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INTERSECTION THEORY OF MANIFOLDS WITH OPERATORS WITH APPLICATIONS TO KNOT THEORY

BY RICHARD C. BLANCHFIELD†*

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Introduction

Let $\mathfrak{M}$ be an oriented combinatorial manifold with boundary, let $\mathfrak{M}^{\ddagger}$ be any of its covering complexes, and let G be any free abelian group of covering transformations. The homology groups of $\mathfrak{M}^{\ddagger}$ are R-modules, where R denotes the integral group ring of the (multiplicative) group G. Each such homology module H has a well-defined torsion sub-module T, and the corresponding Betti module is $B = H/T$.

The automorphism $\gamma \rightarrow \gamma^{-1}$ of the group G extends to a unique automorphism $\alpha \rightarrow \bar{\alpha}$ of the ring $R = \bar{R}$. Following Reidemeister [4] there is defined an intersection S which is a pairing of the homology modules of dual dimension to the ring R, and also a linking V which is a pairing of dual torsion sub-modules to R_0/R, where R_0 is the quotient field of R. Two duality theorems are proved:

(1) S is a primitive pairing to R/π^m of dual Betti modules with coefficients modulo π^m and $\bar{\pi}^m$ respectively, where π is zero or a prime element of R and m is any positive integer.

(2) V is a primitive pairing of dual torsion modules to R_0/R.

These theorems are analogous to the Burger duality theorems [1]; in case R is the ring of integers, they specialize to the Poincaré-Lefschetz duality theorems for manifolds with boundary.

Although dual modules are not, in general, isomorphic, it is demonstrated that if one torsion module has elementary divisors $\Delta_0 \subset \Delta_1 \subset \cdots$ then its dual has elementary divisors $\bar{\Delta}_0 \subset \bar{\Delta}_1 \subset \cdots$. (The numbering differs from that of [2] in that we begin with the first non-zero ideal.)

This result is applied to the maximal abelian covering of a link in a closed 3-manifold. It is proved that the elementary divisors Δ_i of the 1-dimensional torsion module are symmetric in the sense that $\Delta_i = \bar{\Delta}_i$. For the case of a knot or link in Euclidean 3-space, the ideal Δ_0 is generated by the Alexander polynomial, and the symmetry of Δ_0 has been proved previously by Seifert [6] for knots and Torres [7] for links.

The "symmetry" of the Alexander polynomial was proved by Seifert and Torres in a slightly more precise form [8, Cors. 2 and 3]. The problem of similar extra precision in the more general case of a link in an arbitrary closed 3-manifold remains open.

* Died July 25, 1955. The revision of his Princeton 1954 Ph.D. dissertation presented here was made by J. W. Milnor and R. H. Fox. Aside from minor corrections the only changes were a strengthening of Lemma 4.3 and the insertion of Lemma 4.10 and Corollary 5.6.

1. Let $\mathfrak{M}$ be an oriented n-dimensional manifold with boundary $\mathfrak{B}$. Let G be a multiplicative free abelian group of orientation-preserving homeomorphisms of $\mathfrak{M}$ onto itself such that no point of $\mathfrak{M}$ is fixed under any element of G other than the identity. We assume that $\mathfrak{M}$ has been triangulated in such a way that G operates on $\mathfrak{M}$ as a complex; that is to say, the elements of G map cells onto cells preserving the incidence relations. The canonical example of such a manifold is a covering complex of a triangulated manifold where the covering transformations form a free abelian group.

Let $\tilde{\mathfrak{M}}$ be the dual cell complex of $\mathfrak{M}$. Except for those cells of $\mathfrak{M}$ which lie entirely in the boundary $\mathfrak{B}$, each cell of $\mathfrak{M}$ has a dual cell in $\tilde{\mathfrak{M}}$. If the original triangulation of $\mathfrak{M}$ is sufficiently fine, the complex $\tilde{\mathfrak{M}}$ will be a deformation retract of $\mathfrak{M}$. We use $\mathfrak{M}$ for the study of relative homology and $\tilde{\mathfrak{M}}$ for absolute homology.

Let R be the group ring of G with integer coefficients. Define $\bar{\gamma} = \gamma^{-1}$ for all $\gamma \in G$. The mapping $\gamma \rightarrow \bar{\gamma}$ can be extended to an automorphism of R by linearity. Observe that for any $\alpha \in R$, $\bar{\bar{\alpha}} = \alpha$.

Let C_i be the group of i-dimensional chains of $\mathfrak{M}$ modulo $\mathfrak{B}$, and let $\tilde{C}_{n-i}$ be the group of $(n - i)$-dimensional chains of $\tilde{\mathfrak{M}}$, $i = 0, \cdots, n$. The elements of G operate naturally as automorphisms on C_i and $\tilde{C}_{n-i}$, and by linearity we may regard the elements of R as operators on C_i and $\tilde{C}_{n-i}$. Hence [3] we may regard C_i and $\tilde{C}_{n-i}$ as R-modules. Note that C_i and $\tilde{C}_{n-i}$ are free modules in the sense that each one is isomorphic with the direct sum of R with itself a finite number of times.

For any $x \in C_i$ let $\partial_i x$ be the relative boundary of x. Then ∂ is an operator homomorphism of C_i into C_{i-1} and we have the usual relation $\partial_{i-1}\partial_i = 0$ for $i = 1, \cdots, n$. Similarly we have operator homomorphisms $\tilde{\partial}_i : \tilde{C}_i \rightarrow \tilde{C}_{i-1}$. Where no confusion can result, we write ∂_i and $\tilde{\partial}_i$ in abbreviated form as ∂. In the usual way one can define relative and absolute homology modules. As abelian groups these homology modules are the usual homology groups. The operator structure of the modules is inherited from the corresponding structure on the chain modules. The homology modules are invariants of the pair $(\mathfrak{M}, G)$, where we consider two pairs $(\mathfrak{M}, G)$ and $(\mathfrak{M}', G)$ equivalent if there exists a homeomorphism φ of $\mathfrak{M}$ onto $\mathfrak{M}'$ such that $\gamma\varphi = \varphi\gamma$ for all $\gamma \in G$.

Following Reidemeister [4] we introduce an intersection S as follows. Let $\mathfrak{S}$ denote the ordinary intersection of chains (disregarding operators). For $x \in C_i$ and $\tilde{x} \in \tilde{C}_{n-i}$ define

$$S(x, \tilde{x}) = \textstyle\sum_{\gamma \in G} \mathfrak{S}(x, \gamma\tilde{x})\gamma \in R.$$

All but a finite number of terms on the right are zero, so the sum is finite. The following properties of S are easily verified.

(1) $S(x + y, \tilde{x}) = S(x, \tilde{x}) + S(y, \tilde{x})$.

(2) $S(x, \tilde{x} + \tilde{y}) = S(x, \tilde{x}) + S(x, \tilde{y})$.

(3) $S(\alpha x, \bar{\beta}\tilde{x}) = \alpha\beta S(x, \tilde{x})$ for all $\alpha, \beta \in R$.

(4) $S(x, \partial\tilde{x}) = (-1)^i S(\partial x, \tilde{x})$ where $x \in C_i$ and $\tilde{x} \in \tilde{C}_{n-i+1}$.

(5) There exists a pair of dual bases $x_1, \cdots, x_r$ for C_i and $\tilde{x}_1, \cdots, \tilde{x}_r$ for $\tilde{C}_{n-1}$ such that $S(x_i, \tilde{x}_j) = \delta_{ij}$ for $i, j = 1, \cdots, r$.

It follows from (4) that the intersection of a cycle with a bounding cycle is zero, and hence the intersection S gives rise to an intersection of homology classes.

We now abstract the algebraic properties of this setup which are used in the following sections. Let R be a Noetherian ring with unit element 1. We assume that there is given a fixed isomorphism of R onto a ring $\bar{R}$. In the application we will have $R = \bar{R}$, and the relation $\bar{\bar{\alpha}} = \alpha$ will be valid.

Definition. An *R-module* M is an additive abelian group with operators R such that:

(1) $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in R$, $x \in M$

(2) $1 \cdot x = x$ for all $x \in M$.

Since R is a Noetherian ring it follows that if M is finitely generated, then every submodule of M is finitely generated.

Definition. Let M_1 and M_3 be R-modules and let M_2 be an $\bar{R}$-module. Then P is a *pairing* of M_1 and M_2 to M_3 if P assigns to every pair (x_1, x_2), $x_1 \in M_1$, $x_2 \in M_2$, an element $P(x_1, x_2) \in M_3$ such that

(1) $P(x_1 + y_1, x_2) = P(x_1, x_2) + P(y_1, x_2)$ for all $x_1, y_1 \in M_1$, $x_2 \in M_2$

(2) $P(x_1, x_2 + y_2) = P(x_1, x_2) + P(x_1, y_2)$ for all $x_1 \in M_1$, $x_2, y_2 \in M_2$

(3) $P(\alpha x_1, \bar{\beta} x_2) = \alpha\beta P(x_1, x_2)$ for all $\alpha, \beta \in R$, $x_1 \in M_1$, $x_2 \in M_2$.

Definition. Let P be a pairing of M_1 and M_2 to M_3. The *annihilator* A_1 of M_2 is the submodule of all elements $x_1 \in M_1$ for which $P(x_1, x_2) = 0$ for all $x_2 \in M_2$. Similarly the *annihilator* A_2 of M_1 is the submodule of all elements $x_2 \in M_2$ for which $P(x_1, x_2) = 0$ for all $x_1 \in M_1$.

Definition. A pairing P of M_1 and M_2 to M_3 is *primitive* if the annihilators of M_1 and M_2 are both zero.

Definition. Let M_1 be a finitely generated free R-module and let M_2 be a finitely generated free $\bar{R}$-module. Let P be a pairing of M_1 and M_2 to R. Then bases $x_1, \cdots, x_r$ of M_1 and $y_1, \cdots, y_r$ of M_2 are called *dual bases* when $P(x_i, y_j) = \delta_{ij}$ for $i, j = 1, \cdots, r$.

Note that if there exists a pair of dual bases, then the pairing is primitive.

Definition. An n-dimensional *chain complex* with coefficients in R is a system of finitely generated free R-modules and operator homomorphisms

$$C_0(R) \xleftarrow{\partial_1} C_1(R) \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_n} C_n(R)$$

such that $\partial_{i-1}\partial_i = 0$ for $i = 1, \cdots, n$. The submodule of *cycles* $Z_i(R)$ is the kernel of ∂_i, $i = 1, \cdots, n$. $Z_0(R) = C_0(R)$. The submodule of *bounding cycles* $B_i(R)$ is the image of ∂_{i+1}, $i = 0, \cdots, n-1$. $B_n(R) = 0$.

Definition. Let $\{C_i(R)\}$ and $\{\tilde{C}_{n-i}(\bar{R})\}$ be n-dimensional chain complexes with coefficients in R and $\bar{R}$ respectively. Then $\{C_i\}$ and $\{\tilde{C}_{n-i}\}$ are said to be *dual chain complexes* with pairing S if:

(1) S is a pairing of C_i and $\tilde{C}_{n-i}$ to R, for $i = 0, \cdots, n$.

(2) $S(x, \partial\tilde{x}) = (-1)^i S(\partial x, \tilde{x})$, for all $x \in C_i$, $\tilde{x} \in \tilde{C}_{n-i+1}$.

(3) There exists a pair of dual bases for C_i and $\tilde{C}_{n-i}$ for $i = 0, \cdots, n$.

Note that the chain modules introduced earlier form dual chain complexes with the intersection pairing S. For another example, take C_i to be chain modules from some geometric complex and take $\tilde{C}_{n-i}$ to be the i-dimensional cochains. Then the Kronecker product forms a suitable pairing.

2. THEOREM 2.1. *Let P be a pairing of M_1 and M_2 to M_3. Let A_1 be the annihilator of M_2 and let A_2 be the annihilator of M_1. Then P induces a natural primitive pairing P_η of M_1/A_1 and M_2/A_2 to M_3.*

PROOF. Let $\eta_i: M_i \to M_i/A_i$, $i = 1, 2$, be the natural homomorphisms. Define $P_\eta(\eta_1(x_1), \eta_2(x_2)) = P(x_1, x_2) \in M_3$ for all $x_1 \in M_1$, $x_2 \in M_2$. This P_η is well defined because, for $i = 1, 2$:

$$\eta_i(x_i) = \eta_i(y_i) \Rightarrow x_i - y_i \in A_i$$

$$\Rightarrow P(x_1, x_2) - P(y_1, y_2) = P(x_1, x_2) - P(x_1, y_2) + P(x_1, y_2) - P(y_1, y_2)$$

$$= P(x_1, x_2 - y_2) - P(x_1 - y_1, y_2) = 0$$

$$\Rightarrow P(x_1, x_2) = P(y_1, y_2).$$

Trivially P_η is a pairing of M_1/A_1 and M_2/A_2 to M_3. To show that P_η is primitive:

(1) $P_\eta(\eta_1(x_1), \eta_2(x_2)) = 0$ for all $\eta_2(x_2) \in M_2/A_2$
$\Rightarrow P(x_1, x_2) = 0$ for all $x_2 \in M_2 \Rightarrow x_1 \in A_1 \Rightarrow \eta_1(x_1) = 0$.

(2) Similarly $P_\eta(\eta_1(x_1), \eta_2(x_2)) = 0$ for all $\eta_1(x_1) \in M_1/A_1$
$\Rightarrow P(x_1, x_2) = 0$ for all $x_1 \in M_1 \Rightarrow x_2 \in A_2 \Rightarrow \eta_2(x_2) = 0$.

We now assume that R is a unique factorization domain with $R = \bar{R}$ and such that the identity $\bar{\bar{\alpha}} = \alpha$ holds. Let $\{C_i(R)\}$ and $\{\tilde{C}_{n-i}(R)\}$ be dual chain complexes with pairing S. Let π be a prime element of R. Then in a natural way we have dual chain complexes $\{C_i(R/\pi^m)\}$ and $\{\tilde{C}_{n-i}(R/\bar{\pi}^m)\}$ with a pairing which we also call S. It is convenient to regard 0 as a prime so that R/π^m may be the same as R.

If we restrict S to be a pairing of the cycles $Z_i(R/\pi^m)$ and $\tilde{Z}_{n-i}(R/\bar{\pi}^m)$, then this pairing is in general no longer primitive. The object of the remainder of this section is to determine the annihilating submodule of the cycles. The basic tool for this investigation is the consideration of chains with coefficients in local rings.

Let π be a prime element of R and let $\eta: R \to R/\pi^m$, $m > 0$, be the natural homomorphism. We contend that $\eta(\pi)$ generates a prime ideal in R/π^m. Suppose $\eta(\alpha)\cdot\eta(\beta) \in \eta(\pi)(R/\pi^m)$ for some $\alpha, \beta \in R$. Then

$$\eta(\alpha)\cdot\eta(\beta) = \eta(\pi)\cdot\eta(\gamma) \text{ for some } \gamma \in R$$

$$\Rightarrow \alpha\beta = \pi\gamma + \pi^m\delta \text{ for some } \gamma, \delta \in R$$

$$\Rightarrow \pi \text{ divides } \alpha \text{ or } \pi \text{ divides } \beta$$

$$\Rightarrow \eta(\alpha) \in \eta(\pi)(R/\pi^m) \quad \text{or} \quad \eta(\beta) \in \eta(\pi)(R/\pi^m).$$

Therefore $\eta(\pi)$ generates a prime ideal. Let $(R/\pi^m)_{\eta(\pi)}$ denote the local ring at this prime ideal. The ring $(R/\pi^m)_{\eta(\pi)}$ is the ring of quotients of elements of R/π^m where only elements not belonging to the ideal $\eta(\pi)(R/\pi^m)$ are allowed as denominators. In particular the local ring R_π is just the subring of the quotient field of R which consists of those elements that can be written with denominators prime to π. In particular R_0 is the quotient field of R. We always regard R/π^m as a subring of $(R/\pi^m)_{\eta(\pi)}$.

LEMMA 2.2. *There is a natural isomorphism between the rings* $(R/\pi^m)_{\eta(\pi)}$ *and* $R_\pi/\pi^m R_\pi$.

PROOF. We define a mapping $\varphi: R_\pi \to (R/\pi^m)_{\eta(\pi)}$ by setting $\varphi(\alpha/\beta) = \eta(\alpha)/\eta(\beta)$, $\alpha, \beta \in R$ with β prime to π. Note that $\eta(\alpha)/\eta(\beta)$ is admissible as an element of $(R/\pi^m)_{\eta(\pi)}$ because if $\eta(\beta) = \eta(\pi)\eta(\gamma)$ for some $\gamma \in R$, then $\beta = \pi\gamma + \pi^m\delta$ for some $\gamma, \delta \in R$, which contradicts the assumption that β is prime to π. It is easily verified that the mapping φ is a homomorphism of R_π onto $(R/\pi^m)_{\eta(\pi)}$. To determine the kernel of φ let $\alpha/\beta \in R_\pi$. Then

$$\begin{aligned}
&\varphi(\alpha/\beta) = \eta(\alpha)/\eta(\beta) = 0 \\
&\Leftrightarrow \eta(\alpha) = 0 \\
&\Leftrightarrow \alpha = \pi^m\gamma \text{ for some } \gamma \in R \\
&\Leftrightarrow \alpha \in \pi^m R_\pi \\
&\Leftrightarrow \alpha/\beta \in \pi^m R_\pi \text{ because } \beta \text{ is a unit in } R_\pi .
\end{aligned}$$

Hence the kernel of φ is $\pi^m R_\pi$.

The ideals in R_π have a particularly simple structure. Any element in R which is prime to π is a unit in R_π. Hence for $\alpha/\beta \in R_\pi$, α/β belongs to a given ideal if and only if α belongs to the ideal. Let $\pi^r\gamma = \alpha$ where γ is prime to π, $r \geqq 0$. Then α belongs to the given ideal if and only if π^r belongs to the ideal. Hence *the only ideals in* R_π *are the principal ideals generated by some power of* π.

We return now to our dual chain complexes $\{C_i(R)\}$ and $\{\tilde{C}_{n-i}(R)\}$ with the pairing S. These chain complexes may be extended to a pairing (which we also call S) of $C_i(R_\pi)$ and $\tilde{C}_{n-i}(R_{\bar{\pi}})$ to R_π. The dual bases $x_1, \cdots, x_r$ of $C_i(R)$ and $\tilde{x}_1, \cdots, \tilde{x}_r$ of $\tilde{C}_{n-i}(R)$ are also dual bases for $C_i(R_\pi)$ and $\tilde{C}_{n-i}(R_\pi)$ respectively.

Since R_π is a principal ideal ring whose only ideals are $\pi^e R_\pi$, $e = 0, 1, 2, \cdots$, there exists a basis $y_1, \cdots, y_r$ of $C_i(R_\pi)$ such that $\pi^{e_1}y_1, \cdots, \pi^{e_r}y_r$ generate $B_i(R_\pi)$, the submodule of bounding cycles. Expressing the $y_1, \cdots, y_r$ in terms of the $x_1, \cdots, x_r$ and conversely, we have

$$y_i = \textstyle\sum_\nu \alpha_{i\nu}x_\nu \qquad x_i = \sum_\nu \beta_{i\nu}y_\nu$$
$$\textstyle\sum_\nu \alpha_{i\nu}\beta_{\nu j} = \delta_{ij} = \sum_\nu \alpha_{\nu i}\beta_{j\nu}.$$

Let $\tilde{y}_i = \sum_\nu \bar{\beta}_{\nu i}\tilde{x}_\nu$. Then $\tilde{y}_1, \cdots, \tilde{y}_r$ is also a basis for $\tilde{C}_{n-i}(R_{\bar{\pi}})$ because

$$\tilde{x}_i = \textstyle\sum_{\nu,\mu} \bar{\alpha}_{\nu i}\bar{\beta}_{\mu\nu}\tilde{x}_\mu = \sum_\nu \bar{\alpha}_{\nu i}\tilde{y}_\nu .$$

Also $P(y_i, \tilde{y}_j) = P(\sum_\nu \alpha_{i\nu} x_\nu, \sum_\nu \bar{\beta}_{\nu j} \tilde{x}_\nu) = \sum_\nu \alpha_{i\nu}\beta_{\nu j} = \delta_{ij}$ so that $y_1, \cdots, y_r$ and $\tilde{y}_1, \cdots, \tilde{y}_r$ are dual bases.

Now considering the chain complexes with coefficients in $R_\pi/\pi^m R_\pi$ and $R_{\bar{\pi}}/\bar{\pi}^m R_{\bar{\pi}}$ respectively, or, what is the same thing, with coefficients in $(R/\pi^m)_{\eta(\pi)}$ and $(R/\bar{\pi}^m)_{\eta(\bar{\pi})}$, we have dual bases $y_1, \cdots, y_r$ and $\tilde{y}_1, \cdots, \tilde{y}_r$ of $C_i((R/\pi^m)_{\eta(\pi)})$ and $\tilde{C}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$ respectively such that $\eta(\pi^{e_1})y_1, \cdots, \eta(\pi^{e_r})y_r$ generate $B_i((R/\pi^m)_{\eta(\pi)})$, $0 \leqq e_\nu \leqq m - 1$. (We could equally well have chosen the dual bases so that $\eta(\bar{\pi}^{e_1})\tilde{y}_1, \cdots, \eta(\bar{\pi}^{e_r})\tilde{y}_r$ generate $\tilde{B}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$.

LEMMA 2.3. *The annihilator of the submodule of cycles $\tilde{Z}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$ is the submodule of bounding cycles $B_i((R/\pi^m)_{\eta(\pi)})$. Dually the annihilator of $Z_i((R/\pi^m)_{\eta(\pi)})$ is $\tilde{B}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$.*

PROOF. Let $\tilde{z} = \eta(\bar{\beta}_1)\tilde{y}_1 + \cdots + \eta(\bar{\beta}_r)\tilde{y}_r$ be any element of $\tilde{C}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$.

Then

$$\begin{aligned}
&\tilde{z} \in \tilde{Z}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})\\
&\Leftrightarrow \partial\tilde{z} = 0\\
&\Leftrightarrow S(x, \partial\tilde{z}) = 0 \text{ for all } x \in C_{i+1}((R/\pi^m)_{\eta(\pi)})\\
&\Leftrightarrow S(\partial x, \tilde{z}) = 0 \text{ for all } x \in C_{i+1}((R/\pi^m)_{\eta(\pi)})\\
&\Leftrightarrow \tilde{z} \text{ annihilates } B_i((R/\pi^m)_{\eta(\pi)})\\
&\Leftrightarrow S(\eta(\pi^{e_i})y_i, z) = 0 \text{ for } i = 1, \cdots, r\\
&\Leftrightarrow S(\eta(\pi^{e_i})y_i, \eta(\bar{\beta}_i)\tilde{y}_i) = \eta(\pi^{e_i}\beta_i) = 0 \text{ for } i = 1, \cdots, r\\
&\Leftrightarrow \eta(\beta_i) \equiv 0 \pmod{\pi^{n-e_i}} \text{ for } i = 1, \cdots, r\\
&\Leftrightarrow \tilde{z} \text{ is a linear combination of } \eta\bar{\pi}^{n-e_1}\tilde{y}_1, \cdots, \eta\bar{\pi}^{n-e_r}\tilde{y}_r .
\end{aligned}$$

Therefore $\tilde{Z}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$ is generated by $\eta\bar{\pi}^{n-e_1}\tilde{y}_1, \cdots, \eta\bar{\pi}^{n-e_r}\tilde{y}_r$. An element $a = \alpha_1 y_1 + \cdots + \alpha_r y_r \in C_i((R/\pi^m)_{\eta(\pi)})$ annihilates $\tilde{Z}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$

$$\begin{aligned}
&\Leftrightarrow S(a, \eta\bar{\pi}^{n-e_i}\tilde{y}_i) = 0 \text{ for } i = 1, \cdots, r\\
&\Leftrightarrow S(\alpha_i y_i, \eta\bar{\pi}^{n-e_i}\tilde{y}) = \eta(\alpha_i\bar{\pi}^{n-e_i}) = 0 \text{ for } i = 1, \cdots, r\\
&\Leftrightarrow \alpha_i \equiv 0 \pmod{\pi^{e_i}} \text{ for } i = 1, \cdots, r\\
&\Leftrightarrow a \in B_i((R/\pi^m)_{\eta(\pi)}).
\end{aligned}$$

The proof of the dual statement is similar.

LEMMA 2.4. *The annihilator of $\tilde{Z}_{n-i}(R/\bar{\pi}^m)$ in $C_i(R/\pi^m)$ is $B_i((R/\pi^m)_{\eta(\pi)}) \cap C_i(R/\pi^m)$. Dually the annihilator of $Z_i(R/\pi^m)$ is $\tilde{B}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})}) \cap \tilde{C}_{n-i}(R/\bar{\pi}^m)$.*

PROOF. If $x \in B_i((R/\pi^m)_{\eta(\pi)}) \cap C_i(R/\pi^m)$ then x annihilates $\tilde{Z}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$ by the preceding lemma, so, trivially, x annihilates $\tilde{Z}_{n-i}(R/\bar{\pi}^m)$.

Conversely suppose $x \in C_i(R/\pi^m)$ and x annihilates $\tilde{Z}_{n-i}(R/\bar{\pi}^m)$. Let z be any element of $\tilde{Z}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$. Since $\tilde{C}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$ is of finite rank, there exists an element $\eta(\bar{\beta}) \in R/\bar{\pi}^m$ such that $\eta(\bar{\beta})$ is not a multiple of $\eta(\bar{\pi})$ and

$$\eta(\bar{\beta})\tilde{z} \in \tilde{Z}_{n-i}(R/\bar{\pi}^m).$$

Then

$$S(x,\ \eta(\bar{\beta})\tilde{z}) = \eta(\beta)S(x,\ \tilde{z}) = 0 \text{ in } R/\pi^m$$

$$\Rightarrow S(x,\ \tilde{z}) = 0 \text{ in } R/\pi^m \text{ because } \eta(\beta) \text{ is not a multiple of } \eta(\pi).$$

Hence x annihilates $\tilde{Z}_{n-i}((R/\bar{\pi}^m)_{\eta(\bar{\pi})})$, and by the preceding lemma

$$x \in B_i((R/\pi^m)_{\eta(\pi)}).$$

The proof of the dual statement is essentially the same.

DEFINITION. A cycle $z \in Z_i(R/\pi^m)$ is *weakly bounding modulo* π^m if there exists a chain $x \in C_{i+1}(R/\pi^m)$ such that $\partial x = \eta(\alpha)z$ where $\alpha \in R$ is prime to π. Weakly bounding modulo zero is simply called weakly bounding.

LEMMA 2.5. *A cycle $z \in Z_i(R/\pi^m)$ is weakly bounding modulo π^m if and only if $z \in B_i((R/\pi^m)_{\eta(\pi)}) \cap C_i(R/\pi^m)$.*

PROOF. If $\partial x = \eta(\alpha)z$, with $\alpha \in R$ prime to π, then $\partial(\eta(\alpha)^{-1}x) = z$ with coefficients in $(R/\pi^m)_{\eta(\pi)}$, so that $z \in B_i((R/\pi^m)_{\eta(\pi)}) \cap C_i(R/\pi^m)$.

Conversely given $z \in B_i((R/\pi^m)_{\eta(\pi)}) \cap C_i(R/\pi^m)$, there exists a chain $x \in C_{i+1}((R/\pi^m(_{\eta(\pi)})$ such that $\partial x = z$. But for any $x \in C_{i+1}((R/\pi^m)_{\eta(\pi)})$ there exists an $\alpha \in R$ prime to π such that $\eta(\alpha)x \in C_{i+1}(R/\pi^m)$. Hence $\partial(\eta(\alpha)x) = \eta(\alpha)z$ is the desired relation.

We are now in a position to state and prove the main theorem of this section. We have assumed that R is a Noetherian unique factorization domain with an automorphism $\alpha \to \bar{\alpha}$ such that $\bar{\bar{\alpha}} = \alpha$. We have a prime element π in $R(\pi = 0$ is allowed) and dual chain complexes $\{C_i(R)\}$ and $\{\tilde{C}_{n-i}(R)\}$ with pairing S.

DUALITY THEOREM 2.6. *The pairing S defines a primitive pairing of $Z_i(R/\pi^m)/W_i(R/\pi^m)$ and $\tilde{Z}_{n-i}(R/\bar{\pi}^m)/\tilde{W}_{n-i}(R/\bar{\pi}^m)$ to R/π^m, where $Z_i(R/\pi^m)$ and $\tilde{Z}_{n-i}(R/\bar{\pi}^m)$ are the cycles modulo π^m and $\bar{\pi}^m$ respectively, and $W_i(R/\pi^m)$ and $\tilde{W}_{n-i}(R/\bar{\pi}^m)$ are the weakly bounding cycles modulo π^m and $\bar{\pi}^m$ respectively.*

PROOF. It remains only to observe that since $W_i(R/\pi^m)$ is the annihilator of $Z_i(R/\pi^m)$ and $\tilde{W}_{n-i}(R/\pi^m)$ is the annihilator of $\tilde{Z}_{n-i}(R/\bar{\pi}^m)$, we may apply Theorem 2.1.

3. $\{C_i(R)\}$ and $\{\tilde{C}_{n-i}(R)\}$ are dual chain complexes with pairing S. Unless indicated otherwise, all coefficients are in R. Let W_{i-1} and $\tilde{W}_{n-i}$ be the modules of weakly bounding cycles. We now introduce the notion of linking V, which is a pairing of W_{i-1} and $\tilde{W}_{n-i}$ to R_0/R, where R_0 is the quotient field of R. Let $w \in W_{i-1}$ and $\tilde{w} \in \tilde{W}_{n-i}$ and choose chains x and $\tilde{x}$ such that $\partial x = \alpha w$ and $\partial\tilde{x} = \bar{\beta}\tilde{w}$ with $\alpha, \bar{\beta} \neq 0$. Observe first of all that

$$\frac{1}{\alpha}S(x,\ \tilde{w}) = \frac{1}{\alpha\beta}S(x,\ \partial\tilde{x}) = (-1)^i\frac{1}{\alpha\beta}S(\partial x,\ \tilde{x}) = (-1)^i\frac{1}{\beta}S(w,\ \tilde{x}).$$

This shows that $(1/\alpha)S(x,\ \tilde{w})$ is independent of the choice of x and $(-1)^i(1/\beta)S(w,\ \tilde{x})$ is independent of the choice of $\tilde{x}$. Define

$$V(w, \tilde{w}) = \frac{1}{\alpha} S(x, \tilde{w}) = (-1)^i \frac{1}{\beta} S(w, \tilde{x}) \in R_0/R.$$

This V is a pairing of W_{i-1} and $\tilde{W}_{n-i}$ to R_0/R because:

$$\begin{aligned} V(w_1 + w_2, \tilde{w}) &= (-1)^i \frac{1}{\beta} S(w_1 + w_2, \tilde{x}) \\ &= (-1)^i \frac{1}{\beta} [S(w_1, \tilde{x}) + S(w_1, \tilde{x})] = V(w_1, \tilde{w}) + V(w_2, \tilde{w}) \end{aligned} \tag{1}$$

$$\begin{aligned} V(w, \tilde{w}_1 + \tilde{w}_2) &= \frac{1}{\alpha} S(x, \tilde{w}_1 + \tilde{w}_2) = \frac{1}{\alpha} [S(x, \tilde{w}_1) + S(x, \tilde{w}_2)] \\ &= V(w, \tilde{w}_1) + V(w, \tilde{w}_2) \end{aligned} \tag{2}$$

$$V(\gamma w, \bar{\delta} \tilde{w}) = \frac{1}{\alpha} S(\gamma x, \bar{\delta} \tilde{w}) = \frac{1}{\alpha} \gamma \delta S(x, \tilde{w}) = \gamma \delta V(w, \tilde{w}). \tag{3}$$

We take the values of V modulo R so that the bounding cycles will be annihilators and hence V will be a linking of homology classes. The purpose of the remainder of this section is to characterize the complete annihilators of W_{i-1} and $\tilde{W}_{n-i}$.

THEOREM 3.1. *Let D_{i-1} be the submodule of W_{i-1} which is the annihilator of $\tilde{W}_{n-i}$. Then $w \in D_{i-1}$ if and only if w is weakly bounding modulo π^m for all $m > 0$ and all primes π. Dually the annihilator $\tilde{D}_{n-i}$ of W_{i-1} consists of those cycles in $\tilde{W}_{n-i}$ which are weakly bounding modulo $\bar{\pi}^m$ for all $m > 0$ and all primes $\bar{\pi}$.*

PROOF. Suppose $w \notin D_{i-1}$. Then there exists a $\tilde{w} \in \tilde{W}_{n-i}$ such that $V(w, \tilde{w}) \not\equiv 0$ (mod R). Let $\tilde{x}$ be a chain such that $\partial \tilde{x} = \bar{\alpha} \tilde{w}$ with $\bar{\alpha} \neq 0$. Then

$V(w, \tilde{w}) = (-1)^i \frac{1}{\alpha} S(w, \tilde{x}) \not\equiv 0 \pmod{R}$

$\Rightarrow S(w, \tilde{x}) \not\equiv 0 \pmod{\alpha}$

$\Rightarrow S(w, \tilde{x}) \not\equiv 0 \pmod{\pi^m}$ where $\bar{\pi}$ is some prime divisor of $\bar{\alpha}$, and $\bar{\alpha}$ is divisible by $\bar{\pi}^m$ but not by $\bar{\pi}^{m+1}$

$\Rightarrow w$ is not weakly bounding modulo π^m because $\tilde{x}$ is a cycle modulo $\bar{\pi}^m$ with a non-trivial intersection with w modulo π^m.

Conversely suppose w is not weakly bounding modulo π^m for some prime π and some $m > 0$. Then by the duality theorem 2.6 there exists a chain $\tilde{y} \in \tilde{C}_{n-i+1}$ such that $\partial \tilde{y} \equiv 0 \pmod{\bar{\pi}^m}$ and $S(w, \tilde{y}) \not\equiv 0 \pmod{\pi^m}$. But

$$\partial \tilde{y} \equiv 0 \pmod{\bar{\pi}^m} \Rightarrow \partial \tilde{y} = \bar{\pi}^m \tilde{x} \text{ for some } \tilde{x} \in \tilde{C}_{n-i}.$$

Hence $\tilde{x}$ is a weakly bounding cycle such that

$$V(w, \tilde{x}) = (-1)^i (1/\pi^m) S(w, \tilde{y}) \not\equiv 0 \pmod{R}.$$

DUALITY THEOREM 3.2. *The pairing V defines a primitive pairing of W_{i-1}/D_{i-1} and $\tilde{W}_{n-i}/\tilde{D}_{n-i}$ to R_0/R, where D_{i-1} and $\tilde{D}_{n-i}$ are the submodules of W_{i-1} and*

$\tilde{W}_{n-i}$ respectively consisting of all cycles which are weakly bounding modulo π^m for all $m > 0$ and all primes π.

PROOF. Since D_{i-1} and $\tilde{D}_{n-i}$ are the annihilators of $\tilde{W}_{n-i}$ and W_{i-1} respectively, we need only apply Theorem 2.1.

For the next section, we need another characterization of the annihilators D_{i-1} and $\tilde{D}_{n-i}$.

THEOREM 3.3. *The annihilator D_{i-1} of $\tilde{W}_{n-i}$ is the submodule of all elements $w \in W_{i-1}$ for which there exist a finite number of elements $\alpha_1, \cdots, \alpha_r \in R$, with greatest common divisor 1, such that $\alpha_1 w, \cdots, \alpha_r w$ are all bounding cycles.*

PROOF. Suppose we have $\alpha_1, \cdots, \alpha_r$ relatively prime, with $\alpha_1 w, \cdots, \alpha_r w$ all bounding cycles. Then for any $\tilde{x} \in \tilde{W}_{n-i}$:

$$V(\alpha_1 w, \tilde{x}) \equiv \cdots \equiv V(\alpha_r w, \tilde{x}) \equiv 0 \pmod R$$

$$\Rightarrow \alpha_1 V(w, \tilde{x}) \equiv \cdots \equiv \alpha_r V(w, \tilde{x}) \equiv 0 \pmod R$$

$$\Rightarrow V(w, \tilde{x}) \equiv 0 \pmod R$$

$$w \in D_{i-1}.$$

Suppose conversely that $w \in D_{i-1}$. Choose a chain x such that $\partial x = \alpha w$ with $\alpha \neq 0$. Let π be any prime divisor of α. By Theorem 3.1 we know that w is weakly bounding modulo π^m for all $m > 0$. Explicitly we have $x_m \in C_i$, $y_m \in C_{i-1}$, $\alpha_m \in R$, such that $\partial x_m = \alpha_m w + y_m$ with $y_m \equiv 0 \pmod{\pi^m}$ and α_m prime to π for $m > 0$. The chain y_m has a finite number of coefficients, say $\beta_{m1}, \cdots, \beta_{ms}$. Choose k such that β_{11} is divisible by π^{k-1} but not by π^k. For $m \geqq k$ choose γ_m and δ_m relatively prime such that $\gamma_m \beta_{11} + \delta_m \beta_{m1} = 0$. We know that π^m divides β_{m1}, so π^{m-k+1} divides γ_m and hence δ_m is prime to π.

$$\partial(\gamma_m x_1 + \delta_m x_m) = (\gamma_m \alpha_1 + \delta_m \alpha_m) w + (\gamma_m y_1 + \delta_m y_m), \qquad m \geqq k.$$

Since π divides γ_m but not $\delta_m \alpha_m$, we conclude that $(\gamma_m \alpha_1 + \delta_m \alpha_m)$ is prime to π. Also $(\gamma_m y_1 + \delta_m y_m) \equiv 0 \pmod{\pi^{m-k+1}}$. Hence we have a new sequence of equalities of the form $\partial x'_m = \alpha'_m w + y'_m$ with $y'_m \equiv 0 \bmod \pi^{m-k+1})$ and α'_1 prime to π for $m \geqq k$.

Now however the first coefficients in the chains y'_m are all zero. By repeating this process on the new equalities for the second coefficients, and so on, we obtain finally a sequence of equalities with all the coefficients of the y's zero. The first of these equalities is of the form $\partial x_1 = \alpha_1 w$ where α_1 is prime to π.

Similarly for every prime divisor π_j of α, we can find an $x_j \in C_i$ such that $\partial x_j = \alpha_j w$ with α_j prime to π_j. Hence we have elements $\alpha, \alpha_j \in R$ with greatest common divisor 1 and αw, $\alpha_j w$ are all bounding cycles.

There is of course a dual theorem for $\tilde{D}_{n-i}$.

4. Let M be a finitely generated R-module. Then M has a presentation with a finite number of generators $x_1, \cdots, x_n$ and a finite number of defining relations $\alpha_{i1} x_1 + \cdots + \alpha_{in} x_n = 0$, $i = 1, \cdots, m$. Strictly speaking M is a homomorphic image of the free R-module generated by $x_1, \cdots, x_n$ where the kernel of the homomorphism is the submodule generated by $\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n$, $i = 1, \cdots, m$. We write the relations in matrix form as follows:

$$A = \begin{pmatrix} \alpha_{11} & \cdot & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{m1} & \cdot & \alpha_{mn} \end{pmatrix}$$

Let $\Delta_i(A)$, $i = 0, \cdots, n-1$, be the principal ideal generated by the greatest common divisor of the minors of A of order $n - i$. We agree to adjoin rows of zeros to A if necessary for the definition of $\Delta_i(A)$ to make sense. For $i \geqq n$ we define $\Delta_i(A) = R$. It can be shown that if we take any other finite set of generators for M and if we let B be any matrix of defining relations for M, then $\Delta_i(A) = \Delta_i(B)$ for all $i \geqq 0$. Accordingly we define $\Delta_i(M) = \Delta_i(A)$ for $i \geqq 0$. The ideals $\Delta_i(M)$ are invariants of the module M and we call $\Delta_i(M)$ the determinant divisor of M of deficiency i.

Lemma 4.1. *Let M be an R-module with generators $x_1, \cdots, x_n$. Let N be the submodule generated by $x_2, \cdots, x_n$. Then $\Delta_i(N)$ divides $\Delta_i(M)$.*

Proof: Let

$$A = \begin{pmatrix} \alpha_{11} & \cdot & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{m1} & \cdot & \alpha_{mn} \end{pmatrix}$$

be a matrix of defining relations for M. Let B be any $r \times r$ submatrix of A. In order to prove the lemma, it suffices to prove that $\Delta_{n-r}(N)$ divides $|B|\cdot R$. The matrix B involves r rows of A which we call $\rho_1, \cdots, \rho_r$. We may assume that these are the first r rows of A. Let the columns in B be those corresponding to the generators $x_{i_1}, \cdots, x_{i_r}$. We adopt the notation

$$|B| = \begin{vmatrix} \rho_1 \\ \vdots \\ \rho_r \end{vmatrix}_{i_1 \cdots i_r};$$

the subscripts indicating which columns of the matrix

$$\begin{pmatrix} \rho_1 \\ \vdots \\ \rho_r \end{pmatrix}$$

are to be used in forming the determinant. Write $\alpha_{i1} = \delta\alpha'_{i1}$, $i = 1, \cdots, r$, where δ is the greatest common divisor of $\alpha_{11}, \cdots, \alpha_{r1}$. Then $\alpha'_{11}\alpha_{i1} - \alpha'_{i1}\alpha_{11} = 0$. Form

$$B' = \begin{pmatrix} \alpha'_{11}\,\rho_2 - \alpha'_{21}\,\rho_1 \\ \vdots \\ \alpha'_{11}\,\rho_r - \alpha'_{r1}\,\rho_1 \end{pmatrix}$$

which is a matrix of relations involving only $x_2, \cdots, x_r$. Let $B_j = |B'|_{i_1 \cdots i_{j-1} i_{j+1} \cdots i_r}$. Each B_j is a minor of a relation matrix for N, so $\Delta_{n-r}(N)$ divides B_jR.

$\alpha_{1i_1}B_1 - \cdots + (-1)^{r-1}\alpha_{1i_r}B_r$

$$= \begin{vmatrix} \rho_1 \\ \alpha'_{11}\rho_2 - \alpha'_{21}\rho_1 \\ \vdots \\ \alpha'_{11}\rho_r - \alpha'_{r1}\rho_1 \end{vmatrix}_{i_1\cdots i_r} = \begin{vmatrix} \rho_1 \\ \alpha'_{11}\rho_2 \\ \vdots \\ \alpha'_{11}\rho_r \end{vmatrix}_{i_1\cdots i_r} = (\alpha'_{11})^{r-1} \mid B \mid.$$

Therefore $\Delta_{n-r}(N)$ divides $(\alpha'_{11})^{r-1}\mid B \mid \cdot R$.

Similarly $\Delta_{n-r}(N)$ divides $(\alpha'_{j1})^{r-1} \mid B \mid R$, for $j = 1, \cdots, r$. (The same argument applies by renumbering the rows of A.) Since $\alpha'_{11}, \cdots, \alpha'_{r1}$ are relatively prime, $\Delta_{n-r}(N)$ divides $\mid B \mid$.

THEOREM 4.2. *Let M be a finitely generated R-module and let N be a submodule of M. Then $\Delta_i(N)$ divides $\Delta_i(M)$ for all $i \geqq 0$.*

PROOF. Let $x_1, \cdots, x_n$ be generators of M and let $y_1, \cdots, y_m$ be generators of N. Let N_j be the submodule generated by $x_{j+1}, \cdots, x_n, y_1, \cdots, y_m$, for $j = 0, \cdots, n-1$. Then $N = N_n \subset N_{n-1} \subset \cdots \subset N_0 = M$ is a chain of submodules with the property that N_{j+1} is obtained from N_j by dropping one of the generators of N_j. By the preceding lemma $\Delta_i(N_{j+1})$ divides $\Delta_i(N_j)$. Therefore $\Delta_i(N)$ divides $\Delta_i(M)$.

Let M be a finitely generated R-module. We define the dual M^* of M to be the R-module of all homomorphisms $\chi : M \to R_0/R$. Addition and multiplication by an element of R are defined as usual in M, namely $(\chi_1 + \chi_2)(x) = \chi_1(x) + \chi_2(x)$ and $(\alpha\chi)(x) = \alpha\chi(x)$. We wish to study the determinant divisors of M^*. However in general M^* need be finitely generated only when every element of M is of finite order, that is, for every $x \in M$, there exists an $\alpha \in R$, $\alpha \neq 0$, with $\alpha x = 0$. For example take R = integers and M = infinite cyclic group. Then M^* is isomorphic with the rationals mod 1 as an R-module.

LEMMA 4.3. *In order that every element of M be of finite order it is necessary and sufficient that $\Delta_0(M) \neq 0$.*

PROOF. Suppose first that M is generated by $x_1, \cdots, x_n$, and that $\alpha_i x_i = 0$ with $\alpha_i \neq 0$ for $i = 1, \cdots, n$. Let A be the diagonal matrix with α_i as the i^{th} diagonal entry. Then A is an $n \times n$ submatrix of some relation matrix and $\mid A \mid \neq 0$.

Suppose conversely that $\Delta_0(M) \neq 0$. Then if M is generated by n of its elements $x_1, \cdots, x_n$ there must be an $n \times n$ submatrix

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdot & \cdots & \cdot \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}$$

of the corresponding relation matrix such that $\mid A \mid \neq 0$. By multiplying the i^{th} row by the cofactor of α_{ij} and adding, we deduce that $\mid A \mid \cdot x_j = 0$. Hence every element of M is of finite order.

Assume that every element of M has finite order, i.e. that $\Delta_0(M) \neq 0$. In order to find generators and relations for M^*, we study first the case where M has a square matrix of defining relations

$$A = \begin{pmatrix} \alpha_{11} & \cdot & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \cdot & \alpha_{nn} \end{pmatrix}$$

with generators $x_1, \cdots, x_n$. Let

$$A^{-1} = \begin{pmatrix} A_{11} & \cdot & A_{1n} \\ \cdot & \cdot & \cdot \\ A_{n1} & \cdot & A_{nn} \end{pmatrix}; \qquad A^{-1}A = AA^{-1} = \text{identity matrix.}$$

Define a homomorphism χ_i' from the free module $\{x_1, \cdots, x_n\}$ to R_0 by $\chi_i'(x_j) = A_{ji}$, $i, j = 1, \cdots, n$. Then χ_i' gives rise to a homomorphism χ_i of M into R_0/R since

$$\chi_i(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n) = \alpha_{k1}A_{1i} + \cdots + \alpha_{kn}A_{ni} = \delta_{ki} \in R.$$

Let χ be any element of M^*. Then χ is represented by a homomorphism χ' from the free module generated by $x_1, \cdots, x_n$ into the module R_0. Let $\chi'(\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n) = \beta_i \in R$, $i = 1, \cdots, n$. Contention: $\chi = \beta_1\chi_1 + \cdots + \beta_n\chi_n$. For

$$\begin{aligned} \chi'(x_j) &= \chi'((A_{j1}\alpha_{11} + \cdots + A_{jn}\alpha_{n1})x_1) + \cdots + \chi'((A_{j1}\alpha_{1n} + \cdots + A_{jn}\alpha_{nn})x_n) \\ &= A_{j1}\chi'(\alpha_{11}x_1 + \cdots + \alpha_{1n}x_n) + \cdots + A_{jn}\chi'(\alpha_{n1}x_1 + \cdots + \alpha_{nn}x_n) \\ &= A_{j1}\beta_1 + \cdots + A_{jn}\beta_n \\ &= (\beta_1\chi_1' + \cdots + \beta_n\chi_n')(x_j). \end{aligned}$$

Therefore $\chi' = \beta_1\chi_1' + \cdots + \beta_n\chi_n'$.

This shows that $\chi_i, \cdots, \chi_n$ generate M^*. Certainly $\alpha_{1i}\chi_1 + \cdots + \alpha_{ni}\chi_n = 0$, $i = 1, \cdots, n$, are relations among $\chi_1, \cdots, \chi_n$ because

$$\alpha_{ji}\chi_1(x_j) + \cdots + \alpha_{ni}\chi_n(x_j) = \alpha_{1i}A_{j1} + \cdots + \alpha_{ni}A_{jn} = \delta_{ij} \in R.$$

Contention: Any relation is a consequence of the relations $\alpha_{1i}\chi_1 + \cdots + \alpha_{ni}\chi_n = 0$, $i = 1, \cdots, n$. Let $\beta_1\chi_1 + \cdots + \beta_n\chi_n = 0$ be any relation. Let $\gamma_i = \beta_1 A_{i1} + \cdots + \beta_n A_{in} = \beta_1\chi_1(x_i) + \cdots + \beta_n\chi_n(x_i) \in R$. Then

$$\begin{aligned} \gamma_1(\alpha_{11}\chi_1 + \cdots + \alpha_{n1}\chi_n) &+ \cdots + \gamma_n(\alpha_{1n}\chi_1 + \cdots + \alpha_{nn}\chi_n) \\ &= (\beta_1A_{11} + \cdots + \beta_nA_{1n})(\alpha_{11}\chi_1 + \cdots + \alpha_{n1}\chi_n) + \cdots \\ &\qquad + (\beta_1A_{n1} + \cdots + \beta_nA_{nn})(\alpha_{1n}\chi_1 + \cdots + \alpha_{nn}\chi_n) \\ &= \textstyle\sum_{\nu,\mu=1}^{n} (A_{1\nu}\alpha_{\mu1} + \cdots + A_{n\nu}\alpha_{\mu n})\beta_\nu\chi_\mu \\ &= \beta_1\chi_1 + \cdots + \beta_n\chi_n \end{aligned}$$

which shows that $\beta_1\chi_1 + \cdots + \beta_n\chi_n = 0$ is a consequence of the relations $\alpha_{1i}\chi_1 + \cdots \alpha_{ni}\chi_n = 0$, $i = 1, \cdots, n$.

Summarizing. If M has generators $x_1, \cdots, x_n$ with a square non-singular matrix $A = (\alpha_{ij})$ of defining relations, then M^* has generators $\chi_1, \cdots, \chi_n$ with matrix of defining relations ${}^tA$ = transpose of A, where $\chi_i(x_j) = |A|^{-1}$ (cofactor of α_{ij}).

THEOREM 4.4. *Let M be a finitely generated R-module in which every element has finite order. Then $\Delta_i(M) = \Delta_i(M^*)$.*

PROOF. Let A be a matrix of defining relations. Let B be a square submatrix of A of deficiency zero with $|B| \neq 0$. Let M_B be the R-module with the matrix B of defining relations. The module M^* is naturally a submodule of M_B^*, for M^* consists of those elements of M_B^* which happen to be zero on all relations of A. Therefore $\Delta_i(M^*)$ divides $\Delta_i(M_B^*) = \Delta_i(M_B)$. However $\Delta_i(M)$ is generated by the greatest common divisor of the generators of $\Delta_i(M_B)$ as B ranges over the square non-singular submatrices of A of deficiency zero. (This is so because any non-zero $j \times j$ minor of A is also a minor of some non-singular matrix B.) Therefore $\Delta_i(M^*)$ divides $\Delta_i(M)$ for all $i \geqq 0$. Since $\Delta_0(M) \neq 0$ it follows from Lemma 4.3 that every element of M^* is of finite order; replacing M by M^*, $\Delta_i(M^{**})$ divides $\Delta_i(M^*)$.

The elements of M may be regarded in a natural way as homomorphisms of $M^* \to R_0/R$. Thus M is naturally a submodule of M^{**} and $\Delta_i(M)$ divides $\Delta_i(M^{**})$. Combining these divisibility relations, we have $\Delta_i(M) = \Delta_i(M^*)$.

THEOREM 4.5. *Let M_1 be an R-module and let M_2 be an $\bar{R}$-module. Let P be a primitive pairing of M_1 and M_2 to R_0/R. Then $\Delta_i(M_1) = \overline{\Delta_i(M_2)}$.*

PROOF. Let $\bar{M}_2^*$ be the R-module of all conjugate homomorphisms $\bar{\chi}: M_2 \to R_0/R$. By a conjugate homomorphism is meant an additive homomorphism χ satisfying $\bar{\chi}(\bar{\alpha}x_2) = \alpha\bar{\chi}(x_2)$ for all $\bar{\alpha} \in \bar{R}$, $x_2 \in M_2$. The elements of M_1 may be regarded as belonging to $\bar{M}_2^*$ by identifying $x_1 \in M_1$ with the conjugate homomorphism $x_2 \to P(x_1, x_2)$. This identification is one to one because P is primitive. It is easily checked that M is identified with a submodule of $\bar{M}^*$. Hence $\Delta_i(M_1)$ divides $\Delta_i(\bar{M}_2^*)$.

Now if $\bar{\chi}$ is any element of $\bar{M}_2^*$, define $\chi: M_2 \to R_0/R$ by $\chi(x_2) = \overline{\bar{\chi}(x_2)}$. Then χ is an ordinary homomorphism:

$$\chi(\bar{\alpha}x_2) = \overline{\bar{\chi}(\bar{\alpha}x_2)} = \bar{\alpha}\overline{\bar{\chi}(x_2)} = \bar{\alpha}\chi(x_2).$$

Let $\bar{\chi}_1, \cdots, \bar{\chi}_n$ be generators of $\bar{M}_2^*$ with relation matrix $\bar{A}$. Then $\chi_1, \cdots, \chi_n$ will be generators of M_2^* with relation matrix A. Hence $\Delta_i(\bar{M}_2^*) = \overline{\Delta_i(M_2^*)}$ and we have proved that $\overline{\Delta_i(M_2^*)} = \overline{\Delta_i(M_2)}$ is divisible by $\Delta_i(M_1)$.

Now define a new pairing $\bar{P}$ of M_2 and M_1 to R_0/R by $\bar{P}(x_2, x_1) = \overline{P(x_1, x_2)}$, where we now regard M_2 as an R-module and M_1 as an $\bar{R}$-module. Then by the preceding argument $\Delta_i(M_2)$ divides $\overline{\Delta_i(M_1)}$.

COROLLARY 4.6. *Let W_{i-1} and W_{n-i} be the modules of weakly bounding cycles of a pair of dual chain complexes and let D_{i-1} and $\tilde{D}_{n-i}$ be the submodules of cycles which are weakly bounding modulo π^m for all primes π and all $m > 0$. Then $\Delta_j(W_{i-1}/D_{i-1}) = \overline{\Delta_j(\tilde{W}_{n-i}/\tilde{D}_{n-i})}$.*

THEOREM 4.7. *Let M be a finitely generated R-module. Let D be the submodule of M consisting of all elements $x \in M$ for which there exist a finite number of relatively prime elements $\alpha_1, \cdots, \alpha_r \in R$ such that $\alpha_1 x = \cdots = \alpha_r x = 0$. Then $\Delta_i(M) = \Delta_i(M/D)$.*

PROOF. Let A be a matrix of defining relations for M. A matrix for M/D can be obtained by adjoining a finite number of rows $\rho_1, \cdots, \rho_n$ to A. Each ρ_i corresponds to an element of D. Hence for each ρ_i, we have relatively prime elements $\alpha_{i1}, \cdots, \alpha_{ir_i} \in R$ such that $\alpha_{ij}\rho_i = 0$ are consequences of the relations in A. We adjoin all relations $\alpha_{ij}\rho_i$ to A, obtaining a matrix B of defining relations for M. Then

$$C = \begin{pmatrix} B \\ \rho_1 \\ \vdots \\ \rho_n \end{pmatrix}$$

is a matrix of defining relations for M/D. However $\Delta_i(B) = \Delta_i(C)$ because any minor of C involving some of the rows $\rho_1, \cdots, \rho_n$ can be expressed as the greatest common divisor of minors of B involving some of the rows $\alpha_{ij}\rho_i$.

Corollary 4.8. *Let W_{i-1} and $\tilde{W}_{n-i}$ be the modules of weakly bounding cycles of a pair of dual chain complexes and let B_{i-1} and $\tilde{B}_{n-i}$ be the submodules of bounding cycles. Then $\Delta_j(W_{i-1}/B_{i-1}) = \overline{\Delta_j(\tilde{W}_{n-i}/\tilde{B}_{n-i})}$.*

Proof. By Theorem 3.3 the submodules D_{i-1}/B_{i-1} and $\tilde{D}_{n-i}/\tilde{B}_{n-i}$ of W_{i-1}/B_{i-1} and $\tilde{W}_{n-i}/\tilde{B}_{n-i}$ respectively, satisfy the hypothesis of Theorem 4.7. Hence $\Delta_j(W_{i-1}/B_{i-1}) = \Delta_j(W_{n-i}/D_{n-i})$ and $\Delta_j(\tilde{W}_{i-1}/\tilde{B}_{i-1}) = \Delta_j(\tilde{W}_{n-i}/\tilde{D}_{n-i})$.

Theorem 4.9. *Let M be a finitely generated R-module and let N be a submodule of M. Then $\Delta_{i+j}(M)$ divides $\Delta_i(N)\cdot\Delta_j(M/N)$ for all $i, j \geqq 0$.*

Proof. Let $x_1, \cdots, x_n$ be generators of N and let $x_1, \cdots, x_n, y_1, \cdots, y_m$ be generators of M. There exists a matrix of defining relations for M of the form

$$D = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where A is a matrix of defining relations for N, and 0 represents a zero matrix of appropriate size. Then C is a matrix of defining relations for M/N. The product of any $i \times i$ minor of A and any $j \times j$ minor of C may be realized as an $(i + j) \times (i + j)$ minor of D. Hence $\Delta_{i+j}(D)$ divides $\Delta_i(A)\cdot\Delta_j(C)$.

Definition. The *torsion submodule* of a module M is the submodule consisting of the elements of finite order.

Lemma 4.10. *If r is the maximum number of linearly independent elements in a finitely generated module M, and if T is the torsion submodule of M then*

$$\begin{aligned} \Delta_i(M) &= 0 \qquad &&\text{for} \quad i < r, \\ &= \Delta_{i-r}(T) &&\text{for} \quad i \geqq r. \end{aligned}$$

Proof. Let F_0 denote a submodule of M generated by a set of r linearly independent elements. Let $x_1, \cdots, x_n$ generate M/F_0. There exist non-zero elements $\alpha_1, \cdots, \alpha_r$ of R such that $\alpha_i x_i \in F_0$, $i = 1, \cdots, n$. If M is torsion-free, the mapping $x \to \alpha_1\alpha_2 \cdots \alpha_r x$ imbeds M isomorphically in the free module F_0; hence $M \subset F_1$, where F_1 is a free module of rank r. By Theorem 4.2, $\Delta_r(M)$ divides $\Delta_r(F_1) = R$. Thus $\Delta_r(M) = R$ for a torsion-free module M of rank r.

Now let M be an arbitrary finitely generated module and F_0 as above. Consider the submodule M' generated by F_0 and T. By direct calculation

$$\begin{aligned} \Delta_i(M') &= 0 && \text{for} \quad i < r, \\ &= \Delta_{i-r}(T) && \text{for} \quad i \geqq r. \end{aligned}$$

Since M/T is torsion-free, $\Delta_r(M/T) = R$. By Theorem 4.9, $\Delta_i(M)$ divides $\Delta_{i-r}(T)\Delta_r(M/T) = \Delta_{i-r}(T)$ if $i \geqq r$. But, by Theorem 4.2, $\Delta_i(M')$ divides $\Delta_i(M)$, so that $\Delta_i(M) = \Delta_{i-r}(T)$. If $i < r$ then $\Delta_i(M) = 0$ because, by Theorem 4.2, $\Delta_i(M')$ divides $\Delta_i(M)$.

5. Let $\mathfrak{M}$ be a combinatorial 3-manifold with boundary $\mathfrak{B}$, where $\mathfrak{B}$ is the union of μ disjoint tori $\mathfrak{B}_1, \cdots, \mathfrak{B}_\mu$. Such a manifold may arise for example from a tame link of μ components in a closed manifold by removing a suitably small neighborhood of the link. Let $i_*: H_1(\mathfrak{B}) \to H_1(\mathfrak{M})$ be the injection homomorphism of the homology groups with integer coefficients. It will be convenient to think of these groups as written multiplicatively.

THEOREM 5.1. *The rank of $i_*H_1(\mathfrak{B}_i)$ is non-zero for $i = 1, \cdots, \mu$.*

PROOF. Suppose $\mu = 1$ and assume the rank of $i_*H_1(\mathfrak{B})$ is zero. Then $i_*H_1(\mathfrak{B})$ consists entirely of weakly bounding cycles in $H_1(\mathfrak{M})$. Hence $\mathfrak{S}(i_*a, x) = 0$ for all $a \in H_1(\mathfrak{B})$, $x \in H_2(\mathfrak{M}, \mathfrak{B})$, where $\mathfrak{S}$ is the ordinary intersection. This intersection has the property (see [5]), $\mathfrak{S}(i_*a, x) = \pm\mathfrak{S}'(a, \partial x)$, where $\mathfrak{S}'$ is the intersection in $\mathfrak{B}$. Hence $\mathfrak{S}'(a, \partial x) = 0$ for all $a \in H_1(\mathfrak{B})$, $x \in H_2(\mathfrak{M}, \mathfrak{B})$. Since $\mathfrak{S}'$ is primitive, $H_1(\mathfrak{B})$ being free abelian, we must have $\partial x = 0$. In the exact sequence $H_2(\mathfrak{M}, \mathfrak{B}) \xrightarrow{\partial} H_1(\mathfrak{B}) \xrightarrow{i_*} H_1(\mathfrak{M})$ the image of ∂ is 0, so the kernel of i_* is 0. But this is a contradiction of the assumption that $i_*H_1(\mathfrak{B})$ has only torsion elements.

For the case $\mu > 1$, fill in $\mathfrak{B}_2, \cdots, \mathfrak{B}_\mu$ in some way to obtain a manifold $\mathfrak{M}'$ with boundary $\mathfrak{B}_1$. Then the injection homomorphisms $i'_*: H_1(\mathfrak{B}_1) \to H_1(\mathfrak{M}')$ and $i_*: H_1(\mathfrak{B}_1) \to H_1(\mathfrak{M})$ satisfy the relation

$$\text{kernel } i'_* \supset \text{kernel } i_* .$$

Therefore rank $i'_*H_1(\mathfrak{B}_1) \leqq$ rank $i_*H_1(\mathfrak{B}_1)$ and $0 <$ rank $i_*H_1(\mathfrak{B}_1)$.

Similarly $0 <$ rank $i_*H_1(\mathfrak{B}_i)$ for $i = 1, \cdots, \mu$.

Let $\mathfrak{M}^\ddagger$ be the maximal covering complex of $\mathfrak{M}$ which has a free abelian group of covering transformations. The subgroup of $\pi_1(\mathfrak{M})$ corresponding to $\mathfrak{M}^\ddagger$ is the subgroup generated by the elements of finite order modulo commutators. The group G of covering transformations is isomorphic with the 1-dimensional Betti group of $\mathfrak{M}$. Let R be the group ring of G with integer coefficients. The automorphism $R \to \bar{R} = R$ is defined as the unique extension of $\gamma \to \bar{\gamma} = \gamma^{-1}$ for all $\gamma \in G$. The chains of $M^\ddagger$ as R-modules and the relative chains modulo $\mathfrak{B}^\ddagger$ as $\bar{R}$-modules form dual chain complexes with the intersection pairing S as defined in Section 1. As usual we have an exact sequence of homology modules

$$H_1(\mathfrak{B}^\ddagger; R) \xrightarrow{i_*} H_1(\mathfrak{M}^\ddagger; R) \xrightarrow{j_*} H_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R) \xrightarrow{\partial} H_0(\mathfrak{B}^\ddagger; R).$$

DEFINITION. An ideal $\mathfrak{a}$ in R is called *symmetric* if $\mathfrak{a} = \bar{\mathfrak{a}}$. An element of R is called symmetric if it generates a symmetric ideal.

LEMMA 5.2. *There is an element $\alpha \in R$, $\alpha \neq 0$, such that α has only symmetric prime factors and $\alpha \cdot H_i(\mathfrak{B}^\ddagger; R) = 0$ for $i = 1, 0$.*

PROOF. Each surface $\mathfrak{B}_\nu$ is covered by a collection of connected surfaces $\mathfrak{B}_{\nu k}^\ddagger$ in $\mathfrak{M}^\ddagger$. Each $\mathfrak{B}_{\nu k}^\ddagger$ is either a plane or a cylinder. It is not possible for any $\mathfrak{B}_{\nu k}^\ddagger$ to be a torus since in that event $i_* H_1(\mathfrak{B}_\nu)$ would have to have rank zero.

$H_i(\mathfrak{B}^\ddagger; R)$ is generated by the cycles in various $H_i(\mathfrak{B}_{\nu k}^\ddagger; R)$. We are to find an element of R which annihilates all these cycles simultaneously. One of the two generators of $H_1(\mathfrak{B}_\nu)$ must represent a non-torsion element $\alpha_\nu \in H_1(\mathfrak{M})$. The covering transformation $\gamma_\nu \in G$ corresponding to α_ν will slide the non-trivial cycle on $\mathfrak{B}_{\nu k}^\ddagger$ along the cylinder or plane and hence $1 - \gamma_\nu$ annihilates $H_i(\mathfrak{B}_\nu^\ddagger; R)$. Therefore the element $\alpha = \prod_{\nu=1}^{\mu} (1 - \gamma_\nu)$ annihilates $H_i(\mathfrak{B}^\ddagger; R)$. Clearly the prime factors $(1 - \gamma_\nu)$ of α are all symmetric.

DEFINITION. A *torsion module* is a module of weakly bounding cycles modulo bounding cycles. The torsion submodule of a homology module is indicated by replacing the H with a T.

LEMMA 5.3. *The kernel of $\partial: T_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R) \to T_0(\mathfrak{B}^\ddagger; R)$ is $j_* T_1(\mathfrak{M}^\ddagger; R)$.*

PROOF.

$$H_1(\mathfrak{B}^\ddagger; R) \xrightarrow{i_*} H_1(\mathfrak{M}^\ddagger; R) \xrightarrow{j_*} H_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R) \xrightarrow{\partial} H_0(\mathfrak{B}^\ddagger; R).$$

By exactness $j_* T_1(\mathfrak{M}^\ddagger; R)$ is contained in the kernel of ∂. Let $x \in T_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R)$ with $\partial x = 0$. Then there exists a $y \in H_1(\mathfrak{M}^\ddagger; R)$ such that $j_* y = x$. Let $\beta \in R$ such that $\beta x = 0$, $\beta \neq 0$. Then $j_* \beta y = 0$. Hence there exists a $z \in H_1(\mathfrak{B}^\ddagger; R)$ such that $i_* z = \beta y$. By the previous lemma there exists an $\alpha \in R$, $\alpha \neq 0$ such that $\alpha z = 0$. Therefore $\alpha\beta y = 0 \Rightarrow y \in T_1(\mathfrak{M}^\ddagger; R) \Rightarrow x \in j_* T_1(\mathfrak{M}^\ddagger; R)$.

LEMMA 5.4. *If βR divides $\Delta_0(T_0(\mathfrak{B}^\ddagger; R))$, then β is symmetric.*

PROOF. The element α of Lemma 5.2 annihilates $T_0(\mathfrak{B}^\ddagger; R)$, and every divisor of α is symmetric. Hence, for a set of generators $x_1, \cdots, x_n$ of $T_0(\mathfrak{B}^\ddagger; R)$, we have relations $\alpha x_1 = \cdots = \alpha x_n = 0$. The determinant from these rows of a relation matrix is α^n. Therefore $\Delta_0(T_0(\mathfrak{B}^\ddagger; R))$ divides $\alpha^n R$, so any divisor of $\Delta_0(T_0(\mathfrak{B}^\ddagger; R))$ is a divisor of $\alpha^n R$ and hence symmetric.

THEOREM 5.5. *The ideals $\Delta_i(T_1(\mathfrak{M}^\ddagger; R))$ are symmetric for $i = 0, 1, 2 \cdots$.*

PROOF. By Lemma 5.4 all elements dividing $\Delta_0(T_0(\mathfrak{B}^\ddagger; R))$ are symmetric. By Lemma 5.3 $T_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R)/j_* T_1(\mathfrak{M}^\ddagger; R)$ is isomorphic with a submodule of $T_0(\mathfrak{B}^\ddagger; R)$. Hence all elements dividing $\Delta_0(T_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R)/j_* T_1(\mathfrak{M}^\ddagger; R)) = \mathfrak{a}$ are symmetric. By Theorem 4.9, $\Delta_i(T_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R))$ divides $\mathfrak{a} \cdot \Delta_i(j_* T_1(\mathfrak{M}^\ddagger; R))$. By Corollary 4.8, $\Delta_i(T_1(\mathfrak{M}^\ddagger, \mathfrak{B}^\ddagger; R)) = \overline{\Delta_i(T_1(\mathfrak{M}^\ddagger; R))}$. Since $j_* T_1(\mathfrak{M}^\ddagger; R)$ is a homomorphic image of $T_1(\mathfrak{M}^\ddagger; R)$, $\overline{\Delta_i(j_* T_1(\mathfrak{M}^\ddagger; R))}$ divides $\overline{\Delta_i(T_1(\mathfrak{M}^\ddagger; R))}$ which in turn divides $\mathfrak{a} \cdot \Delta_i(j_* T_1(\mathfrak{M}^\ddagger; R))$. Let π be a prime divisor of $\overline{\Delta_i(j_* T_1(\mathfrak{M}^\ddagger; R))}$. Then either π divides $\mathfrak{a}$ in which case π is symmetric, or π and $\bar{\pi}$ must divide

$\Delta_i(j_* T_1(\mathfrak{M}^\ddagger; R))$ to exactly the same power. Hence $\Delta_i(j_* T_1(\mathfrak{M}^\ddagger; R))$ is symmetric. It follows that $\Delta_i(T_1(\mathfrak{M}^\ddagger; R))$ is symmetric since it differs from $\Delta_i(j_* T_1(\mathfrak{M}^\ddagger; R))$ at most by symmetric prime factors.

COROLLARY 5.6. *The ideals* $\Delta_i(H_1(\mathfrak{M}^\ddagger; R))$ *are symmetric for* $i = 0, 1, 2, \cdots$.

PROOF. Theorem 5.5 and Lemma 4.10.

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