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The Analysis of Elliptic Families

II. Dirac Operators, Êta Invariants, and the Holonomy Theorem

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Abstract. In this paper we specialize the results obtained in [BF1] to the case of a family of Dirac operators. We first calculate the curvature of the unitary connection on the determinant bundle which we introduced in [BF1].

We also calculate the odd Chern forms of Quillen for a family of self-adjoint Dirac operators and give a simple proof of certain results of Atiyah-Patodi-Singer on êta invariants.

We finally give a heat equation proof of the holonomy theorem, in the form suggested by Witten [W1, 2].

Contents

Introduction

Let $M \xrightarrow{\pi} B$ be a submersion of the manifold M on the manifold B, with compact even dimensional fibers Z. Let D be a family of first order differential elliptic operators acting along the fibers Z.

In [BF 1], we have shown how to construct a metric and a unitary connection on the determinant bundle λ associated with the family D, thus extending earlier results of Quillen [Q 2], who considered the case of a family of $\bar\partial$ operators on a Riemann surface. In [BF 1], the connection ${}^1\nabla$ on λ was constructed using the superconnection formalism of Quillen [Q 1], which was extended in [B 5] to an infinite dimensional situation. The curvature of ${}^1\nabla$ on λ was also computed in [BF 1] in terms of asymptotic expansions of certain heat kernels.

Our first purpose in this paper is to specialize the results of [BF 1] to the family of Dirac operators considered in [B 5].

Our first main result, which is proved in Theorem 1.21, is that in the setting of [B 5], the curvature of λ is the term of degree 2 in the differential form on B,

$$2i\pi \int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr}\left[\exp - \frac{L}{2i\pi}\right], \tag{0.1}$$

where (0.1) is exactly the differential form which was constructed in [B 5] to represent the Chern character of the difference bundle $\operatorname{Ker} D_+ - \operatorname{Ker} D_-$ naturally associated to D. The proof of this result relies on a surprising link between the natural geometric superconnection considered in [BF 1] and the Levi-Civita superconnection introduced in [B 5].

Our second series of results is related to self-adjoint Dirac operators on odd dimensional manifolds. Let us recall that in [APS 1, 3], Atiyah-Patodi-Singer introduced the êta function $\eta(s)$ associated with a self-adjoint operator D on an odd dimensional manifold M'. They showed that η is holomorphic at 0. When D is a Dirac operator, they proved in [APS 1] that η is holomorphic for $s \geqq -\frac{1}{2}$, by showing how $\eta(0)$ is related to an index problem on a manifold M'' whose boundary is M', and by using local cancellation properties in the heat equation formula for the index on even dimensional manifolds [Gi 1], [ABP]. An alternative proof of this result has been given in [APS 3, p. 84] using Gilkey's theory of invariants [Gi 1], [ABP] for odd dimensional manifolds.

In Sect. 2, we show how a direct approach to the êta invariants of Dirac operators is possible. By using the periodicity of Clifford algebras [ABS] in an elementary form, we show that the local invariant êta function $\eta(s, x)$ is pointwise holomorphic at $s=0$. This is done by introducing a supplementary Grassmann variable z and by a formal transfer of the results of [B 5] in this situation.

Also in [Q 2], Quillen has given a natural candidate to represent the odd Chern classes associated with a family D of self-adjoint operators. We prove that these forms represent the Chern classes when D is a family of Dirac operators. Using the results of [B 5], we calculate the asymptotics of such forms which depend on $t>0$. We exactly obtain again formula (0.1), where Z is now odd dimensional. By noting that the form of degree 1 in (0.1) is the variation of the êta invariant of the family, we thereby obtain a simple proof of the results of Atiyah-Patodi-Singer [APS 3] on the spectral flow of a family of Dirac operators.

Our major concern in this paper is to give a proof of the Witten holonomy theorem [W 1, 2]. Let us recall that in [W 1, 2], Witten has considered the case of a manifold X endowed with a metric g_0. If ψ is a diffeomorphism of X, set $g_1=\psi_* g_0$. Witten considers the family of metrics

$$g_t=(1-t)g_0+tg_1\,, \qquad t\in S_1=R/Z \tag{0.2}$$

and the corresponding family of Dirac operators D_t. He thus constructs the manifold $M\times_\psi S_1$, where $(x,0)$ and $(\psi(x),1)$ are identified. In [W 1], Witten gives an argument showing that if the family D_t has index 0, if $\eta(0)$ is the êta invariant associated with a Dirac operator D' on $M\times_\psi S_1$, then in certain situations, the variation over S_1 of the determinant of the family D_t is given by the formula

$$\delta\,\mathrm{Log}\det D_t=\exp\{-i\pi\eta(0)\}\,. \tag{0.3}$$

In [W 1, 2], Witten was interested in calculating global anomalies in the case where the curvature of the determinant bundle vanishes.

In Sect. 3, we give a rigorous proof of Witten's theorem in the case of the family of Dirac operators considered in [B 5] and in Sect. 1. More precisely, we prove in Theorem 3.16 that if $[\bar\eta]$ is the limit in R/Z of certain refined êta invariants [APS 1, 3] which are obtained by blowing up the metric of B, then the holonomy τ of a loop c is given by

$$\tau=(-1)^{\mathrm{Ind}\,D_+}\exp\{-2i\pi[\bar\eta]\}\,. \tag{0.4}$$

When λ has a curvature equal to 0, it is in general unnecessary to blow up the metric of B. Blowing up the metric of B is equivalent to what Witten calls adiabatic approximation in [W 1].

Again using the periodicity of Clifford algebras, our proof of the holonomy theorem is essentially equivalent to the second proof in [B 5] of the Index Theorem for families of Dirac operators, where the metric of the base B was also blown up. At a technical level, we prove that the imaginary part of our connection ${}^1\nabla$ on λ – which is defined via heat equation – exhibits remarkable cancellations, which match the local cancellations of [B 5] and Sect. 1. Also we have to establish in the course of the proof certain large time estimates on heat kernels. These estimates, as well as certain localization estimates, are obtained using probabilistic methods. More specifically, we use the partial Malliavin calculus of [BM].

The main steps of our proof of the holonomy theorem are closely related to the ideas used in Atiyah-Donelly-Singer [ADS].

Note that our proofs of local cancellations are systematically based on generalized Lichnerowicz formulas with anticommuting variables, which are derived from [B 5, Theorem 3.6].

For an introduction to probability and the Malliavin calculus, we refer to [B 3, BM], and the references therein.

The results which are given here were announced in [BF 2].

I. A Connection on the Determinant Bundle of a Family of Dirac Operators

In [BF 1], we constructed a metric and a unitary connection on the determinant bundle of a family of first order elliptic differential operators. In this section, we will apply this construction to the family of Dirac operators D considered in [B 5]. In particular we prove that the curvature of our connection coïncides with the differential form which was obtained in [B 5] to represent the first Chern class of $\operatorname{Ker} D_+ - \operatorname{Ker} D_-$.

This results generalizes the results obtained by Quillen [Q 1] for the determinant bundle of a family of $\bar{\partial}$ operators over a Riemann surface.

We use the superconnection formalism of Quillen [Q 1] which was extended in [B 5] to an infinite dimensional setting. This permits us to obtain the critical link between the natural geometric superconnection used in [BF 1] to construct a connection on the determinant bundle, and the Levi-Civita superconnection of [B 5].

This section is organized as follows. In a) and b), we recall some well-known results on Clifford algebras and the spin representation [ABS]. In c) and d), we briefly describe the geometric setting of [B 5] and [BF 1]. In e), we calculate a unitary connection on certain infinite dimensional bundles in the setting of [B 5]. This unitary connection plays a key role in [BF 1]. In f), we recall the results of [BF 1]. Finally in g), we compute the curvature of the determinant bundle for a family of Dirac operators.

a) Clifford Algebras: The Even Dimensional Case

R^n denotes the canonical oriented Euclidean space of dimension n. $e_1, \ldots, e_n$ is the canonical oriented orthonormal base of R^n, $dx^1, \ldots, dx^n$ the corresponding dual base.

The Clifford algebra $c(R^n)$ is generated over R by $1, e_1, \ldots, e_n$ and the commutation relations

$$e_i e_j + e_j e_i = -2\delta_{ij}. \tag{1.1}$$

Let $\mathscr{A}(n)$ be the set of (n, n) antisymmetric real matrices. If $A = (a_i^j) \in \mathscr{A}(n)$, we identify A with the element of $c(R^n)$,

$$\tfrac{1}{4} a_i^j e_i e_j, \tag{1.2}$$

and with the element of $\Lambda^2(R^n)$,

$$\tfrac{1}{2} a_j^i dx^i \wedge dx^j. \tag{1.3}$$

Assume first that n is even, so that $n = 2l$. Set

$$\tau = i^l e_1 \ldots e_n. \tag{1.4}$$

Then $\tau^2 = 1$. By [ABS], $c(R^n) \otimes_R \mathbb{C}$ identifies with $\operatorname{End} S_n$, where S_n is a complex Hermitian space of spinors, of dimension 2^l. Set $S_{\pm,n} = \{s; \tau s = \pm s\}$. Then $S_{\pm,n}$ has dimension 2^{l-1}, and $S_n = S_{+,n} \oplus S_{-,n}$.

If $a \in c(R^n)$, let $\operatorname{Tr}[a]$ be the trace of a as an element of $\operatorname{End} S_n$. Set

$$\operatorname{Tr}_s[a] = \operatorname{Tr}[\tau a]. \tag{1.5}$$

Then Tr_s is determined as follows [AB, p. 484]: for $1 \leqq i_1 < i_2 \ldots < i_p \leqq n$, then

$$\mathrm{Tr}_s[e_{i_1} e_{i_2} \ldots e_{i_p}] = 0 \quad \text{if} \quad p < n, \qquad \mathrm{Tr}_s[e_1 \ldots e_n] = (-2i)^l. \tag{1.6}$$

The double cover Spin(n) of SO(n) is naturally embedded in $c(R^n)$. Spin(n) acts unitarily and irreducibly on $S_{+,n}$ and $S_{-,n}$ [ABS, H].

b) Clifford Algebras: The Odd Dimensional Case

Assume now that n is odd, so that $n = 2l+1$. Let φ be the algebra homomorphism from $c(R^n)$ into $c^{\mathrm{even}}(R^{n+1})$ defined by the relation $\varphi(e_i) = e_i e_{n+1}$, $1 \leqq i \leqq n$. Under φ, $c(R^n)$ is isomorphic to $c^{\mathrm{even}}(R^{n+1})$. Then $c(R^n) \otimes_R \mathbb{C}$ identifies with $\mathrm{End}(S_{+,n+1}) \oplus \mathrm{End}(S_{-,n+1})$.

By definition, the space of spinors S_n is identified with $S_{+,n+1}$. $c(R^n)$ acts on S_n. One verifies easily that if $\mathrm{Tr}\, a$ is the trace of $a \in c(R^n)$ acting on S_n, then

$$\mathrm{Tr}[1] = 2^l, \qquad \mathrm{Tr}[e_1 \ldots e_n] = 2^l (-i)^{l+1}, \tag{1.7}$$

and that the trace of the other monomials in $c(R^n)$ is 0.

Since $i^{l+1} e_1 e_2 \ldots e_n$ acts like the identity on S_n, the two formulas in (1.7) are equivalent.

Another construction of S_n is as follows. Set

$$\tau_{n-1} = i^l e_1 \ldots e_{n-1}.$$

Let ψ be the homomorphism of $c(R^n) \otimes_R \mathbb{C}$ into $c(R^{n-1}) \otimes_R \mathbb{C}$ defined by

$$\psi(e_i) = e_i, \qquad 1 \leqq i \leqq n-1, \qquad \psi(e_n) = -i\tau_{n-1}. \tag{1.8}$$

If $a \in c(R^n) \otimes_R \mathbb{C}$, $\psi(a)$ acts naturally on $S_{n-1} = S_{+,n-1} \oplus S_{-,n-1}$. We can then identify S_n and S_{n-1} as representation spaces for $c(R^n) \otimes_R \mathbb{C}$. In particular

$$i^{l+1} \psi(e_1 \ldots e_n) = \tau_{n-1}^2 = 1, \tag{1.9}$$

which fits with (1.7). Spin(n), which double covers SO(n), is naturally embedded in $c(R^n)$ and acts unitarily and irreducibly on S_n.

Remark 1. For n odd, the trace Tr behaves on the odd elements of $c(R^n)$ in exactly the same way as the supertrace Tr_s on the even elements of $c(R^n)$ for n even, i.e. we must saturate all the elements $e_1, \ldots, e_n$ to get a non-zero trace or supertrace. This fact, which is a simple consequence of the periodicity of the Clifford algebras [ABS], will be of utmost importance in the sequel.

c) Description of the Fibered Manifold

We now briefly recall the main results in [B 5, Sect. 1]. B denotes a connected manifold of dimension m. We assume that TB is endowed with a smooth Euclidean scalar product g_B. However the results in [B 5] and in our paper do not depend on g_B.

$n = 2l$ is an even integer. X is a connected compact manifold of dimension n. We assume that X is orientable and spin. M is a $n+m$ dimensional connected manifold. π is a submersion of M onto B, which defines a fibering Z by fibers $Z_y = \pi^{-1}\{y\}$ which are diffeomorphic to X. TZ is the n dimensional subbundle of

TM whose fiber at $x \in M$ is $T_x Z_{\pi(x)}$. We assume that TZ is oriented. $T^H M$ is a smooth subbundle of TM such that $TM = T^H M \oplus TZ$. $T^H M$ is the horizontal part of TM, and TZ the vertical part of TM.

Under π_*, $T_x^H M$ and $T_{\pi(x)} B$ are isomorphic. We lift the scalar product of TB in $T^H M$.

We also assume that TZ is endowed with an Euclidean scalar product g_Z. By assuming that $T^H M$ and TZ are orthogonal, TM is endowed with a metric which we note $g_B \oplus g_Z$. Let $\langle\, , \rangle$ be the corresponding scalar product.

Let O be the $SO(n)$ bundle of oriented orthonormal frames in TZ. We assume that TZ is spin, i.e. the $SO(n)$ bundle $O \xrightarrow{\varrho} M$ lifts to a Spin(n) bundle $O' \xrightarrow{\sigma} O \xrightarrow{\varrho} M$ such that σ induces the covering projection Spin$(n) \to SO(n)$ on each fiber.

F, $F_\pm$ denote the Hermitian bundles of spinors

$$F = O' \times_{\mathrm{Spin}(n)} S_n, \qquad F_\pm = O' \times_{\mathrm{Spin}(n)} S_{\pm, n}. \tag{1.10}$$

d) Connections on TM

Let ∇^B be the Levi-Civita connection of TB. ∇^B lifts into a Euclidean connection on $T^H M$, which we still note ∇^B. ∇^L denotes the Levi-Civita connection of TM for the metric $g_B \oplus g_Z$. P_Z (respectively P_H) denotes the orthogonal projection operators from TM on TZ (respectively $T^H M$). ∇^Z denotes the connection on TZ defined by the relation $U \in TM$, $V \in TZ$, $\nabla_U^Z V = P_Z \nabla_U^L V$. ∇^Z preserves the metric g_Z.

∇ denotes the connection on $TM = T^H M \oplus TZ$, which coïncides with ∇^B on $T^H M$ and with ∇^Z on TZ. We will write $\nabla = \nabla^B \oplus \nabla^Z$. ∇ preserves the metric $g_B \oplus g_Z$.

Definition 1.1. T denotes the torsion of ∇, R the curvature tensor of ∇. R^Z is the curvature of TZ. S is the tensor defined by

$$\nabla^L = \nabla + S. \tag{1.11}$$

Clearly R^Z is the restriction of R to TZ.

For $U \in TM$, $S(U)$ is antisymmetric in End TM. Given $U, V, W \in TM$, we have the well-known relation

$$2\langle S(U)V, W\rangle + \langle T(U,V), W\rangle + \langle T(W,U), V\rangle - \langle T(V,W), U\rangle = 0. \tag{1.12}$$

Let us now recall some results of [B 5, Theorem 1.9].

- T takes its values in TZ.
- If $U, V \in TZ$, $T(U,V) = 0$.
- ∇^Z, T, and the $(3,0)$ tensor $\langle S(\cdot)\cdot, \cdot\rangle$ do not depend on g_B.
- For any $U \in TM$, $S(U)$ sends TZ in $T^H M$.
- For any $U, V \in T^H M$, $S(U)V \in TZ$.
- If $U \in T^H M$, $S(U)U = 0$.

Only the last statement is not explicitly proved in [B 5, Theorem 1.9]. However it immediately follows from (1.12), from the relation $T(U,U) = 0$ and from the fact that T takes its values in TZ.

In the sequel, we will write ∇ instead of ∇^B, ∇^Z.

The connection ∇ on O lifts into a connection on O'. F, $F_\pm$ are then naturally endowed with a unitary connection, which we still note ∇.

ξ is a k-dimensional complex Hermitian bundle on M. We assume that ξ is endowed with a unitary connection ∇^{ξ}, whose curvature tensor is L. The Hermitian bundle $F \otimes \xi$ is naturally endowed with a unitary connection which we note ∇.

e) Connections on Infinite Dimensional Bundles

H^{∞}, $H^{\infty}_{\pm}$ denote the set of C^{∞} sections of $F \otimes \xi$, $F_{\pm} \otimes \xi$ over M. As in [**B** 5, Sect. 2], we will regard H^{∞}, $H^{\infty}_{\pm}$ as being the sets of C^{∞} sections over B of infinite dimensional bundles which we still note H^{∞}, $H^{\infty}_{\pm}$. For $y \in B$, H^{∞}_{y}, $H^{\infty}_{\pm,y}$ are the sets of C^{∞} sections over Z_y of $F \otimes \xi$, $F_{\pm} \otimes \xi$.

Let dx be the Riemannian volume element of Z_y. H^{∞}_{y} is naturally endowed with the Hermitian product

$$\langle h, h' \rangle_y = \int_{Z_y} \langle h, h' \rangle (x) dx . \tag{1.13}$$

For $Y \in TB$, let Y^H be the horizontal lift of Y in $T^H M$. Y^H is characterized by

$$Y^H \in T^H M; \qquad \pi_* Y^H = Y .$$

Definition 1.2. $\tilde{\nabla}$ denotes the connection on H^{∞} which is such that if $Y \in TB$, $h \in H^{\infty}$,

$$\tilde{\nabla}_Y h = \nabla_{Y^H} h . \tag{1.14}$$

By [**B** 5, Proposition 1.11], the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is a first order differential operator acting fiberwise on H^{∞}.

In general, although ∇ is unitary on $F \otimes \xi$, $\tilde{\nabla}$ does not preserve the Hermitian product (1.13) on H^{∞}. However an elementary modification of $\tilde{\nabla}$ permits us to construct a unitary connection on $H^{\infty}_{\pm}$.

$e_1, \dots, e_n$ denotes an orthonormal base of TZ.

Definition 1.3. k is the vector in $T^H M$

$$k = -\tfrac{1}{2} \sum_{1}^{n} S(e_i) e_i . \tag{1.15}$$

$\tilde{\nabla}^u$ is the connection on H^{∞} defined by the relation

$$Y \in TB , \qquad \tilde{\nabla}^u_Y = \tilde{\nabla}_Y + \langle k, Y^H \rangle \tag{1.16}$$

If Y is a vector field on B, the vector field Y^H on M preserves the fibration Z. In particular the divergence $\mathrm{div}_Z(Y^H)$ – which is the infinitesimal action of Y^H on the volume element dx of Z – is well defined at each $x \in M$. One verifies easily that $Y \to \mathrm{div}_Z(Y^H)$ is a tensor.

We first have the technical result.

Proposition 1.4. *For any $Y \in TB$, $x \in M$,*

$$\langle k, Y^H \rangle (x) = \tfrac{1}{2} \mathrm{div}_Z(Y^H)(x) . \tag{1.17}$$

The connection $\tilde{\nabla}^u$ is unitary on H^{∞}. $\tilde{\nabla}^u$ does not depend on the metric g_B.

Proof. By (1.12), we have

$$\langle k, Y^H\rangle = \tfrac{1}{2}\sum_1^n \langle T(Y^H, e_i), e_i\rangle \tag{1.18}$$

$e_1, \ldots, e_n$ can be extended locally into a C^∞ section of O. Clearly

$$T(Y^H, e_i) = \nabla_{Y^H} e_i - \nabla_{e_i} Y^H - [Y^H, e_i] = \nabla_{Y^H} e_i - [Y^H, e_i]. \tag{1.19}$$

Since $\langle e_i, e_i\rangle = 1$, we have

$$\langle \nabla_{Y^H} e_i, e_i\rangle = 0. \tag{1.20}$$

If $L_{Y^H} g_Z$ is the infinitesimal action of Y^H on g_Z, we have

$$0 = Y^H\langle e_i, e_i\rangle = L_{Y^H} g_Z(e_i, e_i) + 2\langle [Y^H, e_i], e_i\rangle. \tag{1.21}$$

From (1.18), (1.21), we find

$$\langle k, Y^H\rangle = \tfrac{1}{4}\sum_1^n L_{Y^H} g_Z(e_i, e_i) = \tfrac{1}{2}\operatorname{div}_Z(Y^H). \tag{1.22}$$

Also, if $h, h' \in H^\infty$,

$$Y\int_Z \langle h, h'\rangle(x)dx = \int_Z [\langle \nabla_{Y^H} h, h'\rangle + \langle h, \nabla_{Y^H} h'\rangle + \operatorname{div}_Z(Y^H)\langle h, h'\rangle](x)dx. \tag{1.23}$$

It is then clear that $\tilde{\nabla}^u$ is unitary. Also by Sect. 1d), $Y \in TM \to \langle k, Y\rangle$ does not depend on g_B. The proposition is proved. □

f) A Connection on the Determinant Bundle of a Family of First Order Elliptic Differential Operators

We now briefly summarize the main results of Bismut and Freed [BF 1] on the construction of a unitary connection on the determinant bundle of a family of first order differential operators.

We will constantly use the superconnection formalism of Quillen [Q 1] which was extended in [B 5] to infinite dimensions. In particular $F\otimes\xi = (F_+\otimes\xi)\oplus(F_-\otimes\xi)$, $H^\infty = H^\infty_+ \oplus H^\infty_-$ are Z_2 graded vector bundles over M and B. $\operatorname{End}(F\otimes\xi)$, $\operatorname{End} H^\infty$ are then naturally Z_2 graded.

For a given $y \in B$, we will always do our computations in the graded tensor product $\operatorname{End} H^\infty_y \hat{\otimes} \Lambda(T^*_y B)$. Locally, we work in $\operatorname{End}_x(F\otimes\xi)\hat{\otimes}\Lambda_{\pi x}(T^*B)$. The sign $\otimes$ will be always omitted.

If A is trace class in $\operatorname{End} H^\infty \hat{\otimes} \Lambda(T^*B)$, its trace Tr and its supertrace $\operatorname{Tr}_s A$ are elements of $\Lambda(T^*B)$. As in [Q 1], we use the convention that if $\omega \in \Lambda(T^*B)$,

$$\operatorname{Tr}\omega A = \omega \operatorname{Tr} A, \qquad \operatorname{Tr}_s \omega A = \omega \operatorname{Tr}_s A. \tag{1.24}$$

For $y \in B$, $D_{+,y}$ is an elliptic first order differential operator which sends $H^\infty_{+,y}$ into $H^\infty_{-,y}$. We assume that $D_{+,y}$ depends smoothly on $y \in B$. $D_{-,y}$ denotes the adjoint of $D_{+,y}$ with respect to the Hermitian product (1.13). Set

$$D_y = \begin{bmatrix} 0 & D_{-,y} \\ D_{+,y} & 0 \end{bmatrix}. \tag{1.25}$$

D is a smooth family of elliptic self-adjoint first order differential operators, which is odd in $\operatorname{End} H^\infty$.

Definition 1.5. λ denotes the complex line bundle over B,

$$\lambda = \det(\operatorname{Ker} D_+)^* \otimes \det(\operatorname{Ker} D_-). \tag{1.26}$$

As shown in [Q 2, BF 1], λ is a well-defined smooth bundle on B, even if B is non-compact. This will be briefly proved in the sequel.

If $\omega \in \Lambda(T^*B)$, $\omega^{(i)}$ denotes the component of ω in $\Lambda^i(T^*B)$. All the asymptotic expansions which we will consider are uniform on the compact subsets of B.

Take $t>0$. $\tilde{\nabla}^u + \sqrt{t}D$ is a superconnection on H^∞. By [B 5, Sect. 2], $\operatorname{Tr}_s[\exp-(\tilde{\nabla}^u+\sqrt{t}D)^2]$ is a C^∞ closed form on B. When B is compact, it represents the (normalized) Chern character $ch_1(\operatorname{Ker} D_+ - \operatorname{Ker} D_-)$.

As $t \downarrow\downarrow 0$, for any $k \in N$, we have the asymptotic expansion,

$$\operatorname{Tr}_s[\exp-(\tilde{\nabla}^u+\sqrt{t}D)^2] = \sum_{-\frac{n}{2}-\left[\frac{m}{2}\right]}^{k} a_j(y)t^j + o(t^k, y). \tag{1.27}$$

The following result is proved in [BF 1, Theorem 1.5].

Proposition 1.6. *The $a_j^{(2)}$ are C^∞ closed purely imaginary 2 forms on B. For $j \neq 0$, $a_j^{(2)}$ is exact.*

In [BF 1], a metric and a connection are constructed on λ. We briefly recall the results of [BF 1].

We have the asymptotic expansion as $t \downarrow\downarrow 0$,

$$\tfrac{1}{2}\operatorname{Tr}\exp(-tD^2) = \sum_{j=-\frac{n}{2}}^{k} A_j t^j + o(t^k, y), \tag{1.28}$$

where the A_j are real C^∞ functions on B. Also

$$d[\tfrac{1}{2}\operatorname{Tr}\exp-tD^2] = -t\operatorname{Tr}[\exp(-tD^2)\tilde{\nabla}^u D D], \tag{1.29}$$

and as $t \downarrow\downarrow 0$,

$$\operatorname{Tr}[\exp-(tD^2)\tilde{\nabla}^u D D] = -\sum_{j=-\frac{n}{2}}^{k} dA_j t^{j-1} + o(t^{k-1}, y). \tag{1.30}$$

Similarly as $t \downarrow\downarrow 0$,

$$\operatorname{Tr}_s[\exp(-tD^2)\tilde{\nabla}^u D D] = -\sum_{j=-\frac{n}{2}}^{k} B_j t^{j-1} + o(t^{k-1}, y). \tag{1.31}$$

The following result in proved in [BF 1, Theorem 1.7].

Proposition 1.7. *The B_j are C^∞ purely imaginary 1 forms on B. Also $dB_j = -2ja_j^{(2)}$. In particular B_0 is closed.*

Take $y_0 \in B$, $a>0$ which is not an eigenvalue of $D_{y_0}^2$. Then a is not an eigenvalue of D^2 on a neighborhood U of y_0. We now follow Quillen [Q 2] and Bismut-Freed [BF 1].

Definition 1.8. K^a_y is the subspace of $H^\infty_{\pm,y}$ which is the direct sum of the eigenspaces of D^2_y corresponding to eigenvalues $<a$.

K^a is a smooth subbundle of H^∞ on U. K^a splits into

$$K^a = K^a_+ \oplus K^a_- . \tag{1.32}$$

Also K^a is stable under D. Let P^a be the orthogonal projection operator on K^a. P^a is a smooth family of regularizing operators which is well defined on U. Set

$$Q^a = I - P^a . \tag{1.33}$$

We also define

$$\lambda^a = \det(K^a_+)^* \otimes \det(K^a_-) . \tag{1.34}$$

λ identifies canonically with λ^a on U. λ^a being a smooth line bundle on U, λ becomes itself a smooth line bundle on B. K^a inherits the Hermitian product (1.13) of H^∞. So λ^a is naturally endowed with a metric $|\,|^a$.

In [BF 1], we modify the metric $|\,|^a$ as in [Q 2] and we simultaneously construct a connection on λ.

Definition 1.9. For $s \in C$, the zêta function $\zeta^a(s)$ is defined by

$$\zeta^a(s) = \frac{1}{2\Gamma(s)} \int_0^{+\infty} t^{s-1} \operatorname{Tr}[e^{-tD^2} Q^a]\, dt . \tag{1.35}$$

Equivalently

$$\zeta^a(s) = \tfrac{1}{2} \operatorname{Tr}[(D^2)^{-s} Q^a] . \tag{1.36}$$

$\zeta^a(s)$ is a meromorphic function, which is holomorphic at $s=0$. μ is a fixed real constant.

Definition 1.10. $\|\,\|^a$ denotes the metric on λ^a which is such that if $l \in \lambda^a$,

$$\|l\|^a = |l|^a \exp\left\{ -\frac{1}{2} \frac{\partial \zeta^a}{\partial s}(0) - \frac{1}{2} \mu A_0 \right\} . \tag{1.37}$$

For $t>0$, γ^a_t, δ^a_t are the C^∞ 1-forms on B,

$$\gamma^a_t = \int_t^{+\infty} \operatorname{Tr}[\exp(-sD^2)(\tilde{\nabla}^u D) D Q^a]\, ds , \qquad \delta^a_t = \int_t^{+\infty} \operatorname{Tr}_s[\exp(-sD^2)(\tilde{\nabla}^u D) D Q^a]\, ds , \tag{1.38}$$

or equivalently

$$\gamma^a_t = -\operatorname{Tr}[\exp(-tD^2) D^{-1} (\tilde{\nabla}^u D) Q^a] , \qquad \delta^a_t = \operatorname{Tr}_s[\exp(-tD^2) D^{-1} (\tilde{\nabla}^u D) Q^a] . \tag{1.39}$$

γ^a_t and δ^a_t are C^∞ 1-forms on U, which are respectively real and purely imaginary. As $t \downarrow\downarrow 0$, we have the expansions

$$\begin{aligned} \gamma^a_t &= \sum_{-\frac{n}{2}}^{-1} dA_j \frac{t^j}{j} + dA_0 \operatorname{Log} t + \gamma^a_0 + O(t, y) , \\ \delta^a_t &= \sum_{-\frac{n}{2}}^{-1} B_j \frac{t^j}{j} + B_0 \operatorname{Log} t + \delta^a_0 + O(t, y) , \end{aligned} \tag{1.40}$$

where γ_0^a, δ_0^a are C^∞ 1-forms on U, which are respectively real and purely imaginary.

The following identities are proved in [BF 1, Theorem 1.15].

Proposition 1.11. *The following identities hold:*

$$d\zeta^a(0) = dA_0\,, \quad \gamma_0^a + \Gamma'(1) dA_0 = -d\left[\frac{\partial}{\partial s}\zeta^a(0)\right],$$

$$\gamma_0^a + \Gamma'(1) dA_0 = -(s\,\mathrm{Tr}[(D^2)^{-s} D^{-1} \tilde{\nabla}^u D Q^a])'(0)\,, \tag{1.41}$$

$$\delta_0^a + \Gamma'(1) B_0 = (s\,\mathrm{Tr}_s[(D^2)^{-s} D^{-1} \tilde{\nabla}^u D Q^a])'(0)\,.$$

dA_0 (respectively $-B_0$) is the residue at $s=0$ of the meromorphic function $\mathrm{Tr}[(D^2)^{-s}D^{-1}\tilde{\nabla}^u DQ^a]$ *(respectively* $\mathrm{Tr}_s[(D^2)^{-s}D^{-1}\tilde{\nabla}^u DQ^a]$*).*

Definition 1.12. ${}^0\nabla^a$ denotes the unitary connection on the bundle K^a over U which is such that if k is a section of K^a,

$${}^0\nabla^a k = P^a \tilde{\nabla}^u k\,. \tag{1.42}$$

${}^0\nabla^a$ induces a connection on λ^a, which is unitary for the metric $|\,|^a$.

Definition 1.13. ${}^1\nabla^a$ is the connection on λ^a,

$${}^1\nabla^a = {}^0\nabla + \tfrac{1}{2}(\gamma_0^a - \delta_0^a) + \tfrac{1}{2}(\Gamma'(1) - \mu)(dA_0 - B_0)\,. \tag{1.43}$$

The main result of Bismut-Freed [BF 1, Theorems 1.11 and 1.18] is as follows.

Theorem 1.14. *Using the canonical identification of λ^a with λ over U, the metrics $\|\,\|^a$ patch into a smooth metric $\|\,\|$ on λ over the manifold B. The connections ${}^1\nabla^a$ patch into a smooth connection ${}^1\nabla$ on λ over B, which is unitary for the metric $\|\,\|$. The curvature of ${}^1\nabla$ is the purely imaginary 2-form $a_0^{(2)}$.*

Remark 2. The rationale for introducing the constant μ in the definition of $\|\,\|$ and ${}^1\nabla$ is the following: Take $b \in R_+^*$. Assume that the family D is replaced by the family bD. Both D and bD have the same determinant bundle λ. However the canonical identifications of λ with λ^a are different. One verifies that $l \in \lambda^a$ should be identified with $b^{\frac{\dim(K_a)}{2}} l \in \lambda^a$.

The metric associated with bD is now $b^{A_0}\|\,\|$. The new connection ${}^1\nabla_b$ on λ is given by

$${}^1\nabla_b = {}^1\nabla + (dA_0 - B_0)\,\mathrm{Log}\, b\,. \tag{1.44}$$

In general ${}^1\nabla_b$ and ${}^1\nabla$ do not coïncide. This is a scaling discrepancy of the connection which we consider.

The introduction of the parameter μ permits us to construct simultaneously all the scaled metrics and connections.

g) The Case of a Family of Dirac Operators: Explicit Computation of the Curvature of the Determinant Bundle

We now assume that D is the family of Dirac operators considered in [B 5]. We briefly recall the definition of D. Remember that the elements of TZ act by Clifford multiplication on $F \otimes \xi$.

$$e_1, \ldots, e_n \text{ is an orthonormal base of } TZ\,.$$

Definition 1.15. D is the family of Dirac operators acting on H^∞

$$D=\sum_1^n e_i\nabla_{e_i}, \tag{1.45}$$

$D_\pm$ denotes the restriction of D to $H^\infty_\pm$.

The family D verifies all the assumptions of Sect. 1f).

We now briefly recall the definition of the Levi-Civita superconnection [B 5, Definition 3.2]. As pointed out in Sect. 1f), we use the formalism of Quillen [Q 1] at a local level. In particular all our computations are done in $c_x(TZ)\hat{\otimes}\Lambda_{\pi x}(T^*B)$. $f_1,\dots,f_m$ is a base of TB, $dy^1\dots dy^m$ the corresponding dual base. We identify $f_1,\dots,f_m$ with their horizontal lifts $f_1^H,\dots,f_m^H$. Also we use $i,j,\dots$ as indices for vertical variables like $e_i, e_j,\dots,\alpha,\beta$ for horizontal variables like $f_\alpha, f_\beta\dots$.

Definition 1.16. For $t>0$, the Levi-Civita superconnection $\tilde{\nabla}^{L,t}+\sqrt{t}D$ associated with the metric $g_B\oplus\dfrac{g_Z}{t}$ is given by

$$\begin{aligned}\tilde{\nabla}^{L,t}+\sqrt{t}D=e_i\left[\sqrt{t}\nabla_{e_i}+\frac{1}{2}\langle S(e_i)e_j,f_\alpha\rangle e_j dy^\alpha+\frac{1}{4\sqrt{t}}\langle S(e_i)f_\alpha,f_\beta\rangle dy^\alpha dy^\beta\right]\\ +dy^\alpha\left[\nabla_{f_\alpha}+\frac{1}{2\sqrt{t}}\langle S(f_\alpha)e_i,f_\beta\rangle e_i dy^\beta\right].\end{aligned}$$

By [B 5, Proposition 3.3] (see also Sect. 1d)), $\tilde{\nabla}^{L,t}+\sqrt{t}D$ does not depend on g_B.

We first compare $\tilde{\nabla}^{L,t}+\sqrt{t}D$ with $\tilde{\nabla}^u+\sqrt{t}D$.

Definition 1.17. A denotes the odd element in $c_x(TZ)\hat{\otimes}\Lambda_{\pi(x)}(T^*B)$

$$A=-\tfrac{1}{4}\sum_{\alpha<\beta}\langle T(f_\alpha,f_\beta),e_i\rangle e_i dy^\alpha dy^\beta. \tag{1.46}$$

Proposition 1.18. *The following identity holds:*

$$\tilde{\nabla}^{L,t}+\sqrt{t}D=\tilde{\nabla}^u+\sqrt{t}D+\frac{A}{\sqrt{t}}. \tag{1.47}$$

Proof. Since ∇^L has zero torsion, for $U, V\in TM$,

$$S(U)V-S(V)U+T(U,V)=0. \tag{1.48}$$

Also $T(e_i,e_j)=0$. We get

$$\sum_{i\neq j}\langle S(e_i)e_j,f_\alpha\rangle e_ie_j=-\tfrac{1}{2}\sum\langle T(e_i,e_j),f_\alpha\rangle e_ie_j=0. \tag{1.49}$$

Using (1.12) and the fact that T takes its values in TZ, we have

$$\tfrac{1}{4}\langle S(e_i)f_\alpha,f_\beta\rangle-\tfrac{1}{2}\langle S(f_\alpha)e_i,f_\beta\rangle=-\tfrac{1}{8}\langle T(f_\alpha,f_\beta),e_i\rangle. \tag{1.50}$$

Equation (1.47) follows from (1.49) and (1.50). □

As shown in [B 5, Sect. 2], $(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2$ and $(\tilde{\nabla}^u+\sqrt{t}D)^2$ are second order elliptic operators acting fiberwise in Z. For $t>0$, $s>0$, let $P_s^{L,t}(x,x')$, $P_s^{u,t}(x,x')$

$(x, x' \in Z_y)$ be the C^∞ kernels associated with the operators $\exp - s(\tilde{V}^{L,t} + \sqrt{t}D)^2$, $\exp - s(\tilde{V}^u + \sqrt{t}D)^2$.

We have the obvious formulas.

$$\begin{aligned} \operatorname{Tr}_s[\exp - (\tilde{V}^{L,t} + \sqrt{t}D)^2] &= \int_Z \operatorname{Tr}_s[P_1^{L,t}(x,x)]\,dx\,, \\ \operatorname{Tr}_s[\exp - (\tilde{V}^u + \sqrt{t}D)^2] &= \int_Z \operatorname{Tr}_s[P_1^{u,t}(x,x)]\,dx\,. \end{aligned} \tag{1.51}$$

Also if E is a complex vector bundle over B, endowed with a connection whose curvature is C, set

$$\operatorname{ch}_1 E = \operatorname{Tr}[\exp - C]\,. \tag{1.52}$$

$\operatorname{ch}_1 E$ represents in cohomology the scaled Chern character of E.

Theorem 1.19. *For any* $t > 0$, $\operatorname{Tr}_s[\exp - (\tilde{V}^{L,t} + \sqrt{t}D)^2]$ *and* $\operatorname{Tr}_s[\exp - (\tilde{V}^u + \sqrt{t}D)^2]$ *are* C^∞ *closed forms on* B *whose common cohomology class does not depend on* t. *If* B *is compact, they represent in cohomology* $\operatorname{ch}_1(\operatorname{Ker} D_+ - \operatorname{Ker} D_-)$. *Moreover*

$$[\operatorname{Tr}_s \exp - (\tilde{V}^u + \sqrt{t}D)^2]^{(2)} = [\operatorname{Tr}_s \exp - (\tilde{V}^{L,t} + \sqrt{t}D)^2]^{(2)}\,. \tag{1.53}$$

Proof. The first part of the Theorem is proved in [B 5, Theorem 2.6, Proposition 2.10]. We now prove (1.53).

By proceeding as in [B 5, Proposition 2.6 and Remark 2.3] – i.e. by using explicitly the C^∞ kernel of $\exp - \left(\tilde{V}^u + \sqrt{t}D + \frac{lA}{\sqrt{t}}\right)^2$, and the vanishing of supertraces on supercommutators in finite dimensions [Q 1] – it is not difficult to prove that

$$\frac{\partial}{\partial l} \operatorname{Tr}_s\left[\exp - \left(\tilde{V}^u + \sqrt{t}D + \frac{lA}{\sqrt{t}}\right)^2\right] = -d \operatorname{Tr}_s\left[\frac{A}{\sqrt{t}} \exp - \left(\tilde{V}^u + \sqrt{t}D + \frac{lA}{\sqrt{t}}\right)^2\right]. \tag{1.54}$$

Equation (1.54) is the fundamental equality which proves that in cohomology, $\operatorname{Tr}_s \exp - \left(\tilde{V}^u + \sqrt{t}D + \frac{lA}{\sqrt{t}}\right)^2$ does not change with l. Also A is of degree 2 in the variables dy^α. Since (1.54) is of even degree, the right-hand side of (1.54) is at least of degree 4. We then find that

$$\frac{\partial}{\partial l}\left[\operatorname{Tr}_s \exp - \left(\tilde{V}^u + \sqrt{t}D + \frac{lA}{\sqrt{t}}\right)^2\right]^{(2)} = 0\,. \tag{1.55}$$

Equation (1.53) is proved. □

Remark 3. Equation (1.53) is equivalent to the relation

$$\int_Z \operatorname{Tr}_s[P_1^{L,t}(x,x)]^{(2)} dx = \int_Z \operatorname{Tr}_s[P_1^{u,t}(x,x)]^{(2)} dx\,. \tag{1.56}$$

The expressions $\operatorname{Tr}_s[P_1^{L,t}(x,x)]$ and $\operatorname{Tr}_s[P_1^{u,t}(x,x)]$ may well be completely different. Their integrals on Z are in the same cohomology class. Moreover in degree 0 and 2, these integrals coïncide.

We now will calculate explicitly the curvature of the determinant bundle λ for the connection ${}^1\nabla$.

Definition 1.20. $\hat{A}$ is the ad $O(n)$ invariant polynomial on $\mathscr{A}(n)$ which is such that if $B \in \mathscr{A}(n)$ has diagonal entries $\begin{bmatrix} 0 & x_i \\ -x_i & 0 \end{bmatrix}$, then

$$\hat{A}(B) = \prod_1^l \frac{\frac{x_i}{2}}{\operatorname{sh}\frac{x_i}{2}}. \tag{1.57}$$

We now have the crucial result.

Theorem 1.21. *For* $-\frac{n}{2} - 1 \leqq j \leqq -1$, $a_j^{(2)} = 0$. *Also*

$$a_0^{(2)} = 2i\pi \left[\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr} \exp - \frac{L}{2i\pi} \right]^{(2)}. \tag{1.58}$$

The curvature of the connection ${}^1\nabla$ *is equal to* $a_0^{(2)}$.

Proof. Let $dx^1 \dots dx^n$ be the oriented volume element in Z. Let φ be the homomorphism on $\Lambda^{\text{even}}(T^*B)$ which to $dy^\alpha dy^\beta$ associates $\frac{dy^\alpha dy^\beta}{2i\pi}$. By [B 5, Theorems 4.12 and 4.16], we know that as $t \downarrow\downarrow 0$, $\varphi[\operatorname{Tr}_s[P_1^{L,t}(x,x)]]\, dx^1 \dots dx^n$ converges uniformly to the term of maximal degree n in the variables $dx^1 \dots dx^n$ in the expression

$$\hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr}\left[\exp - \frac{L}{2i\pi}\right]. \tag{1.59}$$

As in [B 5, Theorem 4.17], we immediately deduce from (1.59) that as $t \downarrow\downarrow 0$

$$\varphi(\operatorname{Tr}_s[\exp - (\tilde{\nabla}^{L,t} + \sqrt{t} D)^2]) \to \int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr}\left[\exp - \frac{L}{2i\pi}\right]. \tag{1.60}$$

Using (1.53), we find

$$\varphi(\operatorname{Tr}_s[\exp - (\tilde{\nabla}^u + \sqrt{t} D)^2])^{(2)} \to \left[\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr} \exp - \frac{L}{2i\pi} \right]^{(2)}. \tag{1.61}$$

Using (1.27) and Theorem 1.14, our theorem is now obvious. □

Remark 4. In general, the local cancellations which explain (1.60) occur in $\operatorname{Tr}_s[P_1^{L,t}(x,x)]$ and not in $\operatorname{Tr}_s[P_1^{u,t}(x,x)]$. The computation of the curvature $a_0^{(2)}$ is then done rather indirectly.

Remark 5. From Proposition 1.7 and Theorem 1.21, we already know that

$$dB_j = 0, \quad j \leqq 0. \tag{1.62}$$

We will prove in Theorem 3.4 that we have the much stronger result

$$B_j = 0, \quad j \leqq 0. \tag{1.63}$$

II. Dirac Operators on Odd Dimensional Manifolds

In this section, we establish certain properties of self-adjoint Dirac operators on odd dimensional manifolds. Families of self-adjoint Dirac operators are also considered.

Our first result concerns the local regularity of the êta function of Dirac operators. Using their results on the index of elliptic operators on manifolds with boundary, Atiyah-Patodi-Singer [APS 1, Theorem 4.2] proved that the êta function $\eta(s)$ of a Dirac operator D is holomorphic for $\operatorname{Re} s > -\frac{1}{2}$. In [APS 3, p. 85] a cancellation mechanism was described in dimension 3 to explain that the pole at $s=0$ of the meromorphic matrix $T_s(x,x)$ – which is the kernel of $D|D|^{-s-1}$ on the diagonal – disappears when calculating $\operatorname{Tr}[T_s(x,x)]$, thereby proving the local regularity of $\eta(s)$ at $s=0$ in dimension 3.

An alternative proof of this result has been given in [APS 3, p. 84] using Gilkey's theory of invariants [Gi 1, ABP] for odd dimensional manifolds.

In Sects. a)–d), we prove that the local êta function $\eta(s,x)$ is holomorphic at $s=0$ by a method which is formally identical to the proof given in [B 5] of the Index Theorem for families. By introducing as an auxiliary Grassman variable z, we establish in b) a Lichnerowicz formula for $\frac{tD^2}{2} - z\sqrt{t}D$. In c), and implicitly using the periodicity of Clifford algebras, we show that $\operatorname{Tr}[D \exp - tD^2]$ is locally $O(t^{1/2})$ as $t\downarrow\downarrow 0$. In d), we prove the local regularity of $\eta(s,x)$ at $s=0$. In e), we briefly calculate the variation of $\eta(0)$ by a heat equation formula [APS 3, p. 75], [ADS, p. 138]. In f), we consider a family of self-adjoint Dirac operators D in odd dimensions. We calculate the odd Chern forms associated with the family D introduced by Quillen [Q 1], by using formally the computations of [B 5]. The formula for these odd Chern forms is strictly identical to the formula obtained in [B 5] for the Chern character of the difference bundle associated with a family of Dirac operators in even dimensions. We thus obtain a simple proof of the result of Atiyah-Patodi-Singer [APS 3] on the spectral flow of a family of Dirac operators, which does not rely on the Index Theorem for manifolds with boundary.

The results obtained in this section will be used in Sect. 3.

a) Assumptions and Notations

M' is a compact connected Riemannian manifold of odd dimension $n=2l+1$, which is oriented and spin. N is the $SO(n)$ bundle of oriented orthonormal frames in TM'. N' is a $\operatorname{Spin}(n)$ bundle over M' which lifts N so that: $N' \underset{\sigma}{\to} N \underset{\varrho}{\to} M$, and σ induces the covering projection $\operatorname{Spin}(n) \underset{\sigma}{\to} SO(n)$ on each fiber. F' is the Hermitian bundle over M

$$F' = N' \times_{\operatorname{Spin}(n)} S_n . \tag{2.1}$$

∇ denotes the Levi-Civita connection on N, which lifts into a connection on N'. TM', F are then naturally endowed with a connection ∇. K is the scalar curvature of M'. ξ is a k-dimensional Hermitian vector bundle, endowed with a unitary connection ∇^{ξ}, whose curvature is L. $F' \otimes \xi$ is a Hermitian bundle, which is naturally endowed with a unitary connection, which we still note ∇. H^{∞} is the set of

C^∞ sections of $F'\otimes\xi$. $e_1, \ldots, e_n$ is an orthonormal base of TM'. D is the Dirac operator acting on H^∞,

$$D = \sum_1^n e_i \nabla_{e_i}. \tag{2.2}$$

b) An Auxiliary Grassmann Variable

z denotes a Grassmann variable which anticommutes with $e_1, \ldots, e_n$ considered as elements of $c(TM')$. If $A(X)$ is a tensor which depends linearly on $X \in TM'$, we use the convention that if $e_1, \ldots, e_n$ is a locally defined C^∞ orthonormal base of TM', then

$$(\nabla_{e_i} + A(e_i))^2 = \sum_1^n (\nabla_{e_i(x)} + A(e_i(x)))^2 - \nabla_{\sum_1^n \nabla_{e_j} e_j(x)} - A\left(\sum_1^n \nabla_{e_j} e_j\right). \tag{2.3}$$

We first prove an elementary identity which extends Lichnerowicz's formula [L, B 4].

Proposition 2.1. *For any $t>0$, the following identity holds:*

$$\frac{tD^2}{2} - z\sqrt{t}D = -\frac{t}{2}\left(\nabla_{e_i} + \sqrt{t}\frac{ze_i}{t}\right)^2 + \frac{tK}{8} + \frac{te_ie_j}{4} \otimes L(e_i, e_j). \tag{2.4}$$

Proof. Clearly

$$-\frac{t}{2}\left(\nabla_{e_i} + \sqrt{t}\frac{ze_i}{t}\right)^2 = -\frac{t}{2}\nabla_{e_i}^2 - z\sqrt{t}D. \tag{2.5}$$

The theorem now obviously follows from Lichnerowicz's formula [L, B 4]. □

Remark 1. As we shall see in Remark 5, Formula (2.4) is a special case of the formula proved in Bismut [B 5, Theorem 3.6], which calculates the curvature of the Levi-Civita superconnection.

c) The Asymptotics of Certain Heat Kernels

dx denotes the volume element of M'. All the considered kernels will be calculated with respect to dx. Let $R(z)$ be the Grassmann algebra generated by 1 and z. All our local computations are done in $(c(TM')\otimes \operatorname{End}\xi)\hat{\otimes} R(z)$.

Definition 2.2. For $t>0$, $P_t(x, x')$ denotes the C^∞ kernel associated with the operator $\exp\left(-\frac{tD^2}{2} + z\sqrt{t}D\right)$.

Clearly

$$\exp\left(-\frac{tD^2}{2} + z\sqrt{t}D\right) = \exp\left(-\frac{tD^2}{2}\right) + z\sqrt{t}D\exp\left(-\frac{tD^2}{2}\right). \tag{2.6}$$

Also we can write

$$P_t(x, x') = P_t^0(x, x') + z\sqrt{t}P_t^1(x, x'). \tag{2.7}$$

$P_t^0(x,x')$ is the kernel associated with $\exp\left(-\frac{tD^2}{2}\right)$ and $P_t^1(x,x')$ the kernel associated with $D\exp\left(-\frac{tD^2}{2}\right)$.

For any $x \in M'$, $P_t(x,x)$ is even in $(c(TM')\otimes \operatorname{End}\xi)\hat{\otimes} R(z)$. $P_t^0(x,x)$ is then even in $c(TM')\otimes \operatorname{End}\xi$, and $P_t^1(x,x)$ is odd in $c(TM')\otimes \operatorname{End}\xi$.

Definition 2.3. For A, $B \in c(TM')\otimes \operatorname{End}\xi$, set

$$\operatorname{Tr}_z(A+zB)=z\operatorname{Tr}B. \tag{2.8}$$

In the right-hand side of (2.8), $\operatorname{Tr}B$ is the trace of B acting on $F'\otimes\xi$.

Clearly

$$\operatorname{Tr}_z[P_t(x,x)]=z\sqrt{t}\operatorname{Tr}[P_t^1(x,x)]. \tag{2.9}$$

Another description of $\operatorname{Tr}_z[P_t(x,x)]$ is as follows. We can write $P_t(x,x)$ in the form

$$P_t(x,x)=\sum_{\substack{i_1<i_2<\ldots<i_p\\ p \text{ even}}} e_{i_1}\cdots e_{i_p}\otimes A_{i_1\ldots i_p}+z\sum_{\substack{i_1<i_2<\ldots<i_p\\ p \text{ odd}}} e_{i_1}\cdots e_{i_p}\otimes B_{i_1\ldots i_p}. \tag{2.10}$$

By (1.7), we know that

$$\operatorname{Tr}_z[P_t(x,x)]=2^l(-i)^{l+1}z\operatorname{Tr}B_{1\ldots n}. \tag{2.11}$$

We now prove the following result.

Theorem 2.4. *As $t\downarrow\downarrow 0$,*

$$\operatorname{Tr}_z[P_t(x,x)]\to 0 \quad \textit{uniformly on } M'. \tag{2.12}$$

There is a C^∞ function $b_{1/2}(x)$ on M' such that as $t\downarrow\downarrow 0$

$$\operatorname{Tr}[P_t^1(x,x)]=b_{1/2}(x)\sqrt{t}+O(t^{3/2},x), \tag{2.13}$$

and $O(t^{3/2},x)$ is uniform on M'.

Proof. As pointed out in Remark 1, the right-hand side of (2.4) has the same structure as the formula proved in [B 5, Theorem 3.6]. More precisely (2.4) coïncides with the formula of [B 5], when assuming that there is one single $dy^\alpha = z$ and that if $\bar{f}$ is the formal vector whose dual variable is z, then

$$\langle S(e_i)e_j, \bar{f}\rangle = -\langle S(e_i)\bar{f}, e_j\rangle = -2\delta_i^j. \tag{2.14}$$

In this context, it follows from (2.14) that

$$\langle \nabla \cdot S(e_i)e_j, \bar{f}\rangle = 0. \tag{2.15}$$

Now (2.11) shows that $\operatorname{Tr}_z[P_t(x,x)]$ is obtained by saturating the Clifford variables $e_1,\ldots,e_n$, i.e. by doing in odd dimensions what is done in [B 5, Sect. 4] in even dimensions.

We can then apply in this context [B 5, Theorem 4.12] which guarantees that as $t\downarrow\downarrow 0$, $\operatorname{Tr}_z[P_t(x,x)]$ has a limit and calculates this limit explicitly in terms of a Brownian bridge w'^1 in T_xM', constructed on a probability space (W, P_1). In [B 5, Theorem 4.12], we find that the term containing z appears in an expression of the type

$$c \int_W \exp\left\{\frac{1}{\sqrt{\pi}} \int_0^1 \langle \nabla_{w_s'^1} S(dw_s'^1) e_i, \bar{f} \rangle dx^i z + \ldots \right\} dP_1(w'^1). \tag{2.16}$$

Using (2.15), we find that the term containing z vanishes. We have proved that as $t \downarrow\downarrow 0$,

$$\mathrm{Tr}_z[P_t(x,x)] \to 0. \tag{2.17}$$

Note that

$$\exp - t\left(\frac{D^2}{2} - zD\right) = \exp\left(-\frac{tD^2}{2}\right) + ztD \exp\left(-\frac{tD^2}{2}\right). \tag{2.18}$$

By using the results of Greiner [Gr, Theorem 1.6.1] on the small time asymptotics of the kernel of the operator $\exp - t\left(\frac{D^2}{2} - zD\right)$, and using also (2.18), we find that C^∞ functions $b_{-\frac{n}{2}-1}, \ldots, b_{-1/2}, b_{1/2}$ exist such that as $t \downarrow\downarrow 0$,

$$\mathrm{Tr}[P_t^1(x,x)] = \frac{b_{-n/2-1}(x)}{t^{n/2+1}} + \ldots + \frac{b_{-1/2}(x)}{t^{1/2}} + b_{1/2}(x) t^{1/2} + O(t^{3/2}, x), \tag{2.19}$$

and $O(t^{3/2}, x)$ is uniform on M'.

Using (2.17), we find that

$$b_{-n/2-1} = \ldots = b_{-1/2} = 0. \tag{2.20}$$

Equation (2.13) is proved. □

Remark 2. As we shall see in Remark 5, Theorem 2.4 can be viewed as a direct consequence of [B 5, Theorem 4.12].

d) Local Regularity of the Êta Invariant

We now closely follow Atiyah-Patodi-Singer [APS 1].

Since D is elliptic and self-adjoint, D has a discrete family of real eigenvalues λ. For $s \in C$, set

$$\eta(s) = \sum_{\lambda \neq 0} \frac{\mathrm{sgn}\,\lambda}{|\lambda|^s}. \tag{2.21}$$

For $\mathrm{Re}\, s > n$, the series defining $\eta(s)$ is absolutely convergent. Also the following identity is easily verified

$$\eta(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{+\infty} t^{\frac{s-1}{2}} \mathrm{Tr}[D \exp - tD^2] dt. \tag{2.22}$$

We now define the local êta function.

Definition 2.5. For $\mathrm{Re}\, s > n$, $x \in M$, set

$$\eta(s,x) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{+\infty} t^{\frac{s-1}{2}} \operatorname{Tr}[P_{2t}^1(x,x)]\,dx\,. \tag{2.23}$$

One verifies easily that as $t \uparrow +\infty$, $\operatorname{Tr}[P_{2t}^1(x,x)]$ decays exponentially and uniformly on M'. Also.

$$\eta(s) = \int_{M'} \eta(s,x)\,dx\,. \tag{2.24}$$

In their proof of the Index Theorem for manifolds with boundary, Atiyah-Patodi-Singer [APS 1, Theorem 4.2] showed that $\eta(s)$ extends into a holomorphic function for $\mathrm{Re}s > -1/2$.

We now refine their result into a local statement on $\eta(s,x)$ [APS 3, p. 84].

Theorem 2.6. *For* $\mathrm{Re}\, s > -2$, $\eta(s,x)$ *is* C^∞ *in* (s,x) *and holomorphic in* s.

Proof. By Theorem 2.4, for $\mathrm{Re}\, s > -2$,

$$\int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}[P_{2t}^1(x,x)]\,dt\,, \tag{2.25}$$

is well defined and holomorphic in s, as well as $\dfrac{1}{\Gamma\left(\frac{s+1}{2}\right)}$. The theorem is proved. □

Remark 3. In [APS 3, p. 85], Atiyah-Patodi-Singer noted that in dimension 3, the kernel $T_s(x,x)$ of $\mathrm{D}|\mathrm{D}|^{-s-1}$ has a pole at $s=0$, but that this pole disappears when considering $\operatorname{Tr}[T_s(x,x)]$. Noting that

$$\eta(s,x) = \operatorname{Tr}[T_s(x,x)]\,, \tag{2.26}$$

this phenomenon should now be fully explained. It is in fact of the same nature as the cancellations observed in the heat equation proof of the Index Theorem. In [APS 3, p. 84], an alternative proof of this result was given using Gilkey's theory of invariants [Gi 1, ABP].

Remark 4. In [Gi 1, 2], Gilkey studied various cases where $\eta(s,x)$ is not holomorphic at $s=0$. Not unexpectedly, some of his examples involve Dirac operators calculated with a connection which is different from the Levi-Civita connection.

e) The Variation of the Êta Invariant

We now make exactly the same assumptions as in Sect. 1c), d), e), g) except that the compact fibers Z have now the odd dimension $n = 2l + 1$. F is instead the bundle of spinors over TZ, H^∞ is the set of C^∞ section over M of $F \otimes \xi$. D is the family of Dirac operators which is still defined as in Definition 1.15. Of course the vector bundles which we consider are no longer Z_2-graded.

Definition 2.7. For $y \in B$, $\eta_y(s)$ is the êta function associated with D_y. h_y is the integer

$$h_y = \dim \operatorname{Ker} D_y. \tag{2.27}$$

$\bar{\eta}_y(s)$ is the function.

$$\bar{\eta}_y(s) = \frac{\eta_y(s) + h_y}{2}. \tag{2.28}$$

If $d \in R$, $[D]$ denotes the image of d in R/Z. As noted in [APS 1, 3], $\bar{\eta}_y(0)$ has integer jumps, and so $[\bar{\eta}_y(0)]$ is a C^∞ function of $y \in B$ with values in R/Z. We now briefly compute $d[\bar{\eta}(0)]$ using a heat equation formula instead of the zêta function formula of [APS 3, Proposition 2.10].

Using again the results of Greiner [Gr, Sect. 1] (see [Gr, Lemma 1.5.5]) which permits us to differentiate the parametrix of the heat kernel, we have the asymptotic expansion

$$\operatorname{Tr}[\tilde{\nabla}^u D \exp - tD^2] = \frac{C_{-n/2}}{t^{n/2}} + \dots + \frac{C_{-1/2}}{t^{1/2}} + O(t^{1/2}, y), \tag{2.29}$$

where $C_{-n/2}, \dots, C_{-1/2}$ are C^∞ 1-forms on B.

Proposition 2.8. *The following identity holds:*

$$d[\bar{\eta}(0)] = -\frac{C_{-1/2}}{\sqrt{\pi}}. \tag{2.30}$$

Proof. As in [APS 3, p. 75 and Proposition 2.11] we can assume that D is invertible on a neighborhood U of $y \in B$. For $\operatorname{Re}(s)$ large enough, using integration by parts, we have

$$\begin{aligned}
\Gamma\left(\frac{s+1}{2}\right) d\eta(s) &= \int_0^{+\infty} t^{\frac{s-1}{2}} \operatorname{Tr}[\tilde{\nabla}^u D \exp(-tD^2) - 2tD^2 \tilde{\nabla}^u D \exp(-tD^2)]\, dt \\
&= \int_0^{+\infty} t^{\frac{s-1}{2}} \left[\operatorname{Tr}(\tilde{\nabla}^u D \exp(-tD^2)) + 2t \frac{\partial}{\partial t} \operatorname{Tr}(\tilde{\nabla}^u D \exp(-tD^2)) \right] dt \\
&= -s \int_0^{+\infty} t^{\frac{s-1}{2}} \operatorname{Tr}[\tilde{\nabla}^u D \exp - tD^2]\, \mathrm{d}t.
\end{aligned} \tag{2.31}$$

The proposition now follows from (2.29). □

f) Odd Chern Forms, Êta Invariant and the Spectral Flow

Although the fibers of Z are now odd dimensional, we entirely adopt the superconnection formalism of Sect. 1f), g). In particular, although $\operatorname{End}(F \otimes \xi)$ is no longer Z_2 graded, we will use instead the Z_2 grading of $(c(T_x Z) \otimes \operatorname{End} \xi) \hat{\otimes} \Lambda_{\pi x}(T^*B)$. We still have

$$e_i dy^\alpha + dy^\alpha e_i = 0. \tag{2.32}$$

We also use the convention (1.24).

The superconnection $\tilde{\nabla}^u + \sqrt{t}D$ is still defined as in Sect. 1f), and the Levi-Civita superconnection as in Definition 1.16 and Proposition 1.18.

$$[\operatorname{Tr}\exp -(\tilde{\nabla}^u + \sqrt{t}D)^2]^{\text{odd}} \quad \text{and} \quad [\operatorname{Tr}\exp -(\tilde{\nabla}^{L,t} + \sqrt{t}D)^2]^{\text{odd}}$$

are then well-defined C^∞ odd forms on B.

The construction of such odd forms is directly inspired from Quillen [Q 1, Sect. 5]. However in the formalism of [Q 1], D, A should be considered as even, and so e_i, D, A commute with dy^α. An extra Clifford variable σ is introduced in [Q 1] – with $\sigma^2 = 1$ – which commutes with D, A and anticommutes with dy^α. In the formalism of [Q 1, Sect. 5], $\tilde{\nabla}^u + \sqrt{t}D$, $\tilde{\nabla}^{L,t} + \sqrt{t}D$ should be replaced by

$$\tilde{\nabla}^u + \sqrt{t}D\sigma, \qquad \tilde{\nabla}^u + \left(\sqrt{t}D + \frac{A}{\sqrt{t}}\right)\sigma.$$

Following [Q 1], if B, C are trace class in $\operatorname{End} H^\infty \otimes \Lambda(T^*B)$, set

$$\operatorname{Tr}_\sigma[B + C\sigma] = \operatorname{Tr} C. \tag{2.33}$$

Note that since elements of $\Lambda(T^*B)$ and $\operatorname{End} H^\infty$ now commute, (2.33) is unambiguously defined.

We claim that

$$\begin{gathered}\operatorname{Tr}_\sigma \exp -(\tilde{\nabla}^u + \sqrt{t}D\sigma)^2 = [\operatorname{Tr}\exp -(\tilde{\nabla}^u + \sqrt{t}D)^2]^{\text{odd}}, \\ \operatorname{Tr}_\sigma \exp -\left(\tilde{\nabla}^u + \left(\sqrt{t}D + \frac{A}{\sqrt{t}}\right)\sigma\right)^2 = \left[\operatorname{Tr}\exp -\left(\tilde{\nabla}^u + \sqrt{t}D + \frac{A}{\sqrt{t}}\right)^2\right]^{\text{odd}}.\end{gathered} \tag{2.34}$$

The key point is to note that $(e_i\sigma)^2 = -1$ and that $e_i\sigma$ anticommutes with dy^α, so that the rules of commutation on the left-hand side of (2.34) become ultimately identical to our rules for the right-hand side. Note that formula (2.34) is not equivalent to Quillen's final formula in [Q 1, Sect. 5], since there, Quillen again assumes that D and dy^α commute.

We now go back to our initial formalism, i.e. assume that e_i and dy^α anticommute. In an infinite dimensional context, the differential forms (2.34) are natural candidates to be representatives in cohomology of the odd Chern classes associated with the index of the family $D \in K^1(B)$. This statement is the analogue of Quillen's formula for a family of Fredholm operators $D \in K^0(B)$, which was proved in [B 5, Sect. 2], when D is a family of Dirac operators.

Definition 2.9. $\hat{A}$ is the ad $O(n)$ invariant polynomial on $\mathscr{A}(n)$, which is such that if B has diagonal entries $\begin{bmatrix} 0 & x_i \\ -x_i & 0 \end{bmatrix}$ and 0, then

$$\hat{A}(B) = \prod_1^l \frac{\frac{x_i}{2}}{\operatorname{sh}\frac{x_i}{2}}. \tag{2.35}$$

$(i)^{1/2}$ is one square root of i, which is fixed once and for all. ψ is the homomorphism of $\Lambda(T^*B)$, which to dy^α associates $dy^\alpha/(2i\pi)^{1/2}$.

Since the fibers Z are odd dimensional, we must make precise our sign conventions, when integrating differential forms along the fiber. If α is a differential form on M which in local coordinates is given by

$$\alpha = dy^{\alpha_1}\dots dy^{\alpha_q}\beta(x)\,dx^1\dots dx^n,$$

we set

$$\int_Z \alpha = dy^{\alpha_1}\dots dy^{\alpha_q}\int_Z \beta(x)\,dx^1\dots dx^n. \tag{2.36}$$

This sign convention will be compatible with the sign convention (1.24).

We now have the following result.

Theorem 2.10. *For any* $t>0$, $(2i)^{1/2}\psi[\operatorname{Tr}\exp-(\tilde{\nabla}^u+\sqrt{t}D)^2]^{\mathrm{odd}}$ *and* $(2i)^{1/2}\psi \cdot \operatorname{Tr}[\exp-(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2]^{\mathrm{odd}}$ *are* C^∞ *differential forms which are closed, whose common cohomology class is independent of* t, *and which both represent the odd Chern classes associated with the family* D. *Also*

$$(2i)^{1/2}\psi[\operatorname{Tr}\exp-(\tilde{\nabla}^u+\sqrt{t}D)^2]^{(1)} = (2i)^{1/2}\psi[\operatorname{Tr}\exp-(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2]^{(1)}, \tag{2.37}$$

and the 1*-forms in* (2.37) *are cohomologous to* $d[\tilde{\eta}(0)]$. *As* $t\downarrow\downarrow 0$, $(2i)^{1/2}[\psi(\operatorname{Tr}\exp -(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2]^{\mathrm{odd}}$ *converges uniformly on the compact subsets of* B *to*

$$\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right)\operatorname{Tr}\exp-\frac{L}{2i\pi}, \tag{2.38}$$

which also represents the odd Chern classes of the family D. *In particular for* $j\leqq -\frac{3}{2}$, $C_j=0$, *and moreover*

$$d[\tilde{\eta}(0)] = \left[\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right)\operatorname{Tr}\exp-\frac{L}{2i\pi}\right]^{(1)}. \tag{2.39}$$

Proof. By proceeding as in [B 5, Propositions 2.9 and 2.10], and by using the formalism of [Q 1] the proof of the first part of the theorem is easy. We now will prove that

$$(2i)^{1/2}\psi[\operatorname{Tr}\exp-(\tilde{\nabla}+\sqrt{t}D)^2]^{\mathrm{odd}} \tag{2.40}$$

represents the odd Chern classes for the family D. This will of course imply the corresponding result for the odd forms considered in the theorem.

We first assume that B is compact. Set $B'=B\times S_1$, $M'=M\times S_1\times S_1$. The mapping $(x,s,v)\in M'\to(\pi x, v)\in B'$ defines a fibration of M' over B', with even dimensional oriented fibers $S_1\times Z$. On $S_1\times S_1$, we consider the Hermitian line bundle which is obtained by identifying $(0,v,X)\in S_1\times S_1\times\mathbb{C}$ and $(1,v,\exp-(2i\pi v)X)$. This line bundle obviously extends into a Hermitian line bundle T on M'. T is naturally endowed with the Hermitian connection $d+2i\pi s\,dv$. For $\varepsilon>0$, $(y,v)\in B'$, we consider the first order differential operator $D'^{\varepsilon}_{(y,v)}$ acting on $F\otimes\xi\otimes T\otimes\mathbb{C}^2$,

$$D'^{\varepsilon}_{(y,v)} = \begin{bmatrix} 0 & \sqrt{\varepsilon}\dfrac{\partial}{\partial s}+D_y \\ -\sqrt{\varepsilon}\dfrac{\partial}{\partial s}+D_y & 0 \end{bmatrix}.$$

D'^{ε} is a family of Dirac operators over B' acting on sections of twisted spinors over the fibers $Z\times S_1$. By [B 5, Theorem 2.6], we know that the differential forms over B',

$$\mathrm{Tr}_s\left[\exp-\left(\tilde{\nabla}+dv\frac{\partial}{\partial v}+2i\pi s\,dv+D'^{\varepsilon}\right)^2\right],\tag{2.41}$$

are closed and represent in cohomology the normalized Chern classes associated with the family D'^{ε}. Moreover these forms are in the same cohomology class as ε varies.

We claim that as $\varepsilon\downarrow\downarrow 0$, the forms (2.41) converge uniformly to the forms

$$2i\sqrt{\pi}\,[\mathrm{Tr}\exp-(\tilde{\nabla}+D)^2]^{\mathrm{odd}}\mathrm{dv}.\tag{2.42}$$

In fact set

$$e_0=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \sigma=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Then (2.41) is equal to

$$\mathrm{Tr}_s\left[\exp\left\{-(\tilde{\nabla}+D\otimes\sigma)^2-\varepsilon\frac{\partial^2}{\partial s^2}-2i\pi\sqrt{\varepsilon}\,1\otimes e_0 dv\right\}\right]$$
$$=-\mathrm{Tr}_s\left[\exp\left\{-(\tilde{\nabla}+D\otimes\sigma)^2-\varepsilon\frac{\partial^2}{\partial s^2}\right\}(2i\pi\sqrt{\varepsilon}1\otimes e_0 dv)\right].\tag{2.43}$$

Using (1.7), we have the relations

$$\mathrm{Tr}[e_1e_2\dots e_n]=(2)^l(-i)^{l+1},$$
$$\mathrm{Tr}_s[(1\otimes e_0)((e_1e_2\dots e_n)\otimes\sigma)]=(-2i)^{l+1}.$$

By proceeding as in [B 5, Theorem 5.3] in a much simpler situation (or by using the same arguments as in Theorem 3.12, in a very simple situation) and also the sign conventions (1.24), it is very easy to obtain the convergence result (2.42).

Equation (2.42) still represents in cohomology the normalized even Chern classes associated with the family D'^{ε}. Since even and odd Chern classes correspond under suspension by integration along the fiber (see [APS 3, p. 82]), by integrating (2.42) in the variable v, and with the adequate normalization, we have proved that (2.40) represents the odd Chern classes associated with the family D.

When B is non-compact, the same result is still true by restriction to compact pieces in B.

Equality (2.37) is trivial. Since $(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2$ is even, if $P_1^{L,t}(x,x')$ is the C^∞ kernel of $\exp-(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2$, $P_1^{L,t}(x,x)$ is even in $(c(TZ)\otimes\mathrm{End}\,\xi)\hat{\otimes}\Lambda(T^*B)$. $\mathrm{Tr}[P_1^{L,t}(x,x)]^{\mathrm{odd}}$ only involves the odd part of $P_1^{L,t}(x,x)$ in $c(TZ)\otimes\mathrm{End}\,\xi$. Also by (1.7), $e_1\dots e_n$ is the only odd monomial in $c(TZ)$ whose trace is non-zero.

This shows that formally, we can use the method and the results of [B 5, Sect. 4] to calculate the asymptotics of $\mathrm{Tr}[P_1^{L,t}(x,x)]$ as $t\downarrow\downarrow 0$. In particular using [B 5, Theorems 4.13 and 4.17] and keeping track of the constants, we obtain (2.38). Also

$$(\tilde{\nabla}^u+\sqrt{t}D)^2=(\tilde{\nabla}^u)^2+\sqrt{t}\,\tilde{\nabla}^u D+tD^2.$$

We then find that

$$[\exp-(\tilde{\nabla}^u+\sqrt{t}D)^2]^{(1)}=[\exp-(\sqrt{t}\tilde{\nabla}^u D+tD^2)]^{(1)}.$$

Using Duhamel's formula, we get

$$\exp\{-(tD^2+\sqrt{t}\tilde{\nabla}^u D)\}$$
$$=\exp(-tD^2)-\int_0^1 \exp\{-s(tD^2+\sqrt{t}\tilde{\nabla}^u D)\}\sqrt{t}\tilde{\nabla}^u D\exp\{-(1-s)tD^2\}ds. \quad (2.44)$$

Iterating (2.44), we find immediately that

$$[\exp-(tD^2+\sqrt{t}\tilde{\nabla}^u D)]^{(1)}=-\int_0^1 \exp\{-stD^2\}\sqrt{t}\tilde{\nabla}^u D\exp\{-(1-s)tD^2\}ds. \quad (2.45)$$

Using (2.45), we get

$$[\operatorname{Tr}\exp-(\tilde{\nabla}^u+\sqrt{t}D)^2]^{(1)}=-\sqrt{t}\operatorname{Tr}[\tilde{\nabla}^u D\exp(-tD^2)]. \quad (2.46)$$

From (2.30), (2.37), (2.46), we immediately deduce that for $j\leqq -\frac{3}{2}$, $C_j=0$, and also (2.39). The statement following (2.37) is now obvious. □

Remark 5. Proposition 2.1 and Theorem 2.4 can be directly derived from [B 5, Theorem 3.6] and from the local convergence result associated with (2.39). In fact let us go back to the assumptions of Sect. 2c). $M'\times R^+$ fibers over R^+ with the fibers M'. For $\varepsilon>0$, we endow the fiber M'_ε with the metric $g_{M'}/\varepsilon^2$ (where $g_{M'}$ is the metric on TM'). The corresponding family of Dirac operators will be εD. The natural connection ∇' on TM' which is constructed as in Sect. 1c) is defined by the relation

$$X\in TM'_\varepsilon, \qquad \nabla'_{\frac{\partial}{\partial\varepsilon}}X=-X/\varepsilon,$$

the covariant differentiation in vertical directions being still given by the Levi-Civita connection of M'. One verifies trivially that the curvature tensor $R^{M'}$ of TM' is such that for $X\in TM'$,

$$R^{M'}\left(X,\frac{\partial}{\partial\varepsilon}\right)=0. \quad (2.47)$$

If S is defined as in Definition 1.1, for X, $Y\in TM'_\varepsilon$, we have

$$\left\langle S(X)\frac{\partial}{\partial\varepsilon}, Y\right\rangle=-\frac{\langle X,Y\rangle_{gM'}}{\varepsilon^3}.$$

Using [B 5, Theorem 3.6], we find that if $\xi=\mathbb{C}$

$$\left(d\varepsilon\frac{\partial}{\partial\varepsilon}-\frac{n}{2}\frac{d\varepsilon}{\varepsilon}+\sqrt{t}\varepsilon D\right)^2=-t\left(\varepsilon\nabla_{e_i}+\frac{1}{2\sqrt{t}}e_i\frac{d\varepsilon}{\varepsilon}\right)^2+t\varepsilon^2K/4. \quad (2.48)$$

The reader will easily check that (2.4) and (2.48) are equivalent. Also using (2.39) and (2.47), we find that in this case $d[\bar{\eta}(0)]=0$. Now Theorem 2.4 is exactly the local version of this result, and this local version also follows from Theorem 2.10. Ultimately we find that in our context, the more natural way of proving that η is holomorphic at $s=0$ is to prove that $[\bar{\eta}(0)]$ is invariant under the scaling of D by using formula (2.39).

We now deduce from Theorem 2.10 the result of Atiyah-Patodi-Singer [APS 3, p. 95] on the spectral flow of a family of Dirac operators.

Let $s \in S_1 = R/Z \to c_s \in B$ be a C^∞ loop in B. Set

$$M' = \pi^{-1}(c).$$

M' is a compact manifold. If $e_1 \dots e_n$ is an oriented base of TZ, we orient M' by $\left(\frac{dc}{ds}, e_1, \dots, e_n\right)$. M' is obviously spin and carries a vector bundle of spinors $F' = F'_+ \oplus F'_-$. The Dirac operator D' acting on sections of $F' \otimes \xi$ over M' is well-defined. $\operatorname{Ind} D'_+$ denotes the index of D'_+ (which is the restriction of D' to the sections of $F'_+ \otimes \xi$).

We now prove again the result of [APS 3, p. 95].

Theorem 2.11. *The following identity holds:*

$$\operatorname{Ind} D'_+ = \int_c d[\bar{\eta}(0)] = \int_{M'} \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr} \exp - \frac{L}{2i\pi}. \tag{2.49}$$

Proof. Using (2.39), and the orientation convention on M', it is clear that

$$\int_c d[\bar{\eta}(0)] = \int_{M'} \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr} \exp - \frac{L}{2i\pi}. \tag{2.50}$$

Also TM' splits into $TM' = T^H M' \oplus TZ$, and $T^H M'$ is trivial. The $\hat{A}$ genus for TM' coïncides with $\hat{A}\left(\frac{R^Z}{2\pi}\right)$.

The Atiyah-Singer Index Theorem shows that

$$\operatorname{Ind} D'_+ = \int_{M'} \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr} \exp - \frac{L}{2i\pi}.$$

The theorem is proved. □

Remark 6. If $\eta(0)$ has a finite number of jumps on S_1, then clearly

$$\int_c d[\bar{\eta}(0)] = -\sum \frac{\Delta \eta(0)}{2}. \tag{2.51}$$

The spectral flow $\sum \frac{\Delta \eta(0)}{2}$ is then equal to $-\operatorname{Ind} D'_+$. In this respect, our sign conventions differ from [APS 3, p. 95] where M' is oriented by $\left(e_1, \dots, e_n, \frac{dc}{ds}\right)$.

Also note that the explicit expression (2.39) is not needed to prove Theorem 2.11. It is enough to know that the forms (2.37) are in the same cohomology class as $d[\bar{\eta}(0)]$ and to do a trivial asymptotics as $t \downarrow\downarrow 0$, similar to what we did in the proof of Theorem 2.10, on the heat equation formula for $\operatorname{Ind} D'_+$. Ultimately the equality of the spectral flow and of $\operatorname{Ind} D'_+$ is a simple consequence of the superconnection algebra.

III. The Holonomy Theorem: A Heat Equation Proof

The purpose of this section is to give a proof of the holonomy Theorem which was suggested by Witten in [W 1, 2]. Namely we calculate the holonomy of the determinant bundle λ over a loop c in B in terms of the limit in R/Z of refined êta invariants of the odd dimensional manifold $M'=\pi^{-1}(c)$, which are obtained by blowing up the metric of B. Formally, the situation is very close to what is done in Bismut [B 5, Sect. 5] in a second proof of the Index Theorem for families. The proof is also closely related to Atiyah-Donelly-Singer [ADS].

The section is organized as follows. After introducing notations in a), we establish in b) a generalized Lichnerowicz formula, which still follows from [B 5, Theorem 3.6]. In c), we construct certain heat kernels along the fibers Z, in order to prove in d) that the differential form δ_t^a introduced in (1.38) converges to δ_0^a as $t\downarrow\downarrow 0$. The proof is obtained by a local cancellation process which matches the local cancellations of [B 5, Sect. 4] and also the local regularity of the êta function proved in Theorems 2.4 and 2.6. In e), if c is a loop in B, we consider the $n+1$ dimensional manifold $M'=\pi^{-1}(c)$ and the Dirac operator D^ε on M' associated with the metric $\frac{g^B}{\varepsilon}\oplus g^Z$. In f), we give a simple geometric proof that if $[\bar{\eta}^\varepsilon(0)]$ is the modified êta invariant of Atiyah-Patodi-Singer [APS 1, 3], which takes its values in R/Z, then as $\varepsilon\downarrow\downarrow 0$, $[\bar{\eta}^\varepsilon(0)]$ has a limit $[\bar{\eta}]$.

In g), we prove that as $\varepsilon\downarrow\downarrow 0$, for t bounded, the local trace of the kernel which is used in formula (2.23) to define $[\bar{\eta}^\varepsilon(0)]$ converges to the local supertrace in the heat kernel formula for δ_t^a in (1.38). The proof of Theorem 3.12 uses three ingredients:

- The local cancellations obtained in Theorem 2.4 and 3.4 to obtain uniformity as $t\downarrow\downarrow 0$. Incidentally, the proof shows how Theorem 3.4 could be deduced from Theorem 2.4.
- Certain probabilistic estimates, which are obtained by the partial Malliavin calculus [BM] and the techniques of [B 2] in order to localize the problem in an arbitrary small neighborhood of a given fiber Z_{y_0}.
- A technique due to Getzler [Ge] which is used to ultimately obtain the required convergence result.

In certain aspects, the proof of Theorem 3.12 should be considered as an expanded treatment of [B 5, Sect. 5].

In h), we prove in Theorem 3.14 that if the family D has index 0 and is invertible over c, we have a uniform exponential decay of the traces of the corresponding heat kernels as $\varepsilon\downarrow\downarrow 0$. This result is technically difficult to prove since it does not follow from trivial bounds on the traces. We use a probabilistic technique, which overcomes the lack of uniform ellipticity in the directions of c, by instead controlling a time depending parabolic equation along the fibers which exhibits a.s. exponential decay in the sense of bounded operators acting on L_2 sections. The exponential decay of the traces is obtained by using the partial Malliavin calculus [BM, B 2] on a finite time interval.

In i), we prove the holonomy Theorem in the form indicated in the introduction. The main difficulty lies in the elimination of zero modes which are unavoidable if $\operatorname{Ind} D_+ \neq 0$. The idea is to deform continuously the family D into a family of pseudo-differential operators, which verifies the assumptions of Theorem 3.14.

Finally, in j), we briefly interpret the process of blowing up the metric of B in terms of the local geometry of the fibered manifold M.

a) Assumptions and Notations

We now go back to the assumptions of Sects. 1c), 1g). In particular in the sequel,

$$D=\begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

will be the family of Dirac operators considered in Sect. 1g).

b) A Generalized Lichnerowicz Formula

$e_1, \ldots, e_n$ is an orthonormal oriented base of TZ. $f_1, \ldots, f_m$, $dy^1, \ldots, dy^m$ are chosen as in Sect. 1g). z is an extra Grassmann variable which anticommutes with the Clifford variables $e_1, \ldots, e_n$ and with the Grassmann variables $dy^1, \ldots, dy^m$. We will use the notation $K^{(0,1)}$ to select the terms in K whose degree in the Grassmann variables dy^α is 0 or 1.

We now prove an extension of the generalized Lichnerowicz formula in Bismut [**B** 5, Theorem 3.6]. By proceeding as in Sect. 2, Remark 5, the reader will easily check that this formula is in fact a direct consequence of [**B** 5, Theorem 3.6].

Theorem 3.1. *For any $t>0$, the following identity holds*

$$\begin{aligned}
&[(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2-2z\sqrt{t}D]^{(0,1)}=[(\tilde{\nabla}^{u}+\sqrt{t}D)^2-2z\sqrt{t}D]^{(0,1)}\\
&=\Bigg[-t\left(\nabla_{e_i}+\frac{1}{2t}\langle S(e_i)e_j, f_\alpha\rangle\sqrt{t}e_j dy^\alpha+\sqrt{t}\frac{ze_i}{t}\right)^2\\
&\quad+t\frac{K}{4}+\frac{t}{2}e_ie_j\otimes L(e_i,e_j)+\sqrt{t}e_i dy^\alpha\otimes L(e_i,f_\alpha)\Bigg]^{(0,1)}. \qquad (3.1)
\end{aligned}$$

Proof. A defined in Definition 1.17 is of degree 2 in the variables dy^α. Using Proposition 1.18, the first part of the identity is obvious. Let $I^{L,t,z}$ be the final expression in (3.1), $I^{L,t}$ the corresponding expression with $z=0$. Clearly

$$I^{L,t,z}=I^{L,t}-2z\sqrt{t}D-\tfrac{1}{2}\langle S(e_i)e_j, f_\alpha\rangle\,(e_j dy^\alpha z e_i+z e_i e_j dy^\alpha)\,. \qquad (3.2)$$

By (1.49), we have

$$\sum_{i \neq j}\langle S(e_i)e_j, f_\alpha\rangle e_ie_j=0\,. \qquad (3.3)$$

Also

$$e_i dy^\alpha z e_i+z e_i e_i dy^\alpha=-dy^\alpha z-z dy^\alpha=0\,, \qquad (3.4)$$

and so

$$I^{L,t,z}=I^{L,t}-2z\sqrt{t}D\,. \qquad (3.5)$$

Now by [**B** 5, Theorem 3.6]

$$[(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2]^{(0,1)}=I^{L,t}\,. \qquad (3.6)$$

Using (3.2)–(3.6), the theorem is proved. □

c) Construction of Certain Heat Kernels

As in Sect. 2c), we construct certain heat kernels using the Grassmann variable z, with the same ultimate purpose of proving local cancellation results.

Definition 3.2. For $t>0$, $R_t(x,x')$ denotes the C^∞ kernel on Z associated with the operator $\exp\left\{-\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}+z\sqrt{t}D\right\}$.

By Theorem 3.1, $R_t^{(0,1)}(x,x')$ is also the kernel for the operator $\exp\left\{-\frac{(\tilde{\nabla}^u+\sqrt{t}D)^2}{2}+z\sqrt{t}D\right\}^{(0,1)}$. $R_t(x,x')$ has the natural decomposition

$$R_t(x,x')=R_t^0(x,x')+z\sqrt{t}R_t^1(x,x'). \tag{3.7}$$

For $x\in M$, $R_t^0(x,x)$ (respectively $R_t^1(x,x)$) is even (respectively odd) in $\mathrm{End}(F\otimes\xi)_x\hat{\otimes}\Lambda_{\pi(x)}(T^*B)$.

The linear functional Tr_s, which is well defined on trace class operators in $\mathrm{End}\,H^\infty\hat{\otimes}\Lambda(T^*B)$ can be naturally extended to trace class operators in $\mathrm{End}\,H^\infty\hat{\otimes}\Lambda(T^*B)\hat{\otimes}R(z)$ in the obvious way. At a local level, the same is true for elements of

$$\mathrm{End}(F\otimes\xi)_x\hat{\otimes}\Lambda_{\pi(x)}(T^*B)\hat{\otimes}R(z).$$

Of course we still use the obvious extension of (1.24) in this situation.

We first prove some useful identities.

Theorem 3.3. *The following identities hold*

$$\begin{aligned}\mathrm{Tr}_s&\left[\exp\left\{-\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}+z\sqrt{t}D\right\}\right]\\ &=\mathrm{Tr}_s\exp-\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}+z\sqrt{t}\,\mathrm{Tr}_s\left[D\exp-\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}\right]\\ &=\int_Z \mathrm{Tr}_s[R_t(x,x)]\,dx\end{aligned}$$

$$\left[\mathrm{Tr}_s D\exp-\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}\right]^{(1)}=-\frac{\sqrt{t}}{2}\mathrm{Tr}_s\left[\exp\left(-\frac{tD^2}{2}\right)\tilde{\nabla}^u DD\right]. \tag{3.8}$$

Proof. Using Duhamel's formula, and the fact that $z^2=0$, we find that

$$\begin{aligned}\exp&\left\{-\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}+z\sqrt{t}D\right\}\\ &=\exp-\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}+\int_0^1\exp\left\{-\frac{s(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}\right\}\\ &\quad\times z\sqrt{t}D\exp\left\{-(1-s)\frac{(\tilde{\nabla}^{L,t}+\sqrt{t}D)^2}{2}\right\}ds.\end{aligned} \tag{3.9}$$

When taking supertraces in (3.9), we can commute $\exp - \frac{s(\tilde{\nabla}^{L,t} + \sqrt{t}D)^2}{2}$ and so obtain the first equality in (3.8). The final equality in this first line is obvious. Also clearly

$$\left[\mathrm{Tr}_s D \exp - \frac{(\tilde{\nabla}^{L,t} + \sqrt{t}D)^2}{2}\right]^{(1)}$$
$$= \left[\mathrm{Tr}_s D \exp - \frac{(\tilde{\nabla}^u + \sqrt{t}D)^2}{2}\right]^{(1)} = \left[\mathrm{Tr}_s D \exp - \frac{(tD^2 + \sqrt{t}\tilde{\nabla}^u D)}{2}\right]^{(1)}. \quad (3.10)$$

By Duhamel's formula we find easily that

$$\left[\exp - \frac{(tD^2 + \sqrt{t}\tilde{\nabla}^u D)}{2}\right]^{(1)} = -\int_0^1 \exp\left(-\frac{stD^2}{2}\right)\frac{\sqrt{t}}{2}\tilde{\nabla}^u D \exp\left(-\frac{(1-s)tD^2}{2}\right) ds. \quad (3.11)$$

The second line of (3.8) immediately follows from (3.10), (3.11). □

d) Local Cancellation Properties of the Connection ${}^1\nabla$

Recall that the differential forms B_j were defined in (1.31). We now prove a cancellation result for the B_j which matches the corresponding result for the $a_j^{(2)}$ proved in Theorem 1.21.

Theorem 3.4. *There is a C^∞ function $C'_{1/2}(x)$ defined on M with values in $\Lambda^1(T^*B)$ (with $C'_{1/2}(x) \in \Lambda_{\pi(x)}(T^*B)$) which is such that as $t\downarrow\downarrow 0$,*

$$[\mathrm{Tr}_s R_t^1(x,x)]^{(1)} = C'_{1/2}(x)\sqrt{t} + O(t^{3/2}, x) \quad (3.12)$$

and $O(t^{3/2}, x)$ is uniform on the compact sets of M. In particular for $j \leq 0$, $B_j = 0$.

Proof. We first study the asymptotics of $\mathrm{Tr}_s[R_t(x,x)]^{(0,1)}$. Set

$$J^t = -t\left(\nabla_{e_i} + \frac{1}{2t}\langle S(e_i)e_j, f_\alpha\rangle\sqrt{t}e_j dy^\alpha + \sqrt{t}\frac{ze_i}{t}\right)^2$$
$$+ \frac{tK}{4} + \frac{t}{2}e_i e_j \otimes L(e_i, e_j) + \sqrt{t}e_i dy^\alpha \otimes L(e_i, f_\alpha). \quad (3.13)$$

Let $R'_t(x,x')$ be the C^∞ kernel associated with $\exp\left(-\frac{J^t}{2}\right)$. By Theorem 3.1, we know that

$$R_t^{(0,1)}(x,x') = R_t'^{(0,1)}(x,x'). \quad (3.14)$$

Equation (3.13) has the same structure as the generalized Lichnerowicz formula of [B 5, Theorem 3.6]. There is a supplementary Grassmann variable z formally associated with a vector $\bar{f}$ and here

$$\begin{aligned} \langle S(e_i)e_j, \bar{f}\rangle &= -\langle S(e_i)\bar{f}, e_j\rangle = -2\delta_i^j, \\ \langle S(e_i)f_\alpha, \bar{f}\rangle &= \langle S(e_i)\bar{f}, f_\alpha\rangle = 0. \end{aligned} \quad (3.15)$$

We now use [B 5, Theorem 4.12] in this situation. We then already know that as $t\downarrow\downarrow 0$, $\mathrm{Tr}_s[R_t'(x,x)]$ has a limit $\mathscr{L}(x)$. Also we know that this limit is expressed as an expectation over the probability space W of a Brownian bridge w'^1 in T_xZ. More precisely

$$\mathscr{L}(x)=\int_W \exp\{K(w'^1)\}\,dP_1(w'^1).$$

Now by (3.15), we find that $\langle \nabla S(e_i)e_j, \bar{f}\rangle=0$. Using [B 5, Theorem 4.12], we find that the Grassmann variable z only appear in $K(w'^1)$ in the form

$$\int_0^1 \langle P_Z S(w'^1)f_\alpha, P_Z S(dw'^1)\bar{f}\rangle\, dy^\alpha z, \tag{3.16}$$

or in the expression obtained by interchanging f_α and $\bar{f}$, which coïncides, up to sign, with (3.16).

Now using (3.15), we get

$$\begin{aligned}&\int_0^1 \langle P_Z S(w'^1)f_\alpha, P_Z S(dw'^1)\bar{f}\rangle\\ &\quad=\sum_{j=1}^n\int_0^1 \langle S(w'^1)f_\alpha, e_j\rangle\langle S(dw'^1)\bar{f}, e_j\rangle = 2\int_0^1\langle S(w'^1)f_\alpha, dw'^1\rangle\\ &\quad=-2\int_0^1\langle S(w'^1)dw'^1, f_\alpha\rangle.\end{aligned}$$

Integrating by parts and using (1.48), we find

$$2\int_0^1 S(w'^1)dw'^1=\int_0^1 S(w'^1)dw'^1-S(dw'^1)w'^1=-\int_0^1 T(w'^1,dw'^1). \tag{3.17}$$

Since $w'^1\in TZ$, $T(w'^1,dw'^1)=0$, and so (3.17) vanishes.

So we find that $\mathscr{L}(x)$ does not contain z. Using (3.7) and (3.14), we see that

$$\lim_{t\downarrow\downarrow 0}\sqrt{t}\,\mathrm{Tr}_s[R_t^1(x,x)]^{(1)}=0. \tag{3.18}$$

Let φ_t' be the homomorphism of $\Lambda(T^*B)\hat{\otimes}R(z)$ which to dy^α, z associates $\sqrt{t}dy^\alpha$, $\sqrt{t}z$. In Sect. 3c), we saw that $R_t^{(0,1)}$ is the kernel of the operator $\left[\exp-\frac{(\tilde{\nabla}^u+\sqrt{t}D)^2}{2}+z\sqrt{t}D\right]^{(0,1)}$. $\varphi_t'R_t^{(0,1)}$ is then the kernel of the operator $\left[\exp t\left(zD-\frac{(\tilde{\nabla}^u+D)^2}{2}\right)\right]^{(0,1)}$.

By Greiner [Gr, Theorem 1.6.1], we have the asymptotic expansion

$$\mathrm{Tr}_s[\varphi_t'R_t^{(0,1)}(x,x)]=\frac{E_{-n/2}(x)}{t^{n/2}}+\ldots+E_0(x)+E_1(x)t+E_2(x)t^2+O(t^3,x).$$

We then find that

$$\sqrt{t}[\mathrm{Tr}_s R_t^1(x,x)]^{(1)}=\frac{E'_{-n/2-1}(x)}{t^{n/2+1}}+\ldots+E_1'(x)t+O(t^2,x),$$

and $O(t^2,x)$ is uniform on compact sets in M. Using (3.18), we find that for $j\leqq 0$, $E_j'=0$. Equation (3.12) is proved.

Also, by Theorem 3.3, we know that

$$\int_Z \mathrm{Tr}_s[R_t^1(x,x)]^{(1)}dx = -\frac{\sqrt{t}}{2}\mathrm{Tr}_s\left[\exp -\frac{tD^2}{2}\tilde{\nabla}^u DD\right]. \tag{3.19}$$

Using (3.12), (3.19), and comparing with (1.31), we find that for $j \leqq 0$, $B_j = 0$. □

Remark 1. By proceeding as in Sect. 2, Remark 5, we could have proved (3.12) as a direct consequence of [B 5, Sect. 4], by using the results of [B 5] on the Index Theorem for families with a new parameter ε included. This is in fact what we implicitly do in (3.15)–(3.17). Equation (3.12) is in fact equivalent to the vanishing of part of a curvature tensor as in (2.47).

Remark 2. The scaling anomaly described in Remark 2 of Sect. 1 has almost disappeared. In (1.44), the connection ${}^1\nabla_b$ is obtained from ${}^1\nabla$ by the gauge transformation $l \in \lambda \to b^{A_0} l$. In particular the holonomy of the determinant bundle λ over loops in B does not depend on the real constant μ introduced in Sect. 1 f), when defining the connection ${}^1\nabla$.

e) The Dirac Operator Over a Lifted Loop

$s \in S_1 = R/Z \to c_s$ is a C^∞ loop in B. Our purpose will now be to calculate the holonomy of the determinant bundle λ over c.

By eventually changing the parametrization of c, and by scaling the metric g_B, we may and we will assume that

$$\left\|\frac{dc}{ds}\right\|_B = 1 .$$

Note that ultimately, all our results will not depend on the metric g_B.

c is naturally oriented by the natural orientation of S_1. M' denotes the manifold $M' = \pi^{-1}(c)$. The dimension of M' is $n' = 2l+1$. Since TZ is even dimensional and oriented, M' is unambiguously oriented.

Let ∇'^L be the Levi-Civita connection on TM'. ∇'^L is obtained by projecting orthogonally ∇^L on TM'. Since the connection ∇ on TZ is the orthogonal projection of ∇^L on TZ, ∇ is also the orthogonal projection of ∇'^L on TZ. This means that the construction of ∇ can in fact be done directly on the manifold M'.

As a consequence, we will temporarily assume that the base manifold B is exactly the loop c. We will still use the notation M'. We otherwise use the same notations as in the previous sections in this new situation, i.e. ∇^L is the Levi-Civita connection on TM', S the tensor defined by the relation $\nabla^L = \nabla + S$, where S acts on TM' etc.... TB is now trivial and spanned by

$$f_1 = \frac{dc}{ds}. \tag{3.20}$$

dy^1 is the Grassmann variable dual to f_1. We also identity f_1 with f_1^H. Clearly $\nabla_{f_1} f_1 = 0$ and more generally

$$\nabla \cdot f_1 = 0. \tag{3.21}$$

By Sect. 1 d), we know that

$$S(f_1)f_1=0, \tag{3.22}$$

and so

$$\nabla^L_{f_1}f_1=0. \tag{3.23}$$

This simply reflects the fact that the integral curves of f_1 in M' are geodesics. Consider on M' the differential equation

$$\frac{dx}{ds}=f_1(x), \qquad x(0)=x_0, \tag{3.24}$$

and set

$$x_s=\psi_s(x_0). \tag{3.25}$$

Take $y_0 \in B$. Then $(s,x)\in R\times Z_{y_0}\to\psi_s(x)\in M'$ is a local diffeomorphism. M' can be identified with $[0,1]\times Z_{y_0}$ and the relation $(0,x)=(1,\psi_1(x))$. In the coordinates (s,x), the metric of M' is given by

$$ds^2+g_{ij}(s,x)dx^i\otimes dx^j. \tag{3.26}$$

Also since B is now of dimension 1, by Proposition 1.18, we have $\tilde{\nabla}^u=\tilde{\nabla}^{L,t}$.

$d'x$ is the volume element of M'. Since dx is the volume element in Z, if dy is the length element of c, we have $d'x=dy\,dx$. The kernels on M' will be calculated with respect to $d'x$.

O still denotes the $SO(n)$ bundle of oriented orthonormal frames in TZ. M' is obviously spin. Using the convention of Sect. 1 b), the bundle of spinors on M' can be identified with $F=F_+\oplus F_-$. By (1.8), f_1 acts on

$$F\otimes\xi=(F_+\otimes\xi)\oplus(F_-\otimes\xi)$$

like $-i\tau$, where τ is the involution defining the grading. In matrix form, f_1 acts on H^∞ as the matrix $\varphi(f_1)$,

$$\varphi(f_1)=\begin{bmatrix}-i & 0\\ 0 & +i\end{bmatrix}. \tag{3.27}$$

This permits us to define the action of f_1 when more general Z_2 graded bundles than $F\otimes\xi$ are considered. This will be the case in the proof of Theorem 3.16.

Any element A of $\operatorname{End}H^\infty\hat{\otimes}c(TB)$ has a unique decomposition

$$A=A_0+A_1f_1; \qquad A_0,A_1\in\operatorname{End}H^\infty. \tag{3.28}$$

One verifies trivially that φ defined by

$$A\in\operatorname{End}H^\infty\hat{\otimes}c(TB)\to\varphi(A)=A_0+A_1\varphi(f_1)\in\operatorname{End}H^\infty$$

is a homomorphism of *ungraded* algebras.

Let $\mathscr{E}$ be the graded algebra $\mathscr{E}=\operatorname{End}H^\infty\hat{\otimes}c(TB)\hat{\otimes}R(z)$. Any $a\in\mathscr{E}$ has a unique decomposition

$$\begin{aligned} a&=a_0+a_1f_1, \qquad a_0,a_1\in\operatorname{End}H^\infty\hat{\otimes}R(z),\\ a&=a'_0+za'_1, \qquad a'_0,a'_1\in\operatorname{End}H^\infty\hat{\otimes}c(TB).\end{aligned} \tag{3.29}$$

If $a \in \mathscr{E}$ is trace class, set

$$\mathrm{Tr}_z a = z\,\mathrm{Tr}[\varphi(a_1')] . \tag{3.30}$$

If $a \in \mathscr{E}^{\text{even}}$ is trace class, a_0 does not contribute to $\mathrm{Tr}_z a$, since in a_0, z factors an odd element of $\mathrm{End}\,H^\infty$. If a_1 is of the form

$$a_1 = a_1^1 + z a_1^2; \qquad a_1^1, a_1^2 \in \mathrm{End}\,H^\infty , \tag{3.31}$$

then

$$\mathrm{Tr}_z a = z\,\mathrm{Tr}\, a_1^2 \varphi(f_1) = -iz\,\mathrm{Tr}_s a_1^2 . \tag{3.32}$$

Also a_1^1 is odd in $\mathrm{End}\,H^\infty$, and so $\mathrm{Tr}_s a_1^1 = 0$. We can also define Tr and Tr_s on $\mathrm{End}\,H^\infty \hat{\otimes} R(z)$ by using the convention

$$\mathrm{Tr}\, zb = z\,\mathrm{Tr}\, b; \qquad \mathrm{Tr}_s zb = z\,\mathrm{Tr}_s b . \tag{3.33}$$

Now in (3.31), a_1^1 is odd and a_1^2 is even in $\mathrm{End}\,H^\infty$. Equation (3.32) implies

$$\mathrm{Tr}_z a = -i\,\mathrm{Tr}_s a_1 . \tag{3.34}$$

In the sequel we will write f_1, A instead of $\varphi(f_1)$, $\varphi(A)$. This will have to be done with some care since φ does not respect the grading. However most of our computations are done in the graded algebra $\mathscr{E}$.

Using the results of Sect. 1 d) and (1.2), we know that when acting on sections of $F \otimes \xi$, ∇ and ∇^L are related by

$$\nabla^L_{e_i} = \nabla_{e_i} + \tfrac{1}{2}\langle S(e_i)e_j, f_1\rangle e_j f_1 , \qquad \nabla^L_{f_1} = \nabla_{f_1} . \tag{3.35}$$

Also $\langle k, f_1\rangle$ is unambiguously defined on M'. This is of course confirmed by Proposition 1.4.

Set

$$\nabla^u_{f_1} = \nabla_{f_1} + \langle k, f_1\rangle . \tag{3.36}$$

We drop the $\sim$ sign in $\tilde{\nabla}^u_{f_1}$ to indicate that $\nabla^u_{f_1}$ is a local operator.

Definition 3.5. For $\varepsilon > 0$, D^ε denotes the operator acting on H^∞

$$D^\varepsilon = \sqrt{\varepsilon}\, f_1 \nabla^u_{f_1} + D . \tag{3.37}$$

D^ε is given in matrix form by

$$D^\varepsilon = \begin{bmatrix} -i\sqrt{\varepsilon}\nabla^u_{f_1} & D_- \\ D_+ & i\sqrt{\varepsilon}\nabla^u_{f_1} \end{bmatrix} . \tag{3.38}$$

We first prove the elementary result.

Proposition 3.6. *D^ε is the self-adjoint Dirac operator associated with the Levi-Civita connection ${}^\varepsilon\nabla^L$ on TM' for the metric $\frac{g_B}{\varepsilon} \oplus g_Z$.*

Proof. We only prove the Proposition for $\varepsilon = 1$. The Dirac operator D' on M' is given by

$$D' = e_i \nabla^L_{e_i} + f_1 \nabla^L_{f_1} .$$

Using (3.35), we find

$$D' = e_i(\nabla_{e_i} + \tfrac{1}{2}\langle S(e_i)e_j, f_1\rangle e_j f_1) + f_1 \nabla_{f_1}.$$

The proof finishes as the proof of Proposition 1.18. □

Definition 3.7. $\eta^\varepsilon(s)$ denotes the êta function for the operator D^ε. h^ε is the integer

$$h^\varepsilon = \dim \operatorname{Ker} D^\varepsilon. \tag{3.39}$$

$\bar{\eta}^\varepsilon(s)$ is defined by

$$\bar{\eta}^\varepsilon(s) = \frac{\eta^\varepsilon(s) + h^\varepsilon}{2}. \tag{3.40}$$

f) Variation of $[\bar{\eta}^\varepsilon(0)]$

We will now calculate $\frac{\partial}{\partial \varepsilon^{1/2}}[\bar{\eta}^\varepsilon(0)]$. R^L is the curvature tensor of TM' for the Levi-Civita connection ∇^L. Similarly $R^{L,\varepsilon}$ is the curvature tensor of TM' for the connection ${}^\varepsilon\nabla^L$.

We will consider S as a one form on TM' with values in antisymmetric tensors on TM'. Using (3.22), we know that

$$S(f_1) = 0. \tag{3.41}$$

We now have the following result:

Theorem 3.8. *The following identity holds:*

$$z\frac{\partial}{\partial \varepsilon^{1/2}}[\bar{\eta}^\varepsilon(0)]_{\varepsilon=1} = \int_{M'} \hat{A}\left(\frac{R^L + zS}{2\pi}\right) \operatorname{Tr}\left[\exp - \frac{L}{2i\pi}\right]. \tag{3.42}$$

Also

$$\lim_{\varepsilon \downarrow\downarrow 0} z\frac{\partial}{\partial \varepsilon^{1/2}}[\bar{\eta}^\varepsilon(0)] = \int_{M'} \hat{A}\left(\frac{R^Z + zS}{2\pi}\right) \operatorname{Tr}\left[\exp - \frac{L}{2i\pi}\right]. \tag{3.43}$$

As $\varepsilon\downarrow\downarrow 0$, $[\bar{\eta}^\varepsilon(0)]$ *converges in* R/Z *to* $[\bar{\eta}]$. $[\bar{\eta}^\varepsilon(0)]$ *is a* C^∞ *function of* $\varepsilon^{1/2}$ *on* $[0,1]$.

Proof. To prove (3.42), we will use formula (2.39). Let P_H, P_Z be the orthogonal projection operators from TM' on TZ, T^HM'. $M' \times R^+$ fibers over R^+ with fiber M'. *For* $\varepsilon \in R^+$, we will note M'_ε the corresponding fiber. We endow TM'_ε with the metric $\frac{g_B}{\varepsilon} \oplus g_Z$. Recall that ${}^\varepsilon\nabla^L$ is the Levi-Civita connection of TM'_ε. On $M' \times R^+$, we consider the connection ∇' on TM' which is defined in the following way:

$$\begin{aligned} &\text{If } X, Y \in TM'_\varepsilon, && \nabla'_X Y = {}^\varepsilon\nabla^L_X Y, \\ &\text{If } Y \in TM'_\varepsilon, && \nabla'_{\frac{\partial}{\partial\varepsilon}} Y = -P_H Y/2\varepsilon. \end{aligned} \tag{3.44}$$

$M' \times R_+$ is naturally endowed with the horizontal subbundle of $T(M' \times R_+)$ which is spanned by $\frac{\partial}{\partial \varepsilon}$. One verifies easily that the connection ∇' on TM' –

considered as a vector bundle on $M' \times R^+$ – preserves the metric of TM' and that ∇' is exactly the connection on TM' which was constructed in Sects. 1d) and 2e) (where TM' was instead TZ).

By proceeding as in [B 5, Eq. (3.10)] it is not difficult to see that if $S^\varepsilon = {}^\varepsilon\nabla^L - \nabla$, then

$$P_Z S^\varepsilon = P_Z S; \qquad P_H S^\varepsilon = \varepsilon P_H S. \tag{3.45}$$

Let R' be the curvature tensor of ∇'. Take $X, Y \in TM'_\varepsilon$. Clearly

$$R'(X, Y) = R^{L,\varepsilon}(X, Y). \tag{3.46}$$

Also, using (3.45), we find

$$\begin{aligned} R'\left(\frac{\partial}{\partial\varepsilon}, X\right) Y &= \nabla'_{\frac{\partial}{\partial\varepsilon}} {}^\varepsilon\nabla^L_X Y - {}^\varepsilon\nabla^L_X \nabla'_{\frac{\partial}{\partial\varepsilon}} Y \\ &= \nabla'_{\frac{\partial}{\partial\varepsilon}} [\nabla_X Y + P_Z S(X) Y + \varepsilon P_H S(X) Y] + {}^\varepsilon\nabla^L_X P_H Y/2\varepsilon \\ &= -P_H \nabla_X Y/2\varepsilon + P_H S(X) Y - P_H S(X) Y/2 + P_H \nabla_X Y/2\varepsilon \\ &\quad + S^\varepsilon(X)(P_H Y)/2\varepsilon \\ &= P_H S^\varepsilon(X) Y/2\varepsilon + S^\varepsilon(X)(P_H Y)/2\varepsilon. \end{aligned} \tag{3.47}$$

Since $S^\varepsilon(X)$ is antisymmetric, it interchanges $T^H M'$ (which is one dimensional) and TZ. From (3.45), we obtain

$$R'\left(\frac{\partial}{\partial\varepsilon}, X\right) Y = S^\varepsilon(X) Y/2\varepsilon. \tag{3.48}$$

Using formula (2.39), we find that

$$z\frac{\partial}{\partial\varepsilon}[\bar{\eta}^\varepsilon(0)] = \int_{M'} \hat{A}\left(\frac{R^{L,\varepsilon} + zS^\varepsilon/2\varepsilon}{2\pi}\right) \operatorname{Tr}\left[\exp - \frac{L}{2i\pi}\right] \tag{3.49}$$

and so

$$z\frac{\partial}{\partial\varepsilon^{1/2}}[\bar{\eta}^\varepsilon(0)] = \int_{M'} \hat{A}\left(\frac{R^{L,\varepsilon} + zS^\varepsilon/\sqrt{\varepsilon}}{2\pi}\right) \operatorname{Tr}\left[\exp - \frac{L}{2i\pi}\right] \tag{3.50}$$

Formula (3.42) is proved.

Clearly, if $\mathring{D}$ is the horizontal differentiation operator associated with ∇,

$$R^{L,\varepsilon} = R + \mathring{D} S^\varepsilon + [S^\varepsilon, S^\varepsilon]. \tag{3.51}$$

If we express $R^{L,\varepsilon}$ on the base $(e_1, \ldots, e_n, \sqrt{\varepsilon} f_1)$, using (3.45), we find

$$R^{L,\varepsilon} = \begin{bmatrix} R^Z + \varepsilon P_Z[S, S] & \varepsilon^{1/2} P_Z \mathring{D} S \\ \varepsilon^{1/2} P_H \mathring{D} S & 0 \end{bmatrix}. \tag{3.52}$$

On the same base, $S^\varepsilon/\sqrt{\varepsilon}$ can be represented in the form

$$S^\varepsilon/\sqrt{\varepsilon} = \begin{bmatrix} 0 & P_Z S \\ P_H S & 0 \end{bmatrix}. \tag{3.53}$$

We then find that as $\varepsilon\downarrow\downarrow 0$,

$$\hat{A}\left(\frac{R^{L,\varepsilon}+zS^{\varepsilon}/\sqrt{\varepsilon}}{2\pi}\right)\to\hat{A}\left(\frac{R^{Z}+zS}{2\pi}\right). \tag{3.54}$$

Equation (3.43) immediately follows from (3.50) and (3.54). Using (3.50), (3.52) and (3.53), it is obvious that $[\bar{\eta}^{\varepsilon}(0)]$ is a smooth function of $\varepsilon^{1/2}$ for $\varepsilon\in[0,1]$. □

Remark 3. Theorem 3.8 makes clear that in general $[\bar{\eta}^{\varepsilon}(0)]$ depends on ε. More precisely $[\bar{\eta}^{\varepsilon}(0)]$ depends explicitly on the tensor T. T vanishes if and only if for $e\in TZ$,

$$\nabla_{f_1}e=[f_1,e], \tag{3.55}$$

or equivalently if f_1 acts isometrically on the fibers Z. M' is then locally and metrically a product. As is clear from (3.52), as $\varepsilon\downarrow\downarrow 0$, we get closer and closer to a product situation.

Remark 4. The general considerations of Atiyah-Patodi-Singer [APS 1, p. 61] show that (3.42) can vanish for purely algebraic reasons. This is for instance the case if ξ is the trivial line bundle and if l is even: the top degree form in the right-hand side of (3.42) vanishes locally. $[\bar{\eta}^{\varepsilon}(0)]$ is then independent of ε. More generally, by using Index Theory with boundary, it is shown in [APS 1] that $[\bar{\eta}^{\varepsilon}(0)]$ is in this case a spin cobordism invariant. Note that we could use instead Theorem 2.10 to obtain the results of [APS 1] on êta invariants.

g) Convergence of Heat Kernels on M' as $\varepsilon\downarrow\downarrow 0$

Recall that our ultimate goal is to prove a formula relating the holonomy of the connection ${}^1\nabla$ on c to $[\bar{\eta}]$. The idea is to use the representation (2.22), (2.23) for $\eta^{\varepsilon}(0)$ and to prove that as $\varepsilon\downarrow\downarrow 0$, the integrand in (2.22) converges to the corresponding integrand which appears in the formula (1.38) defining δ_0.

We first prove two simple identities, which are still special cases of [B 5, Theorem 3.6].

Proposition 3.9. *The following identities hold:*

$$\begin{aligned}
&-\frac{\varepsilon}{2}(f_1\nabla^u_{f_1})^2+z\sqrt{\varepsilon}\,f_1\nabla^u_{f_1}=\frac{\varepsilon}{2}\left(\nabla_{f_1}+\frac{zf_1}{\sqrt{\varepsilon}}+\langle k,f_1\rangle\right)^2,\\
&-\frac{(D^{\varepsilon})^2}{2}+zD^{\varepsilon}=-\frac{\varepsilon(f_1\nabla^u_{f_1})^2}{2}+z\sqrt{\varepsilon}\,f_1\nabla^u_{f_1}-\frac{\sqrt{\varepsilon}\,f_1(\nabla^u_{f_1}D)}{2}-\frac{D^2}{2}+zD.
\end{aligned} \tag{3.56}$$

Proof. The right-hand side of the first line of (3.56) is given by

$$\frac{\varepsilon}{2}\left(\nabla^2_{f_1}+\langle k,f_1\rangle^2+\frac{2z}{\sqrt{\varepsilon}}f_1\nabla_{f_1}+\langle\nabla_{f_1}k,f_1\rangle+2\langle k,f_1\rangle\nabla_{f_1}+\frac{2z}{\sqrt{\varepsilon}}\langle k,f_1\rangle f_1\right). \tag{3.57}$$

Also

$$\begin{aligned}
-\frac{\varepsilon}{2}(f_1\nabla^u_{f_1})^2+z\sqrt{\varepsilon}f_1\nabla^u_{f_1}&=-\frac{\varepsilon}{2}(f_1\nabla_{f_1}+\langle k,f_1\rangle f_1)^2+z\sqrt{\varepsilon}f_1(\nabla_{f_1}+\langle k,f_1\rangle)\\
&=\frac{\varepsilon}{2}(\nabla^2_{f_1}+\langle k,f_1\rangle^2+\langle\nabla_{f_1}k,f_1\rangle+2\langle k,f_1\rangle\nabla_{f_1}+z\sqrt{\varepsilon}f_1(\nabla_{f_1}+\langle k,f_1\rangle).
\end{aligned} \tag{3.58}$$

Comparing (3.57) and (3.58), the first line of (3.56) is proved. Since $Df_1+f_1D=0$, the second line is obvious. □

Definition 3.10. For $\varepsilon>0$, $t>0$, $P_t'^{\varepsilon}(x,x')$, $P_t^{\varepsilon,0}(x,x')$, $P_t^{\varepsilon,1}(x,x')$ denote the C^∞ kernels on M' associated with the operators $\exp\left(-\frac{t(D^\varepsilon)^2}{2}+tzD^\varepsilon\right)$, $\exp-\frac{t(D^\varepsilon)^2}{2}$, $D^\varepsilon\exp-\frac{t(D^\varepsilon)^2}{2}$.

Clearly

$$P_t'^{\varepsilon}(x,x')=P_t^{\varepsilon,0}(x,x')+tzP_t^{\varepsilon,1}(x,x'). \tag{3.59}$$

Also by Theorem 2.4, for $\varepsilon>0$, uniformly on M'

$$\lim_{t\downarrow\downarrow 0}\operatorname{Tr}[P_t^{\varepsilon,1}(x,x)]=0. \tag{3.60}$$

So we can define by continuity the function $\operatorname{Tr}[P_t^{\varepsilon,1}(x,x)]$ at $t=0$.

Definition 3.11. For $t>0$, $R_t'(x,x')$ denotes the C^∞ kernel on Z associated with the operator $\exp\left\{-\frac{t(\tilde{V}^u+D)^2}{2}+tzD\right\}$.

Recall that now $\tilde{V}^u=\tilde{V}^{L,t}$. Also since B is of dimension 1, $(\tilde{V}^u)^2=0$. $R_t'(x,x')$ is the kernel of $\exp\left\{-\frac{t}{2}(D^2+\tilde{V}^uD)+tzD\right\}$.

$R_t'(x,x')$ can be written as

$$R_t'(x,x')=R_t'^0(x,x')+tzR_t'^1(x,x'). \tag{3.61}$$

Comparing with Definition 3.2, we find that

$$[R_t'^1(x,x')]^{(1)}=\sqrt{t}[R_t^1(x,x')]^{(1)}. \tag{3.62}$$

Also by Theorem 3.4, we know that

$$\lim_{t\downarrow\downarrow 0}\operatorname{Tr}_s[R_t^1(x,x)]^{(1)}=0 \quad \text{uniformly on } M'. \tag{3.63}$$

So $\operatorname{Tr}_s[R_t^1(x,x)]^{(1)}$ can also be defined by continuity at $t=0$.

We now prove the first critical step in the proof of the holonomy theorem.

Theorem 3.12. *Take T such that $0<T<+\infty$. Then as $\varepsilon\downarrow\downarrow 0$,*

$$\operatorname{Tr}[P_t^{\varepsilon,1}(x,x)]\to-\frac{i}{\sqrt{2\pi}}\langle\operatorname{Tr}_s[R_t^1(x,x)]^{(1)},f_1\rangle, \tag{3.64}$$

uniformly on $[0,T]\times M'$.

Proof. The proof is divided into two main steps, which we first briefly explain.

- The first step consists in proving that as $\varepsilon\downarrow\downarrow 0$, the kernel $P_t^{\varepsilon,1}(x,x')$ localizes in an arbitrary small neighborhood of the fiber $Z_{\pi x}$. This is done by using a probabilistic representation of the kernel P_t' and the partial Malliavin calculus [BM].

• Once localization is proved, we can now replace M' by $R \times Z_{y_0}$ and assume that out of a small neighborhood of Z_{y_0}, we are metrically in a product situation. We then use a technique of Getzler [Ge] to prove the convergence.

The probabilistic representation of P'_t in the first part of the proof will be essential in the proof of Theorem 3.14, where uniform estimates have to be obtained for arbitrary large t.

Our computation will be done in the graded algebra $\mathscr{E}^{\text{even}}$ defined in Sect. 3e). This means that we work locally in $[c(TM')\hat{\otimes} R(z)]^{\text{even}}$.

Step n° 1. Localization of the Convergence. Proposition 3.9 shows that $-\frac{(D^\varepsilon)^2}{2} + zD^\varepsilon$ is the sum of two operators.

• $-\frac{\varepsilon}{2}(f_1 \nabla^u_{f_1})^2 + z\sqrt{\varepsilon} f_1 \nabla^u_{f_1}$ acts horizontally, i.e. in the directions of f_1.

• $-\frac{\sqrt{\varepsilon}}{2} f_1(\nabla^u_{f_1} D) - \frac{D^2}{2} + zD$ acts vertically, i.e. along the fibers Z.

We now use the idea of [B 5, Sect. 5]. We first construct the semi-group $\exp\left\{-t\left(\frac{\varepsilon}{2}(f_1 \nabla^u_{f_1})^2 + z\sqrt{\varepsilon} f_1 \nabla^u_{f_1}\right)\right\}$ using a Brownian motion y in B. The semi-group $\exp\left\{-\frac{t(D^\varepsilon)^2}{2} + tzD^\varepsilon\right\}$ is then obtained by using a subordination procedure.

Since B identifies with S_1, these constructions will be very simple.

a) Construction of $\exp\left(-\frac{\varepsilon}{2}(f_1 \nabla^u_{f_1})^2 + z\sqrt{\varepsilon} f_1 \nabla^u_{f_1}\right)$

Take $y_0 \in B$. Using the differential equation (3.24), the corresponding group of diffeomorphisms ψ defines the parallel transport of the fiber Z_{y_0} into Z_{y_s}, where $y.$ is any continuous path in B with $y(0) = y_0$. Since B has dimension 1, the holonomy group of this connection is the discrete group generated by the diffeomorphism ψ_1 acting on Z_{y_0}. Similarly, we can parallel transport elements of $H^\infty_{y_0}$ into $H^\infty_{y_s}$ using the connection ∇ or the connection $\tilde{\nabla}^u$. τ^0_s, ${}^u\tau^0_s$ will denote the corresponding parallel transport operators, τ^s_0, ${}^u\tau^s_0$ their inverse. If $x_0 \in Z_{y_0}$, we will note $\tau^0_s x_0 \in Z_{y_s}$ the parallel transport of x_0 along $y.$.

Using Proposition 1.4, we find that if $h \in H^\infty_{y_0}$, ${}^u\tau^0_s h \in \mathrm{H}^\infty_{y_s}$, and moreover if $x \in Z_{y_s}$

$$({}^u\tau^0_s h)(x) = [\operatorname{Jac} \tau^s_0(x)]^{1/2} \tau^0_s h(\tau^s_0 x), \tag{3.65}$$

where $\operatorname{Jac}\tau^s_0(x)$ is the Jacobian of τ^s_0 at x.

Let w be a one dimensional Brownian motion with $w_0 = 0$. Let Q be the probability law of w on $\mathscr{C}(R^+; R)$. Identifying B and $S_1 = R/Z$, consider the differential equation

$$dy = \sqrt{\varepsilon}\, dw_s; \quad y_s \in B, \quad y(0) = y_0. \tag{3.66}$$

Clearly

$$y_s = [y_0 + \sqrt{\varepsilon}\, w_s]. \tag{3.67}$$

Take $x \in Z_{y_0}$. Consider the stochastic differential equation

$$dU = U\left[\frac{zf_1}{\sqrt{\varepsilon}} + \langle k(\tau_s^0 x), f_1\rangle\right]\sqrt{\varepsilon}\,dw\,, \qquad U(0) = I\,. \tag{3.68}$$

U_s is given by the formula

$$U_s = \exp\left\{\int_0^s \langle k(\tau_v^0 x), f_1\rangle \sqrt{\varepsilon}\,dw + zf_1 w_s\right\}. \tag{3.69}$$

By Proposition 1.4, using the relation $z^2 = 0$, we have

$$U_s = [\operatorname{Jac}\tau_s^0(x)]^{1/2}\exp(zf_1 w_s) = [\operatorname{Jac}\tau_s^0(x)]^{1/2}[1 + zf_1 w_s]\,. \tag{3.70}$$

We claim that if $h \in H^\infty$, for $s > 0$,

$$\exp s\left(-\frac{\varepsilon}{2}(f_1 \nabla^u_{f_1})^2 + z\sqrt{\varepsilon} f_1 \nabla^u_{f_1}\right)h(x) = \mathrm{E}[\exp(zf_1 w_s)\,({}^u\tau_0^s h)\,(x)]\,. \tag{3.71}$$

Equation (3.71) is in fact a direct consequence of the first line of formula (3.56), of (3.68)–(3.70) and of Itô's formula [B 3].

In the sequel, we will always assume that $t \leqq T$, $\varepsilon \leqq 1$. The various constants – which in general depend on T – will often be denoted C.

Let Q_{y_0} be the law of w conditional on $y_t = y_0$. Equivalently, Q_{y_0} is the law of w conditional on $w_t = \frac{k}{\sqrt{\varepsilon}}$, $k \in Z$. Let β_t be a standard Brownian motion, with $\beta_0 = 0$. Conditionally on $w_t = \frac{k}{\sqrt{\varepsilon}}$, $w_s (0 \leqq s \leqq t)$ has the same law as $\beta_s - \frac{s}{t}\beta_t + \frac{s}{t}\frac{k}{\sqrt{\varepsilon}}$ [Si, p. 41]. Now for $k \in Z$,

$$Q_{y_0}\left[w_t = \frac{k}{\sqrt{\varepsilon}}\right] = \exp\left(-\frac{k^2}{2\varepsilon t}\right)\Big/\sum_{k'}\exp\left(-\frac{k'^2}{2\varepsilon t}\right). \tag{3.72}$$

Also for any $\eta > 0$ by [IMK, p. 27]

$$Q\left[\sup_{0 \leqq s \leqq t}\sqrt{\varepsilon}\,|\beta_s| \geqq \eta\right] \leqq 2\exp\left(-\frac{\eta^2}{2\varepsilon t}\right). \tag{3.73}$$

Using (3.72), (3.73), it is clear that

$$Q_{y_0}\left[\sup_{0 \leqq s \leqq t}|y_s - y_0| \geqq \eta\right] \leqq C\exp\left(-\frac{\eta^2}{\varepsilon t}\right). \tag{3.74}$$

b) Construction of $\exp\left\{-\frac{t}{2}(D^\varepsilon)^2 + tzD^\varepsilon\right\}$

Take $x \in Z_{y_0}$. In order to prove that as $\varepsilon \downarrow\downarrow 0$, $\operatorname{Tr}[P_t^{\varepsilon,1}(x, x)]$ converges, we will first prove that the kernel $P_t'^{\varepsilon}(x, \cdot)$ concentrates in a small neighborhood of Z_{y_0}, in order to replace the base $B = S_1$ by R.

Let Δ_y^Z be the Laplace-Beltrami operator in the fiber Z_y. We first study the scalar heat kernel p'_t on M' associated with $\exp\frac{t}{2}(\varepsilon f_1^2+\Delta^Z)$ and prove the corresponding concentration result. The proof that $P_t'^{\varepsilon}(x,0)$ concentrates will follow by a subordination procedure.

Recall that O is the $SO(n)$ bundle of oriented orthonormal frames in TZ. O is endowed with the connection ∇. Let $X_1^*, \ldots, X_n^*$ be the standard horizontal vector fields on O along the fibers Z. Along each fiber Z_y, $X_1^*, \ldots, X_n^*$ restrict to the standard horizontal fields of O in the sense of [KN, IV]. f_1^* is the horizontal lift of f_1 in TO for the connection ∇. Let $w'=(w'^1, \ldots, w'^n)$ be a Brownian motion in R^n, which is independent of w. The probability law of w' on $\mathscr{C}(R^+; R^n)$ will be noted P.

Take $x_0 \in Z_{y_0}$, $u_0 \in O_{x_0}$. Consider the stochastic differential equation on $(\mathscr{C}(R^+; R^n) \times \mathscr{C}(R^+; R), P \otimes Q)$,

$$du = X_i^*(u)dw'^i + \sqrt{\varepsilon} f_1^*(u)dw, \qquad u(0)=u_0. \tag{3.75}$$

Set $x_s=\varrho(u_s)$. x_s is a Markov diffusion in M', whose infinitesimal generator is exactly $\frac{1}{2}[\Delta^Z+\varepsilon f_1^2]$. $p'_t(x_0,x)d'x$ is exactly the law of x_t.

We now assume that the law of w is Q_{y_0}. Of course we still suppose that w and w' are independent. Let $p'_t(x)dx$ be the law of x_t in Z_{y_0} conditional on $y_s (0 \leqq s \leqq t)$. Using the partial Malliavin calculus of Bismut-Michel [BM], we know that Q_{y_0} a.s., $p'_t(x)$ is C^∞ on Z_{y_0}.

For given $k \in N$, $q \geqq 1$, we want to establish a uniform bound as $\varepsilon \downarrow\downarrow 0$ of

$$E^{Q_{y_0}}[|p'_t|^q_{\mathscr{C}^k(Z_{y_0}, R)}]. \tag{3.76}$$

To do this, we will explicitly use the method of [BM].

Let v_s be a bounded process taking values in R^n, which is adapted to the filtration $\mathscr{B}(w_h, w'_h | h \leqq s)$. For $l \in R$, consider the stochastic differential equation

$$du^l = X_i^*(u^l)(dw'^i + lv^i ds) + f_1^*(u^l)\sqrt{\varepsilon}\, dw, \qquad u(0)=u_0. \tag{3.77}$$

As in [B 2, Chap. 2], we calculate $\left[\frac{\partial u_t^l}{\partial l}\right]_{l=0}$. Let ω be the connection 1 form on O. Similarly let θ be the R^n valued one form on O

$$X \in TO, \qquad \theta_u(X) = u^{-1}\varrho_* X.$$

Let τ, Ω be the 2 forms on O which are the equivariant representations of T, R^Z. The equation of the connection ∇ on O are given by [KN, IV]

$$d\theta = -\omega \wedge \theta + \tau, \qquad d\omega = -\omega \wedge \omega + \Omega. \tag{3.78}$$

Set

$$\theta_s = \theta\left(\frac{\partial u_s^l}{\partial l}\right)_{l=0}, \qquad \omega_s = \omega\left(\frac{\partial u_s^l}{\partial l}\right)_{l=0}. \tag{3.79}$$

Using (3.77), (3.78) and proceeding as in [B 2, Theorem 2.2], we find that

$$d\theta = v ds + \tau(\sqrt{\varepsilon} f_1^* dw, (u_s\theta)^*) + \omega dw'; \qquad \theta(0)=0,$$
$$d\omega = \Omega((u_s dw')^* + \sqrt{\varepsilon} f_1^* dw, (u_s\theta)^*); \qquad \omega(0)=0.$$

We can then use the rotational invariance of w' under infinitesimal rotations as in [B 2, Theorem 2.2]. We ultimately find that if R^{ic} is the Ricci tensor of Z, if σ is its equivariant representation, the relevant equation to be considered in establishing an integration by parts formula conditional on $y.$ is given by

$$\begin{aligned} d\theta' &= (v - \tfrac{1}{2}\sigma\theta')ds + \tau(\sqrt{\varepsilon} f_1^* dw, (u_s\theta')^*); \qquad \theta'(0) = 0, \\ d\omega' &= \Omega((u_s dw')^* + \sqrt{\varepsilon} f_1^* dw, (u_s\theta')^*); \qquad \omega'(0) = 0. \end{aligned} \tag{3.80}$$

In particular by proceeding as in [BM, Sect. 3] and [B 2, Theorem 2.2], we find that for any $f \in C_b^\infty(M')$,

$$E^{P\otimes Q_{y_0}}[\langle df(x_t), u_t\theta_t'\rangle] = E^{P\otimes Q_{y_0}}\left[f(x_t)\int_0^t \langle v, \delta w'\rangle\right]. \tag{3.81}$$

Observe the critical fact that since u_t maps isometrically R^n into $T_{x_t}Z$, Eqs. (3.80) and (3.81) incorporate the variation of the metric in Z. This is reflected in the fact that τ exactly measures to what extent f_1 does not act isometrically on Z.

Let A_s be the solution of the stochastic differential equation,

$$dA_s = -\tfrac{1}{2}\sigma A ds + \tau(\sqrt{\varepsilon} f_1^* dw, (u_s A)^*); \qquad A(0) = I. \tag{3.82}$$

Fix $k \in N$. To bound uniformly (3.76), by using the Malliavin calculus, it is essentially equivalent to dominate

$$E^{Q_{y_0}}\left[\sup_{0\leqq s\leqq t} |A_s|^q\right] \tag{3.83}$$

with q large enough. Note at this stage that it is essential that Eqs. (3.80), (3.81) incorporate the change of metric on Z, so that the size of the variation of x_t in Z_{y_0} is adequately controlled.

Under Q_{y_0}, w is a Brownian bridge, and this creates some difficulties in the semi-martingale description of w under Q_{y_0} [IW, p. 229] since the stochastic differential equation which drives w under Q_{y_0} has singular coefficients as $s\uparrow\uparrow t$.

Let q_s be the heat kernel of S_1 (for its standard metric). Then note the following facts:

- Q_{y_0} and Q are equivalent on $\mathscr{B}\left(y_s \,\middle|\, 0\leqq s\leqq \frac{t}{2}\right)$, and moreover

$$\frac{dQ_{y_0}}{dQ}\mathscr{B}\left(y_s \,|\, 0\leqq s\leqq \frac{t}{2}\right) = \frac{q_{\varepsilon t/2}(y_{t/2}, y_0)}{q_{\varepsilon t}(y_0, y_0)}. \tag{3.84}$$

It is trivial to verify that (3.84) is uniformly bounded as $\varepsilon\downarrow\downarrow 0$. Also it is standard that

$$E^{P\otimes Q}\left[\sup_{0\leqq s\leqq \frac{t}{2}} |A_s|^q\right] \tag{3.85}$$

is uniformly bounded, and so as $\varepsilon\downarrow\downarrow 0$,

$$E^{P\otimes Q_{y_0}}\left[\sup_{0\leqq s\leqq \frac{t}{2}} |A_s|^q\right] \tag{3.86}$$

is uniformly bounded.

• To estimate (3.83), it is then natural to use time reversal. In fact Q_{y_0} is invariant under time reversal. If we time reverse equation (3.75), we get a stochastic differential equation with a random starting point u_t. However if we write $A_s^{u_0}$ instead of A_s (to note the explicit dependence of A_s on u_0), the Kolmogorov type estimates of [B 1, Chap. I–III] show that

$$E^{Q_{y_0}}\left[\sup_{\substack{0\leq s\leq \frac{t}{2} \\ u_0\in O,\ \varrho u_0\in Z_{y_0}}} |A_s^{u_0}|^q\right], \tag{3.87}$$

is uniformly bounded as $\varepsilon\downarrow\downarrow 0$. The estimates in (3.87) can be obviously time reversed, and so we can uniformly bound $E^{Q_{y_0}}\left[\sup_{\frac{t}{2}\leq s\leq 0} |A_s|^q\right]$. A uniform bound on (3.83) immediately follows.

More generally, as we shall see in more detail in the proof of Theorem 3.16, for $x, x'\in Z_{y_0}$, we can express $P_t'^{\varepsilon}(x,x)$ in the form

$$P_t'^{\varepsilon}(x,x') = q_{\varepsilon t}(y_0,y_0) E^{Q_{y_0}}[C_t(x,\tau_0^t x')^u\tau_0^t \exp z f_1 w_t], \tag{3.88}$$

where C_t is a C^∞ kernel on Z_{y_0}. The kernel $C_t\tau_0^t$ can be constructed by solving a matrix valued stochastic differential equation "subordinated" to $x_s(0\leq s\leq t)$, i.e. calculated over the paths of x. The same estimates as after (3.80) permit us to prove that for $q\geq 1$,

$$\sup_{x_0\in Z_{y_0}} E^{Q_{y_0}}\left[\sup_{x'\in Z_{y_0}} |C_t\tau_0^t(x_0,x')|^q\right] \tag{3.89}$$

is uniformly bounded as $\varepsilon\downarrow\downarrow 0$. Also we have

$$q_{\varepsilon t}(y_0,y_0)\leq \frac{C}{\sqrt{\varepsilon t}}. \tag{3.90}$$

Note the trivial bound for $x\in Z_{y_0}$.

$$|\mathrm{Jac}\,\tau_t^0(x)|\leq c\exp C\left(\sup_{0\leq s\leq t}\sqrt{\varepsilon}\,|w_s|\right).$$

By [IM p. 27], under P, $\sup_{0\leq s\leq t} w_s$ has the same law as $|w_t|$ and so

$$E^P\left[\exp C\sup_{0\leq s\leq t}\sqrt{\varepsilon}|w_s|\right]\leq 2\exp C\varepsilon t. \tag{3.91}$$

By proceeding as after (3.84), we find easily that

$$E^{Q_{y_0}}\left[\exp C\sup_{0\leq s\leq t}\sqrt{\varepsilon}|w_s|\right]\leq c\exp C\varepsilon t. \tag{3.92}$$

Using (3.74), (3.88)–(3.92), we find that for any $\eta>0$,

$$q_{\varepsilon t}(y_0,y_0) E^{Q_{y_0}}\left[\sup_{x\in Z_{y_0}} |C_t(x_0,\tau_0^t x)|\,|\mathrm{Jac}\,\tau_t^0(x_0)|^{1/2} \times 1_{\sup_{0\leq s\leq t}|y_s-y_0|\geq\eta}\right]\leq c\exp -\frac{C}{\varepsilon t}. \tag{3.93}$$

By (3.88), (3.93), we find that as $\varepsilon\downarrow\downarrow 0$, $P_t'^{\varepsilon}(x,x')$ can be adequately evaluated by neglecting the paths y_0 which go to a distance $\geq\eta$ of y_0.

This permits us to trivialize the situation out of a neighborhood of y_0. Namely we can assume that B is replaced by R, that M' is replaced by $R \times Z_{y_0}$, and that if $0 \in R$ is identified with y_0, and if $|y| \geqq \eta$, the fibers Z_y are endowed with a constant metric.

We now will use $dydx^{y_0}$ instead of $dydx^y$ as the base measure on $R \times Z_{y_0}$. This changes the kernel $P_t'^{\varepsilon}$. However $P_t'^{\varepsilon}(x_0, x_0)$ is unchanged.

Step n° 2. The asymptotics of $\operatorname{Tr} P_t'^{\varepsilon}(x_0, x_0)$. The computations which follow will be done in the algebra $\mathscr{E}^{\text{even}}$. In particular zD^ε, $(D^\varepsilon)^2$ and the kernel $P_t'^{\varepsilon}$ should be viewed as elements of $\mathscr{E}^{\text{even}}$. $P_t'^{\varepsilon}(x_0, x)$ is the solution of the partial differential equation

$$\frac{\partial P_t'^{\varepsilon}}{\partial t} = P_t'^{\varepsilon}\left[\frac{(-D^\varepsilon)^2}{2} + zD^\varepsilon\right], \qquad P_0'^{\varepsilon} = \delta_{\{x_0\}} \otimes I. \tag{3.94}$$

For $x \in R \times Z_{y_0}$, with $\pi x = y \in R$, set

$$\bar{P}_t'^{\varepsilon}(x) = P_t'^{\varepsilon}(x) \exp\left(-\frac{zf_1 y}{\sqrt{\varepsilon}}\right) = P_t'^{\varepsilon}(x)\left(1 - \frac{zf_1 y}{\sqrt{\varepsilon}}\right). \tag{3.95}$$

$\bar{P}_t'^{\varepsilon}$ is the solution of the equation

$$\frac{\partial \bar{P}'^{\varepsilon}}{\partial t} = \bar{P}_t'^{\varepsilon} \exp\left(\frac{zf_1 y}{\sqrt{\varepsilon}}\right)\left[\frac{(-D^\varepsilon)^2}{2} + zD^\varepsilon\right] \exp\left(-\frac{zf_1 y}{\sqrt{\varepsilon}}\right), \qquad \bar{P}_0'^{\varepsilon} = \delta_{\{x_0\}} \otimes I \tag{3.96}$$

Clearly zf_1 commutes with $-D^2 + zD$. Also if a is odd in $c(TZ)$, we have

$$zf_1 f_1 a - f_1 a z f_1 = -za + az = -2za.$$

Since $\nabla^u_{f_1} D$ does not act on the variable $y \in R$, we have

$$\exp \frac{zf_1 y}{\sqrt{\varepsilon}}\left[-\frac{\sqrt{\varepsilon} f_1 \nabla^u_{f_1} D}{2}\right] \exp -\frac{zf_1 y}{\sqrt{\varepsilon}} = -\frac{\sqrt{\varepsilon}}{2} f_1 \nabla^u_{f_1} D + zy \nabla^u_{f_1} D. \tag{3.97}$$

Also

$$\exp \frac{zf_1 y}{\sqrt{\varepsilon}}\left(\nabla_{f_1} + \frac{zf_1}{\sqrt{\varepsilon}} + \langle k, f_1 \rangle\right) \exp -\frac{zf_1 y}{\sqrt{\varepsilon}} = \nabla_{f_1} + \langle k, f_1 \rangle. \tag{3.98}$$

Using (3.56), (3.94)–(3.98), we find that $\bar{P}_t'^{\varepsilon}(x)$ is the solution of the equation,

$$\begin{gathered}\frac{\partial \bar{P}_t'^{\varepsilon}}{\partial t} = \bar{P}_t'^{\varepsilon}\left[\frac{\varepsilon}{2}(\nabla_{f_1} + \langle k, f_1 \rangle)^2 - \frac{D^2}{2} + zD - \frac{\sqrt{\varepsilon}}{2} f_1 \nabla^u_{f_1} D + zy \nabla^u_{f_1} D\right], \\ \bar{P}_0'^{\varepsilon} = \delta_{\{x_0\}} \otimes I.\end{gathered} \tag{3.99}$$

Also since $\pi x_0 = 0$, we have

$$\bar{P}_t'^{\varepsilon}(x_0) = P_t'^{\varepsilon}(x_0, x_0). \tag{3.100}$$

We will now transform equation (3.99) according to a procedure due to Getzler [Ge]. Of course, the algebraic situation is much simpler than in [Ge], since we only have one Clifford variable f_1.

We consider $\bar{P}_t'^{\varepsilon}(x)$ as an element of $\mathrm{Hom}((F\otimes\xi)_x, (F\otimes\xi)_{x_0})\hat{\otimes}c(TB)\hat{\otimes}R(z)$. $P_t'^{\varepsilon}(x)$ has a unique decomposition,

$$\bar{P}_t'^{\varepsilon}(x)=Q_t^{\varepsilon}(x)+Q_t'^{\varepsilon}(x)f_1\,, \tag{3.101}$$

where

$$Q_t^{\varepsilon}(x)\,,\qquad Q_t'^{\varepsilon}(x)\in\mathrm{Hom}((F\otimes\xi)_x,(F\otimes\xi)_{x_0})\hat{\otimes}R(z)\,.$$

By (3.34), we have

$$\mathrm{Tr}_z[\bar{P}_t'^{\varepsilon}(x_0)]=-i[\mathrm{Tr}_s Q_t'^{\varepsilon}(x_0)]\,. \tag{3.102}$$

The Grassmann algebra $\Lambda(T^*B)$ is spanned by $1, dy^1$. The operators $dy^1\wedge, i_{f_1}$ both act on $\Lambda(T^*B)$, and also

$$\left(\frac{dy^1}{\sqrt{\varepsilon}}\wedge-\sqrt{\varepsilon}i_{f_1}\right)^2=-1\,. \tag{3.103}$$

In the sequel, we assume that $dy^1\wedge, i_{f_1}$ are odd operators, which anticommute with odd element in $\mathrm{End}\,H^\infty\hat{\otimes}R(z)$. It is then feasible to replace in (3.101) f_1 by $\frac{dy^1}{\sqrt{\varepsilon}}\wedge-\sqrt{\varepsilon}i_{f_1}$. For $(y,x')\in R\times Z_{y_0}$, set

$$\begin{aligned}P_t''^{\varepsilon}(y,x')&=\sqrt{\varepsilon}\left[Q_t^{\varepsilon}(\sqrt{\varepsilon}y,x')+Q_t'^{\varepsilon}(\sqrt{\varepsilon}y,x')\left(\frac{dy^1}{\sqrt{\varepsilon}}\wedge-\sqrt{\varepsilon}\,i_{f_1}\right)\right]\\&=\sqrt{\varepsilon}Q_t^{\varepsilon}(\sqrt{\varepsilon}y,x')+Q_t'^{\varepsilon}(\sqrt{\varepsilon}y,x')\,(dy^1\wedge-\varepsilon i_{f_1})\,.\end{aligned} \tag{3.104}$$

In the coordinates (y,x'), the operator ∇_{f_1} can be written in the form

$$\nabla_{f_1}=\frac{\partial}{\partial y}+\Gamma(y,x')\,,$$

where Γ is a smooth matrix.

Let $\mathscr{L}^{\varepsilon}$ be the differential operator

$$\begin{aligned}\mathscr{L}^{\varepsilon}=\tfrac{1}{2}\left(\frac{\partial}{\partial y}+\sqrt{\varepsilon}\Gamma(\sqrt{\varepsilon}y,x')+\sqrt{\varepsilon}\langle k,f_1\rangle\langle(\sqrt{\varepsilon}y,x')\right)^2-\frac{D^2}{2}(\sqrt{\varepsilon}y,x')\\+zD(\sqrt{\varepsilon}y,x')-\tfrac{1}{2}(dy^1\wedge-\sqrt{\varepsilon}i_{f_1})\nabla_{f_1}^u D(\sqrt{\varepsilon}y,x')+z\sqrt{\varepsilon}y\nabla_{f_1}^u D(\sqrt{\varepsilon}y,x')\,.\end{aligned}$$

$P_t''^{\varepsilon}$ is the solution of the equation

$$\frac{\partial P_t''^{\varepsilon}}{\partial t}=P_t''^{\varepsilon}\mathscr{L}^{\varepsilon};\qquad P_0''^{\varepsilon}=\delta_{\{x_0\}}\otimes I\,. \tag{3.105}$$

Set

$$\mathrm{Tr}_z'[P_t''^{\varepsilon}(0,x_0)]=-i\,\mathrm{Tr}_s[Q_t'^{\varepsilon}(0,x_0)] \tag{3.106}$$

By (3.106), we see that $\mathrm{Tr}_z'[P_t''^{\varepsilon}(0,x_0)]$ is calculated by selecting the term which is a factor of $dy^1\wedge$ in (3.104).

Let $\mathscr{L}$ be the differential operator

$$\mathscr{L}=\frac{1}{2}\frac{\partial^2}{\partial y^2}-\frac{D^2}{2}(0,x')+zD(0,x')-\frac{1}{2}dy^1\wedge\nabla_{f_1}^u D(0,x')\,. \tag{3.107}$$

Clearly as $\varepsilon\downarrow\downarrow 0$, $\mathscr{L}^\varepsilon$ converges to $\mathscr{L}$ in the sense that the smooth coefficients of $\mathscr{L}^\varepsilon$ converge to the coefficients of $\mathscr{L}$ as well as their derivatives, uniformly on the compact subsets of $R\times Z_{y_0}$. Let P''_t be the solution of the equation

$$\frac{\partial P''_t}{\partial t}=P''_t\mathscr{L}; \qquad P''_0=\delta_{\{x_0\}}\otimes I. \tag{3.108}$$

P''_t is trivially given by

$$P''_t(y,x')=\frac{e^{-\frac{|y|^2}{2t}}}{\sqrt{2\pi t}}R'_t(x_0,x'). \tag{3.109}$$

In particular, using (3.61) and (3.109), we find that

$$\mathrm{Tr}'_z[P''_t(0,x_0)]=\frac{-itz}{\sqrt{2\pi t}}\langle[\mathrm{Tr}_s R'^1_t(x_0,x_0)]^{(1)},f_1\rangle.$$

Equivalently using (3.62), we find that

$$\mathrm{Tr}'_z[P''_t(0,x_0)]=-\frac{itz}{\sqrt{2\pi}}\langle[\mathrm{Tr}_s R^1_t(x_0,x_0)]^{(1)},f_1\rangle. \tag{3.110}$$

We claim that for any γ such that $0<\gamma<T$.

$$P''^\varepsilon_t(0,x_0)\to P''_t(0,x_0) \quad \text{uniformly on} \quad [\gamma,T]\times Z_{y_0}. \tag{3.111}$$

The proof of (3.111) can be done using the convergence of $\mathscr{L}^\varepsilon$ to $\mathscr{L}$ and two sorts of arguments.

- One can use the Malliavin calculus as suggested in [B 5, Sect. 5], using a probabilistic representation of P''^ε_t similar to (3.88). We can then directly prove that P''^ε_t and its derivatives remain bounded for $t\geqq\gamma$ on compact sets, and then obtain (3.111).
- Another possibility is to use Duhamel's formula as in Getzler [Ge] in combination with adequate estimates on the vertical part of the kernel.

From (3.100), (3.102), (3.106), (3.110), (3.111), we find that as $\varepsilon\downarrow\downarrow 0$,

$$\mathrm{Tr}[P^{\varepsilon,1}_t(x_0,x_0)]\to\frac{-i}{\sqrt{2\pi}}\langle[\mathrm{Tr}_s R^1_t(x_0,x_0)]^{(1)},f_1\rangle \tag{3.112}$$

uniformly on $[\gamma,T]\times Z_{y_0}$.

Also by Greiner [Gr, Theorem 1.6.1], for $\varepsilon>0$, we have the asymptotic expansion as $t\downarrow\downarrow 0$

$$\mathrm{Tr}'_z[P''^\varepsilon_t(0,x_0)]=\frac{K^\varepsilon_{-\frac{n+1}{2}}(x_0)}{t^{\frac{n+1}{2}}}+\ldots+K^\varepsilon_{1/2}(x_0)t^{1/2}+K^\varepsilon_{3/2}(x_0)t^{3/2}+O(t^{5/2},x_0), \tag{3.113}$$

where the $K^\varepsilon_j\left(-\frac{n+1}{2}\leqq j\leqq 3/2\right)$ are bounded smooth functions on Z_{y_0} and $O(t^{5/2},x_0)$ is uniform on Z_{y_0}. Also since $\mathscr{L}^\varepsilon\to\mathscr{L}$ while staying uniformly elliptic, $O(t^{5/2},x_0)$ is also uniform in $\varepsilon>0$, and the K^ε_j are uniformly bounded as $\varepsilon\downarrow\downarrow 0$.

By (2.13), (3.59), (3.100), (3.102), (3.106) we know that

$$K^{\varepsilon}_{-\frac{n+1}{2}} = \ldots = K^{\varepsilon}_{1/2} = 0\,, \tag{3.114}$$

and so

$$\mathrm{Tr}[P_t^{\varepsilon,1}(x_0, x_0)] = K^{\varepsilon}_{3/2}(x_0)t^{1/2} + O(t^{3/2}, x_0)\,, \tag{3.115}$$

$\mathrm{Tr}[P_t^{\varepsilon,1}(x_0, x_0)]$ then converges to 0 as $t \downarrow\downarrow 0$, uniformly in x_0 and ε. The theorem is proved. □

Remark 5. Incidentally, it should be pointed out that when proving Theorem 3.12, we have proved again Theorem 3.4, by simply using (3.113), (3.114) and the continuous dependence of the $K_j'^{\varepsilon}$ on ε [Gr, Se].

Using Theorem 3.12, we now prove a first fundamental result.

Theorem 3.13. *For any $T > 0$, as $\varepsilon \downarrow\downarrow 0$*

$$\frac{1}{\sqrt{\pi}} \int_0^T \frac{1}{\sqrt{t}} \mathrm{Tr}[D^{\varepsilon} \exp - t(D^{\varepsilon})^2]\, dt \to \frac{i}{2\pi} \int_c \int_0^T \mathrm{Tr}_s[\exp(-tD^2) \tilde{\nabla}^u DD]\, dt\,. \tag{3.116}$$

Proof. By Theorem 3.12, we know that

$$\frac{1}{\sqrt{\pi}} \int_0^T \frac{dt}{\sqrt{t}} \int_{M'} \mathrm{Tr}[P_{2t}^{\varepsilon,1}(x, x)]\, dx \to -\frac{i}{\sqrt{2\pi}} \int_c \int_0^T \frac{dt}{\sqrt{t}} \int_Z \mathrm{Tr}_s[R_{2t}^1(x, x)]^{(1)} dx\,.$$

Using (3.19), we find that (3.116) holds. □

Remark 6. The proof of Theorem 3.12 also shows that if D is instead a general family of first order differential elliptic operators of the type considered in [BF 1] and in Sect. 1f), B_0 which was defined in (1.31) is not only closed, but is also exact. To see this, note that if D^{ε}, η^{ε}, ... are still defined as before, by Atiyah-Patodi-Singer [APS 3], $\eta^{\varepsilon}(s)$ is holomorphic at $s = 0$. This shows that if for $\varepsilon > 0$, we have the expansion

$$\mathrm{Tr}[P_t^{\varepsilon,1}(x_0, x_0)] = \ldots + \frac{K'^{\varepsilon}_{-1/2}}{\sqrt{t}}(x_0) + K'^{\varepsilon}_{1/2}(x_0)\sqrt{t} + \ldots\,, \tag{3.117}$$

then

$$\int_{M'} K'^{\varepsilon}_{-1/2}(x_0, x_0)\, dx_0 = 0\,. \tag{3.118}$$

Now from the fact that $\mathscr{L}^{\varepsilon} \to \mathscr{L}$ and that from Seeley [Se], Greiner [Gr], the coefficients which appear in the small time asymptotics are smooth functions of the local symbol of the considered operators, it is not difficult to find that

$$\int_{M'} K'^{\varepsilon}_{-1/2}(x_0, x_0)\, dx_0 \to -\frac{i}{2\sqrt{\pi}} \int_c B_0\,. \tag{3.119}$$

So we find from (3.118), (3.119) that

$$\int_c B_0 = 0\,, \tag{3.120}$$

and so B_0 is exact. This is a satisfactory result since in full generality, the scaling anomaly described in Sect. 1, Remark 2 is simply equivalent to a gauge transformation.

h) Control of the Integrand of the Êta Invariant as $t\uparrow\uparrow+\infty$

The right-hand side of (3.116) is obviously related to the differential forms δ_0^a defined in Sect. 1f). However we must be able to make $T=+\infty$ in Theorem 3.13.

We prove that this is possible under a special assumption on D.

Theorem 3.14. *Assume that the family of operators D_+ has index 0, and that for every $y\in c$, $\operatorname{Ker} D_y=\{0\}$. Then for $\varepsilon>0$ small enough, $h^\varepsilon=0$. There exists C and $\mu>0$ such that for $\varepsilon>0$ small enough and any $t\geqq 1$*

$$|\operatorname{Tr} D^\varepsilon \exp -t(D^\varepsilon)^2|\leqq C\exp -\mu t\,. \tag{3.121}$$

Proof. Let us first point out that (3.121) is not obvious, since we must take into account the convergence result of Theorem 3.12 – otherwise (3.121) would explode as $\varepsilon\downarrow\downarrow 0$ – while noting that the estimates of Theorem 3.12 are not uniform in T. The idea is to use again the probabilistic construction in the proof of Theorem 3.12 in order to control a time dependent parabolic equation along the fiber whose coefficients are random functions of the Brownian motion $y_.$ on B.

It is then possible to obtain a pointwise exponential decay of the solution in the space of bounded operators on the Hilbert space $H_{y_0}^0$ of L_2 sections of $F\otimes\xi$ over Z_{y_0}. The decay of the corresponding trace is obtained by a method very similar to what is done for deterministic elliptic partial differential equations.

a) $h^\varepsilon=0$ *for ε small enough.* Recall that $d'x$ is the volume element in M'. $f_1\nabla^u_{f_1}$ is clearly a self-adjoint operator. Using Proposition 3.9 with $z=0$, we have for $h\in H^\infty$,

$$\begin{aligned}\int_{M'} |D^\varepsilon h|^2 d'x &= \int_{M'}\left(|Dh|^2 d'x+\varepsilon\int_{M'} |f_1\nabla^u_{f_1}h|^2\right)d'x+\sqrt{\varepsilon}\int_{M'}\langle (f_1\nabla^u_{f_1}D)h,h\rangle d'x\\ &\geqq \int_{M'}|Dh|^2 d'x+\sqrt{\varepsilon}\int_{M'}\langle (f_1\nabla^u_{f_1}D)h,h\rangle d'x\,.\end{aligned} \tag{3.122}$$

$\nabla^u_{f_1}D$ is a first order differential operator which acts fiberwise. If $\|\ \|_y^1$ is a norm in the Sobolev space of order 1 of sections of $F\otimes\xi$ over Z_y, we have

$$\left|\int_{Z_y}\langle (f_1\nabla^u_{f_1}D)h,h\rangle dx\right|\leqq C(\|h\|^1_{Z_y})^2\,. \tag{3.123}$$

Since for every $y\in c$, D_y is invertible, there is a constant $C'>0$ such that for any $y\in c$,

$$\int_{Z_y}|Dh|^2 dx\geqq C'(\|h\|^1_{Z_y})^2\,. \tag{3.124}$$

So if $\sqrt{\varepsilon}\leqq C'/2C$,

$$\int_{M'}|D^\varepsilon h|^2 d'x\geqq (C'-C\sqrt{\varepsilon})\int_{M'}|h(x)|^2 d'x\geqq \frac{C'}{2}\int_{M'}|h|^2 d'x\,. \tag{3.125}$$

h^ε is then equal to 0.

b) *The Asymptotics of* $\operatorname{Tr} D^\varepsilon \exp - t(D^\varepsilon)^2$. We now concentrate on the proof that

$$|\operatorname{Tr} \sqrt{\varepsilon} f_1 \nabla^u_{f_1} \exp - t(D^\varepsilon)^2| \leqq C \exp - \mu t \,. \tag{3.126}$$

We use the notations of the proof of Theorem 3.12. $y_0 \in B$, $x_0 \in Z_{y_0}$ are fixed. U_s is still defined by (3.68), (3.70).

Set

$$H_s = U_s \tau_0^s = \exp z f_1 w_s{}^u \tau_0^s \,.$$

If A_y is a family of operators acting on H_y^∞, we note ${}^uA_s$ the operator acting on $H_{y_0}^\infty$

$${}^uA_s = {}^u\tau_0^s A_{y_s} {}^u\tau_s^0 \,.$$

${}^uA_s$ is unitarily equivalent to A_{y_s}. By proceeding as in (3.97), we find that

$$H_s\left(-\frac{D^2}{2} - \frac{\sqrt{\varepsilon}}{2} f_1 \nabla^u_{f_1} D\right) H_s^{-1} = -\frac{{}^uD^2}{2} - \frac{\sqrt{\varepsilon}}{2} f_1 {}^u\nabla^u_{f_1} D + z\sqrt{\varepsilon}\, w_s{}^u(\nabla^u_{f_1} D) \,. \tag{3.127}$$

Consider the partial differential equation

$$\frac{\partial C}{\partial s} = C\left[-\frac{{}^uD^2}{2} - \sqrt{\varepsilon}\frac{f_1 {}^u\nabla^u_{f_1} D}{2} + z\sqrt{\varepsilon}\, w_s{}^u(\nabla^u_{f_1} D)\right], \qquad C_0 = \delta_{\{x_0\}} \otimes I \,. \tag{3.128}$$

In (3.128), the operator C_s acts on $H_{y_0}^\infty$. Note that since y_s is nowhere differentiable as a function of s, the coefficients of (3.128) are continuous in s, but not smooth. However, by using the method of Treves [T, Chap. III, Sect. 1.3], one finds that since $\frac{{}^uD^2}{2}$ is elliptic, (3.128) has a unique solution, and that for $s > 0$, C_s is a regularizing operator or equivalently that C_s is given by a C^∞ kernel (with respect to dx^{y_0}). Also by using Itô's calculus, Proposition 3.9 and (3.127), one immediately verifies that if $h \in H^\infty$,

$$\exp\left\{-\frac{t(D^\varepsilon)^2}{2} + tz\sqrt{\varepsilon} f_1 \nabla^u_{f_1}\right\} h(x_0) = E^Q[C_t \exp(z f_1 w_t)^u \tau_0^t h](x_0) \,. \tag{3.129}$$

We now disintegrate (3.129). Let S_t be the C^∞ lernel on M' associated with the operator $\exp\left\{-t\frac{(D^\varepsilon)^2}{2} + tz\sqrt{\varepsilon} f_1 \nabla^u_{f_1}\right\}$. Using (3.129) and the fact that as proved by the method of [B 2, Theorem 2.14], a smooth disintegration of the right-hand side of (3.129) is possible, we obtain in particular

$$S_t(x_0, x_0) = q_{\varepsilon t}(y_0, y_0) E^{Q_{y_0}}[C_t(x_0, \tau_0^t x_0) \exp(z f_1 w_t) \tau_0^t [\operatorname{Jac} \tau_0^t]^{1/2}(x_0)] \,. \tag{3.130}$$

The right-hand side of (3.130) should be interpreted in the following way:

- τ_0^t is an element of $\operatorname{Hom}((F \otimes \xi)_{x_0}, (F \otimes \xi)_{\tau_0^t x_0})$.
- Under Q_{y_0}, $\tau_0^t x_0 \in Z_{y_0}$. So $C_t(x_0, \tau_0^t x_0)$ is well defined as an element of

$$\operatorname{Hom}((F \otimes \xi)_{\tau_0^t x_0}, (F \otimes \xi)_{x_0}) \hat{\otimes} c_{y_0}(TB) \hat{\otimes} R(z) \,.$$

So $C_t(x_0, \tau_0^t x_0) \tau_0^t$ should be considered as an element of $\operatorname{End}(F \otimes \xi)_{x_0} \hat{\otimes} c_{y_0}(TB) \hat{\otimes} R(z)$.

By proceeding as in (3.9), we have the identity

$$tz\operatorname{Tr}\left[\sqrt{\varepsilon} f_1 \nabla^u_{f_1} \exp -\frac{t(D^\varepsilon)^2}{2}\right] = \operatorname{Tr}_z\left[\exp\left\{-\frac{t(D^\varepsilon)^2}{2} + tz\sqrt{\varepsilon} f_1 \nabla^u_{f_1}\right\}\right]. \quad (3.131)$$

Also

$$\operatorname{Tr}_z\left[\exp\left(-\frac{t(D^\varepsilon)^2}{2} + tz\sqrt{\varepsilon} f_1 \nabla^u_{f_1}\right)\right] = \int_{M'} \operatorname{Tr}_z[S_t(x,x)]\,d'x$$
$$= \int_B dy \int_{Z_y} \operatorname{Tr}_z[S_t(x,x)]\,dx. \quad (3.132)$$

Using (3.130), we find that

$$\int_{Z_{y_0}} \operatorname{Tr}_z[S_t(x_0,x_0)]\,dx_0 = q_{\varepsilon t}(y_0,y_0) E^{Q_{y_0}} \operatorname{Tr}_z[C_t{}^u\tau_0^t \exp(zf_1 w_t)]. \quad (3.133)$$

In (3.133), $\operatorname{Tr}_z[C_t{}^u\tau_0^t \exp zf_1 w_t]$ is the trace in the sense of (3.30) of the trace class $C_t{}^u\tau_0^t \exp(zf_1 w_t) \in \mathscr{E}^{\text{even}}$. Set

$$\|w\| = \sup_{0\le s\le t} |w_s|.$$

In order to estimate (3.133), we do two transformations on Eq. (3.128):

- We replace z by $\dfrac{z}{1+\sqrt{\varepsilon}\,\|w\|}$ so that the coefficient of ${}^u\nabla^u_{f_1}D$ becomes bounded. We note $\bar{C}_s$ the solution of the new Eq. (3.128).
- We also do Getzler's transformation [Ge]. $\bar{C}_s$ has a unique decomposition

$$\bar{C}_s = \bar{C}_s^0 + \bar{C}_s^1 f_1 \qquad \bar{C}_s^0, \bar{C}_s^1 \in \operatorname{End} H_{y_0} \hat{\otimes} R(z). \quad (3.134)$$

As in the proof of Theorem 3.12, we replace f_1 by $\dfrac{dy^1\wedge}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}\, i_{f_1}$. Set

$$C_s' = \bar{C}_s^0 + \bar{C}_s^1\left(\frac{dy^1\wedge}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}\, i_{f_1}\right). \quad (3.135)$$

C_s' is the solution of the equation

$$\frac{\partial C_s'}{\partial s} = C_s'\left[-\frac{{}^uD^2}{2} - \frac{(dy^1\wedge - \varepsilon i_{f_1})}{2}{}^u\nabla^u_{f_1}D + \frac{z\sqrt{\varepsilon} w_s}{1+\sqrt{\varepsilon}\|w\|}{}^u\nabla^u_{f_1}D\right],$$
$$C_0' = \delta_{\{x_0\}} \otimes I. \quad (3.136)$$

Take a, b which are trace class in $\operatorname{End} H^\infty \hat{\otimes} R(z)$. As in (3.106), we now set

$$\operatorname{Tr}_z'[a + b(dy^1 - \varepsilon i_{f_1})] = -i\operatorname{Tr}_s b. \quad (3.137)$$

By adequately scaling formula (3.133), we get

$$\int_{Z_{y_0}} \operatorname{Tr}_z[S_t(x_0,x_0)]\,dx_0 = \sqrt{\varepsilon}\, q_{\varepsilon t}(y_0,y_0) E^{Q_{y_0}}$$
$$\left[(1+\sqrt{\varepsilon}\|w\|)\operatorname{Tr}_z'\left[C_t'{}^u\tau_0^t \exp\left\{z\left(\frac{dy^1\wedge}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} i_{f_1}\right) w_t/(1+\sqrt{\varepsilon}\|w\|)\right\}\right]\right], \quad (3.138)$$

where in the right-hand side of (3.138), Tr'_z selects terms which contain both z and dy^1.

The idea will now be to control adequately Eq. (3.136) defining C'_s. Set

$$\mathcal{M}_s = -\frac{{}^uD_s^2}{2} - \left(\frac{dy^1 \wedge -\varepsilon i_{f_1}}{2}\right)({}^u\nabla^u_{f_1}D)_s + \frac{z\sqrt{\varepsilon}\,w_s}{1+\sqrt{\varepsilon}\|w\|}({}^u\nabla^u_{f_1}D)_s .$$

Also we will assume that $t \geqq 2$.

For $x \in Z_{y_0}$, let C''_s be the solution of

$$\frac{\partial C''_s}{\partial s} = C''_s \mathcal{M}_s; \qquad C''_1 = \delta_{\{x\}} \otimes 1 \qquad s \geqq 1 . \tag{3.139}$$

Clearly

$$C'_t = C'_1 C''_t . \tag{3.140}$$

Set

$$\int_{Z_{y_0}} \mathrm{Tr}_z[S_t(x_0, x_0)]\,dx_0 = z\varphi_t(y_0) . \tag{3.141}$$

We can then write (3.138) in the form

$$z\varphi_t(y_0) = \sqrt{\varepsilon}\,q_{\varepsilon t}(y_0, y_0) E^{Q_{y_0}}\Bigg[(1+\sqrt{\varepsilon}\,\|w\|) \times \mathrm{Tr}'_z\Bigg[C'_1 C''_t{}^u\tau_0^t \exp\left\{ z\left(\frac{dy^1\wedge}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}\, i_{f_1}\right) w_t/(1+\varepsilon\|w\|)\right\}\Bigg]\Bigg]. \tag{3.142}$$

If a is a linear operator acting on the Hilbert space $H^0_{y_0}$, let $\|a\|_{(\infty)}$, denote the norm of a in the set of bounded operators and $\|a\|_{(1)}$ the norm of a in the set of trace class operators.

We can expand C'_1 in the form

$$C'_1 = a_0 + a_1 z + a_2(dy^1 - \varepsilon i_{f_1}) + a_3 z(dy^1 - \varepsilon i_{f_1}) ,$$

where the a_i are C^∞ kernels on Z_{y_0}. Set

$$\|C'_1\|_{(1)} = \sum_0^3 \|a_j\|_{(1)} .$$

We can define $\|C''_t\|_{(\infty)}$ in exactly the same way. Since ${}^u\tau_0^t$ acts unitarily on $H^0_{y_0}$, we have

$$\|{}^u\tau_0^t\|_{(\infty)} = 1 . \tag{3.143}$$

From (3.140)–(3.143), we find that

$$|\varphi_t(y_0)| \leqq \sqrt{\varepsilon}\,q_{\varepsilon t}(y_0, y_0) E^{Q_{y_0}}\left[\|C'_1\|_{(1)} \|C''_t\|_{(\infty)} \left(1 + \sqrt{\varepsilon}\|w\| + \frac{|w_t|}{\sqrt{\varepsilon}}\right)\right]. \tag{3.144}$$

We now estimate the various terms in the right-hand side of (3.144).

- *Estimation of* $\|C'_1\|_{(1)}$

By proceedings as in (3.140), we can write C'_1 in the form

$$C'_1 = C'_{1/2} \bar{C}'_1 . \tag{3.145}$$

where both $C'_{1/2}$, $\bar{C}_1$ have C^∞ kernels. Since $C'_{1/2}$, $\bar{C}'_1$ are Hilbert-Schmidt operators, if $\| \|_{(2)}$ denotes the Hilbert-Schmidt norm, we have

$$\|C'_1\|_{(1)} \leqq \|C'_{1/2}\|_{(2)} \|\bar{C}'_1\|_{(2)} \leqq C\left(\sup_{x,x'\in Z_{y_0}} |C'_{1/2}(x,x')|\right)\left(\sup_{x,x'\in Z_{y_0}} |\bar{C}'_1(x,x')|\right). \tag{3.146}$$

We claim that for any $p \geqq 1$,

$$E^{Q_{y_0}}\left[\sup_{(x,x')\in Z_{y_0}} |C'_{1/2}(x,x')|^p + \sup_{(x,x')\in Z_{y_0}} |\bar{C}'_1(x,x')|^p\right] \tag{3.147}$$

is uniformly bounded as $\varepsilon \downarrow\downarrow 0$. This cannot be seen directly on Eq. (3.136) since its coefficients can be pointwise very large because of ${}^u\tau_0^s$.

If we instead estimate the kernel of $C'_{1/2}{}^u\tau_0^t$, the methods of the Malliavin calculus described in the proof of Theorem 3.12 – and specifically Eqs. (3.80), (3.81), as indicated before (3.83), we can obtain a uniform bound for

$$E^{Q_{y_0}}\left[\sup_{(x,x')\in Z_{y_0}\times Z_{y_{1/2}}} |C'_{1/2}\tau_0^{1/2}(x,x')|^p\right]. \tag{3.148}$$

Note that since $t \geqq 2$, Q_{y_0} and Q are equivalent on $\mathscr{B} = \mathscr{B}(y_s/0 \leqq s \leqq 1)$, and that $\frac{dQ_{y_0}}{dQ}\mathscr{B} = \frac{q_{\varepsilon(t-1)}(y_1, y_0)}{q_{\varepsilon t}(y_0, y_0)}$ is uniformly bounded as $\varepsilon \downarrow\downarrow 0$ and $t \in [2, +\infty[$, so that in the estimates analogous to (3.83), the problems related to the fact that the stochastic differential equation for y_s is singular at $s = t$ disappear. Also note that with respect to (3.85), we also allow $x \in Z_{y_0}$ to vary. However, the Kolmogorov type estimates of [B 1, Chap. I–III] permit us to include x as a varying parameter. Also we have the trivial bound,

$$\sup_{x\in Z_{y_0}} |\mathrm{Jac}[\tau_0^s](x)| \leqq c \exp\left[C\sqrt{\varepsilon} \sup_{0\leqq h\leqq 1} |w_1|\right]; \quad s \leqq 1. \tag{3.149}$$

The right-hand side of (3.149) is trivially in all the $L_p(Q^{y_0})$. Using Hölder's inequality, we get the required uniform bound on the first term of (3.147). The second term is estimated in the same way.

- *Estimation of* $\|C''_t\|_{(\infty)}$

Let $\bar{H}^\infty_{y_0}$ be the set of linear combinations

$$h = h_0 + zh_1 + dy^1 h_2 + z dy^1 h_3; \quad h_j \in H^\infty_{y_0}, \quad 0 \leqq j \leqq 3.$$

Also for $h \in \bar{H}^\infty_{y_0}$, set

$$|h|^2 = \sum_0^3 |h_j|^2_{H^0_{y_0}}.$$

C''_s acts on $\bar{H}^\infty_{y_0}$ in the obvious way. Let C''^*_s be the adjoint of C''_s. For $h \in \bar{H}^\infty_{y_0}$, set

$$h_s = C''^*_s h.$$

Since ${}^u\tau_0^s$ is unitary and D^2, $\nabla^u_{f_1}D$ are self adjoint, ${}^uD^2$ and ${}^u\nabla^u_{f_1}D$ are self-adjoint. Clearly

$$\frac{d}{ds}|h_s|^2 = 2\,\mathrm{Re}\left\langle\left(-\frac{{}^uD^2}{2} - \frac{{}^u\nabla^u_{f_1}D}{2}(dy^1 \wedge -\varepsilon i_{f_1})^* + \frac{\sqrt{\varepsilon}w_s}{1+\sqrt{\varepsilon}\|w\|}{}^u\nabla^u_{f_1}Dz^*\right)h_s, h_s\right\rangle. \tag{3.150}$$

Since ${}^u\tau_0^s$ is unitary, using (3.124), we have

$$\int_{Z_{y_0}} \langle {}^uD_s^2 h, h\rangle dx = \int_{Z_{y_s}} \langle D_{y_s}{}^u\tau_s^0 h, D_{y_s}{}^u\tau_s^0 h\rangle dx \geqq C' \int_{Z_{y_s}} |{}^u\tau_s^0 h|^2 dx = C' \int_{Z_{y_0}} |h|^2 dx. \tag{3.151}$$

Moreover by (3.123), (3.124), we also have

$$\left|\int_{Z_{y_0}} \left\langle \left[\frac{{}^u\nabla_{f_1}^u D}{2}(dy^1 - \varepsilon i_{f_1})^* + \frac{\sqrt{\varepsilon} w_s}{1+\sqrt{\varepsilon}\|w\|}({}^u\nabla_{f_1} D) z^*\right] h, h\right\rangle dx\right| \leqq C'' \int_{Z_{y_0}} \langle {}^uD^2 h, h\rangle dx. \tag{3.152}$$

Replacing D by kD ($k>0$), we can always assume that $C'' \leqq \frac{C'}{2}$. So using (3.150)–(3.152), we get

$$\frac{d}{ds}|h_s|^2 \leqq -\frac{C'}{2}|h_s|^2. \tag{3.153}$$

By Gronwall's lemma, we find

$$|h_s|^2 \leqq e^{-\frac{C'}{2}(s-1)}|h|^2, \quad s \geqq 1, \tag{3.154}$$

and so

$$\|C_s''^*\|_{(\infty)} \leqq C e^{-\frac{C's}{4}},$$

or equivalently

$$\|C_s''\|_{(\infty)} \leqq C e^{-\frac{C's}{4}} \tag{3.155}$$

Using (3.144), (3.147), (3.155) and Schwarz's inequality, we finally obtain

$$|\varphi_t(y_0)| \leqq \sqrt{\varepsilon}\, q_{\varepsilon t}(y_0, y_0)\left[E^{Q_{y_0}}\left(1 + \varepsilon\|w\|^2 + \frac{|w_t|^2}{\varepsilon}\right)\right]^{1/2} \exp\left(-\frac{C't}{4}\right). \tag{3.156}$$

• *Some Estimates on Brownian Motion*

Let β be a standard Brownian motion in R, with $\beta_0 = 0$. Under Q_{y_0}, and conditionally on $w_t = \frac{k}{\sqrt{\varepsilon}}$, by [Si, p. 41], w_s has the same law as $\beta_s - \frac{s}{t}\beta_t + \frac{s}{t}\frac{k}{\sqrt{\varepsilon}}$. Using (3.156), we find that

$$\begin{aligned} |\varphi_t(y_0)|^2 &\leqq (\sqrt{\varepsilon} q_{\varepsilon t}(y_0, y_0))^2 E^{Q_{y_0}}\left[1 + \varepsilon\|w\|^2 + \frac{|w_t|^2}{\varepsilon}\right] \exp\left(-\frac{C't}{2}\right) \\ &\leqq C\sqrt{\varepsilon} q_{\varepsilon t}(y_0, y_0) \sum_k E^Q\left[1 + k^2 + \varepsilon\|\beta\|^2 + \frac{k^2}{\varepsilon^2}\right] \exp - \left(\frac{C't}{2} + \frac{k^2}{2\varepsilon t}\right). \end{aligned} \tag{3.157}$$

Using Poisson's formula, we have

$$\sqrt{\varepsilon}q_{\varepsilon t}(y_0, y_0) = \frac{\sqrt{\varepsilon}}{\sqrt{2\pi\varepsilon t}} \sum \exp\left(-\frac{k^2}{2\varepsilon t}\right) = \sqrt{\varepsilon} \sum \exp(-2\pi^2 \varepsilon t k^2). \qquad (3.158)$$

It follows from (3.158) that for $t \geqq 2$, $\sqrt{\varepsilon}q_{\varepsilon t}(y_0, y_0)$ is uniformly bounded as $\varepsilon \downarrow\downarrow 0$. Also by scaling β, we find that $E^Q\|\beta\|^2 = ct$. We then find that

$$|\varphi_t(y_0)|^2 \leqq c(1+t) \exp\left(-\frac{C't}{2}\right) \sum_k \exp - \frac{k^2}{2\varepsilon t} + c \exp\left(-\frac{C't}{2}\right) \sum_k \frac{k^2}{\varepsilon^2} \exp - \frac{k^2}{2\varepsilon t}. \qquad (3.159)$$

By (3.158), the first sum in (3.159) grows at most like $t^{1/2}$.

- If $\varepsilon t \leqq 1/2$, the function $x^2 \exp - \frac{x^2}{2\varepsilon t}$ is decreasing on $[1, +\infty[$. Then

$$\begin{aligned} \frac{1}{\varepsilon^2} \sum k^2 \exp\left(-\frac{k^2}{2\varepsilon t}\right) &\leqq \frac{2}{\varepsilon^2} \int_1^{+\infty} x^2 \exp\left(-\frac{x^2}{2\varepsilon t}\right) dx \\ &= \frac{2t^{3/2}}{\varepsilon^{1/2}} \int_{(\varepsilon t)^{-1/2}} \exp\left(-\frac{y^2}{2}\right) y^2 dy \\ &\leqq \frac{ct^{3/2}}{\varepsilon^{1/2}} \int_{(\varepsilon t)^{-1/2}} \exp\left(-\frac{y^2}{4}\right) y dy \\ &= \frac{2ct^{3/2}}{\varepsilon^{1/2}} \exp\left(-\frac{1}{4\varepsilon t}\right) \leqq C' \varepsilon^{1/2} t^{5/2}. \end{aligned} \qquad (3.160)$$

- If $\varepsilon t \geqq 1/2$, since $k^2 \leqq \exp 2|k|$,

$$\begin{aligned} \frac{1}{\varepsilon^2} \sum_k k^2 \exp\left(-\frac{k^2}{2\varepsilon t}\right) &\leqq ct^2 \sum_k \exp\left(2k - \frac{k^2}{2\varepsilon t}\right) \\ &= ct^2 \sum_k \exp\left(-\frac{1}{2\varepsilon t}(k - 2\varepsilon t)^2\right) \exp(2\varepsilon t). \end{aligned} \qquad (3.161)$$

Using Poisson's summation formula, we find

$$\begin{aligned} \frac{1}{\varepsilon^2} \sum_k k^2 \exp - \frac{k^2}{2\varepsilon t} &\leqq ct^2 \exp(2\varepsilon t) \sqrt{\varepsilon t} \sum \exp(-2\pi^2 \varepsilon t k^2) \\ &\leqq ct^{5/2} \exp(2\varepsilon t) \sum_k \exp(-\pi^2 k^2). \end{aligned} \qquad (3.162)$$

From (3.159)–(3.162) we find that

$$|\varphi_t(y_0)|^2 \leqq c(1 + t^{5/2}) \exp\left(2\varepsilon - \frac{C'}{2}\right) t. \qquad (3.163)$$

Using (3.131), (3.132), (3.126) is proved.

The fact that

$$|\mathrm{Tr} D \exp - t(D^\varepsilon)^2| \leqq C \exp - \mu t,$$

can be proved along the same lines. This is left to the reader. □

i) The Holonomy Theorem

We still assume that $B=c$.

Definition 3.15. For $0 \leqq s \leqq 1$, τ_s^0 is the parallel transport operator from λ_{c_0} into λ_{c_s} for the connection ${}^1\nabla$.

τ_1^0 is a complex number τ such that $|\tau|=1$. It does not depend on the origin c_0. Also $\operatorname{Ind} D_+$ will denote the constant integer which is the index of $D_{+,y}$.

We now prove the holonomy theorem.

Theorem 3.16. *The following identity holds*

$$\tau=(-1)^{\operatorname{Ind} D_+} \exp\{-2i\pi[\bar{\eta}]\}. \tag{3.164}$$

Proof. The proof is divided into two steps. In the first step, we suppose the assumptions of Theorem 3.14 are verified. The proof is then a straightforward application of Theorems 3.13 and 3.14.

The second step of the proof is to show that the family D can be continuously modified into a family of pseudodifferential operators D' which verifies the assumptions of Theorem 3.14. Rather unhappily, $\sqrt{\varepsilon} f_1 \nabla_{f_1}^u + D'$ is no longer a pseudodifferential operator on M'. However the probabilistic constructions of the previous sections still apply to the family D'.

We then prove that neither τ nor $[\bar{\eta}]$ change under the continuous deformation of D into D'. The holonomy theorem holding for D' also holds for D.

Step n° 1. We first prove the theorem under the assumptions of Theorem 3.14. In this case, by Theorems 3.13 and 3.14, we have

$$\frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{1}{\sqrt{t}} \operatorname{Tr}[D^\varepsilon \exp - t(D^\varepsilon)^2]\, dt \to \frac{i}{2\pi} \int_c \int_0^{+\infty} \operatorname{Tr}_s[\exp(-tD^2)\tilde{\nabla}^u DD]\, dt. \tag{3.165}$$

The left-hand side of (3.165) is exactly $\eta^\varepsilon(0)$.

Also by Theorem 3.14, for ε small enough, $h^\varepsilon=0$, and so as $\varepsilon \downarrow\downarrow 0$

$$\bar{\eta}^\varepsilon(0) \to \frac{i}{4\pi} \int_c \int_0^{+\infty} \operatorname{Tr}_s[\exp(-tD^2)\tilde{\nabla}^u DD]\, dt. \tag{3.166}$$

Moreover there is $a>0$ such that for any $y \in c$, D_y has no eigenvalue in $[0,a]$. With the notations of Definition 1.8, over c, $K_y^a=\{0\}$ and so $\lambda^a=\mathbb{C}$. Over c, λ has a canonical section σ which is identified with $1 \in \lambda^a$. Clearly, in the sense of Definition 1.12, ${}^0\nabla_{f_1}\sigma=0$, and so, since by Theorem 3.4 $B_0=0$, we find that

$${}^1\nabla_{f_1}\sigma = \langle \tfrac{1}{2}(\gamma_0^a - \delta_0^a) + \tfrac{1}{2}(\Gamma'(1)-\mu)\, dA_0, f_1 \rangle \sigma. \tag{3.167}$$

Also by Proposition 1.11, γ_0^a is exact on c. We then find that

$$\int_c \frac{{}^1\nabla.\sigma}{\sigma} = -1/2 \int_c \delta_0^a. \tag{3.168}$$

Also since on c $K^a=\{0\}$, it is clear that on c

$$\delta_0^a = \int_0^{+\infty} \operatorname{Tr}_s[\exp(-tD^2)\tilde{\nabla}^u DD]\, dt. \tag{3.169}$$

Finally note the straightforward relation

$$\tau = \exp\left\{-\int_c \frac{{}^1\nabla\sigma}{\sigma}\right\}. \tag{3.170}$$

Using (3.166)–(3.170) and the fact that $\operatorname{Ind} D_+ = 0$, we find that (3.164) holds.

Step n° 2. The General Case

a) *Construction of a Family of Index* 0. We here use the notations of [B 5, Sect. 2], but the roles of D_+ and D_- are interchanged. By [AS 3, Proposition 2.2], we know that $q \in N$ and C^∞ sections $s_1, \ldots, s_q$ of $F_+ \otimes \xi$ over M exist such that if $\delta \in R$, if $D'_{-y,\delta}$ is the operator

$$(h, \lambda) \in H^\infty_{-,y} \oplus \mathbb{C}^q \to D'_{-y,\delta}(h, \lambda) = D_{-,y} h + \delta \sum_1^q \lambda_i s_i \in H^\infty_{+,y}, \tag{3.171}$$

then if $\delta \neq 0$, $D'_{-y,\delta}$ is onto.

We endow $\mathbb{C}^q$ with its canonical Hermitian product. The formal adjoint $D'_{+y,\delta}$ of $D'_{-y,\delta}$ is the operator

$$h \in H^\infty_{+,y} \to D'_{+y,\delta} h = (D_{+,y} h, \delta\langle h, s_1\rangle, \ldots, \delta\langle h, s_q\rangle) \in H^\infty_{-,y} \oplus \mathbb{C}^q. \tag{3.172}$$

For $\delta \neq 0$, $\operatorname{Ker} D'_+ = \{0\}$, and $\operatorname{Ker} D'_-$ is a C^∞ bundle over $B \times R/\{0\}$. Also

$$D'_{+y,0} h = (D_{+,y} h, 0, \ldots, 0), \qquad D'_{-y,0} h = D_{-,y} h,$$

and so

$$\operatorname{Ker} D'_{+y,0} = \operatorname{Ker} D_{+,y}, \qquad \operatorname{Ker} D'_{-y,0} = \operatorname{Ker} D_{-,y} \oplus \mathbb{C}^q,$$

Set

$$D' = \begin{bmatrix} 0 & D'_- \\ D'_+ & 0 \end{bmatrix}. \tag{3.173}$$

We can then define the determinant bundle λ' of the family D', which is a line bundle on $B \times R$. Clearly

$$\lambda' = \lambda \quad \text{on} \quad B \times \{0\}. \tag{3.174}$$

Also if $q' = -\operatorname{Ind} D'_+$, then

$$q' = -\operatorname{Ind} D_+ + q, \tag{3.175}$$

and also $q' \geqq 0$.

If $q' > 0$, we allow D'_+ to act on $H^\infty_+ \oplus \mathbb{C}^{q'}$ by the formula

$$(h, \mu) \in H^\infty_+ \oplus \mathbb{C}^{q'} \to D'_+(h, \mu) = D'_+ h \in H^\infty_- \oplus \mathbb{C}^q. \tag{3.176}$$

We endow $\mathbb{C}^{q'}$ with its canonical Hermitian product.

The adjoint D'_- of D'_+ is given by

$$(h, \lambda) \in H^\infty_- \oplus \mathbb{C}^q \to D'_-(h, \lambda) = (D'_-(h, \lambda), 0) \in H^\infty_+ \oplus \mathbb{C}^{q'}. \tag{3.177}$$

Now $\operatorname{Ind} D'_+ = 0$. If λ' is the determinant bundle of D', we still have

$$\lambda' = \lambda \quad \text{on} \quad B \times \{0\}. \tag{3.178}$$

Also for $\delta \neq 0$, $\operatorname{Ker} D'_+ = \mathbb{C}^{q'}$ and $\operatorname{Ker} D'_-$ is a C^∞ bundle.

b) Construction of a Family of Invertible Operators. The parameter space of the family D' is $B \times R$. On $B \times R/\{0\}$, $\operatorname{Ker} D'_+ = \mathbb{C}^{q'}$ and $\operatorname{Ker} D'_-$ is a smooth subbundle of $H^\infty_- \oplus \mathbb{C}^q$ of dimension q'.

Complex bundles over S^1 are trivial. We can then find a smooth trivialization E_+ of $\operatorname{Ker} D'_{-y,1}$ over the loop $B = c$.

For $y \in B$, $E_{+,y}$ is a linear isomorphism from $\mathbb{C}^{q'}$ in $\operatorname{Ker} D'_{-y,1}$. We allow E_+ to act on H^∞_+ by setting $E_+ = 0$ on H^∞_+. E_+ then acts linearly on $H^\infty_+ \oplus \mathbb{C}^{q'}$.

If E_- is the adjoint of E_+, E_- sends $H^\infty_- \oplus \mathbb{C}^q$ into $\mathbb{C}^{q'}$ and is 0 on the orthogonal of $\operatorname{Ker} D'_{-y,1}$ in $H^0_- \oplus \mathbb{C}^q$.

For $(y, \theta) \in B \times R$, set

$$D''_{+y,\theta} = D'_{+y,1} + \theta E_{+,y}; \qquad D''_{-y,\theta} = D'_{-y,1} + \theta E_{-,y},$$
$$D''_{y,\theta} = \begin{bmatrix} 0 & D''_{-y,\theta} \\ D''_{+y,\theta} & 0 \end{bmatrix}. \tag{3.179}$$

Let λ'' be the determinant bundle of the family D''. Clearly

$$D''_{y,0} = D'_{y,1},$$

and so

$$\lambda''_{y,0} = \lambda'_{y,1}. \tag{3.180}$$

c) Extension of the Results of Sects. 1, 2, and 3 to the Families D' and D''. We will show how to extend the results of Sects. 1, 2, and the previous results of Sect. 3 to the family D'. The same arguments hold for the family D''.

We endow $\mathbb{C}^q$ and $\mathbb{C}^{q'}$ with the trivial connections. So $H^\infty_+ \oplus \mathbb{C}^{q'}$ and $H^\infty_- \oplus \mathbb{C}^q$, considered as bundles over $B \times R$, are naturally endowed with a unitary connection.

Let A be a family of linear operators sending $H^\infty_+ \oplus \mathbb{C}^{q'}$ into $H^\infty_- \oplus \mathbb{C}^q$. We write A in matrix form

$$A = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix}.$$

We will say that A is regularizing if A_1 is regularizing in the usual sense [T] and if A_2 is given by a C^∞ distribution. Since A_3 sends $C^{q'}$ in H^∞_-, A_3 is given by a family of C^∞ functions along the fiber Z.

The pseudodifferential calculus can be extended to $H^\infty_+ \oplus C^{q'}$ and $H^\infty_- \oplus C^q$, with this new definition of regularizing operators. Set

$$A_{y,\delta} = D'_{y,\delta} - D'_{y,0}. \tag{3.181}$$

Then $A_{y,\delta}$ is a smooth family of regularizing operators over $B \times R$.

We first briefly show how to extend the results of [BF 1] described in Sect. 1f). By using formally Duhamel's formula, we find that

$$\exp(-tD'^2_{y,\delta}) = \exp-(tD'^2_{y,0}) - \int_0^t \exp(-sD'^2_{y,\delta})\,((D'_{y,\delta})^2 - (D'_{y,0})^2)$$
$$\times \exp(-(t-s)D'^2_{y,0}))\,ds. \tag{3.182}$$

Now $\exp - t(D'_{y,0})^2$ can be evaluated in terms of $\exp(-tD_y^2)$ in an obvious way. Also

$$(D'_{y,\delta})^2-(D'_{y,0})^2=A^2_{y,\delta}+D'_{y,0}A_{y,\delta}+A_{y,\delta}D'_{y,0}, \tag{3.183}$$

and so $(D'_{y,\delta})^2-(D_{y,0})^2$ is regularizing.

This permits us to use an iteration procedure in (3.182) to calculate $\exp(-tD'^2_{y,\delta})$. In particular using (3.183), we find that if $P'^{y,\delta}_t$ is the C^∞ kernel associated with $\exp(-tD'^2_{y,\delta})$, then for any $x\in M$,

$$\mathrm{Tr}_s[P'^{y,\delta}_t(x,x)]-\mathrm{Tr}_s[P'^{y,0}_t(x,x)]=O(t,x). \tag{3.184}$$

It is then not difficult to extend the results of [BF 1] which we described in Sect. 1f) to the family D'.

The determinant bundle λ' is endowed with a metric and unitary connection, which of course restricts to the metric and the connection of λ on $B\times\{0\}$.

We claim that the cancellation result of Theorem 3.4 still holds for the family D'. In fact let $Q^{y,\delta}_t$ be the operator

$$\exp t\left\{-\frac{(D'_{y,\delta})^2}{2}-\frac{1}{2}\tilde{V}^u D'_{y,\delta}+zD'_{y,\delta}\right\}.$$

By Duhamel's formula, we have

$$\begin{aligned}Q^{y,\delta}_t-Q^{y,0}_t=&-t\int_0^1 Q^{y,\delta}_{ts}[\tfrac{1}{2}((D'_{y,\delta})^2-(D'_{y,0})^2)+\tfrac{1}{2}(\tilde{V}^u D'_{y,\delta}-\tilde{V}^u D'_{y,0})\\&-z(D'_{y,\delta}-D'_{y,0})]\,Q^{y,0}_{t(1-s)}ds.\end{aligned} \tag{3.185}$$

As $t\downarrow\downarrow 0$, we find that $(Q^{y,\delta}_t-Q^{y,0}_t)/t$ converges to a regularizing operator, which is of course trace class. In Theorem 3.4, the left-hand side of (3.12) is a quantity where the factors zdy^1 should appear. By iterating (3.185), we find that zdy^1 appears in (3.185) with the factor t^2 and before a regularizing operator. This is just what we need to guarantee that Theorem 3.4 still holds for the family D'.

As indicated in (3.27), we now assume that $\varphi(f_1)$ acts like $-i$ on $H^\infty_+\oplus C^{q'}$, like $+i$ on $H^\infty_-\oplus C^q$.

For $\varepsilon>0$ and over the loop $s\in R/Z\to c'^{\delta}_s=(c_s,\delta)\in B\times R$, we consider the operator

$$D'^{\varepsilon}_\delta=\sqrt{\varepsilon}f_1\nabla^u_{f_1}+D'_{y,\delta}. \tag{3.186}$$

Similarly over the loop $s\in R/Z\to c''^{\theta}_s=(c_s,\theta)\in B\times R$, we consider the operator:

$$D''^{\varepsilon}_\theta=\sqrt{\varepsilon}f_1\nabla^u_{f_1}+D''_{y,\theta}. \tag{3.187}$$

D'^{ε} and D''^{ε} are not pseudodifferential operators, since $A_{y,\delta}$ and E are only fiberwise smooth. A priori such operators do not have êta invariants in the sense of [APS 1, 3].

Still, by using the procedure indicated in the proof of Theorem 3.14, to construct the semi-groups $\exp\{-t(D'^{\varepsilon}_\delta)^2\}$, $\exp\{-t(D''^{\varepsilon}_\delta)^2\}$, we can use a Brownian motion $y.$ on S^1 and integrate a parabolic equation with time depending coefficients in a given fiber, in which the considered operators are truly pseudodifferential operators.

Theorem 3.12 then extends to the families D' and D''. The more difficult point is to obtain the uniform convergence of (3.64) as $t\downarrow\downarrow 0$. However when taking the expansion as $t\downarrow\downarrow 0$ of the trace of the kernel of $D_\delta'^\varepsilon \exp\{-t(D_\delta'^\varepsilon)^2\}$, by proceeding as in (3.185) and using Theorem 3.12 for D^ε, we find that it starts with

$$\frac{1}{\sqrt{4\pi t\varepsilon}} \int_M \operatorname{Tr}[A_\delta(x,x)]d'x + O(\sqrt{t}, x).$$

Since A_δ is odd, $\operatorname{Tr}[A_\delta(x,x)]=0$, and so we get the required uniformity.

It is then not difficult to adapt the proofs of Theorem 3.13 and 3.14. In particular Theorem 3.14 holds for the family $D''_{y,\theta}$ over the curve c''^1.

Since $D_\delta'^\varepsilon$, $D_\theta''^\varepsilon$ are not pseudodifferential operators, we directly define their êta functions by formulas (2.22), (2.23). Let $\eta_\delta'^\varepsilon(s)$, $\eta_\theta''^\varepsilon(s)$ be the corresponding êta functions, which are well defined at $s=0$.

Since for $t>0$, $\exp\{-t(D_\delta'^\varepsilon)^2\}$, $\exp\{-t(D_\theta''^\varepsilon)^2\}$ are regularizing, $\operatorname{Ker} D_\delta'^\varepsilon$, $\operatorname{Ker} D_\theta''^\varepsilon$ are finite dimensional. We can define $\bar\eta_\delta'^\varepsilon(s)$, $\bar\eta_\theta''^\varepsilon(s)$.

Let τ_δ', τ_θ'' be the holonomy of λ', λ'' over the curves c'^δ, c''^θ.

The key step to finish the proof of the theorem is as follows.

Proposition 3.17. $[\bar\eta_\delta'^\varepsilon(0)]$, τ_δ' *(respectively* $[\bar\eta_\theta''^\varepsilon(0)]$, τ_θ''*) do not depend on* δ *(respectively on* θ*).*

Proof. We only prove the Proposition for $[\bar\eta_\delta'^\varepsilon(0)]$, τ_δ'. By Proposition 2.8, $\frac{\partial}{\partial\delta}[\bar\eta_\delta'^\varepsilon(0)]$ is proportional to the finite part as $t\downarrow\downarrow 0$ of

$$\sqrt{t}\operatorname{Tr}\left[\frac{\partial D_\delta'^\varepsilon}{\partial\delta}\exp\left\{-\frac{t(D_\delta'^\varepsilon)^2}{2}\right\}\right].$$

$A=\frac{\partial D_\delta'^\varepsilon}{\partial\delta}$ is a smooth family of odd fiberwise regularizing operators.

We can then use the technique of the proof of Theorem 3.14 to describe the semi-group $\exp-\frac{t(D_\delta'^\varepsilon)^2}{2}$ in conditional form, i.e. by consider first a Brownian motion in c, and by constructing a partial differential equation with random coefficients in the vertical directions. We then find easily that

$$\lim_{t\downarrow\downarrow 0}\sqrt{t}\operatorname{Tr}\left[\frac{\partial D_\delta'^\varepsilon}{\partial\delta}\exp\left\{-\frac{t(D_\delta'^\varepsilon)^2}{2}\right\}\right]=\frac{1}{\sqrt{2\pi\varepsilon}}\int_{M'}\operatorname{Tr}[A(x,x)]dx. \tag{3.188}$$

Since A is odd, $\operatorname{Tr}[A(x,x)]=0$, and so

$$\frac{\partial[\bar\eta_\delta'^\varepsilon(0)]}{\partial\delta}=0. \tag{3.189}$$

$[\bar\eta_\delta'^\varepsilon(0)]$ is then independent of δ and so coïncides with $[\bar\eta_0'^\varepsilon(0)]$.

Let r' be the curvature of λ'. One has the obvious relation

$$\frac{\frac{\partial\tau_\delta'}{\partial\delta}}{\tau_\delta'}=\int_c r'\left(\cdot,\frac{\partial}{\partial\delta}\right). \tag{3.190}$$

Also by Theorem 1.14, r' is the finite part of $[\mathrm{Tr}_s \exp -(\tilde{\nabla}^u + \sqrt{t}D')^2]^{(2)}$. The part of $\mathrm{Tr}_s[\exp -(\tilde{\nabla}^u + \sqrt{t}D')^2]^{(2)}$, which contains the Grassmann variable $d\delta$ is exactly

$$-d\delta \wedge \sqrt{t}\,\mathrm{Tr}_s[A \exp -(\tilde{\nabla}^u + \sqrt{t}D')^2]^{(1)}. \tag{3.191}$$

Since A is trace class, (3.191) obviously converges to 0 as $t \downarrow\downarrow 0$. So by (3.190)

$$\frac{\partial \tau'_\delta}{\partial \delta} = 0.$$

The proposition is proved. □

We now finish the proof of Theorem 3.16. By the first part of the proof applied to the family D'', we know that as $\varepsilon \downarrow\downarrow 0 \frac{\eta_1''^{\varepsilon}(0)}{2}$ has a limit $\bar{\eta}_1''$ and that

$$\tau_1'' = \exp\{-2i\pi[\bar{\eta}_1'']\}.$$

Using Proposition 3.17, it is then clear that as $\varepsilon \downarrow\downarrow 0$, $[\eta_\theta''^{\varepsilon}(0)]$, $[\eta_\delta'^{\varepsilon}(0)]$ have a limit $[\bar{\eta}_1'']$ – which does not depend on θ, δ, and that in particular

$$\tau = \tau'_0 = \exp\{-2i\pi[\bar{\eta}_1'']\}. \tag{3.192}$$

Now since the family $D'_{y,0}$ acts on $(H_+^\infty \oplus C^{q'}) \oplus (H_-^\infty \oplus C^q)$, one finds immediately that for any $\varepsilon \downarrow\downarrow 0$,

$$\dim \operatorname{Ker} D_0'^{\varepsilon} = \dim \operatorname{Ker} D^{\varepsilon} + q' + q,$$

and so

$$\bar{\eta}_0'^{\varepsilon}(0) = \bar{\eta}^{\varepsilon}(0) + 1/2(q' + q). \tag{3.193}$$

We deduce from (3.193) that

$$[\bar{\eta}'_0] = [\bar{\eta}] + [1/2(q - q')]. \tag{3.194}$$

Since $[\bar{\eta}'_0] = [\bar{\eta}_1'']$, (3.164) follows from (3.175) and (3.192). □

j) A Remark on the Metric of B

We now again assume that B is a m dimensional manifold. Let $R^{L,\varepsilon}$ be the curvature tensor of TM for the Levi-Civita connection associated with the metric $\frac{g_B}{\varepsilon} \oplus g_Z$. If R^B is the curvature tensor of TB for the Levi-Civita connection of B, as in (3.52), we can evaluate $R^{L,\varepsilon}$ in terms of R^Z, S, and R^B. More precisely if $f_1, \ldots, f_m$ is an orthonormal base of TB, $e_1, \ldots, e_n$ an orthonormal base of TZ, $R^{L,\varepsilon}$ evaluated on the base $(e_1, \ldots, e_n, \sqrt{\varepsilon} f_1, \ldots, \sqrt{\varepsilon} f_m)$ is given by

$$R^{L,\varepsilon} = \begin{bmatrix} R^Z + \varepsilon P_Z[S,S] & \varepsilon^{1/2} P_Z \mathring{D} S + \varepsilon^{3/2} P_Z[S,S] \\ \varepsilon^{1/2} P_H \mathring{D} S + \varepsilon^{3/2} P_H[S,S] & R^B + \varepsilon P_H \mathring{D} S + \varepsilon P_H S \wedge P_Z S + \varepsilon^2 P_H S \wedge P_H S \end{bmatrix}. \tag{3.195}$$

Using (3.195), it is clear that as $\varepsilon \downarrow\downarrow 0$,

$$\hat{A}\left(\frac{R^{L,\varepsilon}}{2\pi}\right) \to \hat{A}\left(\frac{R^Z}{2\pi}\right) \hat{A}\left(\frac{R^B}{2\pi}\right). \tag{3.196}$$

Since $\hat{A}\left(\frac{R^B}{2\pi}\right)$ only contains forms whose degree is $4q$, we find that as $\varepsilon\downarrow\downarrow 0$,

$$\left[\int_Z \hat{A}\left(\frac{R^{L,\varepsilon}}{2\pi}\right)\operatorname{Tr}\exp-\frac{L}{2i\pi}\right]^{(2)} \to \left[\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right)\operatorname{Tr}\exp-\frac{L}{2i\pi}\right]^{(2)}. \tag{3.197}$$

Now by formula (2.39) the left-hand side of (3.197) is directly related to the variation of $d[\bar{\eta}^\varepsilon(0)]$ when c is made to vary.

In fact by [APS 1], if the metric of $M'=\pi^{-1}(c)$ is product near M', (3.197) appears explicitly when computing $d[\bar{\eta}^\varepsilon(0)]$.

If we were to compute the variation $d[\bar{\eta}^\varepsilon(0)]$ using Theorem 2.10, the proof of Theorem 2.10 being formally identical to the proof of the Index Theorem for families in [B 5], we should blow up the metric of B in directions normal to c.

By making $\varepsilon\downarrow\downarrow 0$, we also blow up to metric of B in the direction tangent to c. Using (3.196), (3.197), we find that if c_s^l is a smooth family of loops in B, then

$$\frac{\partial[\bar{\eta}]}{\partial l} = \int_c i_{\frac{\partial c}{\partial l}}\left[\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right)\operatorname{Tr}\exp-\frac{L}{2i\pi}\right]^{(2)}. \tag{3.198}$$

Also if r is the curvature of ${}^1\tilde{V}$, we have

$$\frac{\partial\tau}{\partial l}\Big/\tau = -\int_c i_{\frac{\partial c}{\partial l}} r.$$

Since $\tau=(-1)^{\operatorname{Ind} D_+}\exp\{-2i\pi[\bar{\eta}]\}$, we find that

$$2i\pi\frac{\partial}{\partial l}[\bar{\eta}] = \int_c i_{\frac{\partial c}{\partial l}} r. \tag{3.199}$$

Of course (3.198) and (3.199) fit whith the formula (1.58) for r.

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