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INVARIANTS OF SELF-LINKING[1]

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By a *multiplication* (with respect to the rationals mod 1) defined over a finite (additive) abelian group G is meant a function L which associates to every ordered pair of elements a, b of G a rational number $L(a, b)$ in such a way that[2]

$$0 \leqq L(a, b) < 1,$$

$$L(a, b + c) \equiv L(a, b) + L(a, c) \qquad (\text{mod } 1),$$

$$L(a + b, c) \equiv L(a, c) + L(b, c) \qquad (\text{mod } 1).$$

We shall deal only with *symmetric multiplications*, i.e. multiplications L which satisfy the condition

$$L(a, b) = L(b, a).$$

An element a of G is called an *annihilator* of L if $L(a, b) = 0$ for every element b of G; a *primitive*[3] multiplication is a (symmetric) multiplication L which has no annihilators other than zero. Multiplications L and L' defined over isomorphic groups G and G' are *equivalent* if there is an isomorphism φ of G on G' such that

$$L(a, b) = L'(\varphi(a), \varphi(b)).$$

(Hence multiplications L and L' defined over the same group G are equivalent if there is an automorphism φ of G satisfying this relation.)

Numerical invariants of equivalence classes of primitive multiplications were constructed by Seifert [5]. This system $\{\sigma\}$ is complete if the order of G is odd, i.e. two multiplications defined on an odd-ordered group G are equivalent if and only if they have the same Seifert invariants σ. In practice there may be some difficulty in calculating Seifert's invariants, especially if one is considering a whole range of groups with multiplications, for the reason that the definition of $\{\sigma\}$ is in terms of a particular choice of a basis for G. In §1 we exhibit a set of invariants $\{\chi\}$ which are invariantly defined. For odd-ordered groups the effectiveness of these new invariants is the same as of those of Seifert: in §2 we show that one can express the system $\{\chi\}$ in terms of the system $\{\sigma\}$, and conversely. Nevertheless the system $\{\chi\}$ has greater flexibility, because of the freedom from choice of basis, and this will be exploited in a subsequent paper.[4]

[1] This paper is substantially the junior paper [8] of R. C. Blanchfield submitted to the Department of Mathematics at Princeton University in February 1949.

[2] To define $L(a, b)$ to be a least non-negative residue seems to be more convenient than to define it to be a residue class.

[2] All multiplications, with the exception of the auxiliary multiplication L^* of §1, considered in this paper are primitive. Only primitive multiplications occur in the applications.

[4] R. H. Fox: *The homology characters of the cyclic coverings of the knots of genus one*. To appear in these Annals.

Over the $(2N + 1)$-dimensional torsion group G of a $(4N + 3)$-dimensional closed oriented manifold $\mathfrak{M}$ a primitive multiplication L, called the *self-linking* (Eigenverschlingung), is defined in a natural way.[5] In order that there be an orientation-preserving topological mapping of such a manifold $\mathfrak{M}$ upon another one $\mathfrak{M}'$ it is necessary not only that the groups G and G' be isomorphic but that their self-linkings L and L' be equivalent. Hence the invariants χ and σ of (G, L) are invariants of $\mathfrak{M}$ (with respect to orientation-preserving homeomorphisms).

When calculating $\{\chi\}$ in concrete cases one is usually presented not with the self-linking L but with a system of fundamental boundary relations and a matrix of intersection numbers. In §3 we derive a formula which enables one to calculate the invariants χ directly without explicitly constructing the self-linking L. This algorithm will be used in the application mentioned above.

The invariants χ were suggested by and are analogous to certain well-known invariants from the arithmetic theory of quadratic forms[6] (with which the self-linking has a more than superficial connection[7]).

1. We consider a finite abelian group G and a primitive multiplication L defined on G. For future reference we note that

(1) $$L\left(\sum_{i=1}^{m} \alpha_i a_i , \sum_{j=1}^{n} \beta_j b_j\right) \equiv \sum_{i,j=1}^{m,n} \alpha_i \beta_j L(a_i , b_j) \qquad (\text{mod } 1)$$

and that

(2) $$\alpha a = 0 \text{ implies } \alpha L(a, b) \equiv 0 \qquad (\text{mod } 1).$$

It is well-known that the finite abelian group G is the direct sum of (non-trivial) subgroups $G_1 , \cdots , G_n$ of respective orders $\tau_1 , \cdots , \tau_n$ such that τ_{i+1} divides τ_i for $i = 1, \cdots , n - 1$. The numbers $\tau_1 \geqq \tau_2 \geqq \cdots \geqq \tau_n > 1$ are called the *torsion coefficients* of G and are uniquely determined by G. It is convenient to define $\tau_r = 1$ for $r = n + 1, n + 2, \cdots$. An n-tuple $(x_1 , \cdots , x_n)$ of elements of G is called a *basis* if x_i generates a cyclic subgroup G_i of order $\tau_i > 1$ and G is the direct sum of $G_1 , \cdots , G_n$. Two bases $(x_1 , \cdots , x_n)$ and $(y_1 , \cdots , y_n)$ are called *dual* if

(3) $$L(x_i , y_j) = \delta_{ij}/\tau_i .$$

The multiplication L being primitive, at least one pair of dual bases can be found [2]. The symbols $x_1 , \cdots , x_n ; y_1 , \cdots , y_n$ will always denote some selected pair of dual bases.

Let $r \leqq n$ be an index for which $\tau_r > \tau_{r+1}$. There is a unique symmetric multiplication L^* over G satisfying the condition

$$L^*(a, b) \equiv \tau_{r+1}L(a, b) \qquad (\text{mod } 1).$$

[5] See [4], [5] or [6]. Self-linking is also defined over the $2N$-dimensional torsion group of a $(4N + 1)$-dimensional manifold, but it is then a *skew-symmetric* multiplication. It has been exhaustively studied by de Rham [3].

[6] See [1].

[7] An n-ary integral quadratic form may be regarded as a multiplication with respect to the integers defined over a free abelian group of rank n.

The annihilators of L^* form a subgroup K of G; let $\bar{G} = G/K$ and denote by $\bar{a}$ the coset of K in G of which a is a member. Define

$$\bar{L}(\bar{a}, \bar{b}) = L^*(a, b),$$

observing that the definition is independent of the choice of the representatives a and b of the respective cosets $\bar{a}$ and $\bar{b}$. It is easily verified that $\bar{L}$ is a primitive multiplication over $\bar{G}$.

LEMMA 1. *The torsion coefficients of* $\bar{G}$ *are* τ_i/τ_{r+1} , $i = 1, \cdots , r$.

PROOF. It is sufficient to prove that K is the subgroup of G generated by the elements $\frac{\tau_1}{\tau_{r+1}} x_1 , \cdots , \frac{\tau_r}{\tau_{r+1}} x_r , x_{r+1} , \cdots , x_n$. To see this we note first of all that

$$L^*\left(\frac{\tau_i}{\tau_{r+1}} x_i , y_j\right) \equiv \tau_i L(x_i , y_j) \equiv 0 \qquad (\text{mod } 1), \; i = 1, \cdots , r,$$

$$L^*(x_i , y_j) \equiv \tau_{r+1} L(x_i , y_j) \equiv 0 \qquad (\text{mod } 1), \, i = r + 1, \cdots , n,$$

by (2), so that the elements $\frac{\tau_1}{\tau_{r+1}} x_1 , \cdots , \frac{\tau_r}{\tau_{r+1}} x_r , x_{r+1} , \cdots , x_n$ are all annihilators of L^*. Then we have only to show that an arbitrary annihilator $a = \sum_{i=1}^n \alpha_i x_i$ can be expressed as a linear combination of these elements. And, in fact,

$$0 \equiv L^*(a, y_j) \equiv \tau_{r+1} L\left(\sum_{i=1}^n \alpha_i x_i , y_j\right) \equiv \tau_{r+1} \frac{\alpha_j}{\tau_j} \qquad (\text{mod } 1)$$

so that $\alpha_j = \gamma_j \tau_j/\tau_{r+1}$, where $\gamma_1 , \cdots , \gamma_n$ are integers, and hence

$$a = \sum_{i=1}^r \gamma_i \cdot \frac{\tau_i}{\tau_{r+1}} \; x_i + \sum_{i=r+1}^n \alpha_i x_i \, .$$

For any two r-tuples $(a_1 , \cdots , a_r)$ and $(b_1 , \cdots , b_r)$, $r \leqq n$, of elements of G we consider the matrix

$$\mathbf{L}^{(r)} = \mathbf{L}(a_1 , \cdots , a_r ; b_1 , \cdots , b_r) = \| L(a_i , b_j) \|_{i,j=1,\cdots,r} .$$

LEMMA 2. $\tau_1 \cdots \tau_r \mid \mathbf{L}^{(r)} \mid$ *is an integer; for any* $r \times r$ *integral matrix* $\mathbf{M}^{(r)}$, $\tau_1 \cdots \tau_r \mid \mathbf{L}^{(r)} + \mathbf{M}^{(r)} \mid$ *is also an integer and* $\tau_1 \cdots \tau_r \mid \mathbf{L}^{(r)} + \mathbf{M}^{(r)} \mid \equiv \tau_1 \cdots \tau_r \mid \mathbf{L}^{(r)} \mid \text{mod } \tau_r$.

PROOF BY INDUCTION ON r: For $r = 1$ the first statement follows from (2) because the order of every element of G divides τ_1 ; the second statement follows immediately because

$$\tau_1 \{ \mid \mathbf{L}^{(1)} + \mathbf{M}^{(1)} \mid - \mid \mathbf{L}^{(1)} \mid \} = \tau_1 \mid \mathbf{M}^{(1)} \mid.$$

We may express the elements $a_1 , \cdots , a_r$ and $b_1 , \cdots , b_r$ in terms of the dual bases $x_1 , \cdots , x_n$ and $y_1 , \cdots , y_n$ obtaining

$$a_i = \sum_{k=1}^n \alpha_{ik} x_k \qquad \text{and} \qquad b_j = \sum_{l=1}^n \beta_{jl} y_l ,$$

where α_{ik} and β_{jl} are integers. From (1) and (3) it follows that there exist integers μ_{ij}, $i, j = 1, \cdots, r$ such that

$$L(a_i, b_j) = \sum_{k=1}^{n} \frac{\alpha_{ik}\beta_{jk}}{\tau_k} + \mu_{ij}.$$

Assuming now the truth of the lemma for indices smaller than r, we note that $| \mathbf{L}^{(r)} + \mathbf{M}^{(r)} | - | \mathbf{L}^{(r)} | = | \mathbf{M}^{(r)} | + \sum_q \pm | \mathbf{L}_q | \cdot | \mathbf{M}_q |$ where $\mathbf{L}_q$ ranges over all the minors of $\mathbf{L}^{(r)}$ of order less than r and greater than 1 and $\mathbf{M}_q$ denotes the minor of $\mathbf{M}^{(r)}$ complementary to $\mathbf{L}_q$. By the inductive hypothesis $\tau_1 \cdots \tau_{r-1} | \mathbf{L}_q |$ is always an integer; since $| \mathbf{M}_q |$ is also always an integer it follows that $\tau_1 \cdots \tau_r(| \mathbf{L}^{(r)} + \mathbf{M}^{(r)} | - | \mathbf{L}^{(r)} |)$ is an integral multiple of τ_r. Applying this result to the case $\mathbf{M}^{(r)} = \| -\mu_{ij} \|_{i,j=1,\cdots,r}$, we see that to prove the first statement it suffices to prove that $\tau_1 \cdots \tau_r | \sum_{k=1}^{n} \frac{\alpha_{ik}\beta_{jk}}{\tau_k} |$ is an integer. But

$$\left| \sum_{k=1}^{n} \frac{\alpha_{ik}\beta_{jk}}{\tau_k} \right| = \sum_{k_1,\cdots,k_r=1}^{n} \left| \frac{\alpha_{ik_j}\beta_{jk_j}}{\tau_{k_j}} \right| = \sum_{k_1,\cdots,k_r=1}^{n} \frac{\beta_{1k_1} \cdots \beta_{rk_r}}{\tau_{k_1} \cdots \tau_{k_r}} | \alpha_{ik_j} |,$$

$| \alpha_{ik_j} |$ vanishes unless the indices $k_1, \cdots, k_r$ are distinct, and $\tau_{k_1} \cdots \tau_{k_r}$ divides $\tau_1 \cdots \tau_r$ if the indices $k_1, \cdots, k_r$ are distinct. Consequently $\tau_1 \cdots \tau_r$ $| \sum_{k=1}^{n} \frac{\alpha_{ik}\beta_{jk}}{\tau_k} |$ must be an integer, and the induction is complete.

The number $\tau_1 \cdots \tau_r | L(a_i, a_j) |_{i,j=1,\cdots,r}$, which according to Lemma 2, must be an integer, will be denoted by $D(a_1, \cdots, a_r)$, $r = 1, \cdots, n$.

THEOREM 1. *Let p be any odd prime divisor of τ_r/τ_{r+1}, where $1 \leqq r \leqq n$. Then there exist r-tuples $(a_1, \cdots, a_r)$ of elements of G for which $D(a_1, \cdots, a_r)$ is not divisible by p. Moreover*[8]

$$\left(\frac{D(a_1, \cdots, a_r)}{p}\right) = \left(\frac{D(b_1, \cdots, b_r)}{p}\right)$$

for any pair of such r-tuples $(a_1, \cdots, a_r)$ and $(b_1, \cdots, b_r)$.

PROOF. We prove this theorem first for the case $r = n$. Accordingly p is now any odd prime divisor of τ_n. To prove the first statement we express the basic elements $y_1, \cdots, y_n$ in terms of the dual basis $x_1, \cdots, x_n$:

$$y_j = \sum_{k=1}^{n} \gamma_{jk} x_k .$$

Then, by (1) and (3),

$$\frac{\delta_{ij}}{\tau_i} = L(x_i, y_j) \equiv \sum_{k=1}^{n} \gamma_{jk} L(x_i, x_k) \qquad (\text{mod } 1),$$

so that, by Lemma 2,

$$1 \equiv | \gamma_{jk} | \cdot D(x_1, \cdots, x_n) \qquad (\text{mod } p),$$

[8] (D/p) denotes the Legendre symbol and is equal to ± 1 according as D is or is not a quadratic residue mod p.

which shows that $D(x_1, \cdots, x_n)$ is not divisible by p. To prove the second statement we consider an arbitrary n-tuple $(a_1, \cdots, a_n)$ of elements of G and express these in terms of $x_1, \cdots, x_n$.

$$a_i = \sum_{k=1}^{n} \alpha_{ik} x_k .$$

Then, by (1),

$$L(a_i, a_j) \equiv \sum_{k,l=1}^{n} \alpha_{ik}\alpha_{jl} L(x_k, x_l) \qquad (\text{mod } 1),$$

so that, by Lemma 2,

$$D(a_1, \cdots, a_n) \equiv |\alpha_{ik}|^2 \cdot D(x_1, \cdots, x_n) \qquad (\text{mod } p).$$

Hence either $|\alpha_{ik}| = 0$ and $D(a_1, \cdots, a_n) \equiv 0 \quad (\text{mod } p)$ or

$$|\alpha_{ik}| \not\equiv 0, D(a_1, \cdots, a_n) \not\equiv 0 \qquad (\text{mod } p)$$

$$\text{and} \qquad \left(\frac{D(a_1, \cdots, a_n)}{p}\right) = \left(\frac{D(x_1, \cdots, x_n)}{p}\right).$$

For $r < n$ we apply the preceding result to the group $\bar{G}$, whose torsion coefficients according to Lemma 1 are $\tau_1/\tau_{r+1}, \cdots, \tau_r/\tau_{r+1}$. Accordingly we define

$$\bar{D}(\bar{a}_1, \cdots, \bar{a}_r) = \frac{\tau_1}{\tau_{r+1}} \cdots \frac{\tau_r}{\tau_{r+1}} |\bar{L}(\bar{a}_i, \bar{a}_j)|_{i,j=1,\cdots,r} .$$

By the special case of the theorem proved in the preceding paragraph applied to the group $\bar{G}$ we see that there exist r-tuples $(a_1, \cdots, a_r)$ of elements of G for which $D(\bar{a}_1, \cdots, \bar{a}_r)$ is not divisible by a given odd prime divisor of τ_r/τ_{r+1}, and that

$$\left(\frac{\bar{D}(\bar{a}_1, \cdots, \bar{a}_r)}{p}\right) = \left(\frac{\bar{D}(\bar{b}_1, \cdots, \bar{b}_r)}{p}\right)$$

for any pair of such r-tuples. To complete the proof of the theorem we have only to observe that

$$\bar{D}(\bar{a}_1, \cdots, \bar{a}_r) = \tau_1 \cdots \tau_r |L(a_i, a_j)|_{i,j=1,\cdots,r} = D(a_1, \cdots, a_r).$$

For any index $r = 1, \cdots, n$ and odd prime divisor p of τ_r/τ_{r+1} we define

$$\chi_r(p) = \left(\frac{D(a_1, \cdots, a_r)}{p}\right), \tag{4}$$

where $(a_1, \cdots, a_r)$ is any one of the r-tuples for which $p \nmid D(a_1, \cdots, a_r)$. This definition is valid by virtue of Theorem 1. Note that the "residue characters" $\chi_r(p)$ are, by their definition, invariants of the group with multiplication (G, L).

2. We now briefly recall the definition [5] of the Seifert invariants and relate them to our invariants χ.

Let p be an odd prime and write $\tau_i = p^{d_i}\tau_i'$, for $i = 1, \cdots, n$, where $d_i \geqq 0$ and p does not divide τ_i'. Let $r_1 < r_2 < \cdots < r_m$ be the indices r for which $\tau_r/\tau_{r+1} \equiv 0 \pmod p$, so that

$$d_1 = \cdots = d_{r_1} > d_{r_1+1} = \cdots = d_{r_2} > \cdots > d_{r_{m-1}+1} = \cdots = d_{r_m} > 0$$

and

$$d_r = 0 \text{ for } r > r_m,$$

where $r_m \leqq n$. The elements of G whose orders are powers of p form a subgroup for which the r_m elements $x_i' = \tau_i' x_i$, $i = 1, \cdots, r_m$ form a basis.

By (2)

$$\| L(x_i', x_j') \|_{i,j=1,\cdots,r_m} = \left\| \frac{\mathrm{A}_{kl}}{p^{\min(d_{r_k}, d_{r_l})}} \right\|_{k,l=1,\cdots,m}$$

where A_{kl} is an integral $(r_k - r_{k-1}) \times (r_l - r_{l-1})$ matrix. Seifert showed that none of the determinants $|\mathrm{A}_{11}|, \cdots, |\mathrm{A}_{mm}|$ are divisible by p. Seifert's invariants of (G, L) are the residue characters

$$\sigma_{r_k}(p) = \left(\frac{|\mathrm{A}_{kk}|}{p}\right), \qquad k = 1, \cdots, m.$$

For $l = 1, \cdots, m$,

$$\prod_{h=1}^{r_l} p^{d_h} \cdot | L(x_i', x_j') |_{i,j=1,\cdots,r_l} = \begin{vmatrix} \mathrm{A}_{11} & p^{d_{r_1}-d_{r_2}} \mathrm{A}_{12} & \cdots & p^{d_{r_1}-d_{r_l}} \mathrm{A}_{1l} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \cdots & p^{d_{r_2}-d_{r_l}} \mathrm{A}_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{A}_{l1} & \mathrm{A}_{l2} & \cdots & \mathrm{A}_{ll} \end{vmatrix}$$

$$\equiv |\mathrm{A}_{11}| \cdot |\mathrm{A}_{22}| \cdots |\mathrm{A}_{ll}| \pmod p.$$

By Lemma 2,

$$\prod_{h=1}^{r_l} \tau_h \cdot | L(x_i', x_j') |_{i,j=1,\cdots,r_l} \equiv \prod_{h=1}^{r_l} \tau_h \cdot | \tau_i' \tau_j' L(x_i, x_j) |_{i,j=1,\cdots,r_l} \pmod p,$$

so that

$$\prod_{h=1}^{r_l} p^{d_h} \cdot | L(x_i' \; x_j') | \equiv \prod_{h=1}^{r_l} p^{d_h} \cdot | \tau_i' \tau_j' L(x_i, x_j) | \pmod p$$

$$\equiv \prod_{h=1}^{r_l} \tau_h' \cdot D(x_1, \cdots, x_{r_l}) \pmod p.$$

Combining these two results we get

$$\prod_{k=1}^{l} \sigma_{r_k}(p) = \prod_{h=1}^{r_l} \left(\frac{\tau_h'}{p}\right) \chi_{r_l}(p), \qquad l = 1, \cdots, m, \tag{5}$$

from which one may compute either the σ from the χ or the χ from the σ.

Thus in particular the residue characters $\chi_r(p)$ form a complete system of invariants if the order of the group G is odd.[9]

3. If one were to set about calculating self-linking invariants for a given $(4N+3)$-dimensional manifold $\mathfrak{M}$, very likely one would first find (cf. [6]) a system of $(2N+1)$-dimensional cycles $A_1, \cdots, A_n$, representing elements $a_1, \cdots, a_n$ of the $(2N+1)$-dimensional torsion group G, a system of $2N$-dimensional chains $B_1, \cdots, B_n$ such that the boundary relations

$$B_i \to \sum_{j=1}^{n} f_{ij} A_j, \qquad i = 1, \cdots, n,$$

determine G, and then the $n \times n$ matrix of intersection numbers $s_{ij} = \mathfrak{S}(B_i, A_j^*)$, $i, j = 1, \cdots, n$, where A_j^* is a cycle homologous to A_j but in a dual subdivision. We shall now find a formula for $D(a_{h_1}, \cdots, a_{h_r})$ in terms of the two (non-singular) matrices $\mathbf{F} = \| f_{ij} \|$ and $\mathbf{S} = \| s_{ij} \|$. In order to obtain $\chi_r(p)$ from $D(a_{h_1}, \cdots, a_{h_r})$ it is only necessary to choose $h_1, \cdots, h_r$ in such a way that $D(a_{h_1}, \cdots, a_{h_r}) \not\equiv 0 \pmod p$.

Let $\mathfrak{B}(A_i, A_j^*)$ denote the linking number of the two cycles A_i and A_j^*. Then

$$s_{ik} = \mathfrak{S}(B_i, A_k^*) = \sum_{j=1}^{n} f_{ij} \mathfrak{B}(A_j, A_k^*).$$

Denoting by F_{ij} the cofactor of f_{ij} in $\mathbf{F}$, we deduce that

$$\begin{aligned} \frac{1}{|\mathbf{F}|} \sum_{i=1}^{n} F_{ij} s_{ik} &= \frac{1}{|\mathbf{F}|} \sum_{i,l=1}^{n} F_{ij} f_{il} \mathfrak{B}(A_l, A_k^*) \\ &= \frac{1}{|\mathbf{F}|} \sum_{l=1}^{n} |\mathbf{F}| \, \delta_{jl} \mathfrak{B}(A_l, A_k^*) \\ &= \mathfrak{B}(A_j, A_k^*). \end{aligned}$$

The self-linking L of G is defined as the multiplication over G that satisfies the condition

$$L(a_j, a_k) \equiv \mathfrak{B}(A_j, A_k^*) \pmod 1.$$

[9] For a group G of even order further invariants, analogous to the "supplementary characters" of [1], may be defined. For example theorem 1 holds with p replaced by 4 or 8 (other powers of 2 are useless) provided $(D/4)$ and $(D/8)$ are properly interpreted. However these invariants do not form a complete system as the following example shows: $\tau_1 = \tau_2 = 4$, $L(x_i, x_j) = \delta_{ij}$, $L'(x_i, x_j) = -\delta_{ij}$, $i, j = 1, 2$. These two (primitive) multiplications are indistinguishable by the character $\chi_2(4)$. But it is easy to see that, as a ranges over G, $L(a, a)$ ranges over O, $1/4$, $2/4$ and $L'(a, a)$ over O, $2/4$, $3/4$. This problem was considered by Van Kampen [7] and a solution was indicated. Nevertheless it seems that a usable set of invariants has yet to be found. In principle a reasonable solution of this problem should exist since the corresponding problem for quadratic forms has been solved.

Hence

$$L(a_j, a_k) \equiv \frac{1}{|\mathbf{F}|}\sum_{i=1}^{n} F_{ij} s_{ik} \qquad \text{(mod 1)}$$

Let us denote, for $1 \leqq i_1 < \cdots < i_r \leqq n$ and $1 \leqq h_1 < \cdots < h_r \leqq n$, by $F_{i_1,\cdots,i_r\,;\,h_1,\cdots,h_r}$ the product of $(-1)^{i_1+\cdots+i_r+h_1+\cdots+h_r}$ and the determinant of the $(n - r) \times (n - r)$ matrix obtained from $\mathbf{F}$ by deleting the i_1th, $\cdots$, i_rth rows and the h_1th, $\cdots$, h_rth columns, and by $(\mathbf{F}\between\mathbf{S})_{h_1\cdots h_r}$ the "hybrid" matrix obtained from $\mathbf{F}$ by replacing its h_lth column by the corresponding column of $\mathbf{S}$ for $l = 1, \cdots, r$. Then

$$D(a_{h_1}, \cdots, a_{h_r}) = \tau_1 \cdots \tau_r \cdot |L(a_{h_j}, a_{h_k})| \qquad j, k = 1, \cdots, r$$

$$\equiv \tau_1 \cdots \tau_r \cdot \left| \frac{1}{|\mathbf{F}|} \sum_{i_j=1}^{n} F_{i_j h_j} s_{i_j h_k} \right| \qquad (\text{mod } \tau_r)$$

(by Lemma 2)

$$\equiv \frac{\tau_1 \cdots \tau_r}{|\mathbf{F}|^r} \sum_{i_1,\cdots,i_r=1}^{n} F_{i_1 h_1} \cdots F_{i_r h_r} |s_{i_j h_k}| \qquad (\text{mod } \tau_r)$$

$$\equiv \frac{\tau_1 \cdots \tau_r}{|\mathbf{F}|^r} \sum_{1 \leqq i_1 < \cdots < i_r \leqq n} |F_{i_j h_k}| \cdot |s_{i_j h_k}| \qquad (\text{mod } \tau_r)$$

$$\equiv \frac{\tau_1 \cdots \tau_r}{|\mathbf{F}|^r} \sum_{1 \leqq i_1 < \cdots < i_r \leqq n} |\mathbf{F}|^{r-1} \cdot F_{i_1 \cdots i_r;\, h_1 \cdots h_r} \cdot |s_{i_j h_k}| \qquad (\text{mod } \tau_r)$$

(by Jacobi's theorem of the adjugate)

$$\equiv \frac{\tau_1 \cdots \tau_r}{|\mathbf{F}|} \cdot |(\mathbf{F}\between\mathbf{S})_{h_1 \cdots h_r}| \qquad (\text{mod } \tau_r)$$

(6)

$$\equiv \frac{\pm 1}{\tau_{r+1} \cdots \tau_n} \cdot |(\mathbf{F}\between\mathbf{S})_{h_1 \cdots h_r}| \qquad (\text{mod } \tau_r).$$

4. The restriction to the n-dimensional group of a $(2n + 1)$-dimensional manifold may be removed by dualizing homology to cohomology and intersection to cup product. Let $\mathfrak{K}$ be a complex of any dimension, H^{n+1} the $(n + 1)$-dimensional cohomology group of $\mathfrak{K}$ with integral coefficients, and $H^{2n+1}(R)$ the $(2n + 1)$-dimensional cohomology group of $\mathfrak{K}$ with coefficient group R = rationals mod 1. A multiplication Q, with respect to $H^{2n+1}(R)$, is defined over the torsion subgroup T of H^{n+1} as follows: If $u, v \in T$ and U, V are cocycles representing u, v respectively there is an n-dimensional cochain W whose coboundary is βV, where β is the order of the element v of T. Then $1/\beta\ (U \smile W)$, regarded as a cochain with coefficients in R, is a cocycle, and, as such, represents an element $Q(u, v)$ of $H^{2n+1}(R)$. It may be shown that $Q(u, v)$ is independent of all the choices made in this construction and that the multiplication Q is symmetric or skew-symmetric according as n is odd or even. If $\mathfrak{R}$ is a $(2n + 1)$-dimensional closed, oriented manifold $H^{2n+1}(R) \approx R$ and duality establishes the equivalence of (T, Q) with (G, L).

To obtain numerical invariants of (T, Q) in the general case one would try to generalize the results of §1 by replacing R by $H^{2n+1}(R)$. Although this might be a useful thing to do we have not attempted it as we do not at present see any immediate application.

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REFERENCES

1. H. J. S. SMITH, *On the orders and genera of quadratic forms containing more than three indeterminates*. Proc. Roy. Soc. London 13 (1864) 199–203; Coll. Math. Papers I, 412–7.
2. L. PONTRJAGIN, *Über den algebraischen Inhalt topologischer Dualitätssätze*. Math. Ann. 105 (1931) 165–205.
3. G. DE RHAM, *Sur l'analysis situs des variétés à n dimensions*, J. Math. Pures Appl. 10 (1931) 156–165.
4. H. SEIFERT and W. THRELFALL, Lehrbuch der topologie, Leipzig, 1934 and New York, 1947.
5. H. SEIFERT, *Verschlingungsinvarianten*, Preuss. Akad. Wiss. Sitzungsber. Berlin 26 (1933) 811–828.
6. H. SEIFERT, *Die Verschlingungsinvarianten der zyklischen Knotenüberlagerung*, Hamburg Abhandl. 11 (1935) 84–101.
7. E. R. VAN KAMPEN, *Invariants derived from looping coefficients*. Amer. J. Math. 60 (1938) 595–610. .
8. R. C. BLANCHFIELD, The linking invariants, Junior Paper, Princeton, 1949.