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TOPOLOGICAL QUANTUM FIELD THEORIES FOR SURFACES WITH SPIN STRUCTURE

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Introduction. A *Topological Quantum Field Theory* (TQFT) in dimension 3 is a functor from a 2 + 1-dimensional cobordism category to a category of modules, satisfying certain axioms. This terminology was introduced by Atiyah [1] following Witten's interpretation [32], in terms of quantum field theory, of the Jones-polynomial invariant of links in the 3-sphere. The TQFT-axioms imply that the functor is determined by its values on closed bordisms. These lie in the ground ring, and are 3-manifold invariants. They are sometimes called *quantum invariants.* Existence of quantum invariants was first proven by Reshetikhin and Turaev [28]. Other constructions of invariants were given, e.g., in [6], [16], [17], [18], [19], [27], [31]. A construction of TQFT-functors based on the invariants of [6] was given in [8].

Refined quantum invariants which depend nontrivially on a choice of spin structure on the manifold were constructed independently by Kirby and Melvin [16], Turaev [29] (using quantum groups), and the first author [5] (using the Kauffman bracket [13]). As a special case, these invariants include (a version of) the well-known classical *Rohlin* (or μ-)*-invariant.*

A formula of [5], [16], [29] asserts that the sum of the spin invariants of a closed 3-manifold is equal to the "unspun" invariant. The question arises of how this generalizes on the level of TQFTs. A partial answer was given already in [8], where it was shown that the V_{8k}-module of a surface is naturally decomposed into a direct sum of submodules associated to spin structures on the surface.

The philosophy of [8] was to start from the 3-manifold invariant, extend it by a universal construction to a functor on a cobordism category, and then prove the TQFT-axioms. In the present paper, we use the same philosophy but start from the spin-refined invariants. The aim is to find out what a "Spin TQFT" should be, and to understand its relationship with the "unspun" theory.

We consider a series of functors V^s_{8k} on 2 + 1-dimensional spin cobordism categories $\mathscr{C}^s_{8k}$. (The indexing by $8k$, $k \geqslant 1$, is for notational coherence with [8].) These functors are constructed from (a suitable renormalization of) the spin invariants of [5]. They are quantization functors (in the sense of [8]), and satisfy surgery properties and (spin) Kauffman relations at a primitive $16k$th root of unity.

There are three main results. The first is that the V^s_{8k}-modules associated to surfaces are free of finite rank (Theorem 7.3). We also give, in the last section,

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explicit dimension formulas depending on the genus of the surface and the Arf invariant of the spin structure.

The second main result concerns the V^s_{8k}-modules associated to disjoint unions. It turns out that the functors V^s_{8k} do not satisfy the usual multiplicativity axiom of TQFT, but an "extended" tensor product formula (Theorem 13.1). Rather than viewing this as a flaw of the theory, we consider this as an interesting new feature.

In the language of [8], the extended multiplicativity property can be naturally expressed in terms of bimodules over algebroids (see Remark 14.3). Another way of describing it is explained in Section 18: One may consider V^s_{8k} as the *even* part of a $\mathbf{Z}/2$-graded functor $\mathbf{V}^s_{8k}$ such that the $\mathbf{V}^s_{8k}$-module of a disjoint union satisfies a tensor product formula in a $\mathbf{Z}/2$-graded sense.

The third main result is a complete answer to the question raised above, i.e., a description of the relationship between the spin functors V^s_{8k} and the "unspun" TQFT-functors V_{8k} of [8]. This is given by a *transfer map* from the unspun theory to the spin theory (Theorem 15.3). The transfer map identifies the spin submodules found in [8] with the "zero graded parts" of the V^s_{8k}-modules. In particular, this means that the spin theory is strictly richer than the unspun theory.

The proofs of these results are organized as follows. On the one hand, we establish structural results about the V^s_{8k}-functors such as the decomposition theorem (Theorem 9.2) and the extended multiplicativity formula (Theorem 13.1). On the other hand, we prove the transfer theorem (Theorem 15.3). Taken together, these determine the spin theory completely, and allow us to deduce the finiteness result from the finiteness result for the unspun theory proved in [8].

An important technical ingredient of this paper is the notion of spin structure *complementary* to a link or graph (see also Gilmer [9]). In Section 2, we develop an obstruction theory for such structures, which we also call "singular" spin structures. As a byproduct, we define in Section 3 the spin Jones-Kauffman skein module and obtain some information on the forgetful map from the spin skein module to the usual one.

The main results of this paper were announced in [21].

Note. As is well known, TQFTs are closely related to representations of mapping class groups. The transfer theorem may be used to compare the representations arising from the spin theory with those of the unspun theory. In [22], this is applied to reprove and generalize a result of G. Wright [33] concerning the Birman-Craggs homomorphisms from the Torelli group to $\mathbf{Z}/2$.

Convention. In this paper, an integer $k \geqslant 1$ will be fixed. Invariants and functors will be generally indexed by $8k$. But the subscript will be quite often omitted, to simplify notation.

1. Basic definitions

Convention. All manifolds are supposed to be compact, smooth, and oriented. The closed interval $[0, 1]$ is denoted by $\mathbf{I}$.

Spin structures and w_2-structures. The notion of a spin structure on an oriented manifold M may be defined homotopy-theoretically as a lift of a classifying map of the stable tangent bundle of M to the classifying space *BSpin*. In this paper, we need to distinguish between an actual lift and its homotopy class. This leads to the definition of *w_2-structures* below (analogous to the definition of p_1-structures in [8]). The name "w_2-structures" is used because the fibration π_{spin}: $BSpin \to BSO$ is induced by a map $BSO \to K(\mathbf{Z}/2, 2)$ corresponding to the second Stiefel-Whitney class w_2.

Let γ_{Spin} denote the pull-back $\pi^{\star}_{\text{spin}}(\gamma_{SO})$ of the universal stable vector bundle γ_{SO} over BSO. Recall that if M is a manifold (always supposed to be oriented, by the convention above), there is a "classifying" map $f: M \to BSO$ and an orientation-preserving isomorphism between the stable tangent bundle of M, τ_M, and $f^{\star}(\gamma_{SO})$. Equivalently, there is a bundle morphism F from τ_M to γ_{SO}, which is an orientation-preserving linear isomorphism on each fiber.

Definition 1.1. A w_2-structure on a manifold M is a bundle morphism from the stable tangent bundle of M, τ_M, to γ_{Spin}, which is an orientation-preserving linear isomorphism on each fiber. A spin structure on M is a homotopy class of w_2-structures.

This definition of spin structure is equivalent to the usual one, given, e.g., in [26], [15].

If manifolds M_1, M_2 with w_2-structures σ_1, σ_2 are given, we say that an orientation-preserving diffeomorphism $\varphi: M_1 \to M_2$ *preserves w_2-structures* if $\sigma_2 \circ d\varphi = \sigma_1$, where $d\varphi$ is the (stable) tangent map of φ.

Note. A w_2-structure on a manifold M determines a Riemannian metric on the stable tangent bundle of M, obtained by pulling back the canonical metric on γ_{SO}.

Orientation reversal. Given a w_2-structure on a closed manifold Σ, we define the orientation reversed w_2-structure on $-\Sigma$ (the manifold Σ with reversed orientation) by composing with the automorphism of τ_Σ which is the orthogonal reflection in the direction of the first stabilization.

If M has nonempty boundary ∂M, using the "outward normal vector first" convention, $\tau_{\partial M}$ is identified with the restriction of τ_M to ∂M. If a w_2-structure is given on M, its *restriction* to the boundary ∂M is defined using this identification. Notice that if the opposite normal vector were used, one would get precisely the orientation-reversed w_2-structure on $-\partial M$.

If a w_2-structure σ is given on ∂M, then we call *relative spin structure* a homotopy class (rel. boundary) of w_2-structures on M extending σ.

If a relative spin structure on M is given, we define the *orientation-reversed relative spin structure* as follows. Pick a unit vector field in the stable tangent bundle of M extending the outward normal vector field on ∂M, and consider

the w_2-structure obtained by composing a w_2-structure representing the given relative spin structure with the automorphism of τ_M which is the orthogonal reflection in the direction of the chosen vector field. By obstruction theory, such a vector field exists and is unique up to homotopy (rel. boundary). Therefore the induced relative spin structure on $-M$ is well defined. It depends only on the given relative spin structure on M.

Observe that with this definition, orientation reversal commutes with restriction to the boundary.

Links, graphs, and complementary spin structures. We are going to study "singular" w_2- or spin structures on 2- or 3-manifolds; in the definition below (see also [9]), the link appears as the singularity of the structure. Here link in a 3-manifold M stands for smoothly embedded circles, or segments ending tranversally in the boundary of M.

Definition 1.2. Let K be a link in a 2- or 3-manifold (possibly with boundary). A spin structure complementary to K is a spin structure on $M - K$ which does not extend across any component of K.

A w_2-structure complementary to K is a w_2-structure on $M - K$ such that the induced spin structure is complementary to K.

A w_2- or spin structure complementary to a link will sometimes be called a *singular* w_2- or spin structure.

In the following, we will speak of a *banded link* in a 3-manifold M for a finite number of smoothly embedded oriented copies of either $S^1 \times [-\varepsilon, \varepsilon]$, or $\mathbf{I} \times [-\varepsilon, \varepsilon]$ ending tranversally in the boundary of M. A finite collection of oriented segments in a surface will also be called a banded link; thus the boundary of a 3-manifold with banded link is a surface with banded link.

The notions of spin and w_2-structures complementary to a banded link are defined in the obvious way.

A *banded* 3-*valent graph* in a 3-manifold M is a graph G, contained in an oriented surface $SG \subset M$, such that:

—G meets ∂M transversally in the set of vertices of G of degree 1;

—every vertex of G contained in the interior of M is of degree 2 or 3;

—the surface SG is a regular neighborhood of G in SG, and $SG \cap \partial M$ is a regular neighborhood of $G \cap \partial M$ in $SG \cap \partial M$. (Note that $SG \cap \partial M$ is a banded link in ∂M.)

Colorings. Recall that we have fixed an integer $k \geqslant 1$. A *color* is an integer i such that $0 \leqslant i \leqslant 4k - 2$.

A coloring of a banded link is an assignment of a color to each component of the link. A w_2- or spin structure complementary to a colored link is defined to be one complementary, in the sense previously defined, to the odd-colored sublink.

An admissible coloring of a 3-valent banded graph is an assignment of a color

to each edge, in such a way that

colors adjacent to a 2-valent vertex are equal;
colors a, b, c adjacent to a 3-valent vertex are subject to the admissibility condition $|a - b| \leqslant c \leqslant a + b$, $a + b + c$ even.

A w_2- or spin structure complementary to a colored banded graph is one complementary, in the sense previously defined, to the odd-colored subgraph (which is in a natural way a banded link).

Glueing and cobordism categories. Let Σ_1, Σ_2 be closed surfaces. A *cobordism* from Σ_1 to Σ_2 is a triple (M, i_1, i_2) where M is a 3-manifold,[1] and $i_1\colon -\Sigma_1 \to \partial M$ and $i_2\colon \Sigma_2 \to \partial M$ are orientation-preserving diffeomorphic embeddings whose images are disjoint and cover ∂M. If (M', i_1', i_2') is another cobordism from Σ_2 to Σ_3, M and M' can be glued together to yield a cobordism M'' from Σ_1 to Σ_3. Note that the diffeomorphism class (rel. boundary) of M'' is determined by the diffeomorphism classes (rel. boundary) of M and M'. In this way, one obtains a *category*, whose objects are closed surfaces, whose morphisms are *diffeomorphism classes (rel. boundary)* of 3-dimensional cobordisms, and composition is given by glueing. Taking diffeomorphism classes ensures that composition is well defined and associative, and the product manifold, $\Sigma \times \mathbf{I}$ (with the obvious inclusions of Σ into the boundary), represents the identity morphism of Σ.

The spin cobordism category. A collection of objects and (co)bordisms "as above" will be called a *cobordism category*. We shall only be concerned with cobordism categories where objects and morphisms are as above, but enriched with additional structure. For example, we may consider an extension of the above to the case where the objects (surfaces) are equipped with w_2-structures, and the cobordisms are equipped with relative spin structures. (Here, we demand that the inclusions i_j preserve the w_2-structures.) If the surfaces Σ_1, Σ_2, Σ_3 are equipped with w_2-structures, and the cobordisms M, M' are equipped with relative spin structures, then there is a well-defined relative spin structure on the composite cobordism M''. Hence, one obtains again a category, which might be called the *spin cobordism category*.

Convention. When speaking of a cobordism M from Σ_1 to Σ_2, we shall from now on, for notational reasons, omit mention of the inclusions i_j, and pretend ∂M is equal to $-\Sigma_1 \amalg \Sigma_2$.

p_1-structures [8]. Let $\pi_X\colon X \to BSO$ be a map, and let γ_X be the pull-back of the universal bundle γ_{SO} to X. A (precise) X-structure on a manifold N is a bundle morphism from the stable tangent bundle of N to γ_X, which is an orientation-preserving linear isomorphism on each fiber. We have already discussed the case

[1] Recall that by convention, all manifolds are assumed compact and oriented.

$X = BSpin$, where the corresponding X-structure is called a w_2-structure. Consider the space X_{p_1} which is the homotopy fiber of the map $BSO \to K(\mathbf{Z}, 4)$ corresponding to the first Pontryagin class p_1. Then an X_{p_1}-structure is called a p_1-structure. (Another example for X is the classifying space of the trivial group, where the corresponding structure is a (stable) framing.) The notions of *relative p_1-structure, orientation reversal,* and *p_1-cobordism category* are defined as for w_2-structures.

The cobordism categories $\mathscr{C}^s_{8k}$. Recall that we have fixed an integer $k \geqslant 1$. The main subject of this paper is the study of certain functors on the cobordism category $\mathscr{C}^s = \mathscr{C}^s_{8k}$ defined as follows. An object of $\mathscr{C}^s$ is a quadruple $\Sigma = (\Sigma, \alpha, l, \sigma)$, where

—Σ is a closed oriented surface;

—α is a p_1-structure on Σ;

—l is a colored banded link in Σ;

—σ is a w_2-structure which is complementary to l.

If $\Sigma_i = (\Sigma_i, \alpha_i, l_i, \sigma_i)$, $(i = 1, 2)$, are objects of $\mathscr{C}^s$, a cobordism from Σ_1 to Σ_2 is a quadruple $M = (M, \tilde{\alpha}, L, \tilde{\sigma})$, where

—M is a cobordism from Σ_1 to Σ_2 in the usual sense;

—$\tilde{\alpha}$ is a relative p_1-structure on M;

—L is a colored banded trivalent graph in M realizing an embedded cobordism from l_1 to l_2;

—$\tilde{\sigma}$ is a relative spin structure which is complementary to L.

A *morphism* in $\mathscr{C}^s$ is a diffeomorphism class (rel. boundary) of cobordisms, where the diffeomorphisms are required to preserve the relative spin and p_1-structure.

Remarks. (a) There is a "geometric" cobordism category which is defined as above, except that uncolored banded links are used. This uncolored cobordism category is included in the colored one, by coloring each component with the color 1. Although we could define the spin theory without using colors, we need them to compute the theory, and to express the multiplicativity property.

(b) Forgetting the w_2- or spin structures defines a forgetful functor from $\mathscr{C}^s_{8k}$ to the "unspun" cobordism category used in [8].

(c) One could use a slightly different definition of the cobordism category, using framings instead of w_2- together with p_1-structures. (This would lead naturally to Bass-Sullivan's notion of S^1-singular framed manifolds, see e.g., [4].) The resulting theory should be essentially the same.

(d) We have introduced w_2-structures on surfaces because it is convenient when it comes to glueing cobordisms together, and natural (at least from the point of view of obstruction theory). But one could use a weaker notion where one would use spin structures on surfaces, plus a trivialization at a point in each component.

2. Obstruction theory for singular spin structures. It is well known [26], [15] that a manifold M admits spin structures if and only if its second Stiefel-Whitney class $w_2(M)$ is zero, and in this case the set of spin structures is affinely isomorphic to $H^1(M; \mathbf{Z}/2)$.

The generalization to the relative case is as follows. Let Σ be a submanifold of M, with trivialized normal bundle. The stable tangent bundle τ_Σ is then identified with the restriction over Σ of τ_M. In this context a w_2-structure on M induces one on Σ, and the extension problem makes sense.

PROPOSITION 2.1. *Suppose Σ is a submanifold of M with trivialized normal bundle and σ is a w_2-structure on Σ; then σ extends over M if and only if the obstruction $w_2(\sigma) \in H^2(M, \Sigma; \mathbf{Z}/2)$ is zero, and in this case the set of homotopy classes (relative to Σ) of extensions is affinely isomorphic to $H^1(M, \Sigma; \mathbf{Z}/2)$.*

We now consider singular w_2- and spin structures, i.e., those which are complementary to a link. It turns out that there is also an obstruction theory which nicely parallels the usual one.

Notation. Suppose (Σ, l) is the boundary of a 3-manifold-link pair (M, K). Assume we are given a w_2-structure σ on $\Sigma - l$, complementary for l. We denote by $\mathrm{Spin}(M|K, \sigma)$ the set of relative spin structures on $M - K$, extending σ, and complementary for K (homotopy classes, relative to $\Sigma - l$, of extensions of σ).

PROPOSITION 2.2. (a) *If it is not empty, the set* $\mathrm{Spin}(M|K, \sigma)$ *is affinely isomorphic to $H^1(M, \Sigma; \mathbf{Z}/2)$.*

(b) *There exists an obstruction $w_2(M|K, \sigma) \in H^2(M, \Sigma; \mathbf{Z}/2)$ which is zero if and only if* $\mathrm{Spin}(M|K, \sigma)$ *is not empty. Furthermore, every class in $H^2(M, \Sigma; \mathbf{Z}/2)$ can be realized by a convenient K (the w_2-structure σ being fixed).*

(c) *Let Σ' be an embedded surface in M, such that $\partial\Sigma' \subset \Sigma - l$. Suppose Σ' and K are in general position; then the evaluation of the obstruction $w_2(M|K, \sigma)$ on $[\Sigma', \partial\Sigma']$ is $\Sigma' \cdot K + q_\sigma(\partial\Sigma')$. (Here, $\Sigma' \cdot K$ is the algebraic intersection number* mod 2, *and q_σ denotes the quadratic form on $H_1(\Sigma - l; \mathbf{Z}/2)$ associated to σ.)*

Remark 2.3. We recall the relationship between spin structures on surfaces and quadratic forms. (See [2], [10].) Let Σ be a (not necessarily closed) oriented surface. Every class in $H_1(\Sigma; \mathbf{Z}/2)$ can be represented by an embedded circle γ on Σ. Let P_Σ be the oriented frame bundle and $\tilde{\gamma}$ be the lift of γ given by the tangent vector. A spin structure s on Σ can be seen as an element $c_s \in H^1(P_\Sigma; \mathbf{Z}/2)$ which is not zero on the fiber (see [26], [15]). The assignment $q_s(\gamma) = 1 + c_s(\tilde{\gamma})$ defines a quadratic form q_s on $H_1(\Sigma; \mathbf{Z}/2)$ inducing the intersection form (i.e., q_s: $H_1(\Sigma; \mathbf{Z}/2) \to \mathbf{Z}/2$ satisfies $q_s(a + b) = q_s(a) + q_s(b) + a \cdot b$). This induces a bijective correspondence between spin structures on Σ and such quadratic forms.

Alternatively, $q_s(\gamma)$ is the value of γ (equipped with the restriction of s) in the spin cobordism group $\Omega_1^{\mathrm{spin}} = \mathbf{Z}/2$. In other words, $q_s(\gamma) = 0$ if and only if $(\gamma, s_{|\gamma})$ bounds *some* compact oriented spin 2-manifold. (This description of $q_s(\gamma)$ is valid for any (i.e., possibly nonconnected) embedded curve γ.)

Proof of 2.2. Consider the following diagram whose line and columns are exact. (The group of coefficients is $\mathbf{Z}/2$.)

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & H^1(M,\Sigma) \arrow[d, "j_1^*"] & & & \\
0 \arrow[r] & H^1(M,\Sigma-l) \arrow[r, "i_2^*"] \arrow[d] & H^1(M-K,\Sigma-l) \arrow[dddll, "\delta"] & & \\
 & 0 & & & \\
 & & & 0 \arrow[d] & \\
 & & & H^2(M,\Sigma) \arrow[d, "j_1^*"] & \\
 & & H^2(M,M-K) \arrow[r, "i_1^*"] & H^2(M,\Sigma-l) \arrow[r, "i_2^*"] \arrow[d, "j_2^*"] & H^2(M-K,\Sigma-l) \\
 & & & H^2(\Sigma,\Sigma-l) & \\
\end{tikzcd}$$

The group $H^2(M, M-K)$ is isomorphic to $(\mathbf{Z}/2)^{\#K}$ (where $\#K$ is the number of components of K), and has a standard basis. Let g be the sum of the basis elements. A w_2-structure $\tilde{\sigma}$ on $M-K$ is complementary for K if and only if $w_2(M, \tilde{\sigma}) \in H^2(M, M-K)$ is equal to g. Moreover if $\tilde{\sigma}$, $\tilde{\sigma}'$ are two w_2-structures on $M-K$ extending σ, one has $w_2(M, \tilde{\sigma}') - w_2(M, \tilde{\sigma}) = \delta([\tilde{\sigma}'] - [\tilde{\sigma}])$. ($[\tilde{\sigma}]$ denotes the homotopy class of extensions represented by $\tilde{\sigma}$.) We deduce that the set $\mathrm{Spin}(M|K, \sigma)$, if not empty, is affinely isomorphic to $\mathrm{Ker}(\delta) \simeq H^1(M, \Sigma)$ (assertion (a)).

Since σ is complementary for l, the class $i_1^*(g) - w_2(M, \sigma) \in H^2(M, \Sigma - l)$ lies in the kernel of j_2^*. We define $w_2(M|K, \sigma) \in H^2(M, \Sigma)$, by $j_1^*(w_2(M|K, \sigma)) = i_1^*(g) - w_2(M, \sigma)$. If there exists an extension $\tilde{\sigma}$ complementary for K, then $j_1^*(w_2(M|K, \sigma)) = i_1^*(g) - i_1^*(w_2(M, \tilde{\sigma})) = 0$, and $w_2(M|K, \sigma) = 0$. Now suppose conversely that $w_2(M|K, \sigma) = 0$; then $w_2(M-K, \sigma) = i_2^*(w_2(M, \sigma)) = i_2^* i_1^*(g) = 0$, so there exists an extension $\tilde{\sigma}_0$ over $M-K$. Observe that $i_1^*(w_2(M, \tilde{\sigma}_0)) = i_1^*(g)$; choose $x \in H^1(M-K, \Sigma - l)$, such that $\delta(x) = g - w_2(M, \tilde{\sigma}_0)$; then $[\tilde{\sigma}_0] + x$ belongs to $\mathrm{Spin}(M|K, \sigma)$. To complete the proof of assertion (b), it remains to see that every class can be obtained. This is obtained using Poincaré duality.

Denote by l' the link $K \cap \Sigma'$ in Σ'. The obstruction $w_2(M|K, \sigma)$ evaluated on $[\Sigma', \partial\Sigma']$ is zero if and only if there exists over Σ' an extension of $\sigma_{|_{\partial\Sigma'}}$ complementary for l'. Such an extension exists if and only if the number of components of l' has parity $q_\sigma(\partial\Sigma')$. This gives (c), and completes the proof of the proposition. □

For later use, we note the following special case. The set $\mathrm{Spin}(\Sigma \times \mathbf{I} | l \times \mathbf{I}, \sigma \times \partial\mathbf{I})$ of relative spin structures on the identity cobordism $(\Sigma, l) \times \mathbf{I}$ has a group structure given by the composition of cobordisms. By Proposition 2.2, this group is isomorphic to $H^1(\Sigma \times \mathbf{I}, \Sigma \times \partial\mathbf{I}; \mathbf{Z}/2) = H^0(\Sigma; \mathbf{Z}/2)$. (The element $0 \in H^0(\Sigma; \mathbf{Z}/2)$ corresponds to the product extension.)

Now $\mathrm{Spin}(\Sigma \times \mathbf{I} | l \times \mathbf{I}, \sigma \times \partial \mathbf{I})$ acts by glueing on the set $\mathrm{Spin}(M|K, \sigma)$. Thus we have a natural action of $H^0(\Sigma; \mathbf{Z}/2)$ on $\mathrm{Spin}(M|K, \sigma)$.

LEMMA 2.4. *Under this action, the augmentation class* $1 \in H^0(\Sigma; \mathbf{Z}/2)$ *acts as the identity on* $\mathrm{Spin}(M|K, \sigma)$, *hence the action factors through the reduced group* $\tilde{H}^0(\Sigma; \mathbf{Z}/2)$.

Proof. The action corresponds to the connecting homomorphism $H^0(\Sigma; \mathbf{Z}/2) \to H^1(M, \Sigma; \mathbf{Z}/2)$, and since the class 1 comes from $H^0(M; \mathbf{Z}/2)$, the result follows. □

3. A spin refinement of the Jones-Kauffman module

The Kauffman skein relations. We suppose A is an invertible element in a commutative ring Λ. Let M be a closed 3-manifold. The Jones-Kauffman skein module of M, denoted by $\mathscr{K}(M)$, is the Λ-module freely generated by banded links in M, quotiented by ambient isotopy and the Kauffman relations [13]:

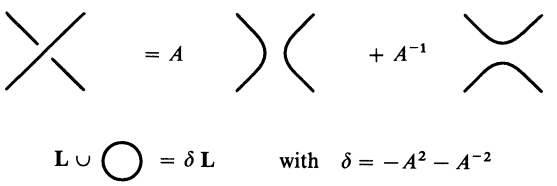

FIGURE 1

(where $\mathbf{L}$ is any banded link). For example, one has $\mathscr{K}(S^3) \approx \Lambda$, the isomorphism being given by the Kauffman bracket.

Note. As is well known, the Kauffman bracket is essentially equivalent to Jones's original V-polynomial [12]. But for our purposes, the Kauffman bracket is more convenient.

The following is an extension of the above definition to the case of singular spin structures.

Definition 3.1. Let M be a closed 3-manifold. The spin Jones-Kauffman module of M, denoted by $\mathscr{K}^s(M)$, is the Λ-module freely generated by the pairs (K, s), where K is a banded link in M, and s is a spin structure complementary for K, quotiented by ambient isotopy and the usual Kauffman relations.

Observe that the Kauffman relations make sense in this context, because if a w_2-structure σ, complementary for the link, is given outside the small ball, the extension over the complement of the two segments (or the circle) in the ball is unique up to homotopy; and if we change σ by a homotopy outside the ball, the homotopy extends (see Proposition 2.2).

Definition 3.2. If M is a 3-manifold with boundary, given a w_2-structure σ complementary for a banded link l in ∂M, the relative spin Jones-Kauffman module, denoted by $\mathscr{K}^s(M, l, \sigma)$, is the Λ-module freely generated by the $(K, \tilde{\sigma})$, where K is a banded link in M meeting ∂M tranversally in $\partial K = l$, and $\tilde{\sigma}$ is a homotopy class of extensions of σ complementary to K, quotiented by ambient isotopy (rel. boundary) and the Kauffman relations.

Forgetting the relative spin structure defines a map from $\mathscr{K}^s(M, l, \sigma)$ to the usual relative skein module, denoted by $\mathscr{K}(M, l)$. This is the same as quotienting out by the natural action of the group $H^1(M, \partial M; \mathbf{Z}/2)$ on $\mathscr{K}^s(M, l, \sigma)$. For $u \in H^2(M, \partial M; \mathbf{Z}/2)$, let $\mathscr{K}_u(M, l, \sigma)$ be the submodule of $\mathscr{K}(M, l)$ generated by those K with obstruction $w_2(M|K, \sigma)$ equal to u. Using Proposition 2.2 (c), one has the following.

PROPOSITION 3.3. *The relative skein module $\mathscr{K}(M, l)$ is the direct sum of the submodules $\mathscr{K}_u(M, l, \sigma)$, for $u \in H^2(M, \partial M; \mathbf{Z}/2)$. The image of the forgetful map is precisely $\mathscr{K}_0(M, l, \sigma)$.*

Remark 3.4. (1) Note that it follows from Proposition 2.2 (b) that $\mathscr{K}_u(M, l, \sigma)$ is nonzero for all u.

(2) The direct sum decomposition of $\mathscr{K}(M, l)$ is independent of the w_2-structure σ on $\Sigma - l$. Only the indexing of the submodules depends on it; in fact, it depends only on the induced spin structure $[\sigma]$, in the following way. If $[\sigma'] = [\sigma] + x$ with $x \in H^1(\Sigma; \mathbf{Z}/2)$, then

$$\mathscr{K}_u(M, l, \sigma') = \mathscr{K}_{u+\delta(x)}(M, l, \sigma),$$

where $\delta\colon H^1(\Sigma; \mathbf{Z}/2) \to H^2(M, \Sigma; \mathbf{Z}/2)$ is the coboundary.

Example. The classical Jones-Kauffman module of the solid torus $D^2 \times S^1$ is isomorphic to the Λ-algebra $\Lambda[z]$. Here, z is represented by $J \times S^1$, where $J \subset D^2$ is a small embedded interval. We denote by $\Lambda[z]^{\text{even}}$ and $\Lambda[z]^{\text{odd}}$ respectively the even and odd part of $\Lambda[z]$. If σ is a w_2-structure on $S^1 \times S^1$ (without link), the module $\mathscr{K}^s(D^2 \times S^1, \sigma)$ is isomorphic (as a module) to $\Lambda[z]^{\text{even}}$ or $\Lambda[z]^{\text{odd}}$, depending whether the w_2-structure σ extends over $D^2 \times S^1$ or not.

Note. An interesting example is $M = S^2 \times S^1$. Its spin Jones-Kauffman module is computed in [23].

4. Kirby calculus for singular spin structures. The Kirby calculus [14] describes 3-manifolds in terms of banded links in the 3-sphere. It is well known how to describe spin structures in terms of characteristic sublinks (see [5], [16]). In this section, we generalize this to the case of singular spin structures.

Assume that M is presented as the result of *surgery* on a framed link L in the 3-sphere S^3. Thus, $M = \partial W_L$ where W_L is the 4-ball B^4 with 2-handles added along the components L_i of $L \subset \partial B^4 = S^3$. Up to isotopy, any banded link in M

may be assumed to lie in $S^3 - L$. If K is such a banded link, we ask for existence of spin structures complementary to K in M. To such a spin structure, σ, corresponds a *characteristic sublink* of L denoted by C_σ. It is defined as follows. A component of L belongs to C_σ if and only if the restriction of σ to $S^3 - (L \cup K)$ does *not* extend across that component.

PROPOSITION 4.1. *The correspondence $\sigma \mapsto C_\sigma$ defines a bijection between the set of spin structures complementary to K in M, and the solutions of the following equation (the components of L are denoted by L_i)*

$$\forall i \quad L_i \cdot (C \cup K) = L_i \cdot L_i \qquad (\mathrm{mod}\ 2).$$

Here the point between two distinct components stands for the linking number in S^3, and $L_i \cdot L_i$ is the framing coefficient.

Remark 4.2. The equation above can be seen as a linear system in $(\mathbf{Z}/2)^{\# L}$. It can be written

$$\forall i \quad \sum_j b_{ij} c_j = b_{ii} + k_i,$$

where $b_{ij} = L_i \cdot L_j$, $k_i = L_i \cdot K$ and $c_j = 1$ whenever L_j belongs to C. The numbers $c_j \in \mathbf{Z}/2$ are called the *characteristic coefficients.*

Assuming the proposition for a moment, let us denote by $M_{(L,C,K)}$ the 3-manifold obtained by surgery on the banded link L equipped with the spin structure complementary to K corresponding to the characteristic sublink C.

THEOREM 4.3. *The two manifolds $M_{(L,C,K)}$ and $M_{(L',C',K')}$ are spin diffeomorphic if and only if (L, C, K) and (L', C', K') are equivalent under the relation generated by the following move.*

Add to L an unknotted component L_i with framing coefficient $\varepsilon = \pm 1$ and change $L \cup K$ by giving a twist (right or left handed, depending on the sign of ε) to the part of $L \cup K$ lying in a regular neighborhood of a disk bounded by L_i. The characteristic coefficient of the new component is $c_i = 1 + lk(L_i, C \cup K)$; the other characteristic coefficients remain unchanged.

This follows in standard fashion from Kirby's theorem [14]. Details are left to the reader. (The case where $K = \varnothing$ is treated in [5]. For the extension to the general case, one may consult [7, p. 44].)

We now prove Proposition 4.1. A spin structure on a manifold M can also be defined as a cohomology class in $H^1(P_M; \mathbf{Z}/2)$, where P_M is the oriented frame bundle (see [26], [15]). If M is 3-dimensional, we can evaluate this cohomology class on a banded knot, which represents a 1-cycle in P_M. (The result does not depend on the orientation of the core of the banded knot.) Note that the evaluation of this cohomology class on a trivial banded knot with writhe zero is not zero, but 1.

LEMMA 4.4. *Given a link $L = (L_1, \ldots, L_n)$ in S^3 and a sequence $v_1, \ldots, v_n$ in $\mathbf{Z}/2$, there exists a unique spin structure σ on $S_3 - L$ such that for all j, the evaluation of σ on the meridian m_j (viewed as a banded knot in the obvious way using the meridinal disk) is v_j.*

Proof. Let σ_0 be the spin structure on $S^3 - L$ induced by the unique spin structure on S^3. Then $\mathrm{Spin}(S^3 - L) = \sigma_0 + H^1(S^3 - L; \mathbf{Z}/2)$, and $<\sigma_0 + c, m_j> = 1 + <c, \mu_j>$, where μ_j is the core of the banded meridian m_j. The lemma follows. □

Remark. Observe that the evaluation of σ on a banded knot γ in $S^3 - L$ is $1 + \gamma \cdot \gamma + \sum_j (1 - v_j)(\gamma \cdot L_j)$.

Proof of 4.1. Spin structures complementary to K in M are in one-to-one correspondence with cohomology classes on the oriented frame bundle of $S^3 - (L \cup K)$ which take the value 0 on the meridians around components of K, and the value 1 on the preferred longitude of the components of L. Using the lemma, we may describe such a spin structure by a sequence of v_j's. The v_j's corresponding to K are zero, and the remaining v_j's must satisfy conditions given by the remark above. These conditions are equivalent to the condition in the statement of 4.1, since the v_i corresponding to a component L_i of L is zero if and only if L_i is in the characteristic sublink C corresponding to the spin structure. □

5. The invariant $\theta_{8k}^{\mathrm{spin}}(M)$. In this section, we generalize the invariants of [5] to 3-manifolds with banded link and complementary spin structure.[2] Although we can define a spin invariant for each root of unity whose order is not congruent to 8 modulo 16, we suppose here that A is a primitive $16k$th root of unity. The reason is that in other cases, the invariant depends only weakly on the spin structure (see [5]).

The coefficient ring. Let $k \geqslant 1$ be a fixed integer. We suppose that Λ is a ring (commutative, with unit, but not necessarily an integral domain) in which we are given the following data.

(a) A primitive $16k$th root of unity A.
(b) An element κ such that $\kappa^6 = A^{-6-4k}$. (This element will be used for the p_1-normalization of the invariant.)
(c) An involutive automorphism which sends A and κ to their inverses.

We suppose moreover that

(d) 2 and k are invertible in Λ.

A natural choice for Λ is (φ_d is the d-indexed cyclotomic polynomial)

$$\mathbf{Z}\left[A, \kappa, \frac{1}{2k}\right] \Big/ (\varphi_{16k}(A), \kappa^6 - A^{-6-4k}).$$

[2] This generalization was stated without proof already in [7].

The metabracket. Given a banded link L, one can replace each of its components L_i by k_i parallels, and then evaluate the Kauffman bracket of this cabled link. Since $\mathscr{K}(D^2 \times S^1) = \Lambda[z]$, this yields a multilinear form $\Lambda[z]^{\otimes n} \to \Lambda$, which we call the metabracket of L (see [5], [6], [18], [19]). If $L = (L_1, \ldots, L_v)$, the metabracket of L will be denoted by $\langle L_1(\,), \ldots, L_v(\,)\rangle$. The notation $\langle L_1(a_1), \ldots, L_v(a_v)\rangle$, where $a_1, \ldots, a_v$ are polynomials in the variable z, means that each component of L_i is replaced by the formal cabling corresponding to a_i; the Kauffman bracket of such a thing is obtained expanding linearly. For $\varepsilon = \pm 1$, $\langle U_\varepsilon(\,)\rangle$ stands for the metabracket associated with the trivial knot with framing ε. The metabracket associated with the zero-framed unknot is simply denoted $\langle\ \rangle$ as usual. We denote by $\langle H(\,,\,)\rangle$ the metabracket associated with the Hopf link, where each component has framing zero.

The invariant θ^{spin}. Following [5], define elements Ω_0, $\Omega_1 \in \mathscr{K}(D^2 \times S^1) = \Lambda[z]$ by

$$\Omega_0 = \sum_{i=0}^{k-1} \langle e_{2i}\rangle e_{2i} \qquad \Omega_1 = (-A^2 - A^{-2})^{-1} z\Omega_0,$$

where the Chebychev polynomials $e_j \in \Lambda[z]$ are defined by $e_{-1} = 0$, $e_0 = 1$, and $ze_j = e_{j-1} + e_{j+1}$.

Remark 5.1. The elements Ω_i are the "reduced" elements denoted by $\tilde{\omega}_i$ in [5]. As explained there, one may also use $(1/2)\sum_{i=0}^{2k-1} \langle e_{2i}\rangle e_{2i}$ in place of Ω_0, and $(1/2)\sum_{i=0}^{2k-2} \langle e_{2i+1}\rangle e_{2i+1}$ in place of Ω_1. (This will be used in the proof of Lemma 16.1.)

Let (L, C_σ, K) be a surgery presentation for the closed 3-manifold M equipped with a banded link K and a complementary spin structure σ. Thus $M = \partial W_L$ and C_σ is the characteristic sublink as defined in 4.1. Let (b_+, b_-) be the signature of the intersection form on $H_2(W_L; \mathbf{R})$. Using Theorem 4.3, the methods of [5] generalize to give the following.

THEOREM 5.2. *The expression*

$$\theta_{8k}^{\text{spin}}(M, K, \sigma) = \frac{\langle C_\sigma(\Omega_1), (L - C_\sigma)(\Omega_0), K\rangle}{\langle U_1(\Omega_1)\rangle^{b_+}\langle U_{-1}(\Omega_1)\rangle^{b_-}}$$

is a topological invariant of (M, K, σ).

Remark. One has $\langle\Omega_0\rangle = \langle\Omega_1\rangle = -2k/(A^2 - A^{-2})^2$ and $\langle U_\varepsilon(\Omega_0)\rangle = 0$. The computation of $\langle U_\varepsilon(\Omega_1)\rangle$ reduces to a Gauss sum:

$$\langle U_\varepsilon(\Omega_1)\rangle = (A^{2\varepsilon} - A^{-2\varepsilon})^{-1} A^{-3\varepsilon} g_{8k}(\varepsilon) \quad \text{with} \quad g_{8k}(\varepsilon) = \sum_{i=0}^{2k-1} A^{4\varepsilon j^2}.$$

For $A = e^{i\pi/8k}$, $g_{8k}(1) = \overline{g_{8k}(-1)} = \sqrt{k}(1 + i)$.

Remark 5.3. The normalization of $\theta_{8k}^{\text{spin}}$ is as in [5]. If $k = 1$, the invariant is a version of the classical Rohlin invariant $\mu(M)$.[3] In fact, $\theta_8^{\text{spin}}(M) = (-A)^{-3\mu(M)}$ (where M has empty link).

The Jones-Wenzl idempotents f_i. (See [18], [19], [24], [8].) Let us denote by ${}_nT_m$ the Jones-Kauffman module $\mathscr{K}(D^2 \times \mathbf{I}, l)$, where l is a standard banded link with n components in $D^2 \times 0$ and m components in $D^2 \times 1$. Thus, an element of ${}_nT_m$ is a linear combination of (n, m)-tangles, taken modulo skein relations. Comp ition of tangles makes the collection of these Λ-modules into a Λ-algebroid. In particular, if $n = m$, ${}_nT_n$ is a Λ-algebra, called the Temperley-Lieb algebra on n strings. If $i \leqslant 4k - 1$, this algebra has a unique central idempotent f_i with the property that f_i annihilates all tangles that factor through a (j, j)-tangle with $j < i$. (These idempotents were first considered (in a different language) by Jones [11] and Wenzl [30].)

Extension to 3-manifolds with colored graphs. It is clear that θ^{spin} extends to a well-defined linear invariant on the spin Jones-Kauffman module $\mathscr{K}^s(M)$. We extend the invariant also to colored banded 3-valent graphs G in M with complementary spin structure σ, by setting

$$\theta^{\text{spin}}(M, G, \sigma) = \theta^{\text{spin}}(E(M, G, \sigma)),$$

where $E(M, G, \sigma) \in \mathscr{K}^s(M)$ is the *expansion* of (G, σ) defined as follows.

Each i-colored edge of the graph is replaced by i parallel banded segments, and a copy of the Jones-Wenzl idempotent f_i is inserted. The admissibility condition ensures that all these segments can be joined together in a natural way to get a link, more precisely a linear combination of links. (See Section 4 of [8] for more details.) Furthermore, the complementary spin structure σ given outside the graph has, for each link which appears in this linear combination, a well-defined extension complementary to this link. The resulting element of $\mathscr{K}^s(M)$ is the expansion of (G, σ).

Remark 5.4. Expanding an i-colored circle component (which is viewed as an i-colored edge with the two vertices identified) is the same as cabling that component by the Chebychev polynomial e_i.

6. The invariant $< M >_{8k}^{\text{spin}}$. The functors V_{8k}^s on the cobordism categories $\mathscr{C}_{8k}^s$ will be defined by a universal construction from an invariant $< >_{8k}^{\text{spin}}$ of closed bordisms in $\mathscr{C}_{8k}^s$. This invariant is just a slight modification of $\theta_{8k}^{\text{spin}}$, but depends on the p_1-structure. The use of this modified invariant gives nice surgery properties for the functors V_{8k}^s.

[3] Recall that $\mu(M)$ takes values in $\mathbf{Z}/16$: it associates to a closed oriented 3-manifold with spin structure, M, the signature mod 16 of an oriented spin 4-manifold with boundary equal to M.

The invariant $\sigma(\alpha)$. Let α be a p_1-structure on a closed 3-manifold M. Suppose $M = \partial W_L$ is given by surgery on the banded link L in S^3. Let $p_1(L, \alpha) \in \mathbf{Z}$ be the obstruction to extending α to W_L, evaluated on the fundamental class $[W_L] \in H_4(W_L, M; \mathbf{Z})$. As in [8], we define $\sigma(\alpha) = 3$ signature$(W_L) - p_1(L, \alpha)$, which, by Hirzebruch's signature formula, is a well-defined integer (see Atiyah [3]).

In the ring Λ (see Section 5), we set[4]

$$\eta_s = \kappa^3 \langle U_1(\Omega_1) \rangle^{-1} = \kappa^3 (A^2 - A^{-2}) A^3 g_{8k}(1)^{-1}.$$

In the definition below, the closed bordism $M = (M, \alpha, K, \sigma)$ in $\mathscr{C}^s$ is the disjoint union of the connected bordisms $(M_i, \alpha_i, K_i, \sigma_i)$, $1 \leqslant i \leqslant b_0$; α_i is a p_1-structure on M_i, and σ_i is a spin structure complementary for the colored graph K_i. Let b_1 denote the first Betti number of M.

Definition 6.1.

$$< M >_{8k}^{\text{spin}} = \eta_s^{b_0 + b_1} \prod_{i=1}^{b_0} \kappa^{\sigma(\alpha_i)} \theta_{8k}^{\text{spin}}(M_i, K_i, \sigma_i).$$

PROPOSITION 6.2. *The invariant* $< M >^{\text{spin}}$ *is multiplicative:*

$$< M \amalg M' >^{\text{spin}} = < M >^{\text{spin}} < M' >^{\text{spin}} \qquad \textit{and} \qquad < \varnothing >^{\text{spin}} = 1$$

and involutive:

$$< -M >^{\text{spin}} = \overline{< M >^{\text{spin}}}.$$

Proof. Multiplicativity is obvious. Involutivity follows from $\bar{\eta}_s = \eta_s$ and the involutivity of θ^{spin}, which is proved in routine fashion from the behaviour of the Kauffman bracket under mirror images.

Remark 6.3. If a connected closed bordism $M = (M, \alpha, K, \sigma)$ has a surgery presentation which is a banded link L in S^3 with characteristic sublink C, then

$$< M >^{\text{spin}} = \eta_s \kappa^{-p_1(L, \alpha)} \langle C_\sigma(\omega_1), (L - C_\sigma)(\omega_0), K \rangle,$$

where

$$\omega_0 = \lambda \Omega_0, \qquad \omega_1 = \lambda \Omega_1$$

with $\lambda = (\eta_s \langle \Omega_0 \rangle)^{-1}$.

Remark 6.4. For $\varepsilon \in \{0, 1\}$, let $S^2 \times S^1_\varepsilon$ be $S^2 \times S^1$ equipped with the empty link, a standard p_1-structure (with σ-invariant zero), and a spin structure which

[4] The notation η_s is used because η_s is different from the number η used in [8]: one has $\eta_s = 2\eta$.

extends to $S^2 \times D^2$ if and only if $\varepsilon = 0$. Then for both $\varepsilon = 0$ and $\varepsilon = 1$, one has

$$<S^2 \times S^1_\varepsilon>^{\text{spin}} = \eta_s\langle\omega_\varepsilon\rangle = 1.$$

7. The functors V^s_{8k}

Definition of $V^s_{8k}(\Sigma)$. Let Σ be an object of the category $\mathscr{C}^s_{8k}$. Denote by $\mathscr{V}^s(\Sigma)$ the Λ-module freely generated by the set of all bordisms M such that $\partial M = \Sigma$. The formula

$$<M, M'>^{\text{spin}}_\Sigma = <M \cup_\Sigma (-M')>^{\text{spin}}$$

extends to a sesquilinear form $<\,,\,>^{\text{spin}}_\Sigma$ on $\mathscr{V}^s(\Sigma)$.

LEMMA 7.1. *The form* $<\,,\,>^{\text{spin}}_\Sigma$ *is hermitian.*

This follows essentially from the involutivity of the invariant $<\,>^{\text{spin}}$. A detailed proof will be given in the next section.

Definition 7.2. The module $V^s(\Sigma)$ is defined to be the quotient of $\mathscr{V}^s(\Sigma)$ by the radical of the form $<\,,\,>^{\text{spin}}_\Sigma$.

The induced hermitian form on $V^s(\Sigma)$ is denoted again by $<\,,\,>^{\text{spin}}_\Sigma$; it is nondegenerate by definition. The class of M in $V^s(\Sigma)$ is denoted by $Z^s(M)$. Thus, for all bordisms M, we have an element $Z^s(M) \in V^s(\partial M)$. The multiplicativity of the invariant $<\,>^{\text{spin}}$ implies that $V^s(\varnothing)$ is identified with Λ, with $Z^s(\varnothing)$ corresponding to $1 \in \Lambda$, and, more generally, $Z^s(M)$ corresponding to $<M>^{\text{spin}} \in \Lambda$ whenever M is a closed bordism. Furthermore, if M is a cobordism from Σ to Σ', glueing along Σ defines a homomorphism, denoted by Z^s_M, from $V^s(\Sigma)$ to $V^s(\Sigma')$.

Note. The above is an adaptation of the universal construction of [8] to the invariants defined in 6.1. The pair (V^s, Z^s) is a quantization functor on the category $\mathscr{C}^s_{8k}$, as defined in [8, Section 1.A].

Note. Glueing along Σ defines a natural map

$$V^s(-\Sigma \amalg \Sigma') \to \operatorname{Hom}(V^s(\Sigma), V^s(\Sigma')).$$

This map sends $Z^s(M)$ to Z^s_M. (This map need not be an isomorphism; see Remark 12.2.)

In the next sections we are going to compute the theories (V^s, Z^s). The following theorem will be proven in Section 17.

THEOREM 7.3 (Finiteness). *For all objects* Σ *in* $\mathscr{C}^s$, *the module* $V^s(\Sigma)$ *is free and finitely generated, and the hermitian form* $<\,,\,>^{\text{spin}}_\Sigma$ *is unimodular.*

Let $V^s(\Sigma)^*$ denote the dual module $\operatorname{Hom}(V^s(\Sigma), \Lambda)$.

COROLLARY 7.4. *The map $V^s(-\Sigma) \to V^s(\Sigma)^*$ (defined by glueing) is an isomorphism.*

Proof. For all bordisms M, M' with boundary Σ, we have

$$< -M, -M' >_{-\Sigma}^{\text{spin}} = < M', M >_{\Sigma}^{\text{spin}} = \overline{< M, M' >_{\Sigma}^{\text{spin}}}.$$

Hence orientation reversal extends to a well-defined isomorphism from $\overline{V^s(\Sigma)}$ (the conjugate module) to $V^s(-\Sigma)$. The map $V^s(-\Sigma) \to V^s(\Sigma)^*$ is equal to the inverse of this isomorphism, followed by the adjoint map $\overline{V^s(\Sigma)} \to V^s(\Sigma)^*$. Unimodularity means that the adjoint map is an isomorphism, whence the result. □

A precise description of $V^s(\Sigma)$ will be given. An interesting point will be to discuss multiplicativity. We shall see in Section 12 that the obvious map $V^s(\Sigma_1) \otimes V^s(\Sigma_2) \to V^s(\Sigma_1 \amalg \Sigma_2)$ need not be an isomorphism. However, we will show that the theory satisfies some kind of $\mathbf{Z}/2$-graded multiplicativity property.

8. Glueing and orientation reversal. Let Σ be an object of $\mathscr{C}^s$. As already said in Section 2, the set of relative spin structures on $\Sigma \times \mathbf{I}$ is canonically identified with $H^0(\Sigma; \mathbf{Z}/2)$. So, to each $c \in H^0(\Sigma; \mathbf{Z}/2)$ is associated a cobordism $(\Sigma \times \mathbf{I})_c$, between Σ and itself (we choose the trivial extension of the p_1-structure).

The following technical construction is a key point in understanding the tricky sign questions involved with orientation reversal of relative spin structures. Let M be a bordism with boundary ∂M. Recall that we have defined the orientation-reversed bordism, $-M$, with boundary $-\partial M$, by composing the w_2- and p_1-structure with the orthogonal reflection in the direction of a unit vector field in the stable tangent bundle of M which extends the *outward* normal vector on ∂M.

LEMMA 8.1. *Assume ∂M is decomposed as a disjoint union of components $\Sigma \amalg \Sigma'$. Let M' be the bordism, with boundary $-\partial M$, obtained from M by composing the w_2- and p_1-structure on M with the orthogonal reflection in the direction of a unit vector field which extends the outward normal vector on Σ', and the inward normal vector on Σ. Then*

$$M' = (-M) \underset{-\Sigma}{\cup} ((-\Sigma) \times \mathbf{I})_1,$$

where $1 \in H^0(\Sigma; \mathbf{Z}/2)$ is the augmentation class.

Proof. We may assume that the vector fields used for constructing $-M$ and M' coincide except in a collar $\Sigma \times \mathbf{I}$ of Σ, where the first one, v, say, is constant in the time direction, and the second one, v', say, is inward at both ends. Our problem is to compare the w_2- and p_1-structure on $(-\Sigma) \times \mathbf{I}$ obtained by using the reflections in the direction of v and v', respectively. Using v we get the cobordism $((-\Sigma) \times \mathbf{I})_0$ which acts trivially. A moment's thought shows that using v' we get $((-\Sigma) \times \mathbf{I})_1$. This proves the lemma. □

Note. The point here is that the reflection in the direction of the vector field $t \mapsto (\cos \pi t, \sin \pi t)$ in the stable tangent bundle of the interval $\mathbf{I}$ (where $\mathbf{R} \times 0$ is the tangent direction, and $0 \times \mathbf{R}$ is the direction of the first stabilization), gives the nontrivial relative spin structure on $-\mathbf{I}$. Note, however, that the induced relative p_1-structure is trivial, so that this technical problem can be ignored for p_1-structures.

Example 8.2. In particular, the above argument shows that

$$-((\Sigma \times \mathbf{I})_0) = ((-\Sigma) \times \mathbf{I})_1.$$

Note that orientation reversal is not functorial. The following corollary to Lemma 8.1 describes the defect.

COROLLARY 8.3. *Let M_1 be a cobordism from Σ_1 to Σ_2, and M_2 be a cobordism from Σ_2 to Σ_3. Then*

$$-(M_1 \underset{\Sigma_2}{\cup} M_2) = (-M_1) \underset{-\Sigma_2}{\cup} ((-\Sigma_2) \times \mathbf{I})_1 \underset{-\Sigma_2}{\cup} (-M_2),$$

where $1 \in H^0(-\Sigma_2; \mathbf{Z}/2)$ is the augmentation class.

Remark 8.4. In the special case where Σ_1 or Σ_3 is empty, Lemma 2.4 implies that the insertion of $((-\Sigma_2) \times \mathbf{I})_1$ does not change the relative spin structure, and hence can be omitted.

Note. It is possible to define orientation reversal in such a way as to get a functor, by making the definition of the orientation-reversed manifold dependent on the way it is regarded as a cobordism. For simplicity, we have chosen not to do this here.

Proof of Lemma 7.1. We must show that the form $<\,,\,>_\Sigma^{\text{spin}}$ is hermitian. Let M, M' be bordisms with boundary Σ. By the involutivity of the invariant $<\,>^{\text{spin}}$, it suffices to show that the spin and p_1-structure on $M' \cup_\Sigma (-M)$ is equal to the orientation reversal to the spin and p_1-structure on $M \cup_\Sigma (-M')$. This follows from Corollary 8.3 (with $M_1 = M$, $M_2 = -M'$) and the remark following it. □

9. The decomposition theorem. Let Σ be an object of $\mathscr{C}^s$. For $c \in H^0(\Sigma; \mathbf{Z}/2)$, the cobordism $(\Sigma \times \mathbf{I})_c$ (defined in the previous section) induces an endomorphism τ_c on $V^s(\Sigma)$. If an element of $V^s(\Sigma)$ is represented by a bordism M with $\partial M = \Sigma$, then $\tau_c(M)$ is represented by $M \cup_\Sigma (\Sigma \times \mathbf{I})_c$.

PROPOSITION 9.1. *τ_c is an isometry.*

The proof is deferred to the end of this section.

Thus, we have an isometric action τ of the group $H^0(\Sigma; \mathbf{Z}/2)$ on $V^s(\Sigma)$. Lemma 2.4 shows that, under this action, the augmentation class $1 \in H^0(\Sigma; \mathbf{Z}/2)$ acts as the identity on $V^s(\Sigma)$. Hence the action factors through the reduced group $\tilde{H}^0(\Sigma; \mathbf{Z}/2)$.

This action of $\tilde{H}^0(\Sigma; \mathbf{Z}/2)$ immediately gives the following decomposition of $V^s(\Sigma)$. For $a \in \tilde{H}_0(\Sigma; \mathbf{Z}/2)$ we set

$$V^s(\Sigma; a) = \{x \in V^s(\Sigma): \forall c\ \tau_c(x) = (-1)^{c(a)}x\}.$$

THEOREM 9.2 (Decomposition theorem). *There is a canonical orthogonal decomposition*

$$V^s(\Sigma) = \bigoplus_{a \in \tilde{H}_0(\Sigma;\, \mathbf{Z}/2)} V^s(\Sigma; a).$$

Remark 9.3. Consider an object Σ' obtained from Σ by homotoping the w_2-structure. This homotopy defines a cobordism from Σ to Σ' which induces an isomorphism $V^s(\Sigma; a) \approx V^s(\Sigma'; a)$ for all $a \in \tilde{H}_0(\Sigma; \mathbf{Z}/2)$. In general, this isomorphism *depends* on the chosen homotopy. But, by their very definition, the "zero-graded" parts $V^s(\Sigma; 0)$ and $V^s(\Sigma'; 0)$ are *canonically* isomorphic.

It remains to prove Proposition 9.1. It is a consequence of the following more general lemma.

LEMMA 9.4. *Suppose C is a cobordism between Σ_0 and Σ_1, whose underlying manifold is $\Sigma \times \mathbf{I}$; then Z_C^s: $V^s(\Sigma_0) \to V^s(\Sigma_1)$ is an isometric isomorphism.*

Proof. First we observe that w_2- or p_1-structures on $\Sigma \times \mathbf{I}$ are nothing but homotopies between the w_2- or p_1-structures on Σ_0 and Σ_1. Let C' be the cobordism between Σ_1 and Σ_0 corresponding to the inverse homotopies; then the cobordism $C \cup_{\Sigma_1} C'$ is equivalent to the trivial $(\Sigma_0 \times \mathbf{I})_0$. We deduce that Z_C and $Z_{C'}$ are inverse isomorphisms.

Suppose that M and M' are bordisms with boundary Σ_0, then $<Z_C^s(M), Z_C^s(M')>_{\Sigma_1}^{\text{spin}}$ is the invariant of the manifold $M \cup_{\Sigma_0} C \cup_{\Sigma_1} C'' \cup_{\Sigma_0} (-M')$, where C'' is obtained from C by composing the structures with the reflection in the time direction. Using the diffeomorphism

$$\Sigma \times \mathbf{I} \to \Sigma \times \mathbf{I}$$

$$(x, t) \mapsto (x, 1 - t),$$

we can replace C'' by the cobordism C', and then forget $C \cup_{\Sigma_1} C'$. Whence the equality

$$<Z_C^s(M), Z_C^s(M')>_{\Sigma_1}^{\text{spin}} = <M, M'>_{\Sigma_0}^{\text{spin}},$$

which shows that Z_C^s is an isometry. □

10. Surgery axioms. Let S^3 denote a closed bordism whose underlying 3-manifold is the 3-sphere (with empty link), with a standard w_2- and p_1-structure

(extending to D^4). Since $<S^3>^{\text{spin}} = \eta_s$, we have

(S0) $<S^3>^{\text{spin}}$ *is invertible in* Λ.

Assume $S^0 \times D^3$ and $D^1 \times S^2$ are equipped with their product orientations and some relative spin and p_1-structure, such that $\partial(S^0 \times D^3) = \partial(D^1 \times S^2) = S^0 \times S^2$, and such that the structure on the 3-sphere $S^0 \times D^3 \cup_{S^0 \times S^2} -D^1 \times S^2$ extends to the 4-ball. (This fixes the relative p_1-structure on $D^1 \times S^2$, but allows both relative spin structures.)

(S1) *(Index-1 surgery)* $Z^s(S^0 \times D^3) = \eta_s Z^s(D^1 \times S^2) \in V^s(S^0 \times S^2)$.

Proof. It suffices to check that an index-1 surgery multiplies the invariant of a closed bordism M by η_s^{-1}. Performing a surgery of index 1 on M means either replacing two components of M by their connected sum (in this case, an argument as in Lemma 2.4 shows that the spin structure on the connected sum does not depend on the relative spin structure on $D^1 \times S^2$), or replacing a component M_i by its connected sum with $(S^2 \times S^1)_0$ or $(S^2 \times S^1)_1$ (which have the same invariant; see Remark 6.4). Using the fact that the invariant θ^{spin} is multiplicative under connected summing, one checks that in each case one gets the required factor. □

Notation. Let $S^1_\varepsilon \times S^1_{\varepsilon'}$ be the torus $S^1 \times S^1$ equipped with a fixed w_2-structure such that the corresponding quadratic form (see 2.3) takes the value ε on the meridian $S^1 \times \{1\}$ and the value ε' on the longitude $\{1\} \times S^1$.

Assume $S^1 \times D^2$ and $D^2 \times S^1$ are equipped with their product orientations so that $\partial(-S^1 \times D^2) = \partial(D^2 \times S^1) = S^1 \times S^1$. Fix relative p_1-structures respectively on $D^2 \times S^1$ and $-S^1 \times D^2$ such that the structure on the 3-sphere $D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2$ extends to the 4-ball. We denote by $D^2 \times S^1_\varepsilon$ (respectively, $-S^1_\varepsilon \times D^2$) the p_1-manifold above equipped with a w_2-structure which extends the one given on $S^1_0 \times S^1_\varepsilon$ (respectively, on $S^1_\varepsilon \times S^1_0$).

(S2) *(Index-2 surgery) For* $\varepsilon \in \{0, 1\}$, *the element* $Z^s(D^2 \times S^1_\varepsilon) \in V^s(S^1_0 \times S^1_\varepsilon)$ *lies in the submodule generated by links and complementary (relative) spin structures, in the solid torus* $-S^1 \times D^2$.

Proof. Consider the expression for the invariant $<M>^{\text{spin}}$ given in Remark 6.3. The surgered manifold M is obtained by taking out a tubular neighborhood isomorphic to $-S^1 \times D^2$ of each component L_i of L, and replacing it by $D^2 \times S^1_\varepsilon$ (ε depends on whether L_i is characteristic or not). Formula 6.3 says that if, instead of doing surgery, L_i is cabled by ω_ε, then the same invariant is obtained. It follows that $Z^s(D^2 \times S^1_\varepsilon)$ is represented by ω_ε, whence (S2). □

Note. For $\varepsilon = 0$, the above is surgery in the spin category, whereas for $\varepsilon = 1$, it is "characteristic" surgery. One can show that existence of ω_0 satisfying the spin surgery formula implies existence of ω_1 satisfying the characteristic surgery formula.

The following is a key result for the computation of $V^s(\Sigma)$.

LEMMA 10.1. *Let Σ be an object of $\mathscr{C}^s$, and N be a connected 3-manifold with boundary Σ. Then $V^s(\Sigma)$ is generated by the elements $Z^s(M)$, where the bordism M has N as underlying manifold. Furthermore, if N' is another connected 3-manifold with boundary Σ, then an element $\sum \lambda_i Z^s(M_i) \in V^s(\Sigma)$ is zero if and only if $\sum \lambda_i < Z^s(M_i), Z^s(M') >_\Sigma^{\text{spin}}$ is zero for all bordisms M' with N' as underlying manifold.*

Proof. This follows from the surgery axioms. Details are omitted, as the proof is analogous to the proof of Proposition 1.9 of [8]. □

Remark 10.2. Let Σ be an object of $\mathscr{C}^s$, with banded link l, complementary w_2-structure σ, and p_1-structure α. Let N be a 3-manifold with boundary Σ, equipped with a relative p_1-structure $\tilde{\alpha}$ extending α. There is a well-defined map from the spin Jones-Kauffman module $\mathscr{K}^s(N, l, \sigma)$ (see Section 3) to $V^s(\Sigma)$, sending $(K, \tilde{\sigma})$ to $Z^s(N, \tilde{\alpha}, K, \tilde{\sigma})$. (This follows from the fact that the invariant $< >^{\text{spin}}$, considered as a function of a banded link K in a fixed manifold, satisfies the Kauffman relations.) This map depends affinely on the relative p_1-structure $\tilde{\alpha}$ on N, in the sense that the set of relative p_1-structures on N is affinely isomorphic to $H^3(N, \Sigma; \mathbf{Z})$, and changing $\tilde{\alpha}$ by a cohomology class c multiplies the map by κ^d, where d is the evaluation of c on the fundamental class $[N, \partial N]$.

Note. In Section 3, the spin Jones-Kauffman module $\mathscr{K}^s(N, l, \sigma)$ was defined only for uncolored links. This definition can be extended to colored links and graphs using the notion of expansion explained above. We leave this to the reader, as it is not needed in the sequel.

Convention. To simplify language, a bordism M in $\mathscr{C}^s$ with underlying manifold N will from now on be called a *structure* in N. Also, all manifolds are assumed to be equipped with p_1-structures which will not be mentioned explicitly any more.

11. Some examples. We denote by $S^2(i_1, \ldots, i_\nu)$ a sphere with a ν-component banded link colored by $(i_1, \ldots, i_\nu)$ ($i_1 + \cdots + i_\nu$ must be even). We can compute the following examples. Note that in (a)–(e) below, the complementary relative spin structure is unique.

(a) $V^s(S^2(i)) = \begin{cases} \Lambda & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$

A generator of $V^s(S^2(0)) = V^s(S^2)$ is represented by the standard 3-ball with empty link.

(b) $V^s(S^2(i_1, i_2)) = \begin{cases} \Lambda & \text{if } i_1 = i_2 \\ 0 & \text{otherwise.} \end{cases}$

A generator of $V^s(S^2(i, i))$ is represented by the standard 3-ball equipped with a standard graph having just one edge colored by i.

A triple (i_1, i_2, i_3) is said to be $8k$-*admissible* if it satisfies the admissibility condition $|i_1 - i_2| \leqslant i_3 \leqslant i_1 + i_2$ and $i_1 + i_2 + i_3$ even, and if, moreover, $i_1 + i_2 + i_3 < 8k - 2$.

(c) $V^s(S^2(i_1, i_2, i_3)) = \begin{cases} \Lambda & \text{if } (i_1, i_2, i_3) \text{ is } 8k\text{-admissible} \\ 0 & \text{otherwise.} \end{cases}$

If (i_1, i_2, i_3) is $8k$-admissible, a generator of $V^s(S^2(i_1, i_2, i_3))$ is represented by the standard 3-ball equipped with a standard "trivalent vertex."

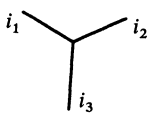

FIGURE 2

Proof. Statements (a)–(c) are proved as follows. By the first part of Lemma 10.1, the V^s-module in question is generated by structures in the 3-ball. Applying the Kauffman relations and taking into account the defining property of the idempotents f_i, we see that it has a generator as claimed. Using now the second part of the lemma, we see that it suffices to compute the hermitian product of this generator with structures in a complementary 3-ball. Applying Kauffman relations again, we just have to test the generator with itself. Up to multiplication by $<S^3>^{\text{spin}}$, this amounts to computing the bracket of a certain colored graph in S^3. Case (a) is obvious. In case (b), the graph is an i-colored circle, and its bracket is $<i> = (-1)^i[i+1]$, where $[n] = (A^{2n} - A^{-2n})/(A^2 - A^{-2})$. In case (c), the graph is a θ-shaped graph with edges colored by i_1, i_2, i_3, and the result is (see [24])

$$<i_1, i_2, i_3> = (-1)^{\alpha+\beta+\gamma} \frac{[\alpha+\beta+\gamma+1]![\alpha]![\beta]![\gamma]!}{[i_1]![i_2]![i_3]!},$$

where α, β, γ are determined by $i_1 = \beta + \gamma$, $i_2 = \gamma + \alpha$, $i_3 = \alpha + \beta$, and $[n]! = [1][2]\cdots[n]$. Since $<i>$ is invertible in Λ for all colors i, and $<i_1, i_2, i_3>$ is invertible precisely in the required range, and zero otherwise, the V^s-module in question is as asserted. □

(d) (Fusion law) In the module $V^s(S^2(i_1, i_2, i_1, i_2))$, one has

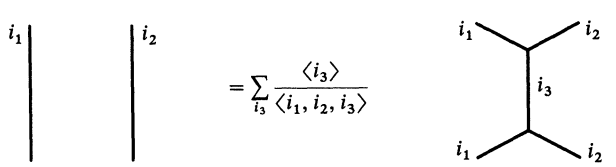

FIGURE 3

(The sum is over those i_3 such that (i_1, i_2, i_3) is $8k$-admissible.)

Proof. An argument as in the proof of Theorem 3.4 of [8], together with case (c) above, shows that the graphs appearing on the right-hand side represent orthogonal generators of $V^s(S^2(i_1, i_2, i_1, i_2))$. Computing the hermitian form as above (using [24]) gives the required formula.

(e) (Band move) As a special case, in the module $V^s(S^2(4k-2, 4k-2, 4k-2, 4k-2))$, one has

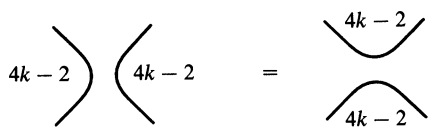

FIGURE 4

12. The computation of $V^s(-S^2(i_1) \amalg S^2(i_2))$. While the formulas in the preceding section follow from standard Kauffman bracket computations, the computation of $V^s(-S^2(i_1) \amalg S^2(i_2))$ is specific to the spin theory.

If i is even, the manifold $S^2(i) \times \mathbf{I}$ has two different relative complementary spin structures (use Proposition 2.2). We let $(S^2(i) \times \mathbf{I})_0$ denote the product extension and $(S^2(i) \times \mathbf{I})_1$ the other one. (This notation was already used in Section 8.)

PROPOSITION 12.1. $V^s(-S^2(i_1) \amalg S^2(i_2)) = \begin{cases} \Lambda & \text{if } i_1 = i_2 = 0 \text{ or } i_1 = i_2 = 4k-2 \\ 0 & \text{otherwise.} \end{cases}$

If $i = 0$, a generator is given by $Z^s((S^2 \times \mathbf{I})_0) = Z^s((S^2 \times \mathbf{I})_1)$.

If $i = 4k-2$, a generator is given by $Z^s((S^2(4k-2) \times \mathbf{I})_0)$, which is equal to $-Z^s((S^2(4k-2) \times \mathbf{I})_1)$.

Proof. First of all, we may suppose i_1, i_2 are even (otherwise there is no complementary spin structure on $S^2(i_j)$). By Lemma 10.1, the module $V^s(-S^2(i_1) \amalg S^2(i_2))$ is generated by bordisms M, with $S^2 \times \mathbf{I}$ as underlying manifold. One can suppose that the graph is contained in a tube $D^2 \times \mathbf{I}$. By example (b) of Section 11, it follows that $V^s(-S^2(i_1) \amalg S^2(i_2))$ is zero except if $i_1 = i_2$. Moreover $V^s(-S^2(i) \amalg S^2(i))$ is generated by bordisms whose underlying manifold is $S^2(i) \times \mathbf{I}$, i.e., by $(S^2(i) \times \mathbf{I})_0$ and $(S^2(i) \times \mathbf{I})_1$. We must compute the hermitian product

$$<(S^2(i) \times \mathbf{I})_\varepsilon, (S^2(i) \times \mathbf{I})_{\varepsilon'}>^{\text{spin}}_{-S^2(i) \amalg S^2(i)}.$$

Note that the "closure" of $(S^2(i) \times \mathbf{I})_\varepsilon$ is $S^2(i) \times S^1_{1+\varepsilon}$, where the subscripts are taken modulo 2 (see, e.g., the discussion of spin structures on the circle in [15]). As already observed in Example 8.2, $-((S^2(i) \times \mathbf{I})_\varepsilon)$ is equal to $((-S^2(i)) \times \mathbf{I})_{1+\varepsilon}$.

Hence, the hermitian product is equal to

$$<(S^2(i) \times \mathbf{I})_\varepsilon \cup_{S^2 \times \{0,1\}} ((-S^2(i)) \times \mathbf{I})_{1+\varepsilon'}>^{\text{spin}} = <S^2(i) \times S^1_{\varepsilon+\varepsilon'}>^{\text{spin}}$$

$$= \eta_s \langle H(\omega_{\varepsilon+\varepsilon'}, e_i)\rangle$$

$$= \begin{cases} 1 & \text{if } i = 0 \\ (-1)^{\varepsilon+\varepsilon'} & \text{if } i = 4k-2 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the module $V^s(-S^2(i) \amalg S^2(i))$ is zero except if $i = 0$ or $i = 4k - 2$, and in these cases, it is free of rank 1, generated by $Z^s((S^2(i) \times \mathbf{I})_0) = \pm Z^s((S^2(i) \times \mathbf{I})_1)$. (The sign is plus, if $i = 0$, and minus, if $i = 4k - 2$.) This completes the proof. □

Notation. We denote $S^2(4k-2)$ by $\hat{S}^2$.

Remark 12.2. Since $V^s(\hat{S}^2) = 0$ and $V^s(-\hat{S}^2 \amalg \hat{S}^2) = \Lambda$, the multiplicativity axiom of TQFT does not hold for the functor V^s. Also, if $M = \hat{S}^2 \times \mathbf{I}$, then $Z^s(M)$ generates $V^s(-\hat{S}^2 \amalg \hat{S}^2) = \Lambda$, but Z^s_M lives in $\mathrm{Hom}(V^s(\hat{S}^2), V^s(\hat{S}^2)) = 0$.

As a consequence, one has the following description of the decomposition of $V^s(-S^2 \amalg S^2)$ and $V^s(-\hat{S}^2 \amalg \hat{S}^2)$ given in Theorem 9.2.

COROLLARY 12.3. $V^s(-S^2 \amalg S^2; a) = \begin{cases} \Lambda & \textit{if } a = 0 \\ 0 & \textit{otherwise} \end{cases}$

$$V^s(-\hat{S}^2 \amalg \hat{S}^2; a) = \begin{cases} 0 & \textit{if } a = 0 \\ \Lambda & \textit{otherwise.} \end{cases}$$

This follows from the following lemma.

LEMMA 12.4. *Let Σ be an object of $\mathscr{C}^s$ having a component equal to $\pm S^2$ (resp. $\pm \hat{S}^2$). Let $c \in H^0(\Sigma; \mathbf{Z}/2)$ be the class dual to that component. Then τ_c is the identity (resp. minus the identity) of $V^s(\Sigma)$.*

Proof. Since $V^s(-\Sigma)$ is $\overline{V^s(\Sigma)}$ (the conjugate module), we may suppose the component is S^2 (resp. $\hat{S}^2$). The action τ_c is given by glueing $(S^2 \times \mathbf{I})_1$ (resp. $(\hat{S}^2 \times \mathbf{I})_1$). As shown in Proposition 12.1, one has $Z^s((S^2 \times \mathbf{I})_1) = Z^s((S^2 \times \mathbf{I})_0)$, and $Z^s((\hat{S}^2 \times \mathbf{I})_1) = -Z^s((\hat{S}^2 \times \mathbf{I})_0)$, whence the lemma. □

COROLLARY 12.5. $V^s(\hat{S}^2 \amalg \hat{S}^2 \amalg \hat{S}^2) = 0$.

Proof. Let c be the class dual to one component. By the preceding lemma, τ_c is minus the identity. Since the augmentation class 1 acts trivially, τ_c is equal to τ_{1-c} which acts as the identity. The result follows. □

Remark 12.6. Using an orientation-reversing diffeomorphism φ of the 2-sphere, one can construct a cobordism from $-\hat{S}^2$ to $\hat{S}^2$. Lemma 9.4 shows that for all objects Σ, this cobordism induces an isometric isomorphism

$$V^s(\Sigma \amalg -\hat{S}^2) \xrightarrow{\approx} V^s(\Sigma \amalg \hat{S}^2).$$

However, there does not seem to be a *canonical* choice for such a cobordism. (Observe that one needs to choose a homotopy between the canonical structure on $-\hat{S}^2$, and the structure on $\hat{S}^2$ transported by $d\varphi$.) Therefore, we prefer not to identify $\hat{S}^2$ and $-\hat{S}^2$.

13. Extended multiplicativity. Using $\hat{S}^2$ we can formulate the extended multiplicativity property.

THEOREM 13.1 (Extended multiplicativity). *For all objects Σ, Σ' in the category $\mathscr{C}^s$ the map*

$$V^s(\Sigma) \otimes V^s(\Sigma') \oplus V^s(\Sigma \amalg \hat{S}^2) \otimes V^s(-\hat{S}^2 \amalg \Sigma') \to V^s(\Sigma \amalg \Sigma')$$

(defined by glueing) is an isomorphism. Moreover, this isomorphism is an isometry, if the sum is considered as an orthogonal sum.

Proof of 13.1. We first show that we have an epimorphism. We choose a cobordism N (respectively N'), from $-\Sigma$ to S^2 (respectively from S^2 to Σ'). By Lemma 10.1, elements of $V^s(\Sigma \amalg \Sigma')$ are represented in $N'' = N \cup_{S^2} N'$. Let M'' be a bordism, with N'' as underlying manifold. The colored banded graph in M'' (supposed to meet S^2 tranversally) induces a colored banded link in S^2. We claim that bordisms M'' which induce a 1-component link on S^2 generate $V^s(\Sigma \amalg \Sigma')$. To see this, we can use the graphical identity (fusion law) given in example (d) of Section 11 to modify the graph near the sphere. Proposition 12.1 shows that bordisms inducing a 1-component link colored by i, $i \neq 0$ and $i \neq 4k-2$, give zero. Surjectivity follows.

To show injectivity, it suffices to show that the hermitian forms are preserved. We are going to compute $<M_1'', M_2''>^{\text{spin}}_{\Sigma \amalg \Sigma'}$ in three cases.

Case 1. If each M_i'' is a disjoint union of bordisms M_i, M_i', such that $\partial M_i = \Sigma$, $\partial M_i' = \Sigma'$, then clearly

$$<M_1'', M_2''>^{\text{spin}}_{\Sigma \amalg \Sigma'} = <M_1, M_2>^{\text{spin}}_{\Sigma} <M_1', M_2'>^{\text{spin}}_{\Sigma'}.$$

Case 2. If M_1'' is a disjoint union of bordisms M_1, M_1', such that $\partial M_1 = \Sigma$, $\partial M_1' = \Sigma'$, and M_2'' is a union over $\hat{S}^2$ of bordisms M_2, M_2', such that $\partial M_2 = \Sigma \amalg \hat{S}^2$, $\partial M_2' = -\hat{S}^2 \amalg \Sigma'$, then we must show $<M_1'', M_2''>^{\text{spin}}_{\Sigma \amalg \Sigma'} = 0$. But this is clear since $V^s(\hat{S}^2) = 0$.

Case 3. If each M_i'' is a union over $\hat{S}^2$ of bordisms M_i, M_i', such that $\partial M_i = \Sigma \amalg \hat{S}^2$, $\partial M_i' = -\hat{S}^2 \amalg \Sigma'$, then we must show that

$$<M_1'', M_2''>^{\text{spin}}_{\Sigma \amalg \Sigma'} = <M_1, M_2>^{\text{spin}}_{\Sigma \amalg \hat{S}^2} <M_1', M_2'>^{\text{spin}}_{-\hat{S}^2 \amalg \Sigma'}.$$

Observe that $<M_1'', M_2''>^{\text{spin}}_{\Sigma \amalg \Sigma'}$ is *minus* the invariant of the closed bordism shown in the figure below. The minus sign comes from 8.3 and the fact that $Z^s(((-\hat{S}^2) \times \mathbf{I})_1) = -Z^s(((-\hat{S}^2) \times \mathbf{I})_0)$.

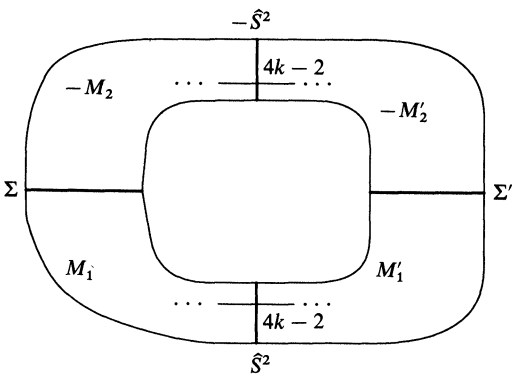

FIGURE 5

Hence, we must show that the invariant of this closed bordism is *minus* the invariant of the following closed bordism:

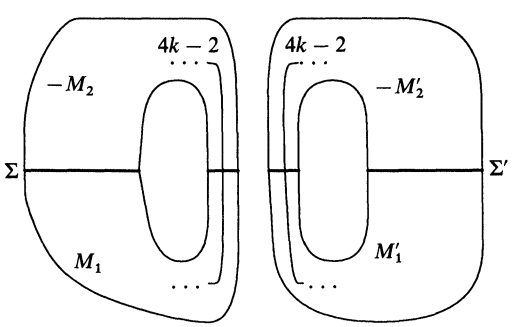

FIGURE 6

To see why this is true, observe that the difference between the two closed bordisms under consideration is that we have replaced a cobordism from $-\hat{S}^2 \amalg \hat{S}^2$ to itself by another one. The first one is the "horizontal" identity cobordism $C_1 = ((-\hat{S}^2 \amalg \hat{S}^2) \times \mathbf{I})_0$. The second, "vertical," one is $C_2 = (-\hat{S}^2 \times \mathbf{I})_0 \amalg (\hat{S}^2 \times \mathbf{I})_0$

viewed as a cobordism from $-\hat{S}^2 \amalg \hat{S}^2$ to $\varnothing$ composed with a cobordism from $\varnothing$ to $-\hat{S}^2 \amalg \hat{S}^2$.

Thus, the result is a consequence of the following.

LEMMA 13.2. *One has* $Z^s_{C_2} = -Z^s_{C_1}$.

Proof. The space $V^s(-\hat{S}^2 \amalg \hat{S}^2)$ is generated by $Z^s((\hat{S}^2 \times \mathbf{I})_0)$. By pasting C_2 onto the bordism $(\hat{S}^2 \times \mathbf{I})_0$ we get the disjoint union

$$(\hat{S}^2 \times S^1)_1 \amalg (\hat{S}^2 \times \mathbf{I})_0 .$$

Since $<(\hat{S}^2 \times S^1)_1>^{\text{spin}} = -1$ (see the computation in the proof of Proposition 12.1), we have $Z^s_{C_2} = -\mathrm{Id}_{V^s(-\hat{S}^2 \amalg \hat{S}^2)} = -Z^s_{C_1}$, as asserted. □

14. Description of $V^s(\Sigma; a)$

Notation. For $v, v' \in \{\varnothing, \hat{S}^2\}$, we put

$${}_vV^s(\Sigma)_{v'} = V^s(-v \amalg \Sigma \amalg v').$$

Let $\Sigma = \coprod_{i=1}^{l} \Sigma_i$ be a disjoint union of connected objects Σ_i. To each sequence $v = (v_1, \ldots, v_{l-1}) \in \{\varnothing, \hat{S}^2\}^{l-1}$, we associate $a(v) = \sum_{i=1}^{l-1} v(v_i)([\Sigma_{i+1}] - [\Sigma_i]) \in \tilde{H}_0(\Sigma; \mathbf{Z}/2)$, where $v(\varnothing) = 0$ and $v(\hat{S}^2) = 1$.

THEOREM 14.1. *The glueing map*

$${}_{\varnothing}V^s(\Sigma_1)_{v_1} \otimes {}_{v_1}V^s(\Sigma_2)_{v_2} \otimes \cdots \otimes {}_{v_{l-1}}V^s(\Sigma_l)_{\varnothing} \to V^s(\Sigma)$$

is an isometric injection, with image equal to $V^s(\Sigma; a(v))$.

Proof. It follows by induction from Theorem 13.1 that the map

$$\bigoplus_v {}_{\varnothing}V^s(\Sigma_1)_{v_1} \otimes {}_{v_1}V^s(\Sigma_2)_{v_2} \otimes \cdots \otimes {}_{v_{l-1}}V^s(\Sigma_l)_{\varnothing} \to V^s(\Sigma)$$

is an isometric isomorphism. In Lemma 14.2 below, we are going to compute the action of $H^0(\Sigma; \mathbf{Z}/2)$, and we shall show that the image denoted W_v of the summand corresponding to a given v is included in $V^s(\Sigma; a(v))$. As $v \mapsto a(v)$ is one-to-one, the theorem will be proved. □

LEMMA 14.2. *We have*

$$\forall x \in W_v \ \forall c \in H^0(\Sigma; \mathbf{Z}/2) \quad \tau_c(x) = (-1)^{c(a(v))} x .$$

Proof. Let $c_i \in H^0(\Sigma; \mathbf{Z}/2)$ be the class dual to the component Σ_i. The action of c_i is given by glueing $(\Sigma_i \times \mathbf{I})_1$. We have to compute the element represented by

the manifold below:

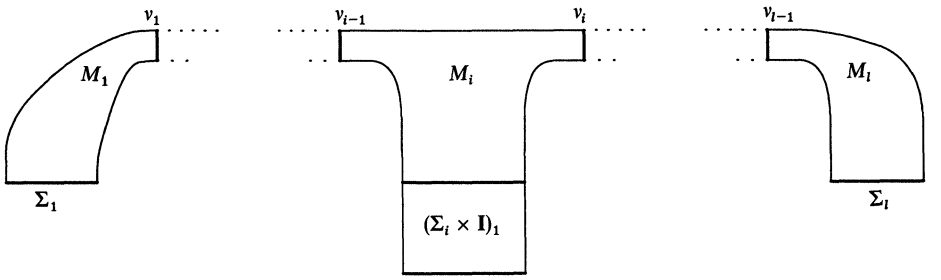

FIGURE 7

The action of c_i can be computed locally, i.e., it suffices to compute $\tau_{c_i}(x_i)$ for $x_i \in {}_{v_{i-1}}V^s(\Sigma_i)_{v_i}$. Since the augmentation class $1 \in H^0(-v_{i-1} \amalg \Sigma_i \amalg v_i; \mathbf{Z}/2)$ acts trivially, the action is also given by glueing $-(v_{i-1} \times \mathbf{I})_1 \amalg (v_i \times \mathbf{I})_1$. The case $v_{i-1} = v_i = \varnothing$ is trivial. In the case $\{v_{i-1}, v_i\} = \{\varnothing, \hat{S}^2\}$, we get $\tau_{c_i}(x_i) = -x_i$ by Lemma 12.4, and in the case $v_{i-1} = v_i = \hat{S}^2$, the formula is $\tau_{c_i}(x_i) = (-1)^2 x_i = x_i$. This completes the proof. □

Remark 14.3. In the language of bimodules over algebroids (see [8]), Theorem 13.1 may be expressed as follows. Define an algebroid Δ with two objects $\varnothing$ and $\hat{S}^2$ by setting

$$ {}_v\Delta_{v'} = {}_vV^s(\varnothing)_{v'} = V^s(-v \amalg v'), \quad v, v' \in \{\varnothing, \hat{S}^2\}. $$

The obvious glueing makes ${}_-V^s(\Sigma)_-$ into a Δ-bimodule. The algebroid Δ is completely reduced, i.e., one has ${}_v\Delta_v = \Lambda$, and ${}_v\Delta_{v'} = 0$ if $v \neq v'$. Theorem 13.1 says that glueing induces an isomorphism

$$ {}_-V^s(\Sigma)_- \underset{\Delta}{\otimes} {}_-V^s(\Sigma')_- \xrightarrow{\approx} {}_-V^s(\Sigma \amalg \Sigma')_- . $$

In the language of [8], this means that a certain algebroid $\Delta^s(\varnothing)$, whose objects are all objects of $\mathscr{C}^s$, and which is naturally associated to the functor V^s, is Morita equivalent to the completely reduced algebroid Δ.

15. The transfer theorem. In this section, we describe a natural transfer map from the TQFT-functor (V_{8k}, Z_{8k}) constructed in [8] (here called the *unspun theory*) to the functor (V^s_{8k}, Z^s_{8k}) (called the *spin theory*).

The functor V_{8k} of [8] has the same coefficient ring as the functor V^s_{8k}, and is defined on a cobordism category $\mathscr{C}_{8k}$ which is obtained from $\mathscr{C}^s_{8k}$ by omitting the w_2-structures.

As before, we fix $k \geqslant 1$ and write V for V_{8k}, V^s for V^s_{8k}, etc. In this and the next section, it will be convenient to use the notation Σ for an object of the cobordism

category $\mathscr{C}$ (rather than $\mathscr{C}^s$), and to denote an object of $\mathscr{C}^s$ by (Σ, σ) (where σ is a w_2-structure complementary to the link in Σ). The spin structure induced by σ will be denoted by $s = [\sigma]$.

We recall the following result of [8].

THEOREM 15.1 [8]. *Let Σ be an object of the cobordism category $\mathscr{C}$, and let l be the (colored) link in Σ. There is a natural orthogonal decomposition*

$$V(\Sigma) = \bigoplus_{\mathrm{Spin}(\Sigma|l)} V(\Sigma, s)$$

where $\mathrm{Spin}(\Sigma|l)$ *is the set of spin structures complementary to l in Σ.*

Remark 15.2. In [8], the submodules were denoted by $V_{8k}(\Sigma, q)$, with q running through the quadratic forms on $H_1(\Sigma - l; \mathbf{Z}/2)$ inducing the intersection form. Recall that those correspond bijectively to spin structures on $\Sigma - l$ (see 2.3). It was noted in Remark 7.11 of [8] that the submodule is zero if the corresponding spin structure is not complementary to l.

Suppose now that σ is a (singular) w_2-structure over an object Σ in $\mathscr{C}$, and $s = [\sigma]$ is the corresponding spin structure. Now we are given two modules, namely $V(\Sigma, s)$ and $V^s(\Sigma, \sigma)$. The (b) statement of the next theorem identifies $V(\Sigma, s)$ with $V^s((\Sigma, \sigma); 0)$ (the fixed part under the action of $H^0(\Sigma; \mathbf{Z}/2)$).

THEOREM 15.3 (Transfer theorem). *Let Σ be an object of the cobordism category $\mathscr{C}$, and let l be the (colored banded) link in Σ. Let σ be a w_2-structure on Σ complementary to l, and let $s = [\sigma]$ denote the induced spin structure.*

(a) *There exists a well-defined linear map*

$$\Phi_\sigma\colon V(\Sigma) \to V^s(\Sigma, \sigma)$$

with the following property. For every bordism M in $\mathscr{C}$ such that $\partial M = \Sigma$,

$$\Phi_\sigma(Z(M)) = 2^{-b_0(M,\Sigma)} \sum_{\tilde{\sigma} \in \mathrm{Spin}(M|K,\sigma)} Z^s(M, \tilde{\sigma}),$$

where K denotes the colored banded graph in M with $\partial K = l$, and $\mathrm{Spin}(M|K, \sigma)$ *is the set of relative spin structures complementary to K (see Section 2).*

(b) *This map factors through the orthogonal projection $\pi_s\colon V(\Sigma) \to V(\Sigma, s)$, and induces an isomorphism*

$$\Phi_\sigma\colon V(\Sigma, s) \xrightarrow{\approx} V^s((\Sigma, \sigma); 0).$$

(c) *Moreover, the hermitian forms satisfy*

$$\forall x, y \in V(\Sigma, s) \quad \langle x, y \rangle_\Sigma = 2^{-b_0(\Sigma)} \langle \Phi_\sigma(x), \Phi_\sigma(y) \rangle^{\mathrm{spin}}_{(\Sigma,\sigma)}.$$

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