CHAPTER 34

Topological Methods

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Contents

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1. Introduction

In this chapter we discuss some of the ways in which topology has been used in combinatorics. The emphasis is on methods for solving genuine combinatorial problems that initially do not involve any topology – rather than on more theoretical aspects of the combinatorics–topology connection – and the selection of material reflects this aim.

The chapter is divided into two parts. In part I several examples are presented which illustrate different uses of topology in combinatorics. In part II we have gathered a number of tools which have proven useful for dealing with the topological structure found in combinatorial situations. Also, a brief review of relevant parts of combinatorial topology is given. Part II, which begins with section 9, is intended mainly for reference purposes.

Among the examples in part I one can discern at least four ways in which topology enters the combinatorial sphere. Of course, it is in the nature of such comments that no rigid demarcation lines could or should be drawn. Also other connections exist between topology and combinatorics that follow different paths.

(i) In the first three examples (sections 2–4) topology enters in the following way. First a relevant simplicial complex is identified in the combinatorial context. Then it is shown that this complex has sufficiently favorable properties to allow application of some theorem of algebraic topology, which implies the combinatorial conclusion.

(ii) A different approach is seen in section 5 and in Bárány's proof in section 4. There a combinatorial configuration is represented in concrete fashion in $\mathbb{R}^d$ or on the d-sphere, and a topological result (Borsuk's Theorem) has the desired effect on the configuration.

(iii) The case of oriented matroids (section 7) is unique. For these combinatorial objects there is a *topological representation theorem*, saying that oriented matroids are the same thing as arrangements of certain codimension one subspheres in a sphere. Of course, in this situation the topological perspective is always at hand as an alternative way of looking at these objects. Some nontrivial properties of oriented matroids find particularly simple proofs in this way.

(iv) The need for homotopy results in combinatorics sometimes arises as follows. Say we want to define some property $\mathcal{P}$ at all vertices of a connected graph $G = (V, E)$. We start by defining $\mathcal{P}$ at some root node r, and then give a rule for how to define $\mathcal{P}$ at v's neighbors, having already defined it at $v \in V$. The problem of consistency arises: Can different paths from r to v lead to different definitions of $\mathcal{P}$ at v? One strategy for dealing with this is to define "elementary homotopies", meaning certain pairs of paths which can be exchanged without affecting the result (usually such pairs form small circuits such as triangles and squares). Then we need a "homotopy theorem" saying that any path from r to v can be deformed into any other such path using elementary homotopies. Tutte's and Maurer's homotopy theorems (section 6) are of this kind. From a topological point of view, the "elementary homotopies" mean that certain 2-cells are attached

to the graph, and the homotopy theorem then says that the resulting 2-complex is simply connected.

Being topologically k-connected has a direct combinatorial meaning for $k = 0$ (connected), and, as we have seen, also for $k = 1$ (simply connected). The way that higher connectivity influences combinatorics is more subtle; see the examples in sections 4 and 6.

In section 8 a glimpse is given of the Hard Lefschetz Theorem and its applications to combinatorics found by R. Stanley. The question here is of finding a complex projective variety whose topology (in the form of its cohomology ring) is relevant to the combinatorics at hand. This rarefied method has found a few striking applications. Since it deals more with algebraic-geometric matters (the topology is somewhat subordinate), section 8 is rather loosely connected with the rest of the chapter.

Topological reasoning plays an important role in connection with several other topics in discrete mathematics not treated here. Among these, let us mention: embeddings of graphs in surfaces (see chapter 5 by Thomassen), convex polytopes (see chapter 18 by Klee and Kleinschmidt and also Bayer and Lee 1993), arrangements of subspaces (see Orlik and Terao 1992 and Björner 1994a), group-related incidence geometries (diagram geometries, chamber systems, posets of subgroups) (see Buekenhout 1995, Ronan 1989 and Webb 1987), computational geometry and realization spaces (see Bokowski and Sturmfels 1989), lower bounds for decision and computation trees (see chapter 32 by Alon and also Björner 1994a).

Notation and terminology is explained in part II. We treat simplicial complexes and posets almost interchangeably. The order complex of a poset and the poset of faces of a complex – these two constructions take posets to complexes and vice versa, and no ambiguity can arise from the topological point of view.

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PART I. EXAMPLES

2. Evasive graph properties

By a *graph property* we shall understand a property of graphs which is isomorphism-invariant: if $G_1 \cong G_2$ then G_1 has the property if and only if G_2 does. The following discussion will concern simple graphs having some fixed vertex set V. These graphs can be identified with the various subsets of $\binom{V}{2}$. Also, it is convenient to identify a graph property with the subset of the power set $\mathbf{2}^{\binom{V}{2}}$ which consists of all graphs having the property. A graph property $\mathcal{P} \subseteq \mathbf{2}^{\binom{V}{2}}$ is called *monotone* if it is preserved under deletion of edges. It is called *trivial* if either $\mathcal{P} = \emptyset$ or $\mathcal{P} = \mathbf{2}^{\binom{V}{2}}$.

In section 4.5 of chapter 23 by Bollobás the concept of *complexity* (sometimes called "argument complexity") of graph properties is discussed. Also, *evasive* graph properties are defined as those of maximal complexity. The following result (stated as Theorem 4.5.5 in chapter 23) confirms for prime-power number of vertices n a well-known conjecture.

Theorem 2.1 (Kahn, Saks and Sturtevant 1984). *Let $n = p^k$ where p is a prime. Then every non-trivial monotone property of graphs with n vertices is evasive.*

We will sketch the proof of Kahn et al. to show the way in which topology is used.

Suppose that card $V = p^k$, p prime, and that $\mathcal{P} \neq \emptyset$ is a monotone nonevasive graph property. $\mathcal{P}$ is a family of subsets of $\binom{V}{2}$ closed under the formation of subsets – i.e., a simplicial complex. The conclusion we want to draw is that $\mathcal{P}$ is trivial, which, since $\mathcal{P} \neq \emptyset$, must mean that $\binom{V}{2} \in \mathcal{P}$ – i.e., topologically $\mathcal{P}$ is the full simplex.

These two facts are crucial:

2.2. The geometric realization $\|\mathcal{P}\|$ is contractible.

2.3. There exists a group Γ of simplicial automorphisms of $\mathcal{P}$ which acts transitively on $\binom{V}{2}$ and which has a normal p-subgroup Γ_1, such that Γ/Γ_1 is cyclic.

For (2.2) one argues that the monotone property $\mathcal{P}$ is not evasive in the algorithmic sense defined above if and only if as a simplicial complex $\mathcal{P}$ is nonevasive in the recursive sense of (11.1). By (11.1) all nonevasive complexes are contractible.

The group Γ needed in (2.3) is constructed as follows. Identify V with the finite field $\mathrm{GF}(p^k)$. Let $\Gamma = \{x \mapsto ax + b \mid a, b \in \mathrm{GF}(p^k), a \neq 0\}$ and $\Gamma_1 = \{x \mapsto x + b \mid b \in \mathrm{GF}(p^k)\}$. The assumption that $\mathcal{P}$ is an isomorphism-invariant property of graphs on V means that if γ is any permutation of V – in particular, if $\gamma \in \Gamma$ – then $A \in \mathcal{P}$ if and only if $\gamma(A) \in \mathcal{P}$. Hence, Γ is a group of simplicial automorphisms of $\mathcal{P}$. One checks that Γ is doubly transitive on $V = \mathrm{GF}(p^k)$ and that the subgroup Γ_1 has the required properties.

By a theorem of Oliver (1975), any action of a finite group Γ, having a subgroup Γ_1 with the stated properties, on a finite $\mathbb{Z}_p$-acyclic simplicial complex must have stationary points. Since our complex $\mathcal{P}$ is $\mathbb{Z}_p$-acyclic (being contractible), this means that there exists some point $x \in \|\mathcal{P}\|$ such that $\gamma(x) = x$ for all $\gamma \in \Gamma$. The point x is carried by the relative interior of a unique face $G \in \mathcal{P}$ (the lowest-dimensional face containing it), and the fact that x is stationary implies that $\gamma(G) = G$ for all $\gamma \in \Gamma$. But since Γ is transitive on $\binom{V}{2}$ this is impossible unless $G = \binom{V}{2}$. Hence, $\binom{V}{2} \in \mathcal{P}$, and we are done.

It has been conjectured that *all* non-trivial monotone graph properties are evasive. This conjecture remains open for all non-prime-power $n \geq 10$; the $n = 6$ case was verified by Kahn et al. (1984). The evasiveness conjecture has been proven also for the case of bipartite graphs by Yao (1988), using the topological method.

3. Fixed points in posets

A poset P is said to have the *fixed-point property* if every order-preserving self-map $f: P \to P$ has a fixed point $x = f(x)$. It was shown by A. C. Davis and A. Tarski that a lattice has the fixed-point property if and only if it is complete (meaning that meets and joins exist for subsets of arbitrary cardinality). It has long been an open problem to find some characterization of the finite posets which have the fixed-point property. See Rival (1985) for references to work in this area. In the absence of such a characterization efforts have been directed toward finding nontrivial classes of finite posets which have the fixed point property. For this the Lefschetz fixed-point theorem has proved to be useful.

Let L be a finite lattice and $z \in L$. Then y is said to be a *complement* of z, written $y \perp z$, if $y \wedge z = \hat{0}$ and $y \vee z = \hat{1}$. Let $\mathcal{C}o(z) = \{y \in L \mid y \perp z\}$. The lattice L is called *complemented* if $\mathcal{C}o(z) \neq \emptyset$ for all $z \in L$.

A finite lattice L has the fixed point property, as is easy to see. It is more interesting to look at the *proper part* $\bar{L} = L - \{\hat{0}, \hat{1}\}$ of the lattice, which may or may not have the fixed point property. This is also natural from the point of view of lattice automorphisms, for which every nontrivial fixed point must lie in $\bar{L}$.

Theorem 3.1 (Baclawski and Björner 1979, 1981). *Let L be a finite lattice and $z \in \bar{L}$. Then the poset $\bar{L} - \mathcal{C}o(z)$ has the fixed point property. In particular, if L is noncomplemented then $\bar{L}$ has the fixed point property.*

By Theorem 10.15 the order complex $\Delta(\bar{L} - \mathcal{C}o(z))$ is contractible, and therefore by Lefschetz's Theorem 13.4 it has the *topological* fixed point property. From this the result easily follows.

For example, let L be a finite Boolean lattice of order n. Then $\bar{L}$ has $(n-1)!$ fixed-point-free automorphisms, but the removal of any one element from $\bar{L}$ leads to a poset with the fixed point property.

The preceding argument is, of course, applicable to *any* $\mathbb{Q}$-acyclic finite poset [see (11.1) for some other combinatorially defined classes of such]. Also, with this method one can prove more about the combinatorial structure of the *fixed-point sets* $P^f = \{x \in P \mid x = f(x)\}$ than merely that they are nonempty.

Let $f: P \to P$ be an order-preserving mapping of a finite $\mathbb{Q}$-acyclic poset. Then the Möbius function μ computed over P^f augmented by new bottom and top elements must equal zero: $\mu(P^f) = 0$. This follows from the Hopf trace formula, see (13.5) and the comments following it. A consequence is that for instance two or more incomparable points cannot alone form a fixed-point set in an acyclic poset. For other finite posets with the fixed point property such fixed-point sets are, however, possible.

Similarly, let $g: P \to P$ be an order-reversing mapping of a finite $\mathbb{Q}$-acyclic poset. Then the Hopf trace formula (13.2) specializes to $\mu(P_g) = 0$, where $P_g = \{x \in P \mid x = g^2(x) \leqslant g(x)\}$. In particular, if no $x \in P$ satifies $x = g^2(x) < g(x)$ then g has a *unique* fixed point. See Baclawski and Björner (1979) for further details and examples.

4. Kneser's Conjecture

Consider the collection of all n-element subsets of a $(2n+k)$-element set. $n \geqslant 1, k \geqslant 0$. It is easy to partition this collection into $k+2$ classes so that every pair of n-sets within the same class has nonempty intersection. Can the same be done with only $k+1$ classes? M. Kneser conjectured in 1955 that the answer is negative, and this was later confirmed by L. Lovász.

Theorem 4.1 (Lovász 1978). *If the n-subsets of a $(2n+k)$-element set are partitioned into $k+1$ classes, then some class will contain a pair of disjoint n-sets.*

Lovász's proof relies on Borsuk's Theorem 13.6 and homotopical connectivity arguments. Soon after Lovász's breakthrough a simpler way of deducing Kneser's Conjecture from Borsuk's Theorem was discovered by Bárány (1978). However, Lovász's proof method is applicable also to other situations and hence perhaps of greater general interest. See also chapter 24 by Frankl for a discussion of this result.

Let us first sketch Bárány's proof. By a theorem of Gale (1956) (see also Schrijver 1978), for $n, k \geqslant 1$ there exist $2n+k$ points on the sphere S^k such that any open hemisphere contains at least n of them. Partition the n-subsets of these points into classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k$. For $0 \leqslant i \leqslant k$, let $\mathcal{O}_i$ be the set of all points $x \in S^k$ such that the open hemisphere around x contains an n-subset from the class $\mathcal{C}_i$. Then $(\mathcal{O}_i)_{0 \leqslant i \leqslant k}$ gives a covering of S^k by open sets. Part (i) of Borsuk's Theorem 13.6 implies that one of the sets, say $\mathcal{O}_k$, contains antipodal points. But the open hemispheres around these points are disjoint and both contain n-subsets from the class $\mathcal{C}_k$. Hence, $\mathcal{C}_k$ contains a pair of disjoint n-sets.

For Lovász's proof it is best to think of the problem in graph-theoretic terms. Define a graph $\mathrm{KG}_{n,k}$ as follows: The vertices are the n-subsets of some fixed $(2n+k)$-element set X and the edges are formed by the pairs of disjoint n-sets. Then Theorem 4.1 can be reformulated: *The Kneser graph $\mathrm{KG}_{n,k}$ is not $(k+1)$-colorable.*

For any graph $G = (V, E)$ let $\mathcal{N}(G)$ denote the simplicial complex, called the *neighborhood complex*, whose vertex set is V and whose simplices are those sets of vertices which have a common neighbor (i.e., $A \in \mathcal{N}(G)$ iff there exists $v \in V$ such that $\{v, a\} \in E$ for all $a \in A$). The topology of this complex has surprising combinatorial content.

Theorem 4.2 (Lovász 1978). *For any finite graph G, if $\mathcal{N}(G)$ is $(k-1)$-connected, then G is not $(k+1)$-colorable.*

To prove Theorem 4.1 it will then suffice to show that $\mathcal{N}(\mathrm{KG}_{n,k})$ is $(k-1)$-connected. This can be done as follows. Let $P = \{A \subseteq X \mid n \leqslant \operatorname{card} A \leqslant n+k\}$. Ordered by containment P is a subposet of the Boolean lattice $B(X)$ of all subsets of X. $B(X)$ is shellable (11.10) (iv), hence by (11.2) and Theorem 11.14 P is $(k-1)$-connected. Let C be the crosscut of n-element sets. By Theorem 10.8, P and the

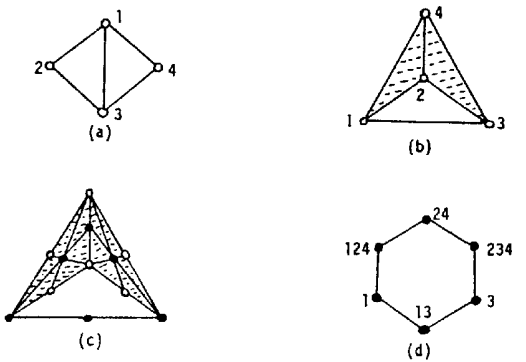

Figure 1.

crosscut complex $\Gamma(P, C)$ are homotopy equivalent. It follows that $\Gamma(P, C)$, which is the same thing as $\mathcal{N}(\mathrm{KG}_{n,k})$, is also $(k-1)$-connected.

The known proofs for Theorem 4.2 are more involved. A very elegant functorial argument was given by Walker (1983a), which we will sketch here in briefest possible fashion. The same general argument was also found by Lovász (unpublished lecture notes) as a variation of his original proof.

Let $G = (V, E)$ be a finite graph. The mapping $\nu : \mathcal{N}(G) \to \mathcal{N}(G)$ defined by $\nu(A) = \{v \in V \mid \{v, a\} \in E \text{ for all } a \in A\}$ has the properties

$$\text{(i)} \quad A \subseteq B \text{ implies } \nu(A) \supseteq \nu(B), \qquad \text{and (ii)} \quad \nu^2(A) \supseteq A.$$

Let $\tilde{\mathcal{N}}(G)$ denote the order complex of the poset of fixed points of ν^2 ordered by containment. Thus, $\tilde{\mathcal{N}}(G)$ is a subcomplex of the barycentric subdivision of $\mathcal{N}(G)$. In fact, the subspace $\|\tilde{\mathcal{N}}(G)\|$ is by Corollary 10.12 a strong deformation retract of $\|\mathcal{N}(G)\|$, so $\tilde{\mathcal{N}}(G)$ and $\mathcal{N}(G)$ are of the same homotopy type. This construction is illustrated in fig. 1, where part (a) shows a graph G, (b) the neighborhood complex $\mathcal{N}(G)$, (c) its barycentric subdivision, and (d) the retract complex $\tilde{\mathcal{N}}(G)$.

Property (i) of the mapping $\nu : \mathcal{N}(G) \to \mathcal{N}(G)$ shows that ν restricts to a simplicial mapping $\nu : \tilde{\mathcal{N}}(G) \to \tilde{\mathcal{N}}(G)$, and from property (ii) it follows that $\nu^2 = \text{identity}$. Hence, $(\tilde{\mathcal{N}}(G), \nu)$ (or, to be precise, $(\|\tilde{\mathcal{N}}(G)\|, \|\nu\|)$) is an antipodality space. Furthermore, it can be shown that every graph map (mapping of the nodes which takes edges to edges) $g : G_1 \to G_2$ induces an equivariant map $\tilde{g} : \tilde{\mathcal{N}}(G_1) \to \tilde{\mathcal{N}}(G_2)$. As these facts suggest, the construction $\tilde{\mathcal{N}}(\cdot)$ sets up a functor from the category of finite graphs and graph maps to the category of antipodality spaces and homotopy classes of equivariant maps, see Walker (1983a). For the example illustrated in fig. 1(d), the induced antipodal mapping of $\tilde{\mathcal{N}}(G)$ coincides with its antipodal map $x \mapsto -x$ as a circle.

For K_{k+1}, the complete graph on $k+1$ vertices, one sees that $\tilde{\mathcal{N}}(K_{k+1}) = \mathcal{N}(K_{k+1})$ is combinatorially the barycentric subdivision of the boundary of a k-simplex. It is also easy to verify that, as an antipodality space, $(\tilde{\mathcal{N}}(K_{k+1}), \nu)$ is isomorphic to the sphere $(\mathbf{S}^{k-1}, \alpha)$ with its standard antipodality map $\alpha(x) = -x$.

We now have all the ingredients for a proof of Theorem 4.2. Suppose that a graph G is $(k+1)$-colorable. This is clearly equivalent to the existence of a graph map $G \to K_{k+1}$. Hence, we deduce the existence of an equivariant map $\tilde{\mathcal{N}}(G) \to \tilde{\mathcal{N}}(K_{k+1}) \cong \mathbf{S}^{k-1}$. So by part (v) of Borsuk's Theorem 13.6, we conclude that $\tilde{\mathcal{N}}(G)$, and hence $\mathcal{N}(G)$, is not $(k-1)$-connected.

Schrijver (1978) has shown, using Bárány's method, that the conclusion of Theorem 4.1 remains true for the class of n-subsets that contain no consecutive elements $i, i+1$ in circular order (mod $2n+k$), and that this class is minimal with this property. A different application of Theorem 4.2 is given in Lovász (1983).

The following generalized "Kneser" conjecture was made by P. Erdős in 1973 and has recently been proved.

Theorem 4.3 (Alon, Frankl and Lovász 1986). *Let $n, t \geq 1$ and $k \geq 0$. If the n-subsets of a $(tn + (t-1)k)$-element set are partitioned into $k+1$ classes, then some class will contain t pairwise disjoint n-sets.*

The proof is analogous to Lovász's proof of Theorem 4.1. For general t-uniform hypergraphs H a suitable *neighborhood complex* $\mathcal{C}(H)$ is defined. It is shown that if t is a prime and $\mathcal{C}(H)$ is $(k(t \quad 1) \quad 1)$-connected then H is not $(k+1)$-colorable. To prove this for odd primes t the Bárány–Shlosman–Szűcs Theorem 13.8 is used rather than Borsuk's Theorem. It can be shown by an elementary argument that if Theorem 4.3 is valid for two values of t then it is also valid for their product. Hence one may assume that t is prime. See Alon et al. (1986) for the details.

Theorem 4.3 has been further generalized by Sarkaria (1990) to involve "j-wise disjoint" instead of "pairwise disjoint" families of n-sets. The proof uses a generalized Borsuk–Ulam theorem and the deleted join construction for simplicial complexes (defined in section 9).

5. Discrete applications of Borsuk's Theorem

One of the most famous consequences of Borsuk's Theorem 13.6 is undoubtedly the Ham Sandwich Theorem 13.7. This result, or some version of the "ham sandwich" argument which leads to it (outlined in connection with Theorem 13.7), can be used in certain combinatorial situations to prove that composite configurations can be split in a balanced way. Two examples of this, due to N. Alon and coauthors, will be given in this section. Also, we discuss how Borsuk's Theorem and its

generalizations have been used in connection with results of "Tverberg" type. For other applications of Borsuk's Theorem to combinatorics, see Bárány and Lovász (1982), Yao and Yao (1985), and section 4. Surveys of this topic are given by Alon (1988), Bárány (1993) and Bogatyi (1986).

Suppose that $2n$ points are given in general position in the plane $\mathbb{R}^2$, half colored red and the other half blue. It is an elementary problem to show that the red points can be connected to the blue points by n nonintersecting straight-line segments. A quick argument goes like this. Of the $n!$ ways to match the blue and red points using straight-line segments, choose one which minimizes the sum of the lengths. If two of its lines intersect, they could be replaced by the sides of the quadrilateral that they span, and a new matching of even shorter length would result. No such elementary proof is known for the following generalization to higher dimensions.

Theorem 5.1 (Akiyama and Alon 1989). *Let A be a set of $d \cdot n$ points in general position (no more than d points on any hyperplane) in $\mathbb{R}^d$. Let $A = A_1 \cup A_2 \cup \cdots \cup A_d$ be a partition of A into d pairwise disjoint sets of size n. Then there exist n pairwise disjoint $(d-1)$-dimensional simplices, such that each simplex intersects each set A_i in one of its vertices, $1 \leqslant i \leqslant d$.*

The idea of Akiyama and Alon is to surround each point $p \in A$ by a small ball of radius ε, where ε is small enough that no hyperplane intersects more than d such balls. Give each ball a uniform mass distribution of measure $1/n$. Then each color class $A_i, 1 \leqslant i \leqslant d$, is naturally associated with its n balls, forming a measurable set of measure 1. By the Ham Sandwich Theorem 13.7 there exists a hyperplane H which simultaneously bisects each color class. If n is odd, then H must intersect at least one ball from each A_i. General position immediately implies that H must intersect precisely one ball from each A_i, and in fact bisect this ball. By induction on n, the points on each side of H can now be assembled into disjoint simplices, and finally the points in H form one more such simplex. The argument if n is even is similar, but in that case H might have to be slightly moved to divide the points correctly for the induction step.

The next example has a more "applied" flavor. Suppose that k thieves steal a necklace with $k \cdot n$ jewels. There are t kinds of jewels on it, with $k \cdot a_i$ jewels of type $i, 1 \leqslant i \leqslant t$. The thieves want to divide the necklace fairly between them, wasting as little as possible of the precious metal in the links between jewels. They need to know in how many places they must cut the necklace? If the jewels of each kind appear contiguously on the opened necklace, then at least $t(k-1)$ cuts must be made. This number of cuts in fact always suffices. (Of course, what the thieves *really* need is a fast algorithm for where to place these cuts.)

Theorem 5.2 (Alon and West 1986, Alon 1987). *Every open necklace with $k \cdot a_i$ beads of color $i, 1 \leqslant i \leqslant t$, can be cut in at most $t(k-1)$ places so that the resulting segments can be arranged into k piles with exactly a_i beads of color i in each pile, $1 \leqslant i \leqslant t$.*

The idea for the proof is to turn the situation into a continuous problem by placing the open necklace (scaled to length 1) on the unit interval, and then to

use a "ham-sandwich"-type argument there. For $k = 2$ this was done in Alon and West (1986) using Borsuk's Theorem. The extension to general k was achieved in Alon (1987) using the Bárány–Shlosman–Szűcs Theorem 13.8.

Radon's Theorem, a well-known result in convexity theory, says that any collection of $d+2$ points in $\mathbb{R}^d$ can be split into two nonempty blocks whose convex hulls have nonempty intersection. This was generalized by Tverberg (1966) as follows: *For all $p \geqslant 2$ and $d \geqslant 1$, any set of $(p-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into p blocks $B_1, \ldots, B_p$ so that $\operatorname{conv}(B_1) \cap \cdots \cap \operatorname{conv}(B_p) \neq \emptyset$.* For a quite short proof of Tverberg's Theorem, see Sarkaria (1992). Results of the Radon–Tverberg type have generated a lot of interest, and recent work shows that in many cases such results rely on topological foundations that lead to formulations more general than the original ones in terms of convexity. See Eckhoff (1979) and Bárány (1993) for surveys of results of this kind.

Radon's theorem can be obtained as a consequence of Borsuk's Theorem, as was shown by Bajmóczy and Bárány (1979). Here is the connection. Let Δ^d denote the d-dimensional simplex. Bajmóczy and Bárány prove that there exists a continuous map $g: S^d \to \Delta^{d+1}$ such that the supports of $g(x)$ and $g(-x)$ are disjoint for every $x \in S^d$. Suppose now that Radon's Theorem is false; say it fails for the points $y_1, \ldots, y_{d+2}$ in $\mathbb{R}^d$. Define $f: \Delta^{d+1} \to \mathbb{R}^d$ by sending the ith vertex of Δ^{d+1} to y_i and extending linearly. Then the map $f \circ g: S^d \to \mathbb{R}^d$ would violate the Borsuk–Ulam Theorem 13.6 (ii).

In the preceding argument the map f could as well be an arbitrary continuous map (i.e., not necessarily linear). In a similar way, using Theorem 13.8 instead of Borsuk's Theorem, Bárány, Shlosman and Szűcs (1981) proved the following "topological Tverberg theorem": *Suppose that $f: \Delta^N \to \mathbb{R}^d$ is a continuous mapping, where $N = (p-1)(d+1)$ and p is prime. Then there exist p pairwise disjoint faces $\sigma_1, \ldots, \sigma_p$ of Δ^N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_p) \neq \emptyset$.* It is still unknown whether the restriction to prime p is needed here in the non-linear case. See Sarkaria (1991b) for even more general results of this kind.

The following result has the general flavor of Tverberg's Theorem, and goes in an opposite direction from Theorem 5.1.

Theorem 5.3 (Živaljević and Vrećica 1992). *Let $A = A_1 \cup A_2 \cup \cdots \cup A_{d+1}$ be a set of points in $\mathbb{R}^d$ partitioned into $d+1$ pairwise disjoint sets (color classes) of size $|A_i| \geqslant 4n-1$. Then there exist n pairwise disjoint $(d+1)$-subsets $B_1, \ldots, B_n$ of A such that $|A_i \cap B_j| = 1$ for all i, j and $\operatorname{conv}(B_1) \cap \cdots \cap \operatorname{conv}(B_n) \neq \emptyset$.*

The proof for this "colored Tverberg theorem" uses a Borsuk–Ulam-type result for free $\mathbb{Z}_p$-actions, p prime, which establishes the non-existence of an equivariant map from a certain "configuration space" of sufficiently high connectivity to a sphere of appropriate dimension.

It has been conjectured by Bárány and Larman that $|A_i| \geqslant n$ suffices in Theorem 5.3. This has been proven for $d = 2$ by Bárány and Larman and for $n = 2$ by Lovász, whose proof uses Borsuk's theorem. See Živaljević and Vrećica (1992) for

these references and for a fuller discussion of the status of this "colored Tverberg problem".

6. Matroids and greedoids

This section and the next are devoted to certain topological aspects of matroids and of two related structures – oriented matroids and greedoids. For the basic definitions see chapter 9 by Welsh. Additional topological facts about matroid complexes and geometric lattices are mentioned in (11.10); see also Björner (1992).

Basis complexes and partitions of graphs

The following result was proven by E. Győry and L. Lovász in response to a conjecture by A. Frank and S. Maurer.

Theorem 6.1 (Lovász 1977, Győry 1978). *Let $G = (V, E)$ be a k-connected graph, $\{v_1, v_2, \ldots, v_k\}$ a set of k vertices, and $n_1, n_2, \ldots, n_k$ positive integers with $n_1 + n_2 + \cdots + n_k = |V|$. Then there exists a partition $\{V_1, V_2, \ldots, V_k\}$ of V such that $v_i \in V_i, |V_i| = n_i$ and V_i spans a connected subgraph of $G, i = 1, 2, \ldots, k$.*

The proof of Lovász uses topological methods, that of Győry does not. At the end of this section Lovász's proof will be outlined for the case $k = 3$ in order to illustrate its use of topological reasoning. It relies on the connectivity of a certain polyhedral complex associated with certain forests in G. Similar complexes can be defined over the bases of a matroid, and more generally over the bases of a greedoid. The greedoid formulation contains the others as special cases, and we shall use it to develop the general result.We begin by recalling the definition.

A set system $(E, \mathcal{F}), \mathcal{F} \subset 2^E$, is called a *greedoid* if the following axioms are satisfied:

(G1) $\emptyset \in \mathcal{F}$,

(G2) for all nonempty $A \in \mathcal{F}$ there exists an $x \in A$ such that $A - x \in \mathcal{F}$,

(G3) if $A, B \in \mathcal{F}$ and $|A| > |B|$, then there exists an $x \in A - B$ such that $B \cup x \in \mathcal{F}$.

If also the extra condition (G4) is satisfied, then $(E, \mathcal{F})$ is called an *interval greedoid*:

(G4) if $A \subset B \subset C$ where $A, B, C \in \mathcal{F}$ and $A \cup x, C \cup x \in \mathcal{F}$ for some $x \in E - C$, then also $B \cup x \in \mathcal{F}$.

The sets in $\mathcal{F}$ are called *feasible* and the maximal feasible sets *bases*. All bases have the same cardinality r, which is the *rank* of the greedoid.

The only examples which will be of concern here are *matroids* (feasible sets = independent sets) and *branching greedoids* of rooted graphs (feasible sets = edge sets which form a tree containing the root node). Both are interval greedoids. For other examples and further information about greedoids, see chapter 9 by Welsh and the expository accounts Korte, Lovász and Schrader (1991) and Björner and Ziegler (1992).

The feasible sets of a greedoid do not form a simplicial complex other than in the matroid case. However, a useful topology is given by (the order complex of) the poset $\bar{\mathcal{F}} = \mathcal{F} - \{\emptyset\}$, ordered by inclusion. A greedoid $(E, \mathcal{F})$ is called *k-connected* if for each $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ with $A \subseteq B, |B - A| = \min(k, r - |A|)$ and such that $C \in \mathcal{F}$ for every $A \subseteq C \subseteq B$. Matroids are r-connected, and the branching greedoid of a k-connected rooted graph is k-connected.

Proposition 6.2 (Björner, Korte and Lovász 1985). *Let $(E, \mathcal{F})$ be a k-connected interval greedoid ($k \geq 2$). Then the poset of feasible sets $(\bar{\mathcal{F}}, \subseteq)$ is (topologically) $(k-2)$-connected.*

This result follows from (11.10) (iii) via Theorem 10.8, since for the crosscut C of minimal elements in $\bar{\mathcal{F}}$ the crosscut complex $\Gamma(\bar{\mathcal{F}}, C)$ is a matroid complex of rank $\geq k$.

Let $\mathcal{B}$ be the collection of all bases in a greedoid $(E, \mathcal{F})$ of rank r. Two bases B_1 and B_2 are *adjacent* if $B_1 \cap B_2 \in \mathcal{F}$ and $|B_1 \cap B_2| = r - 1$. Attaching edges between all adjacent pairs we get a graph with vertex set $\mathcal{B}$, the *basis graph*.

The shortest circuits in the basis graph can be explicitly described. There are two kinds of triangles and one kind of square (quadrilateral):

6.3. Three bases $A \cup x, A \cup y, A \cup z$, where $A \in \mathcal{F}, |A| = r - 1$, span a triangle of the first kind.

6.4. Three bases $A \cup x \cup y, A \cup x \cup z, A \cup y \cup z$, where $A \in \mathcal{F}, |A| = r - 2$, span a triangle of the second kind.

6.5. Four bases $A \cup x \cup u, A \cup x \cup v, A \cup y \cup u, A \cup y \cup v$, where $A \in \mathcal{F}, |A| = r - 2$, span a square.

For branching greedoids triangles of the second kind cannot occur.

Now, attach a 2-cell (a "membrane") into each triangle and square. This gives a 2-dimensional regular cell complex $\mathcal{K}$, which we call the *basis complex*.

It is a straightforward combinatorial exercise to check that the basis complex of any 2-connected greedoid of rank ≤ 2 is 1-connected (i.e., connected and simply connected). For rank 2 (the only non-trivial case) this follows directly from the exchange axiom (G3). In higher ranks the following is true.

Theorem 6.6 (Björner, Korte and Lovász 1985). *The basis complex $\mathcal{K}$ of any 3-connected interval greedoid is 1-connected.*

In order to illustrate some of the tools given in part II, we give a short proof of this. Let P be the poset of closed cells of $\mathcal{K}$ ordered by inclusion, and let Q be the top three levels of $(\bar{\mathcal{F}}, \subset)$, i.e., the feasible sets of ranks $r \quad 2, r - 1$ and r. Let $f : P \to Q$ be the order-reversing map which sends each cell τ to the intersection of the bases which span τ. By Proposition 6.2 and Lemma 11.12 the poset Q is 1-connected, so by Theorem 10.5 we only have to check that the fibers $f^{-1}(Q_{\geq A})$

are 1-connected for all $A \in Q$. But if $r(A) = r - i$, $i = 0,1,2$, then $f^{-1}(Q_{\geqslant A})$ is the basis complex of the rank i greedoid obtained by contracting A, and we have already checked that basis complexes of rank $\leqslant 2$ greedoids are 1-connected.

Let $P = B_1B_2\cdots B_d$ and $Q = B_dB_{d+1}\cdots B_g$ be paths in the basis graph of a matroid, and let $PQ = B_1B_2\cdots B_dB_{d+1}\cdots B_g$ be their concatenation. Say that paths $P'Q$ and PRQ differ by an *elementary homotopy* if R is of the form $BCB, BCDB$ or $BCDEB$ with $B = B_d$.

Theorem 6.7 (Maurer 1973). *Let P and P' be any two paths with the same endpoints in the basis graph of a matroid. Then P can be transformed into P' via a sequence of elementary homotopies.*

Maurer's "Homotopy Theorem" 6.7 is clearly a combinatorial reformulation of Theorem 6.6 in the matroid case. An application to oriented matroids will be given in the next section.

The time has come to return to Theorem 6.1. The following outline of the proof for the $k = 3$ case is quoted from Lovász (1979) (with some adjustments in square brackets to better suit the present discussion):

"So let G be a 3-connected graph, $v_1, v_2, v_3 \in V(G)$ and $n_1 + n_2 + n_3 = |V(G)|$. Take a new point a and connect it to v_1, v_2, and v_3. Consider the topological space $\mathcal{K}$ constructed for this new graph G'. [In our language, $\mathcal{K}$ is the basis complex of the branching greedoid determined by the rooted graph (G', a). This greedoid, whose bases are the spanning trees of G', is 3-connected.] For each spanning tree T of G', let $f_i(T)$ denote the number of points in T accessible from a along the edge $(a, v_i)(i = 1,2)$. Then the mapping

$$f : T \mapsto (f_1(T), f_2(T))$$

maps the vertices of $\mathcal{K}$ onto lattice points of the plane. Let us subdivide each quadrilateral 2-cell in $\mathcal{K}$ by a diagonal into two triangles; in this way we obtain a triangulation $\tilde{\mathcal{K}}$ of $\mathcal{K}$. Extend f affinely to each such triangle so as to obtain a continuous mapping of $\mathcal{K}$ into the plane. Obviously, the image of $\mathcal{K}$ is contained in the triangle $\Delta = \{x \geqslant 0, y \geqslant 0, x + y \leqslant n\}$. We are going to show that the mapping is onto Δ.

"Let us pick three spanning trees, T_1, T_2, T_3 first such that $f(T_1) = (n,0), f(T_2) = (0,n), f(T_3) = (0,0)$. Obviously, such trees exist. Next, by applying [the fact that the basis graph of a 2-connected greedoid is connected] to the graph $G' - (a, v_3)$, we select a polygon P_{12} in $\mathcal{K}$ connecting T_1 to T_2 and having $f_3(x) = 0$ at all points. Thus $f(P_{12})$ connects $(n,0)$ to $(0,n)$ along the side of the triangle Δ with these endpoints. Let P_{23} and P_{31} be defined analogously.

"By Theorem 6.6, $P_{12} + P_{23} + P_{31}$ can be contracted in $\mathcal{K}$ to a single point. Therefore, $f(P_{12}) + f(P_{23}) + f(P_{31})$ can be contracted in $f(\mathcal{K})$ to a single point. But 'obviously' (or, rather, by applying the well-known fact [Brouwer's Theorem 13.1] that the boundary of a triangle cannot be contracted to a single point in the triangle with one interior point taken out), $f(\mathcal{K})$ must cover the whole triangle Δ. So in particular the point (n_1, n_2) belongs to the image of $\mathcal{K}$, and therefore it belongs to

the image of a triangle of $\tilde{\mathcal{K}}$. But it is easy to see that this implies that (n_1, n_2) is the image of one of the vertices of $\tilde{\mathcal{K}}$; i.e., there exists a spanning tree T with

$$f_1(T) = n_1, \quad f_2(T) = n_2.$$

The three components of $T - a$ now yield the desired partition of $V(G)$."

Theorem 6.6 is a special case of a more general result saying that for any k-connected interval greedoid a certain higher-dimensional basis complex is $(k - 2)$-connected. This more general result implies Theorem 6.1 for arbitrary k by extension of the ideas we have just seen in the $k = 3$ case. See Lovász (1977) and Björner, Korte and Lovász (1985) for complete details.

Tutte's Homotopy Theorem

A matroid is called *regular* if it can be coordinatized over every field. In Tutte (1958) a characterization is given of regular matroids in terms of forbidden minors. The proof relies in an essential way on a "Homotopy Theorem", expressing the 1-connectivity of certain 2-dimensional complexes. Tutte's Homotopy Theorem was also used by R. Reid and R. Bixby to prove the forbidden minor characterization for representability over GF(3). More recently other proofs of these results, avoiding use of the Homotopy Theorem, have been found by P. Seymour and others. See chapter 10 by Seymour for an up-to-date account.

Tutte's Homotopy Theorem seems to be the oldest topological result of its kind in combinatorics. Unfortunately it is quite technical both to state in full and to prove. Here we shall state the Homotopy Theorem in sufficient detail that the nature of the result can be understood. Complete details can be found in Tutte (1958) and Tutte (1965).

Let L be a finite geometric lattice of rank r, and write L^i for the set of flats of rank i; so L^{r-1} is the set of copoints, L^{r-2} the colines and L^{r-3} the coplanes. Flats $X \subset L$ will be thought of as subsets of the point set L^1 via $\tilde{X} = \{p \in L^1 \mid p \leqslant X\}$.

Given any point $a \in L^1$ we define a graph $\mathrm{TG}(L, a)$ on the vertex set $L^{r-1}_{\not\geqslant a} = \{X \in L^{r-1} \mid X \not\geqslant a\}$ as follows: two copoints X and Y "off a" (i.e., in the set $L^{r-1}_{\not\geqslant a}$) span an edge if $X \wedge Y$ is a coline and $\tilde{X} \cup \tilde{Y} \neq L^1 - a$. On this graph we construct a 2-dimensional regular cell complex $\mathrm{TC}(L, a)$ by attaching 2-cells into the triangles and squares of the following kinds:

6.8. Triangles $XYZX$ for which $\mathrm{rk}(X \wedge Y \wedge Z) \geqslant r - 3$.

6.9. Squares $XYZTX$ for which $\mathrm{rk}(P) = r - 3$, where $P = X \wedge Y \wedge Z \wedge T$, and *either* the coline $P \vee a$ is covered by exactly two copoints *or else* the interval $[P, \hat{1}]$ is isomorphic to the lattice of flats of the Fano matroid F_7 minus one of its points.

If L has no minor isomorphic to F_7^*, the dual of the Fano matroid, then (6.8) and (6.9) describe all the 2-cells of the *Tutte complex* $\mathrm{TC}(L, a)$. [This means that for use in representation theory the definition (6.8)–(6.9) of $\mathrm{TC}(L, a)$ is sufficient.] In general it is necessary to attach 2-cells also into certain squares $XYZTX$ for

which $\mathrm{rk}(X \wedge Y \wedge Z \wedge T) = r - 4$. The definition of these squares (of the "corank 4 kind") is fairly complicated, so we refrain from describing them here.

Theorem 6.10 (Homotopy Theorem, Tutte 1958). *The complex* $\mathrm{TC}(L,a)$ *is* 1-*connected.*

The combinatorial meaning of Theorem 6.10 is that any two copoints X and Y "off a" can be connected "off a" by a path in the Tutte graph $\mathrm{TG}(L,a)$, and that any two such paths differ by a sequence of *elementary homotopies* of type $XYX, XYZX$ as in (6.8), or $XYZTX$ as in (6.9) or of the corank 4 kind. (Compare the discussion preceding Theorem 6.7.)

The given formulation of the Homotopy Theorem differs in form but not in content from the statement in Tutte (1958). Tutte has remarked about his theorem (Tutte 1979, p. 446) that "the proof ... is long, but it is purely graph-theoretical and geometrical in nature. I am rather surprised that it seems to have acquired a reputation for extreme difficulty". No significant simplification of the original proof seems to be known, other than in special cases. One such case is if $X \cup Y \neq L^1 - a$ for all pairs X, Y of copoints "off a" such that $X \wedge Y$ is a coline. Then the top three levels of $L - [a, \hat{1}]$ form a poset which is 1-connected by (11.10) (iv), (11.2) and Theorem 11.14, and the 1-connectivity can be transferred to $\mathrm{TC}(L,a)$ by an application of the Fiber Theorem 10.5, similar to the proof of Theorem 6.6. A simpler and more conceptual proof of Tutte's Theorem in full strength would be of definite interest.

Unfortunately the available space does not permit a thorough explanation of how Theorem 6.10 is used in representation theory. Here is a briefest possible sketch of the idea. Tutte's proof of sufficiency for his characterization of regular matroids runs by induction on the size of the ground set (that is why it is of interest to delete the point a). Roughly speaking, the "regular" coordinatization lives on the copoints, and its value at the new point a is extended from one copoint in $L^{r-1}_{\not\ge a}$ to another via paths in the Tutte graph $\mathrm{TG}(L,a)$. The Homotopy Theorem is then needed to check that different paths do not lead to contradictions. A similar idea is illustrated in greater detail in the proof of Theorem 7.6.

7. Oriented matroids

Two topics from the theory of oriented matroids will be discussed in this section. Most important is the topological representation theorem of Folkman and Lawrence (1978), which states that every oriented matroid can be realized by an arrangement of pseudospheres. As an application we show how such realizations lead to quick proofs of some combinatorial properties of rank 3 oriented matroids. Second, we sketch (following Las Vergnas 1978) how Maurer's Homotopy Theorem 6.7 can be used to deduce the existence of a determinantal sign function.

Oriented matroids are defined in chapter 9 by Welsh. Since we will use a slightly different formulation of the concept (due to Folkman and Lawrence 1978) and need to refer to the linear case for motivation, we will start with a quick review of the basics, which will also serve to fix notation. More extensive treatments can be found in the monographs Bachem and Kern (1992) and Björner, Las Vergnas, Sturmfels, White and Ziegler (1993).

Let E be a finite set with a fixed-point free involution $x \mapsto x^*$ (i.e., $x^* \neq x = x^{**}$ for all $x \in E$). Write $A^* = \{x^* \mid x \in A\}$, for subsets $A \subseteq E$. An *oriented matroid* $\mathcal{O} = (E, *, \mathcal{C})$ is such a set together with a family $\mathcal{C}$ of nonempty subsets such that

(OM1) $\mathcal{C}$ is a clutter (i.e., $C_1 \neq C_2$ implies $C_1 \not\subseteq C_2$ for all $C_1, C_2 \in \mathcal{C}$);

(OM2) if $C \in \mathcal{C}$ then $C^* \in \mathcal{C}$ and $C \cap C^* = \emptyset$;

(OM3) if $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2^*$ and $x \in C_1 \cap C_2^*$, then there exists $D \in \mathcal{C}$ such that $D \subseteq C_1 \cup C_2 - \{x, x^*\}$.

The sets in $\mathcal{C}$ are called *circuits* of the oriented matroid $\mathcal{O}$. For elements $x \in E$ let $\bar{x} = \{x, x^*\}$, and let $\bar{A} = \{\bar{x} \mid x \in A\}, A \subseteq E$, and $\bar{\mathcal{C}} = \{\bar{C} \mid C \in \mathcal{C}\}$. The system $\bar{\mathcal{C}}$ satisfies the usual matroid circuit-exchange axioms, so $\bar{\mathcal{O}} = (\bar{E}, \bar{\mathcal{C}})$ is a matroid, called the *underlying matroid* of $\mathcal{O}$. Not all matroids arise from oriented matroids in this way; those that do are called *orientable*. A subset $B \subseteq E$ is called a *basis* of $\mathcal{O}$ if $\bar{B}$ is a basis of $\bar{\mathcal{O}}$. The *rank* of $\mathcal{O}$ equals the rank of $\bar{\mathcal{O}}$. Without significant loss of generality we will make the tacit assumption in what follows that all oriented matroids are *simple*, meaning that no circuit has fewer than three elements.

The fundamental models for oriented matroids are sets of vectors in $\mathbb{R}^d$ and the relation of positive linear dependence (or, more generally, positive linear dependence of vectors over *any* ordered field). Suppose that E is a finite subset of $\mathbb{R}^d - \{0\}$ such that $E = -E$, and if $x \neq y$ in E are parallel then $y = -x$. For $x \in E$ let $x^* = -x$. A subset $A \subseteq E$ is *positive linearly dependent* if $\sum_{x \in A} \lambda_x x = 0$ for some real coefficients $\lambda_x \geq 0$, not all equal to zero. Let $\mathcal{C}$ be the family of all inclusion-wise minimal positive linearly dependent subsets of E, except those of the form $\{x, x^*\}, x \in E$. Equivalently, $\mathcal{C}$ consists of all subsets of E which form the vertex set of a simplex of dimension ≥ 2 containing the origin in its relative interior. Oriented matroids $(E, *, \mathcal{C})$ which arise in this way are called *linear* (or, *realizable*) over $\mathbb{R}$. Not all oriented matroids are isomorphic to linear ones.

Topological Representation Theorem

To pave the way for the Representation Theorem for oriented matroids it is best to look at the linear case for motivation. The Representation Theorem in fact says that intuition gained from the linear case is going to be essentially correct (modulo some topological deformation which cannot be too bad) for general oriented matroids.

Let E be a finite subset of $\mathbb{R}^d - \{0\}$ such that $E = -E$, and let $\mathcal{O} = (E, *, \mathcal{C})$ be the linear oriented matroid as previously discussed. For each $e \in \bar{E} = \{\bar{x} = \{x, x^*\} \mid x \in E\}$, let H_e be the hyperplane orthogonal to the line spanned by e. The *arrangement of hyperplanes* $\mathcal{H} = \{H_e \mid e \in \bar{E}\}$ contains all information about $\mathcal{O}$, since one can go from H_e back to a pair of opposite normal vectors, and the definition of the sets which form circuits in $\mathcal{O}$ (i.e., the sets in $\mathcal{C}$) is independent of the length of vectors. By intersecting with the unit sphere S^{d-1} we can alterna-

tively look at the *arrangement of spheres* $\mathscr{S} = \{H_e \cap S^{d-1} \mid e \in \bar{E}\}$, which is merely a collection of equatorial $(d-2)$-spheres inside the $(d-1)$-sphere. Clearly: *linear oriented matroids* (up to reorientation), *arrangements of hyperplanes* and *arrangements of spheres* are the same thing.

When thinking about a linear oriented matroid $(E, *, \mathscr{C})$ as an arrangement of spheres it is useful to visualize elements $x \in E$ as closed hemispheres $\bar{H}_x = \{y \in S^{d-1} \mid (y,x) \geqslant 0\}$. Then a subset $A \subseteq E$ belongs to $\mathscr{C}$ if and only if $A \cap A^* = \emptyset$ and A is minimal such that $\bigcup_{x \in A} \bar{H}_x = S^{d-1}$.

We shall need the following terminology. A *sphere* $\sum$ is a topological space for which there is a homeomorphism $f : S^j \to \sum$ with the standard j-sphere $S^j = \{x \in \mathbb{R}^{j+1} \mid \|x\| = 1\}$, for some $j \geqslant 0$. A *pseudosphere* S in $\sum$ is any image $S = f(\{x \in S^j \mid x_{j+1} = 0\})$ under such a homeomorphism. [In the topological literature pseudospheres are known as "tamely embedded (or, flat) codimension-one subspheres", cf. Rushing (1973).] The two *sides* (or, pseudohemispheres) of S are $S^+ = f(\{x \in S^j \mid x_{j+1} \geqslant 0\})$ and $S^- = f(\{x \in S^j \mid x_{j+1} \leqslant 0\})$. Clearly, S is the intersection of its two sides, which are homeomorphic to balls.

The crucial definition is this: An *arrangement of pseudospheres* $(\bar{E}, \mathscr{A})$ in S^{d-1} is a finite collection $\mathscr{A} = \{S_e \mid e \in \bar{E}\}$ of distinct pseudospheres S_e in S^{d-1} such that

(AP1) Every nonempty intersection $S_A = \bigcap_{e \in A} S_e$, $A \subseteq \bar{E}$, is a sphere.

(AP2) For every nonempty intersection S_A and all $e \in \bar{E}$, either $S_A \subseteq S_e$ or $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

This definition is due to Folkman and Lawrence (1978). They actually required more, but the additional assumptions in their definition were proved to be redundant by Mandel (1982).

In analogy with the linear case (arrangement of spheres), an arrangement of pseudospheres $(\bar{E}, \mathscr{A})$ gives rise to a system $\mathscr{O}(\mathscr{A}) = (E, *, \mathscr{C})$ as follows: put $E = \{S_e^+ \mid e \in \bar{E}\} \cup \{S_e^- \mid e \in \bar{E}\}$, let $(S_e^+)^* = S_e^-$ and vice versa, and define $\mathscr{C}$ to be the collection of the minimal subsets $A \subseteq E$ such that $\bigcup A = S^{d-1}$ and $A \cap A^* = \emptyset$. It turns out that $\mathscr{O}(\mathscr{A})$ is an oriented matroid (in spite of the topological deformations). What is more surprising is that the construction leads to *all* oriented matroids. We call an arrangement $\mathscr{A}$ *essential* if $\bigcap \mathscr{A} = \emptyset$.

Theorem 7.1 (Representation Theorem, Folkman and Lawrence 1978).

(i) *If $\mathscr{A}$ is an arrangement of pseudospheres in S^{d-1}, then $\mathscr{O}(\mathscr{A})$ is an oriented matroid. Furthermore, if $\mathscr{A}$ is essential then rank $\mathscr{O}(\mathscr{A}) = d$.*

(ii) *If $\mathscr{O}$ is an oriented matroid of rank d, then $\mathscr{O} = \mathscr{O}(\mathscr{A})$ for some essential arrangement of pseudospheres in S^{d-1}.*

(iii) *The mapping $\mathscr{A} \to \mathscr{O}(\mathscr{A})$ induces a one-to-one correspondence between rank d oriented matroids and essential arrangements of pseudospheres in S^{d-1}, up to natural equivalence relations.*

The proof of this result is quite involved. For part (ii) a poset is first constructed from the oriented matroid, and then it is shown using Theorem 12.6 that this poset is the poset of faces of some regular cell complex $\mathscr{C}$. This complex $\mathscr{C}$ provides the

$(d-1)$-sphere and various subcomplexes the $(d-2)$-subspheres forming the arrangement. The sphere $\mathscr{C}$ is constructible (Edmonds and Mandel 1978, Mandel 1982), and even shellable (Lawrence 1984), which implies that the whole construction of $\mathscr{C}$ and the relevant subcomplexes can be carried out in piecewise linear topology. In particular, this means that no topological pathologies need to be dealt with in representations of oriented matroids. Complete proofs of Theorem 7.1 can be found in Folkman and Lawrence (1978), Mandel (1982), and Björner, Las Vergnas, Sturmfels, White and Ziegler (1993).

The Representation Theorem shows that oriented matroids of rank 3 correspond to arrangements of "pseudocircles" on the 2-sphere or, in the projective version, arrangements of pseudolines in the real projective plane. This representation can be used for quick proofs of some combinatorial properties as in the following application.

Theorem 7.2. *Let M be an orientable matroid of rank 3. Then:*

(i) *M has a 2-point line,*

(ii) *if the points of M are 2-colored there exists a monochromatic line.*

Here is how Theorem 7.2 follows from Theorem 7.1. Represent the points of M as pseudocircles on the 2-sphere. Then lines are maximal collections of pseudocircles with nonempty intersection (which is necessarily a 0-sphere, i.e., two points). The arrangement of pseudocircles gives a graph G whose vertices are the points of intersection and edges the segments of pseudocircles between such points. Since this graph lies embedded in S^2 it is planar, and since $\mathrm{rk}(M) = 3$ it is simple. We need the following lemma.

Lemma 7.3. *For any planarly embedded simple graph:*

(i) *some vertex has degree at most five,*

(ii) *if the edges are 2-colored then there exists a vertex around which the edges of each color class are consecutive in the cyclic ordering induced by the embedding.*

Part (i) is a well-known consequence of Euler's formula (cf. chapter 5 by Thomassen). Part (ii) is also a consequence of Euler's formula, but not as well known. It was used by Cauchy in the proof of his Rigidity Theorem for 3-dimensional convex polytopes.

To finish the proof of Theorem 7.2, look at the graph G determined by the arrangement of pseudocircles. If all lines in M have at least 3 points, then every vertex in G will have degree at least 6, in violation of (i). If the pseudocircles are 2-colored and through every intersection point there is at least one pseudocircle of each color, then the induced coloring of the edges of G will violate (ii).

The proof of the first part of Theorem 7.2, a generalization of the Sylvester–Gallai Theorem (see chapter 17 by Erdős and Purdy), has been known since the 1940s in the linear case. The following strengthening by Csima and Sawyer (1993) also uses pseudoline representation: *The number of 2-point lines in M is at least* $\frac{6}{13}(\mathrm{card}\ M)$. The proof of the second part, due to G.D. Chakerian in the linear

case, was rediscovered by Edmonds, Lovász and Mandel (1980), who also observed the generalization to oriented matroids.

Basis signatures

Just like ordinary matroids, oriented matroids can be characterized in several ways. We shall discuss a characteristic property of the set of bases $\mathscr{B}$ of an oriented matroid, namely that a determinant can be defined up to sign (but not magnitude). This was first shown by Las Vergnas (1978). Characterizations of oriented matroids in terms of signed bases were also discovered by J. Bokowski, A. Dress, L. Gutierrez-Novoa and J. Lawrence.

Let us review some essential features of the function $\delta : \tilde{\mathscr{B}} \to \{+1, -1\}$, taking *ordered* bases of a *linear* oriented matroid $(E, *, \mathscr{C}), E \subset \mathbb{R}^d$, to the sign of their determinants. A function η can be defined for certain pairs of ordered bases β and β' in $\mathbb{R}^d$ as follows:

7.4. Suppose β and β' are permutations of the same basis B. Let $\eta(\beta, \beta') = +1$ if they are of the same parity and $= -1$ otherwise.

7.5. Suppose $\beta = x_1 x_2 \cdots x_{r-1} y$ and $\beta' = x_1 x_2 \cdots x_{r-1} z$ with $y \neq z$. Let $\eta(\beta, \beta') = +1$ if y and z are on the same side of the hyperplane spanned by $\{x_1, \ldots, x_{r-1}\}$, and $= -1$ otherwise.

Now, once we choose an ordered basis β_0 and put $\det(\beta_0) := +1$, the function $\det(\beta)$ and its sign $\delta(\beta)$ is determined for all ordered bases β by the usual rules of linear algebra. But the function $\delta(\beta)$ is also *combinatorially* determined, because any pair of ordered bases can be connected by a chain of steps of type (7.4) or (7.5) and we have: *If β and β' are ordered bases as in* (7.4) *or* (7.5) *then* $\delta(\beta) = \eta(\beta, \beta') \cdot \delta(\beta')$.

The preceding discussion points the way how to generalize the determinantal sign function to all oriented matroids. First, to cast (7.5) in a form which is more compatible with the axiom system (OM 1)–(OM 3), we replace it by the following reformulation:

7.5′. Suppose $\beta = x_1 x_2 \cdots x_{r-1} y$ and $\beta' = x_1 x_2 \cdots x_{r-1} z$ with $y \neq z$, and if $y \neq z^*$ let $\{C, C'\}$ be the unique pair of circuits such that in the underlying matroid $\{\bar{y}, \bar{z}\} \subseteq C \subseteq \{\bar{x}_1, \ldots, \bar{x}_r, \bar{y}, \bar{z}\}$. Put $\eta(\beta, \beta') = +1$ if one of y and z lies in C and the other in C', and put $\eta(\beta, \beta') = -1$ otherwise.

Theorem 7.6 (Las Vergnas 1978). *Let $\tilde{\mathscr{B}}$ be the set of ordered bases of an oriented matroid, and let $\beta_0 \in \tilde{\mathscr{B}}$. There exists a unique function $\delta : \tilde{\mathscr{B}} \to \{+1, -1\}$ such that $\delta(\beta_0) = +1$ and if $\beta, \beta' \in \tilde{\mathscr{B}}$ are related as in* (7.4) *or* (7.5′) *then $\delta(\beta) = \eta(\beta, \beta') \cdot \delta(\beta')$.*

The proof runs as follows. Define a graph on the vertex set $\tilde{\mathscr{B}}$ by connecting pairs $\{\beta, \beta'\}$ which are related as in (7.4) or (7.5′) by an edge. The graph is clearly connected, and there is a projection $\pi : \tilde{\mathscr{B}} \to \mathscr{B}$ to the basis graph $\mathscr{B}$ of the underlying matroid. Now, put $\delta(\beta_0) := +1$, and for $\beta \in \tilde{\mathscr{B}}$ define

$$\delta(\beta) := \prod_{i=1}^{n} \eta(\beta_{i-1}, \beta_i)$$

for some choice of path $\beta_0, \beta_1, \ldots, \beta_n = \beta$ in $\tilde{\mathscr{B}}$. The proof is complete once we show that this definition is independent of the choice of path from β_0 to β. If P_1 and P_2 are two such paths then by Theorem 6.7 their projections $\pi(P_1)$ and $\pi(P_2)$ in the basis graph differ by a sequence of elementary homotopies. Thus the checking is reduced to verifying

$$\prod_{i=1}^{k} \eta(\alpha_{i-1}, \alpha_i) = 1$$

for closed paths $\alpha_0, \alpha_1, \ldots, \alpha_k = \alpha_0$ in $\tilde{\mathscr{B}}$ whose projection in $\mathscr{B}$ is an edge BCB, triangle $BCDB$ or square $BCDEB$. However, the basis configurations which give triangles or squares in the basis graph are explicitly characterized in (6.3)–(6.5), and this way the checking is brought down to a manageable size. See Las Vergnas (1978) for further details.

8. Discrete applications of the Hard Lefschetz Theorem

One of the most esoteric results to have found applications in combinatorics is the Hard Lefschetz Theorem. It was used by R. Stanley to prove the Erdős–Moser conjecture (chapter 32 by Alon) and to show necessity in the characterization of f-vectors of simplicial convex polytopes (chapter 18 by Klee and Kleinschmidt).

In this section we will state the Hard Lefschetz Theorem and briefly explain how it is used for these applications. The presentation follows Stanley (1980a,b, 1983b, 1985, 1989). Other applications appear in Stanley (1987a,b).

Unfortunately, concepts must be used here which go beyond what is reviewed and explained in part II of this chapter. In particular we must assume some familiarity with the singular cohomology ring of a topological space, and with a few basic notions of algebraic geometry (projective varieties, smoothness, etc.). See Hartshorne (1977) for this.

Let X be a smooth irreducible complex projective variety of complex dimension d, and let $H^*(X) = H^0(X) \oplus H^1(X) \oplus \cdots \oplus H^{2d}(X)$ denote its singular cohomology ring with real coefficients. Recall that if $\omega \in H^i(X)$ and $\tau \in H^j(X)$ then $\omega \cdot \tau \in H^{i+j}(X)$. Being projective, we may intersect X with a generic hyperplane H of an ambient projective space. By a standard construction in algebraic geometry the subvariety $X \cap H$ represents a cohomology class $\omega \in H^2(X)$.

Theorem 8.1 (The Hard Lefschetz Theorem). *Let X and $\omega \in H^2(X)$ be as above, and let $0 \leqslant i \leqslant d$. Then the linear map $H^i(X) \to H^{2d-i}(X)$ given by multiplication by ω^{d-i} is an isomorphism of vector spaces.*

See Stanley (1983b) for references to various proofs of this theorem (the first rigorous one is due to W. Hodge). Note that the fact that $H^i(X)$ and $H^{2d-i}(X)$ are isomorphic is known already from Poincaré duality. Thus the point of the theorem is entirely the existence of a special cohomology class ω with such favorable multiplicative properties. Whereas Poincaré duality is a purely topological phenomenon (valid for all compact orientable manifolds, and in various versions also more generally), the Hard Lefschetz Theorem uses smoothness in an essential way. There is not (as far as is known) any intrinsically topological construction of a good cohomology class ω that would make Theorem 8.1 valid for some reasonable class of topological manifolds. Nevertheless, the Hard Lefschetz Theorem has been extended to some more general classes of varieties, e.g., to Kähler manifolds in differential topology and to V-varieties (nonsmooth varieties with finite quotient singularities, e.g., the toric varieties of simplicial polytopes discussed below).

Stanley's (1980a) proof of the Erdős–Moser conjecture is outlined in section 9 of chapter 32 by Alon. Referring to the discussion there, and using the same notation, we will now indicate how Theorem 8.1 is used.

For a certain poset $M(n)$ of rank $N = \binom{n+1}{2}$ and with rank-level sets $M(n)_i, i = 0, 1, \ldots, N$, let V_i be the real vector space with basis $M(n)_i$. For the proof it is needed to construct linear mappings $\varphi_i : V_i \to V_{i+1}$ such that the composition $\varphi_{N-i-1} \circ \varphi_{N-i-2} \circ \cdots \circ \varphi_i : V_i \to V_{N-i}$ is invertible, for $0 \leqslant i \leqslant [N/2]$, and if $x \in M(n)_i$ and $\varphi_i(x) = \sum_{y \in M(n)_{i+1}} c_y \cdot y$, then $c_y \neq 0$ implies $y > x$.

Take the special orthogonal group $G = \mathrm{SO}_{2n+1}(\mathbb{C})$ and let P be the maximal parabolic subgroup corresponding to the simply-laced part of its Dynkin diagram. Then G/P is a smooth irreducible complex projective variety having a cell decomposition (in a certain algebraic-geometric sense) such that the poset of closed cells is isomorphic to $M(n)$. This cell decomposition of G/P (induced by the Bruhat decomposition of G) has cells only in even dimensions, and we may identify $M(n)_i$ with the set of $2i$-dimensional cells and conclude that $V_i \cong H^{2i}(G/P)$. The relevance of Theorem 8.1 is now becoming clear; indeed, letting the linear mapping $\varphi_i : V_i \to V_{i+1}$ be multiplication with ω, all required properties turn out to hold.

The poset $M(n)$ is a member of a class of finite rank-symmetric posets arising as Bruhat order on Weyl groups and on their quotients modulo parabolic subgroups. Using Theorem 8.1, Stanley (1980a) showed that all such posets are rank-unimodal and satisfy a strong form of the Sperner property.

Many of the results of Stanley (1980a), including the proof of the Erdős–Moser conjecture, can be proven with just linear algebra, see Proctor (1982). This is done, essentially, by rewriting the first proof (including a proof of the Hard Lefschetz Theorem) as concretely as possible and throwing out all mention of algebraic geometry.

We now turn to the characterization of f-vectors of simplicial polytopes. This application of Theorem 8.1 uses more of its content. The fact that the linear mappings φ_i constructed above are given by multiplication is irrelevant for the previous argument, whereas the global multiplicative structure of $H^*(X)$ is essential in what follows.

We refer to chapter 18 by Klee and Kleinschmidt for definitions relating to simplicial d-polytopes P and their h-vectors $h(P) = (h_0, h_1, \ldots, h_d)$. As observed there, every simplicial polytope in $\mathbb{R}^d$ is combinatorially equivalent to one with vertices in $\mathbb{Q}^d$.

Let P be a d-dimensional convex polytope with vertices in $\mathbb{Q}^d$. There is a general construction (see Ewald 1995, Fulton 1993 or Oda 1988) which associates with P an irreducible complex projective variety $X(P)$ of complex dimension d, called a *toric variety*. This variety is in general not smooth, not even in the simplicial case.

Suppose now that P is simplicial. Then the following is true [work of V.I. Danilov, J. Jurkiewicz, M. Saito and others; see the cited books or Stanley (1983b, 1985, 1987a)]:

(i) the cohomology of $X(P)$ vanishes in all odd dimensions, and $\dim_{\mathbb{R}} H^{2i}(X(P)) = h_i(P)$, for $i = 0, 1, \ldots, d$.

(ii) $H^*(X(P))$ is generated (as an algebra over $\mathbb{R}$) by $H^2(X(P))$,

(iii) the Hard Lefschetz Theorem 8.1 holds for $X = X(P)$ and the class of a hyperplane section $\omega \in H^2(X)$.

It follows from (iii) that the mapping $H^{2i}(X) \to H^{2(i+1)}(X)$ given by multiplication with ω is injective if $i < d/2$ and surjective if $i \geqslant [d/2]$. Therefore, taking the quotient of the cohomology ring

$$H^*(X) = \oplus_{i=0}^{d} H^{2i}(X)$$

by the ideal generated by ω, we get a graded ring

$$R = H^*(X)/\langle\omega\rangle = \oplus_{i=0}^{[d/2]} R_i,$$

where $R_i = H^{2i}(X)/\omega H^{2i-2}(X)$, for $i \geqslant 1$, and $R_0 = H^0(X) \cong \mathbb{R}$. Furthermore, R is generated by R_1 [by (ii)], and $\dim_{\mathbb{R}} R_i = h_i - h_{i-1}$ [by (i) and (iii)]. This shows that $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ is an "$\mathcal{O}$-sequence", as defined in Theorem 6.2 of chapter 18 by Klee and Kleinschmidt. As explained after Theorem 6.5 of that chapter, this is precisely what needs to be shown to complete the proof of necessity of the characterization of f-vectors of simplicial polytopes.

A more elementary (and self-contained) proof of necessity has recently been found by McMullen (1993). He replaces the cohomology ring of the toric variety by a certain subalgebra of the polytope algebra and proves the needed analog of the Hard Lefschetz Theorem using convex geometry.

In Stanley (1987a) sharp lower bounds are given for the differences $h_i - h_{i-1}$, $1 \leqslant i \leqslant [d/2]$, for a centrally symmetric simplicial d-polytope. The proof involves the interaction between the Hard Lefschetz Theorem and a finite group action.

The toric variety $X = X(P)$ of a *non-simplicial* polytope P with rational vertices is unfortunately more difficult to use for combinatorial purposes. For instance, $\dim_{\mathbb{R}} H^i(X)$ may depend on the embedding of P and not only on its combinatorial type, and cohomology may fail to vanish in odd dimensions. However, the *intersection cohomology* (of middle perversity) $IH^*(X)$, defined by M. Goresky and R. MacPherson, turns out to be combinatorial and to satisfy a module version of the hard Lefschetz theorem. This leads to some interesting information for general

rational polytopes, such as Theorem 6.8 of chapter 18 by Klee and Kleinschmidt. See Stanley (1987b) for more information.

PART II. TOOLS

The rest of this chapter is devoted to a review of some definitions and results from combinatorial topology that have proven to be particularly useful in combinatorics. The material in sections 9 (simplicial complexes), 12 (cell complexes) and 13 (fixed-point and antipodality theorems) is of a very general nature and detailed treatments can be found in many topology books. Specific references will therefore be given only sporadically. Most topics in sections 10 and 11, on the other hand, are of a more special nature, and more substantial references (and even some proofs) will be given.

Many of the results mentioned have been discussed in a large number of papers and books. When relevant, our policy has been to reference the original source (when known to us) and some more recent papers that contribute simple proofs, extensions or up-to-date discussion (a subjective choice). We apologize for any inaccuracy or omission that may unintentionally have occurred.

9. Combinatorial topology

This section will review basic facts concerning simplicial complexes. Good general references are Munkres (1984a) and Spanier (1966). Basic notions such as (*topological*) *space*, *continuous map* and *homeomorphism* will be considered known. Throughout this chapter, every map between topological spaces is assumed to be continuous, even if not explicitly stated.

Simplicial complexes and posets

9.1. An *(abstract) simplicial complex* $\Delta = (V, \Delta)$ is a set V (the *vertex set*) together with a family Δ of nonempty finite subsets of V (called *simplices* or *faces*) such that $\emptyset \neq \sigma \subseteq \tau \in \Delta$ implies $\sigma \in \Delta$. Usually, $V = \bigcup \Delta$ (shorthand for $V = \bigcup_{\sigma \in \Delta} \sigma$) so V can be suppressed from the notation.

The *dimension* of a face σ is $\dim \sigma = \operatorname{card} \sigma - 1$, the *dimension* of Δ is $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$. A d-dimensional complex is *pure* if every face is contained in a d-*face* (i.e., d-dimensional face). The complex consisting of all nonempty subsets of a $(d+1)$-element set is called the d-*simplex*.

Note that our definition allows the empty complex $\Delta = \emptyset$. It is, by convention, (-1)-dimensional. [Remark: The definition of a simplicial complex (with *nonempty* faces) that we use here is the standard one in topology. In combinatorics it is usually more convenient to allow the empty set as a face of a complex; in particular, this is consistent with the definition of reduced homology.]

Let $\Delta^k = \{k\text{-faces of } \Delta\}$ and $\Delta^{\leqslant k} = \bigcup_{j \leqslant k} \Delta^j$, for $k \geqslant 0$. The elements of $\Delta^0 = V$ and Δ^1 are called *vertices* and *edges*, respectively. If Δ is pure d-dimensional the elements of Δ^d are called *facets* (or *chambers*). $\Delta^{\leqslant k}$ is the k-*skeleton* of Δ. It is a *subcomplex* of Δ.

A (*geometric*) *simplicial complex* is a polyhedral complex in $\mathbb{R}^d$ [in the sense of (12.1)] whose cells are geometric simplices (the convex hull of affinely independent point-sets). If Γ is a geometric simplicial complex then the family of extreme-point-sets of cells in Γ form an abstract simplicial complex $\Delta(\Gamma)$ which is finite. Conversely, if $\Delta \neq \emptyset$ is a d-dimensional finite abstract simplicial complex then there exist geometric simplicial complexes Γ in $\mathbb{R}^{2d+1}$ such that $\Delta(\Gamma) \cong \Delta$. The underlying space $\bigcup \Gamma$ of any such Γ, unique up to linear homeomorphism, is called the *geometric realization* (or *space*) of Δ, denoted by $\|\Delta\|$. Conversely, Δ is called a *triangulation* of the space $\|\Delta\|$, and of every space homeomorphic to it. Thus, abstract and geometric simplicial complexes are equivalent notions in the finite case (and more generally, when finite-dimensional, denumerable and locally finite). The geometric realization $\|\Delta\|$ of arbitrary infinite abstract simplicial complexes Δ can be constructed as in Spanier (1966).

A *simplicial map* $f: \Delta_1 \to \Delta_2$ is a mapping $f: \Delta_1^0 \to \Delta_2^0$ such that $f(\sigma) \in \Delta_2$ for all $\sigma \in \Delta_1$. By affine extension across simplices it induces a continuous map $\|f\|: \|\Delta_1\| \to \|\Delta_2\|$.

Whereas the rectilinear realization of all d-dimensional simplicial complexes in $\mathbb{R}^{2d+1}$ is easy to prove (and $2d+1$ is best possible), the existence in special cases of rectilinear and of topological realizations in spaces $\mathbb{R}^j$, for $d < j \leqslant 2d$, are difficult and much studied problems. For $d = 1$ this is the question of planarity of graphs (see chapter 5 by Thomassen), for rectilinear embeddings when $d \geqslant 2$, see, e.g., Bokowski and Sturmfels (1989) and the references found therein, and for topological embeddings see Rushing (1973). It is for instance not known whether every triangulation of the 2-dimensional torus has a rectilinear embedding into $\mathbb{R}^3$. A classical result concerning topological embeddings is the van Kampen–Flores Theorem (from 1932–33), which says that the d-skeleton of a $(2d+2)$-simplex does not embed into $\mathbb{R}^{2d}$. Sarkaria (1991b) gives an up-to-date discussion of this result in a setting which also includes the topological Radon–Tverberg theorems discussed in section 5, see also Sarkaria (1991a).

9.2. Let $P = (P, \leqslant)$ be a *poset* (partially ordered set). A totally ordered subset $x_0 < x_1 < \cdots < x_k$ is called a *chain* of *length* k. The supremum of this number over all chains in P is the *rank* (or *length*) of P. If all maximal chains have the same finite length then P is *pure*. P is a *lattice* if every pair of elements $x, y \in P$ has a least upper bound (*join*) $x \vee y$ and a greatest lower bound (*meet*) $x \wedge y$.

For $x \in P$, let $P_{\geqslant x}, P_{>x}, P_{<x}, P_{\leqslant x}$ be defined by $P_{\geqslant x} = \{y \in P : y \geqslant x\}$, etc. For $x \leqslant y$ define the *open interval* $(x, y) = P_{>x} \cap P_{<y}$ and the *closed interval* $[x, y] = P_{\geqslant x} \cap P_{\leqslant y}$. A *bottom element* $\hat{0}$ and a *top element* $\hat{1}$ in P are elements satisfying $\hat{0} \leqslant x$ (respectively $x \leqslant \hat{1}$) for all $x \in P$. If both $\hat{0}$ and $\hat{1}$ exist, P is *bounded*. Then

$\bar{P} = P - \{\hat{0}, \hat{1}\}$ denotes the *proper part* of P. For arbitrary poset P, $\hat{P} = P cup \{\hat{0}, \hat{1}\}$ denotes P extended by new top and bottom elements (so, card $(\hat{P} \backslash P) = 2$).

Let P be a pure poset of rank r. For $x \in P$, let $r(x) = \text{rank}(P_{\leqslant x})$. The *rank function* $r: P \to \{0, 1, \ldots, r\}$ is bijective on each maximal chain. It decomposes P into *rank levels* $P^i = \{x \in P: r(x) - i\}, 0 \leqslant i \leqslant r$.

9.3. The *face poset* $P(\Delta) = (\Delta, \subseteq)$ of a simplicial complex Δ is the set of faces ordered by inclusion. The *face lattice* of Δ is $\hat{P}(\Delta) = P(\Delta) \cup \{\hat{0}, \hat{1}\}$. It is a lattice. $P(\Delta)$ is pure iff Δ is pure, and rank $P(\Delta) = \dim \Delta$.

The *order complex* $\Delta(P)$ of a poset P is the simplicial complex on vertex set P whose k-faces are the k-chains $x_0 < x_1 < \cdots < x_k$ in P. A poset map $f: P_1 \to P_2$ which is *order-preserving* [$x \leqslant y$ implies $f(x) \leqslant f(y)$] or *order-reversing* [$x \leqslant y$ implies $f(x) \geqslant f(y)$] is simplicial $f: \Delta(P_1) \to \Delta(P_2)$, and therefore induces a continuous map $\|f\|: \|\Delta(P_1)\| \to \|\Delta(P_2)\|$. The definition of $\Delta(P)$ goes back to Aleksandrov (1937).

For a simplicial complex Δ, $\text{sd}\Delta = \Delta(P(\Delta))$, is called the *(first) barycentric subdivision* (due to its geometric version). A basic fact is that Δ and $\text{sd}\Delta$ are homeomorphic. Therefore, passage between simplicial complexes and posets via the mappings $P(\cdot)$ and $\Delta(\cdot)$ does not affect the topology, and from a topological point of view simplicial complexes and posets can be considered to be essentially equivalent notions.

The geometric realization $\|P\| = \|\Delta(P)\|$ associates a topological space with every poset P. In this chapter, whenever we make topological statements about a poset P we have the space $\|P\|$ in mind.

There exists at least one other way of associating a useful topology with a poset P (also due to Aleksandrov 1937), namely, let the *order-ideals* (subsets $A \subset P$ satisfying $x \leqslant y \in A$ implies $x \in A$) be the open sets of a topology on P. Denote this space $T(P)$. For instance, for the poset depicted to the right in fig. 2 (section 12), $T(P)$ is a space with exactly ten open sets, whereas $\Delta(P)$ is homeomorphic to the 2-sphere. For the *ideal topology* $T(\cdot)$ the continuous maps are precisely the order-preserving maps and homotopy [see (9.10)] has a direct combinatorial meaning. For instance, $T(P)$ is contractible iff P is dismantlable in the sense of (11.1); see Stong (1966). The ideal topology $T(P)$ is relevant for sheaf cohomology over posets (Baclawski 1975, Yuzvinsky 1987) and has surprising connections with the order complex topology $\Delta(P)$ (McCord 1966).

9.4. Let T be a topological space, $\approx$ an equivalence relation on T, and $\pi: T \to T/\approx$ the projection map. The quotient $T/\approx$ is made into a topological space by letting $A \subseteq T/\approx$ be open iff $\pi^{-1}(A)$ is open in T. If $S_i, i \in I$, are pairwise disjoint subsets of T, then $T/(S_i)_{i \in I}$ denotes the *quotient space* obtained by identifying the points within each set $S_i, i \in I$. For example, $\text{cone}(T) = T \times [0,1]/(T \times \{1\})$ is the *cone over* T, and $\text{susp}(T) = T \times [0,1]/(T \times \{0\}, T \times \{1\})$ is the *suspension* of T. The d-ball modulo its boundary is homeomorphic to the d-sphere: $B^d/S^{d-1} \cong S^d$.

If $(T_i, x_i)_{i \in I}, x_i \in T_i$, is a family of pointed pairwise-disjoint spaces, then the *wedge* of this family is $\bigcup_{i \in I} T_i / (\bigcup_{i \in I} \{x_i\})$. The *join* of two spaces T_1 and T_2 is the space $T_1 * T_2 = T_1 \times T_2 \times [0,1] / (\{(t, x, 0) \mid x \in T_2\}, \{(y, s, 1) \mid y \in T_1\})_{t \in I_1, s \in I_2}$.

The *join* of two simplicial complexes Δ_1 and Δ_2 (with $\Delta_1^0 \cap \Delta_2^0 = \emptyset$) is the complex $\Delta_1 * \Delta_2 = \Delta_1 \cup \Delta_2 \cup \{\sigma \cup \tau \mid \sigma \in \Delta_1 \text{ and } \tau \in \Delta_2\}$. Further, the *cone* over Δ and *suspension* of Δ are the complexes $\text{cone}(\Delta) = \Delta * \Gamma_1$, $\text{susp}(\Delta) = \Delta * \Gamma_2$, where Γ_i is the 0-dimensional complex with i vertices, $i = 1, 2$. There is a homeomorphism

$$\|\Delta_1 * \Delta_2\| \cong \|\Delta_1\| * \|\Delta_2\|. \tag{9.5}$$

[In case Δ_1 and Δ_2 are not locally finite the topology of the right-hand side may need to be modified to the associated compactly generated topology, see Walker (1988).] In particular, $\|\text{cone}(\Delta)\| \cong \text{cone}(\|\Delta\|)$ and $\|\text{susp}(\Delta)\| \cong \text{susp}(\|\Delta\|)$.

The join of two complexes Δ_1 and Δ_2 has the following geometric realization. First realize Δ_1 and Δ_2 in the same space $\mathbb{R}^d$, with d sufficiently large, so that two distinct line segments $[x_1, x_2]$ and $[y_1, y_2]$ with $x_1, y_1 \in \|\Delta_1\|$ and $x_2, y_2 \in \|\Delta_2\|$ never intersect in an interior point. Then take the union of all such line segments (with the topology induced as a subspace of $\mathbb{R}^d$) – this gives $\|\Delta_1 * \Delta_2\|$.

The *p-fold deleted join* $\Delta_*^{(p)}$ of a simplicial complex Δ is defined as follows. Let $\Delta_1, \ldots, \Delta_p$ be disjoint copies of Δ with isomorphisms $f_i: \Delta_i \to \Delta$. Then $\Delta_*^{(p)}$ is the subcomplex of $\Delta_1 * \cdots * \Delta_p$ consisting of all faces $\sigma_1 \cup \cdots \cup \sigma_p$ such that $f_i(\sigma_i) \cap f_j(\sigma_j) = \emptyset$ for all $i \neq j$. For combinatorial uses of this construction see Sarkaria (1990, 1991a,b) and Živaljević and Vrećica (1992).

The *direct product* $P \times Q$ of two posets is the Cartesian product set ordered by $(x, y) \leqslant (x', y')$ if $x \leqslant x'$ in P and $y \leqslant y'$ in Q. The *join* (or *ordinal sum*) $P * Q$ of two posets is their disjoint union ordered by making each element of P less than each element of Q and otherwise keeping the given orderings within P and Q. Clearly, $\Delta(P * Q) = \Delta(P) * \Delta(Q)$.

There are the following homeomorphisms (Quillen 1978, Walker 1988):

$$\|P \times Q\| \cong \|P\| \times \|Q\|, \tag{9.6}$$

$$\|(P \times Q)_{< (x,y)}\| \cong \|P_{<x}\| * \|Q_{<y}\|, \tag{9.7}$$

$$\|((x, y), (x', y'))\| \cong \text{susp}(\|(x, x')\| * \|(y, y')\|),$$
$$\text{if } x < x' \text{ in } P \text{ and } y < y' \text{ in } Q. \tag{9.8}$$

(Again, special care has to be taken with the topology of the right-hand sides if the participating order complexes are not locally finite.)

9.9. Let Δ be a simplicial complex and $\sigma \in \Delta \cup \{\emptyset\}$. Then define the subcomplexes: *deletion* $\text{dl}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset\}$, *star* $\text{st}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\}$ and *link* $\text{lk}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$. Clearly, $\text{dl}(\sigma) \cap \text{st}(\sigma) = \text{lk}(\sigma)$ and $\sigma * \text{lk}(\sigma) = \text{st}(\sigma)$. If $\sigma \in \Delta^0$ then also $\text{dl}(\sigma) \cup \text{st}(\sigma) = \Delta$; and $\text{dl}(\emptyset) = \text{st}(\emptyset) = \text{lk}(\emptyset) = \Delta$.

Homotopy and homology

9.10. Two mappings $f_0, f_1 : T_1 \to T_2$ of topological spaces are *homotopic* (written $f_0 \sim f_1$) if there exists a mapping (called a *homotopy*) $F : T_1 \times [0,1] \to T_2$ such that $F(t,0) = f_0(t)$ and $F(t,1) = f_1(t)$ for all $t \in T_1$. (Remember that all mappings between topological spaces are assumed to be continuous.) The spaces T_1 and T_2 are *of the same homotopy type* (or *are homotopy equivalent*) if there exist mappings $f_1 : T_1 \to T_2$ and $f_2 : T_2 \to T_1$ such that $f_2 \circ f_1 \sim \mathrm{id}_{T_1}$ and $f_1 \circ f_2 \sim \mathrm{id}_{T_2}$. Denote this by $T_1 \simeq T_2$. A space which is homotopy equivalent to a point is called *contractible*.

Let $S^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ and $B^d = \{x \in \mathbb{R}^d \mid \|x\| < 1\}$ denote the standard $(d-1)$-*sphere* and d-*ball*, respectively. Note that $S^{-1} = \emptyset$, $S^0 = \{\text{two points}\}$ and $B^0 = \{\text{point}\}$. The class of spheres and balls is closed under the operation of taking joins (up to homeomorphism): $S^a * S^b \cong S^{a+b+1}$, $B^a * B^b \cong B^a * S^b \cong B^{a+b+1}$.

A space T is *k-connected* if for all $0 \leqslant i \leqslant k$ each mapping $f : S^i \to T$ can be extended to a mapping $\hat{f} : B^{i+1} \to T$ such that $\hat{f}(x) = f(x)$ for all $x \in S^i$. In particular, 0-connected means arcwise connected. The property of being k-connected is a *homotopy invariant* (i.e., is transferred to other spaces of the same homotopy type). S^d is $(d-1)$-connected but not d-connected (see Theorem 13.1), B^d is contractible. It is convenient to define the following degenerate cases: (-1)-*connected* means "nonempty", and every space (whether empty or not) is *k-connected* for $k \leqslant -2$.

A simplicial complex Δ is contractible iff Δ is k-connected for all $k \geqslant 0$ (or equivalently, for all $0 \leqslant k < \dim \Delta$). (The corresponding statement for general spaces is false in the nontrivial direction.) Furthermore, a simplicial complex is k-connected iff its $(k+1)$-skeleton is k-connected.

Let $\pi_i(T) = \pi_i(T, x)$ denote the set of homotopy classes of maps $f : S^i \to T$ such that $f((1,0,\ldots,0)) = x$, from the pointed i-sphere to a pointed topological space $(T, x), x \in T$, $i \geqslant 0$. For $i \geqslant 1$ there exists a composition that makes $\pi_i(T)$ into a group, the *ith homotopy group* of T (at the point x). For $i \geqslant 2$, the group $\pi_i(T)$ is Abelian. $\pi_1(T)$ is the *fundamental group*, and T is *simply connected* if $\pi_1(T) = 0$. The space T is k-connected iff $\pi_i(T, x) = 0$ for all $0 \leqslant i \leqslant k$ and $x \in T$. So, 1-connected means simply connected and arcwise connected.

9.11. For the definitions of *simplicial homology groups* $H_i(\Delta, G)$ and *reduced simplicial homology groups* $\tilde{H}_i(\Delta, G)$ of a complex Δ with coefficients in an Abelian group G, we refer to Munkres (1984a) or Spanier (1966).

Let $\tilde{H}_i(\Delta) = \tilde{H}_i(\Delta, \mathbb{Z})$. The degenerate case

$$\tilde{H}_i(\emptyset) \cong \begin{cases} \mathbb{Z}, & i = -1, \\ 0, & i \neq -1, \end{cases}$$

should be noted. For $\Delta \neq \emptyset$, $\tilde{H}_i(\Delta) = 0$ for all $i < 0$ and all $i > \dim \Delta$, and $\tilde{H}_0(\Delta) \cong \mathbb{Z}^{c-1}$, where c is the number of connected components of Δ. $H_i(\Delta) = \tilde{H}_i(\Delta)$ for all $i \neq -1, 0$; $H_{-1}(\Delta) = 0$ and $H_0(\Delta) \cong \tilde{H}_0(\Delta) \oplus \mathbb{Z}$.

Let Δ_1 and Δ_2 be finite complexes and assume that at least one of $\tilde{H}_p(\Delta_1)$ and $\tilde{H}_q(\Delta_2)$ is torsion-free when $p + q = i - 1$. Then

$$\tilde{H}_{i+1}(\Delta_1 * \Delta_2) \cong \bigoplus_{p+q=i} (\tilde{H}_p(\Delta_1) \otimes \tilde{H}_q(\Delta_2)). \tag{9.12}$$

The same decomposition holds (without any restriction) for reduced homology with coefficients in a field. See Milnor (1956) or chapter V of Cooke and Finney (1967) for further details.

For a finite simplicial complex Δ let $\beta_i = \operatorname{rank} H_i(\Delta) = \dim_{\mathbb{Q}} H_i(\Delta, \mathbb{Q})$, $i \geqslant 0$. The *Betti numbers* β_i satisfy the *Euler–Poincaré formula*

$$\sum_{i \geqslant 0} (-1)^i \operatorname{card}(\Delta^i) = \sum_{i \geqslant 0} (-1)^i \beta_i. \tag{9.13}$$

Either side of (9.13) can be taken as the definition of the *Euler characteristic* $\chi(\Delta)$. The *reduced Euler characteristic* is $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$. Formula (9.13) is valid with $\beta_i = \dim_{\boldsymbol{k}} H_i(\Delta, \boldsymbol{k})$ for an arbitrary field $\boldsymbol{k}$, although the individual integers β_i may depend on $\boldsymbol{k}$. Additional relations exist between the face-count numbers $f_i = \operatorname{card}(\Delta^i)$ and the Betti numbers β_i (Björner and Kalai 1988). Much is known about the *f-vectors* $f(\Delta) = (f_0, f_1, \ldots)$ for various special classes of complexes Δ. See chapter 18 by Klee and Kleinschmidt for the important case of polytope boundaries, and Björner and Kalai (1989) for a survey devoted to more general classes of complexes.

The Möbius function of a (locally) finite poset is defined in chapter 21 by Gessel and Stanley. Theorem 13.4 of that chapter (due to P. Hall) can in view of (9.13) be restated as

$$\mu(x, y) = \tilde{\chi}(\Delta((x, y))), \quad \text{if } x < y, \tag{9.14}$$

where the right-hand side denotes the reduced Euler characteristic of the order complex of the open interval (x, y). This connection between the Möbius function and topology, first pointed out by Rota (1964) and Folkman (1966), has many interesting ramifications.

9.15. Two complexes of the same homotopy type have isomorphic homology groups in all dimensions. A complex Δ is *k-acyclic over G* if $\tilde{H}_i(\Delta, G) = 0$ for all $i \leqslant k$. So, (-1)-*acyclic* means nonempty and 0-*acyclic* means nonempty and connected. Further, Δ is *acyclic* over G (or simply "G-acyclic" if confusion cannot arise) if $\tilde{H}_i(\Delta, G) = 0$ for all $i \in \mathbb{Z}$. When G is suppressed from the notation we always mean $G = \mathbb{Z}$.

We now list some relations between homotopy properties and homology of a complex Δ, which are frequently useful. They are consequences of the theorems of Hurewicz and Whitehead (see Spanier 1966).

9.16. Δ is k-connected iff Δ is k-acyclic (over $\mathbb{Z}$) and simply connected, $k \geqslant 1$.

9.17. Δ is contractible iff Δ is $\mathbb{Z}$-acyclic and simply connected.

9.18. If Δ is simply connected, $\tilde{H}_i(\Delta) = 0$ for $i \neq d > 1$, and $\tilde{H}_d(\Delta) \simeq \mathbb{Z}^k$, then Δ is homotopy equivalent to a wedge of k d-spheres.

9.19. Assume $\dim \Delta = d \geq 0$. Then Δ is $(d-1)$-connected iff Δ is homotopy equivalent to a wedge of d-spheres.

[Remark: The analogues of (9.17)–(9.19) may fail for non-triangulable spaces.]

9.20. If Δ_1 is k_1-acyclic and Δ_2 is k_2-acyclic then $\Delta_1 * \Delta_2$ is $(k_1 + k_2 + 2)$-acyclic. This follows from (9.12). Using (9.16) it implies that if Δ_i is k_i-connected then $\Delta_1 * \Delta_2$ is $(k_1 + k_2 + 2)$-connected. (For this, see also Milnor 1956.)

10. Combinatorial homotopy theorems

In this section we collect some tools for manipulating homotopies and the homotopy type of complexes and posets, which have proven to be useful in combinatorics. Parallel tools for homology exist in most cases. We begin with some elementary lemmas.

Suppose Δ is a simplicial complex and T a space. Let $C : \Delta \to 2^T$ be order-preserving (i.e., $C(\sigma) \subseteq C(\tau) \subseteq T$, for all $\sigma \subseteq \tau$ in Δ). A mapping $f : \|\Delta\| \to T$ is *carried* by C if $f(\|\sigma\|) \subseteq C(\sigma)$ for all $\sigma \in \Delta$. Let $k \in \mathbb{Z}_+ \cup \{\infty\}$.

Lemma 10.1 (Carrier Lemma). *Assume that $C(\sigma)$ is $\min(k, \dim(\sigma))$-connected for all $\sigma \in \Delta$. Then:*

(i) *if $f, g : \|\Delta^{\leq k}\| \to T$ are both carried by C, then $f \sim g$,*

(ii) *there exists a mapping $\|\Delta^{\leq k+1}\| \to T$ carried by C.*

In particular, if $C(\sigma)$ is always contractible then $\|\Delta\|$ can replace the skeleta in (i) and (ii) ($k = \infty$ case). Carrier lemmas of various kinds are common in topology. For proofs of this version, see Lundell and Weingram (1969) or Walker (1981b).

Lemma 10.2 (Contractible Subcomplex Lemma). *If Δ_0 is a contractible subcomplex of a simplicial complex Δ, then the projection map $\|\Delta\| \to \|\Delta\|/\|\Delta_0\|$ is a homotopy equivalence.*

This is a consequence of the homotopy extension property for simplicial pairs [for more details see Brown (1968) or Björner and Walker (1983)].

Lemma 10.3 (Gluing Lemma). *Examples of simple gluing results for simplicial complexes Δ_1 and Δ_2 are:*

(i) if Δ_1 and Δ_2 are contractible, then $\Delta_1 \cup \Delta_2 \simeq \Delta_2$,

(ii) if Δ_1 and Δ_2 are k-connected and $\Delta_1 \cap \Delta_2$ is $(k-1)$-connected, then $\Delta_1 \cup \Delta_2$ is k-connected,

(iii) if Δ_1 and Δ_2 are k-connected, then so are also Δ_1 and Δ_2.

Such results are often special cases of the theorems in this section, especially Theorem 10.6. Otherwise they can be deduced from the Mayer–Vietoris long exact sequence (for k-acyclicity) and the Seifert–van Kampen theorem (for simply-connectedness), using (9.16) and (9.17).

A general principle for gluing homotopies appears in Brown (1968, p. 240) and Mather (1966). It gives a convenient proof for part (i) of the following lemma. For part (ii) use Lemma 10.2. A more general method for gluing homotopies (the "diagrams of spaces" technique) appears in Ziegler and Živaljević (1993).

Lemma 10.4. *Let $\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_n$ be a simplicial complex with subcomplexes Δ_i, and assume that $\Delta_i \cap \Delta_j \subseteq \Delta_0$ for all $1 \leq i < j \leq n$.*

(i) *If Δ_i is contractible for all $1 \leq i \leq n$, then*

$$\Delta \simeq \Delta_0 \cup \bigcup_{i=1}^{n} \operatorname{cone}(\Delta_0 \cap \Delta_i)$$

(i.e., raise a cone independently over each subcomplex $\Delta_0 \cap \Delta_i$).

(ii) *If Δ_i is contractible for all $0 \leq i \leq n$, then*

$$\Delta \simeq \operatorname{wedge}_{1 \leq i \leq n} \operatorname{susp}(\Delta_0 \cap \Delta_i).$$

Some of the following results concern *simplicial* maps $f : \Delta \to P$ from a simplicial complex Δ to a poset P. Such a map sends vertices of Δ to elements of P in such a way that each $\sigma \in \Delta$ is mapped to a chain in P. In particular, an order-preserving or order-reversing mapping of posets $Q \to P$ is of this type.

Theorem 10.5 (Fiber Theorem, Quillen 1978, Walker 1981b). *Let $f : \Delta \to P$ be a simplicial map from a simplicial complex Δ to a poset P.*

(i) *Suppose all fibers $f^{-1}(P_{\geq x}), x \in P$, are contractible. Then f induces homotopy equivalence between Δ and P.*

(ii) *Suppose all fibers $f^{-1}(P_{\geq x}), x \in P$, are k-connected. Then Δ is k-connected if and only if P is k-connected.*

Proof. Suppose that all fibers are contractible. Then the mapping $C(\sigma) = f^{-1}(P_{\geq \min \sigma})$, $\sigma \in \Delta(P)$, is a contractible carrier from $\Delta(P)$ to $\|\Delta\|$. By Lemma 10.1 (ii) there exists a continuous map $g : \Delta(P) \to \Delta$ carried by C, i.e., $g(\|\sigma\|) \subseteq \|f^{-1}(P_{\geq \min \sigma})\|$, for every chain $\sigma \in \Delta(P)$. One sees that g is a homotopy inverse to f as follows, using Lemma 10.1 (i): $C'(\sigma) = \|P_{\geq \min \sigma}\|$, $\sigma \in \Delta(P)$, is contractible and carries $f \circ g$ and id_P, and $C''(\pi) = \|f^{-1}(P_{\geq \min f(\pi)})\|$, $\pi \in \Delta$, is contractible and carries $g \circ f$ and id_Δ. Hence, $f \circ g \sim \mathrm{id}_P$ and $g \circ f \sim \mathrm{id}_\Delta$.

The second part is proved analogously by passing to $(k+1)$-skeleta and using k-connected carriers in Lemma 10.1. □

The *nerve* of a family of sets $(A_i)_{i \in I}$ is the simplicial complex $\mathcal{N} = \mathcal{N}(A_i)$ defined on the vertex set I so that a finite subset $\sigma \subseteq I$ is in $\mathcal{N}$ precisely when $\cap_{i \in \sigma} A_i \neq \emptyset$.

Theorem 10.6 (Nerve Theorem, Borsuk 1948, Björner et al. 1985, 1994). *Let Δ be a simplicial complex (or, a regular cell complex) and $(\Delta_i)_{i\in I}$ a family of subcomplexes such that $\Delta = \bigcup_{i\in I} \Delta_i$.*

(i) *Suppose every nonempty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \cdots \cap \Delta_{i_t}$ is contractible. Then Δ and the nerve $\mathcal{N}(\Delta_i)$ are homotopy equivalent.*

(ii) *Suppose every nonempty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \cdots \cap \Delta_{i_t}$ is $(k-t+1)$-connected. Then Δ is k-connected if and only if $\mathcal{N}(\Delta_i)$ is k-connected.*

Proof. For convenience, assume that the covering of Δ by the Δ_i's is *locally finite*, meaning that each vertex of Δ belongs to only finitely many subcomplexes Δ_i. (The case of more general coverings requires a slightly different argument.)

Let $Q = P(\Delta)$ and $P = P(\mathcal{N}(\Delta_i))$ be the face posets. Define a mapping $f: Q \to P$ by $\pi \mapsto \{i \in I \mid \pi \in \Delta_i\}$. Clearly f is order-reversing, so $f: \Delta(Q) \to P$ is simplicial. The fiber at $\sigma \in P$ is $f^{-1}(P_{\geqslant\sigma}) = \bigcap_{i\in\sigma} \Delta_i$. Part (i) now follows from Theorem 10.5. Also, if all nonempty finite intersections are k-connected, part (ii) follows the same way. In the stated generality, part (ii) is proved in Björner et al. (1994). □

The Nerve Theorem has several versions for coverings of a topological space by subspaces. The earliest of these seem to be due to Leray (1945) and Weil (1952). Discussions of results of this kind can be found in Wu (1962) and McCord (1967). We state here a version which seems suitable for use in combinatorics. An application to oriented matroids appears in Edelman (1984).

Theorem 10.7 (Nerve Theorem, Weil 1952, Wu 1962, McCord 1967). *Let X be a triangulable space and $(A_i)_{i\in I}$ a locally finite family of open subsets (or a finite family of closed subsets) such that $X = \bigcup_{i\in I} A_i$. If every nonempty intersection $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_t}$ is contractible, then X and the nerve $\mathcal{N}(A_i)$ are homotopy equivalent.*

By *locally finite* is meant that each point of X lies in at most finitely many sets A_i. We warn that Theorem 10.7 is false for locally finite coverings by *closed* sets and also for too general spaces X. For a counterexample in the first case, take X to be the unit circle and $A_i = \{e^{2\pi i t} \mid 1/(i+1) \leqslant t \leqslant 1/i\}, i = 1, 2, \ldots$. In the second case one can, e.g., let X be the wedge of two topologist's combs A_1 and A_2 [as in Spanier (1966, Ex. 5, p. 56)].

The conclusions in part (ii) of Theorems 10.5 and 10.6 can be strengthened: In Theorem 10.5, if all fibers are k-connected, then f induces isomorphisms of homotopy groups $\pi_j(\Delta) \cong \pi_j(P)$, for all $j \leqslant k$. Consequently, if in Theorem 10.6 all nonempty finite intersections $\Delta_{i_1} \cap \Delta_{i_2} \cap \cdots \cap \Delta_{i_t}$ are k-connected, then $\pi_j(\Delta) \cong \pi_j(\mathcal{N}(\Delta_i))$, for all $j \leqslant k$. A similar k-connectivity version of Theorem 10.7 appears in Wu (1962).

Let P be a poset. A subset $C \subseteq P$ is called a *crosscut* if (1) C is an antichain, (2) for every finite chain σ in P there exists some element in C which is comparable to each element in σ, (3) if $A \subseteq C$ is *bounded* (here meaning that A has an upper bound or a lower bound in P) then the join $\vee A$ or the meet $\wedge A$ exists in P. For instance, the atoms of a lattice L of finite length form a crosscut in L and in $\bar{L}$.

A crosscut C in P determines the simplicial complex $\Gamma(P, C)$ consisting of the bounded subsets of C.

Theorem 10.8 (Crosscut Theorem, Rota 1964, Folkman 1966, Björner 1981). *The crosscut complex $\Gamma(P, C)$ and P are homotopy equivalent.*

Proof. For $x \in C$, let $\Delta_x = \Delta(P_{\leqslant x} \cup P_{\geqslant x})$. Then $(\Delta_x)_{x\in C}$ is a covering of $\Delta(P)$, by condition (2), and every nonempty intersection is a cone, by condition (3), and hence contractible. Since $\Gamma(P, C) = \mathcal{N}(\Delta_x)$, Theorem 10.6 implies the result. □

The neighborhood complex of a graph defined in section 4 is a special kind of nerve complex. The following result gives a special decomposition property of neighborhood complexes of bipartite graphs.

Theorem 10.9 (Bipartite Relation Theorem, Dowker 1952, Mather 1966). *Suppose $G = (V_0, V_1, E)$, $E \subseteq V_0 \times V_1$, is a bipartite graph, and let $\Delta_i, i = 0, 1$, be the simplicial complex whose faces are all finite subsets $\sigma \subseteq V_i$ that have a common neighbor in V_{1-i}. Then Δ_0 and Δ_1 are homotopy equivalent.*

Proof. First delete any isolated vertices from G. This does not affect Δ_0 and Δ_1. Now, for every $x \in V_1$ let Δ_x consist of all finite subsets of $\{y \in V_0 \mid (y, x) \in E\}$. Then $(\Delta_x)_{x\in V_1}$ is a covering of Δ_0 with contractible nonempty intersections. The nerve of this covering is Δ_1, so Theorem 10.6 applies. □

Theorems 10.6 (i), 10.8 and 10.9 are equivalent in the sense that either one implies the other two. The following is a variation of the Fiber Theorem 10.5.

Theorem 10.10 (Ideal Relation Theorem, Quillen 1978). *Let P and Q be posets and suppose that $R \subseteq P \times Q$ is a relation such that $(x, y) \leqslant (x', y') \in R$ implies that $(x, y) \in R$. (That is, R is an order ideal in the product poset.) Suppose furthermore that $R_x = \{y \in Q \mid (x, y) \subset R\}$ and $R_y = \{x \in P \mid (x, y) \in R\}$ are contractible for all $x \in P$ and $y \in Q$. Then P and Q are homotopy equivalent.*

Proof. By symmetry it suffices to show that P and R are homotopy equivalent. By Theorem 10.5 it suffices for this to show that the fiber $\pi^{-1}(P_{\geqslant x})$ is contractible for all $x \in P$, where $\pi: R \to P$ is the projection map $\pi(x, y) = x$. Let $F_x = \pi^{-1}(P_{\geqslant x}) = \{(z, y) \in R \mid z \geqslant x\}$, and let $\rho: F_x \to R_x$ be the projection $\rho(z, y) = y$. Now, $\rho^{-1}((R_x)_{\geqslant y}) = \{(z, w) \in F_x \mid w \geqslant y\} = \{(z, w) \in R \mid (z, w) \geqslant (x, y)\}$ is a cone and hence contractible, for all $y \in R_x$. So by the Fiber Theorem F_x is homotopy equivalent to R_x, which by assumption is contractible. (Remark: There is also an obvious k-connectivity version of this result.) □

Theorem 10.11 (Order Homotopy Theorem, Quillen 1978). *Let $f, g: \Delta \to P$ be simplicial maps from a simplicial complex Δ to a poset P. If $f(x) \leqslant g(x)$ for every vertex x of Δ, then f and g are homotopic.*

Proof. For each face $\sigma \in \Delta$, let $C(\sigma) = f(\sigma) \cup g(\sigma)$. The minimal element in the chain $f(\sigma)$ is below every other element in $C(\sigma)$. So the order complex of $C(\sigma)$ is a cone, and hence contractible. Since C carries both f and g, these maps are homotopic by Lemma 10.1. □

Corollary 10.12. *Let $f: P \to P$ be an order-preserving map such that $f(x) \geqslant x$ for all $x \in P$. Then f induces homotopy equivalence between P and $f(P)$.*

If also $f^2(x) = f(x)$ for all $x \in P$ (f is then called a *closure operator* on P) then $f(P)$ is a strong deformation retract of P. The hypotheses of Theorem 10.11 and Corollary 10.12 can be weakened to that $f(x)$ and $g(x)$ [resp., $f(x)$ and x] are comparable for all x.

Call a poset P *join-contractible (via p)*, if for some element $p \in P$ the join (least upper bound) $p \vee x$ exists for all $x \in P$. Define *meet-contractible* in dual fashion.

Corollary 10.13 (Quillen 1978). *If P is join-contractible then P is contractible.*

Proof. Since $x \leqslant p \vee x \geqslant p$, for all $x \in P$, Theorem 10.11 shows that $\mathrm{id} \sim p \vee \mathrm{id} \sim p$, i.e., the identity map on P is homotopic to the constant map p. □

The following is a consequence of Corollary 10.12, and also of Theorem 10.8.

Corollary 10.14. *Let L be a lattice of finite length and A the set of its atoms. Let $J = \{\vee B \mid B \subseteq A\}$. Then $\bar{L}$ and $\bar{L} \cap J$ are homotopy equivalent.*

Proof. The mapping $f(x) = \vee(A \cap L_{\leqslant x})$ satisfies $f^2(x) = f(x) \leqslant x$ for all $x \in \bar{L}$. Now use Corollary 10.12. □

The set of complements $\mathcal{C}\mathrm{o}(z)$ of an element z in a bounded lattice L is defined in section 3. Recall that $\bar{L} = L - \{\hat{0}, \hat{1}\}$.

Theorem 10.15 (Homotopy Complementation Theorem, Björner and Walker 1983). *Let L be a bounded lattice and $z \in \bar{L}$.*

(i) *The poset $\bar{L} - \mathcal{C}\mathrm{o}(z)$ is contractible. In particular, if L is noncomplemented then $\bar{L}$ is contractible.*

(ii) *If $\mathcal{C}\mathrm{o}(z)$ is an antichain, then*

$$\bar{L} \simeq \operatorname*{wedge}_{y \in \mathcal{C}\mathrm{o}(z)} \operatorname{susp}(\bar{L}_{<y} * \bar{L}_{>y}).$$

Proof. For each chain σ in $P = \bar{L} - \mathcal{C}\mathrm{o}(z)$, let $C(\sigma) = \{x \in P \mid x \geqslant z\} \cup \{y \in P \mid y \leqslant \max \sigma\}$. Either $z \vee \max \sigma$ exists in P, in which case $C(\sigma)$ is meet-contractible via it, or else $z \wedge \max \sigma$ exists, and $C(\sigma)$ is join-contractible via it. So, C is contractible and carries the constant map z as well as id_P. Therefore by Lemma 10.1 $z \sim \mathrm{id}_P$, which proves part (i). Part (ii) then follows by Lemma 10.4 (ii). □

Suppose that L is a bounded lattice whose proper part is not contractible. Then by part (i) every element x has a complement in L. This conclusion can be strengthened in the following way: [Lovász and Schrijver (unpublished)] Every chain $x_0 < x_1 < \cdots < x_k$ in $\bar{L}$ has a complementing chain $y_0 \geqslant y_1 \geqslant \cdots \geqslant y_k$ (i.e., $x_i \perp y_i$ for $0 \leqslant i \leqslant k$). Here one can even demand that each complement y_i is a join of atoms (assuming that atoms exist, which is the case, e.g., if L is of finite length).

A more general poset version of Theorem 10.15 is given in Björner (1994b). There the antichain assumption is dropped from part (ii) at the price of a more complicated description of the right-hand side as a quotient space of a wedge indexed by pairs $x \leqslant y$ in $\mathcal{C}\mathrm{o}(z)$.

11. Complexes with special structure

Some special properties of complexes that are frequently encountered in combinatorics, and which express a certain simplicity of structure, will be reviewed.

Collapsible and shellable complexes

11.1. Let Δ be a simplicial complex, and suppose that $\sigma \in \Delta$ is a proper face of exactly one simplex $\tau \in \Delta$. Then the complex $\Delta' = \Delta \backslash \{\sigma, \tau\}$ is obtained from Δ by an *elementary collapse* (and Δ is obtained from Δ' by an *elementary anticollapse*). Note that $\Delta' \simeq \Delta$. If Δ can be reduced to a single point by a sequence of elementary collapse steps, then Δ is *collapsible*.

The class of *nonevasive* complexes is recursively defined as follows: (i) a single vertex is nonevasive, (ii) if for some $x \in \Delta^0$ both $\mathrm{lk}_\Delta(x)$ and $\mathrm{dl}_\Delta(x)$ are nonevasive, then so is Δ.

The following logical implications are strict (i.e., converses are false):

$$\text{cone} \Longrightarrow \text{nonevasive} \longrightarrow \text{collapsible} \Longrightarrow \text{contractible} \Longrightarrow \mathbb{Z}\text{-acyclic}.$$

Furthermore, for an arbitrary field $\boldsymbol{k}$:

$$\mathbb{Z}\text{-acyclic} \Longrightarrow \boldsymbol{k}\text{-acyclic} \Longrightarrow \mathbb{Q}\text{-acyclic} \Longrightarrow \tilde{\chi} = 0,$$

and $\mathbb{Z}$-acyclic $\Longleftrightarrow \mathbb{Z}_p$-acyclic for all prime numbers p.

Nonevasive complexes were defined by Kahn et al. (1984) to model the notion of argument complexity discussed in section 2. A complex Δ is nonevasive iff for all $F \subseteq \Delta^0$ it is possible in less than card Δ^0 questions of the type "Is $x \in F$?" to decide whether $F \in \Delta$.

Collapsibility has long been studied in combinatorial topology. Noteworthy is the fact that two simply connected finite complexes Δ and Δ' are homotopy equivalent iff a sequence of elementary collapses and elementary anticollapses can transform Δ into Δ' (see Cohen 1973). In particular, the contractible complexes are precisely the complexes that collapse/anticollapse to a point.

An element x in a poset P is *irreducible* if $P_{>x}$ has a least element or $P_{<x}$ a greatest element. A finite poset is *dismantlable* if successive removal of irreducibles leads to a single-element poset. A dismantlable poset is nonevasive. A topological characterization of dismantlable posets of Stong (1966) is mentioned in (9.3). A *directed* poset (for all $x, y \in P$ there exists $z \in P$ such that $x, y \leqslant z$) is contractible.

11.2. Let Δ be a pure d-dimensional simplicial complex, and suppose that the k-face σ is contained in exactly one d-face τ. Then the complex $\Delta' = \Delta \backslash \{\gamma \mid \sigma \subseteq \gamma \subseteq \tau\}$ is obtained from Δ by a (k,d)-*collapse*. If $\sigma \neq \tau$, then $\Delta' \simeq \Delta$. If Δ can be reduced to a single d-simplex by a sequence of (k,d)-collapses, $0 \leqslant k \leqslant d$, then Δ is *shellable*.

A pure simplicial complex Δ is *vertex-decomposable* if (i) $\Delta = \emptyset$, or (ii) Δ consists of a single vertex, or (iii) for some $x \in \Delta^0$ both $\mathrm{lk}_\Delta(x)$ and $\mathrm{dl}_\Delta(x)$ are vertex-decomposable. For example, every simplex and simplex-boundary is vertex-decomposable. The class of *constructible* complexes is defined by: (i) every simplex and $\emptyset$ is constructible, (ii) if Δ_1, Δ_2 and $\Delta_1 \cap \Delta_2$ are constructible and $\dim \Delta_1 - \dim \Delta_2 = 1 + \dim(\Delta_1 \cap \Delta_2)$, then $\Delta_1 \cup \Delta_2$ is constructible.

The following logical implications between these properties of a pure d-dimensional complex are strict:

$$\text{vertex-decomposable} \Longrightarrow \text{shellable} \Longrightarrow \text{constructible}$$
$$\Longrightarrow (d-1)\text{-connected}.$$

The first implication and the definition of vertex-decomposable complexes are due to Provan and Billera (1980). The concept of shellability has an interesting history going back to the 19th century, see Grünbaum (1967). Constructible complexes were defined by M. Hochster, see Stanley (1977).

Shellability is usually regarded as a way of *putting together* (rather than collapsing – taking apart) a complex. Therefore the following alternative definition is more common: A finite pure d-dimensional complex Δ is *shellable* if its d-faces can be ordered $\sigma_1, \sigma_2, \ldots, \sigma_t$ so that $(\delta\sigma_1 \cup \cdots \cup \delta\sigma_{k-1}) \cap \delta\sigma_k$ is a pure $(d-1)$-dimensional complex for $2 \leqslant k \leqslant t$, where $\delta\sigma_j = 2^{\sigma_j} \backslash \{\emptyset, \sigma_j\}$ is the boundary complex of σ_j. Equivalently, for all $1 \leqslant i < k \leqslant t$ there exists $j < k$ such that $\sigma_i \cap \sigma_k \subseteq \sigma_j \cap \sigma_k$ and $\dim(\sigma_j \cap \sigma_k) = d-1$. In words, the requirement is that the kth facet σ_k intersects the union of the preceding ones along a part of its boundary which is a union of maximal proper faces of σ_k. Such an ordering of the facets is called a *shelling*.

If $\sigma \in \Delta$ and Δ is a shellable (or constructible) complex, then so is $\mathrm{lk}_\Delta(\sigma)$. Shellability is also preserved by some other constructions on complexes and posets such as Theorem 11.13. Several basic properties of simplicial shellability (also for infinite complexes) are reviewed in Björner (1984b). Shellability of cell complexes is discussed in Danaraj and Klee (1974) and Björner (1984a); see also chapter 18 by Klee and Kleinschmidt. To establish shellability of (order complexes of) posets, a special method exists called *lexicographic shellability*. See Björner (1980) and Björner and Wachs (1983, 1994) for details. The notions of shellability and vertex-decomposability and most of their useful properties can easily be generalized to non-pure complexes, see Björner and Wachs (1994).

11.3. Simplicial PL spheres and PL balls are defined in (12.2), (PL. = piecewise linear). The property of being PL is a combinatorial property – whether a geometric simplicial complex Δ is PL depends only on the abstract simplicial complex Δ.

For showing that specific complexes are homeomorphic to spheres or balls, the following result is frequently useful.

Theorem 11.4. *Let Δ be a constructible d-dimensional simplicial complex.*
(i) *If every $(d-1)$-face is contained in exactly two d-faces, then Δ is a PL sphere.*
(ii) *If every $(d-1)$-face is contained in one or two d-faces, and containment in only one d-face occurs, then Δ is a PL ball.*

Theorem 11.4 follows from some basic PL topology such as the facts quoted in (12.2). For shellable Δ it appears implicitly in Bing (1964) and explicitly in Danaraj and Klee (1974).

If Δ is a triangulation of the d-sphere (or any manifold) and $\sigma \in \Delta^k$, then $\mathrm{lk}_\Delta(\sigma)$ has the same homology as the $(d-1-k)$-sphere. If $\sigma \in \Delta^0$, then there is even homotopy equivalence between $\mathrm{lk}_\Delta(\sigma)$ and S^{d-1}. However, if Δ is a PL d-sphere and $\sigma \in \Delta^k$, then $\mathrm{lk}_\Delta(\sigma)$ is itself a PL $(d-1-k)$-sphere.

Cohen–Macaulay complexes

11.5. Let $\boldsymbol{k}$ be a field or the ring of integers $\mathbb{Z}$. A finite-dimensional simplicial complex Δ is *Cohen–Macaulay over* $\boldsymbol{k}$ (written CM/$\boldsymbol{k}$ or CM if $\boldsymbol{k}$ is understood or irrelevant) if $\mathrm{lk}_\Delta(\sigma)$ is $(\dim \mathrm{lk}_\Delta(\sigma) - 1)$-acyclic over $\boldsymbol{k}$ for all $\sigma \in \Delta \cup \{\emptyset\}$. Further, Δ is *homotopy-Cohen–Macaulay* if $\mathrm{lk}_\Delta(\sigma)$ is $(\dim \mathrm{lk}_\Delta(\sigma) - 1)$-connected for all $\sigma \in \Delta \cup \{\emptyset\}$.

The following implications are strict:

$$\text{constructible} \Longrightarrow \text{homotopy-CM} \Longrightarrow \mathrm{CM}/\mathbb{Z} \Rightarrow \mathrm{CM}/\boldsymbol{k} \Rightarrow \mathrm{CM}/\mathbb{Q},$$

for an arbitrary field $\boldsymbol{k}$. Furthermore, $\mathrm{CM}/\mathbb{Z} \Longleftrightarrow \mathrm{CM}/\mathbb{Z}_p$ for all prime numbers p. The first implication follows from the fact that constructibility implies $(d-1)$-connectivity and is inherited by links, the second implication follows from (9.15), and the rest via the Universal Coefficient Theorem. In particular, shellable complexes are homotopy-CM.

An important aspect of *finite* CM-complexes Δ is that they have an equivalent ring-theoretic definition. Suppose that $\Delta^0 = \{x_1, x_2, \ldots, x_n\}$, and consider the ideal I in the polynomial ring $\boldsymbol{k}[x_1, x_2, \ldots, x_n]$ generated by monomials $x_{i_1} x_{i_2} \ldots x_{i_k}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \notin \Delta$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n, k \geqslant 1$. Let $\boldsymbol{k}[\Delta] = \boldsymbol{k}[x_1, \ldots, x_n]/I$, called the *Stanley–Reisner ring* (or *face ring*) of Δ. Then Δ is CM/$\boldsymbol{k}$ iff the ring $\boldsymbol{k}[\Delta]$ is Cohen–Macaulay in the sense of commutative algebra (Reisner 1976). An

exposition of the ring-theoretic aspects of simplicial complexes, and their combinatorial use, can be found in Stanley (1983a). There other ring-theoretically motivated classes of complexes, such as *Gorenstein complexes* and *Buchsbaum complexes*, are also discussed. Other approaches to the ring-theoretic aspects of complexes and to Reisner's theorem can be found in Baclawski and Garsia (1981) and Yuzvinsky (1987). See also section 5 of chapter 41 on Combinatorics in Pure Mathematics.

Cohen–Macaulay complexes and posets were introduced around 1974–75 in the work of Baclawski (1976, 1980), Hochster (1977), Reisner (1976) and Stanley (1975, 1977). The notion of homotopy-CM first appeared in Quillen (1978). Björner, Garsia and Stanley (1982) give an elementary introduction to CM posets. A notable combinatorial application of Cohen–Macaulayness is Stanley's proof of tight upper bounds for the number of faces that can occur in each dimension for triangulations with n vertices of the d-sphere (Stanley 1975, 1983a; see also chapter 18 by Klee and Kleinschmidt.) An application to lower bounds is given in Stanley (1987a).

11.6. Define a pure d-dimensional complex Δ to be *strongly connected* (or *dually connected*) if each pair of facets $\sigma, \tau \in \Delta^d$ can be connected by a sequence of facets $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \tau$, so that $\dim(\sigma_{i-1} \cap \sigma_i) = d \quad 1$ for $1 \leqslant i \leqslant n$.

Proposition 11.7. *Every CM complex is pure and strongly connected.*

This follows from the following lemma, which is proved by induction on $\dim \Delta$:

Let Δ be a finite-dimensional simplicial complex, and assume that $\mathrm{lk}_\Delta(\sigma)$ is connected for all $\sigma \in \Delta \cup \{\emptyset\}$ such that $\dim(\mathrm{lk}_\Delta(\sigma)) \geqslant 1$. Then Δ is pure and strongly connected.

The property of being CM is *topologically invariant*: whether Δ is CM/$\boldsymbol{k}$ or not depends only on the topology of $\|\Delta\|$. This is implied by the following reformulation of CM-ness, due to Munkres (1984b).

Theorem 11.8. *A finite-dimensional complex Δ is* CM/$\boldsymbol{k}$ *iff its space $T = \|\Delta\|$ satisfies: $\tilde{H}_i(T, \boldsymbol{k}) = H_i(T, T \backslash p, \boldsymbol{k}) = 0$ for all $p \in T$ and $i < \dim \Delta$.*

In this formulation $\tilde{H}_i$ denotes reduced singular homology and H_i relative singular homology with coefficients in $\boldsymbol{k}$. A consequence of Theorem 11.8 is that if M is a triangulable manifold (with or without boundary) and $\tilde{H}_i(M) = 0$ for $i < \dim M$, then every triangulation of M is CM. For instance: (1) every triangulation of the d-sphere, d-ball or $\mathbb{R}^d$ is CM/$\mathbb{Z}$, but not necessarily homotopy-CM (beware: homotopy-CM is *not* topologically invariant), (2) a triangulation of real projective d-space is CM/$\boldsymbol{k}$ iff char $\boldsymbol{k} \neq 2$.

11.9. The definition of Cohen–Macaulay posets (posets P such that $\Delta(P)$ is CM) deserves a small additional comment. Let P be a poset of finite rank and σ: $x_0 < x_1 < \cdots < x_k$ a chain in P. Then $\mathrm{lk}_{\Delta(P)}(\sigma) = P_{<x_0} * (x_0, x_1) * \cdots * (x_{k-1}, x_k) * P_{>x_k}$. It therefore follows from (9.20) that P is CM [resp. homotopy-CM] iff every open interval (x, y) in $\hat{P}$ is $(\mathrm{rank}(x, y) - 1)$-acyclic [resp. $(\mathrm{rank}(x, y) - 1)$-connected].

Some uses of Cohen–Macaulay posets in commutative algebra are discussed in section 5 of chapter 41 on Combinatorics in Pure Mathematics.

11.10. An abundance of shellable and CM simplicial complexes appear in combinatorics. Only a few important examples can be mentioned here.

(i) The boundary complex of a simplicial convex polytope is shellable (Bruggesser and Mani 1971, Danaraj and Klee 1974; see also chapter 18 by Klee and Kleinschmidt). Every simplicial PL sphere is the boundary of a shellable ball (Pachner 1986). There exist non-shellable triangulations of the 3-ball (M.E. Rudin) and of the 3-sphere (see below). Shellability of spheres and balls is surveyed in Danaraj and Klee (1978).

(ii) The following implications are valid for any simplicial sphere: constructible $\Longrightarrow PL \Longrightarrow$ homotopy-CM. The 5-sphere admits triangulations that are non-homotopy-CM (R.D. Edwards, see Daverman 1986), and also PL triangulations that are non-constructible (Mandel 1982). Every triangulation of the 3-sphere is PL, but all are not shellable (Lickorish 1991, see also Vince 1985). Face lattices of regular complex polytopes are CM (Orlik 1990).

(iii) The complex of independent sets in a matroid is constructible (Stanley 1977) and vertex-decomposable (Provan and Billera 1980). More generally, the complex generated by the basis-complements of a greedoid is vertex-decomposable (Björner, Korte and Lovász 1985). Complexes arising from matroids are discussed in Björner (1992).

(iv) Every semimodular (in particular, every geometric or modular) lattice of finite rank is CM (Folkman 1966) and shellable (Björner 1980). For any element $x \neq \hat{0}$ in a geometric lattice L, the poset $L \backslash [x, \hat{1}]$ is shellable (Wachs and Walker 1986).

(v) Tits buildings are CM (Solomon–Tits, see Brown 1989 or Ronan 1989) and shellable (Björner 1984b). The topology of more general group-related geometries has been studied by Ronan (1981), Smith (1988), Tits (1981) and others with a view to uses in group theory. See Buekenhout (1995) and Ronan (1989) for general accounts.

(vi) The poset of elementary Abelian p-subgroups of a finite group was shown by Quillen (1978) to be homotopy-CM in some cases. See also Stong (1984). The full subgroup lattice of a finite group G is shellable (or CM) iff G is supersolvable (Björner 1980). Various posets of subgroups have been studied from a topological point of view. See Thévenaz (1987), Webb (1987) and Welker (1994) for a guide to this literature.

Induced subcomplexes

Connectivity, Cohen–Macaulayness, etc., are under certain circumstances inherited by suitable subcomplexes. For a simplicial complex Δ and $A \subseteq \Delta^0$, let $\Delta_A = \{\sigma \in \Delta \mid \sigma \subseteq A\}$ (the *induced subcomplex* on A).

Lemma 11.11. *Let Δ be a finite-dimensional complex, and $A \subseteq V = \Delta^0$. Assume that $\mathrm{lk}_\Delta(\sigma)$ is k-connected for all $\sigma \in \Delta_{V \backslash A}$. Then Δ_A is k-connected iff Δ is k-connected.*

Lemma 11.12. *Let P be a poset of finite rank and A a subset. Assume that $P_{>x}$ is k-connected for all $x \in P\backslash A$. Then A is k-connected iff P is k-connected.*

Proof. These lemmas are equivalent. We start with Lemma 11.12. Let $f: A \to P$ be the embedding map. For $x \in P$,

$$f^{-1}(P_{\geqslant x}) = \begin{cases} A_{\geqslant x}, & \text{if } x \in A, \\ P_{>x} \cap A, & \text{if } x \notin A. \end{cases}$$

Now, $A_{\geqslant x}$ is contractible (being a cone), and $P_{>x} \cap A$ is k-connected by induction on rank(P). The result therefore follows by Theorem 10.5 (ii).

To prove Lemma 11.11, let $P = P(\Delta)$ and $Q = \{\tau \in \Delta \mid \tau \cap A \neq \emptyset\} \subseteq P$. Since $P_{>\sigma} \cong P(\mathrm{lk}_\Delta(\sigma))$ is k-connected for all $\sigma \in P\backslash Q$, Lemma 11.12 applies. On the other hand, by Corollary 10.12 the map $f(\tau) = \tau \cap A$ on Q induces homotopy equivalence between Q and $f(Q) = P(\Delta_A)$. □

The homology versions of Lemmas 11.11 and 11.12, obtained by using k-acyclicity throughout, can be proven by a parallel method. Also, if the hypothesis "k-connected" were replaced by "contractible" in these lemmas, then the conclusion would be that Δ_A and Δ (resp. A and P) are homotopy equivalent.

Theorem 11.13. *Let Δ be a pure d-dimensional simplicial complex, $A \subseteq \Delta^0$ and $1 \leqslant m \leqslant d$. Suppose that $\mathrm{card}(A \cap \sigma) = m$ for every facet $\sigma \in \Delta^d$. If Δ is CM/$\boldsymbol{k}$, homotopy-CM or shellable, then the same property is inherited by Δ_A.*

For CM-ness this result was proven in varying degrees of generality by Bacławski (1980), Munkres (1984b), Stanley (1979) and Walker (1981a). It follows easily from Lemma 11.11. For shellability, proofs appear in Björner (1980, 1984b).

Suppose that Δ is a pure d-dimensional simplicial complex and that there exists a mapping $t: \Delta^0 \to \{0, 1, \ldots, d\}$ which restricts to a bijection on each facet $\sigma \in \Delta^d$. Then Δ is called *completely balanced* (or *numbered*, or *colored*) with *type-map* t. For instance, the order complex of a pure poset is completely balanced with type-map $t = rank$ [cf. (9.2)], and also building-like incidence geometries (Buekenhout 1995) give rise to completely balanced complexes. CM complexes of this kind were studied by Stanley (1979) and others.

For each $J \subseteq \{0, 1, \ldots, d\}$, the *type-selected subcomplex* $\Delta_{(J)} = \Delta_{t^{-1}(J)}$ is the induced subcomplex on $t^{-1}(J) \subseteq \Delta^0$. Theorem 11.13 shows that if Δ is CM then $\Delta_{(J)}$ is also CM and hence (card $J - 2$)-acyclic. A certain converse is also true in the sense of the following result, which gives an alternative characterization of the CM property for completely balanced complexes. It is due to Bacławski and Garsia (1981) in the finite CM case, and to J. Walker (letter to the author, 1981) in general including the homotopy case.

Theorem 11.14. *Let Δ be a pure d-dimensional completely balanced complex. Then Δ is* CM/$\boldsymbol{k}$ *[resp., homotopy-CM] if and only if $\Delta_{(J)}$ is* (card $J - 2$)*-acyclic over* $\boldsymbol{k}$ *[resp.,* (card $J - 2$)*-connected] for all $J \subseteq \{0, 1, \ldots, d\}$.*

12. Cell complexes

Most classes of cell complexes differ from the simplicial case in that a purely combinatorial description of these objects *as such* cannot be given. However, the two classes defined here, polyhedral complexes and regular CW complexes, are sufficiently close to the simplicial case to allow a similar combinatorial approach in many cases. For simplicity only *finite* complexes will be considered.

Good general references for polyhedral complexes are Grünbaum (1967) and Hudson (1969), and for cell complexes Cooke and Finney (1967) and Lundell and Weingram (1969). Cell complexes are also discussed in many books on algebraic topology such as Munkres (1984a) and Spanier (1966).

Polyhedral complexes and PL topology

12.1. A *convex polytope* π is a bounded subset of $\mathbb{R}^d$ which is the solution set of a finite number of linear equalities and inequalities. Any nonempty subset obtained by changing some of the inequalities to equalities is a *face* of π. Equivalently, $\pi \subseteq \mathbb{R}^d$ is a convex polytope iff π is the convex hull of a finite set of points in $\mathbb{R}^d$. See chapter 18 by Klee and Kleinschmidt for more information about convex polytopes.

A *polyhedral complex* (or *convex cell complex*) Γ is a finite collection of convex polytopes in $\mathbb{R}^d$ such that (i) if $\pi \in \Gamma$ and σ is a face of π then $\sigma \in \Gamma$, and (ii) if $\pi, \tau \in \Gamma$ and $\pi \cap \tau \neq \emptyset$ then $\pi \cap \tau$ is a face of both π and τ. The members of Γ are called *cells*. The *underlying space* of Γ is $\|\Gamma\| = \bigcup \Gamma$, with the topology induced as a subset of $\mathbb{R}^d$. If every cell in Γ is a *simplex* (the convex hull of an affinely independent set of points) then Γ is called a (*geometric*) *simplicial complex*. The *dimension* of a cell equals the linear dimension of its affine span, and $\dim \Gamma = \max_{\pi \in \Gamma} \dim \pi$. Further terminology, such as *vertices, edges, facets, pure, k-skeleton, face poset, face lattice*, etc., is defined just as in the simplicial case, see (9.1) and (9.3).

12.2. A polyhedral complex Γ_1 is a *subdivision* of another such complex Γ_2 if $\|\Gamma_1\| = \|\Gamma_2\|$ and every cell of Γ_1 is a subset of some cell of Γ_2. The abstract simplicial complex $\Delta(P(\Gamma))$, i.e., the order complex of Γ's face poset, has geometric realizations (by choosing as new vertices an interior point in each cell) that subdivide Γ. Every polyhedral complex can be simplicially subdivided without introducing new vertices.

Let Σ^d denote the complex consisting of a geometric d-simplex and all its faces, and let $\delta\Sigma^d$ denote its boundary. These complexes provide the simplest triangulations of the d-ball and the $(d-1)$-sphere, respectively. A polyhedral complex Γ is called a PL *d-ball* (or PL *$(d-1)$-sphere*) if it admits a simplicial subdivision whose face poset is isomorphic to the face poset of some subdivision of Σ^d (resp. $\delta\Sigma^d$). This is equivalent to saying that there exists a homeomorphism $\|\Gamma\| \to \|\Sigma^d\|$ (resp.

$\|\Gamma\| \to \|\delta\Sigma^d\|$) which is induced by a simplicial map defined on some subdivision (a *piecewise linear*, or PL, *map*). The boundary complex of a convex d-polytope is a PL $(d-1)$-sphere.

The PL property is mainly of technical interest. Several properties of balls and spheres that are desirable, and would in many cases seem intuitively "obvious", hold only in the PL case. Some examples are: (1) (Newman's Theorem) *the closure of the complement of a PL d-ball lying in a PL d-sphere is itself a PL d-ball*; (2) *the union of two PL d-balls, whose intersection is a PL* $(d-1)$*-ball lying in the boundary of each, is a PL d-ball*; (3) *the link of any face in a PL sphere is itself a PL sphere* (cf. remark following Theorem 11.4). All these statements would be false with "PL" removed.

See Hudson (1969) for proofs and further information about PL topology. Mandel (1982) develops basic PL topology from a combinatorial perspective.

Regular cell complexes

12.3. By "cell complex" we will here understand what in topology is usually called a "finite CW complex".

Let X be a Hausdorff space. A subset σ is called an *open d-cell* if there exists a mapping $f: \boldsymbol{B}^d \to X$ whose restriction to the interior of the d-ball is a homeomorphism $f: \mathrm{Int}(\boldsymbol{B}^d) \to \sigma$. The *dimension* $\dim \sigma = d$ is well-defined by this. The closure $\bar{\sigma}$ is the corresponding *closed cell*. It is true that $f(\boldsymbol{B}^d) = \sigma$, but $\bar{\sigma}$ is not necessarily homeomorphic to $\boldsymbol{B}^d$. We write $\dot{\sigma} = \bar{\sigma} \backslash \sigma$.

A *cell complex* $\mathscr{C}$ is a finite collection of pairwise disjoint sets together with a Hausdorff topology on their union $\|\mathscr{C}\| = \bigcup \mathscr{C}$ such that:

(i) each $\sigma \in \mathscr{C}$ is an open cell in $\|\mathscr{C}\|$, and

(ii) $\dot{\sigma} \subseteq \mathscr{C}^{<\dim \sigma}$ (the union of all cells in $\mathscr{C}$ of dimension less than $\dim \sigma$), for all $\sigma \in \mathscr{C}$.

Then $\mathscr{C}$ is also called a *cell decomposition* of the space $\|\mathscr{C}\|$. Furthermore, $\mathscr{C}$ is *regular* if each mapping $f: \boldsymbol{B}^d \to \|\mathscr{C}\|$ defining the cells can be chosen to be a homeomorphism on *all* of $\boldsymbol{B}^d$. Then, of course, every closed cell $\bar{\sigma}$ is homeomorphic to a ball. (However, it is not enough for the definition of a regular complex to only require that every closed cell is homeomorphic to a ball. The smallest example showing this has three vertices, three edges and one 2-cell.)

The cell decomposition of the d-sphere into one 0-cell and one d-cell (a point and its complement in $\boldsymbol{S}^d$) is not regular. Every polyhedral complex is a regular cell complex (the relative interiors of the convex polytopes are the open cells). Regular cell complexes are more general than polyhedral complexes in several ways. For instance, it is allowed that the intersection of two closed cells can have nontrivial topological structure.

12.4. From now on only regular cell complexes will be considered. Define the *face poset* $P(\mathscr{C})$ as the set of all closed cells ordered by containment. The following two

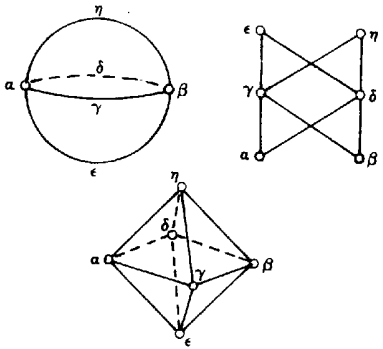

Figure 2.

particular properties make a regular complex $\mathscr{C}$ favorable from a combinatorial point of view (see Cooke and Finney 1967 or Lundell and Weingram 1969 for proofs):

(i) *The boundary* $\dot{\sigma}$ *of each cell* $\sigma \in \mathscr{C}$ *is a union of cells* (*a subcomplex*). Hence, the situation resembles that of polyhedral complexes: each closed d-cell $\bar{\sigma}$ is homeomorphic to $\boldsymbol{B}^d$, and its boundary $\dot{\sigma}$ (homeomorphic to $\boldsymbol{S}^{d-1}$) has a regular cell decomposition provided by the cells that intersect $\dot{\sigma}$.

(ii) $\|\mathscr{C}\| \cong \|\Delta(P(\mathscr{C}))\|$, i.e., *the order complex of* $P(\mathscr{C})$ *is homeomorphic to* $\|\mathscr{C}\|$. Geometrically this means that regular cell complexes admit "barycentric subdivisions". From a combinatorial point of view it means that regular cell complexes can be interpreted as a class of posets without any loss of topological information.

Because of (i), regular cell complexes can be characterized in the following way: A family of balls (homeomorphs of $\boldsymbol{B}^d, d \geq 0$) in a Hausdorff space X is the set of closed cells of a regular cell complex iff the interiors of the balls partition X and the boundary of each ball is a union of other balls. This is what Mandel (1982) calls a "ball complex".

An important consequence of (ii) is that a d-dimensional regular cell complex $\mathscr{C}$ can always be "realized" in $\mathbb{R}^{2d+1}$ by a simplicial complex, so that every closed cell in $\mathscr{C}$ is a triangulated ball (a cone over a simplicial sphere).

For a detailed discussion of regular cell complexes from a combinatorial point of view, see section 4.7 of Björner et al. (1993). Figure 2 shows a regular cell decomposition $\mathscr{C}$ of the 2-sphere, its face poset $P(\mathscr{C})$, and its simplicial representation $\Delta(P(\mathscr{C}))$, where each original 2-cell is triangulated into four triangles.

12.5. Given a finite poset P, does there exist a regular cell complex (or even a polyhedral complex) $\mathscr{C}$ such that $P \cong P(\mathscr{C})$; and if so, what is its topology and how can $\mathscr{C}$ be constructed from P? This question is discussed in Björner (1984a) and Mandel (1982) from different perspectives. One answer is that P is isomorphic to the face poset of some regular cell complex iff $\Delta(P_{<x})$ is homeomorphic to a sphere for all $x \in P$. However, since it is known that simplicial spheres cannot be recognized algorithmically this is not a fully satisfactory answer. The question of how to recognize the face posets of polyhedral complexes is one version of the Steinitz problem (see chapter 18 by Klee and Kleinschmidt).

For the cellular interpretation of posets the following result, derivable from Theorem 11.4, has proven useful in practice. See Björner (1984a) for further details. Let us call a poset P *thin* if every closed interval of rank 2 has four elements (two "in the middle"). Also, $P \cup \{\hat{0}\}$ will denote P with a new minimum element $\hat{0}$ adjoined, and $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ as usual.

Theorem 12.6. *Let P be a pure finite poset of rank d. Assume that $\Delta(P)$ is constructible.*

(i) *If $P \cup \{\hat{0}\}$ is thin, then $P \cong P(\mathscr{C})$ for some regular cell complex $\mathscr{C}$ homotopy equivalent to a wedge of d-spheres.*

(ii) *If $\hat{P}$ is thin, then $P \cong P(\mathscr{C})$ for some regular cell decomposition of the d-sphere.*

13. Fixed-point and antipodality theorems

The topological fixed-point and antipodality theorems of greatest use for combinatorics will be reviewed. We start by stating four equivalent versions of the oldest of them: Brouwer's fixed-point theorem (from 1912). Proofs and references to original sources for all otherwise unreferenced material in this section can be found in many topology books, e.g., in Dugundji and Granas (1982). Recall that mappings between topological spaces are always assumed to be continuous.

Theorem 13.1 (Brouwer's Theorem). (i) *Every mapping $f: \boldsymbol{B}^d \to \boldsymbol{B}^d$ has a fixed point $x = f(x)$.*

(ii) *$\boldsymbol{S}^{d-1}$ is not a retract of $\boldsymbol{B}^d$ (i.e., no mapping $\boldsymbol{B}^d \to \boldsymbol{S}^{d-1}$ leaves each point of $\boldsymbol{S}^{d-1}$ fixed).*

(iii) *$\boldsymbol{S}^{d-1}$ is not $(d-1)$-connected.*

(iv) *$\boldsymbol{S}^{d-1}$ is not contractible.*

Brouwer's Theorem is implied by the following combinatorial lemma of Sperner (1928), see also Cohen (1967): *If the vertices of a triangulation of $\boldsymbol{S}^{d-1}$ are colored with d colors, then there cannot be exactly one $(d-1)$-face whose vertices use all d colors.* Sperner's Lemma was generalized by Lovász (1980): *If the vertices of a $(d-1)$-dimensional manifold are labeled by elements from some rank-d loopless matroid, then there cannot be exactly one $(d-1)$-face whose vertices form a basis of the matroid.* A further generalization and an application to hypergraphs appear in Lindström (1981). Sperner's Lemma is of practical use for the design of fixed-point-finding algorithms in connection with applications of Brouwer's Theorem, see Todd (1976).

It is well known that Brouwer's Theorem for $d = 2$ implies that there is no draw in the 2-person game HEX. Actually the implication goes the other way as well. Gale (1979) defines a d-person d-dimensional HEX game, and proves that for each $d \geqslant 2$ the Brouwer Theorem 13.1 is equivalent to the impossibility of a draw in d-dimensional HEX.

We turn next to the (Hopf-)Lefschetz fixed-point theorem (from 1927–28), which gives a vast generalization of Theorem 13.1. Lefschetz' Theorem and the closely related trace formula of Hopf will be stated in simplicial versions.

Let Δ be a nonempty simplicial complex and $f: \|\Delta\| \to \|\Delta\|$ a continuous map. The *Lefschetz number* $\Lambda(f)$ is defined by $\Lambda(f) = \sum_{i \geq 0} (-1)^i \operatorname{trace}(f_i^*)$, where $f_i^*: \tilde{H}_i(\Delta, \mathbb{Q}) \to \tilde{H}_i(\Delta, \mathbb{Q})$ is the induced mapping on i-dimensional reduced homology. (We use $\mathbb{Q}$-coefficients throughout here for simplicity; other fields may of course be used instead.) Note that $f \sim g$ implies $\Lambda(f) = \Lambda(g)$ (since homotopic maps induce identical maps on homology), in particular if f is *null-homotopic* (meaning homotopic to a constant map) then $\Lambda(f) = 0$. Also, if Δ is $\mathbb{Q}$-acylic then $\Lambda(f) = 0$ for all self-maps f.

Now, suppose that $f: \Delta \to \Delta$ is simplicial, and say that a face $\tau \in \Delta$ is *fixed* if $f(\tau) = \tau$ as a set. Let $\varphi_i^+(f)$ [resp. $\varphi_i^-(f)$]be the number of fixed i-faces whose orientation is preserved [resp. reversed]. Here we consider the orientation of $\tau = \{x_0, x_1, \ldots, x_i\}$ to be preserved if the permutations $x_0, x_1, \ldots, x_i$ and $f(x_0), f(x_1), \ldots, f(x_i)$ have the same parity. The following is a special case of the *Hopf trace formula*:

$$\Lambda(f) + 1 - \sum_{i \geq 0} (-1)^i \, |\varphi_i^+(f) - \varphi_i^-(f)|. \tag{13.2}$$

Notice that for $f = \mathrm{id}$ formula (13.2) specializes to the Euler–Poincaré formula (9.13).

One sees from (13.2) that if f has no fixed face, then $\Lambda(f) = -1$. Using simplicial approximation and compactness the following is deduced.

Theorem 13.3 (Lefschetz's Theorem). *If $f: \|\Delta\| \to \|\Delta\|$ is a mapping such that $\Lambda(f) \neq -1$, then f has a fixed point.*

The following two consequences of Theorem 13.3 generalize Brouwer's Theorem in different directions.

Corollary 13.4. *Let T be a compact triangulable space.*

(a) *Every null-homotopic self-map of T has a fixed point.*

(b) *If T is $\mathbb{Q}$-acyclic, then every self-map of T has a fixed point.*

The following consequence of the Hopf trace formula is useful in some combinatorial situations. Let once more $f: \Delta \to \Delta$ be a simplicial mapping of a simplicial

complex Δ. Assume that a face $\tau \in \Delta$ is fixed if and only if τ is *point-wise* fixed [i.e., $f(\tau) = \tau$ implies $f(x) = x$ for all $x \in \tau$]. One may then define the *fixed subcomplex* $\Delta^f = \{\tau \in \Delta \mid f(\tau) = \tau\}$, which coincides with the induced subcomplex on the set of fixed vertices, and (13.2) specializes to

$$\Lambda(f) = \tilde{\chi}(\Delta^f). \tag{13.5}$$

One situation where this is used (see, e.g., Curtis, Lehrer and Tits 1980) is in connection with groups acting on finite complexes, where (13.5) says that the "Lefschetz character" has a topological interpretation as the reduced Euler characteristic of the fixed subcomplex. Another such situation (see Bacławski and Björner 1979 and section 3 of this chapter) is when $f: P \to P$ is an order-preserving poset map, in which case (13.5) can be rewritten $\Lambda(f) = \mu(P^f)$, the right-hand side denoting the value of the Möbius function computed over the subposet of fixed points augmented with a new $\hat{0}$ and $\hat{1}$ [cf. (9.14)].

The following definitions will now be needed. Let p be a prime. By a $\mathbb{Z}_p$-*space* we understand a pair (T, ν) where T is a topological space and $\nu: T \to T$ is a fixed-point free continuous mapping of order p (i.e., $\nu^p = \mathrm{id}$). A mapping $f: T_1 \to T_2$ of $\mathbb{Z}_p$-spaces $(T_i, \nu_i), i = 1, 2$, is *equivariant* if $\nu_2 \circ f = f \circ \nu_1$. A $\mathbb{Z}_2$-space is often called an *antipodality space*. The standard example is (S^d, α), the d-sphere with its antipodal map $\alpha(x) = -x$.

We state five equivalent versions of the antipodality theorem of Borsuk (1933).

Theorem 13.6 (Borsuk's Theorem).

(i) *If S^d is covered by $d+1$ subsets, all closed or all open, then one of these must contain a pair of antipodal points.* (Borsuk–Liusternik–Schnirelman)

(ii) *For every continuous mapping $f: S^d \to \mathbb{R}^d$ there exists a point x such that $f(x) = f(-x)$.* (Borsuk–Ulam)

(iii) *For every odd [$f(-y) = -f(y)$ for all y] continuous mapping $f: S^d \to \mathbb{R}^d$ there exists x for which $f(x) = 0$.* (Borsuk–Ulam)

(iv) *There exists no equivariant map $S^n \to S^d$, if $n > d$.*

(v) *For any d-connected antipodality space T, there exists no equivariant map $T \to S^d$.*

Borsuk's Theorem is implied by a certain combinatorial lemma of A.W. Tucker, much like Brouwer's Theorem is implied by Sperner's Lemma. See Freund and Todd (1981) for a statement and proof of Tucker's Lemma and further references. In Theorem 13.6 (v) it suffices to assume that T is d-acyclic over $\mathbb{Z}_2$, see Walker (1983b).

Steinlein (1985) gives an extensive survey of generalizations, applications and references related to Borsuk's Theorem. Applications to combinatorics are surveyed by Alon (1988), Bárány (1993) and Bogatyi (1986); see also sections 4 and 5 of this chapter.

The following extension of the Borsuk–Ulam Theorem appears in Yang (1955): *For every mapping $S^{dn} \to \mathbb{R}^d$ there exist n mutually orthogonal diameters whose $2n$ endpoints are mapped to the same point.* The same paper also gives references to the following related theorem of Kakutani–Yamabe–Yujobô: *For every mapping $S^n \to \mathbb{R}$ there exist $(n+1)$ mutually orthogonal radii whose $(n+1)$ endpoints are mapped to the same point.* An interesting consequence of the last result is that every compact convex body $K \subset \mathbb{R}^{n+1}$ is contained in an $(n+1)$-cube C such that every maximal face of C touches K [for each $x \in S^n$ let $f(x)$ be the minimal distance between two parallel hyperplanes orthogonal to the vector x and containing K between them].

Suppose E_1 and E_2 are two bounded and measurable subsets of $\mathbb{R}^2$. Identify $\mathbb{R}^2$ with the affine plane $A = \{(\xi, \eta, 1)\}$ in $\mathbb{R}^3$, and for each $x \in S^2$ let $f_i(x)$ be the measure of that part of E_i which lies on the same side as x of the plane H_x through the origin orthogonal to x, for $i = 1, 2$. The Borsuk–Ulam Theorem implies that $f_1(x) = f_1(-x)$ and $f_2(x) = f_2(-x)$ for some $x \in S^2$, which means that the line $A \cap H_x$ bisects both E_1 and E_2. This "ham sandwich" argument generalizes to arbitrary dimensions and leads to the following consequence of the Borsuk–Ulam Theorem.

Corollary 13.7 ("Ham Sandwich Theorem"). *Given d bounded and Lebesgue measurable sets in $\mathbb{R}^d$ there exists some affine hyperplane that simultaneously bisects them all.*

Also Corollary 13.7 has several generalizations and related results. The case when $k \leqslant d$ bounded and measurable sets are given is covered by the following result of Živaljević and Vrećica (1990): *Let $\mu_1, \mu_2, \ldots, \mu_k$ be a collection of σ-additive probability measures defined on the σ-algebra of all Borel sets in $\mathbb{R}^d$, $1 \leqslant k \leqslant d$. Then there exists a $(k-1)$-dimensional affine subspace $A \subseteq \mathbb{R}^d$ such that for every closed halfspace $H \subseteq \mathbb{R}^d$ and every $i = 1, 2, \ldots, k$, $A \subseteq H$ implies $\mu_i(H) \geqslant 1/(d-k+2)$.* For $k = d$ this specializes to a measure-theoretic version of the Ham Sandwich Theorem (see also Hill 1988), and for $k = 1$ it gives a theorem of Rado (1946) which says that for any measurable $E \subseteq \mathbb{R}^d$ there exists a point $x \in \mathbb{R}^d$ such that every halfspace containing x contains at least a $1/(d+1)$-fraction of E.

We end by stating a useful generalization of the Borsuk–Ulam Theorem to $\mathbb{Z}_p$-spaces for $p > 2$. First a few definitions, see Bárány et al. (1981) for complete details. Let p be a prime and $n \geqslant 1$. Take p disjoint copies of the $n(p-1)$-dimensional ball and identify their boundaries. Call this space $X_{n,p}$. There exists a mapping $\nu: S^{n(p-1)-1} \to S^{n(p-1)-1}$ of the identified boundary which makes it into a $\mathbb{Z}_p$-space. Extend this mapping to $X_{n,p}$ as follows. If (y, r, q) denotes the point of $X_{n,p}$ from the qth ball with radius r and $S^{n(p-1)-1}$-coordinate y, then put $\nu(y, r, q) = (\nu y, r, q+1)$, where $q+1$ is reduced modulo p. This mapping ν makes $X_{n,p}$ into a $\mathbb{Z}_p$-space. [Note that $(X_{n,2}, \nu) \cong (S^n, \alpha)$.]

Theorem 13.8 (Bárány, Shlosman and Szűcs 1981). *For every continuous mapping $f: X_{n,p} \to \mathbb{R}^n$ there exists a point x such that $f(x) = f(\nu x) = \cdots = f(\nu^{p-1} x)$.*

Some applications of Theorem 13.8 are mentioned in sections 4 and 5.

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Part IV
Applications