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Nerves, fibers and homotopy groups ☆

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Abstract

Two theorems are proved. One concerns coverings of a simplicial complex Δ by subcomplexes. It is shown that if every t-wise intersection of these subcomplexes is $(k-t+1)$-connected, then for $j \leqslant k$ there are isomorphisms $\pi_j(\Delta) \cong \pi_j(\mathcal{N})$ of homotopy groups of Δ and of the nerve $\mathcal{N}$ of the covering.

The other concerns poset maps $f : P \to Q$. It is shown that if all fibers $f^{-1}(Q_{\leqslant q})$ are k-connected, then f induces isomorphisms of homotopy groups $\pi_j(P) \cong \pi_j(Q)$, for all $j \leqslant k$.

1. Introduction

The Nerve Theorem, usually attributed to Borsuk [3], and the poset Fiber Theorem of Quillen [6] are important and versatile tools in topological combinatorics. For pointers to their many uses we refer to the survey [1]. Both theorems assert the homotopy equivalence of two simplicial complexes under suitable conditions.

In this paper, we sharpen both results to versions (summarized in the abstract and stated more carefully as Theorems 2 and 6) that assert the isomorphism of homotopy groups up to a certain dimension. The "classical" versions are obtained as consequences, and some other related versions (due to Quillen and others) as special cases, see the remarks following the proofs.

Our proofs are elementary, the main ingredients being a homotopy carrier lemma and simplicial approximation. The idea to use such a carrier lemma to prove the original version of the Fiber Theorem is due to Walker [7].

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2. Preliminaries

We use standard notions of topology, see e.g. [4]. In particular, continuous functions are called *maps*, and a space T is said to be k-*connected* ($k \geqslant 0$) if, for every $0 \leqslant r \leqslant k$, every map of the r-sphere into T is homotopic to a constant map (or, equivalently, it can be continuously extended across the interior of the $(r+1)$-ball). Thus, 0-connected means "arcwise connected", and 1-connected means "arcwise connected and simply connected". Extending the definition, we let (-1)-connected mean "nonempty", and agree to consider every space (empty or not) to be k-connected for every $k \leqslant -2$.

Notational distinction between an abstract simplicial complex Δ and its geometric realization $||\Delta||$ will be made only if the meaning is otherwise not clear from context. The k-skeleton of Δ is denoted by $\Delta^{(k)}$.

Let Δ be a simplicial complex and T a space. A function C taking faces σ of Δ to subspaces $C(\sigma)$ of T is a *carrier* if $C(\sigma) \subseteq C(\tau)$ for all $\sigma \subseteq \tau$ in Δ. A map $f : ||\Delta|| \to T$ is *carried* by C if $f(||\sigma||) \subseteq C(\sigma)$ for all $\sigma \in \Delta$.

The following is a connectivity version of [5, Theorem II.9.2, p. 76; 7, Lemma 2.1], see also [1, Lemma 10.1]. Let $k \in \mathbb{N} \cup \{\infty\}$.

Lemma 1 (Carrier Lemma). (i) *Assume that* $C(\sigma)$ *is* $\dim(\sigma)$-*connected for all* $\sigma \in \Delta^{(k)}$. *Then any two maps* $f, g : ||\Delta^{(k)}|| \to T$ *that are both carried by* C *are homotopic*: $f \sim g$.

(ii) *Assume that* $C(\sigma)$ *is* $(\dim(\sigma) - 1)$-*connected for all* $\sigma \in \Delta^{(k)}$. *Then there exists a map* $||\Delta^{(k)}|| \to T$ *carried by* C.

Proof. The proof is quite straightforward from the definitions. For details, see the proof of [5, Theorem II.9.2] or [7, Lemma 2.1]. □

3. Fibers

By *poset map* we mean a map $f : P \to Q$ of posets (partially ordered sets) that is order-preserving ($x \leqslant_P y \Rightarrow f(x) \leqslant_Q f(y)$). With a poset P we associate the (abstract) simplicial complex $\Delta(P)$ (the *order complex*) whose faces are the c hains (totally ordered subsets) of Δ. This way of associating a topological space to a poset is quite common in combinatorics. See [1] for more details, examples and references.

In order to unburden notation we do not always distinguish notationally between P, $\Delta(P)$, and $||\Delta(P)||$, if context makes it clear whether we are referring to a poset, its order complex, or its geometric realization. Without real loss of generality we assume that our posets are *connected* (each pair $x, y \in P$ is linked via a path $x \leqslant z_1 \geqslant z_2 \leqslant \cdots \geqslant z_{n-1} \leqslant z_n \geqslant y$), in order to make homotopy groups $\pi_j(P)$ independent of basepoint.

The following result was stated without proof in [1, p. 1850].

Theorem 2. *Let P and Q be connected posets and $f: P \to Q$ a poset map. Suppose that the fiber $f^{-1}(Q_{\leqslant q})$ is k-connected for all $q \in Q$. Then the induced map on homotopy groups $f_j^*: \pi_j(P) \to \pi_j(Q)$ is an isomorphism for all $j \leqslant k$.*

Proof. For $\sigma \in \Delta^{(k+1)}(Q)$ (i.e., the $(k+1)$-skeleton of $\Delta(Q)$), let $C(\sigma) = f^{-1}(Q_{\leqslant \max \sigma})$. Then C is a k-connected carrier from $\Delta^{(k+1)}(Q)$ to subcomplexes of $\Delta(P)$. Hence, by Lemma 1(ii) there exists a map $g: ||\Delta^{(k+1)}(Q)|| \to ||\Delta(P)||$ carried by C.

The idea now is to show that the induced map on homotopy groups $g_j^*: \pi_j(Q) \to \pi_j(P)$ is an isomorphism for all $j \leqslant k$, and moreover its inverse is f_j^*.

Claim 1. $g_j^*: \pi_j(Q) \to \pi_j(P)$ *is injective for all* $j \leqslant k$.

For $\sigma \in \Delta^{(k+1)}(Q)$ let $D(\sigma) = Q_{\leqslant \max \sigma}$. Then D is a carrier from $\Delta^{(k+1)}(Q)$ to subcomplexes of $\Delta(Q)$, and $D(\sigma)$ is always contractible (being a cone). Let *id* be the identity map on $\Delta^{(k+1)}(Q)$. Now, the two maps *id* and $f \circ g: ||\Delta^{(k+1)}(Q)|| \to ||\Delta(Q)||$ are both carried by D. Hence, by Lemma 1(i) they are homotopic. This implies that $f_j^* \circ g_j^* = (f \circ g)_j^* = id_j^*: \pi_j(\Delta^{(k+1)}(Q)) \to \pi_j(\Delta(Q))$, which is an isomorphism for $j \leqslant k$ (since the homotopy groups in dimensions up to k live on the $(k+1)$-skeleton). Hence, g_j^* is injective.

Claim 2. $g_j^*: \pi_j(Q) \to \pi_j(P)$ *is surjective for all* $j \leqslant k$.

Let $r \in \pi_j(P)$. Via simplicial approximation we can assume that there is a simplicial subdivision Σ of the j-sphere such that $r: ||\Sigma|| \to ||\Delta^{(k)}(P)||$ is a simplicial map (in the original homotopy class). Let $t = f \circ r: ||\Sigma|| \to ||\Delta^{(k)}(Q)||$; so, in particular, $t \in \pi_j(Q)$. We will show that $r = g^*(t)$ by again using Lemma 1(i).

For $\tau \in \Sigma$, let $E(\tau) = f^{-1}(Q_{\leqslant \max t(\tau)})$. Then E is a k-connected carrier from the j-dimensional complex Σ to subcomplexes of $\Delta(P)$. Both maps r and $g \circ t = g \circ f \circ r$ are carried by E. Hence these maps are homotopic: $r \sim g \circ t$, i.e., $r = g^*(t)$. $\square$

Remark 3. If the assumption is strengthened to "*suppose that the fiber $f^{-1}(Q_{\leqslant q})$ is contractible for all $q \in Q$*", then the conclusion "*$f_j^*: \pi_j(P) \to \pi_j(Q)$ is an isomorphism for all $j \in \mathbb{N}$*" implies, via Whitehead's theorem [4, p. 486; 5, p. 125], that f induces homotopy equivalence $P \simeq Q$. This consequence of the theorem is the original version due to Quillen [6, Proposition 1.6].

Remark 4. If the conclusion is weakened to "*then P is k-connected if and only if Q is k-connected*" the theorem specializes to another result of Quillen's [6, Proposition 7.6].

Remark 5. If the assumption is weakened to "*suppose that the fiber $f^{-1}(Q_{\leqslant q})$ is* $\min(k, \dim(Q_{<q}))$*-connected for all $q \in Q$*", then the following weaker conclusion can

be drawn: "$f_j^*: \pi_j(P) \to \pi_j(Q)$ *is surjective for all* $j \leqslant k$". The proof is the same, up to but not including Claim 2.

4. Nerves

A CW complex is said to be *regular* if every attaching map can be chosen to be a homeomorphism on the entire cell being attached (not just on its interior), cf. [5, Chapter III]. Simplicial complexes are examples of regular CW complexes.

The *nerve* of a family of sets $(X_i)_{i \in I}$ is the (abstract) simplicial complex $\mathcal{N}(X_i)$ defined on the vertex set I by the rule that a finite set $\sigma \subseteq I$ is in $\mathcal{N}(X_i)$ if and only if $\bigcap_{i \in \sigma} X_i \neq \emptyset$.

Theorem 6. *Let Δ be a connected regular CW complex and $(\Delta_i)_{i \in I}$ a family of subcomplexes such that $\Delta = \bigcup_{i \in I} \Delta_i$. Suppose that every non-empty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \cdots \cap \Delta_{i_t}$ is $(k - t + 1)$-connected, $t \geqslant 1$. Then there is a map $f: ||\Delta|| \to ||\mathcal{N}||$, where $\mathcal{N}$ denotes the nerve $\mathcal{N}(\Delta_i)$, inducing isomorphisms of homotopy groups $f_j^*: \pi_j(\Delta) \cong \pi_j(\mathcal{N})$ for all $j \leqslant k$.*

Proof. Let $P = \mathcal{F}(\Delta)$ and $Q = \mathcal{F}(\mathcal{N})$ denote the respective face posets (i.e., closed cells ordered by containment). It is a fact that taking the order complex $\Delta(\mathcal{F}(\Delta))$ one obtains a simplicial complex homeomorphic to Δ (a generalized barycentric subdivision). See the discussion in [1, pp. 1860–1861] or [5, pp. 78–83] for more details about this construction.

Define the order-reversing poset map $f: P \to Q$ by $f(\sigma) = \{i \in I \mid \sigma \in \Delta_i\}$. Thus, $f: \Delta(P) \to \Delta(Q)$ is a simplicial map inducing a continuous map $f: ||\Delta|| \cong ||\Delta(P)|| \to ||\Delta(Q)|| \cong ||\mathcal{N}||$. This is the map "$f$" of the theorem.

Let $Q^{(k+1)}$ be the truncation of Q to dimensions $0, 1, \ldots, k+1$, i.e., the face poset of the $(k+1)$-skeleton of $\mathcal{N}$. For $\sigma \in \Delta(Q^{(k+1)})$ let $C(\sigma) = f^{-1}(Q_{\geqslant \min \sigma}) = \bigcap_{i \in \min \sigma} \Delta_i$. Then C is a $(k - \dim(\min \sigma))$-connected carrier from $\Delta(Q^{(k+1)})$ to subcomplexes of $\Delta(P)$. Note that $\dim(\min \sigma) + \dim(\sigma) \leqslant k + 1$, since $\sigma \in \Delta(Q^{(k+1)})$. Hence, the carrier C is in particular $(\dim(\sigma) - 1)$-connected. By Lemma 1(ii) there exists a map $g: ||\Delta(Q^{(k+1)})|| \to ||\Delta(P)||$ carried by C.

As in the previous proof the idea now is to show that the induced map on homotopy groups $g_j^*: \pi_j(Q) \to \pi_j(P)$ is an isomorphism for all $j \leqslant k$, and moreover its inverse is f_j^*. The arguments are similar but a bit more involved.

Claim 1. *$g_j^*: \pi_j(Q) \to \pi_j(P)$ is injective for all $j \leqslant k$.*

For $\sigma \in \Delta(Q^{(k+1)})$ let $D(\sigma) = Q_{\geqslant \min \sigma}$. Then D is a contractible carrier from $\Delta(Q^{(k+1)})$ to subcomplexes of $\Delta(Q)$. Let *id* be the identity map on $Q^{(k+1)}$. The two maps *id* and $f \circ g: ||\Delta(Q^{(k+1)})|| \to ||\Delta(Q)||$ are both carried by D.

Hence, by Lemma 1(i) they are homotopic. This implies that $f_j^* \circ g_j^* = (f \circ g)_j^* = id_j^* : \pi_j(Q^{(k+1)}) \to \pi_j(Q)$, which is an isomorphism for $j \leqslant k$. Hence, g_j^* is injective.

Claim 2. *$g_j^* : \pi_j(Q) \to \pi_j(P)$ is surjective for all $j \leqslant k$.*

Let $r \in \pi_j(P)$. Via simplicial approximation we can assume that there is a simplicial subdivision Σ of the j-sphere such that $r : \|\Sigma\| \to \|\Delta(P)\|$ is a simplicial map (in the original homotopy class). Then also $f \circ r : \|\Sigma\| \to \|\Delta(Q)\|$ is simplicial.

Let $u : \hat{\Sigma} \to \mathcal{N}^{(k)}$ be a simplicial approximation to $f \circ r$, where $\hat{\Sigma}$ is some sufficiently fine iterated barycentric subdivision of Σ. Finally, let $t = \mathrm{sd}\, u : \mathrm{sd}\, \hat{\Sigma} \to \mathrm{sd}\, \mathcal{N}^{(k)} = \Delta(Q^{(k)}) \hookrightarrow \Delta(Q^{(k+1)})$, obtained by one more barycentric subdivision. Then $t \in \pi_j(Q)$. We will show that $r = g^*(t)$.

For $\tau \in \mathrm{sd}\, \hat{\Sigma}$, let $E(\tau) = f^{-1}(Q_{\geqslant \min t(\tau)})$. Then E is a $(k - \dim(\min t(\tau)))$-connected carrier from the j-dimensional complex $\mathrm{sd}\, \hat{\Sigma}$ to subcomplexes of $\Delta(P)$. Note that $\dim(\min t(\tau)) + \dim(\tau) \leqslant k$, since $\dim\,(u(v)) \leqslant \dim\,(v)$ for all faces v of $\hat{\Sigma}$. Hence, the carrier E is in particular $(\dim(\tau))$-connected.

By construction, E carries $g \circ t : \mathrm{sd}\, \hat{\Sigma} \to \Delta(P)$. Furthermore, E also carries $r : \mathrm{sd}\, \hat{\Sigma} \to \Delta(P)$. This is equivalent to the statement: for every $\tau \in \mathrm{sd}\, \hat{\Sigma}$, all elements of the chain $f \circ r(\tau)$ are above $\min t(\tau)$ in the partial order of Q. This follows from what it means for u to be a simplicial approximation to $f \circ r$, see e.g. [4, p. 251].

Thus, both maps r and $g \circ t$ are carried by E. Hence, by Lemma 1(i) these maps are homotopic: $r \sim g \circ t$, i.e., $r = g^*(t)$. □

Remark 7. If the assumption is strengthened to "*suppose that every non-empty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \cdots \cap \Delta_{i_t}$ is contractible*", then the conclusion "*$f_j^* : \pi_j(\Delta) \cong \pi_j(\mathcal{N}(\Delta_i))$ for all $j \in \mathbb{N}$*" implies, via Whitehead's theorem [4, p. 486; 5, p. 125], that f induces homotopy equivalence $\Delta \simeq \mathcal{N}(\Delta_i)$. This consequence of the theorem is the usual form of Borsuk's [3] Nerve Theorem.

Remark 8. If the conclusion is weakened to "*then Δ is k-connected if and only if $\mathcal{N}(\Delta_i)$ is k-connected*" the theorem specializes to [2, Lemma 1.2].

Example 9. An application of the classical Nerve Theorem in combinatorics is to give an easy proof of the homotopy version of the Crosscut Theorem. See [1, Theorem 10.8] for this (including references). Applying Theorem 6 in the same way gives various "k-connected crosscut theorems". As an example, for $k = 1$ one can conclude this:

Let A be the set of minimal elements of a poset P. Suppose that P and all nonempty pairwise intersections $P_{>x} \cap P_{>y}$, for $x \neq y$ in A, are connected. Define a 2-dimensional simplicial complex Γ on the vertex set A as having edges $\{x, y\}$ when $P_{>x} \cap P_{>y} \neq \emptyset$ and 2-faces $\{x, y, z\}$ when $P_{>x} \cap P_{>y} \cap P_{>z} \neq \emptyset$. Then there is isomorphism of fundamental groups: $\pi_1(P) \cong \pi_1(\Gamma)$.

This results from covering $\Delta(P)$ with the family of subcomplexes $(\Delta(P_{\geq x}))_{x \in A}$ and considering the nerve.

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