The Lawrence-Krammer representation

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Abstract The Lawrence-Krammer representation of the braid groups recently came to prominence when it was shown to be faithful by myself and Krammer. It is an action of the braid group on a certain homology module $H_2(\tilde{C})$ over the ring of Laurent polynomials in q and t. In this paper we describe some surfaces in \tilde{C} representing elements of homology. We use these to give a new proof that $H_2(\tilde{C})$ is a free module. We also show that the (n-2,2) representation of the Temperley-Lieb algebra is the image of a map to relative homology at $t = -q^{-1}$, clarifying work of Lawrence.

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1 Introduction

The Lawrence-Krammer representation is the action of the braid group B_n on a certain homology module $H_2(\tilde{C})$ over the ring $\Lambda = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$. It was introduced by Lawrence [Law90], and recently came to prominence when it was shown to be faithful for n = 4 [Kra00], and then for all n [Big01], [Kra02]. Since then, a number of papers have appeared which closely examine certain aspects of the representation. See [PP01], [Son02], and [Bud02]. We now continue in this tradition, with the specific goal of understanding the connection with the Temperley-Lieb algebra.

This paper was partly motivated by an attempt to clarify two points from [Big01]. First, the pairing between a "noodle" and a "fork" involved an algebraic intersection number between two non-compact surfaces in \tilde{C} . Such a thing is not necessarily well-defined, so I gave an indirect proof of the existence

of a certain closed surface corresponding to a fork. I now have an explicit description of this surface, which will be given in Section 3.4. In fact it is possible to define the pairing without reference to this surface by using results from [Kaw96, Appendix E]. (Thanks to Won Taek Song, whose paper [Son02] drew my attention to this.)

Second, to compute matrices for the representation, I tensored $H_2(\tilde{C})$ with a field containing Λ . The resulting vector space contains $H_2(\tilde{C})$, but strictly speaking, the action of B_n on this vector space should not be called the Lawrence-Krammer representation. In Section 4 we give a new proof that $H_2(\tilde{C})$ is a free Λ -module. This is originally due to Paoluzzi and Paris [PP01], but our proof uses an explicit description of surfaces representing elements of a free basis of $H_2(\tilde{C})$. The correct matrices for the Lawrence-Krammer representation could be computed from this, but we do not do so since they are quite complicated.

By addressing these two issues from [Big01] we also shed new light on the work of Lawrence [Law90]. There, the Lawrence-Krammer representation was used to give a topological interpretation to a representation of the Temperley-Lieb algebra. The idea is to specialise t to the value $-q^{-1}$, at which point the Lawrence-Krammer representation becomes reducible and the desired Temperley-Lieb representation appears as a quotient. Somewhat complicated methods are used in [Law90] to define the required quotient. In Section 5 we show that it is simply the image of a map to a certain relative homology module. The significance of the value $t = -q^{-1}$ can then be seen to follow from the existence of certain kind of surface in \tilde{C} that maps to a multiple of 1 + qt in relative homology.

Throughout this paper n is a positive integer, D is the unit disk centred at the origin in the complex plane, $-1 < p_1 < \ldots < p_n < 1$ are real numbers, and $D_n = D \setminus \{p_1, \ldots, p_n\}$ is the n-times punctured disk. The braid group B_n is the mapping class group of D_n . We also assume familiarity with the presentation of B_n using Artin generators σ_i , and the interpretation of B_n as the fundamental group of a configuration space in the plane.

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2 Definitions

2.1 The Lawrence-Krammer representation

We briefly review the definition of the Lawrence-Krammer representation as given in [Big01]. Let C be the space of unordered pairs of distinct points in D_n . Let $c_0 = \{d_1, d_2\}$ be a basepoint in C, where d_1 and d_2 are distinct points on the boundary of the disk.

We define a homomorphism

$$\Phi \colon \pi_1(C, c_0) \to \langle q \rangle \oplus \langle t \rangle$$

as follows. Suppose $\alpha: I \to C$ is a closed loop in C representing an element of $\pi_1(C, c_0)$. By ignoring the puncture points we can consider α as a loop in the space of unordered pairs of points in the disk, and hence as a braid in B_2 . Let b be the exponent of this braid in the Artin generator σ_1 . Similarly, the map

$$s \mapsto \{p_1, \ldots, p_n\} \cup \alpha(s)$$

determines a braid in B_{n+2} . Let b' be the the exponent sum of this braid in the Artin generators σ_i of B_{n+2} . Note that b and b' have the same parity. Let $a = \frac{1}{2}(b'-b)$. We define

$$\Phi(\alpha) = q^a t^b.$$

Let \tilde{C} be the connected covering space of C such that $\pi_1(\tilde{C}) = \ker(\Phi)$. Fix a choice of \tilde{c}_0 in the fibre over c_0 . The homology group $H_2(\tilde{C})$ admits a Λ -module structure. Suppose f is a homeomorphism from D_n to itself, representing an element of B_n . The induced map f_* from $\pi_1(C, c_0)$ to itself has the property

$$\Phi \circ f_* = \Phi.$$

It follows that f has a unique lift

$$\tilde{f}: (\tilde{C}, \tilde{c}_0) \to (\tilde{C}, \tilde{c}_0).$$

Further, the induced map

$$\tilde{f}_*: H_2(\tilde{C}) \to H_2(\tilde{C})$$

is a Λ -module automorphism. We define the Lawrence-Krammer representation to be this action of B_n on $H_2(\tilde{C})$.

2.2 An intersection pairing

We now introduce some relative homology modules and an intersection pairing. For $\epsilon > 0$, let ν_{ϵ} be the set of points $\{x, y\} \in C$ such that either x and y are within distance ϵ of each other, or at least one of them is within distance ϵ of a puncture point. Let $\tilde{\nu}_{\epsilon}$ be the preimage of ν_{ϵ} in \tilde{C} . The relative homology modules $H_2(\tilde{C}, \tilde{\nu}_{\epsilon})$ are nested by inclusion. Let

$$H_2(\tilde{C}, \tilde{\nu}) = \lim_{\epsilon \to 0} H_2(\tilde{C}, \tilde{\nu}_{\epsilon}),$$

and

$$H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu}) = \lim_{\epsilon \to 0} H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu}_{\epsilon}).$$

The braid group B_n acts on these, and on $H_2(\tilde{C}, \partial \tilde{C})$, by Λ -module automorphisms.

For $x \in H_2(\tilde{C})$ and $y \in H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu})$ let $(x \cdot y) \in \mathbb{Z}$ denote the standard intersection number. We define an intersection pairing

$$\langle \cdot, \cdot \rangle \colon H_2(\tilde{C}) \times H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu}) \to \Lambda$$

by

$$\langle x,y\rangle = \sum_{a,b\in \mathbf{Z}} (x\cdot q^a t^b y) q^a t^b.$$

A similar definition gives a pairing

$$\langle \cdot, \cdot \rangle' \colon H_2(\tilde{C}, \tilde{\nu}) \times H_2(\tilde{C}, \partial \tilde{C}) \to \Lambda.$$

To prove that these are well defined requires some elementary homology theory. See [Kaw96, Appendix E].

For
$$x \in H_2(\tilde{C})$$
, $y \in H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu})$, $\sigma \in B_n$, and $\lambda \in \Lambda$, we have

$$\langle \sigma x, \sigma y \rangle = \langle x, y \rangle,$$

and

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle = \langle x, \overline{\lambda}y \rangle,$$

where $\bar{\lambda}$ is the image of λ under the automorphism of Λ taking q to q^{-1} and t to t^{-1} . Similar identities hold for $\langle \cdot, \cdot \rangle'$.



Figure 1: $s_{i,j}$ for j = i + 2 and j > i + 2.

2.3 The Temperley-Lieb algebra

A good introduction to the Temperley-Lieb algebra and its representation theory is [Wes95]. We review some of the basics here.

Let R be a domain containing an invertible element q with a square root $q^{\frac{1}{2}}$. The Temperley-Lieb algebra $TL_n(R)$ is the R-algebra given by generators $1, e_1, \ldots, e_{n-1}$ and relations

- $e_i e_j = e_j e_i$ if |i j| > 1,
- $e_i e_j e_i = e_i$ if |i j| = 1,
- $e_i^2 = (-q^{\frac{1}{2}} q^{-\frac{1}{2}})e_i$.

We simply write TL_n if R is understood. There is a map from B_n to TL_n given by $\sigma_i \mapsto 1 + q^{\frac{1}{2}} e_i$. Thus any representation of TL_n gives rise to a representation of B_n .

Let M be the left-ideal of TL_n generated by e_1e_3 , and let N be the left-ideal generated by $\{e_5, \ldots, e_{n-1}\}$. The representation of TL_n corresponding to the partition (n-2,2) is the TL_n -module $S = M/(M \cap N)$. This is called S(n,2) in [Wes95]. It is a free R-module with basis

$$s_{i,j} = (e_i \dots e_3 e_2)(e_{j-1} \dots e_5 e_4)(e_1 e_3)$$

for $1 \le i < j \le n$ such that $j > \max(i+1,3)$. There is a diagrammatic interpretation of TL_n , in which $s_{i,j}$ corresponds to the diagram shown in Figure 1.

3 Some surfaces

We describe some immersed surfaces in \tilde{C} representing elements of homology and relative homology. In each case we will describe a map from a surface to Cwhich can be lifted to a map to \tilde{C} . We will not specify a choice of lift, and we will not pay much attention to issues of orientation. Thus the resulting element of homology or relative homology will only be defined up to multiplication by a unit in Λ . This will be sufficient for our purposes.

3.1 Squares and triangles

We describe some properly embedded surfaces representing elements of relative homology modules such as $H_2(\tilde{C}, \tilde{\nu})$. These will be represented by one or two embedded edges in the disk.

Suppose $\alpha_1, \alpha_2: I \to D$ are disjoint embeddings of the interior of I into D_n , and map the endpoints of I to puncture points (not necessarily injectively). Let f be the map from the interior of $I \times I$ to C given by $f(x,y) = \{\alpha_1(x), \alpha_2(y)\}$. A lift of f to \tilde{C} will represent an element of $H_2(\tilde{C}, \tilde{\nu})$. Similarly, we can define an element of $H_2(\tilde{C}, \partial \tilde{C})$ corresponding to a pair of disjoint edges in D_n with endpoints on ∂D . Finally, we can define an element of $H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu})$ corresponding to a pair of edges which have a mixture of endpoints on ∂D and on puncture points. For all of these examples, call the resulting element of relative homology the square corresponding to the edges α_1 and α_2 .

Now suppose $\alpha: I \to D$ is an embedding of the interior of I into D_n , and maps the endpoints of I to the puncture points. Define a map

$$f: \{(x, y) \in I \times I : 0 < x < y < 1\} \to C$$

by $f(x,y) = \{f(x), f(y)\}$. A lift of f to \tilde{C} will represent an element of $H_2(\tilde{C}, \tilde{\nu})$. Similarly, we can obtain an element of $H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu})$ if we allow one or both endpoints of the edge to be on ∂D . For all of these examples, call the resulting element of relative homology the *triangle* corresponding to the edge α .

3.2 A genus one surface

Suppose $\alpha_1, \alpha_2: S^1 \to D_n$ are disjoint figure-eights, each going around two puncture points, as in Figure 2. Define a map f from the torus $S^1 \times S^1$ to C



Figure 2: A torus and a square.



Figure 3: The disk B and edges $\alpha_1(I)$ and $\alpha_2(I)$.

by $f(x,y) = \{\alpha_1(x), \alpha_2(y)\}$. Both the meridian and longitude of the torus are mapped into the kernel of Φ , so f lifts to a map

$$\tilde{f}: S^1 \times S^1 \to \tilde{C}.$$

We obtain an element $[\tilde{f}]$ of $H_2(\tilde{C})$. The image of $[\tilde{f}]$ in $H_2(\tilde{C}, \tilde{\nu})$ is $(1-q)^2$ times the square as indicated in Figure 2.

3.3 A genus two surface

Suppose $\alpha_1, \alpha_2: S^1 \to D_n$ are figure-eights such that α_1 passes around p_i and p_j , α_2 passes around p_j and p_k , and α_1 intersects α_2 twice, as in Figure 4. Let $B \subset D$ be a disk centred at p_j , containing the two points of intersection, and meeting each of α_1 and α_2 in a single edge. For i = 1, 2, let $I_i \subset S^1$ be the interval $\alpha_i^{-1}(B)$. For convenience, assume $I_1 = I_2$, and identify both with I = [0, 1], oriented as shown in Figure 3.

Let T be the closure of

$$(S^1 \times S^1) \setminus (I_1 \times I_2).$$

Note that for $(x, y) \in T$ we have $\alpha_1(x) \neq \alpha_2(y)$. We can therefore define $f: T \to C$ by $f(x, y) = \{\alpha_1(x), \alpha_2(y)\}.$

Let f_0 be the restriction of f to $\partial T = \partial (I \times I)$. For $s \in I$, let f_s be the composition of f_0 with an anticlockwise rotation of B by an angle of $s\pi$ about



Figure 4: A genus two surface and a square.

the centre p_j . We can assume B has a rotational symmetry so that $f_1(x, y) = f_0(y, x)$ for all $(x, y) \in \partial(I \times I)$. Thus f_0 and f_1 represent the same loops, but with opposite orientations.

We now build a closed genus two surface Σ_2 by gluing together $\partial T \times I$ and two copies of T as follows. First glue $\partial T \times I$ to T by $((x, y), 0) \sim (x, y)$. Then glue $\partial T \times I$ to a second copy T' of T by $((x, y), 1) \sim (y, x)$. Let Σ_2 be the surface so obtained. Let $g: \Sigma_2 \to C$ be given by $g|_T = g|_{T'} = f$ and $g|_{(\partial T \times \{s\})} = f_s$.

The fundamental group of Σ_2 is generated by the meridian and longitude of T and T'. Each of these is mapped by g into the kernel of Φ , so g lifts to a map $\tilde{g}: \Sigma_2 \to \tilde{C}$. We obtain an element $[\tilde{g}]$ of $H_2(\tilde{C})$.

We now compute the image of $[\tilde{g}]$ in $H_2(\tilde{C}, \tilde{\nu})$. For any $\epsilon > 0$ we can assume that the disk *B* has radius less than ϵ , and thus *g* maps $\partial T \times I$ into ν_{ϵ} . Then $\tilde{g}|_T$ represents an element of $H_2(\tilde{C}, \tilde{\nu}_{\epsilon})$. This element is $(1-q)^2$ times a square. It remains to figure out how $\tilde{g}|_{T'}$ is related to $\tilde{g}|_T$.

Consider the path $\delta: I \to \Sigma_2$ given by

$$\delta(s) = ((0,0), s) \in \partial T \times I.$$

This goes from $(0,0) \in T$ to $(0,0) \in T'$. Now $g \circ \delta$ is a loop in C in which the pair of points switch places by an anticlockwise rotation through an angle of π around p_j . Thus $\Phi(g \circ \delta) = qt$. It follows that $\tilde{g}|_{T'} = qt \circ \tilde{g}|_T$.

Also note that T' and T inherit the same orientation from Σ_2 . This is because the ends of the annulus $\partial T \times I$ were attached with opposite orientations. We conclude that the image of $[\tilde{g}]$ in $H_2(\tilde{C}, \tilde{\nu})$ is $(1-q)^2(1+qt)$ times a square, as shown in Figure 4.

3.4 A genus three surface

Suppose $\alpha_1, \alpha_2: S^1 \to D_n$ are figure-eights, both passing around p_i and p_j , and intersecting transversely at four points, as in Figure 5. We construct a map from

$$(\bullet, \bullet)$$
 $\mapsto (1-q)^2(1+qt)(1-t)$ \bullet .

Figure 5: A genus three surface and a triangle.

a genus three surface Σ_3 into \tilde{C} by a slight modification of the procedure used above for Σ_2 . This time the surface T will be a torus with two disks removed, one for each of p_i and p_j . Two annuli are then needed to glue together two copies of T. We obtain a genus three surface Σ_3 and a map $g: \Sigma_3 \to C$. The image of the longitude, meridian, and both boundary components of T all lie in the kernel of Φ , so g lifts to a map $\tilde{g}: \Sigma_3 \to \tilde{C}$. This represents an element $[\tilde{g}]$ of $H_2(\tilde{C})$.

The image of $[\tilde{g}]$ in $H_2(\tilde{C}, \tilde{\nu})$ is $(1+qt)(1-q)^2$ times a square, which in turn is $(1+qt)(1-q)^2(1-t)$ times a triangle, as shown in Figure 5.

4 A basis

In this section, we give a new proof of the following theorem.

Theorem 4.1 $H_2(\tilde{C})$ is a free Λ -module of rank $\binom{n}{2}$.

This is originally due to Paoluzzi and Paris [PP01]. Our proof gives a more explicit description of surfaces representing elements of a free basis for $H_2(\tilde{C})$. We use the following basic result about $H_2(\tilde{C})$.

Lemma 4.2 The vector space $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ has dimension $\binom{n}{2}$. The natural map from $H_2(\tilde{C})$ to $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ is injective.

The proof uses a 2-complex that is homotopy equivalent to C. Various methods for constructing such a complex can be found in [Law90], [Big01], [PP01], and [Bud02].

4.1 Proof of the theorem

We now prove Theorem 4.1. For $1 \leq i < j \leq n$, we define $v'_{i,j} \in H_2(\tilde{C}, \tilde{\nu})$ as follows. If j - i > 2, let $v'_{i,j}$ be the square corresponding to the edges $[p_i, p_{i+1}]$

and $[p_{j-1}, p_j]$. If j-i=2 and i>1, let $v'_{i,j}$ be the square corresponding to the edge $[p_i, p_{i+1}]$ and an edge from p_{i-1} to p_j whose interior lies in the lower half plane. Let $v'_{1,3}$ be the square corresponding to the edges $[p_1, p_2]$ and $[p_2, p_3]$. If j-i=1, let $v'_{i,j}$ be the triangle corresponding to the edge $[p_i, p_j]$.

For $1 \leq i < j \leq n$, let $v_{i,j}$ be an element of $H_2(\tilde{C})$ whose image in $H_2(\tilde{C}, \tilde{\nu})$ is

- $(1-q)^2(1+qt)(1-t)v'_{i,j}$ if j=i+1,
- $(1-q)^2(1+qt)v'_{i,j}$ if i=1 and j=3,
- $(1-q)^2 v'_{i,j}$ otherwise.

Such an element exists by Section 3. We will show that the $v_{i,j}$ form a free basis for $H_2(\tilde{C})$.

Every $x_{i,j} \in H_2(\tilde{C}, \partial \tilde{C})$ be the square corresponding to a pair of vertical edges, one passing just to the right of p_i , the other passing just to the left of p_j , and both having endpoints on ∂D .

Lemma 4.3 The following pairings are units in Λ .

- $\langle v'_{i,j}, x_{i,j} \rangle'$ for $1 \le i < j \le n$,
- $\langle v'_{i,i+2}, x_{i-1,i+1} \rangle'$ for $i = 2, \dots, n-2$,
- $\langle v'_{i,i+2}, x_{i,i+1} \rangle'$ for $i = 2, \dots, n-2$.

All other pairings $\langle v'_{i',j'}, x_{i,j} \rangle'$ are zero.

Proof Project the surfaces representing $v'_{i',j'}$ and $x_{i,j}$ to C. If the resulting surfaces intersect transversely at precisely one point then $\langle v'_{i',j'}, x_{i,j} \rangle'$ is a unit in Λ . If they do not intersect at all then the pairing is zero.

For convenience, choose lifts and orientations so that

$$\langle v'_{i,j}, x_{i,j} \rangle' = 1$$

for $1 \leq i < j \leq n$.

From Lemma 4.3 it is easy to show that the $v'_{i,j}$ are linearly independent. Hence the $v_{i,j}$ are linearly independent. By Lemma 4.2, the $v_{i,j}$ span $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ as a $\mathbf{Q}(q,t)$ -module. It remains to show that they span $H_2(\tilde{C})$ as a Λ -module. In other words, we must prove the following.

Lemma 4.4 Let $c_{i,j} \in \mathbf{Q}(q,t)$ for $1 \le i < j \le n$ be such that

$$v = \sum_{1 \le i < j \le n} c_{i,j} v_{i,j}$$

lies in $H_2(\tilde{C})$. Then $c_{i,j} \in \Lambda$ for all $1 \leq i < j \leq n$.



Figure 6: The function representing $x_{n-1,n}$ breaks into triangles.

We use the following facts about the $x_{i,j}$.

Lemma 4.5 $x_{n-1,n}$ is a multiple of (1-q)(1+qt)(1-t) in $H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu})$.

Proof Let $\alpha: I \to D$ be a straight edge from ∂D to p_n . Let $\alpha_1, \alpha_2: I \to D_n$ be disjoint edges that have endpoints on ∂D and pass anticlockwise around α , such that α_1 encloses α_2 . Define $f: I \times I \to C$ by $f(x, y) = \{\alpha_1(x), \alpha_2(y)\}$. Then f lifts to a map \tilde{f} which represents $x_{2,3}$.

We can assume that for all $s \in I$ the four points $\{\alpha_i(s), \alpha_i(1-s)\}$ lie within distance ϵ of each other, where $\epsilon > 0$ is small. Then f maps the lines $\{(x, x)\}$, $\{(x, -x)\}, \{(\frac{1}{2}, y)\}, \{(x, \frac{1}{2})\}$ into ν_{ϵ} . These lines cut $I \times I$ into eight pieces. The restriction of \tilde{f} to each of these pieces represents an element of $H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu_{\epsilon}})$. Each of these elements is a multiple of the triangle corresponding to α , as shown in Figure 6. Combining these, we see that $x_{n-1,n}$ is (1-q)(1+qt)(1-t) times the triangle corresponding to α .

Lemma 4.6 For all $1 \leq i < j \leq n$, $x_{i,j}$ is a multiple of $(1-q)^2$ in $H_2(\tilde{C}, \partial \tilde{C} \cup \tilde{\nu})$.

Proof Use a similar argument to Lemma 4.5, as suggested by Figure 7.

Proof of Lemma 4.4 First consider the case n = 3. Then

$$\langle v, x_{2,3} \rangle = (1-q)^2 (1+qt)(1-t)c_{2,3}.$$

By Lemma 4.6, $x_{2,3}$ is a multiple of $(1-q)^2$, and hence of $(1-\bar{q})^2$. Since $\langle \cdot, \cdot \rangle$ is sesquilinear, it follows that $(1+qt)(1-t)c_{2,3} \in \Lambda$. Similarly, Lemma 4.5 implies that $(1-q)c_{2,3} \in \Lambda$. Since Λ is a unique factorisation domain, it follows that $c_{2,3} \in \Lambda$. By a symmetrical argument, $c_{1,2} \in \Lambda$. Now

$$v - (c_{1,2}v_{1,2} + c_{2,3}v_{2,3})$$

$$(\mathbf{i}, \mathbf{i}, \mathbf{i}) = (1 - q)(1 - q^{-1}) \left((\mathbf{i}, \mathbf{i}, \mathbf{i}) + q (\mathbf{i}, \mathbf{i}, \mathbf{i}) + \dots \right)$$

Figure 7: $x_{2,4}$ is a multiple of $(1-q)^2$.

still lies in $H_2(\tilde{C})$, so we can reduce to the case $v = c_{1,3}v_{1,3}$. The following holds up to multiplication by a unit in Λ .

$$\langle v, \sigma_2 x_{2,3} \rangle = (1-t)(1-q)^2(1+qt)c_{1,3}.$$

Now $\sigma_2 x_{2,3}$ is a multiple of (1-t)(1-q)(1+qt). Thus $(1-q)c_{1,3} \in \Lambda$. But

 $\langle v, x_{1,3} \rangle = (1-q)^2 (1+qt)c_{1,3},$

so $(1+qt)c_{1,3} \in \Lambda$. Thus $c_{1,3} \in \Lambda$.

Now suppose n > 3. For $i = 1, \ldots, n - 2$ we have

$$\langle v, x_{i,n} \rangle = (1-q)^2 c_{i,n}$$

By Lemma 4.6, $c_{i,n} \in \Lambda$. Also

$$\langle v, x_{n-1,n} \rangle = (1-q)^2 (1+qt)(1-t)c_{n-1,n}.$$

By Lemma 4.6, $(1+qt)(1-t)c_{n-1,n} \in \Lambda$. By Lemma 4.5, $(1-q)c_{n-1,n} \in \Lambda$. Thus $c_{n-1,n} \in \Lambda$. We can now subtract the terms $c_{i,n}v_{i,n}$, and so reduce to the case $c_{i,n} = 0$ for all $i = 1, \ldots, n-1$. Then v represents an element of homology of the preimage in \tilde{C} of the space of unordered pairs of distinct points in an (n-1)-times punctured disk. The result now follows by induction on n.

This completes the proof of Theorem 4.1.

4.2 The Krammer representation

There is some confusion as to the exact definition of the "Lawrence-Krammer representation". In an attempt to clarify the situation, we now compare and contrast a slightly different representation which we will call the "Krammer representation". Let the Krammer representation be the following action of B_n on a free Λ -module V of rank $\binom{n}{2}$ with basis $\{F_{i,j} : 1 \leq i < j \leq n\}$.

$$\sigma_i(F_{j,k}) = \begin{cases} F_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qF_{i,k} + (q^2 - q)F_{i,j} + (1 - q)F_{j,k} & i = j - 1, \\ F_{j+1,k} & i = j \neq k - 1, \\ qF_{j,i} + (1 - q)F_{j,k} + (1 - q)qtF_{i,k} & i = k - 1 \neq j, \\ F_{j,k+1} & i = k, \\ -tq^2F_{j,k} & i = j = k - 1. \end{cases}$$

This action is given in [Kra00], except that Krammer's -t is my t. It is also in [Big01], but with a sign error. The name "Krammer representation" was chosen because Krammer seems to have initially found this independently of Lawrence and without any use of homology.

The vector spaces $\mathbf{Q}(q,t) \otimes V$ and $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ are isomorphic representations of B_n , by [Big01, Theorem 4.1]. The isomorphism is given by

 $F_{i,j} \mapsto (\sigma_{i-1} \dots \sigma_2 \sigma_1) (\sigma_{j-1} \dots \sigma_3 \sigma_2) v_{1,2}.$

However it is shown in [PP01] that the representations V and $H_2(\tilde{C})$ are not isomorphic for $n \geq 3$. The distinction becomes important when we specialise qand t to values which are not algebraically independent, as we will in the next section.

It is possible to compute the matrices for the Lawrence-Krammer representation with respect to the basis $\{v_{i,j}\}$. We will not do this since they are quite complicated. They must be conjugate to those of the Krammer representation when considered as matrices over $\mathbf{Q}(q,t)$, but not over Λ .

I conjecture that the Krammer representation is the action of B_n on $H_2(\tilde{C}, \tilde{\nu})$. This can be reduced to showing that $\mathbf{Q}(q,t) \otimes H_2(\tilde{C}, \tilde{\nu})$ has dimension $\binom{n}{2}$.

5 A representation of the Temperley-Lieb algebra

The aim of this section is to prove the following.

Theorem 5.1 Let R be a domain containing invertible elements q and t, and let

$$\iota \colon R \otimes H_2(C) \to R \otimes H_2(C, \tilde{\nu})$$

be induced by the natural map from homology to relative homology. If 1 + qt = 0, q has a square root, $q^2 \neq 1$, and $q^3 \neq 1$ then the image of ι is the representation of $\text{TL}_n(R)$ corresponding to the partition (n-2,2).

The main example to keep in mind is $R = \mathbf{Z}[q^{\pm \frac{1}{2}}]$ with $t = -q^{-1}$.

In [Law90], Lawrence constructed the representation corresponding to the partition (n - k, k) for $k = 1, \ldots, \lfloor n/2 \rfloor$. This was generalised still further in [Law96] to give the Hecke algebra representation corresponding to any partition of n. Theorem 5.1 only covers the case (n - 2, 2), but also has a number of advantages over the work of Lawrence. Firstly, it gives a more elementary description of the required quotient of $R \otimes H_2(\tilde{C})$. Secondly, it works over a fairly general ring, whereas Lawrence worked over \mathbf{C} and used the matrices we gave in Section 4.2 for the "Krammer representation". Finally, our proof gives an explicit isomorphism between the two representations. It is to be hoped that these advantages can also be generalised to an arbitrary partition of n.

5.1 A basis

For $1 \leq i < j \leq n$, let $v_{i,j}$, $v'_{i,j}$ and $x_{i,j}$ be as defined in Section 4. Then

$$\iota(v_{i,j}) = \begin{cases} 0 & \text{if } j = i+1 \text{ or } j = 3, \\ (1-q)^2 v'_{i,j} & \text{otherwise.} \end{cases}$$

The pairing $\langle \cdot, \cdot \rangle'$ can be extended to a map

$$\langle \cdot, \cdot \rangle' \colon (R \otimes H_2(\tilde{C}, \tilde{\nu})) \times H_2(\tilde{C}, \partial \tilde{C}) \to R.$$

Lemma 4.3 still holds, so the $v'_{i,j}$ are linearly independent in $R \otimes H_2(\tilde{C}, \tilde{\nu})$. Thus the image of ι is the free module with basis

$$\{(1-q)^2 v'_{i,j} : j > \max(i+1,3)\}.$$

Let H be the free module with basis

$$\{v'_{i,j}: j > \max(i+1,3)\}$$

To ease the notation, we will work with H instead of the image of ι , since the two are isomorphic.

Let K be the field of fractions of R. Then $K \otimes H$ is a vector space of dimension n(n-3)/2, and contains H as an embedded submodule. It is sometimes easier to prove that a relation holds in H by proving that it holds in $K \otimes H$.

5.2 The Hecke algebra

We now prove that H is a representation of the Hecke algebra.

Definition 5.2 The Hecke algebra $\mathcal{H}_n(R)$, or simply \mathcal{H}_n , is the *R*-algebra given by generators $1, \sigma_1, \ldots, \sigma_{n-1}$ and relations

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if |i j| > 1,
- $\sigma_i \sigma_j \sigma_i = \sigma_i \sigma_j \sigma_i$ if |i j| = 1,
- $(\sigma_i 1)(\sigma_i + q) = 0.$

Thus \mathcal{H}_n is the group algebra RB_n of the braid group modulo the relations

$$(\sigma_i - 1)(\sigma_i + q) = 0.$$

Lemma 5.3 $(\sigma_i - 1)(\sigma_i + q)$ acts as the zero map on *H*.

Proof Since the σ_i are all conjugate, we need only check the case i = n - 1. To do this, we find a basis for $K \otimes H$ consisting of eigenvectors of σ_{n-1} , each having eigenvalue 1 or -q.

The elements $v'_{i,j}$ for $\max(i+1,3) < j \leq n-2$ are linearly independent eigenvectors of σ_{n-1} with eigenvalue 1. There are (n-2)(n-5)/2 of these.

Let α be a simple closed curve based at p_{n-2} enclosing p_{n-1} and p_n . For $i = 1, \ldots, n-4$, let u_i be the square corresponding to the edges $[p_i, p_{i+1}]$ and α . Let u_{n-3} be the square corresponding to the edge $[p_{n-3}, p_{n-2}]$ and a simple closed curve based at p_{n-4} enclosing p_{n-3}, \ldots, p_n . These are eigenvalues of σ_{n-1} with eigenvalue 1. For $i, i' = 1, \ldots, n-3$, the following identities hold up to multiplication by a unit in Λ .

- $\langle u_i, x_{i,n} \rangle' = 1 q$,
- $\langle u_i, x_{i',n} \rangle' = 0$ for $i' \neq i$,
- $\langle v'_{i,j}, x_{i',n} \rangle' = 0$ for $1 \le i < j \le n 2$.

Thus the u_i are linearly independent and not in the span of $\{v'_{i,j} : j \leq n-2\}$. Now let u be the square corresponding to α and a simple closed curve based at p_{n-3} enclosing α . The following identities hold up to multiplication by a unit in Λ .

- $\langle u, x_{n-2,n-1} \rangle' = (1-q^2)(1+q^2t)(1-t),$
- $\langle u_i, x_{n-2,n-1} \rangle' = 0$ for $i = 1, \dots, n-3$,
- $\langle v'_{i,j}, x_{n-2,n-1} \rangle' = 0$ for $j \le n-2$.



Figure 8: $\sigma_{n-1}^{-1}x_{i,n} = x_{i,n-1} - q^{-1}x_{i,n}$.

We conclude that

$$\mathcal{B}'_1 = \{v_{i,j} : \max(i+1,3) < j \le n-2\} \cup \{u_1, \dots, u_{n-3}\} \cup \{u\}$$

is a linearly independent set of eigenvectors of σ_{n-1} with eigenvalue 1. Each $v \in \mathcal{B}_1$ lies in H, as can be seen by constructing a surface in \tilde{C} whose image in $H_2(\tilde{C}, \tilde{\nu})$ is $(1-q)^2 v$.

For i = 1, ..., n-3, note that $v'_{i,n}$ must be an eigenvector of σ_{n-1} , because the function σ_{n-1} can be chosen to fix setwise the square in C whose lift represents $v'_{i,n}$. To find the eigenvalue, note that

$$\sigma_{n-1}^{-1}x_{i,n} = x_{i,n-1} - q^{-1}x_{i,n},$$

as shown in Figure 8, so

$$\langle \sigma_{n-1}v'_{i,n}, x_{i,n} \rangle = \langle v'_{i,n}, \sigma_{n-1}^{-1}x_{i,n} \rangle = -q \langle v'_{i,n}, x_{i,n} \rangle.$$

Thus

$$\mathcal{B}_{-q} = \{v'_{i,n} : i = 1, \dots, n-3\}.$$

is a set of eigenvectors of σ_{n-1} with eigenvalue -q. They are linearly independent, and since $-q \neq 1$, they are not in the span of \mathcal{B}_1 . By a dimension count, $\mathcal{B}_1 \cup \mathcal{B}_{-q}$ is a basis for $K \otimes H$. Each element of this basis is annihilated by $(\sigma_{n-1}-1)(\sigma_{n-1}+q)$, so we are done.

5.3 The Temperley-Lieb algebra

We now prove that H is a representation of the Temperley-Lieb algebra TL_n . Consider the map from \mathcal{H}_n to TL_n given by $\sigma_i \mapsto 1 + q^{\frac{1}{2}} e_i$. Using the presentation of the TL_n given in Section 2.3, one can check that this is well-defined and has kernel generated by

$$z_{i,j} = \sigma_i \sigma_j \sigma_i - \sigma_i \sigma_j - \sigma_j \sigma_i + \sigma_i + \sigma_j - 1$$

for |i - j| = 1.



Figure 9: The arcs ξ_1 and γ used to define y_1 .

Lemma 5.4 $z_{i,j}$ acts as the zero map on H.

Proof The $z_{i,j}$ are all conjugate to each other, so it suffices to prove that $z_{n-2,n-1}$ acts as the zero map on H. We have

$$z_{n-2,n-1} = (\sigma_{n-2}\sigma_{n-1} - \sigma_{n-1} + 1)(\sigma_{n-2} - 1)$$

= $(\sigma_{n-1}\sigma_{n-2} - \sigma_{n-2} + 1)(\sigma_{n-1} - 1).$

Thus it suffices to find a basis for $K \otimes H$, each of whose elements is an eigenvector of either σ_{n-1} or σ_{n-2} with eigenvalue 1.

Let \mathcal{B}_1 be the linearly independent set of eigenvectors of σ_{n-1} with eigenvalue 1 as defined in the proof of Lemma 5.3.

Let α' be a simple closed curve based at p_n and enclosing p_{n-1} and p_{n-2} . For $i = 1, \ldots, n-4$, let w_i be the square corresponding to the edges $[p_i, p_{i+1}]$ and α' . Let w_{n-3} be the square corresponding to α' and a simple closed curve based at p_{n-3} and enclosing α' . The w_i all lie in H, and are eigenvectors for σ_{n-2} with eigenvalue 1.

For i = 1, ..., n-3, let $\xi_i: I \to D_n$ be a vertical edge with endpoints on ∂D , passing just to the right of p_i , and let $\gamma: S^1 \to D_n$ be a figure-eight going around p_{n-1} and p_n in a small regular neighbourhood of $[p_{n-1}, p_n]$. See Figure 9. Define

$$f_i: I \times S^1 \to C$$

by $f_i(x, y) = \{\xi_i(x), \gamma(y)\}$. This lifts to \tilde{C} . Let y_i be the corresponding element of $H_2(\tilde{C}, \partial \tilde{C})$. For $i, j = 1, \ldots, n-3$, the following holds up to multiplication by a unit in Λ .

$$\langle w_i, y_i \rangle' = \begin{cases} 1 - q^3 & i = j \neq n - 3\\ (1 - q^3)(1 - q^3 t^2) & i = j = n - 3\\ 0 & i \neq j \end{cases}$$

Further, $\langle v, y_i \rangle' = 0$ for all $v \in \mathcal{B}_1$. Thus

$$\{w_1,\ldots,w_{n-3}\}\cup\mathcal{B}_1$$

is linearly independent. By a dimension count, it forms a basis of $K \otimes H$. Each vector in this basis is annihilated by $z_{n-2,n-1}$, so we are done.

5.4 The (n-2,2) representation

We have shown that H can be considered as a representation of TL_n . It remains to show that it is isomorphic to the representation S corresponding to the partition (n-2,2). Recall that S is the TL_n -module $M/(M \cap N)$, where M is the left-ideal of TL_n generated by e_1e_3 and N is the left-ideal generated by $\{e_5, \ldots, e_{n-1}\}$.

Let ψ : $\operatorname{TL}_n \to H$ be the unique map such that $\psi(1) = v'_{1,4}$. Then

$$\psi(e_3) = q^{-\frac{1}{2}}(\sigma_3 - 1)(v'_{1,4}) = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})v'_{1,4}.$$

Similarly $\psi(e_1) = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})v'_{1,4}$. Thus

$$\psi(e_1e_3) = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 v'_{1,4}$$

Let $\psi': M \to N$ be given by

$$\psi' = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-2} \psi|_M.$$

For $i = 5, \ldots, n-1$ we have that

$$\psi(e_i) = q^{-\frac{1}{2}}(\sigma_i - 1)(v'_{1,4}) = 0.$$

Thus $\psi(N) = 0$, so $\psi'(M \cap N) = 0$. We therefore have an map $\phi: S \to H$ induced by ψ' .

Recall that S has a basis

$$s_{i,j} = (e_i \dots e_2)(e_{j-1} \dots e_4)(e_1 e_3)$$

for $1 \le i < j \le n$ such that $j > \max(i+1,3)$. We will show that $\phi(s_{i,j}) = v'_{i,j}$ up to multiplication by a unit in Λ .

Lemma 5.5 For j = 5, ..., n we have $(\sigma_{j-1} - 1)v'_{1,j-1} = v'_{1,j}$ up to multiplication by a unit in Λ .

Proof We can assume that the edge $\sigma_{j-1}([p_{j-2}, p_{j-1}])$ passes within ϵ of p_{j-1} . Then we see that

$$\sigma_{j-1}v'_{1,j-1} = \lambda v_{1,j-1} + \mu v_{1,j}$$

for some units $\lambda, \mu \in \Lambda$. To show that $\lambda = 1$, note that $\sigma_{j-1}x_{1,j-1} = x_{1,j-1}$, so

$$\langle \sigma_{j-1} v'_{1,j-1}, x_{1,j-1} \rangle' = \langle \sigma_{j-1} v'_{1,j-1}, \sigma_{j-1} x_{1,j-1} \rangle' = \langle v'_{1,j-1}, x_{1,j-1} \rangle'.$$

Lemma 5.6 For i = 2, ..., j - 1 we have $(\sigma_i - 1)v'_{i-1,j} = v'_{i,j}$ up to multiplication by a unit in Λ .

Proof In the case i < j - 1, the proof is almost identical to that of the previous lemma. Suppose i = j - 1. Since everything takes place in a four-times punctured disk, we can assume j = n = 4. Then H has a basis $\{v'_{1,4}, v'_{2,4}\}$, so

$$\sigma_2 v_{1,4}' = \lambda v_{1,4}' + \mu v_{2,4}'$$

for some $\lambda, \mu \in \Lambda$. To see that $\lambda = 1$, use the same argument as in the proof of the previous lemma. To see that μ is a unit in Λ , note that it is equal to $\langle \sigma_2 v'_{1,4}, x_{2,4} \rangle'$, and the surfaces representing $\sigma_2 v'_{1,4}$ and $x_{2,4}$ intersect transversely exactly once in C.

By these two lemmas we have $\phi(s_{i,j}) = v'_{i,j}$ up to multiplication by a unit in Λ . This implies that ϕ is an isomorphism, which completes the proof of Theorem 5.1.

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