Abelian group of rank 2 (generators x,v) and hence cannot be a free group. By Corollary 8.6 it follows that N is not finitely related! This example shows that the assumption "N is almost finitely presented" in Corollary 8.6 cannot be replaced by "N is finitely generated".

Notice, however, that many groups of cohomology dimension 2 do have the property that every finitely generated subgroup is finitely presented. This was proved by G.P.Scott [51] for all 3-manifold groups (hence in particular for all knot groups) and by Karass and Solitar [37a] for the 1-relator groups of the form $\langle x_1,x_2,\dots,x_n, y_1, y_2,\dots,y_m;\ w(x_i,\dots,x_n) = w^1(y_1,\dots,y_m)\rangle$, the so called " pinched" one relator groups. Whether or not <u>all</u> 1-relator groups have this property is still an open question.

2) Let G be the group of a knot and G^1 its commutator subgroup. One knows that $cdG \leq 2$ and $G/G^1 \cong \mathbf{Z}$. If G^1 is finitely generated then, by Scott's result [51], G^1 is finitely presented and hence, by Corollary 8.6, free. This is a well known result in knot theory (cf. [44]), Theorem 4.51).

<u>Corollary 8.7</u>. Let G be a finitely generated group of cohomology dimension ≤ 2 and N a finitely generated free normal subgroup in G. Then G/N is finitely generated free-by-finite, and

G is the fundamental group of a finite graph of free groups (all of whose edges and vertices contain N as a subgroup of finite index).

Proof. Straightforward from Theorem 8.3. The finite graph of finite groups of Q lifts to a finite graph of groups $N \trianglelefteq S_i \leq G$, with $|S_i/N| < \infty$. By Serre's Theorem (Theorem 5.11) $cdS_i = cdN = 1$, hence the S_i's are free by Theorem 7.6.

Exercise Prove that every finitely generated normal subgroup in a group G with $cd_R G = 1$ is either finite or of finite index in G.

8.3 Centres. Clearly the cohomology dimension of a group G is a bound for the torsion-free rank of the Abelian subgroups of G. Here we show that central subgroups are subject to further restrictions.

Theorem 8.8. Let G be a non-Abelian group of finite cohomology dimension n, with centre Z and commutator subgroup G'. Then one has

(a) $cd\ Z \leq n-1$,

(b) if Z is free-Abelian of rank n-1 then G' is free.

Proof. First we remark that if A is a free-Abelian group of rank r then $H^k(A; \mathbb{Z}A) = 0$ for $k \neq r$ and $H^r(A; \mathbb{Z}A) \cong \mathbb{Z}$. The proof of this goes by induction on r; cf. also Chapter III. It follows that the assumptions (i), (ii), (iii) in Theorems 8.2 and 8.3 are fulfilled if N is free Abelian of rank n or $n-1$, respectively.

(a) Assume $cdZ = n$. Then G/Z is periodic; for an element of infinite order in G/Z would give rise to a subgroup of the form $Z \times \mathbb{Z} \leq G$, but $cd(Z \times \mathbb{Z}) = n+1$ (Theorem 5.5). Let N be a finitely generated central subgroup of maximal rank ($= n$ or $n-1$, cf. Theorem 7.10). Then Theorems 8.2 and 8.4 tell that G/N is locally free-by-finite. But the short exact sequence $Z/N \rightarrowtail G/N \twoheadrightarrow G/Z$ shows that G/N is also periodic, hence G/N is in fact locally finite. It follows that G/Z is locally finite, and this implies by Schur's Theorem that G' is locally finite. Since G is torsion-free we conclude $G' = 1$, i.e., G is Abelian.

(b) Assuming that Z is free-Abelian of rank $n-1$ implies, by Theorem 8.4, that G/Z is locally free-by-finite. Since the homology functor $H_2(-; \mathbb{Z})$ commutes with direct limits (Proposition 4.8) this implies that $H_2(G/Z; \mathbb{Z})$ is periodic. Now, consider the following part of the 5-term exact sequence

$$\ldots \to H_2(G/Z; \mathbb{Z}) \xrightarrow{\delta} Z \to G/G' \to \ldots \quad .$$

As Z is torsion-free δ is the zero-map hence $Z \cap G' = 1$. Thus G contains $Z \times G'$ as a subgroup. Since Z is finitely generated we can apply Theorem 5.5 (with remark 2),

$cd(Z \times G') = cdZ + cdG' = n-1 + cdG' \leq cdG = n$. We conclude that $cdG' \leq 1$, hence G' must be free by the Stallings-Swan result Theorem 7.6. □

Corollary 8.9. The centre Z of a non-Abelian group G of cohomology dimension 2 is cyclic. If $Z \neq 1$ then the commutator subgroup G' is free.

Remarks. 1) Using the subgroup Theorem for HNN-groups only Karrass-Pietrowski-Solitar [38] prove that if G is a non-Abelian subgroup of a torsion-free one relator group then the centre of G is cyclic. Much more detailed information is available when G is itself a one relator group with non-trivial centre, cf [49].

2) The fact that knot groups with non-trivial centre have free commutator subgroup is a well-known result in knot theory, cf.[44].

8.4. Amalgamated products. Here we apply the main results of Section 6 to low dimensional situations. The first result is due to Karrass-Solitar [37] (cf also [47]).

Proposition 8.10. Let G be the amalgamated product $G = G_1 *_S G_2$, where both G_1 and G_2 are finitely generated and contain S as a subgroup of finite index. If $cd_{\mathbb{Q}} G \leq 1$ (in particular, if G is free-by-finite) then G_1 and G_2 are finite.

Proof. Since $cd\, G_i \le 1$, G_1 and G_2 are of type (FP) over $\mathbb{Q}$. By Corollary 6.5 it follows that $cd_{\mathbb{Q}} G_i + 1 = cd_{\mathbb{Q}} G \le 1$, hence $cd_{\mathbb{Q}} G_i = 0$; i.e. $|G_i| < \infty$ for $i = 1,2$. □

Remark. Notice that, in particular, G cannot be a non-trivial free group.

Theorem 8.11. Let G be the amalgamated product $G = G_1 *_S G_2$, where both G_1 and G_2 are (almost) finitely presented and contain S as a subgroup of finite index. If $cdG \le 2$ then G_1 and G_2 are free (and hence G is finitely presented).

Proof. Since $cdG_i \le 2$, G_i is of type (FP) for $i = 1,2$, and Corollary 6.5 applies. It follows $cdG_i \le 1$ hence G_i is free by Theorem 7.6, $i = 1,2$. □

Exercise: Prove the analogous results for HNN-groups.

8.5 Results on the derived series. The aim of this subsection is to add some more results on groups of cohomology dimension 2, in order to make this aspect reasonably complete. Theory and results are due to Ralph Strebel see [59a] and [59b], where many other results are to be found.

Definition. We denote by $\mathfrak{D}(R)$ the class of all groups G with the property that the functor $(R\otimes_{RG}-)$ detects monomorphisms between RG-projective modules; i.e. if P and Q are RG-projective modules and $\psi\colon P \to Q$ a homomorphism such that $R\otimes_{RG}\psi$ is injective, then so is ψ.

Remark. There is a most useful reduction: If the functor $(R\otimes_{RG}-)$ detects monomorphisms between finitely generated free modules, then it detects monomorphisms between arbitrary projective modules. This follows from the fact that every free module is the union of its finitely generated free direct summands and that every projective module is a direct summand in a free module. The details are left as an exercise.

Proposition 8.12. The class $\mathfrak{D}(R)$ has the following closure properties:

(a) it is subgroup closed,

(b) it is extension closed,

(c) it is closed with respect to arbitrary direct products,

(d) it is closed with respect to direct limits,

(e) if G has a (transfinite) descending series
$G = G_0 \rhd G_1 \rhd \ldots \rhd G_\omega \rhd G_{\omega+1} \rhd \ldots \rhd G_\alpha = 1$ with $G_\beta/G_{\beta+1} \in \mathfrak{D}(R)$ for all β, then $G \in \mathfrak{D}$.

Proof. (a) Let $S \le G$, $G \in \mathfrak{D}(R)$. Since tensoring with $RG\otimes_{RS}-$ converts RS-projectives to RG-projectives and preserves injections, $S \in \mathcal{X}(R)$.

(b) Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups with $N, Q \in \mathcal{X}(R)$ and let $\psi\colon A \to B$ an RG-map between projective modules, such that $R\otimes_{RG}\psi$ is injective. Since $R\otimes_{RG}\psi = R\otimes_{RQ}(R\otimes_{RN}\psi)$ it follows that $R\otimes_{RN}\psi$ and hence ψ are injections.

(c) We are not going to use this and leave it as an exercise. Notice that the countable case follows from (e).

(d) Let $G = \varinjlim G_i$, $G_i \in \mathcal{X}(R)$, and let $\psi\colon E \to F$ be an RG-homomorphism between finitely generated free RG-modules. Choose a basis for E and for F. Then ψ is given by a matrix Φ which involves only finitely many elements of G, so that there is some G_m such that Φ can be lifted to G_m. We can restrict the direct limits to $i \geq m$, so that $\underset{\sim}{\Phi}$ can compatibly be lifted to all G_i. Clearly $R\otimes_{G_i}\Phi$ coincides with $R\otimes_G\underset{\sim}{\Phi}$ for all i, hence all "lifts" define monomorphisms. It follows that their direct limit $\underset{\sim}{\Phi}$ defines a monomorphism.

(e) We use transfinite induction. If α is not a limit ordinal, then $G \in \mathcal{X}(R)$ follows from the inductive hypothesis and (b). So let α be a limit ordinal. Let A, B be free RG-modules and $\psi\colon A \to B$ an RG-map such that $R\otimes_{RG}\psi$ is injective. One has a commutative diagram

$$(*)\qquad \begin{array}{ccc} A & \longrightarrow & \prod_{\beta<\alpha} (R\otimes_{RG_\beta} A) \\ \psi\downarrow & & \downarrow \prod (R\otimes_{RG_\beta}\psi) \\ B & \longrightarrow & \prod_{\beta<\alpha} (R\otimes_{RG_\beta} B) \end{array}$$

Since $R\otimes_{RG}\psi = R\otimes_{RG/G_\beta}(R\otimes_{RG_\beta}\psi)$ is injective, the inductive hypothesis on G/G_β implies that $R\otimes_{RG_\beta}\psi$ is injective for all $\beta < \alpha$. Now, the observation that $RG \to \prod_\beta R(G/G_\beta)$ is a monomorphism shows that the horizontal maps in (*) are monomorphisms. This implies that ψ is a monomorphism. □

<u>Remark</u>. (a) and (c) imply that $\mathfrak{D}(R)$ is residually closed.

<u>Proposition 8.13</u>. The infinite cyclic group is in $\mathfrak{D}(R)$ for every commutative ring R.

<u>Proof</u>. Let G be $\cong \mathbf{Z}$, A and B projective RG-modules and $\psi: A \to B$ an RG-map. One has $\bigcap_n \mathfrak{g}^n = 0$ (e.g. [30] p.56) and hence $\bigcap_n \mathfrak{g}^n A = 0 = \bigcap_n \mathfrak{g}^n B$. ψ induces maps

$$\psi_n: \mathfrak{g}^n A/\mathfrak{g}^{n+1}A \longrightarrow \mathfrak{g}^n B/\mathfrak{g}^{n+1}B .$$

Now, $\mathfrak{g}^n M/\mathfrak{g}^{n+1}M \cong \mathfrak{g}^n/\mathfrak{g}^{n+1} \otimes_{RG} M$ for every projective module M, hence injectivity of $R\otimes_{RG}\psi$ implies that all maps $\psi_n = \mathfrak{g}^n/\mathfrak{g}^{n+1}\otimes_{RG}\psi$ are injective, and therefore that ψ itself is injective. □

<u>Corollary 8.14</u>. If a group G admits a descending (transfinite) series

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \ldots \rhd G_\omega \rhd G_{\omega+1} \rhd \ldots \rhd G_\alpha = 1$$

all of whose factors $G_\beta/G_{\beta+1}$ are torsion-free Abelian groups, then G lies in $\mathfrak{D}(R)$ for every commutative ring R.

Remarks. 1) It follows, in particular, that $\mathfrak{D}(R)$ contains all torsion-free nilpotent groups and all free groups.

2) A further consequence is that $\mathfrak{D}(R)$ is closed under arbitrary free products. Indeed, let $G,H \in \mathfrak{D}(R)$ and consider the canonical map $G * H \twoheadrightarrow G \times H$. It is well known that the kernel is free (hence in $\mathfrak{D}(R)$) and thus Proposition 8.12(b) yields $G * H \in \mathfrak{D}(R)$. Arbitrary free products are direct limits of finite free products, hence the assertion follows from Proposition 8.12 (d).

Definition. We denote by $\mathcal{E}(R)$ the class of all groups G admitting an RG-projective resolution $\cdots \to P_2 \xrightarrow{\partial_2} P_1 \to P_0 \twoheadrightarrow R$ whose second differential ∂_2 has the property that $R\otimes_{RG}\partial_2$ is a monomorphism.

If G is a group in $\mathcal{E}(R)$ then clearly $H_2(G; R) = 0$, where R is regarded as a trivial G-module. Moreover, if G is a group with $cd_R G \leq 2$, then $G \in \mathcal{E}(R)$ if and only if $H_2(G; R) = 0$, and those are the groups we primarily have in mind; for further examples of groups in $\mathcal{E}(R)$ see Proposition 8.16.

Exercise and remark. Use the Universal Coefficients Theorem to prove that the following statements are equivalent:

(i) $G \in \mathcal{E}(R)$ for every commutative ring R

(ii) $G \in \mathcal{E}(\mathbb{Z})$ and $G/[G,G]$ is torsion-free.

This shows, in particular, that knot groups G are in $\mathcal{E}(R)$ for all R, since one has $cdG \le 2$, $G/[G,G] \cong \mathbb{Z}$ and $H_2(G;\mathbb{Z}) = 0$.

Now comes the main result of this section. It provides some information on the derived series $G = G^0, G', G'', \ldots, G^{(\omega)}, G^{(\omega+1)}, \ldots$ of a group G which belongs to $\mathcal{E}(R)$ for all R.

Theorem 8.15. If the group G is in $\mathcal{E}(R)$ for all commutative rings R then one has:

(a) For every ordinal α, $G^{(\alpha)} \in \mathcal{E}(R)$; in particular $G^{(\alpha)}/G^{(\alpha+1)}$ is torsion-free and $H_2(G^{(\alpha)};\mathbb{Z}) = 0$.

(b) The smallest ordinal α with $G^{(\alpha)} = G^{(\alpha+1)}$ is equal to 0, 1, 2 or a limit ordinal λ.

Proof. Let $\ldots \to P_2 \xrightarrow{\partial_2} P_1 \to P_0 \twoheadrightarrow R$ be an RG-projective resolution such that $R \otimes_{RG} \partial_2$ is a monomorphism. Let $G^{(\alpha)} = N$ and $G/G^{(\alpha)} = Q$ and assume that (a) holds for all β with $\beta < \alpha$. Then Q admits a descending series with torsion-free Abelian factors and terminal 1, hence $Q \in \mathcal{D}(R)$ for all R. Now, $R \otimes_{RG} \partial_2 = R \otimes_{RQ} (R \otimes_{RN} \partial_2)$, hence $R \otimes_{RN} \partial_2$ is an injection, i.e., $N = G^{(\alpha)} \in \mathcal{E}(R)$, which proves (a).

If $N = G^{(\alpha)} = G^{(\alpha+1)}$ then we have $H_1(N;\mathbb{Z}) = 0 = H_2(N,\mathbb{Z})$, hence

$$R \otimes_{RN} P_2 \xrightarrow{R \otimes_{RN} \partial_2} R \otimes_{RN} P_1 \longrightarrow R \otimes_{RN} P_0 \twoheadrightarrow R$$

is exact and hence the beginning of an RQ-projective resolution. But since $Q \in \mathfrak{D}(R)$ we have that $R \otimes_{RN} \partial_2$ is injective, hence $cd_R Q \leq 2$ for all R.

From Theorem 7.10 it follows that solvable subgroups of Q are of Hirsch number ≤ 2. If α is finite then Q itself is solvable of degree ≤ 2, i.e., $\alpha = 0$, 1 or 2. If α is infinite put $\alpha = \lambda + n$ where λ is a limit ordinal and $0 \leq n < \infty$, and let $G^{(\lambda)}/G^{(\alpha)} = K \triangleleft Q$. Any element $x \in Q$, $x \notin K$ together with K generates a solvable subgroup of Hirsch number $1 + hK$, hence $hK \leq 1$, i.e., K is torsion-free Abelian of rank ≤ 1. So K has Abelian automorphism group, hence $[Q', K] = 1$. If $K \neq 1$ then Q' is a group of cohomology dimension ≤ 2 with non-trivial centre, hence Q'' is free by Corollary 8.9., contradicting $Q'' \geq K$. It follows that $K = 1$, i.e., $\alpha = \lambda$. □

In order to outline the whole power of Theorem 8.15, we conclude by showing that a large number of finitely presented groups belong to $\mathcal{E}(\mathbb{Z})$.

Let $G = \langle x_1, x_2, \ldots, x_n; r_1, r_2, \ldots, r_m \rangle$ a finitely presented group. One has always (see e.g. [30], p.172)

$$(*) \quad n - m \leq h(G/G') - d(H_2(G;\mathbb{Z})),$$

where h denotes the Hirsch number and d the minimal number of generators. G is said to be <u>efficient</u> if there is a presentation for G, such that $(*)$ is actually an equality.

Proposition 8.16. Let G be an efficient group with $H_2(G;\mathbb{Z}) = 0$. Then $G \in \mathcal{E}(\mathbb{Z})$.

Proof. Let $K \rightarrowtail F \twoheadrightarrow G$ be a presentation for G, where F is free of rank n, K is generated as a normal subgroup by m elements, and assume that $(\ast)$ is an equality. The beginning of a G-free resolution can be constructed using the exact sequence which appeared in the proof of Proposition 2.2.

$$\begin{array}{ccccccccc} & & M & & & & & & \\ & & \downarrow & \searrow^{\partial_2} & & & & & \\ 0 & \longrightarrow & K/K' & \longrightarrow & \mathbb{Z}G \otimes_F \mathfrak{f} & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} \end{array} ,$$

where M is the free G-module of rank m and the vertical arrow maps its free generators to the given relators. Tensoring this with $(\mathbb{Z}\otimes_G -)$ and modifying the right end yields the diagram

$$\begin{array}{ccccccccc} & & M_G & & & & & & \\ & & \pi \downarrow & \searrow^{\mathbb{Z}\otimes_G \partial_2} & & & & & \\ 0 \to H_2(G;\mathbb{Z}) & \longrightarrow & K/[F,K] & \longrightarrow & F/F' & \longrightarrow & G/G' & \longrightarrow & 0 \end{array}$$

whose horizontal row is exact. Now, $H_2(G;\mathbb{Z}) = 0$ implies that $K/[F,K]$ is a free-Abelian group of rank $n-h(G/G')$ which is $= m$ since G is efficient. M_G is free-Abelian of rank m as well, hence the epimorphism π is an isomorphism and $\mathbb{Z}\otimes_G \partial_2$ is injective.□

8.6. Appendix: Yet another homological finiteness criterion. In this section we make up leeway by proving R.Strebel's finiteness criterion [59] quoted in the proof of Proposition 8.3.

We start with some general remarks. Let Λ be an arbitrary ring with unit, M and A left Λ-modules, $M^* = \mathrm{Hom}_\Lambda(M,\Lambda)$, and consider the natural map of Section 3.1.

$$\phi : M^* \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(M,A).$$

The following Lemma gives necessary and sufficient conditions for a Λ-homomorphism $f: M \to A$ to be in the image of ϕ.

Lemma 8.17. (a) f is in the image of ϕ if and only if f factors through a finitely generated free module F, $f: M \to F \to A$.

(b) If A is projective, then f is in the image of ϕ if and only if f(M) is a finitely generated submodule of A.

Proof. (a) Assume that $f = \phi\left(\sum_{i=1}^{n} f_i \otimes a_i\right)$, $f_i \in M^*$, $a_i \in A$. Let F be the free Λ-module over $e_1, e_2, \ldots, e_n$, define $f': M \to F$ by $f'(m) = \sum f_i(m)e_i$ and $f'': F \to A$ by $f''(e_i) = a_i$, and check that $f = f'' \circ f'$. Conversely, assume that f factors through F, $f = f'' \circ f$. Then define $f_i \in M^*$ by $f'(m) = \sum f_i(m)e_i$ and check that $f = \phi\left(\sum f_i \otimes f''(e_i)\right)$.

(b) Let E be a free Λ-module containing A as a direct summand. If f(M) is finitely generated, then f(M) is contained in a finitely generated free submodule F of E and f factors as $f: M \twoheadrightarrow f(M) \rightarrowtail F \rightarrowtail E \twoheadrightarrow A$. □

Next we note that the natural map ϕ induces - just as in the group ring case in Section 5.1 - natural homomorphisms

$$\phi^k : \mathrm{Ext}^k_\Lambda\ (M,\Lambda)\otimes_\Lambda A \longrightarrow \mathrm{Ext}^k_\Lambda\ (M,A)$$

for every Λ-module A and all $k\in\mathbb{Z}$. Here we shall always assume that A is a free Λ-module. Then it is easy to see, using the fact that $\mathrm{Ext}^k_\Lambda(M,-)$ is an additive functor, that ϕ^k is monomorphic for all k and it is an isomorphism when A is finitely generated.

Lemma 8.18 Let $K \rightarrowtail P \twoheadrightarrow M$ be a short exact sequence of Λ-modules such that P is projective and K is finitely generated. If ϕ^o and ϕ^1 are epimorphic for all free Λ-modules A then P is finitely generated.

Proof. One has the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & M^*\otimes_\Lambda A & \longrightarrow & P^*\otimes_\Lambda A & \longrightarrow & K^*\otimes_\Lambda A & \longrightarrow & \mathrm{Ext}^1_\Lambda(M,\Lambda)\otimes_\Lambda A \\
 & \phi^o\downarrow & & \phi^o{}_P\downarrow & & \phi^o{}_K\downarrow & & \downarrow\phi^1 \\
0 \longrightarrow & \mathrm{Hom}_\Lambda(M,A) & \longrightarrow & \mathrm{Hom}_\Lambda(P,A) & \longrightarrow & \mathrm{Hom}_\Lambda(K,A) & \longrightarrow & \mathrm{Ext}^1_\Lambda(M,A)
\end{array}$$

From Lemma 8.17(a) it is obvious that $\phi^o{}_P$ is an isomorphism; hence $\phi^o{}_K$ is an isomorphism by the 5-Lemma. Assume $A = P\oplus Q$. Then the injection $\iota: P \rightarrowtail A$ factors over a finitely generated free module F, $P \xrightarrow{\alpha} F \xrightarrow{\beta} A \xrightarrow{\pi} P$, hence $P = \pi\beta\ (F)$ is finitely generated.

Proposition 8.19. Let $n \geq 1$ be an integer and assume that the Λ-module M has a projective resolution $\underline{\underline{P}} \twoheadrightarrow M$ which is finitely generated in dimension $n+1$. If the natural map

$$\phi^n : \mathrm{Ext}^n_\Lambda\ (M,\Lambda) \otimes_\Lambda A \longrightarrow \mathrm{Ext}^n_\Lambda\ (M,A)$$

is epimorphic for every free Λ-module A then there is a projective resolution $\underline{\underline{P}}' \twoheadrightarrow M$ which is finitely generated both in dimension $n+1$ and dimension n.

Proof. We consider the part $P_{n+1} \xrightarrow{\partial} P_n \xrightarrow{\partial} P_{n-1}$ of the given projective resulution. By naturality of ϕ the following diagram is commutative.

$$\begin{array}{ccccc} P^*_{n-1} \otimes_\Lambda P_n & \longrightarrow & P^*_n \otimes_\Lambda P_n & \longrightarrow & P^*_{n+1} \otimes_\Lambda P_n \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ \mathrm{Hom}_\Lambda(P_{n-1},P_n) & \longrightarrow & \mathrm{Hom}_\Lambda(P_n,P_n) & \longrightarrow & \mathrm{Hom}_\Lambda(P_{n+1},P_n) \end{array} .$$

Replacing P_n and P_{n-1}, if necessary, we may assume that P_n is free. Then ∂P_{n+1} lies in a finitely generated direct summand K of P_n. Let $\pi : P_n \to P_n$ denote the endomorphism which projects onto the submodule K. By Lemma 8.17(b) $\pi = \phi(u)$ for some $u \in P_n{}^* \otimes_\Lambda P_n$. Now, $(1-\pi)\partial = 0$, i.e. $1-\pi$ is a cocycle. From the assumpton that ϕ^n is an epimorphism for every free module A it follows readily that the same holds for every projective module A; in particular, $\mathrm{Ext}^n_\Lambda(M,\Lambda) \otimes_\Lambda P_n \to \mathrm{Ext}^n_\Lambda(M,P_n)$ is an epimorphism, hence there is some element $v \in P^*_n \otimes_\Lambda P_n$ and a homomorphism $\sigma \in \mathrm{Hom}_\Lambda(P_{n-1},P_n)$ such that $1-\pi = \phi(v)+\sigma_o\partial$, hence one has

$$(*) \qquad 1 = \phi(u+v)+\sigma_o\partial \quad .$$

The image of $\phi(w+v)$ is a finitely generated submodule of P_n and hence, by Lemma 8.18, lies in a finitely generated direct summand $P_n{}'$ of P_n. By $(*)$, $\sigma\,\partial\, P_n{}' \le P_n{}'$, hence $\partial P_n{}' \le \sigma^{-1}P_n{}'$ and the maps ∂ and σ induce

$$P_n/P_n{}' \xrightarrow{\partial_*} P_{n-1}/\sigma^{-1}P_n{}' \xrightarrow{\sigma_*} P_n/P_n{}' .$$

σ_* is plainly a monomorphism and $(*)$ implies that $\sigma_*\partial_*$ is the identity, hence σ_* and ∂_* are isomorphisms. The diagram

$$\begin{array}{ccccccccc}
 & & & P_n' & \longrightarrow & \sigma^{-1}P_n' & & & \\
 & & \nearrow & \downarrow & & \downarrow & \searrow & & \\
\cdots \to & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \to \cdots \\
 & & & \downarrow & & \downarrow & & & \\
 & & & P_n/P_n' & \tilde{\to} & P_{n-1}/\sigma^{-1}P_n' & & &
\end{array}$$

visualizes the result obtained so far. Now the bypass via the top arrows yields the new projective resolution $\underline{P}' \twoheadrightarrow M$. Indeed P_n' is finitely generated and projective; P_n/P_n' is projective, hence so is $P_{n-1}/\sigma^{-1}P_n'$ and $\sigma^{-1}P_n' = P_{n-1}'$, and exactness follows by easy diagram chasing. □

<u>Theorem 8.20</u>. Let M be a Λ-module of finite projective dimension. Then the following statements are equivalent:

(i) M is of type (FP).

(ii) The functor $\mathrm{Ext}^k_\Lambda(M,-)$ commutes with exact colimits for all $k \geq 0$.

(iii) The functor $\mathrm{Ext}^k_\Lambda(M,-)$ commutes with direct sums for all $k \geq 0$.

(iv) The natural map $\nu : \oplus\mathrm{Ext}^k_\Lambda(M,\Lambda) \to \mathrm{Ext}^k(M, \oplus\Lambda)$ is an isomorphism for all $k \geq 0$ and arbitrary direct sums $\oplus\Lambda$ of copies of Λ.

Proof. (i) $\Rightarrow$ (ii) is contained in Corollary 1.6. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are trivial. It remains to prove that (iv) $\Rightarrow$ (i). Let $m = \text{pr.dim}\, M$. There is a projective resolution of length m, $0 \to P_m \to P_{m-1} \to \dots \to P_0 \twoheadrightarrow M$. The limiting homomorphism ν clearly coincides with the maps ϕ^k for the free module $F = \oplus \Lambda$, hence, by Proposition 8.19, one can assume that P_m is finitely generated. Iterating this argument and terminating with Lemma 8.18 yields the result.

Corollary 8.21. A group G is of type (FP) over R if and only if firstly: $cd_R G < \infty$ and secondly: the natural maps $\nu: \oplus_I H^k(G; RG) \to H^k(G; \oplus_I RG)$ are isomorphisms for all $k \geq 1$ and all direct sums of $|I| = \max(\aleph_0, |R|)$ copies of RG.

Remark The assumption that $\text{pr.dim}\, M < \infty$ and $cd_R G < \infty$ in Theorem 8.20 and Corollary 8.21 is essential. To see this let G be the free-Abelian group of rank $\aleph_0$. If N is a (free-Abelian normal) subgroup of rank n then we know (cf. the proof of Theorem 8.8; or Chapter III) that $H^k(N; F) = H^k(N; \mathbb{Z}N) \otimes_N F = 0$ for $k \neq n$ and every free G-module F. It follows that $H^k(G; F) = 0$ for all $k < n$. But G contains subgroups of arbitrarily large rank, hence $H^k(G; F) = 0$ for all $k \in \mathbb{Z}$. Thus the map $\nu: \oplus H^k(G; \mathbb{Z}G) \to H^k(G; \oplus \mathbb{Z}G)$ is the trivial isomorphism for all k, despite the fact that G is not even finitely generated.

The remark that Strebel's criterion Theorem 8.20 does not hold for type $(FP)_\infty$ might look like a slight disadvantage from a theoretical point of view. However, due to the fact that direct sums are much easier to deal with than arbitrary direct limits or direct products, it is often much better for explicit applications than Corollary 1.6, as we have seen in the proof of Theorem 8.8.

CHAPTER III

9. Duality Groups

9.1 Preliminary remark. In Section 5.1 we have obtained information on cohomology groups with projective coefficients and homology groups with injective coefficients for groups of type $(FP)_\infty$. In general it is not possible to deduce information for arbitrary coefficients from these results (try a finite group!). However, if G is of type (FP) we do get such information in terms of spectral sequences (cf. Theorems 3.2 and 3.3). In particular, in the top dimension one finds:

Lemma 9.1. Let G be a group of type (FP) over R, $n = cd_R G$, and let C denote the right RG-module $H^n(G; RG)$. Then the natural maps (**) of Section 5.1 provide isomorphisms

$$\phi^n: C \otimes_{RG} A \xrightarrow{\simeq} H^n(G; A), \qquad \psi_n: H_n(G;B) \xrightarrow{\simeq} \mathrm{Hom}_{RG}(C, B),$$

for every left RG-module A and right RG-module B.

Proof. We give a direct proof not referring to the spectral sequences of Section 3.1. Let $K \rightarrowtail F \twoheadrightarrow A$ be a short exact sequence of RG-modules, with F RG-free. By naturality of ϕ^n we get the commutative diagram

$$\begin{array}{ccccccc} C \otimes_{RG} K & \to & C \otimes_{RG} F & \to & C \otimes_{RG} A & \to & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \delta & & \\ H^n(G; K) & \to & H^n(G; F) & \to & H^n(G; A) & \to & 0 \end{array} \quad \phi^n$$

whose rows are exact. By Proposition 5.3 β is an isomorphism, implying that δ is epimorphic. This holds for arbitrary A, hence α is epimorphic as well. By the 5-Lemma, we conclude that δ is an isomorphism. The second part of the assertion is dual. □

Exercise and remark. Use Proposition 8.19 to prove that $\phi^n: C \otimes_{RG} A \to H^n(G; A)$ $(C \neq 0)$ is an isomorphism for every RG-module A if and only if (i) $cd_R G = n$, (ii) $C \simeq H^n(G; RG)$, and (iii) there is an RG-projective resolution $\underline{P} \twoheadrightarrow \mathbb{Z}$ which is finitely generated in dimension n.

See [6] for further results on the maps ϕ^n and ψ_n of Lemma 9.1.

9.2 Duality groups

Definition. A group G is said to be a duality group of dimension n over R if there is a (right) RG-module C such that one has natural isomorphisms

$$H^k(G; A) \simeq H_{n-k}(G; C \otimes_R A)$$

for all $k \in \mathbb{Z}$ and all RG-modules A. Hereby G acts diagonally on the tensor product $C \otimes_R A$.

If G is a duality group, the module C which occurs in the definition is called "the dualizing module of G".

Let G be a duality group of dimension n and with dualizing module C. Then one has obviously

(a) $\quad \mathrm{cd}_R G \leq n.$

In particular G is a torsion-free group. Next notice that the functor $H^k(G; -) \approx H_{n-k}(G; C \otimes_R -)$ commutes with direct limits for all $k \geq 0$; by Proposition 2.4 this implies

(b) $\quad G$ is of type $(FP)_\infty$ over R.

Now, let us consider an induced RG-module $A = L \otimes_R RG$. By definition of duality we get for all $k \in \mathbb{Z}$,

$$H^k(G; A) \approx H_{n-k}(G; C \otimes_R A);$$

But $C \otimes_R (L \otimes_R RG)$ is isomorphic, by Lemma 2.9, to the induced RG-module $(C \otimes_R L) \otimes_R RG$. Therefore we find

$$\text{(c)} \qquad H^k(G; L \otimes_R RG) = \begin{cases} 0 & \text{if } k \neq n \\ C \otimes_R L & \text{if } k = n, \end{cases}$$

for every R-module L. It is easily checked that, by naturality, the isomorphism $H^n(G; L \otimes_R RG) \approx C \otimes_R L$ is actually a right G-module isomorphism. For $L = R$ we get

$$\text{(d)} \qquad \begin{aligned} H^k(G; RG) &= 0 && \text{if } k \neq n, \\ H^n(G; RG) &\approx C && (\neq 0), \end{aligned}$$

hence $\mathrm{cd}_R G = n$, and both n and C are determined by G. Finally we claim

(e) $\quad C$ is flat as an R-module.

Proof. Let $L' \rightarrowtail L \twoheadrightarrow L''$ be a short exact sequence of R-modules. Since RG is R-free, the sequence

$$L' \otimes_R RG \rightarrowtail L \otimes_R RG \twoheadrightarrow L'' \otimes_R RG$$

is still exact and hence gives rise to the exact sequence

$$H^{n-1}(G; L'' \otimes_R RG) \to H^n(G; L' \otimes_R RG) \to H^n(G; L \otimes_R RG) \to H^n(G; L \otimes_R RG) \to 0.$$

By formula (c) we thus get the short exact sequence

$$0 \to C \otimes_R L' \to C \otimes_R L \to C \otimes_R L'' \to 0,$$

i.e., C is a flat module over R.

Now we shall see that the statements (a), (b), (d) and (e) are also sufficient for G to be a duality group.

Theorem 9.2. A group G is a duality group of dimension n over R if and only if the following three conditions hold:

(i) G is of type (FP) over R

(ii) $H^k(G; RG) = 0$ for $k \neq n$

(iii) $H^n(G; RG)$ is flat as an R-module.

Proof. It remains to show that (i), (ii) and (iii) imply that G is a duality group of dimension n. Let

$$0 \to P_n \to P_{n-1} \to \dots \to P_0 \twoheadrightarrow R$$

be a resolution of the trivial G-module R by finitely generated projective RG-modules. Condition (ii) then implies that

$$0 \to P_0^* \to P_1^* \to \dots \to P_n^* \twoheadrightarrow C,$$

$P_i^* = \mathrm{Hom}_{RG}(P_i, RG)$, is a finite projective resolution of the right RG-module $C = H^n(G; RG)$. By Proposition 3.1 we now have natural isomorphisms

$$\phi: \ P_k^* \otimes_{RG} A \xrightarrow{\sim} \mathrm{Hom}_{RG}(P_k, A)$$

for all $k \in \mathbf{Z}$ and all RG-modules A, whence using (iii) and Lemma 9.3(a) below,

$$H^k(G; A) \cong \mathrm{Tor}_{n-k}^{RG}(C, A) \cong H_{n-k}(G; C \otimes_R A),$$

i.e. G is a duality group. □

<u>Lemma 9.3</u>. Let G be a group, A a left RG-module and B and C right RG-modules. Then for all $k \in \mathbf{Z}$ the following holds

(a) If C is R-flat one has natural isomorphisms
$H_k(G; C \otimes_R A) \cong \mathrm{Tor}_k^{RG}(C, A)$.

(b) If C is R-projective one has natural isomorphisms
$H^k(G; \mathrm{Hom}_R(C,B)) \cong \mathrm{Ext}_{RG}^k(C,B)$.

Hereby G acts diagonally on $C \otimes_R A$ and $\mathrm{Hom}_R(C,B)$ respectively.

<u>Proof</u>. Let $\underline{P} \twoheadrightarrow R$ be an RG-projective resolution of the trivial G-module R. Then $\underline{P} \otimes_R C \twoheadrightarrow C$ is an RG-flat (resp. RG-projective)

resolution and can be used to compute $\mathrm{Tor}_k^{RG}(C,A)$ and $\mathrm{Ext}_{RG}^k(C,A)$ respectively. Moreover one has the obvious natural isomorphisms $(\underline{P} \otimes_R C) \otimes_{RG} A \simeq \underline{P} \otimes_{RG} (C \otimes_R A)$, and $\mathrm{Hom}_{RG}(\underline{P} \otimes_R C, B) \simeq \mathrm{Hom}_{RG}(\underline{P}, \mathrm{Hom}_R(C,B))$, whence the lemma. □

9.3. Inverse duality. There is an obvious dualization of the definition of a duality group.

Definition. A group G is said to be an inverse duality group of dimension n over R if there is a (right) RG-module D such that one has natural isomorphisms

$$H_k(G; B) \simeq H^{n-k}(G; \mathrm{Hom}_R(D,B))$$

for all $k \in \mathbf{Z}$ and all RG-modules B. Hereby G acts diagonally on $\mathrm{Hom}_R(D,B)$.

Let G be an inverse duality group of dimension n, with "inverse dualizing module" D. As in the duality case, we wonder what properties such a group has. Obviously we have

(a) $\mathrm{hd}_R G \leq n$.

Next, the functor $H_k(G; -) \simeq H^{n-k}(G; \mathrm{Hom}_R(C,-)$ commutes with direct products for all $k \geq 0$; this implies by Proposition 2.4

(b) G is of type $(FP)_\infty$ over R,

hence, by Theorem 4.6 $\mathrm{cd}_R G = \mathrm{hd}_R G \leq n$. Now, consider a coinduced RG-module $B = \mathrm{Hom}_R(RG, L)$. By the definition of inverse duality we get for all $k \in \mathbf{Z}$

$$H_k(G;\ B) \;=\; H^{n-k}(G;\ \mathrm{Hom}_R(D,B));$$

but $\mathrm{Hom}_R(D,B)$ is isomorphic, by Lemma 2.9, to the coinduced RG-module $\mathrm{Hom}_R(RG,\mathrm{Hom}_R(D,L))$. Therefore we find

$$\text{(c)}\quad H_k(G;\mathrm{Hom}_R(RG,L)) \;=\; \begin{cases} 0 & \text{if } k \neq n \\ \mathrm{Hom}_R(D,L) & \text{if } k = n. \end{cases}$$

Using (c) and arguments dual to those in the duality case, it follows that $\mathrm{Hom}_R(D,-)$ is an exact functor, hence

(d) D is projective as an R-module.

Next we claim that (b) and (c) imply

(e) $H^k(G;\ RG) = 0$ for $k \neq n$.

Proof. Let L be an injective R-module. Then the coinduced module $\mathrm{Hom}_R(RG,\ L)$ is RG-injective and hence, by Proposition 5.4(c) one has an isomorphism

$$\begin{aligned} \psi : H_k(G;\ \mathrm{Hom}_R(RG,L)) &\approx \mathrm{Hom}_{RG}(H^k(G;\ RG),\ \mathrm{Hom}_R(RG,\ L)) \\ &\approx \mathrm{Hom}_R(H^k(G;\ RG) \otimes_{RG} RG,\ L) \\ &\approx \mathrm{Hom}_R(H^k(G;\ RG),\ L) \end{aligned}$$

for all $k \in \mathbb{Z}$. Thus $\mathrm{Hom}_R(H^k(G;\ RG),\ L) = 0$ for all $k \neq n$ and all injective R-modules L. Since every R-module can be embedded in an injective R-module, this implies $H^k(G; RG) = 0$ for $k \neq n$. □

Finally, we claim that there is an isomorphism

(f) $D \simeq H^n(G; RG)$ as (right) RG-modules.

Proof. Let $C = H^n(G; RG)$. By Lemma 9.1 one has a natural isomorphism $\mathrm{Hom}_{RG}(C, B) \simeq H_n(G; B)$. It follows, by inverse duality, that $\mathrm{Hom}_{RG}(C, -)$ and $\mathrm{Hom}_{RG}(D, -)$ are naturally equivalent functors. This implies that C and D are isomorphic RG-modules. □

Theorem 9.4. A group G is an inverse duality group of dimension n over R if and only if the following three conditions hold:

(i) G is of type (FP) over R

(ii) $H^k(G; RG) = 0$ for $k \neq n$

(iii) $H^n(G; RG)$ is projective as an R-module.

Corollary 9.5. The group G is an inverse duality group over R if and only if G is a duality group whose dualizing module is R-projective.

Proof. (of Theorem 9.4). We have proved that inverse duality implies (i) - (iii). Conversely assume that the conditions (i) - (iii) hold for G and let

$$0 \to P_n \to P_{n-1} \to \dots \to P_0 \twoheadrightarrow R$$

be a finite RG-projective resolution. Then

$$0 \to P_0^* \to P_1^* \to \ldots \to P_n^* \twoheadrightarrow C$$

is a finite projective resolution of the (right) RG-module $C = H^n(G; RG)$. By Proposition 3.1 we now have natural isomorphisms

$$\psi: B \otimes_{RG} P_k \xrightarrow{\sim} \mathrm{Hom}_{RG}(P_k^*, B) \quad ,$$

for all $k \in \mathbf{Z}$ and all RG-modules B, whence, using (iii) and Lemma 9.2 (b),

$$H_k(G; B) \simeq \mathrm{Ext}_{RG}^{n-k}(C, B) \simeq H^{n-k}(G; \mathrm{Hom}_R(C, B)),$$

i.e. G is an inverse duality group. □

Remarks. 1) Examples of duality groups shall be discussed in Section 9.8. We do not know of an example of a duality group whose dualizing module is not even R-free.

2) One direction of Theorems 9.2 and 9.4 could have been proved by referring to the spectral sequences of Theorems 3.2 and 3.3. In fact these spectral sequences should be regarded as a more general version of the duality and inverse duality isomorphisms.

9.4 <u>The cap-product</u>. Let G be an arbitrary group. and $\underline{P} \twoheadrightarrow R$, $\underline{Q} \twoheadrightarrow R$ RG-projective resolutions of the trivial G-module R. For left RG-modules A and right RG-modules B, we consider the double complex homomorphism

$$\cap\colon \mathrm{Hom}_R(A,B) \otimes_{RG}(\underline{P} \otimes_R \underline{Q}) \to \mathrm{Hom}_R(\mathrm{Hom}_{RG}(\underline{P}, A), B \otimes_{RG} \underline{Q}),$$

given by $\cap(h \otimes p \otimes q)(f) = h(f(p)) \otimes q$, $h \in \mathrm{Hom}_R(A, B)$, $p \in \underline{P}$, $q \in \underline{Q}$, $f \in \mathrm{Hom}_{RG}(\underline{P}, A)$. Hereby G acts diagonally on $\mathrm{Hom}_R(A, B)$.

We shall use the notation $(\cap\, e)(f) = e \cap f$, $e = h \otimes p \otimes q$ and f as above. With ∂ denoting the boundary in the chain complexes $B \otimes_{RG} \underline{Q}$ and $\mathrm{Tot}(\mathrm{Hom}_R(A,B) \otimes_{RG}(\underline{P} \otimes_R \underline{Q}))$, and δ denoting the coboundary in the cochain complex $\mathrm{Hom}_{RG}(P, A)$, one finds the formula

$$\partial(e \cap f) = (-1)^{\deg f}(\partial e \cap f) + e \cap \delta f,$$

which shows that $\cap$ defines a homomorphism in homology. Notice that $\underline{P} \otimes_R \underline{Q} \twoheadrightarrow R$ with diagonal G-action is an RG-projective resolution, too, such that we have obtained maps

$$(*) \qquad \cap\colon \; H_n(G; \mathrm{Hom}_R(A,B)) \to \mathrm{Hom}_R(H^k(G; A), H_{n-k}(G; B))$$

for every pair of integers n, k. This is the cap-product. One can show that it does not depend upon the choice of $\underline{P}$ and $\underline{Q}$; it is natural in A, B and G and commutes (up to a sign) with connecting homomorphisms. For $n = k$ the cap-product map

$$\cap\colon H_n(G; \mathrm{Hom}_R(A, B)) \to \mathrm{Hom}_R(H^n(G; A), B_G)$$

coincides with the map induced by evaluation

$$w: \mathrm{Hom}_R(A, B) \otimes_{RG} P_n \to \mathrm{Hom}_R(\mathrm{Hom}_{RG}(P_n, A), B_G),$$

$w(h \otimes p)(f) = h(f(p)) \otimes_{RG} R$, with $\underline{P} \twoheadrightarrow R$ an arbitrary RG-projective resolution, and $h \in \mathrm{Hom}_R(A, B)$, $p \in P_n$, $f \in \mathrm{Hom}_{RG}(P_n, A)$.

We shall use two slightly different versions of the cap-product $(*)$. To obtain these, firstly replace B by the diagonal G-module $C \otimes_R A$, C a right RG-module, and compose $(*)$ with the homomorphism induced by $\alpha: C \to \mathrm{Hom}_R(A, C \otimes_R A)$, $\alpha(c)(a) = c \otimes a$, $c \in C$, $a \in A$. This yields the cap-product maps

$$(**) \qquad \cap: H_n(G; C) \to \mathrm{Hom}_R(H^k(G; A), H_{n-k}(G; C \otimes_R A)).$$

Secondly, replace A in $(**)$ by the diagonal G-module $\mathrm{Hom}_R(C, B)$, B a right RG-module, and compose $\cap$ with the homomorphism induced by evaluation $C \otimes_R \mathrm{Hom}_R(C, B) \to B$. This yields the modified cap-product

$$(***) \qquad \pitchfork: H_n(G; C) \to \mathrm{Hom}_R(H^k(G; \mathrm{Hom}_R(C, B)), H_{n-k}(G; B)).$$

Finally, it is sometimes useful to write the maps $(**)$ and $(***)$ in the form

$$\cap: H_n(G; C) \otimes_R H^k(G; A) \to H_{n-k}(G; C \otimes_R A)$$

$$\pitchfork: H_n(G; C) \otimes_R H^k(G; \mathrm{Hom}_R(C, B)) \to H_{n-k}(G; B),$$

just using the fact that $- \otimes_R X$ is left adjoint to $\mathrm{Hom}_R(X, -)$.

In the definition of duality groups, we did not require that the duality isomorphisms commute with connecting homomorphisms nor with maps induced by homomorphisms in the group argument. However, the next result shows that duality isomorphism can be given by a cap-product, and those are, of course, natural in any reasonable sense.

Theorem 9.5. Let G be a duality group of dimension n over R with dualizing module $C = H^n(G; RG)$. Then there is a "fundamental class" $e \in H_n(G; C)$ with the property that the cap-product with e produces isomorphisms

$$(e\cap -): \quad H^k(G; A) \xrightarrow{\cong} H_{n-k}(G; C \otimes_R A),$$

for every RG-module A and all $k \in \mathbb{Z}$. Moreover, if C is R-projective then the $\pitchfork$-product with e produces isomorphisms

$$(e\pitchfork -): H^k(G; \operatorname{Hom}_R(C, B)) \xrightarrow{\cong} H_{n-k}(G; B),$$

for every RG-module B and all $k \in \mathbb{Z}$.

Proof. G is of type (FP) over R with $cd_R G = n$ so that by Lemma 9.1 one has the isomorphisms

$$\phi: C \otimes_{RG} A \xrightarrow{\cong} H^n(G; A), \qquad \psi: H_n(G; B) \xrightarrow{\cong} \operatorname{Hom}_{RG}(C, B).$$

Let $e = \psi^{-1}(Id_C) \in H_n(G; C)$. The cap-product $H^n(G; A) \to C \otimes_{RG} A$ is given by

$$e \cap \pi(f) = (C \otimes_{RG} f)(e), \quad f \in \operatorname{Hom}_{RG}(P_n, A) \ ,$$

where $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R$ is an RG-projective resolution and π denotes the projection $\mathrm{Hom}_{RG}(P_n, A) \twoheadrightarrow H^n(G; A)$. Let $\sum c_i \otimes p_i \in C \otimes_{RG} P_n$ be a cycle representing e. Then we get for $A = RG$ and $f = \lambda \in P_n^*$,

$$e \cap \pi(f) = (C \otimes_{RG} \lambda)(e) = \sum c_i \otimes \lambda(p_i)$$

$$= \sum c_i \lambda(p_i) = \psi(\sum c_i \otimes p_i)(\lambda) = \psi(e)(\lambda).$$

Thus one has for every $c \in C$

$$(*) \qquad e \cap c = \psi(e).$$

Now, for an arbitrary element $a \in A$ let $\alpha : RG \to A$ denote the homomorphism given by $\alpha(1) = a$. Naturality of ϕ and $(e \cap -)$ gives rise to the commutative diagram

$$\begin{array}{ccccc} C \otimes_{RG} RG & \xrightarrow{\phi = Id} & H^n(G; RG) & \xrightarrow{(e\cap -)} & C \otimes_{RG} RG \\ \alpha_* \downarrow & & \alpha_* \downarrow & & \alpha_* \downarrow \\ C \otimes_{RG} A & \xrightarrow[\phi]{} & H^n(G; A) & \xrightarrow[(e\cap -)]{} & C \otimes_{RG} A \end{array} .$$

By $(*)$ the composite of the top row is the identity of C; since $a \in A$ was arbitrary, this shows that the composite of the bottom row is the identity of $C \otimes_{RG} A$, hence $(e \cap -) = \phi^{-1}$ is an isomorphism.

Next let $K \rightarrowtail F \twoheadrightarrow A$ be a short exact sequence of RG-modules with F RG-free. Since C is R-flat $C \otimes_R K \rightarrowtail C \otimes_R F \twoheadrightarrow C \otimes_R A$ is still a short exact sequence and hence naturality of the cap-product yields the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\ldots \to H^{n-1}(G;\ F) & \to & H^{n-1}(G;\ A) & \to & H^n(G;\ K) & \to & H^n(G;\ F) & \to & \ldots \\
(e\cap -)\ \downarrow & & \downarrow & & \downarrow \wr & & \downarrow \wr & & \\
\ldots \to 0 & \to & H_1(G; C \otimes_R A) & \to & H_0(G; C \otimes_R K) & \to & H_0(G; C \otimes_R F) & \to & \ldots
\end{array}$$

Since $H^{n-1}(G;\ F) = \oplus H^{n-1}(G;\ RG) = 0$ by Theorem 9.2 it follows that $(e\cap -)\colon H^{n-1}(G;\ A) \to H_1(G;\ C \otimes_R A)$ is an isomorphism, too, and we can iterate the argument.

Now, we consider the modified cap-product $(e \pitchfork -)$: $\mathrm{Hom}_{RG}(C,\ B) \to H_n(G;\ B)$. We claim that one has always

$$e \pitchfork h = H_n(G;\ h)\ (e), \quad h \in \mathrm{Hom}_{RG}(C,\ B), \quad e \in H_n(G;\ C).$$

Indeed, if $\sum c_i \otimes p_i \in C \otimes_{RG} P_n$ is a cocycle representing e, then $e\cap h = \sum (c_i \otimes h) \otimes p_i \in (C \otimes_R \mathrm{Hom}_R(C,\ B)) \otimes_{RG} P_n$, whence $e \pitchfork h = \sum h(c_i) \otimes p_i$, as asserted.

For any $f \in \mathrm{Hom}_{RG}(C,\ B)$, naturality of $(e \pitchfork -)$ and ψ yield the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{RG}(C,\ C) & \xrightarrow{(e\ \frown\ -)} & H_n(G;\ C) & \xrightarrow{\psi} & \mathrm{Hom}_{RG}(C,\ C) \\ f_* \downarrow & & f_* \downarrow & & f_* \downarrow \\ \mathrm{Hom}_{RG}(C,\ B) & \xrightarrow[(e\ \frown\ -)]{} & H_n(G;\ B) & \xrightarrow[\psi]{} & \mathrm{Hom}_{RG}(C,\ B) \end{array} .$$

If $\psi(e) = \mathrm{Id}_C$ it follows that the composite of the top row maps Id_C to Id_C; but as f was arbitrary this implies that the composite of the bottom row is the identity on $\mathrm{Hom}_{RG}(C,\ B)$, i.e., $(e\frown -) = \psi^{-1}$ is an isomorphism.

Finally, let $B \rightarrowtail I \twoheadrightarrow Q$ be a short exact sequence of RG-modules with I RG-injective. If C is R-projective then $\mathrm{Hom}_R(C,\ B) \rightarrowtail \mathrm{Hom}_R(C,\ I) \twoheadrightarrow \mathrm{Hom}_R(C,\ Q)$ is still a short exact sequence and one can use arguments dual to those above to show that $(e\frown -)$: $H^k(G;\ \mathrm{Hom}_R(C,\ B)) \to H_{n-k}(G;\ B)$ is an isomorphism for all k and all B. □

9.5 <u>The dualizing module</u>. Here we show that the dualizing module of a duality group has always some special features. First we list three general properties which summarize results of Sections 8.2 and 8.3.

<u>Proposition 9.6</u>. Let G be a duality group of dimension n over R and let $C = H^n(G;\ RG)$ be its dualizing module. Then one has:

(a) C is flat as an R-module.

(b) C is of type (FP) over RG, and $\mathrm{pr.dim}_{RG} C = n$.

(c) $\mathrm{Ext}^k_{RG}(C, RG) = 0$ for $k \neq n$, and $\cong R$ for $k = n$.

(d) Every RG-endomorphism of C is multiplication with a scalar $r \in R$.

Proof. (a) is assertion (iii) of Theorem 9.2. By Theorem 9.2(i) there is a finite projective resolution $0 \to P_n \to P_{n-1} \to \dots \to P_0 \twoheadrightarrow R$, and by (ii) its dual is a finite projective resolution for C, $0 \to P_0^* \to P_1^* \to \dots \to P_n^* \twoheadrightarrow C$; moreover, $\mathrm{Tor}_n^{RG}(C, R) = H_n(G; C) \cong H^0(G; R) = R$, whence (b). Using the above resolution $\underline{P}^* \twoheadrightarrow C$ in order to compute $\mathrm{Ext}^k_{RG}(C, RG)$ yields (c), since $\underline{P}^{**} \cong \underline{P}$. Finally, by Lemma 9.1 one has $H_n(G; B) \cong \mathrm{Hom}_R(C, B)$ for every RG-module B and thus obtains an R-isomorphism.

$$R = H^0(G; R) \cong H_n(G; C) \cong \mathrm{Hom}_{RG}(C, C) .$$

It is not hard to check that this is actually a ring homomorphism, whence (d). □

For the next results we need restrictions on the ring R.

Proposition 9.7. Let G be a duality group over R with dualizing module C. Then the following holds:

(a) If R has no zero divisors then C is indecomposable (with respect to direct sums) as an RG-module.

(b) If R is a p.i.d. (= principal ideal domain) and R_0 its field of fractions then either C is $\simeq R$ as an R-module or one has $\dim_{R_0}(V \otimes_R R_0) = \infty$ for every non-trivial RG-submodule $V \leq C$.

Proof. By Proposition 9.6(d) $\mathrm{End}_{RG}(C) \simeq R$. If $C = A \oplus B$, $A \neq 0 \neq B$, then $\mathrm{End}_{RG}(C)$ has the zero divisors $(A \oplus 0)(0 \oplus B) = 0$. (b) shall be proved later; it relies on the following result:

Theorem 9.8 (F.T.Farrell [26]). Let K be a field, n an integer ≥ 0 and assume that the group G meets the following properties:

(i) $\mathrm{cd}_K G$ is finite,

(ii) G is of type $(FP)_n$ over K,

(iii) $H^k(G; KG) = 0$ for all $0 \leq k \leq n-1$

(iv) $H^n(G; KG)$ contains a non-trivial G-invariant subspace V of finite dimension over K.

Then $\mathrm{cd}_K G = n$, hence G is of type (FP) over K, and $H^n(G; KG)$ is $\simeq K$ as a K-module.

Proof. As the case $n = 0$ is trivial we may assume that $n \geq 1$. Let $P_n \to P_{n-1} \to \ldots \to P_0 \twoheadrightarrow K$ be the beginning of a KG-projective resolution of the trivial G-module K with P_i finitely generated for all $0 \leq i \leq n$. $H^k(G; KG) = 0$ for all $0 \leq k \leq n-1$ implies that we get a finite projective resolution

$$0 \to P_0^* \to P_1^* \to \ldots \to P_n^* \twoheadrightarrow M$$

where M is the cokernel of $P_{n-1}^* \to P_n^*$. In particular $\text{pr.dim}_{KG} M \le n$. The submodule $V \le H^n(G; KG)$ is also a submodule of M, and the short exact sequence $V \rightarrowtail M \twoheadrightarrow M/V$ gives rise to the long exact sequence

$$\ldots \to \text{Ext}^m_{KG}(M, A) \to \text{Ext}^m_{KG}(V, A) \to \text{Ext}^{m+1}_{KG}(M/V, A) \to \ldots \quad .$$

Let $m = \text{cd}_K G$ and let $A = L \otimes_K KG$ be a free KG-module with $H^m(G; A) \neq 0$. By Lemma 9.3(b), $\text{Ext}^{m+1}_{KG}(M/V, A) = 0$ and $\text{Ext}^m_{KG}(V, A) \cong H^m(G; \text{Hom}_K(V, A))$ (here we use the assumption that K is a field). As $\dim_K V$ is finite, Proposition 3.1 and Lemma 2.9 yield isomorphism

$$\text{Hom}_K(\overset{\swarrow\searrow}{V, A}) \cong \overset{\searrow}{V^*} \otimes_K \overset{\searrow}{A} \qquad\qquad (V^* = \text{Hom}_K(V, K))$$

$$\cong (V^* \otimes_K L) \otimes_K \overset{\searrow}{KG} ,$$

where the arrows $\searrow$ indicate the G-action. It follows that $\text{Hom}_K(V, A)$ is the direct sum of $s = \dim_K V$ copies of A, hence $\text{Ext}^m_{KG}(V, A) \neq 0$ and hence $\text{Ext}^m_{KG}(M, A) \neq 0$, i.e., $m \le \text{pr.dim}_{KG} M \le n$. But we have also $H^n(G; KG) \neq 0$, whence $m = n = \text{cd}_K G$.

Now we know that G is of type (FP) over K with $H^k(G; KG) = 0$ for $k \neq n$, hence G is an inverse duality group by Theorem 9.4. Moreover, repeating the argument above with

$M = C = H^n(G; KG)$ and $A = KG$ we obtain an epimorphism

$$\mathrm{Ext}^n_{KG}(C, KG) \twoheadrightarrow \bigoplus_{s \text{ copies}} H^n(G; KG) .$$

But by Lemma 9.3(b) and inverse duality

$$\mathrm{Ext}^n_{KG}(C, KG) \simeq H^n(G;\mathrm{Hom}_K(C, KG)) \simeq H_0(G; KG) \simeq K,$$

hence $s = 1$ and $H^n(G; KG) \simeq K$.

<u>Remark</u>. Let G be a finitely generated infinite group with $\mathrm{cd}_K G < \infty$. Then, by Theorem 9.8., one has $\dim_K H^1(G; KG) = 0, 1$ or ∞.

Let G, in addition, be finitely presented. If $H^1(G; KG) \neq 0$ then, by Theorem 7.1, G has an α-decomposition $G = G_1 *_F G_2$ or a β-decomposition $G = G_1 *_{F,\sigma}$ ($|F| < \infty$), and the Mayer-Vietoris sequences show that $H^2(G_1; KG)$ is a direct summand of $H^2(G; KG)$. But G_1 is again of type $(FP)_2$, hence $H^2(G_1; KG) \simeq H^2(G_1; KG_1) \otimes_{KG_1} KG \simeq H^2(G_1; KG_1) \otimes_K KG/G_1$. This shows that $\dim_K H^2(G; KG) = 0$ or ∞. On the other hand, if $H^1(G; KG) = 0$ then Theorem 9.8 applies so that we have in any case $\dim_K H^2(G, KG) = 0, 1$ or ∞.

Using more subtle topological arguments one can show that both assertions hold even without assuming that $\mathrm{cd}_K G < \infty$. This is due to Hopf [32] for $H^1(G; KG)$ and to Farrell[26] for $H^2(G; KG)$. It would be interesting to know whether these results can be generalized to higher dimensions.

Proof (of Proposition 9.6(b)) Let R be a p.i.d. Let R_p denote either its field of fractions if $p = 0$ or the prime field $R_p = R/pR$ if p is a prime element of R. Let G be a duality group of dimension n over R, $C = H^n(G; RG)$. G is of type (FP) over every field R_p, $p \geq 0$, and by Corollary 3.6 we have $H^k(G; R_pG) = 0$ for all $k \neq n$ and RG-isomorphisms $H^n(G; R_pG) \simeq C \otimes_R R_p$.

Let $V \leq C$ be an RG-submodule with $0 < \dim_{R_0}(V \otimes_R R_0) < \infty$. Theorem 9.8 applied to $V \otimes_R R_0 \leq H^n(G; R_0G)$ yields $C \otimes_R R_0 \simeq R$; since C is R-flat this implies that $C \leq R_0$. Without loss of generality we may assume that $1 \in C$. Then every R-automorphism of C is multiplication with an element $r \in R_0$ with $rC = C$. Since $0 \neq H^n(G; R_pG) = C/pC$ for every prime $0 \neq p \in R$ this implies that r must be a unit in R. As C is finitely generated as an RG-module it now follows that C is finitely generated as an R-module; but since R is a p.i.d. this means that C is R-free, whence $C \simeq R$. □

Remark. The case when the dualizing module C of a duality group is $\simeq R$ is particularly interesting, and we introduce the following terminology: If G is a duality group of dimension n over R whose dualizing module $C = H^n(G; RG)$ has its underlying R-module $\simeq R$, then G is called a Poincaré duality group. If C is actually isomorphic to the trivial G-module R then the Poincaré duality group G is said to be oriented, otherwise non-oriented. Poincaré duality groups will be investigated in Section 9.10.

<u>Exercise</u>. Let $G = \langle x_1, x_2, \ldots, x_n; r = 1 \rangle$ be a torsion-free one relator group. Use the resolution $0 \to K/[K,K] \to \mathbb{Z}G \otimes_F \mathcal{I} \to \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ (where $K \to F \twoheadrightarrow G$ is the 1-relator presentation, c.f. the proof of Proposition 2.2) together with Lyndon's Theorem that $K/[K,K] \cong \mathbb{Z}G$, $r[K,K] \mapsto 1$, to prove that one has an isomorphism of right G-modules

$$H^2(G; \mathbb{Z}G) \cong \mathbb{Z}G / \sum \pi\left(\frac{\partial r}{\partial x_i}\right) \mathbb{Z}G.$$

(notation of Section 2.3). This formula will be used in Section 9.8, Example 6.

9.6 Extensions

<u>Theorem 9.9</u>. Let G be a group without R-torsion and let $S \leq G$ be a subgroup of finite index in G. Then G is a duality group over R if and only if S is.

<u>Proof</u>. By Proposition 5.7 $H^k(G; RG) \cong H^k(S; RS)$ for all $k \in \mathbb{Z}$, and by Theorem 5.11 $cd_R G = cd_R S$. Moreover, G is of type $(FP)_\infty$ if and only if S is, so that the assertion follows from Theorem 9.2.

<u>Remark</u>. Notice that the dualizing modules of G and S are isomorphic as RS-modules. In particular, G is a Poincaré duality group if and only if S is.

<u>Theorem 9.10</u>. Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups. Assume that N and Q are duality groups of dimension n

and q respectively over R. Then G is a duality group of dimension $n+q$ over R, and for the dualizing modules one has an RG-isomorphism

$$H^{n+q}(G;\ RG) \simeq H^q(Q;\ RQ) \otimes_R H^n(N;\ RN),$$

where G acts diagonally on the right hand side.

Proof. N and Q are of type (FP) over R hence so is G. The LHS-spectral sequence

$$E_2^{r,s} = H^r(Q;\ H^s(N;\ RG)) \Rightarrow H^{r+s}(G;\ RG)$$

collapses since $H^s(N;\ RG) = 0$ for $s \neq n$. Moreover by Proposition 5.4 and Lemma 5.6 one has RG-module isomorphisms

$$H^n(N;\ RG) \simeq H^n(N;\ RN) \overset{\swarrow\ \searrow}{\otimes_{RN}} RG \simeq H^n(N;\ RN) \otimes_R RQ,$$

i.e., $H^n(N;\ RG)$ is an induced RQ-module. It follows that $H^k(G;\ RG) = 0$ for $k \neq n+q$ and that $H^{n+q}(G;\ RG) = H^q(Q;\ RQ) \otimes_R H^n(N,\ RN)$, so that the assertion follows by Theorem 9.2. □

The next result is the converse of the above extension theorem. For technical reasons we need R to be a p.i.d.

Theorem 9.11. Let R be a p.i.d. Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of groups with $cd_R Q < \infty$ and N of type

$(FP)_\infty$ over R. If G is a duality group over R then so are N and Q.

The first step in the proof of Theorem 9.11 is to prove the assertion under the additional hypothesis that R be a field:

<u>Proof</u> (of Theorem 9.11 for R = K a field). The duality group G is of type (FP) hence so is Q by Proposition 2.7. Now consider the LHS -spectral sequence

$$E_2^{r,s} = H^r(Q, H^s(N; KG)) \Rightarrow H^{r+s}(G; KG).$$

By Proposition 5.4 and Lemma 5.6 one has KG-isomorphisms $H^s(N; KG) \simeq H^s(N; KN) \otimes_{KN} KG \simeq H^s(N; KN) \otimes_K KG$ and hence

$$E_2^{r,s} \simeq H^r(Q; KQ) \otimes_K H^s(N; KN), \quad r,s \in \mathbb{Z}$$

(here we have used that K is a field). Now, let n and q be the least integers with $H^n(N; KN) \neq 0$ and $H^q(Q; KQ) \neq 0$ respectively. Then $E_2^{r,s} = 0$ if either $r < q$ or $s < n$, hence a "corner argument" yields

$$E_2^{q,n} \simeq H^q(Q; KQ) \otimes_K H^n(N; KN) \simeq H^{q+n}(G; KG).$$

It follows that $H^{q+n}(G; KG) \neq 0$ (here we use again that K is a field) and hence $q + n = cd_R G$. Of course $q \leq cd_K Q$, $n \leq cd_R N$, and by Theorem 5.5 $cd_K N + cd_K Q = cd_K G$. Thus $q = cd_K Q$ and $n = cd_K N$, and the assertion follows by Theorem 9.2. □

Throughout the remainder of Section 9.6, R will be a p.i.d. R_i will denote either its field of fractions if $i = 0$ or the field $R_p = R/pR$ if $i = p$ is a prime element of R. If A is an R-module we use the notation ${}_pA$ for the submodule of all p-torsion elements and $t(A)$ for the submodule of all torsion elements of A.

Lemma 9.12 Let G be a group of type (FP) over R with $n = cd_R G > cd_{R_0} G$. Then there is a prime element $p \in R$ such that $H^{n-1}(G; R_pG) \neq 0$ and $H^n(G; R_pG) \neq 0$.

Proof. By assumption one has a finite RG-projective resolution $0 \to P_n \to P_{n-1} \to \dots \to R$, hence $H^n(G; RG) = \operatorname{coker}(P^*_{n-1} \to P^*_n)$ is finitely generated as an RG-module. On the other hand one has by Corollary 3.6 $0 = H^n(G; R_0G) \cong H^n(G; RG) \otimes_R R_0$, hence $H^n(G; RG)$ is an R-torsion module. It follows that $H^n(G; RG)$ is bounded, i.e., $I = \{c \in R \mid cH^n(G; RG) = 0\} \neq 0$. I is an ideal in R and hence generated by one element c_0: and since $H^n(G; RG) \neq 0$ there is at least one prime element $p \in R$ dividing c_0. Now, the short exact sequence $RG \xrightarrow{p} RG \twoheadrightarrow R_pG$ gives rise to the exact sequence

$$\dots \to H^{n-1}(G; R_pG) \to H^n(G; RG) \xrightarrow{p_*} H^n(G; RG) \to H^n(G; R_pG) \to \dots$$

p_* is multiplication by p and hence neither an epimorphism nor a monomorphism, whence the result. □

Proposition 9.13. Let G be a group of type (FP) over R. Then G is a duality group of dimension n over R if and only if

G is a duality group of dimension n over all fields R_i ($i = 0$ or prime).

Proof. If G is a duality group over R, then clearly G is a duality group over all fields R_i. Conversely, assume that G is a duality group of dimension n over all R_i. Then $cd_R G = n$ ($= cd_{R_0} G$), since otherwise one could find, by Lemma 9.12, a prime $p \in R$ with $H^{n-1}(G; R_p G) \neq 0 \neq H^n(G; R_p G)$, contradicting the assumption that G be a duality group over all R_i's. Next, notice that one has by Corollary 3.6 $0 = H^k(G; R_0 G) \simeq H^k(G; RG) \otimes_R R_0$ for all $k \neq n$, hence $H^k(G; RG)$ is an R-torsion module for all $k \neq n$. On the other hand, Corollary 3.6 yields also $0 = H^k(G; R_p G) \twoheadrightarrow \operatorname{Tor}_1^R (H^{k+1}(G; RG), R_p)$ for all $k \neq n$ and all prime elements $p \in R$, i.e., $H^k(G; RG)$ is R-torsion-free for all $k \neq n+1$. It follows that $H^k(G; RG) = 0$ for $k \neq n$ and $H^n(G; RG)$ is R-flat, whence G is a duality group over R by Theorem 9.2. □

Proof (of Theorem 9.11) Groups of type (FP) over R are of type (FP) over R_i, and duality groups over R are duality groups of the same dimension over all fields R_i. Thus Theorem 9.11 which is already proved over a field implies that N and Q are duality groups over all fields R_i. We claim that $cd_{R_p} Q = cd_{R_0} Q$ and $cd_{R_p} N = cd_{R_0} N$ for all prime elements $p \in R$. To see this notice first that one has always $cd_{R_p} Q \leq cd_R Q$ and $cd_{R_p} N \leq cd_R N$. Moreover, in the present situation $cd_R Q = cd_{R_0} Q$ and $cd_R N = cd_{R_0} N$, for otherwise the conclusion of Lemma 9.12 would contradict duality over all fields R_p. Thus

$$(\ast) \quad cd_{R_p} Q \le cd_{R_0} Q, \qquad cd_{R_p} N \le cd_{R_0} N, \quad \text{all primes } p.$$

But Theorem 5.5 and duality of G yield $cd_{R_p} Q + cd_{R_p} N =$ $= cd_{R_p} G = cd_{R_0} G = cd_{R_0} Q + cd_{R_0} N$, and this shows that the inequalities $(\ast)$ are actually equalities. Therefore we may apply Proposition 9.13 to conclude that G is a duality group over R. □

9.7. Amalgamated products and HNN-extensions. Here we discuss conditions under which amalgamated products or HNN-extensions of duality groups are again duality groups. We shall start with the following necessary dimension relation:

Proposition 9.14. Let G be a duality group of dimension n over R. Assume that G is a non-trivial amalgamated product $G = G_1 *_S G_2$ or an HNN-extension $G = G_1 *_{S,\sigma}$. Then, in either case, one has the relation

$$n-1 \le cd_R S \le cd_R G_i \le n.$$

Proof. Assume $cd_R S < n-1$. Then one has also $hd_R S < n-1$ and hence the Mayer-Vietoris sequences yield

$$H_n(G; B) \simeq \bigoplus_i H_n(G_i; B)$$

for every RG-module B. In particular for $B = C = H^n(G; RG)$ this implies that (without loss of generality) $H_n(G_1; C) \neq 0$. On the other hand we have

$$H_n(G_1; C) \simeq H_n(G; C \otimes_{RG_1} RG) \simeq H_n(G; C \otimes_R R(G/G_1)) \simeq H^0(G; R(G/G_1)),$$

but $H^0(G; R(G/G_1)) = 0$ unless $|G: G_1|$ is finite which is impossible in either case. This proves $n-1 \leq cd_R S$; the other implications are trivial. □

Remark. We shall see that all dimension combinations which comply with Proposition 9.14 actually do occur.

Proposition 9.15. Let G be an amalgamated product $G = G_1 *_S G_2$ or an HNN-extension $G = G_1 *_{S,\sigma}$. Assume that G_1 and G_2 are duality groups of dimension n and that S is a duality group of dimension $n-1$ over R. Then, in either case, G is a duality group of dimension n over R. For the dualizing modules one has the short exact sequence of RG-modules

$$H^{n-1}(S; RS) \otimes_{RS} RG \rightarrowtail H^n(G; RG) \twoheadrightarrow \oplus H^n(G_i; RG_i) \otimes_{RG_i} RG$$

Proof. Use the Mayer-Vietoris sequences and Theorem 9.2. □

Proposition 9.16. Let G be a non-trivial amalgamated product $G = G_1 *_S G_2$ or an HNN-extension $G = G_1 *_{S,\sigma}$. Assume that G_1, G_2 and S are duality groups of dimension $n-1$ over R. Then one has:

(a) If $cd_R G = n-1$ then G is a duality group of dimension $n-1$ over R and one has the short exact sequence of RG-modules

$$H^{n-1}(G;\ RG) \rightarrowtail \oplus\, H^{n-1}(G_i;\ RG_i) \otimes_{RG_i} RG \twoheadrightarrow H^{n-1}(S;\ RS) \otimes_{RS} RG.$$

(b) If S is of finite index in both G_1 and G_2 or, in the second case, if both S and $\sigma(S)$ are of finite index in G_1, then G is a duality group of dimension n and one has a short exact sequence of RG-modules which splits over R

$$\oplus\, H^{n-1}(G_i;\ RG_i) \otimes_{RG_i} RG \rightarrowtail H^{n-1}(S;\ RS) \otimes_{RS} RG \twoheadrightarrow H^{n}(G;\ RG)$$

<u>Proof</u>. (a) The Mayer-Vietoris sequences yield $H^k(G;\ RG) = 0$ for $k \neq n-1$ and also $H^{n-2}(G;\ RG \otimes_R L) = 0$ for every R-module L. As shown in the proof of Theorem 9.2 this implies that G is a duality group of dimension $n-1$.

(b) follows from the Mayer-Vietoris sequences together with the split monomorphisms of Theorems 6.3 and 6.6. □

<u>Exercise</u>. Prove a "mixed version" of Propositions 9.15 and 9.16 for $G = G_1 *_S G_2$ with $cd_R G_1 = cd_R S+1$ and $|G_2 : S| < \infty$.

<u>Remarks</u> (a) The examples 3) - 8) of the next Section 9.8 illustrate Propositions 9.15 and 9.16.

(b) The converse of Proposition 9.16(b) is false: There are most interesting cases where $G = G_1 *_S G_2$ with $|G_1:S| = |G_2:S| = \infty$, G_1, G_2, S duality groups of the same dimension $n-1$, but G a duality group of dimension n. This situation occurs e.g. if (G_1, S) and (G_2, S) are "Poincaré duality pairs of dimension n" in the sense of [13]. We are not going to touch the relative theory here, but an explicit example is given in Section 9.8, Example 9).

9.8 Examples, Low dimensions. We are now going to see that being a duality group is by no means an eccentric property, particularly among low dimensional groups.

Proposition 9.17. (a) G is a duality group of dimension 0 over R if and only if $|G|$ is finite and invertible in R.

(b) G is a duality group of dimension 1 over R if and only if G is finitely generated and $cd_R G = 1$.

(c) G is a duality group of dimension 2 over R, if and only if G is almost finitely presented, $\alpha\beta$-indecomposable*) and $cd_R G = 2$.

Proof. (a) is obvious from Theorem 9.2 and Proposition 4.12, and so is (b). As for (c), it is a consequence of Proposition 9.14 that all duality groups of dimension ≥ 2 are $\alpha\beta$-indecomposable. Conversely, if a finitely generated group G is $\alpha\beta$-indecomposable then, by Theorem 7.1, $H^1(G; RG) = 0$ for every ring R. This shows that $H^1(G; L \otimes_R RG) = 0$ for every cyclic R-module L; using a (module)-extension argument one then gets the same result for all

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*) See Section 7.1

finitely generated R-modules L. Finally if G is almost finitely presented then, by Theorem 1.3, the functor $H^1(G;-)$ commutes with the direct limit, hence $H^1(G; L \otimes_R RG) = 0$ for all R-modules L. Also, by Lemma 9.1, one has natural isomorphisms $H^2(G; L \otimes_R RG) \cong H^2(G; RG) \otimes_R L$; and as shown in Section 9.2(e) these two facts imply that $H^2(G; RG)$ is flat as an R-module. By Theorem 9.2 this shows that G is a duality group over R. □

Remarks. 1) It follows from (b) that every finitely generated free-by-finite group without R-torsion is a duality group of dimension 1 over R. Whether or not the converse of this holds is still open.

2) From (c) it follows that every (almost) finitely presented group G of cohomology dimension cdG = 2 (over $\mathbb{Z}$) is the free product of a finitely generated free group F and a finite number of duality groups G_i of dimension 2, $G = F * G_1 * G_2 * \cdots * G_m$.

3) Higher dimensional duality groups can be constructed by using Theorem 9.10 or Proposition 9.16(b).

Examples (of duality groups over $\mathbb{Z}$). 1) Every poly-(finitely generated free) group is a duality group.

2) Every torsion-free polycyclic group is a Poincaré duality group.

3) $G = \langle x,y;\ x^y = x^2 \rangle$ is an HNN-group over $\langle x \rangle$ with $\sigma: x \mapsto x^2$, hence G is a duality group of dimension 2 by Proposition 9.16(b).

4) $G = \langle x,y,z;\ x^y = x^2,\ y^z = y^2 \rangle$ is the free product of two copies of 3) amalgamated along y and x respectively. So G is a duality group of dimension 2 by Proposition 9.15.

5) The subgroup generated by x and z in 4) is in fact freely generated by x and z. Let G be the free product of two copies of 4) amalgamated along $x \mapsto z$, $z \mapsto x$. Then G is Higman's group

$$G = \langle w,x,y,z;\ w^x = w^2,\ x^y = x^2,\ y^z = y^2,\ z^w = z^2 \rangle .$$

By Proposition 9.15 G is a duality group of dimension 2.

<u>Remark and exercise</u>: Higman's group G has the property that all of its proper subgroups are of infinite index. Use the Mayer-Vietoris sequences to prove that $H_i(G;\mathbb{Z}) = 0 = H^i(G;\mathbb{Z})$ for every $i \neq 0$.

6) Let $G = \langle a,b,c,d;\ [a,b][c,d] = r = 1 \rangle$. G is a torsion-free one-relator group, hence $\mathrm{cd}G \leq 2$. By the exercise at the end of Section 9.5 $H^2(G;\mathbb{Z}G)$ is isomorphic to the quotient of $\mathbb{Z}G$ modulo the right ideal I generated by the images of the Fox derivatives. Using the notation of Section 2.3 but abandoning π by abuse of notation one has

$$\frac{\partial r}{\partial a} = 1 - aba^{-1} \qquad \frac{\partial r}{\partial b} = a - aba^{-1}b^{-1}$$

$$\frac{\partial r}{\partial c} = dcd^{-1}c^{-1} - d \qquad \frac{\partial r}{\partial d} = dcd^{-1} - 1 \quad ,$$

whence $I \subseteq \mathfrak{J}$. Now, checking that the following equations hold

$$1-a = \frac{\partial r}{\partial a}(1-a) - \frac{\partial r}{\partial b} b \qquad 1-b = \frac{\partial r}{\partial a}(ab-b+1) + \frac{\partial r}{\partial b}(b-b^2)$$

$$1-d = -\frac{\partial r}{\partial d}(1-d) + \frac{\partial r}{\partial c} c \qquad 1-c = \frac{\partial r}{\partial d}(dc-c+1) + \frac{\partial r}{\partial c}(c-c^2) ,$$

shows that actually $\dot{I} = \mathfrak{J}$, hence $H^2(G; \mathbb{Z}G) = \mathbb{Z}$ as G-modules. It follows that G is αβ-indecomposable (cf Remark to Theorem 9.8), hence $H^1(G; \mathbb{Z}G) = 0$ and G is an orientable Poincaré duality group of dimension 2.

7) Let g be an integer ≥ 1, and consider the group

$$G_g = \langle a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g ; [a_1,b_1][a_2,b_2]\dots[a_g,b_g] = 1 \rangle ,$$

the fundamental group of an oriented surface of genus g. It is well known that if $g \geq 2$ then G_g is contained in G_2 as a subgroup of finite index. It follows by Theorem 9.9 that G_g is an orientable Poincaré duality group of dimension 2 for all $g \geq 1$ $(G_1 = \mathbb{Z} \times \mathbb{Z})$. Let

$$H_g = \langle x_1, x_2, \dots, x_{g+1} ; x_1^2 x_2^2 \dots x_{g+1}^2 = 1 \rangle$$

be the fundamental group of a non-orientable surface. Then H_g is torsion-free and contains G_g as a subgroup of index 2, hence H_g is again a Poincaré duality group of dimension 2. The fact that the relator of H_g is not a product of commutators shows that $H_2(H_g; \mathbb{Z}) = 0$

(see [30] , p.170), hence H_g is not orientable.

8) A trivial example for the situation of Proposition 9.16(a) is $G = \langle x,y,z;-\rangle = \langle x,y;-\rangle *_{y=w} \langle z,w;-\rangle$.

9) The group $G = G_2$ of 6) and 7) is the free product of the two free groups $\langle a,b;-\rangle$, $\langle c,d;-\rangle$ with amalgamated cyclic subgroups generated by $[a,b]$ and $[d,c]$ respectively. As the amalgamated subgroup is of infinite index in both factors we are in the situation described in remark b) at the end of Section 9.7.

Remark. Notice that if G is a duality group which is obtained from the trivial group by applying finitely often the constructions of Theorems 9.9, 9.10, 9.15 and 9.16 to groups already obtained, then the dualizing module of G is R-free. I do not know whether all duality groups are obtainable this way.

9.9. Topological remarks. In this section we briefly mention some topological aspects of duality groups.

A CW-complex X is said to be a duality complex of formal dimension n if firstly X is dominated by a finite CW-complex (i.e., there is a map $Y \to X$ with a homotopy right inverse) and secondly there is a local coefficient system C on X and an element $e \in H_n(X; C)$ such that the cap-product with e yields isomorphisms

$$e \cap - : H^k(X; A) \xrightarrow{\sim} H_{n-k}(G; C \otimes A)$$

for every local coefficient system A and all $k \in \mathbb{Z}$. If the Abelian group C_x is infinite cyclic for every $x \in X$ then the duality complex X is a Poincaré complex as defined by C.T.C. Wall [62].

If, in particular, X is an Eilenberg-MacLane complex (i.e. aspherical, i.e., $\pi_i(X) = 0$ for $i \neq 1$) then the (co)homology with local coefficient systems is the group (co)homology of $G = \pi_1(X)$ and hence G is a duality group. Moreover, the fact that X is dominated by a finite CW-complex implies that G is finitely presented. Conversely, if G is a finitely presented group of type $(FP)_\infty$ then, by Theorem 1.9, G admits an Eilenberg-MacLane complex $X = K(G, 1)$ with finitely many cells in each dimension, and $cdG < \infty$ implies that X is actually finitely dominated. By Theorem 9.5 it now follows that X is a duality complex. We summarize:

<u>Proposition 9.18</u>. An Eilenberg-MacLane complex $K(G, 1)$ is a duality space if and only if G is a finitely presented duality group.

Now let M be a compact closed connected manifold of dimension m. Then M satisfies Poincaré duality with local coefficients; so if $\pi_i(M) = 0$ for $i \geq 2$ then $G = \pi_1(M)$ is a finitely presented Poincaré duality group. All known examples of Poincaré complexes which are not homotopy equivalent to a closed manifold are essentially simply connected; therefore it is conceivable that in fact every Poincaré duality group is the fundamental group of a closed aspherical manifold.

The most important source of Poincaré duality groups are the discrete subgroups of real Lie-groups. Let G be a real Lie-group and K a maximal compact subgroup of G. Then $X = G/K$ is (diffeomorphic to) $\mathbb{R}^n$, $n = \dim G - \dim K$. Every torsion-free discrete subgroup $\Gamma \leq G$ operates properly on X, i.e., every $x \in X$ has an open neighbourhood U with $U \cap \gamma(U) = \emptyset$ for every $1 \neq \gamma \in \Gamma$, so that the manifold X/Γ is an Eilenberg-MacLane complex $K(\Gamma,1)$. Moreover X/Γ is compact (with empty boundary) if and only if G/Γ is compact. Thus we have proved

Proposition 9.19. Let Γ be a torsion-free discrete subgroup of a real Lie-group G. If G/Γ is compact then Γ is a Poincaré duality group.

Now we turn to the non-Poincaré duality case. Assume that $M = K(G,1)$ is a compact connected m-dimensional manifold with non-empty boundary ∂M. Poincaré duality for the pair $(M, \partial M)$ with local coefficients yields

$$H^k(M; \mathbb{Z}G) \cong H_{m-k}(M \bmod \partial M; \mathbb{Z}G)$$

$$\cong H_{m-k}(\tilde{M} \bmod \partial\tilde{M}; \mathbb{Z}),$$

where $\tilde{M}$ denotes the universal covering complex of M. Notice that this holds in the non-orientable case, too, since the twisted action on $\mathbb{Z}G$ is isomorphic to the original one. Now since $\widetilde{M}$ is contractible one has $H_i(\widetilde{M}; \mathbb{Z}) = 0$ for all $i \neq 0$, hence the homology sequence for $(\widetilde{M}, \partial\widetilde{M})$ yields

$$H_{m-k}(\widetilde{M} \bmod \partial\widetilde{M}; \mathbb{Z}) = H_{m-k-1}(\partial\widetilde{M}; \mathbb{Z})$$

for all $k \in \mathbb{Z}$ (reduced homology for $k=m-1$). Thus applying Theorem 9.2 we have proved

Theorem 9.19. Let G be a group admitting an Eilenberg-MacLane complex M which is a compact connected m-dimensional manifold with non-empty boundary ∂M. If the (reduced) integral homology groups $H_i(\partial\widetilde{M})$ are $=0$ for $i \neq q$ and $H_q(\partial\widetilde{M})$ is torsion-free over $\mathbb{Z}$ then G is a duality group of dimension $n = m-q-1$.

Remarks. 1) As a special case of Theorem 9.19, assume that $M = K(G, 1)$ is a compact manifold with boundary $\partial M = K(S,1)$ and that S embeds into G. Then $\partial\widetilde{M}$ is the disjoint union of copies of the contractible space $\widetilde{\partial M}$, hence $H_i(\partial\widetilde{M}) = 0$ for all $i \neq 0$. G permutes the components of $\partial\widetilde{M}$ which are in (natural) one-to-one correspondence with the coset space G/S, hence the reduced group $H_0(\partial\widetilde{M})$ is isomorphic, as a right G-module, with $\ker(\mathbb{Z}(G/S) \twoheadrightarrow \mathbb{Z})$. Any non-cyclic knot group G provides an explicit example for this situation: the closed complement of the knot in S^3 is a $K(G,1)$ by Papakyriakopoulos' Theorem, and the fundamental group of its boundary torus embeds into G (see [48]).

2) Borel-Serre [14] have shown that every torsion-free arithmetic subgroup of an algebraic $\mathbb{Q}$-group is a duality group. This is established by constructing an Eilenberg-MacLane space K(G, 1) which is a compact manifold whose boundary has the homotopy type of a bouquet of spheres, i.e., the situation is exactly that of Theorem 9.19. Notice that Theorem 9.10 and Proposition 9.14 now yield some information on the structure of arithmetic groups.

9.10. Poincaré duality groups. In this section we collect some special results on Poincaré duality groups. Recall that these are those duality groups G whose dualizing module $C = H^n(G; \mathbb{Z}G)$ has its underlying additive group infinite cyclic. If C is actually the trivial G-module $\mathbb{Z}$ then G is called orientable, otherwise non-orientable. If G is non-orientable then it follows by Theorem 9.9 (with Remark) that the kernel N of the action on C is an orientable subgroup of index 2 in G. All other subgroups of index 2 are non-orientable, hence N is characteristic in G.

Obviously $\mathbb{Z}$ is the only Poincaré duality group of dimension 1 and is orientable. Poincaré duality groups of dimension 2 have been found in Section 9.8, Example 7, namely the fundamental groups of all 2-dimensional closed surfaces of genus ≥ 1. Whether or not this is a complete list of all 2-dimensional Poincaré duality groups is an open question; but there is some evidence that this might be the case as we shall see now.

Let G be an n-dimensional Poincaré duality group. Then $H_n(G; \mathbb{Z}) = \mathbb{Z}$ and $H^n(G; \mathbb{Z}) = \mathbb{Z}$ if G is orientable and $H_n(G; \mathbb{Z}) = 0$ and $H^n(G; \mathbb{Z}) = \mathbb{Z}_2$ otherwise. Also, by the universal-Coefficients Theorem $H^n(G; \mathbb{Z}) \simeq \mathrm{Hom}(H_n(G; \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}\,(H_{n-1}(G; \mathbb{Z}), \mathbb{Z})$, hence the torsion part of $H_{n-1}(G; \mathbb{Z})$ is $=0$ if G is orientable and $= \mathbb{Z}_2$ otherwise. Moreover, if G is orientable and $n = 2k$, then the cup-product yields a $(-1)^k$-commutative non-degenerate quadratic form $H^k(G; \mathbb{Q}) \otimes H^k(G; \mathbb{Q}) \to H^n(G; \mathbb{Q}) = \mathbb{Q}$, which implies, if k is odd, that the dimension of $H^k(G; \mathbb{Q}) \simeq H_k(G; \mathbb{Q}) \simeq H_k(G; \mathbb{Z}) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space is even. For $n = 2$ we conclude that $H_2(G; \mathbb{Z}) = \mathbb{Z}$ and $H_1(G; \mathbb{Z}) = \mathbb{Z}^{2g}$ if G is orientable and $H_2(G; \mathbb{Z}) = 0$, $H_1(G; \mathbb{Z}) = \mathbb{Z}^g \oplus \mathbb{Z}_2$ if G is non-orientable, where g is an integer ≥ 0. Thus every 2-dimensional duality group has the homology of a (uniquely determined) closed surface.

<u>Remark</u>. Whether or not the case $g = 0$ occurs is not known: Joel Cohen [20] has shown that if X is a <u>finite</u> Poincaré complex of formal dimension 2 with $H_1(X) = 0$ or $= \mathbb{Z}_2$ then X has the homotopy type of the 2-sphere or the 2-dimensional projective space respectively, hence X cannot be an Eilenberg-MacLane space. But I do not know the answer when X is merely finitely dominated.

Further evidence in favour of the conjecture that 2-dimensional Poincaré duality groups are surface groups is given in the following two results which we mention without proofs.

Theorem 9.20 (Dyer-Vasquez [23]) Every finitely presented orientable 2-dimensional Poincaré duality group which is not perfect is residually finite.

Theorem 9.21 (Farrell [25], [26]) Every subgroup of a 2-dimensional Poincare duality group is either locally free or of finite index.*

Notice that Theorem 9.20 proves the conjecture in the genus $g = 1$ case. Indeed if G is orientable one has a map $\phi: G \twoheadrightarrow \mathbb{Z} \times \mathbb{Z}$ inducing an isomorphism $\phi_*: H_*(G;\mathbb{Z}) \to H_*(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z})$ from which one deduces using the method of Stallings [56a] and Stammbach [57a] that ϕ is an isomorphism. The non-orientable case follows from the orientable case since $G = \langle x,y;\; x^{-1}yx = y^{-1} \rangle$, the fundamental group of the Klein bottle, is the only torsion-free non-Abelian group which contains $\mathbb{Z} \times \mathbb{Z}$ as a subgroup of index 2.

Proposition 9.22. Let G be a Poincaré duality group of dimension n. Then every subgroup $S \leq G$ is either of homology dimension $hdS \leq n-1$ or of finite index in G.

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* See Appendix, Theorem 10.

Proof. It is sufficient to consider the orientable case. Then one has for all right S-modules B,

$$H_n(S; B) \simeq H_n(G; B \otimes_S \mathbb{Z}G)$$

$$\simeq H^0(G; B \otimes_S \mathbb{Z}G)$$

but $B \otimes_S \mathbb{Z}G$ has no fixed elements $\neq 0$ unless S has finite index in G.

Remark. If $n = 2$ and S is (almost) finitely presented then by Theorem 4.6 cdS = hdS, hence Proposition 2.22 together with Stallings' result yields a weaker version of Theorem 2.21. It would be interesting to have a cohomology version of Proposition 2.22.

Now we discuss the solvable Poincaré duality groups. From the Theorems 9.9 and 9.10, it follows immediately that every torsion-free polycyclic group is a Poincaré duality group. We shall prove now that the converse holds, i.e.,

Theorem 9.23. Every solvable Poincaré duality group is polycyclic.

Proof. Let G be a solvable Poincaré duality of dimension m. G contains an orientable subgroup G_1 of index ≤ 2. G_1 is torsion-free and all its Abelian subgroups are of rank $\leq m$, hence G_1 has a unique maximal nilpotent normal subgroup $N \trianglelefteq G_1$ and G_1/N contains a finitely generated free-Abelian subgroup G_2/N of finite index (cf. [15a] or [0]). Let $n = hdN$, $r = hd(G_2/N)$; by Theorem 7.10 we know that the homological dimensions of torsion-free solvable groups coincide with the corresponding Hirsch numbers, so that $m = n+r$. Since G_2 is an orientable Poincaré duality group it follows by a corner argument in the Lyndon-Hochschild-Serre spectral sequence

$$\mathbb{Z} = H_m(G_2; \mathbb{Z}) \cong H_r(G_2/N; H_n(N; \mathbb{Z}))$$

$$\cong H^0(G_2/N; H_n(N; \mathbb{Z})).$$

By Proposition 7.12 $H_n(N;\mathbb{Z})$ is isomorphic to a subgroup of the additive group of $\mathbb{Q}$, hence we obtain that $H_n(N;\mathbb{Z}) \cong \mathbb{Z}$. But this implies, again by Proposition 7.12, that N is polycyclic, hence G_2, G_1 and G are polycyclic. □

Remark. F.E.A. Johnson [35] has shown that every torsion-free polycyclic-by-finite group is in fact the fundamental group of some compact closed aspherical manifold.

The following is a purely group theoretic criterion in order to decide whether or not a given (torsion-free) polycyclic group G is orientable: Take an invariant series

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \ldots \rhd G_d = 1$$

with Abelian quotients $Q_k = G_{k-1}/G_k$, $1 \leq k \leq d$ ($G_k \lhd G$). Conjugation with $x \in G$ induces automorphisms ϕ_k on the "torsion-free parts" (i.e., on the quotients Q_k/torsion) of all factors Q_k. Let

$$\mathrm{sign}(x) = \det(\phi_1)\ \det(\phi_2)\ldots\det(\phi_d)\ .$$

Then $\mathrm{sign}(x) = \pm 1$; one can show that sign(x) does not depend upon the choice of the invariant series, and G is orientable if and only if $\mathrm{sign}(x) = 1$ for all $x \in G$. If G is non-orientable then the set of all elements $x \in G$ with $\mathrm{sign}(x) = +1$ form the unique orientable subgroup of index 2.

Remark. Finitely generated solvable groups of finite cohomology dimension are necessarily of type A_4 in the sense of Malcev, i.e., they admit a finite series all of whose factors are either torsion-free Abelian of finite rank or finite. Now, being polycyclic the Poincaré duality groups are very special among the finitely generated solvable groups of type A_4. This contrasts the following result on solvable duality groups the proof of which shall be published in collaboration with Gilbert Baumslag.

Theorem 9.24. Every finitely generated solvable group of type A_4 can be embedded in a finite extension of a solvable duality group.

Exercise. For a group G of type (FP) the "naive" Euler characteristic is defined by

$$\chi\,(G) = \sum_{k=0}^{\infty} (-1)^k \dim H_k(G;\mathbb{Q})$$

One can show that $\chi(S) = |G:S|\,\chi(G)$ for every subgroup S of finite index in G (cf. K.S. Brown: Euler characteristics of discrete groups and G-spaces; Inventiones math.27 (1974), 229-264). Prove that if G is a Poincaré duality group of dimension n, then the following holds:

(a) if G is orientable and $n = 2k+1$ then $\chi(G) = 0$;

(b) if G is orientable and $n = 4k+2$ then $\chi(G)$ is even;

(c) if $\chi(G) \neq 0$ then G is co-Hopfian, i.e., G does not contain a proper subgroup $S < G$ with $S \cong G$.

Remarks and comments.

We add a few scattered remarks and comments on the contents of each section, mostly concerning the origin of results or proofs.

1.3 The Tor-part of Theorem 1.3 was proved in [11], the Ext-part is due to K.S.Brown [15]. The Theorem should also be compared with R.Strebel's finiteness criterion in [59]. (cf.Section 8.6). Proposition 1.5 is of course well known and usually proved by Shanuel's Lemma.

1.4 Ignorant of Brown's paper [15], J.C. Hausmann and myself stumbled on Theorem 1.9 and Corollary 1.12 in Vancouver, August 1974. Our proof was based directly on Wall [62].

2.1 The terminology "almost finitely presented" was introduced by Stallings [55] in a slightly different (topological) sense. Stallings shows that if G is almost finitely presented (over $\mathbb{Z}$) in the sense of Section 2.1, then G is almost finitely presented in his (topological) sense.

2.3 The free differential calculus was introduced by R. H. Fox [28].

2.4 A Mayer-Vietoris sequence for integral homology of an amalgamated product was obtained topologically by

Stallings [54]; the general sequences appear in Barr-Beck [2], Ribes [50] and Swan [60]. Our combinatorial proof of Proposition 2.8 (Swan [60], Lemma 2.1) seems to be new.

2.5 Proposition 2.11 and the Mayer-Vietoris sequences for HNN-groups have appeared in [7].

2.6 The groups A_n, B_n appear in [8]. The group B_2 was originally constructed as a candidate for a counter example to the Novikov Conjecture on the homotopy invariance of higher signatures [9].

3.2 Universal Coefficient spectral sequences seem to be folk-lore. Theorems 3.3 and 3.4 follow from the spectral Universal Coefficient Theorem 29 , p.100, or from the (cohomology version of the) spectral sequence of Dold [22]; cf. also F.Ischebeck [33].

4.2 Theorem 4.3 is due to Berstein [3].

5.3 This is the content of [6], Section 3. The present proof of the main result (Theorem 5.8) is considerably simpler than the original one.

5.4 A detailed proof of Serre's Theorem (including the verification of all signs) is given in D.Cohen's Notes [18].

Theorem 5.13 (the homology version of Serre's Theorem) seems to be new.

6.2, 6.3 Theorems 6.3, 6.6 and Corollaries 6.5, 6.7 are new. A weaker version of the Theorems is to be found in [12] Lemma 4.5 and [7] Lemma 5.4, respectively.

7.1 The concept of a "fundamental group of a graph of groups" was introduced by Bass-Serre in their theory of groups acting on a tree [53].

7.3 The first results on the (co)homology dimension of solvable groups are due to K.W. Gruenberg[30]: a complete description of cd for nilpotent and polycyclic groups (Theorem 7.13) and a necessary and a sufficient condition for the finiteness of cd for arbitrary solvable groups. Then U.Stammbach [57] has obtained $cd_{\mathbb{Q}} G = hG$ for arbitrary solvable groups, and Theorem 6.9 has first been proved by Fel'dman [27], (cf. also [4]). Theorem 7.15 and Corollary 7.16 are the main results of [4].

8.1 Is an improved version of [8], Section 5. The fact that we need not assume that G is of type (FP) in Theorem 8.4 (cf. [8], Corollary 6.5) was pointed out by R.Strebel [59].

8.3 Theorem 8.8(a) has been proved by Swan in the case of a finitely presented group of cd ≤ 2, cf. [30], p.156, where other references and results on the centre of groups with finite cd are to be found. Theorem 8.8 has also been proved for knot groups (see [44], Theorem 5.4.3 and 5.4.4), and for subgroups of torsion-free one-relator groups [38]. More precise results are available for torsion-free one-relator groups with non-trivial centre, cf. [49].

8.5 All results are to be found in [59a] and/or [59b]. Just Theorem 8.15 (b) is slightly improved by excluding the case $\alpha = \lambda+1$ (λ = limit ordinal).

9.1 - 9.4 Duality groups have been investigated in [10]. The treatment here follows [6] but makes use of the considerable simplifications due to result that duality groups are always of type (FP) (Brown [15], Strebel [59]). Inverse duality made its first appearance in [6], cf. also [26].

9.6 Theorem 9.11 is the main first result of [8].

9.7 is a slightly improved version of [12] and [7].

9.10 Poincaré duality groups have independently been investigated by Johnson-Wall [36] and myself [5]. Theorem 9.23 was proved in [5].

Some recent developments.

(added April 1981)

I. Groups of type $(FP)_m$

1. More striking examples than those constructed in Section 2.6 have recently been given by U. Stuhler [88]: Let K be a function field of transcendence degree 1 over a finite field, S a finite, non-empty set of places of K , and $\mathcal{O}_S \subset K$ the ring of S-integers. Then one has

Theorem 1 (Stuhler [86]). PGL $(2, \mathcal{O}_S)$ is of type $(FP)_m$ if and only if $|S| \geq m + 1$.

It has been conjectured that - more generally - any "S-arithmetic subgroup $\mathbf{G}(\mathcal{O}_S)$ of a simple algebraic group in the function field case" is of type $(FP)_m$ if and only if $|S| + \text{rk}\mathbf{G} \geq m + 2$. In addition to Stuhler's solution in the rank 1 case the conjecture has essentially been verfied for m = 2 by Behr [65], Rehmann-Soulé [86], and Hurrelbrink [81]. (Type $(FP)_2$ and finite presentability seem to coincide for those groups).

2. Type $(FP)_2$. Whether groups of type $(FP)_2$ are, in general, finitely presented is still an open question. Stuhler [86] shows that the answer is positive for the groups PGL $(2, \mathcal{O}_S)$ in Theorem 1. Moreover, we have

Theorem 2 (Bieri-Strebel [71]). Metabelian groups of type $(FP)_2$ (over some commutative ring R with 1) are finitely presented.

Ralph Strebel observed (see [71]) that Theorem 2 is sharp in the sense that there are 3-step soluble groups which are of type $(FP)_2$ over all fields but are not of type $(FP)_2$ over $\mathbb{Z}$ (and hence not finitely presented). Let $H \leq GL(4, \mathbb{Z}[\frac{1}{2}])$ be Abels' group [63], consisting of all groups upper triangular matrices with positive units in the diagonal. The centre Z of H is contained in the commutator subgroup of H , hence the homology 5-term exact sequence for $G = H/Z$ yields an epimorphism $H_2(G; \mathbb{Z}) \twoheadrightarrow Z$. But Z is isomorphic to the additive group of $\mathbb{Z}[\frac{1}{2}]$; hence $H_2(G; \mathbb{Z})$ cannot be finitely generated and so G is not of type $(FP)_2$ over $\mathbb{Z}$.

On the other hand Abels [63] has shown that H is finitely presented. Let $H = F/R$ where F is a finitely generated free group and $R \trianglelefteq F$ the normal closure of a finite subset of F, $G = F/R$. Then $Z \simeq R/N$ tensored, over $\mathbb{Z}$, with any field K yields an exact sequence

$$N/N' \otimes K \longrightarrow R/R' \otimes K \longrightarrow Z \otimes K \longrightarrow 0,$$

from which we conclude that $R/R' \otimes K$ is finitely generated as a KG-module, i.e., G is of type $(FP)_2$ over K.

3. Metabelian groups. Let

$$(*) \quad A \rightarrowtail G \twoheadrightarrow Q$$

be a short exact sequence of groups with G finitely generated and both A and Q Abelian. It is natural to ask for necessary and sufficient conditions, in terms of the Q-module A and possibly the of extension class of $(*)$ in $H^2(Q;A)$, for G to be of type $(FP)_m$. This problem was attacked in [71], [72] and also in [69]. This problem is still open, in general, but we now know that its solution is going to involve fairly deep connections between valuations of fields, convexity arguments, and the theorie of groups acting on simplicial complexes. I shall sketch the main results obtained so far.

Let Q be a finitely generated Abelian group. By a valuation of Q we mean a homomorphism $v: Q \longrightarrow \mathbb{R}$ into the additive group of $\mathbb{R}$; two valuations are equivalent if they coincide up to a positive constant scalar multiple. Then the set $S(Q)$ of all equivalence classes $[v]$ of non-trivial valuations v is called the valuation sphere; it can be identified with the unit sphere $S^{n-1} \subset \mathbb{R}^n$, where n is the $\mathbb{Z}$-rank of Q. Now, let R be a commutative ring with unity. In [71] and [72] we attach to every finitely generated RQ-module A a subset $\Sigma_A \subseteq S(Q)$ as follows: for every point $[v] \in S(Q)$ we consider the monoid $Q_v = \{q \in Q \mid v(g) \geq 0\} \subseteq Q$ and we define

$$\Sigma_A = \{[v] \mid A \text{ is finitely generated over } RQ_v\}.$$

If we wish to emphasize the ground ring we write $\Sigma_A(R)$ for Σ_A. One can show that Σ_A is always open in $S(Q)$.

Theorem 3 (Bieri-Strebel [71]). The metabelian group G in (*) is of type $(FP)_2$ if and only if $\Sigma_A(\mathbb{Z})$ together with its antipodal set $-\Sigma_A(\mathbb{Z})$ covers the sphere $S(Q)$.

Another way to express the condition $S(Q) = \Sigma_A \cup -\Sigma_A$ is to say that the set theoretic complement $\Sigma_A^c = S(Q) \setminus \Sigma_A$ contains no antipodal points; or, equivalently, that every pair of 2 points in Σ_A^c is contained in an open hemisphere. I suspect that this is the form in which Theorem 3 might generalize to:

Conjecture. The metabelian group G in (*) is of type $(FP)_m$ if and only if every m-point subset of $\Sigma_A^c(\mathbb{Z})$ is contained in an open hemisphere.

The main result of [69] establishes one of the implications "over a field".

Theorem 4 (Bieri-Groves [69]. If the metabelian group G in (*) is of type $(FP)_m$ over a field K then every m-point subset of $\Sigma_{A\otimes K}^c(K)$ is contained in an open hemisphere.

4. Type $(FP)_\infty$. Margulis [83] has shown that no subgroup of finite index in $SL(n,\mathbb{Z})$, $n \geq 3$, is a non-trivial amalgamated product. This answers the problem on p.36 in the negative. $SL(n,\mathbb{Z})$ is not contained in the class C.

Our knowledge concerning the influence of type $(FP)_\infty$ on the internal structure of a group is still meagre. I suspect that if G is a group of type $(FP)_\infty$ then the centre of G is finitely generated and the torsion-free subgroups of G are of finite cohomology dimension. Partial results in this direction are

Theorem 5 (Bieri [66]). The centre of a $\mathbb{Q}$-linear group ot type $(FP)_\infty$ is finitely generated.

This has been generalized by Alperin and Shalen [64] to subgroups G of GL(K), where K is a field of characteristic O and the Hirsch numbers of the unipotent subgroups of G are bounded.

Theorem 6 (Bieri-Groves [69]). Metabelian groups of type $(FP)_\infty$ are torsion-free-by-finite and of finite Abelian section rank.

In particular, such groups are of finite Hirsch number, whence the result that torsion-free metabelian groups of type $(FP)_\infty$ are, in fact, of type (FP).

II. Groups of finite cohomology dimension

5. linear groups. Serre's result [52] that every finitely generated torsion-free subgroup of GL($\mathbb{Q}$) has finite cohomology dimension was mentioned in the introduction (in fact Serre gave formulas for the precise cohomology dimension of various arithmetic and S-arithmetic groups [52], [14], [73]). This result has been generalized to

Theorem 7 (Alperin-Shalen [64]). Let R be a finitely generated integral domain of characteristic O and G a torsion-free subgroup of $GL_n(A)$, $n > o$. Then $cdG < \infty$ if and only if there is an upper bound for the Hirsch number of the unipotent subgroups of G.

6. Soluble groups. In Section 7.4 we asked whether soluble groups G with $cdG = hG < \infty$ are necessarily of type (FP). Some progress towards a solution of this problem has been made: D.Gildenhuys [79] gave a positive answer in the case when hG = 2 and hereby completed the classification of soluble groups with cd 2. (The same result was obtained by R.L. Snider [unpublished]). Further evidence in favour of a positive solution is the following nice result which generalizes Theorem 7.14 and Corollary 7.16.

Theorem 8 . (Gildenhuys-Strebel [8o]) (a) The class of all countable soluble groups G with cd $G = hG < \infty$ is closed with respect to taking homomorphic images

(b) Every torsion-free soluble group G with $cdG = hG < \infty$ is finitely generated.

7. Improved application. Using M. Dunwoody's general solution for the problem of classifying all groups G with $cd_R G \leq 1$ (see [76]) enables one to improve several results of Sections 8.1 and 8.2. Instead of Theorem 8.4 one can prove, e.g.: Let G be a finitely generated group with cd $G \leq n < \infty$, and $N \triangleleft G$ a normal subgroup of type (FP) with $H^i(N; \mathbb{Z}N) = O$ for $o \leq i \leq n-2$. Then G/N is free-by-finite, and if G/N is infinite then cd $N = n-1$ (see [67]). As in Section 8.2 this is a purely group theoretic

statement if n = 2: If G is a finitely generated group with cd $G \leq 2$ and $1 \neq N$ G a finitely presented normal subgroup with infinite index then both N and G/N are free-by-finite. More precise information is available if G is a one-relator-group: Using an Euler characteristic argument one obtains in addition, that G is torsion-free, generated by 2 elements, and either N is cyclic or G/N is cyclic-by-finite. (see [67], [84]).

III, Duality groups

8. Farrell-Tate Cohomology. Duality groups have turned out to be central for Farrell's extension of Tate-Cohomology to groups which are virtually of type (FP). Good accounts of the Farrell theory, which we cannot go into here, are given in [78], [74], or in the forthcoming book of K.Brown [75].

9. Poincaré duality groups and pairs. In [68] the notion of Poincaré duality group was extended to pairs (G, $\underline{S}$) where G is a group and $\underline{S}$ a finite family of subgroups of G. One motivation to di so is this: There is a procedure of "pasting Poincaré duality pairs together" in terms of amalgamated products and/or HNN-extension, which is the precise analog of pasting manifolds-with-boundary along homeomorphic boundary components.This leads to new Poincaré duality pairs and eventually to (absolute) Poincaré duality groups. Conversely, one can try to cut a given Poincaré duality group into pairs which are easier to investigate. In this way Eckmann and Müller were able to prove

Theorem 9 (Eckmann-Müller [77]). Let G be a Poincaré duality group of dimension 2. If G/G' is infinite then G is isomorphic to the fundamental group of a closed surface.

It is not known whether two dimensional Poincaré duality groups with fine abelianization exist. For such a group G one would have $|G/G'| \leq 2$, and one can show that G' would not contain a proper subgroup of finite index. By Strebel's result below it follows that every subgroup of G not equal to G or to G' would be free!

Theorem 1o (Strebel [87]). The subgroups of infinite index in a two-dimensional Poincaré duality group are free.

Let G be a group having a normal series $G = G_o \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots \trianglerighteq G_m = 1$ such that every factor G_i/G_{i+1} is isomorphic to the fundamental group of a closed surface. F.E.A. Johnson [82] has shown that G contains a subgroup of finite index which is isomorphic to the fundamental group of a smooth closed aspherical manifold (of dimension 2m). It is not known whether the same holds for G itself. Similar results hold when the factors G_i/G_{i+1} are certain discrete cocompact subgroups of Lie-Groups. [83]

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