Cobordism and Concordance of Knots

Vincent Blanlœil

to V., V. & A.

孔子说:
“温习过去所学的知识,能有新体会,新发展,这样就可以当老师了.”

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Chapter 1

Introduction

"... the theory of "Cobordisme" which has,
within the few years of its existence,
led to the most penetrating insights into
the topology of differentiable manifolds."
H. Hopf,
International Congress of Mathematics, 1958.

1.1 History

In the early fifties Rohlin [127] and Thom [149] studied the *cobordism groups* of manifolds. At the 1958 International Congress of Mathematicians in Edinburgh, René Thom received a Fields Medal for his development of cobordism theory.

Then, Fox and Milnor [43, 44] were the first to study *cobordism of knots*, i.e., cobordism of embeddings of the circle S^1 into the 3-sphere S^3 ; knot cobordism is slightly different from the general cobordism, since its definition is more restrictive. After Fox and Milnor, Kervaire [72] and Levine [89] studied embeddings of the n-sphere S^n (or homotopy n-spheres) into the $(n+2)$-sphere S^{n+2}, and gave classifications of such embeddings up to cobordism for $n \geq 2$. Moreover, Kervaire defined group structures on the set of cobordism classes of n-spheres embedded in S^{n+2}, and on the set of concordance classes of embeddings of S^n into S^{n+2}. The structures of these groups for $n \geq 2$ were clarified by Kervaire [72], Levine [89, 90] and Stoltzfus [147].

Note that embeddings of spheres were studied only in the codimension two case, since in the PL category Zeeman [169] proved that all such embeddings in codimension greater than or equal to three are unknotted, and Stallings [144] proved that it is also true in the topological category (here, one needs to assume the locally flatness condition), provided that the ambient sphere has dimension greater than or equal to five. In the smooth category Haefliger [52] proved that a cobordism of spherical knots in codimension greater than or equal to three implies isotopy.

Later, people studied embeddings of manifolds, which are not necessary homeomorphic to spheres, into codimension two spheres. One motivation comes from the topology of complex hypersurfaces near isolated singular points. More precisely, Milnor [109] showed that, in a neighborhood of an isolated singular point, a complex hypersurface is homeomorphic to the cone over the *algebraic knot* associated with the singularity. Hence, the embedded topology of a complex hypersurface around an isolated singular point is given by the algebraic knot, which is a special case of a *fibered knot.* After Milnor's work, the class of fibered knots has been recognized as an important class of knots to study. Usually algebraic knots are not homeomorphic to spheres, and this motivated the study of embeddings of general manifolds (not necessarily homeomorphic to spheres) into spheres in codimension two. Moreover, in the beginning of the seventies, Lê [85] proved that isotopy and cobordism are equivalent for 1-dimensional algebraic knots. Lê proved this for the case of *connected* (or spherical) algebraic 1-knots, and the generalization to arbitrary algebraic 1-knots follows easily (for details, see §7.1).

During Arcata's symposium of pure mathematics in 1974, Durfee [38] listed several unsolved problems about algebraic knots ; and after Lê's result concerning one dimensional algebraic knots, the following question seems natural

Problem 5([38]): Are cobordant algebraic knots (with K homeomorphic to a sphere) isotopic?

But we had to wait about twenty years for an answer when Du Bois and Michel [35] gave the first examples of algebraic spherical knots that are cobordant but are not isotopic. These examples motivated the classification of fibered knots up to cobordism.

1.1.1 Contents

This book is organized as follows. In Chapter 1 we give several apropos definitions to the cobordism theory of knots. The *Seifert form* associated with a knot is also introduced.

In Chapter 2 we introduce Morse function and handle decomposition of manifolds. Then we prove the h-cobordism Theorem and explain surgeries on manifolds.

In Chapter 3 we review the classifications of (simple) spherical $(2n-1)$-knots with $n \geq 2$ up to isotopy and up to cobordism.

In Chapter 4 we review nice properties of fibered knots.

In Chapter 5 we define algebraic cobordism and we clarify this definition with several explicit examples. Then we prove that this relation is an equivalence relation on the set of unimodular bilinear forms defined on free **Z**-modules of finite rank.

In Chapter 6 we present the classifications of simple fibered $(2n-1)$-knots with $n \geq 3$ up to isotopy and up to cobordism, and we introduce the *algebraic cobordism* of integral bilinear forms.

In Chapter 7 we review the properties of algebraic 1-knots and present the classification theorem of algebraic 1-knots up to cobordism due to Lê [85].

In Chapter 8 we study cobordism of 3-dimensional knots, and we introduce the notion of Spin cobordism.

In Chapter 9 we define the pull back relation for knots which naturally arises from the viewpoint of the codimension two surgery theory.

In Chapter 10 we present several relevant examples concerning the notions introduced in the previous chapters.

In Chapter 11 we study embedded surfaces in S^4

In Chapter 12 we prove that embeddings of simply connected and closed 4-manifolds in S^6 are all concordant.

In Chapter 13 we present the most general topological background in which we can study cobordism of knots, and we extend the result about 3-knots to a larger class.

in Chapter 14 we list several open problems related to the cobordism theory of non-spherical knots.

With all the results collected in this book, we have classifications of knots up to cobordism in any dimensions. Only the classical case of one dimensional knots, and the case of three dimensional knots remain to have complete classifications.

Some chapters of this book are made of a series of lectures for graduate students in Louis Pasteur university of Strasbourg during the academic year 2006-2007. The purpose of these lectures was to give the opportunity to students to learn topology of high dimensional manifolds while studying knot cobordism.

Many proofs and results in this book are coming from papers written before on the subject, and published in different journals. I want to thank here all my co-authors.

1.1.2 Notations

We will work in the smooth category, but sometimes manifolds might have corners. When a manifold M has boundary we denote it by ∂M. Moreover, if M is an oriented manifold with boundary we use the outward first convention to orient its boundary ∂M. All the homology and cohomology theory used have integer coefficients. The symbol $\cong$ denotes a diffeomorphism between manifolds or an isomorphism between algebraic objects. An embedding of a manifold K in a manifold M is denoted by $K \hookrightarrow M$. The closure of X is denoted by $\overline{X}$, and its interior is denoted by $\overset{\circ}{X}$ or by $\mathrm{Int}X$. We denote by ${}^tA$ the transpose of a matrix A.

1.2 Definitions

In this section we introduce knot cobordism. We also present some detailed constructions in order to give to the reader a precise idea of the subject.

Since our aim is to study cobordism and concordance of codimension two embeddings of manifolds which are not necessarily homeomorphic to spheres, we define *knots* as follows.

Definition 1.1. Let K be a closed n-dimensional manifold embedded in the $(n+2)$-dimensional sphere S^{n+2}. We suppose that K is

$(k-2)$-connected if $n = 2k-1$ and $k \geq 2$, or

$(k-1)$-connected if $n = 2k$ and $k \geq 1$.

When K is orientable, we further assume that it is oriented. Then we call K or its (oriented) isotopy class an *n-knot*, or simply a *knot*.

An n-knot K is *spherical* if K is a homotopy n-sphere.

Remark 1.2. With our definition, one dimensional knots may have several connected components. But spherical 1-knots are connected and diffeomorphic to S^1, see Figure 1.1 and Figure 1.2.

We impose a connectivity condition in Definition 1.1, this is first motivated by the usual definition of *algebraic knot* (see Definition 1.13), and second because we will need connectivity conditions to perform embedded surgeries.

In order to define, and compute, invariants of isotopy and cobordism classes of knots, we will need some algebraic data associated with knots like *Seifert forms* and *Alexander polynomials*. In the classical knot theory, i.e., the case of spherical 1-knots, it is usual to make combinatorial computations associated with crossing of planar representations. We will have another approach, in

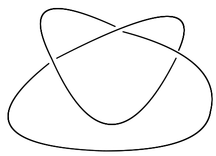


Figure 1.1. The trefoil knot is a spherical 1-knot

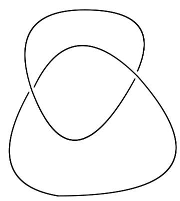


Figure 1.2. The Hopf link is not a spherical 1-knot

a sense may be more algebraic, since we will do computations using integral bilinear forms.

The first step is to define *Seifert manifolds* associated with knots.

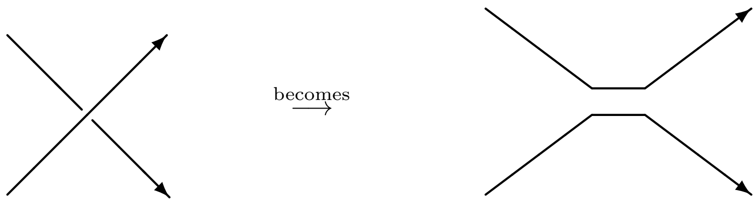
1.2.1 Seifert manifolds associated with knots

Proposition 1.3. *For every oriented n-knot K with $n \geq 1$, there exists a compact oriented $(n+1)$-dimensional submanifold V of S^{n+2} having K as boundary. Such a manifold V is called a* Seifert manifold *associated with K. When K is a one dimensional knot, the manifold V is usually called a* Seifert surface.

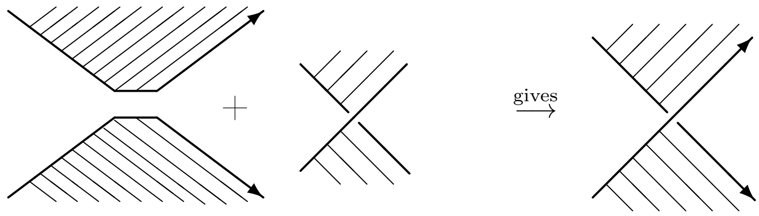
Remark 1.4. Seifert manifolds are not unique. For a given Seifert manifold of dimension k, one can construct a new one by doing its connected sum with a compact closed k-manifold embedded in S^{k+1}.

Proof. The construction of Seifert surfaces associated with 1-knots is elementary, see [129], for example.

Start by assigning an orientation to each component of the knot, and then choose a regular projection into the plane. Around each crossing do the following modification:



Then the regular projection of K has become a disjoint collection of oriented S^1 embedded in the plane. Each one bounds a disk, and by pushing the interior of these disks off the plane in the three sphere they can be made disjoint. The orientations of the S^1 induce orientations of disks. Hence we can connect these oriented disks at each crossing with half twisted strips in order to form an embedded, 2-manifold in S^3, whose boundary is K as depicted bellow:



This construction gives the desired surface, embedded in S^3, which has the knot as boundary. When K is not spherical it is moreover necessary to do the oriented connected sum of the connected components of the surfaces we just constructed.

For general dimensions, the existence of a Seifert manifold associated with a n-knot K can be proved by using the obstruction theory as follows.

Let $p : \tau_K \to K$ be the normal bundle of $K \hookrightarrow S^{n+2}$, and let $p_0 : \tau_K^0 \to K$ be the bundle p without the zero section, i.e., for all $x \in K$ the fibers satisfy $p_0^{-1}(x) = p^{-1}(x) \setminus \{0\}$. A global orientation for τ_K means that we choosed a prefered generator μ of $\mathrm{H}^2(\tau_K, \tau_K^0)$.

The zero section of the bundle τ_K is an embedding of K in τ_K, moreover K is a deformation retract of τ_K and $p^* : \mathrm{H}^2(K) \overset{\cong}{\to} \mathrm{H}^2(\tau_K)$ is an isomorphism.

Let us denote i the inclusion map of $(\tau_K, \varnothing)$ into $(\tau_K, \tau_K \setminus K)$, which induces the morphism $i^* : \mathrm{H}^2(\tau_K, \tau_K \setminus K) \to \mathrm{H}^2(\tau_K)$ in cohomology.

Recall that the Euler class $e(\tau_K) = {p^*}^{-1} \circ i^*(\mu)$ of the normal bundle is an obstruction to having a nonzero normal section.[1]

Let $T_K \overset{\tau}{\cong} K \times D^2$ be an open tubular neighborhood of K in S^{n+2}. The 2-disk bundle T_K is diffeomorphic to τ_K and we have the following commutative diagram

[1]Since K is a n-knot then we have $e(\tau_K) \in \mathrm{H}^2(K) = 0$ as soon as K is 2-connected. Then we already have that τ_K is trivial for $n \geq 5$.

$$\begin{array}{ccccc}
\mathrm{H}^2(S^{n+2}, S^{n+2} \setminus K) & \xrightarrow[\cong]{\epsilon^*} & \mathrm{H}^2(T_K, T_K \setminus K) & \xrightarrow[\cong]{\varphi^*} & \mathrm{H}^2(\tau_K, \tau_K^0) \\
{\scriptstyle j^*}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle i^*} \\
0 = \mathrm{H}^2(S^{n+2}) & \xrightarrow{\nu^*} & \mathrm{H}^2(K) & \xrightarrow[\cong]{p^*} & \mathrm{H}^2(\tau_K)
\end{array}$$

Where $\mathrm{H}^2(S^{n+2}, S^{n+2} \setminus K) \overset{\epsilon^*}{\cong} \mathrm{H}^2(T_K, T_K \setminus K)$ is given by the excision, and the morphisms j^* and ν^* are induced by inclusions.

Since $e(\tau_K) = {p^*}^{-1} \circ i^*(\mu)$, the commutativity of the diagram gives

$$e(\tau_K) = {p^*}^{-1} \circ i^*(\mu) = \nu^* \circ j^* \circ {\epsilon^*}^{-1} \circ {\varphi^*}^{-1}(\mu) = 0.$$

So the normal bundle of $K \hookrightarrow S^{n+2}$ is trivial.

Let $N_K \overset{\overline{\tau}}{\cong} K \times D^2$, the closure of T_K in S^{n+2}, be a closed tubular neighborhood of K in S^{n+2}, and

$$\Phi : \partial N_K \overset{\cong}{\to} K \times S^1 \overset{pr_2}{\to} S^1$$

the composite of the restriction of $\overline{\tau}$ to the boundary of N_K and the projection pr_2 to the second factor. Using the exact sequence

$$H^1(S^{n+2} \setminus T_K) \to H^1(\partial N_K) \to H^2(S^{n+2} \setminus T_K, \partial N_K),$$

associated with the pair $(S^{n+2} \setminus T_K, \partial N_K)$, we see that the obstruction to extending Φ to $\widetilde{\Phi} : S^{n+2} \setminus T_K \to S^1$ lies in the cohomology group

$$H^2(S^{n+2} \setminus T_K, \partial N_K) \cong H_n(S^{n+2} \setminus T_K).$$

By Alexander duality we have

$$H_n(S^{n+2} \setminus T_K) \cong H^1(K),$$

which vanishes if $n \geq 4$, since K is simply connected for $n \geq 4$. When $n \leq 3$, we can show that by choosing the trivialization τ appropriately, the obstruction in question vanishes. Therefore, a desired extension $\widetilde{\Phi}$ always exists. Now, for a regular value y of $\widetilde{\Phi}$, the manifold $\widetilde{\Phi}^{-1}(y)$ is a submanifold of S^{n+2} with boundary being identified with $K \times \{y\}$ in $K \times S^1$. The desired Seifert manifold associated with K is obtained by gluing a small collar $K \times [0,1]$ to $\widetilde{\Phi}^{-1}(y)$. □

Let us now recall the classical definition of Seifert forms of odd dimensional oriented knots, which were first introduced in [140] and play an important role in the study of knots cobordism.

Definition 1.5. Suppose that V is a compact oriented $2n$-dimensional submanifold of S^{2n+1}, and let G be the quotient of $H_n(V)$ by its $\mathbf{Z}$-torsion. The *Seifert form* associated with V is the bilinear form $\mathfrak{A} : G \times G \to \mathbf{Z}$ defined as follows

$$\begin{array}{rcl}
\mathfrak{A} : G \times G & \longrightarrow & \mathbf{Z} \\
(x, y) & \mapsto & \mathfrak{A}(x, y) = l_{S^{2n+1}}(\xi_+, \eta).
\end{array}$$

where $l_{S^{2n+1}}(.,.)$ denotes the linking number of chains in S^{2n+1}, the two n-chains ξ and η are representing the cycles x and y respectively, and ξ_+ is the n-chain η pushed off V into the positive normal direction to V in S^{2n+1}.

Recall that the linking number of two n-chains ξ and η in S^{2n+1} is given by the algebraic intersection number in S^{2n+1} of a $(n+1)$-chain Θ, which bounds ξ in S^{2n+1}, and η (resp. by the algebraic intersection number in S^{2n+1} of ξ and a $(n+1)$-chain Ω, which bounds η in S^{2n+1}) ; or by the algebraic intersection number in D^{2n+2} of a $(n+1)$-chain Θ', which bounds ξ in D^{2n+2}, and a $(n+1)$-chain Ω', which bounds η in D^{2n+2}.

By definition a *Seifert form* associated with an oriented $(2n-1)$-knot K is the Seifert form associated with V, where V is a Seifert manifold associated with K. A matrix representative of a Seifert form with respect to a basis of G is called a *Seifert matrix*.

Remark 1.6. One can as well define the Seifert form $\mathfrak{A}'(x,y)$ to be the linking number of ξ and η_+ instead of ξ_+ and η, where ξ_+ is the n-cycle ξ pushed off V into the positive normal direction to V in S^{2n+1}. There is no essential difference between the two forms $\mathfrak{A}$ and $\mathfrak{A}'$. However some formulas may take different forms.

More precisely, for a given n-chain ξ in F we denote by ξ_- the n-chain ξ pushed off V into the negative normal direction to V in S^{2n+1}. Then we have

$$l_{S^{2n+1}}(\xi,\eta_+) = l_{S^{2n+1}}(\xi_-,\eta),$$

and recall

$$l_{S^{2n+1}}(\xi,\eta) = (-1)^{n+1} l_{S^{2n+1}}(\eta,\xi).$$

According to these formulas we get

$$\begin{array}{rcl} \mathfrak{A}(x,y) & = & l_{S^{2n+1}}(\xi_+,\eta) \\ \mathfrak{A}(x,y) & = & (-1)^{n+1} l_{S^{2n+1}}(\eta,\xi_+) \\ \mathfrak{A}(x,y) & = & (-1)^{n+1}\mathfrak{A}'(y,x) \end{array}$$

So if A is the Seifert matrix associated with $\mathfrak{A}$ and A' is the Seifert matrix associated with $\mathfrak{A}'$ we have $A' = (-1)^{n+1}\, {}^tA$

Let us illustrate the above definition in the case of the trefoil knot. First consider the Seifert manifold F associated with the trefoil knot as depicted in Fig. 1.3, where "+" indicates the positive normal direction. Note that $\operatorname{rank}\big(H_1(V)\big) = 2$. We denote by ξ and η the 1-cycles which represent the generators of $H_1(F)$. Then, with the aid of Fig. 1.3, we see that the Seifert matrix for the trefoil knot is given by

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Definition 1.7. Let $n \geq 1$. We say that an $(2n-1)$-knot is *simple* if it admits an $(n-1)$-connected Seifert manifold.

Let K be a simple knot with an $(n-1)$-connected Seifert manifold F. The Universal coefficient Theorem [16] states that the following short exact sequence is exact

$$0 \to \operatorname{Ext}\big(\mathrm{H}_{k-1}(F,K)\big) \to \mathrm{H}^k(F,K) \to \operatorname{Hom}\big(\mathrm{H}_k(F,K)\big) \to 0.$$

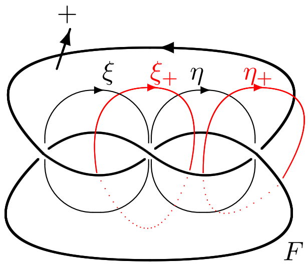


Figure 1.3. Computing a Seifert matrix for the trefoil knot

Since F is an $(n-1)$-connected Seifert manifold, then $\mathrm{Ext}\big(\mathrm{H}_{n-1}(F,K)\big) = 0$ and the group $\mathrm{H}^n(F,K)$ is torsion free. But by Poincaré-Lefschetz duality we have $\mathrm{H}^n(F,K) \cong \mathrm{H}_n(F)$. Hence $\mathrm{H}_n(F)$ is torsion free.

In the following, when a $(2n-1)$-knot is simple, we consider an $(n-1)$-connected Seifert manifold associated with this knot unless otherwise specified.

When $n \geq 2$, the long exact sequence associated with a simple $(2n-1)$-knot K and its $(n-1)$-connected Seifert manifold F, induces the following short exact sequence

$$0 \to H_n(K) \to H_n(F) \overset{S_*}{\to} H_n(F,K) \to H_{n-1}(K) \to 0 \tag{1.1}$$

where the homomorphism S_* is induced by the inclusion. Let

$$\widetilde{\mathfrak{P}} : H_n(F,K) \overset{\cong}{\to} \mathrm{Hom}_{\mathbf{Z}}(H_n(F), \mathbf{Z})$$

be the composite of the Poincaré-Lefschetz duality isomorphism and the universal coefficient isomorphism.

If we denote by $\mathfrak{S}$ the intersection pairing[2]

$$\mathfrak{S} : \mathrm{H}_n(F) \times \mathrm{H}_n(F) \to \mathbf{Z},$$

then for all $(a,b) \in \mathrm{H}_n(F) \times \mathrm{H}_n(F)$ we have $\mathfrak{S}(a,b) = \big(\widetilde{\mathfrak{P}} \circ S_*(b)\big)(a)$.

Proposition 1.8. *Let K be a simple $(2n-1)$-knot with an $(n-1)$-connected Seifert manifold F. Let $\mathfrak{A}$ be the Seifert form associated with F and $\mathfrak{S}$ the intersection pairing. If we denote by A the Seifert matrix and by S the matrix representative of $\mathfrak{S}$, then $S = A + (-1)^n\, {}^tA$.*

Proof. Let $0 < \varepsilon << 1$. First we identify a regular tubular neighborhood of F in S^{2n+1} with $F \times [-\varepsilon, \varepsilon]$. For each $t \in [-\varepsilon, \varepsilon]$ we define a diffeomorphism

$$i_t : F \to S^{2n+1}$$

which is a translation by a vector of length t in the positive normal direction when t is positive, and in the negative normal direction when t is negative. Remark that for a n-chain γ we have

$$\gamma_+ = i_\varepsilon(\gamma). \tag{1.2}$$

[2]Sometimes called intersection form.

where γ_+ was introduced in Remark 1.6 in order to compute the matrix of Seifert forms.

Let x and y be two n-cycle in $\mathrm{H}_n(F)$, set $x = [\xi]$ and $y = [\eta]$ for two n-chains ξ and η.

As consequence of Equation 1.2 we get

$$l_{S^{2n+1}}\big(\xi, i_\varepsilon(\eta)\big) = l_{S^{2n+1}}\big(i_{-\varepsilon}(\xi), \eta\big).$$

Let $\Lambda = \bigcup_{t\in[-\varepsilon,\varepsilon]} i_t(\xi) \cong \xi \times [-\varepsilon, \varepsilon]$ the oriented $(n+1)$-chain in S^{2n+1} with $\partial\Lambda = \big(i_\varepsilon(\xi) - i_{-\varepsilon}(\xi)\big)$ since we use the outward first convention for the orientation of the boundary of an oriented manifold. The intersection of ξ and η in F is equal to the intersection of Λ and η in S^{2n+1}, this implies the following equalities

$$\begin{array}{rcl}
\mathfrak{S}(x,y) & = & l_{S^{2n+1}}(\partial\Lambda, \eta) \\
\mathfrak{S}(x,y) & = & l_{S^{2n+1}}\big(i_\varepsilon(\xi), \eta\big) - l_{S^{2n+1}}\big(i_{-\varepsilon}(\xi), \eta\big) \\
\mathfrak{S}(x,y) & = & \mathfrak{A}(x,y) - (-1)^{n+1} l_{S^{2n+1}}\big(i_\varepsilon(\eta), \xi\big) \\
\mathfrak{S}(x,y) & = & \mathfrak{A}(x,y) + (-1)^n \mathfrak{A}(y,x)
\end{array}$$

This implies the desired relation between matrices. □

Remark 1.9. Intersection forms $\mathfrak{S}$ are $(-1)^n$-symmetrical, contrary to Seifert forms, which are not generally symmetrical. For example see the matrix of the trefoil knot we computed with the aid of Fig. 1.3.

Let us now focus on cobordism and concordance classes of knots.

Definition 1.10. Two n-knots K_0 and K_1 in S^{n+2} are said to be *cobordant* if there exists a properly embedded $(n+1)$-dimensional manifold X of $S^{n+2} \times [0,1]$ such that

1. X is diffeomorphic to $K_0 \times [0,1]$, and
2. $\partial X = (K_0 \times \{0\}) \cup (K_1 \times \{1\})$.

The manifold X is called a *cobordism* between K_0 and K_1. When the knots are oriented, we say that K_0 and K_1 are *oriented cobordant* (or simply *cobordant*) if there exists an oriented cobordism X between them such that

$$\partial X = (-K_0 \times \{0\}) \cup (K_1 \times \{1\}),$$

where $-K_0$ is obtained from K_0 by reversing the orientation.

Recall that a manifold with boundary Y embedded in a manifold X with boundary is said to be *properly embedded* if $\partial Y = \partial X \cap Y$ and Y is transverse to ∂X.

It is clear that isotopic knots are always cobordant. However, the converse is not true in general (see Fig. 1.5). For explicit examples, see §10.

We also introduce the notion of *concordance* for embedding maps as follows.

Definition 1.11. Let K be a closed n-dimensional manifold. We say that two embeddings $f_i : K \to S^{n+2}$, $i = 0, 1$, are *concordant* if there exists a proper embedding

$$\Phi : K \times [0,1] \to S^{n+2} \times [0,1]$$

such that

$$\Phi|_{K\times\{i\}} = f_i : K \times \{i\} \to S^{n+2} \times \{i\}\ ,\ i = 0, 1.$$

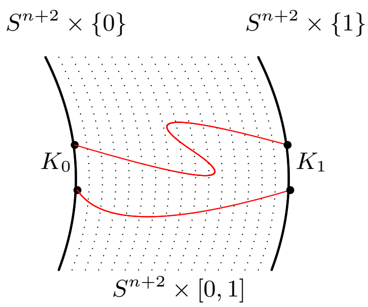


Figure 1.4. A cobordism between K_0 and K_1

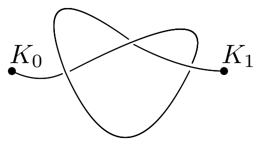


Figure 1.5. A cobordism which is not an isotopy

Where an embedding map $\varphi : Y \to X$ between manifolds with boundary is said to be *proper* if $\partial Y = \varphi^{-1}(\partial X)$ and Y is transverse to ∂X.

Remark 1.12. Concordant knots are cobordant, but the converse is not true in general. See Theorem 3.14 for the spherical case and Remark 11.8 for non spherical examples of 2-knots.

Cobordant knots are diffeomorphic. Hence, to have a cobordism between two given knots, we need to have topological information about the knots. Since a simple fibered $(2n-1)$-knot is the boundary of the closure of a fiber, which is an $(n-1)$-connected Seifert manifold associated with the knot, by considering the above exact sequence (1.1) we can use the kernel and the cokernel of the homomorphism S^* to get topological data of the knot. Note that in the case of spherical knots, these considerations are not necessary since S_* and S^* are isomorphisms.

1.3 Complex hypersurfaces isolated singularities and fibered knots

We are motivated by the study of the topology of isolated singularities of complex hypersurfaces, let us be more precise.

Let

$$f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$$

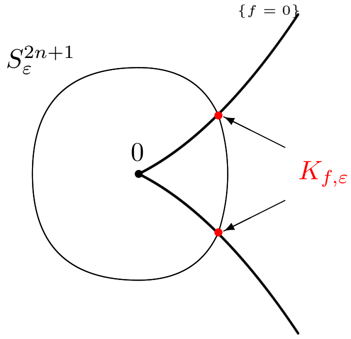


Figure 1.6. The algebraic knot $K_{f,\varepsilon}$ associated to the singularity at 0 of a germ f

be a holomorphic function germ with an isolated singularity at the origin. If $\varepsilon > 0$ is sufficiently small, then in [109] Milnor proved that

$$K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$$

is a $(2n-1)$-dimensional manifold which is naturally oriented and $(n-2)$-connected, where S_ε^{2n+1} is the sphere in $\mathbf{C}^{n+1}$ of radius ε centered at the origin. Furthermore, its (oriented) isotopy class in $S_\varepsilon^{2n+1} = S^{2n+1}$ does not depend on the choice of ε (see [109]).

Definition 1.13. We call K_f the *algebraic knot* associated with the isolated singularity at 0 of f.

Fortunately, algebraic knots are some knots in the sense of Definition 1.1. Moreover, Milnor proved that the pair

$$(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2})$$

is homeomorphic to the cone over the pair

$$(S_\varepsilon^{2n+1}, K_f).$$

Hence the algebraic knot completely determines the local embedded topological type of $f^{-1}(0)$ near the origin, where D_ε^{2n+2} is the disk in $\mathbf{C}^{n+1}$ of radius ε centered at the origin.

In [109], Milnor considered only polynomial functions. However it is known that a holomorphic function germ with an isolated critical point is topologically equivalent to a polynomial function germ.

Moreover, the complement of an algebraic knot K_f in the sphere S^{2n+1} admits a fibration, called Milnor fibration, over the circle S^1, and the closure of each fiber is a compact $2n$-dimensional oriented $(n-1)$-connected submanifold of S^{2n+1} which has K_f as boundary.

Then we define

Definition 1.14. We say that an oriented n-knot K is *fibered* if there exists a smooth fibration

$$\phi : S^{n+2} \setminus K \to S^1$$

and a trivialization

$$\tau : N_K \to K \times D^2$$

of a closed tubular neighborhood N_K of K in S^{n+2} such that $\phi|_{N_K \setminus K}$ coincides with $\pi \circ \tau|_{N_K \setminus K}$, where

$$\pi : K \times (D^2 \setminus \{0\}) \to S^1$$

is the composition of the projection to the second factor and the obvious projection $D^2 \setminus \{0\} \to S^1$. Note that then the closure of each fiber of ϕ in S^{n+2} is a compact $(n+1)$-dimensional oriented manifold whose boundary coincides with K. We shall often call the closure of each fiber simply a *fiber*.

Furthermore, for $n \geq 1$ we say that a fibered $(2n-1)$-knot K is *simple* if each fiber of ϕ is $(n-1)$-connected.

The definition of fibered knots gives a topological framework for algebraic knots associated with isolated singularities.

Though the notion of fibered knot it is much more restrictive, it gives additional nice properties, like *monodromy* and *variation map* see Chapter 4, which are very useful.

When K is a fibered knot, the closure of a fiber is always a Seifert manifold associated with K. In the following, for a fibered $(2n-1)$-knot, we use the Seifert form associated with a fiber unless otherwise specified.

1.4 Alexander polynomial

The Alexander polynomial associated with a knot K was initially defined for spherical 1-knots, and was computed with a combinatorial presentation of 1-knots, i.e., crossings. But, with the aid of a Seifert form associated with a knot, it is possible to define Alexander polynomials for knots of every dimension.

Let K a $(2n-1)$-knot, with $n \geq 1$. Set A be a Seifert form for K associated with a Seifert manifold F. The polynomial

$$\Delta_A(t) = \det\bigl(tA + (-1)^n \, {}^tA\bigr)$$

of $\mathbf{Z}[t, t^{-1}]$, defined up to units of $\mathbf{Z}[t, t^{-1}]$, is called the *Alexander polynomial* of K.

We define the Alexander polynomial up to units of $\mathbf{Z}[t, t^{-1}]$ since the Seifert manifold associated with the knot is not unique. More precisely, the connected sum of a Seifert manifold with a closed oriented manifold of same dimension will change the Alexander polynomial by a product of a unit of $\mathbf{Z}[t, t^{-1}]$.

For the study of fibered knots, if we restrict to Seifert forms associated with a fiber of the fibration, then this polynomial is uniquely defined. Moreover, in that case, the Alexander polynomial will be the characteristic polynomial of the monodromy (see Chapter 4, section 4.1.3).

We will see later that the Alexander Polynomial is a very powerful tool to study the embedded topology of knots. For instance cobordant one dimensional algebraic knots have same Alexander polynomial, see Lê [85].

Chapter 2

h-cobordism Theorems and surgeries on manifolds

Macbeth ...— What is the night?
Lady Macbeth Almost at odds with morning, which is which.
Macbeth Act III, sc IV

The goal of this Chapter is to gives clues to prove the h-cobordism Theorem. In fact we will explain how to prove a slightly more general theorem, which is called s-cobordism Theorem. We choose to give the proof of the s-cobordism theorem because of the similarity of the proofs, though we need to consider Whitehead torsions to prove the s-cobordism Theorem. The first step is to introduce Morse theory and handlebody decomposition for manifolds. In conclusion of this Chapter we will describe modifications of manifolds called *surgeries*.

2.1 Morse functions and handle decompositions of manifolds

In this section we recall briefly some classical results on Morse theory, we refer to [106] and [98] for detailed proofs.

We will consider functions defined on manifolds. Let M^n be a n-dimensional manifold with $n \in \mathbf{N}^*$, recall that we only consider smooth manifolds. A function $f : M \to \mathbf{R}$ is smooth if there exists a local coordinate system $(x_1, \dots, x_n)$ around each point p of M in which f is C^∞. By opposition we define

Definition 2.1. A point $p_0 \in M$ is a *critical* point, or a *singular* point, of the function $f : M \to \mathbf{R}$ if $\dfrac{\partial f}{\partial x_i}(p_0) = 0,\ i = 1, \dots, n.$

It is easy to check that this definition does not depend on the choice of a coordinate system.

Definition 2.2. We say that a critical point p_0 of f is *non-degenerate* if the determinant

$$H_f(p_0) = \det \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(p_0) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(p_0) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(p_0) & \dots & \frac{\partial^2 f}{\partial x_n^2}(p_0) \end{pmatrix}$$

is not zero, and it is *degenerate* if $H_f(p_0) = 0$. We call $H_f(p_0)$ the *Hessian* of f at the critical point p_0.

Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two coordinate systems, and set

$$J(p_0) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1}(p_0) & \ldots & \frac{\partial x_1}{\partial y_n}(p_0) \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1}(p_0) & \ldots & \frac{\partial x_n}{\partial y_n}(p_0) \end{pmatrix},$$

which is usually called the *Jacobian matrix* of the coordinate transformation evaluated at p_0.

If we denote by $H_f^x(p_0)$ the Hessian of f in the coordinate system $x = (x_1, \ldots, x_n)$, then by direct computation we get

$$H_f^y(p_0) = {}^tJ(p_0) H_f^x(p_0) J(p_0).$$

Definition 2.3. A real number c is called a *critical value* of a $f : M \to \mathbf{R}$ if there exists a critical point $p_0 \in M$ such that $f(p_0) = c$.

Since the Jacobian of the coordinate transformation at a point p_0 has a non-zero determinant, then we have

$$\det H_f^y(p_0) = \det\big({}^tJ(p_0)\big) \det\big(H_f^x(p_0)\big) \det\big(J(p_0)\big).$$

But the determinant of the Jacobian of any coordinate transformation at a point p_0 has a non-zero determinant. Hence $\det H_f^y(p_0) \neq 0$ if and only if $\det H_f^x(p_0) \neq 0$, and the property of a critical point of a function being non-degenerate or degenerate does not depend on the choice of a coordinate system at p_0.

Definition 2.4. A function $f : M \to \mathbf{R}$ is called a *Morse function* if every critical point of f is non-degenerate.

Theorem 2.5 (Morse Lemma). *Let p_0 be a non-degenerate critical point of $f : M \to \mathbf{R}$. Then there exists a local coordinate system $(x_1, \ldots, x_n)$ at p_0 such that with respect to these coordinates f has the form*

$$-x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2 + f(p_0)$$

Sylvester's law implies that $0 \leq \lambda \leq n$ is well defined and do not depend on the choice of the coordinate system. Since λ depends only on the function f and the critical point p_0, then we define

Definition 2.6. The integer λ is called the *index* of the non-degenerate critical point p_0 of the function f.

Proof of Morse Lemma. Without loss of generality one can assume that $f(p_0) = 0$, and let $(x_1, \ldots, x_n)$ be a local coordinate system around the origin p_0. Since $f(p_0) = 0$, then according to the fundamental Theorem of calculus one can find n smooth functions $h_i(x) = \displaystyle\int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$, $i = 1, \ldots, n$ such that

$$f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i \, h_i(x_1, \ldots, x_n).$$

With this decomposition we get $\dfrac{\partial f}{\partial x_i}(0,\dots,0) = h_i(0,\dots,0)$ for $i = 1,\dots,n$.

Now, since the origin p_0 in the local coordinate system $(x_1,\dots,x_n)$ is a critical point for the function f, then we have $h_i(0,\dots,0) = 0$ for $i = 1,\dots,n$. As made before for f, for each $h_i, i = 1,\dots,n$ one can find n smooth functions $h_{i,j}, j = 1,\dots,n$ such that

$$h_i(x_1,\dots,x_n) = \sum_{j=1}^{n} x_j\, h_{i,j}(x_1,\dots,x_n).$$

Putting these decompositions all together, we get

$$f(x_1,\dots,x_n) = \sum_{i,j=1}^{n} x_i x_j\, h_{i,j}(x_1,\dots,x_n),$$

setting $H_{i,j} = \frac{h_{i,j}+h_{j,i}}{2}$ gives $H_{i,j} = H_{j,i}$ and the following quadratic representation of f

$$f(x_1,\dots,x_n) = \sum_{i,j=1}^{n} x_i x_j\, H_{i,j}(x_1,\dots,x_n). \tag{2.1}$$

We will now reduce this representation to the wanted one using the Gauss algorithm on quadratic forms.

The computation of the second order partial derivative of 2.1 gives

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0,\dots,0) = 2\, H_{i,j}(0,\dots,0).$$

Since p_0 is a non-degenerate critical point of the function f, then we have $\det H_f(p_0) = \det H_f^x(0,\dots,0) = \det\big(H_{i,j}(0,\dots,0)\big)_{i,j} \neq 0$. Moreover, up to a change of local coordinates, we can assume that

$$\frac{\partial^2 f}{\partial x_1^2}(0,\dots,0) \neq 0,$$

hence since the functions $H_{i,j}$ are continuous this gives $H_{1,1} \neq 0$ (eventually on a smaller neighborhood of p_0 than the one of the local coordinate system).

Now for an appropriate choice of local coordinate $(X_1, x_2, \dots, x_n)$ the function f is of the form

$$f(X_1, x_2, \dots, x_n) = \pm X_1^2 + \varphi(x_2,\dots,x_n) \tag{2.2}$$

with $\varphi(x_2,\dots,x_n)$ a quadratic form with $n-1$ variables $x_2,\dots,x_n$. By induction on the number of variables one can reduce the function f to the desired form. □

Corollary 2.7. *Let $f : M \to \mathbf{R}$ be a Morse function. Any non-degenerate critical point of f is isolated, and when M is a compact n-manifold f admits finitely many critical points.*

Proof. According to Morse Lemma, in a small coordinate neighborhood of a critical point p_0, the function f is of the form $-x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2 + f(p_0)$. So the origin, i.e., the point p_0, is the only critical point in the coordinate neighborhood of p_0. Recall that for a Morse function any critical point is non-degenerate.

Assume that the Morse function f admits infinitely many distinct critical points $(p_i)_{i\in\mathcal{I}}$ where $\mathcal{I}$ is an infinite set. Since non-degenerated critical points are isolated there exists disjoint open sets $(U_i)_{i\in\mathcal{I}}$ such that $U_i \subset M$ contains only one critical point p_i. First construct $U \subset M$ an open set such that for all i in $\mathcal{I}$ the point p_i is not in U, then the infinite cover

$$M \subset U \bigcup_{i\in\mathcal{I}} U_i$$

can't be reduced to a finite one. This is in contradiction with the hypothesis of compactness for M.

Finally the Morse function f admits only finitely many critical points. $\square$

Now we will see that every function $f : M \to \mathbf{R}$ on a compact manifold can be approximate by a Morse function.

Definition 2.8. Let M be a compact manifold, and let $\varepsilon > 0$ be a real. A function $f : M \to \mathbf{R}$ is a C^2_ε-*approximation* of a function $\varphi : M \to \mathbf{R}$ if there exists a compact covering $M \subset \bigcup_{i=1,\dots,m} Y_i$ and on each compact $Y_i \subset M$, $i = 1, \dots, m$ the following hold

1. $\forall y \in Y_i \ |f(y) - g(y)| < \varepsilon$,
2. $\forall y \in Y_i \ |\frac{\partial f(y)}{\partial x_j} - \frac{\partial g(y)}{\partial x_j}| < \varepsilon, \ j = 1, \dots, n$,
3. $\forall y \in Y_i \ |\frac{\partial^2 f(y)}{\partial x_j \partial x_k} - \frac{\partial^2 g(y)}{\partial x_j \partial x_k}| < \varepsilon, \ j,k = 1, \dots, n$.

Theorem 2.9 (Existence of Morse functions). *Let M be a compact manifold without boundary, and $f : M \to \mathbf{R}$ a smooth function. Then for each real $\varepsilon > 0$ there exists a Morse function ψ on M which is a C^2_ε-approximation of f. Moreover one can assume that the critical values associated with distinct critical points of ψ are distinct.*

We refer to [98] for a detailed proof of this Theorem.

Using Morse functions defined on a manifold M, we will explain now how to construct some particular tangent vector fields on M. These vector fields make easier to understand the behavior of the manifold around the critical points of the Morse functions.

Before, recall, that for a given vector $v \in T_pM$ the directional derivative of a function $f : M \to \mathbf{R}$ can be defined as follows. Let $c(\tau) = \big(x_1(\tau), \dots, x_n(\tau)\big)$ be a curve in M such that $c(0) = p$ and $\dfrac{\mathrm{d}c}{\mathrm{d}t}(0) = v$. Then the directional derivative of f in the direction v at p is the real function defined on M

$$v.f = \sum_{i=1}^{n} \frac{\mathrm{d}x_i}{\mathrm{d}t}(0) \frac{\partial f}{\partial x_i}.$$

When X is a tangent vector field on M, i.e., to each point p in M we associate a tangent vector $X(p)$ in $T_p(M)$, we extend this definition. We compute the

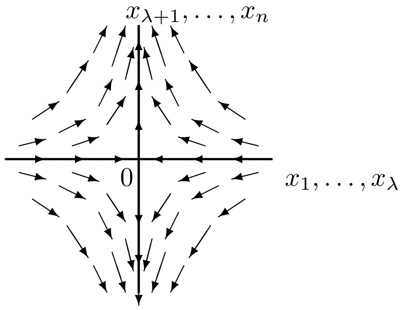


Figure 2.1. The gradient vector field of $x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$

directional derivative of f in the direction $X(p)$ at p. Then we can differentiate f with respect to X as well. A tangent vector field is defined by

$$X(p) = \sum_{i=1}^{n} \xi_i(p)\Big(\frac{\partial}{\partial x_i}\Big)_p,$$

where $\xi_i(p)$ are smooth functions defined on a coordinate system at p for $i = 1, \ldots, n$. Then set

$$\big(X.f\big)(p) = \Big(\sum_{i=1}^{n} \xi_i(p)\Big(\frac{\partial}{\partial x_i}\Big)_p.f\Big)(p)$$

Now let us consider the gradient vector field of a Morse function $f : M \to \mathbf{R}$ in a small neighborhood of a critical point for f. We saw that in an appropriate local coordinate system $(x_1, \ldots, x_n)$ the function f has the form

$$-x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

Its gradient vector field is

$$\nabla_f = -2x_1\frac{\partial}{\partial x_1} - \ldots - 2x_\lambda\frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}} + \ldots + 2x_n\frac{\partial}{\partial x_n}$$

Remark that $\nabla_f.f = \sum_{i=1}^{n}(\frac{\partial f}{\partial x_i})^2 \geq 0$, and $\big(\nabla_f.f\big)(p) > 0$ when p is not a critical point of the Morse function f. This inequality means that locally the gradient vector field of f follows a direction into which f is increasing.

This induces the following definition.

Definition 2.10. We say that a vector field X on M is a *gradient like* vector field for the Morse function $f : M \to \mathbf{R}$ if

1. $\big(X.f\big)(p) > 0$ for any non-critical point $p \in M$,
2. around any critical point of f there exists an appropriate coordinate system such that $X = \nabla_f$.

Theorem 2.11. *Let $f : M \to \mathbf{R}$ be a Morse function on a compact manifold. Then there exists a gradient like vector field on M.*

A way to prove this Theorem is to glue all together gradient vector fields of f defined on a finite number of coordinate neighborhoods. We refer to [98] for a detailed proof.

We illustrate the utility of gradient like vector fields with the two following Propositions.

Proposition 2.12. *Let $f : M \to \mathbf{R}$ be a Morse function. If the function f has no critical value in a real interval $[\alpha, \beta]$, then the manifold*

$$M_{[\alpha,\beta]} = \Big\{ p \in M | \alpha \leq f(p) \leq \beta \Big\}$$

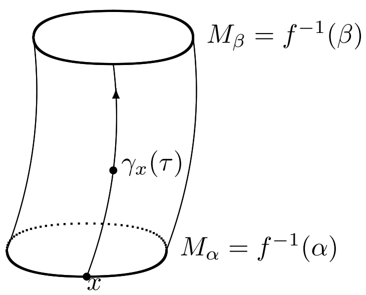
is diffeomorphic to the product $f^{-1}(\alpha) \times [\alpha, \beta]$, and M_α is diffeomorphic to M_β.

Proof. Let X be a gradient like vector field of f. Since f has no critical point on $M_{[\alpha,\beta]}$, then $\big(X.f\big)(p) > 0$ for all $p \in M_{[\alpha,\beta]}$. Set $Y = \dfrac{1}{X.f} X$ a vector field on $M_{[\alpha,\beta]}$, and let $\gamma_x(\tau)$ the integral curve of Y which start at $x \in f^{-1}(\alpha)$.

Since $\dfrac{d}{dt} f\big(\gamma_x(\tau)\big) = Y.f = 1$, then the integral curve $\gamma_x(\tau)$ starts at $x \in M_\alpha$ when $\tau = 0$ and is reaching M_β when $\tau = \beta - \alpha$. We know that the integral curves $\gamma_x(\tau)$ depend smoothly on both x and τ and two distinct integral curves never meet, hence the map

$$\begin{array}{rccc} h: & M_\alpha \times [0, \beta - \alpha] & \to & M_{[\alpha,\beta]} \\ & (x, \tau) & \mapsto & h(x, \tau) = \gamma_x(\tau) \end{array}$$

is a diffeomorphism.



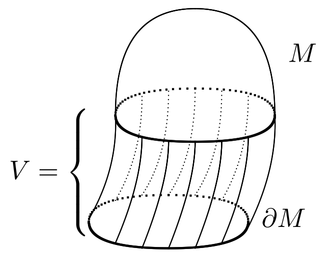
$\square$

Proposition 2.13 (Existence of collar neighborhood). *Let M be a manifold with compact boundary ∂M. Then there exists a neighborhood V of ∂M in M, which is diffeomorphic to $\partial M \times [0, 1)$.*

Proof. First glue two copies of M along their boundary ∂M to get a smooth closed manifold $W = M \cup_\partial M$. Then if $f : W \to \mathbf{R}$ is a Morse function on W, up to change one can suppose that f has no critical value in a neighborhood of 0 and $f(\partial M) = 0$. Then we have

$$M = W_{f \geq 0} = \{p \in W | 0 \leq f(p)\}.$$

Hence we may assume that there exists a Morse function $f : M \to \mathbf{R}^+$ on M such that $f^{-1}(0) = \partial M$ and 0 is not a critical value. As in the previous Proposition, one can construct a gradient like vector field, for which integral curves give the desired diffeomorphism.



□

2.1.1 Handle decompositions of manifolds

In this subsection we will use Morse functions to describe handle decompositions of compact manifolds.

Let $f : M \to \mathbf{R}$ be a Morse function on a compact n-manifold M with a critical point at $p_0 \in M$ of index λ, and set

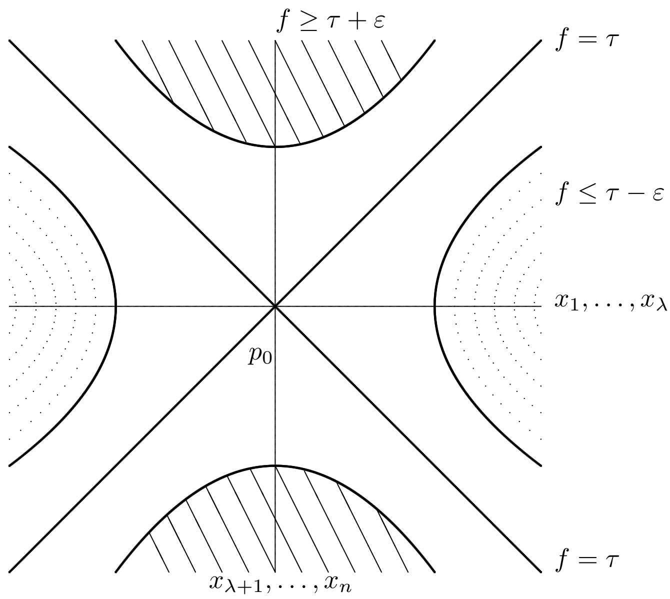
$$M_{\leq \tau} = \{p \in M | f(p) \leq \tau\}.$$

We will describe the changes of $M_{\leq \tau}$ when $\tau \in]c-\varepsilon, c+\varepsilon[$ where $\varepsilon > 0$ is a real such that $c = f(p_0)$ is the only critical value of f in $]c-\varepsilon, c+\varepsilon[$.

As seen before, in a local coordinate system around p_0, the function f is of the form

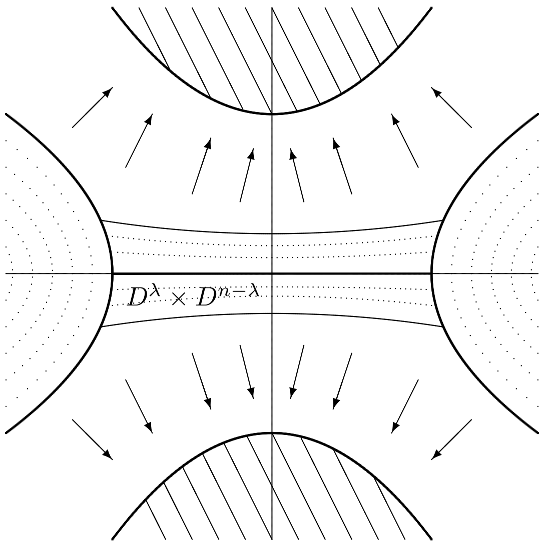
$$-x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

In the following picture we illustrated the behavior of f on M in a small coordinate neighborhood of the critical point p_0, we made a normal projection of a small neighborhood of the critical point p_0 of the manifold M onto $\mathbf{R}^n$. The shaded areas correspond to the set points of M for which the value of f is greater or equal to $\tau + \varepsilon$, the doted areas correspond to the set of points of M for which the value of f is lower or equal to $\tau - \varepsilon$.



Definition 2.14. The product manifold $D^\lambda \times D^{n-\lambda}$ is called a λ-*handle*, and the λ-disk $D^\lambda \times \{0\} \subset D^\lambda \times D^{n-\lambda}$ is called the *core* of the handle.

In the following picture we glued a λ-handle $D^\lambda \times D^{n-\lambda}$, along $D^{\lambda-1} \times D^{n-\lambda}$, to the boundary of the set of points of M for which f takes value lower or equal to $\tau - \varepsilon$.



With the gradient like vector field depicted by the arrows on the picture, one can see that, after smoothing, the manifold $M_{\leq \tau-\varepsilon} \cup D^\lambda \times D^{n-\lambda}$ is diffeomorphic to $M_{\leq \tau+\varepsilon}$.

Remark 2.15. Let $c_1, \ldots, c_k$ the distinct critical values of a Morse function $f : M \to \mathbf{R}$ defined on a compact manifold M without boundary. Let $\varepsilon > 0$ a real small enough, then the following hold

1. $M_{\leq c_1-\varepsilon} = \varnothing$,
2. $M_{\leq c_1+\varepsilon} = D^n$, is a 0-handle,
3. $M_{\leq c_k+\varepsilon} = M$.

Let X be a n-manifold with non-empty boundary, and let

$$\varphi : S^{\lambda-1} \times D^{n-\lambda} \to \partial X$$

be an embedding. Using φ we can attach a λ-handle to X. Set

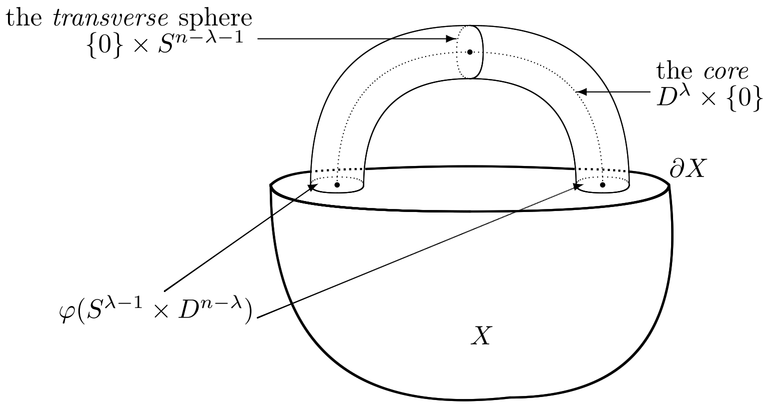
$$Y = X \cup_\varphi (D^\lambda \times D^{n-\lambda}),$$

which is the manifold obtained from X by gluing the λ-handle $D^\lambda \times D^{n-\lambda}$ to ∂X along $\varphi(S^{\lambda-1} \times D^{n-\lambda})$. After smoothing corners if necessary we can assume that Y is smooth.

Definition 2.16. We say that Y is obtained by attaching a λ-handle to X, and φ is called the *attaching map* of the λ-handle. We will use the notation

$$Y = X \cup (\varphi^\lambda).$$

The disk $D^\lambda \times \{0\}$ is called the *core* of the λ-handle, and the sphere $\{0\} \times S^{n-\lambda-1}$ is called the *transverse* sphere of the λ-handle.



Remark 2.17. Sometimes, the transverse sphere to a handle is called a *belt sphere*.

When we attach several handles to X, we use the same notation, e.g.

$$Y = X \cup (\varphi^\lambda) \cup (\psi^\mu).$$

But beware of this description the order of attaching is important, so it should be written

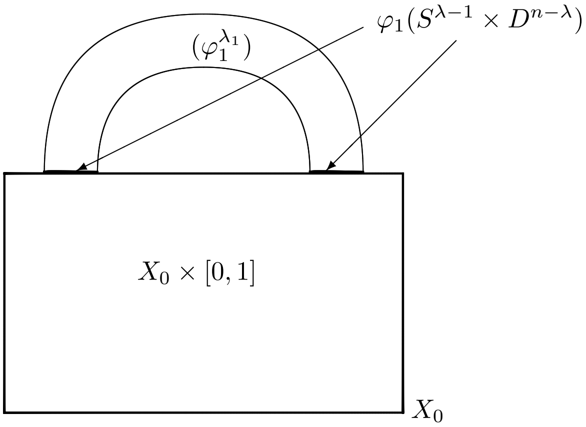
$$Y = \Big(X \cup (\varphi^\lambda)\Big) \cup (\psi^\mu),$$

meaning that first the λ-handle is attached to ∂X and then the μ-handle is attached to $\partial(X \cup (\varphi^\lambda))$.

Definition 2.18. A manifold obtained from D^n by attaching handles of various indices is called a *handlebody*.

When the boundary of a compact manifold X is of the form $X_0 \coprod X_1$, then it is sometimes more convenient to give a handle decomposition in which we attach the first handles to a collar neighborhood of the component $X_0 \subset \partial X$ of the boundary.

To do that, it is enough to start with a Morse function $f : X \to \mathbf{R}$ which maps X_0 to $f(X_0) = 0$, X_1 to $f(X_1) = 1$ and such that all the critical values $\lambda_1, \ldots, \lambda_k$ of $f_{|\overset{\circ}{X}}$ are in $]0, 1[$. Then the first handle, corresponding to the first critical value λ_1 of f, must be attach to a collar neighborhood of X_0 (see the following picture).



Then using this Morse function we have a handle decomposition for X as stated in the following Proposition

Proposition 2.19 (Handle decomposition of boundary manifolds). *Let X be a compact manifold with boundary $\partial X = X_0 \coprod X_1$. Then X possesses a handlebody decomposition up to diffeomorphism*

$$X = X_0 \times [0,1] \bigcup_{i=1,\ldots,m} (\varphi_i^{\lambda_i}).$$

Remark 2.20. When $\partial X = \varnothing$ the statement remains valid since in that case the first handle must be of index 0 and the last one must be of index n. The process start with a collection of n-disks, the 0-handles, then handles of index greater or equal to one are glued on these disks.

The decomposition given in Theorem 2.19 is not unique. So we will try to find good decompositions for our purpose. First we have to describe modifications of handlebody decompositions which do not change the diffeomorphism type. The goal is to find decompositions with less handles, and as few as possible of distinct indexes of handles. Note that all the following lemmas are due to Smale [142], see [71] and [94] as well for proofs.

Lemma 2.21 (Isotopy lemma). *Let X be a manifold of dimension n such that its boundary ∂X is $X_0 \coprod X_1$. Let $\varphi, \psi : S^{\lambda-1} \times D^{n-\lambda} \to X_1$ be two isotopic embeddings. Then there exists a diffeomorphism between $X \cup (\varphi)$ and $X \cup (\psi)$ which is the identity on X_0.*

Proof. The idea of the proof is to find an ambient isotopy on X which is identity on X_0. It induces a diffeomorphism h on X with $h \circ \varphi = \psi$, and then a diffeomorphism between $X \cup (\varphi)$ and $X \cup (\psi)$. $\square$

Remark 2.22. We sometimes call isotopy between attaching map of handles *sliding* of handles. This terminology comes from the fact that we can illustrate this isotopy by the moving of one handle to the other by the sliding of the gluing set.

In the following, for two handle decompositions

$$X = X_0 \times [0,1] \bigcup_{i=1,\ldots,m} (\varphi_i^{\lambda_i}),$$

$$X = X_0 \times [0,1] \bigcup_{i=1,\ldots,m} (\psi_i^{\lambda_i}),$$

of X, we will construct diffeormorphism of X which is the identity on $X_0 \times \{0\}$.

Definition 2.23. We say that the two handle decompositions

$$X = X_0 \times [0,1] \bigcup_{i=1,\ldots,m} (\varphi_i^{\lambda_i}),$$

$$X = X_0 \times [0,1] \bigcup_{i=1,\ldots,m} (\psi_i^{\lambda_i}),$$

of X, are diffeormorphic together *relatively* to X_0 when the diffeomorphism is the identity on $X_0 \times \{0\}$.

Lemma 2.24. *Let X be a manifold of dimension n such that its boundary ∂X is $X_0 \coprod X_1$. If $\lambda \leq \mu$ are some positive integers, then $X_0 \times [0,1] \cup (\psi^\mu) \cup (\varphi^\lambda)$ is diffeomorphic to $X_0 \times [0,1] \cup (\varphi_\star^\lambda) \cup (\psi^\mu)$ relatively to X_0 for an appropriate attaching map $\varphi_\star$.*

Proof. The inequality of dimensions $(\lambda-1)+(n-\mu-1) < n-1$ holds, so up to an isotopy $\varphi(S^{\lambda-1} \times \{0\})$ does not meet the transverse sphere of the μ-handle. Hence one can find an embedding

$$\varphi_\star : S^{\lambda-1} \times D^{n-\lambda} \to \partial(X \cup (\psi^\mu))$$

which does not meet the image of ψ in ∂X, namely $\psi(S^{\mu-1} \times D^{n-\mu})$. By Lemma 2.21 we have that

$$X_0 \times [0,1] \cup (\psi^\mu) \cup (\varphi^\lambda)$$

is diffeomorphic to

$$X_0 \times [0,1] \cup (\varphi_\star^\lambda) \cup (\psi^\mu).$$

□

Remark 2.25. Let $\lambda \leq \mu$, and let $X_0 \times [0,1] \cup (\varphi^\lambda) \cup (\psi^\mu)$ the manifold obtained by attaching two handles. Note that the attaching map of the μ-handle

$$\psi : S^{\mu-1} \times D^{n-\mu} \to \partial\big(X \cup (\varphi^\lambda)\big)$$

may not be isotopic to an embedding

$$\psi_\star : S^{\mu-1} \times D^{n-\mu} \to \partial\Big(X \setminus \big(\varphi(S^{\lambda-1} \times D^{n-\lambda})\big)\Big).$$

This means that the formula $X_0 \times [0,1] \cup (\psi^\mu) \cup (\varphi^\lambda)$ may be meaningless (up to diffeomorphism as well) in this situation, since the attaching map ψ may not be defined (up to isotopy) on $X_0 \times \{1\}$. Hence the order in which handles appear is very important and changing this order must be done carefully.

Let us consider the manifold Y obtained from $X_0 \times [0,1]$ by adding two handles of consecutive index, say λ and $\lambda+1$. If φ and ψ are the attaching maps one can write

$$Y = X_0 \times [0,1] \cup (\varphi^\lambda) \cup (\psi^{\lambda+1}).$$

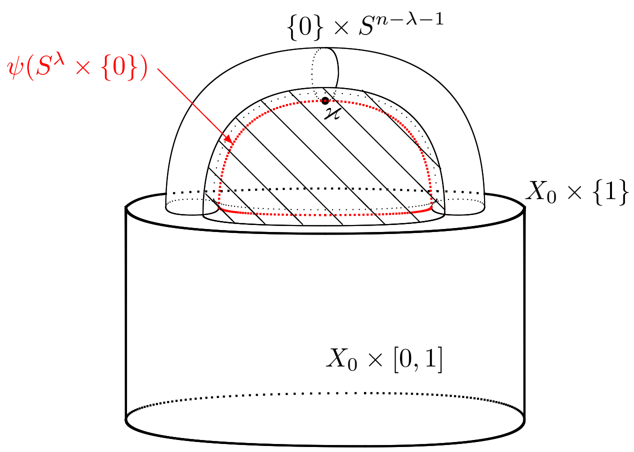
Assume that $\psi(S^\lambda \times \{0\})$ meets the transverse sphere of the λ-handle, namely $\{0\} \times S^{n-\lambda-1}$, transversally in exactly one point $\varkappa$. Let $\mathcal{U}$ be a small neighbourhood of the transverse sphere $\{0\} \times S^{n-\lambda-1}$ in the λ-handle (φ^λ). Then one can find an isotopy between $D^{n-\lambda} \times \mathcal{U}$ and the λ-handle. Then we have

$$\psi\big(S^\lambda \times \{0\}\big) \cap (\varphi^\lambda) = D^\lambda \times \{\varkappa\}.$$

Then it is technical, but not difficult, to check that the n-manifold

$$D^\lambda \times D^{n-\lambda} \cup_{\psi_{|S^\lambda \times D^{n-\lambda})\cap D^\lambda \times D^{n-\lambda}}} D^{\lambda+1} \times D^{n-\lambda-1},$$

which is the gluing of the λ-handle and the $(\lambda+1)$-handle along $\psi(S^\lambda \times D^{n-\lambda}) \cap D^\lambda \times D^{n-\lambda}$, is homeomorphic to the contractible manifold D^n. This implies that X and Y are diffeomorphic. The following picture illustrates this cancellation phenomenon.



We proved

Lemma 2.26 (Cancellation Lemma). *Let X_0 be a manifold without boundary, and let $Y = X_0 \times [0,1] \cup (\varphi^\lambda) \cup (\psi^{\lambda+1})$ such that $\psi(S^\lambda \times \{0\})$ meets the transverse sphere of the λ-handle transversally in exactly one point. Then $X_0 \times [0,1]$ and Y are diffeomorphic.*

Using this Lemma, if needed, one can change a handle decomposition and add two handles with consecutive indexes. First choose an embedded n-disk D in $X_0 \times \{0\}$. Then construct an embedding

$$\varphi : S^\lambda \times D^{n-\lambda} \to D$$

and an embedding

$$\psi : S^{\lambda+1} \times D^{n-\lambda-1} \to \partial\big(X \cup (\varphi^\lambda)\big)$$

such that $\psi(S^\lambda \times \{0\})$ meets the transverse sphere of the λ-handle transversally in exactly one point. According to the Cancellation Lemma 2.26 the manifolds $X_0 \times [0,1]$ and $X_0 \times [0,1] \cup (\varphi^\lambda) \cup (\psi^{\lambda+1})$ are diffeomorphic relatively to X_0.

Let us describe how to remove a λ-handle. The first step is to construct a $(\lambda+1)$-handle with a transversality condition with the λ-handle which allows cancellation. Then construct a handle of index $\lambda+2$ such that the two handles of indexes $\lambda+1$ and $\lambda 2$ are canceling together.

Now, up to technical assumptions we are ready to eliminate a λ-handle and replace it by a $(\lambda+2)$-handle as stated in the next Lemma.

First we have to fix some notations. Suppose that we have a handle decomposition of a manifold

$$Y = X_0 \times [0,1] \bigcup_{i=1}^{p_1} (\varphi_i^1) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n),$$

then we denote

- $Y^q = X_0 \times [0,1] \bigcup_{i=1}^{p_1} (\varphi_i^1) \dots \bigcup_{i=1}^{p_q} (\varphi_i^q)$, the manifold obtained from $X_0 \times [0,1]$ after the gluing of handles of index less or equal to q,
- $\hat{\partial} Y^q = \partial Y^q \setminus \coprod_{i=1}^{p_{q+1}} \varphi_i^{q+1}(S^q \times \overset{\circ}{D}{}^{n-1-q})$

Lemma 2.27. *Let X_0 be a $(n-1)$-manifold without boundary and $1 \le \lambda \le n-3$. Fix a handle decomposition of $Y = X_0 \times [0,1] \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$, with no handle of index strictly less than λ.*

Let $1 \le k \le p_\lambda$ be a fixed integer. Suppose that there exists an embedding $\psi^{\lambda+1} : S^\lambda \times D^{n-1-\lambda} \to \hat{\partial} Y^\lambda$ such that

1. *${\psi^{\lambda+1}}_{|S^\lambda \times \{0\}}$ is isotopic in ∂Y^λ to an embedding $\xi^{\lambda+1} : S^\lambda \times \{0\} \to \partial Y^\lambda$ which meets the transverse sphere of the handle (φ_k^λ) and is disjointed from the transverse spheres of the handles $(\varphi_i^\lambda)_{\substack{i=1,\dots,p_\lambda \\ i \ne k}}$*
2. *${\psi^{\lambda+1}}_{|S^\lambda \times \{0\}}$ is isotopic in $\partial Y^{\lambda+1}$ to an embedding of S^λ into a $(n-1)$-disk $D^{n-1} \subset \partial Y^{\lambda+1}$.*

Then Y is diffeomorphic, relatively to X_0, to a manifold which has the following handle decomposition

$$X_0 \times [0,1] \bigcup_{\substack{i=1,\dots,p_\lambda \\ i \ne k}} (\varphi_i^\lambda) \bigcup_{i=1}^{p_{\lambda+1}} (\overline{\varphi}_i^{\lambda+1}) \cup (\psi^{\lambda+2}) \bigcup_{i=1}^{p_{\lambda+2}} (\overline{\varphi}_i^{\lambda+2}) \dots \bigcup_{i=1}^{p_n} (\overline{\varphi}_i^n)$$

Proof. All the technical assumptions made in this Lemma allow to add first a new $(\lambda+1)$-handle $(\psi^{\lambda+1})$ which cancel with the handle (φ_k^λ), second to glue a new $(\lambda+2)$-handle $(\psi^{\lambda+2})$ which cancel with $(\psi^{\lambda+1})$. With the second assumption made in the statement, the gluing of the two handles $(\psi^{\lambda+1})$ and $(\psi^{\lambda+2})$ can be made in a $(n-1)$-disk embedded in $\hat{\partial} Y^{\lambda+2}$.

Then according to the Isotopy and Cancellation Lemmas (2.21 and 2.26) we can find the appropriate embeddings $\{\overline{\varphi} i_{i_k}^k | k = \lambda+1, \dots, n \ ; \ i_k = 1, \dots, p_k\}$ to give the desired handle decomposition of a manifold which is diffeomorphic to Y relatively to X_0. □

This Lemma will be very useful to prove the h-cobordism Theorem. But first we have to introduce a CW-complex associated with handle decompositions of manifolds. This CW-complex will allow us to compute the Whitehead torsion that appears in the s-cobordism Theorem.

2.1.2 CW-complex and handlebodies

In this subsection, we briefly recall some elementary properties of relative CW-complexes, and then we will construct a CW-complex which is associated with the handlebody decomposition of a manifold.

Let us denote by $X^{(0)}$ a set of discrete points. Let $n \geq 1$ be an integer. If the set $X^{(n-1)}$ has been defined, then consider $\{\psi_\alpha\}_{\alpha \in \mathcal{A}_n}$ a set of maps $\psi_\alpha : S^{n-1} \to X^{(n-1)}$. Set

$$X^{(n)} = X^{(n-1)} \cup \Big(\bigcup_{\psi_\alpha} D^n_\alpha \Big)_{\alpha \in \mathcal{A}_n}$$

be the gluing of $X^{(n-1)}$ and some n-dimensional disks along their boundaries $\partial D^n_\alpha \cong S^{n-1}$ with the maps ψ_α.

This induces a filtration

$$X^{(0)} \subset X^{(1)} \subset \ldots \subset X^{(n)} \subset \ldots,$$

the path components of $X^{(n)} \setminus X^{(n-1)}$ are called *open n-cells*, the maps ψ_α are called *attaching maps*, and the maps $\Psi_\alpha : D_n \to X^{(n)}$ induced by ψ_α are called *characteristic maps.*

The set

$$X = \bigcup_{n \in \mathcal{N}} X^{(n)}$$

is called a CW-complex. When $\mathcal{N}$ is not finite, then a set is open in X if its intersection with each $X^{(n)}$ is open in $X^{(n)}$. The letter C stands for CLOSURE FINITE and the letter W stands for WEAK TOPOLOGY. A set is *open* if its intersection with each $X^{(n)}$ is open in $X^{(n)}$.

Remark 2.28. An open n-cell is open in $X^{(n)}$, but usually is not an open set in X.

The image of a characteristic map is a compact subset of X, which is sometimes called a *closed cell*, but usually is not homeomorphic to D^n.

A *relative* CW-complex (X, A) consists of a pair of topological spaces $A \subset X$, such that X is obtained from A by gluing λ-cells, with $\lambda \geq 1$, as we did for CW-complexes. The associated filtration is

$$A = X^{(\lambda-1)} \subset X^{(\lambda)} \subset \ldots \subset X^{(n)} \subset \ldots.$$

Let (X, A) be a relative CW-complex. Assume that X is arcwise connected[1] and set $\pi = \pi_1(X)$. Let $\rho : \widetilde{X} \to X$ be the universal covering of X, and set $\widetilde{X}^{(q)} = \rho^{-1}\big(X^{(q)}\big)$ and $\widetilde{A} = \rho^{-1}(A)$. Then $(\widetilde{X}, \widetilde{A})$ is a relative CW-complex with the filtration $\widetilde{A} \subset \widetilde{X}^{(1)} \subset \ldots \subset \widetilde{X}^{(n)} \subset \ldots$.

[1]This assumption is only made in order to avoid considerations about base points and simplify the argument.

Recall that the homology of the relative CW-complex $(\widetilde{X}, \widetilde{A})$ can be computed using a $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{X}, \widetilde{A})$. The q^{th} $\mathbf{Z}[\pi]$-chain module is the singular homology $\mathrm{H}_q(\widetilde{X}^{(q)}, \widetilde{X}^{(q-1)})$ and the π-action is coming from the covering transformations, the q^{th} differential is then given by the composite map

$$\mathrm{H}_q(\widetilde{X}^{(q)}, \widetilde{X}^{(q-1)}) \xrightarrow{\partial_q} \mathrm{H}_{q-1}(\widetilde{X}^{(q-1)}) \xrightarrow{i_q} \mathrm{H}_{q-1}(\widetilde{X}^{(q-1)}, \widetilde{X}^{(q-2)}),$$

where ∂_q is the q^{th} boundary map associated with the homology long exact sequence of the pair $(\widetilde{X}^{(q)}, \widetilde{X}^{(q-1)})$ and i_q is induced by the inclusion.

If we denote by β_i the image of a generator of $\mathrm{H}_q(D^q, S^{q-1}) \cong \mathbf{Z}$ under the map $(\Psi_i^q, \psi_i^q)_q : \mathrm{H}_q(D^q, S^{q-1}) \to C_q(\widetilde{X}, \widetilde{A}) = \mathrm{H}_q(\widetilde{X}^{(q)}, \widetilde{X}^{(q-1)})$, then the set $\{\beta_i\}_{i \in \mathcal{A}_q}$ is a $\mathbf{Z}[\pi]$-basis for $C_q(\widetilde{X}, \widetilde{A})$. We call this basis the *cellular* basis.

Recall that the homology of a relative CW-complex is given by the homology of the $\mathbf{Z}[\pi]$-chain complex we just defined, i.e.,

$$\mathrm{H}_*(X, A) \cong H_*\big(C_*(X, A)\big).$$

Let M be a closed $(n-1)$-manifold. Now suppose we have a handle decomposition of a manifold

$$Y = M \times [0,1] \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}) \ldots \bigcup_{i=1}^{p_n} (\varphi_i^n),$$

where the λ-handles are attached on $M \times \{0\}$. We denote by Y^q the manifold

$$Y^q = M \times [0,1] \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}) \ldots \bigcup_{i=1}^{p_q} (\varphi_i^q)$$

obtained from $M \times [0,1]$ by adding handles of index less or equal to q.

Let us denote $M \times \{0\}$ by M_0. Then we construct by induction over $q = \lambda, \ldots, n$ a sequence of spaces $X^{(q)}$ with a filtration

$$M_0 \subset X^{(\lambda)} \subset \ldots \subset X^{(n)} = \mathfrak{X}$$

such that $(\mathfrak{X}, M_0)$ is a relative CW-complex. We define the attaching maps of the relative CW-complex $(\mathfrak{X}, M \times \{0\})$ using the attaching maps of the handlebody decomposition of Y.

More precisely set

$$f_{\lambda-1} : Y^{\lambda-1} = M \times [0,1] \to X^{(\lambda-1)} = M_0$$

the projection, which is a homotopy equivalence.

Assume that, for $q \geq \lambda$, the set $X^{(q-1)}$ is constructed and there exists a homotopy equivalence $f_{q-1} : Y^{q-1} \to X^{(q-1)}$. Then define the attaching maps $f_{q-1} \circ \varphi^q_{i|S^{q-1} \times \{0\}}$ for $i = 1, \ldots, p_q$ to construct Y^q. Now consider the relative CW-complex $(\mathcal{Y}^q, Y^q)$, where $\mathcal{Y}^q$ is constructed from Y^q by adding q-cells with the attaching maps $\big\{\varphi^q_{i|S^{q-1} \times \{0\}}\big\}_{i=1,\ldots,p_q}$. One can see that both $X^{(q)}$ and Y^q are homotopically equivalent to $\mathcal{Y}^q$, hence there exists a homotopy equivalence $f_q : Y^q \to X^{(q)}$ such that $f_{q|Y^{q-1}} = f_{q-1}$.

Denote by $\rho : \widetilde{Y} \to Y$ the universal covering of Y with $\pi = \pi_1(Y)$ as covering transformations group. Set $\widetilde{Y}^q = \rho^{-1}(Y^q)$. As done before in the general context of relative CW-complex, one can associate a $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{Y}, \widetilde{M_0})$. The q^{th} $\mathbf{Z}[\pi]$-chain module is the singular homology $\mathrm{H}_q(\widetilde{Y}^{(q)}, \widetilde{Y}^{(q-1)})$ and the q^{th} differential is then given by the composite map

$$\mathrm{H}_q(\widetilde{Y}^{(q)}, \widetilde{Y}^{(q-1)}) \xrightarrow{\partial_q} \mathrm{H}_{q-1}(\widetilde{Y}^{(q-1)}) \xrightarrow{i_q} \mathrm{H}_{q-1}(\widetilde{Y}^{(q-1)}, \widetilde{Y}^{(q-2)}),$$

where ∂_q is the q^{th} boundary map associated with the homology long exact sequence of the pair $(\widetilde{Y}^{(q)}, \widetilde{Y}^{(q-1)})$ and i_q is induced by the inclusion.

Since the maps $f_q : Y^q \to X^{(q)}$ constructed before are homotopy equivalences, then we get an isomorphism of $\mathbf{Z}[\pi]$-chain complexes

$$C_*(\widetilde{Y}, \widetilde{M_0}) \overset{\Theta}{\cong} C_*(\widetilde{\mathfrak{X}}, \widetilde{M_0}).$$

Moreover each handle of index q with attaching map φ_i^q for $i = 1, \ldots, p_q$ determines an element $[\varphi_i^q] \in C_q(\widetilde{Y}, \widetilde{M_0})$. And the basis $\{[\varphi_i^q]\}_{i=1,\ldots,p_q}$ of $C_q(\widetilde{Y}, \widetilde{M_0})$ maps to the cellular basis of $C_*(\widetilde{\mathfrak{X}}, \widetilde{M_0})$ under Θ.

Now we are ready to prove the h-cobordism Theorem.

2.2 *h*-cobordism Theorem

First let us state the h-cobordism Theorem due to Smale.

Theorem 2.29 (h-cobordism [142]). *Let M_1 and M_2 be two closed oriented and simply connected manifolds of dimension $n \geq 5$. If there exists an oriented compact manifold W with ∂W diffeomorphic to the disjoint union of M_1 and $-M_2$, and each component of ∂W is a deformation retract of W then W is diffeomorphic to $M_1 \times [0,1]$.*

The manifold $-M_2$ is the manifold M_2 with the reversed orientation.

Remark 2.30. As an important consequence we have that the two manifolds M_1 and M_2 are diffeomorphic to each other.

Remark that the inclusions $M_i \hookrightarrow W$, for $i = 1, 2$, are homotopy equivalences. And the letter h in h-cobordism is for *homotopy equivalence*.

The h-cobordism Theorem can be reformulated as follows.

Theorem 2.31 (h-cobordism). *Let M_1 and M_2 be two closed oriented and simply connected manifolds of dimension $n \geq 5$. If there exists an oriented compact manifold W with ∂W diffeomorphic to the disjoint union of M_1 and $-M_2$, and $\mathrm{H}_*(W, M_1) = 0$ then W is diffeomorphic to $M_1 \times [0,1]$.*

Remark 2.32. In the second statement of the h-cobordism Theorem it is equivalent to replace $\mathrm{H}_*(W, M_1) = 0$ by $\mathrm{H}_*(W, M_2) = 0$.

More precisely, when $\mathrm{H}_*(W, M_1) = 0$ the universal coefficient Theorem implies $\mathrm{H}^*(W, M_1) \cong \mathrm{Hom}\big(\mathrm{H}_*(W, M_1)\big) = 0$, and by Poincaré duality we get $\mathrm{H}_*(W, M_2) = 0$. Similarly $\mathrm{H}_*(W, M_2) = 0$ implies $\mathrm{H}_*(W, M_1) = 0$.

Assuming Theorem 2.29 one can prove Theorem 2.31.

Proof of Theorem 2.31. First remark that if M_1 and M_2 are both deformation retracts of W then we have $\mathrm{H}_*(W, M_1) = 0$, and $\mathrm{H}_*(W, M_2) = 0$ as well.

Second when $\pi_1(M_1) = 0$, $\pi_1(W, M_1) = 0$ and $\mathrm{H}_*(W, M_1) = 0$ then, according to the relative Hurewicz isomorphism Theorem (see [16]), we have $\pi_i(W, M_1) = 0$ for $i \in \mathbf{N}$. Then one can construct a deformation retraction from W to M_1. As explained in Remark 2.31 the nullity of $\mathrm{H}_*(W, M_1)$ implies $\mathrm{H}_*(W, M_2) = 0$, and M_2 is, by the same argument, a deformation retract of W. □

The h-cobordism Theorem is crucial for the study of cobordism classes of high dimensional knots. It concerns simply connected manifolds, but this connectivity condition is automatic for knots of dimension greater or equal to 2.

In the subsection 2.2.1 we will prove an extension to non-simply connected manifolds called *s-cobordism theorem.* Though we will not need this extension for the study of knot cobordism, we choose to give this proof since the core of the proof is the same of the proof of the h-cobordism Theorem and is essentially made of Smale's lemmas .

The s-cobordism Theorem was proved by Barden in [4], by Mazur in [99] and by Stallings (who never published his proof). For additional details we refer to Kervaire's paper [71] devoted to a detailed proof of this Theorem.

2.2.1 s-cobordism Theorem

Theorem 2.33 (s-cobordism Theorem). *Let M_1 and M_2 be two closed oriented and connected manifolds of dimension $n \geq 5$, and let $\pi = \pi_1(M_1)$ the fundamental group of M_1. If there exists an oriented compact manifold W with ∂W diffeomorphic to the disjoint union of M_1 and $-M_2$, and each component of ∂W is a deformation retract of W then W is diffeomorphic to $M_1 \times [0,1]$ if and only if the Whitehead torsion $\tau(W, M_1) \in \mathrm{Wh}(\pi)$ vanishes.*

To make this statement understandable we have to define briefly Whitehead groups and Whitehead torsion, see [151] for details.

Whitehead groups. Let π be a group, and let $GL\big(n, \mathbf{Z}[\pi]\big)$ the group of invertible matrices of order n on the group ring $\mathbf{Z}[\pi]$. We denote by $GL\big(\mathbf{Z}[\pi]\big)$ the set of disjoint union $\displaystyle\bigcup_{n\in\mathbf{Z}} GL\big(n, \mathbf{Z}[\pi]\big)$, it is the set of invertible matrices of arbitrary size with entries in $\mathbf{Z}[\pi]$.

Let us denote by $E^n_{i,j}$ a $n \times n$ matrix with all entries 0 except for a 1 in the (i,j) spot; and by $\Delta^n_i(\gamma)$ a $n \times n$ diagonal matrix with entries on the diagonal equal to 1 except for γ in the (i,i) spot. If I_n denotes the identity matrix of rank n, then an *elementary matrix* is a matrix of the form $(I_n + aE^n_{i,j})$, with $a \in \mathbf{Z}[\pi]$; and let $E\big(\mathbf{Z}[\pi]\big)$ be the subgroup of $GL\big(\mathbf{Z}[\pi]\big)$ generated by the elementary matrices.

It is not difficult to show that $E\big(\mathbf{Z}[\pi]\big)$ is the commutator subgroup of $GL\big(\mathbf{Z}[\pi]\big)$, and any subgroup of $GL\big(\mathbf{Z}[\pi]\big)$ which contains $E\big(\mathbf{Z}[\pi]\big)$ is a normal subgroup of $GL\big(\mathbf{Z}[\pi]\big)$.

Let us consider the subgroup $\pm\pi$ of $\mathbf{Z}[\pi]$ of trivial units, namely

$$\big\{p \mid p \in \pi\big\} \cup \big\{-p \mid p \in \pi\big\} = \pm\pi < \mathbf{Z}[\pi].$$

Then we define $I_{\pm\pi}$ to be the set

$$I_{\pm\pi} = \{M \in GL(\mathbf{Z}[\pi]) \mid M = \Delta_i^n(\gamma) \text{ with } \gamma \in \pm\pi, \text{ or } M \in E(\mathbf{Z}[\pi]\}.$$

In $I_{\pm\pi}$ we collected the matrices of $E\big(\mathbf{Z}[\pi]\big)$ and the matrices of the form

$$\begin{pmatrix} I_i & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & I_j \end{pmatrix}$$

with $\gamma \in \pm\pi$.

Hence the group E_π, which is generated by the matrices of $I_{\pm\pi}$, is a normal subgroup of $GL\big(\mathbf{Z}[\pi]\big)$.

Definition 2.34. The whitehead group $\mathrm{Wh}(\pi)$ is the abelian quotient group $GL\big(\mathbf{Z}[\pi]\big)_{/E_\pi}$.

In the following we will use another definition of $\mathrm{Wh}(\pi)$, which is more complicated but more convenient for our purpose. On $GL(\mathbf{Z}[\pi])$ we define an equivalence relation, denoted by $\mathcal{R}$, generated by the elementary operations listed below.

Let A be a matrix in $GL\big(\mathbf{Z}[\pi]\big)$,

1. multiply the i-th row of A from left by $\pm\gamma$ with $\gamma \in \pi$;
2. add the i-th row to j-th row of A;
3. change the matrix $A \in GL\big(n, \mathbf{Z}[\pi]\big)$ to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;
4. change the matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL\big(n+1, \mathbf{Z}[\pi]\big)$ to A (this is the inverse of the previous item).

Remark 2.35. We do not use column operations in our definition, i.e., right product with elementary matrices. Because if two matrices A and B are related together with column and row operations, then there exist two matrices E_1 and E_2, which are product of elementary matrices, such that $I_n = E_1 \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix} B^{-1} E_2$. But this means that $E_2^{-1} = E_1 \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix} B^{-1}$, and then $I_n = E_2\, E_1 \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix} B^{-1}$. This implies that A and B are related together only using row operations.

One can define a product on classes of matrices in $GL\big(\mathbf{Z}[\pi]\big)_{/\mathcal{R}}$. We denote by $[A] \in GL\big(\mathbf{Z}[\pi]\big)_{/\mathcal{R}}$ the class of a matrix $A \in GL\big(\mathbf{Z}[\pi]\big)$. Let $[A]$ and $[B]$ be in $GL\big(\mathbf{Z}[\pi]\big)_{/\mathcal{R}}$, then there exist two integers i and j (may be equal to 0) such that the two matrices $A \oplus I_i$ and $B \oplus I_j$ are invertible matrices of same rank. We define

$$[A].[B] = \big[(A \oplus I_i).(B \oplus I_j)\big].$$

The neutral element is given by $[I_n]$ for any positive integer n. The inverse of $[A]$ is given by $[A^{-1}]$. One can prove that $\Big(GL\big(\mathbf{Z}[\pi]\big)_{/\mathcal{R}}, .\Big)$ is an abelian group, and $\mathrm{Wh}(\pi)$ is the quotient $GL\big(\mathbf{Z}[\pi]\big)_{/\mathcal{R}}$.

Proposition 2.36. *These two definitions of Whitehead groups are equivalent together.*

See [29] for this equivalence.

In the following we will denote by A both a matrix in $GL\big(\mathbf{Z}[\pi]\big)$ and its class in $\mathrm{Wh}(\pi)$.

Whitehead torsion. We will define the Whitehead torsion of a pair (X, Y) when both X and Y are CW-complexes such that Y is a deformation retract of X. But Whitehead torsion may be defined algebraically for acyclic chain complexes over a ring R under some assumptions for R, we refer to [151] and [108] for detailed expositions on Whitehead torsion.

Since the inclusion $Y \hookrightarrow X$ is a homotopy equivalence, then it induces an isomorphism of fundamental groups $\pi_1(Y) \cong \pi_1(X) = \pi$, provided we choose a base point in Y. Let us consider again the universal covering $\widetilde{\rho} : \widetilde{X} \to X$, it induces the covering $\widetilde{\rho}_{|\widetilde{Y}} : \widetilde{Y} \to Y$ and the subcomplex $\widetilde{Y}$ is a deformation retract of $\widetilde{X}$. Therefore the $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{X}, \widetilde{Y})$ of length n is acyclic. Recall that π acts on $C_*(\widetilde{X}, \widetilde{Y})$, and this makes it a free chain complex over $\mathbf{Z}[\pi]$; each $\mathbf{Z}[\pi]$-module $C_q(\widetilde{X}, \widetilde{Y})$ equiped with the cellular basis $\mathcal{B}_q = \{\beta_i\}_{i \in \mathcal{A}_q}$ see § 2.1.2.

1. First assume that for all integer $0 \leq q \leq n$ the $\mathbf{Z}[\pi]$-module $\operatorname{Im} d_q$ is free. Since the complex is acyclic, then we have the short exact sequences

$$0 \to \operatorname{Im} d_q \to C_q(\widetilde{X}, \widetilde{Y}) \xrightarrow{d_q} \operatorname{Im} d_{q-1} \to 0.$$

By exactness of the last short sequences we get sections s_q of d_q, then set $\mathcal{I}^\star_{q-1} = s_q(\mathcal{I}_q)$ the image of the basis $\mathcal{I}_{q-1}$ of $\operatorname{Im} d_{q-1}$ under s_q. Note that, since for any distinct integers i and j the two $\mathbf{Z}[\pi]$-modules $\mathbf{Z}[\pi]^i$ and $\mathbf{Z}[\pi]^j$ are not isomorphic, then the juxtaposition of the two basis $\mathcal{I}_q$ and $\mathcal{I}^\star_{q-1}$ is a basis of $C_q(\widetilde{X}, \widetilde{Y})$. Set $T_{\mathcal{I}_q \mathcal{I}^\star_{q-1} \to \mathcal{B}_q}$ the transition matrix from $\mathcal{I}_q \mathcal{I}^\star_{q-1}$ to $\mathcal{B}_q$.

The following product matrix

$$\tau = \prod_{i=0}^{n} T^{(-1)^{i+1}}_{\mathcal{I}_q \mathcal{I}^\star_{q-1} \to \mathcal{B}_q}$$

is invertible.

Moreover one can prove that its class in $\mathrm{Wh}(\pi)$ does not depend on the choices of the basis and is invariant under cellular subdivisions. According to these facts when for all integer $0 \leq q \leq n$ the $\mathbf{Z}[\pi]$-module $\operatorname{Im} d_q$ are free, then we define the *torsion* of the complex $C_*(\widetilde{X}, \widetilde{Y})$ to be the class of τ in $\mathrm{Wh}(\pi)$.

2. When the $\mathbf{Z}[\pi]$-module $\operatorname{Im} d_q$ are not free we have the following Lemma

Lemma 2.37. *For all integers $0 \leq q \leq n$ there exists a free $\mathbf{Z}[\pi]$-module F_q such that the $\mathbf{Z}[\pi]$-module $\operatorname{Im} d_q \oplus F_q$ is free.*

Proof. Note that $\operatorname{Im} d_0 = C_0(\widetilde{X}, \widetilde{Y})$ is free.

We will prove the property by induction on q. Assume there exists an integer $k \geq 0$ for which there exists a free $\mathbf{Z}[\pi]$-module F_k such that the $\mathbf{Z}[\pi]$-module $\operatorname{Im} d_k \oplus F_k$ is free.

Since the $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{X}, \widetilde{Y})$ of length n is acyclic, then we have the following short exact sequence

$$0 \to \operatorname{Im} d_{k+1} \to C_k(\widetilde{X}, \widetilde{Y}) \oplus F_k \xrightarrow{d_k \oplus Id} \operatorname{Im} d_k \oplus F_k \to 0.$$

The last $\mathbf{Z}[\pi]$-module is free, hence there exists a section σ_k for $d_k \oplus Id$. The $\mathbf{Z}[\pi]$-module $\sigma_q(\operatorname{Im} d_k \oplus F_k)$ is free, and $\operatorname{Im} d_{k+1} \oplus \sigma_q(\operatorname{Im} d_k \oplus F_k) = C_k(\widetilde{X}, \widetilde{Y}) \oplus F_k$ as well. □

Let us denote by $C_*^q(F)$ the free based acyclic $\mathbf{Z}[\pi]$-chain complex associated with a free based $\mathbf{Z}[\pi]$-module F, which has $d_q : F \to F$ as the only non-trivial differential

$$\ldots \to 0 \to F \xrightarrow{d_q} F \to 0 \to \ldots.$$

Define a new $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{X}, \widetilde{Y}) \bigoplus_{k=0}^n C_*^k(F_k)$. Since in this free acyclic $\mathbf{Z}[\pi]$-chain complex the image of the differential are some free $\mathbf{Z}[\pi]$-modules, then we can compute its torsion as just made before. One can prove that the torsion of this complex does not depend on the choices made on the free $\mathbf{Z}[\pi]$-modules F_q for $q = 0, \ldots, n$.

We define the torsion $\tau(X, Y)$ to be the torsion of the $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{X}, \widetilde{Y}) \bigoplus_{k=0}^n C_*^k(F_k)$.

Come back to the statement of the s-cobordism Theorem. Assume that W is an oriented compact manifold with boundary $\partial W \cong M_1 \coprod -M_2$, such that both M_1 and M_2 are deformation retracts of W. To a handlebody decomposition

$$W = M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \ldots \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \ldots \bigcup_{i=1}^{p_n} (\varphi_i^n),$$

one can associate first a $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{W}, \widetilde{M_1})$ and second a relative CW-complex $(\widetilde{\mathfrak{X}}, \widetilde{M_1})$ such that the $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{\mathfrak{X}}, \widetilde{M_1})$ is isomorphic to $C_*(\widetilde{W}, \widetilde{M_1})$.

Since M_1 is a deformation retract of W, in the relative CW complex $(\widetilde{\mathfrak{X}}, \widetilde{M_1})$ we have that $\widetilde{M_1}$ is a deformation retract of $\widetilde{\mathfrak{X}}$ as well. Hence $\tau(\widetilde{\mathfrak{X}}, \widetilde{M_1})$ is well defined, and the torsion $\tau(W, M_1)$ is by definition equal to the torsion $\tau(\widetilde{\mathfrak{X}}, \widetilde{M_1})$.

Simple homotopy equivalence. When the map $f : E \to F$ is a homotopy equivalence between CW-complexes, then F is a deformation retract of the mapping cylinder

$$M_f = \big(X \times [0,1]\big) \coprod Y_{/(x,1)} \sim f(x)$$

of f.

We define the *Whitehead torsion of* f, denoted by $\tau(f) \in \mathrm{Wh}\big(\pi_1(Y)\big)$, to be the image of the torsion $\tau(M_f, Y) \in \mathrm{Wh}\big(\pi_1(M_f)\big)$ in $\mathrm{Wh}\big(\pi_1(Y)\big)$ under the isomorphism between $\mathrm{Wh}\big(\pi_1(M_f)\big) \xrightarrow{\cong} \mathrm{Wh}\big(\pi_1(Y)\big)$ induced by the isomorphism $\pi_1(M_f) \xrightarrow{\cong} \pi_1(Y)$.

This torsion is well defined, and when two cellular homotopy equivalences between two CW-complexes are homotopic the torsion are equal.

Definition 2.38. We say that a homotopy equivalence $f : X \to Y$ of finite CW-complexes is *simple* if the torsion $\tau(f)$ vanishes in $\mathrm{Wh}\big(\pi_1(Y)\big)$.

This definition extends to homotopy equivalences between smooth manifolds.

Remark 2.39. In the statement of the s-cobordism Theorem the inclusions $M_i \hookrightarrow W$ are simple homotopy equivalences. The letter s in s-cobordism refers to $s$imple homotopy equivalence.

Proof of the s-cobordism Theorem. To prove the s-cobordism Theorem we need some technical Lemmas. There exists many written proofs of these crucial Lemmas in the literature, see Lück [94] and Kervaire [71].

Let us fix some notations. In the following we will consider handle decompositions of a manifold W which has $M_1 \coprod M_2$ as boundary.

$$W = M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \dots \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n).$$

Then we will denote

$$W^\lambda = M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \dots \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda)$$

the manifold obtained from $M_1 \times [0,1]$ after the gluing of handles of indexes less or equal to λ, and

$$\hat{\partial} W_+^\lambda = \partial W^\lambda \setminus \Big(\coprod_{i=1}^{p_{\lambda+1}} \varphi_i^{\lambda+1} (S^\lambda \times \overset{\circ}{D}{}^{n-1-\lambda}) \coprod M \times \{0\} \Big)$$

the upper boundary of W^λ without the gluing sets of handles of index $\lambda + 1$.

Lemma 2.40. *Let W be an oriented compact n-manifold with $n \geq 6$ and ∂W is diffeomorphic to the disjoint union of two compact $(n-1)$-manifolds M_1 and M_2. Suppose that each component of ∂W is a deformation retract of W, then W is diffeomorphic to*

$$M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\varphi_i^2) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$$

relatively to M_1.

Proof. Let $M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$ be a handle decomposition of W. To prove this Lemma we have to show that we can remove the handle of indexes 0 and 1.

Recall that to add a 0-handle we make the disjoint union with a n-disk. But since W is connected there exists almost one 1-handle joining $M_1 \times [0,1]$ to this n-disk. Up to isotopy all the gluing sets of 1-handles, which are not in the 0-handles, are in $M_1 \times \{1\}$, hence the order of attaching 1-handles is not important. So if (φ_1^0) is the first 0-handle, one can assume that the 1-handle (φ_1^1) is joining $M_1 \times [0,1]$ to (φ_1^0). But the gluing of (φ_1^1) with (φ_1^0) is homeomorphic to a n-disk since we only attach one connected component of the boundary of the 1-handle to the 0-handle. These two handles (φ_1^1) and (φ_1^0) are canceling together, so we can remove the 0-handle (φ_1^0) and the 1-handle (φ_1^1). Finally one may assume that there is no 0-handle.

The handle decomposition of W became

$$M_1 \times [0,1] \bigcup_{i=1}^{p_1-p_0} (\varphi_i^1) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n).$$

Since $\hat{\partial} W_+^0$ consists only in $M \times \{1\}$ with $2\,p_1$ disks of dimension $(n-1)$ removed, then $\pi_1(\hat{\partial} W_+^0) = \pi_1(M \times \{1\})$. Moreover $M_1 \times \{1\}$ is a deformation retract of W, so $\pi_1(\hat{\partial} W_+^0)$ maps surjectively onto $\pi_1(W)$. Let

$$\phi_1^1 : D^1 \times D^{n-1} \to W^1$$

be the embedding of the 1-handle (φ_1^1) . Consider now $[\sigma] \in \pi_1(W^1)$ given by the homotopy class of

$$\sigma = \phi_1^1\big(D^1 \times \{0\}\big) \cup \gamma_{\varphi_1^1(S^0 \times \{0\})}$$

the gluing , along their boundary of the core the 1-handle and a path γ, which join in $\hat{\partial} W_+^0$ the two points of $\varphi_1^1(S^0 \times \{0\})$. By construction $[\sigma]$ is not equal to 0 in $\pi_1(W^1)$; but since $\pi_1(W) \cong \pi_1(M_1)$, then $[\sigma]$ must be 0 in $\pi_1(W)$. This means that σ is null-homotopic in W. Because of the dimensions, one can find some attaching maps $\{\varphi_i'^2\}_{i=1,\dots,p_2}$ isotopic to $\{\varphi_i^2\}_{i=1,\dots,p_2}$ such that for all $i = 1, \dots, p_2$ the images of $\varphi_i'^2$ do not meet the loop σ. Hence one can construct an embedding

$$\phi : S^1 \to \hat{\partial} W^1$$

such that

$$\big[\phi(S^1)\big] = [\sigma]$$

and $\phi(S^1)$ meets the transverse sphere of (φ_1^1) transversally in exactly one point. Since σ is null-homotopic in W, then ϕ is null-homotopic in W and in ∂W^2 as well. This means that the image of ϕ bounds an immersed 2-disk, and twice the dimension of this disk is strictly less than the dimension of ∂W^2, which is 5. According to Whitney's embedding Theorem, this homotopy can be realized with an embedding of a 2-disk in ∂W^2. This means that one can extend ϕ to an embedding $\Phi : S^1 \times D^{n-1} \to \partial W^2$ which is isotopic to a trivial embedding in ∂W^2. By construction Φ fulfills the hypothesis of Lemma 2.27, so we can eliminate the first 1-handle in the decomposition of W. By induction we get the desired decomposition

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\overline{\varphi}_i^2) \bigcup_{i=1}^{p_3'} (\overline{\varphi}_i^3) \dots \bigcup_{i=1}^{p_n} (\overline{\varphi}_i^n)$$

□

Remark 2.41. In the proof we strongly used the assumption $n \geq 6$ to smooth immersed disks to embedded disks.

As a consequence of this Lemma one can give a description of the $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{W}, \widetilde{M_1})$ in term of homotopy groups, see § 2.1.2 for the definition of this complex, where we have identified $M_1 \times \{0\}$ to M_1, the manifold $\widetilde{W}$ is the universal covering of W and $\pi = \pi_1(W)$.

First we fix a base point in $M_1 \times \{0\}$ and a lift of that point in $\rho^{-1}(W)$, all the homotopy groups will be considered with respect to these base points. Now we define the $\mathbf{Z}[\pi]$-chain complex

$$\pi_*(W^*, W^{*-1}) = \begin{cases} 0 & \text{if } q \leq 1, \\ \pi_q(W^q, W^{q-1}) & \text{if } q \geq 2. \end{cases}$$

The differentials are given by the composite maps

$$\pi_q(W^q, W^{q-1}) \overset{\partial_q}{\to} \pi_{q-1}(W^{q-1}) \overset{i_{q-1}}{\to} \pi_{q-1}(W^{q-1}, W^{q-2})$$

where ∂_q is a boundary operator, and i_{q-1} is induced by the inclusion.

For all $q \geq 1$ the group $\pi_1(\widetilde{W}^{q-1})$ is trivial, then the relative Hurewicz homeomorphism $\pi_q(\widetilde{W}^q, \widetilde{W}^{q-1}) \to \mathrm{H}_q(\widetilde{W}^q, \widetilde{W}^{q-1})$ is an isomorphism. Moreover the covering maps $\widetilde{\rho}_q : \widetilde{W}^q \to W^q$ induce the isomorphisms

$$\pi_q(\widetilde{W}^q, \widetilde{W}^{q-1}) \cong \pi_q(W^q, W^{q-1}).$$

Finally we get an isomorphism of $\mathbf{Z}[\pi]$-chain complexes

$$C_*(\widetilde{W}, \widetilde{M_1}) \cong \pi_*(W^*, W^{*-1}).$$

Each basis element $[\varphi_i^q] \in C_q(\widetilde{W}, \widetilde{M_1})$, associate with the attaching maps of the handles, can be considered as an element of $\pi_q(W^q, W^{q-1})$ with this isomorphism. It corresponds to the element given by the homotopy class of the mapping $\big(D^q \times \{0\}, \varphi^q(S^{q-1} \times \{0\}\big) \hookrightarrow (W^q, W^{q-1})$.

In the following Lemma we give conditions which ensure that the embedding of a sphere meets suitably the transverse spheres of a handle decomposition.

Lemma 2.42. *Let W be a compact n-manifold with $n \geq 6$ and ∂W is diffeomorphic to the disjoint union of two compact $(n-1)$-manifolds M_1 and M_2. Suppose that W is diffeomorphic to*

$$M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\varphi_i^2) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$$

relatively M_1.

Fix $\lambda \in \{1, \dots, n-3\}$ and $k \in \{1, \dots, p_\lambda\}$. Let $f : S^\lambda \to \hat{\partial} W_+^\lambda$ be an embedding. Then the following are equivalent

1. *There exists an embedding $g : S^\lambda \to \hat{\partial} W_+^\lambda$ isotopic to f which meets the transverse spheres of the λ-handle (φ_k^λ) transversally in exactly one point and is disjoint from the transverse spheres of the λ-handles $\big\{(\varphi_i^\lambda)\big\}_{i \neq k}$,*

2. *For any lift* $\widetilde{f} : S^\lambda \to \widetilde{W}^\lambda$ *of* f *under* $\widetilde{\rho}_{|\widetilde{W}^\lambda}$*; if* $[\widetilde{f}]$ *denotes the image of* f *under the composite map* $\pi_\lambda(\widetilde{W}^\lambda) \to \pi_\lambda(\widetilde{W}^\lambda, \widetilde{W}^{\lambda-1}) \to \mathrm{H}_\lambda(\widetilde{W}^\lambda, \widetilde{W}^{\lambda-1})$*, then there exists* $\gamma \in \pi$ *such that* $[\widetilde{f}] = \pm\gamma[\varphi_k^\lambda]$.

Proof. When the transversality conditions of the first statement are fulfilled, the second follows easily.

Let us explain the converse. Because of dimensions the image of f meets the set of transverse spheres of the λ-handles only in a finite number of points, set

$$\operatorname{Im} f \bigcap \Big\{\{0\} \times S_i^{n-\lambda-1}\Big\}_{i=1,\dots,p_\lambda} = \big\{x_{i,1}, \dots, x_{i,n_i}\big\}_{i=1,\dots,p_\lambda}.$$

Fix $* \in \operatorname{Im} f$ a base point in W, and in each transverse sphere $\{0\} \times S_i^{n-\lambda-1}$ fix a base point $*_i$, for $i = 1, \dots, p_\lambda$, such that $*_i \notin \big\{x_{i,1}, \dots, x_{i,n_i}\big\}_{i=1,\dots,p_\lambda}$.

Now let

$$c_{i,j} : [0,1] \to S^\lambda$$

be a path such that for all $(i,j) \in \{1, \dots, p_\lambda\} \times \{1, \dots, n_i\}$ we have $f \circ c_{i,j}(0) = *$ and $f \circ c_{i,j}(1) = x_{i,j}$. Let

$$b_{i,j} : [0,1] \to W^\lambda$$

be a path such that for all $(i,j) \in \{1, \dots, p_\lambda\} \times \{1, \dots, n_i\}$ we have $b_{i,j}(0) = x_{i,j}$ and $b_{i,j}(1) = *_i$. And let

$$a_i : [0,1] \to W^\lambda$$

be a path such that for all $i \in \{1, \dots, p_\lambda\}$ we have $a_i(0) = *_i$ and $a_i(1) = *$.

Now let $l_{i,j}$ a loop base in $*$, which is the composite path of $f(c_{i,j})$, $b_{i,j}$ and a_i. if we denote by $\gamma_{i,j}$ the homotopy class of $l_{i,j}$ in $\pi = \pi_1(W, *)$, then we have

$$[\widetilde{f}] = \sum_{i=1}^{p_\lambda} \sum_{j=1}^{n_i} \epsilon_{i,j}\, \gamma_{i,j}\, [\varphi_i^\lambda]$$

where $\epsilon_{i,j} = \pm 1$.

We assume that there exists $\gamma \in \mathbf{Z}[\pi]$ such that $[\widetilde{f}] = \pm\gamma[\varphi_k^\lambda]$, but since the set $\big\{[\varphi_i^\lambda]\big\}_{i=1,\dots,p_\lambda}$ is a basis of $\mathrm{H}_\lambda(\widetilde{W}^\lambda, \widetilde{W}^{\lambda-1})$ then, for $i \neq k$, we can associate the elements of $\big\{x_{i,1}, \dots, x_{i,n_i}\big\}_{i=1,\dots,p_\lambda}$ by pairs such that for each pair, say (x_{i,j_1}, x_{i,j_2}), we have $\epsilon_{i,j_1}\, \epsilon_{i,j_2} = -1$. This means that the loop, which is the composite path of $f(c_{i,j_1})$, b_{i,j_1}, the inverse of b_{i,j_2} and the inverse of $f(c_{i,j_2})$ is null-homotopic in $\hat{\partial} W_+^\lambda$.

Now, since $n \geq 6$, then one can apply the Whitney trick (see [162]) to modify f with an isotopy, and get new embedding of S^λ in $\hat{\partial} W_+^\lambda$ with the two intersection points x_{i,j_1} and x_{i,j_2} removed and no change to the other intersection points with the transverse spheres.

By induction we get the first statement with $\gamma = \pm \sum_{j=1}^{n_k} \epsilon_{k,j}\, \gamma_{k,j}$. □

Lemma 2.43. *Let* $f : S^\lambda \to \hat{\partial} W_+^\lambda$ *be an embedding , and let* $\{x_j\}_{j=1,\dots,p_{\lambda+1}}$ *be a set of elements of* $\mathbf{Z}[\pi]$.

An embedding $g : S^\lambda \to \hat{\partial} W_+^\lambda$ *is isotopic to* f *if and only if to each lift* $\widetilde{f} : S^\lambda \to \widetilde{W}^\lambda$ *of* f *under* $\widetilde{\rho}_{|\widetilde{W}^\lambda}$ *one can find a lift* $\widetilde{g} : S^\lambda \to \widetilde{W}^\lambda$ *of* g *such that in* $\mathrm{H}_\lambda(\widetilde{W}^\lambda, W^{\lambda-1})$ *we have*

$$[\widetilde{g}] = [\widetilde{f}] + \sum_{j=1}^{p_{\lambda+1}} x_j \, d_{\lambda+1}[\varphi_j^{\lambda+1}]$$

where $d_{\lambda+1}$ *is the* $(\lambda+1)$*-differential of the complex* $C_*(\widetilde{W}, \widetilde{M}_1)$*.*

This Lemma is more or less proved in Smale's work [142], for a proof see [71] or [94].

Lemma 2.44. *Let* W *be an oriented compact* n*-manifold with* $n \geq 6$ *and* ∂W *is diffeomorphic to the disjoint union of two compact* $(n-1)$*-manifolds* M_1 *and* M_2*. Suppose that each component of* ∂W *is a deformation retract of* W*, then for any* $\lambda \in \{2, \ldots, n-3\}$ *there exists a handlebody decomposition of* W *of the form*

$$M_1 \times [0,1] \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}).$$

Proof. We saw that handles of indexes 0 and 1 can be removed so we start with a handle decomposition for W of the form

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\varphi_i^2) \ldots \bigcup_{i=1}^{p_n} (\varphi_i^n).$$

Now we will show that we can decrease p_q by one provided that $p_r = 0$ for $r \leq q-1$ and $q \leq n-3$.

Start with a decomposition

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_q} (\varphi_i^q) \ldots \bigcup_{i=1}^{p_n} (\varphi_i^n).$$

As done before the trick is to attach a new $(q+1)$-handle, which cancel with (φ_1^q), and a new $(q+2)$-handle such that the two new handles cancel together. To do that we will use Lemma 2.27.

Let

$$\Psi^{q+1} : S^{q+1} \times D^{n-q-1} \to \hat{\partial} W_+^q$$

be an embedding such that its image is included in a n-disk $D^n \subset \hat{\partial} W_+^q$.

Since the inclusion $M_1 \hookrightarrow W$ is a homotopy equivalence, then the $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{W}, \widetilde{M}_1)$ is acyclic. But we assume that there is no k-handle with $k \leq q-1$ in the handle decomposition for W, hence the the $\mathbf{Z}[\pi]$-module $C_{q-1}(\widetilde{W}, \widetilde{M}_1) = \mathrm{H}_{q-1}(\widetilde{W}^{q-1}, \widetilde{W}^{q-2})$ is trivial. So the $(q+1)$-differential of the complex $C_*(\widetilde{W}, \widetilde{M}_1)$, namely $d_{q+1} : C_{q+1}(\widetilde{W}, \widetilde{M}_1) \to C_q(\widetilde{W}, \widetilde{M}_1)$, is surjective. This implies that there exists a set $\{x_k\}_{i=1,\ldots,p_{q+1}}$ of elements in $\mathbf{Z}[\pi]$, such that

$$\mathrm{H}_q(\widetilde{W}^q, \widetilde{W}^{q-1}) \ni [\varphi_1^q] = \sum_{i=1}^{p_{q+1}} x_i \, d_{q+1}\big([\varphi_i^{q+1}]\big).$$

According to Lemma 2.43, one can find an embedding

$$\psi^{q+1} : S^{q+1} \times D^{n-q-1} \to \hat{\partial} W_+^{q+1},$$

which is isotopic to Ψ^{q+1} in $\hat{\partial} W_+^{q+1}$, such that

$$[\psi^{q+1}_{|S^q \times \{0\}}] = [\Psi^{q+1}_{|S^q \times \{0\}}] + \sum_{i=1}^{p_{q+1}} x_i \, d_{q+1}\big([\varphi_i^{q+1}]\big).$$

But $[\psi^{q+1}_{|S^q \times \{0\}}] = [\varphi_1^q]$ since $[\Psi^{q+1}_{|S^q \times \{0\}}]$ is null-homotopic in $\hat{\partial} W_+^{q+1}$. Moreover, according to Lemma 2.42 the embedding $\psi^{q+1}_{|S^q \times \{0\}}$ is isotopic in $\hat{\partial} W_+^{q+1}$ to an embedding $S^q \to \hat{\partial} W_+^{q+1}$ which meets the transverse sphere of (φ_1^q) transversally exactly in one point and do not meet the transverse spheres of the other q-handles.

We can apply Lemma 2.27 to find a new handle decomposition

$$W \cong M_1 \times [0,1] \bigcup_{i=2}^{p_q} (\varphi_i^q) \bigcup_{i=2}^{p_{q+1}} (\overline{\varphi}_i^{q+1}) \cup (\psi^{q+2}) \bigcup_{i=2}^{p_{q+2}} (\overline{\varphi}_i^{q+2}) \ldots \bigcup_{i=1}^{p_n} (\overline{\varphi}_i^n),$$

and the number of q-handle decreased by one. By induction we can remove all q-handles.

Now using the dual handle decomposition for W, i.e., the handle decomposition associated with the Morse function $-f$ instead of f which start with $M_2 \times [0,1]$; we have the following decomposition

$$W \cong M_2 \times [0,1] \bigcup_{i=1}^{p_0} (\phi_i^n) \ldots \bigcup_{i=1}^{p_\lambda} (\phi_i^{n-\lambda}) \ldots \bigcup_{i=1}^{p_n} (\phi_i^0).$$

As just explained before one can remove handles of indexes less or equal to $n - \lambda - 2$ in this decomposition, and

$$W \cong M_2 \times [0,1] \bigcup_{i=1}^{p_0} (\phi_i^n) \ldots \bigcup_{i=1}^{p_{\lambda+1}} (\phi_i^{n-\lambda-1}).$$

If we take again the dual handle decomposition of the last one, then one can find a handle decomposition for W of the form

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\Phi_i^0) \ldots \bigcup_{i=1}^{p_{\lambda+1}} (\Phi_i^{\lambda+1}).$$

Now we remove all handles of indexes less or equal to $\lambda - 1$ in the last decomposition and we get the desired result

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}).$$

□

We are ready to finish the proof of the s-cobordism Theorem.

Proof of s-cobordism Theorem. With the previous Lemma 2.44 we can assume that the manifold W admits a handle decomposition of the form

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p} (\varphi_i^\lambda) \bigcup_{i=1}^{p} (\varphi_i^{\lambda+1}).$$

The number of handle is the same since we assume that both M_1 and M_2 are deformation retracts of W.

The acyclic $\mathbf{Z}[\pi]$-chain complex $C_*(\widetilde{W}, \widetilde{M}_1)$ has only one differential which is non zero, namely

$$d_{\lambda+1} : \mathrm{H}_{\lambda+1}(\widetilde{W}_{\lambda+1}, \widetilde{W}_\lambda) \to \mathrm{H}_\lambda(\widetilde{W}_\lambda, \widetilde{W}_{\lambda-1}).$$

Let D be the matrix of the isomorphism $d_{\lambda+1}$ with respect to the basis $\{[\varphi_i^{\lambda+1}]\}_{i=1,\dots,p}$ of $C_{\lambda+1}(\widetilde{W}, \widetilde{M}_1) = \mathrm{H}_{\lambda+1}(\widetilde{W}_{\lambda+1}, \widetilde{W}_\lambda)$ and the basis $\{[\varphi_i^\lambda]\}_{i=1,\dots,p}$ of $C_\lambda(\widetilde{W}, \widetilde{M}_1) = \mathrm{H}_\lambda(\widetilde{W}_\lambda, \widetilde{W}_{\lambda-1})$. The entries $d_{i,j} \in \mathbf{Z}[\pi]$ of the matrix D, for $i,j = 1,\dots,p$, are defined by the equations

$$d_{\lambda+1}\big([\varphi_i^{\lambda+1}]\big) = \sum_{j=1}^{p} d_{i,j}\,[\varphi_j^\lambda].$$

By definition, the Whitehead torsion $\tau(W, M_1)$ is given by the class of the matrix D in $\mathrm{Wh}(\pi)$.

Let us give the geometrical interpretation of the four elementary operations which generate the Whitehead group described in Definition 2.34 and Proposition 2.36, when these operations are made on the matrix D we just defined.

1. The multiplication of the k-th row of D by $\pm\gamma$ with $\gamma \in \mathbf{Z}[\pi]$ correspond to the modification of the lift in $\widetilde{W}^{\lambda+1}$ of φ_k^λ. But according to Lemma 2.43 this corresponds to the gluing of a new λ-handle $(\varphi_k'^\lambda)$ instead of (φ_k^λ). The resulting manifold is diffeomorphic to W.

2. Similarly to the previous item, the addition to the k-th row of the j-th row of D can be realized by the gluing of a new λ-handle which is isotopic to (φ_k^λ).

3. This operation corresponds to the gluing of a new λ-handle (ψ^λ) and a new $(\lambda+1)$-handle $(\psi^{\lambda+1})$ in a n-disk of $\hat{\partial} W_+^{\lambda+1}$, such that these handles are canceling together.

4. This operation is the converse of the previous one, when we do it we just remove to handles, which are canceling together, from the handle decomposition of W.

Since all of the modifications on the matrix D correspond to modifications of the handle decomposition of W which do not change W up to diffeomorphism, then we see that the Whitehead torsion $\tau(W, M_1)$ vanishes if and only if W admits a handle decomposition in which all handles can be removed, and then $W \cong M_1 \times [0,1]$. □

Proposition 2.45. *The s-cobordism Theorem implies the h-cobordism Theorem.*

Proof. Recall that any invertible matrix over the integers can be reduced by elementary operations to the identity matrix. So all the matrices in $GL(\mathbf{Z})$ are equivalent in $\mathrm{Wh}(\mathbf{Z})$ which is trivial. When the manifolds M_1 and M_2 are simply connected, then $\mathrm{Wh}(\pi) = \{0\}$ and the s-cobordism Theorem implies the h-cobordism Theorem. □

2.2.2 The relative case

The notion of *relative h-cobordism* was introduced by Heafliger [51].

Definition 2.46. Two pairs (M_1, V_1) and (M_2, V_2) of manifolds with $V_i \subset M_i$ for $i = 1, 2$ are *h-cobordant* if there exists a pair of manifold (M, V) with $V \subset M$ such that $\partial M = M_1 - M_2$, $\partial V = V_1 - V_2$ and the inclusion $M_i \hookrightarrow M$, $V_i \hookrightarrow V$ are homotopy equivalences for $i = 1, 2$.

Then the h-cobordism and s-cobordism theorems can be extended to the relative case.

2.3 Stabilized *h*-cobordism, and *h*-cobordism Theorem for 3-manifolds

During the proof, we saw that the h-cobordism Theorem is valid when the dimensions of manifolds are greater or equal to five. In this section we present the stabilized h-cobordism Theorem for four dimensional manifolds.

First recall that in [156] Wall proved that if (W, M, M') is an h-cobordism between closed, simply connected 4-manifolds M and M', then $M\#\big(\#_k S^2 \times S^2\big)$ is diffeomorphic to $M'\#\big(\#_k S^2 \times S^2\big)$ for some positive integer k.

Then in [83] Lawson extended the proof to 4-manifolds which are not simply connected.

Theorem 2.47. *Suppose $(W; M_+, M_-)$ is a smooth s-cobordism whose boundary cobordism from ∂M_+ to ∂M_- has a product structure. Then for some integer k the k-fold stabilization of W by connected sum along an arc with $(S^2 \times S^2) \times [0, 1]$ has a product structure extending the one given on the boundary.*

2.4 Surgery on manifolds

In this section we describe modifications on manifolds called *surgeries.* We introduce them now since they are related to handle gluing. When we give handlebody decompositions of manifolds we attach handles in order to give some descriptions of the manifolds, but when we do surgeries we attach handles to kill some homology classes.

Start with a n-manifold X, and let $\psi : S^k \times D^{n-k} \to X$ be an embedding for $0 < k < n$. Set Y be the manifold obtained from X as follows

$$Y = X \setminus \big(\psi(S^k \times D^{n-k}\big) \cup_\partial \big(D^{k+1} \times S^{n-k-1}\big),$$

where the gluing is given by the identification of boundaries.

Definition 2.48. We say that Y is obtained from X after a *surgery* on $\psi(S^k)$.

When the manifold X is embedded in a manifold W, if there exists an embedding

$$\phi : D^{k+1} \times S^{n-k-1} \to \big(W \setminus X\big) \cup \big(\psi(S^k \times S^{n-k-1})\big)$$

such that $\phi(S^k \times S^{n-k-1}) = \psi(S^k \times S^{n-k-1})$, then we say that the manifold

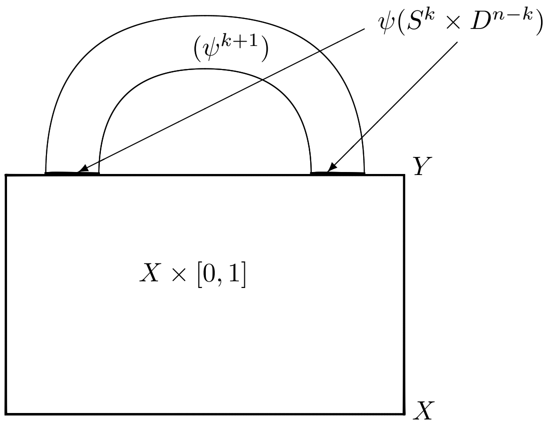
$$Y = X \setminus \big(\psi(S^k \times D^{n-k}\big) \cup_\partial \phi(D^{k+1} \times S^{n-k-1}),$$

is obtained from X after an *embedded surgery* on $\psi(S^k)$.

In fact surgeries can be described with handles gluing. The manifold Y constructed by surgery on $\psi(S^k)$ can be viewed as the upper boundary of

$$X \times [0,1] \cup (\psi^{k+1}),$$

as depicted bellow



Modifications of manifolds with surgeries change homology groups. More precisely a surgery on $\psi(S^k)$ in a manifold X gives a manifold Y in which the homology class of $\psi(S^k)$ is zero. So if $\psi(S^k)$ is a n-dimensional chain which represents a non trivial homology class in X, then the rank of the k^{th} homology group of Y may not be equal to those of X. Moreover if $\psi(S^k)$ is a n-dimensional chain which represents a trivial homology class in X then a surgery on $\psi(S^k)$ must add some homology class of dimension not equal to n.

Anyway, using Mayer-Vietoris exact sequences associated with decomposition of manifolds like

$$X \setminus \big(\psi(S^k \times D^{n-k}\big) \cup_\partial \big(D^{k+1} \times S^{n-k-1}\big),$$

one can compute exactly how a surgery modifies the homology of X.

As a reference we cite [154]

We will combine surgeries and h-cobordism Theorem to construct cobordism of knots. More precisely, to prove that two knots K_0 and K_1 are cobordant, we need to find a manifold X such that $\partial X = K_0 \coprod K_1$ and $X \cong K_0 \times [0,1]$. A way to do that is to start with a manifold Z such that $\partial Z = K_0 \coprod K_1$ and do some surgeries on Z to get a manifold X with $\mathrm{H}_*(X, K_0) = 0$ and then apply the h-cobordism Theorem to get $X \cong K_0 \times [0,1]$.

Chapter 3

Spherical knots

"Die Mathematiker sind eine Art Franzosen:
Redet man zu ihnen, so übersetzen sie es in ihre
Sprache, und dann ist es alsbald etwas anderes."
J.W. von Goethe,
- Maximen und Reflexionen

In this chapter, we consider the case of spherical knots. In the sixties, Kervaire and Levine gave classifications of spherical knots up to cobordism, we will recall some of their results in the following.

Unless specified all knots in this chapter are simple spherical $(2n-1)$-knots.

3.1 S-equivalence

The Seifert form is the main tool to study cobordisms of odd dimensional spherical knots. Since spherical knots are not in general fibered, then there exists many distinct Seifert manifolds for a given spherical knot. Before going further, the first step is to know what happen on Seifert forms when we change the Seifert manifolds associated with a spherical knot. In [91] Levine described the possible modifications on Seifert forms of a spherical simple knot corresponding to alterations of Seifert manifolds.

For a given $(2n-1)$-knot K embedded in S^{2n+1} let us consider two Seifert manifolds V_1 and V_2 associated with K. One can suppose that $V_i \times \{i\}$ is embedded in $S^{2n+1} \times \{i\} \hookrightarrow S^{2n+1} \times [0,1]$ for $i = 0,1$. We denote by $\mathfrak{A}_i$ the Seifert form associated with V_i, and by $\mathfrak{S}_i = \mathfrak{A}_i + (-1)^n\, {}^t\mathfrak{A}_i$ the intersection form associated with V_i for $i = 0,1$.

Recall that intersection forms of spherical knots are unimodular.

With similar arguments as those used to prove that every knot bounds an embedded Seifert manifold, one can see that there is no obstruction to construct an embedded submanifold W of $S^{2n+1} \times [0,1]$ such that

$$\partial W = V_0 \cup K \times [0,1] \cup V_1.$$

Then the handle decomposition associated with a Morse function

$$f : W \to [0,1]$$

shows that V_0 and V_1 are related each other by embedded surgeries.

Remark 3.1. The manifold W is very useful to construct submodules on which the Seifert forms vanish. More precisely, we will se that when two n-cycle α and β in $\mathrm{H}(V_0) \oplus \mathrm{H}(V_1)$ are null-homologous in $\mathrm{H}_n(W)$ then

$$\big(A_0 \oplus -A_1\big)(x, y) = 0.$$

To prove this equality, remark that the positive direction of the normal bundle of $V_0 \coprod V_1$ in S^{2n+1} extend to a positive direction of the normal bundle of W in $S^{2n+1} \times [0,1]$. Set ξ and η some $(n+1)$-chains in W such that $[\partial\xi] = x$ and $[\partial\eta] = y$ and ξ_{+W} the chain ξ pushed out W in the positive normal direction in $S^{2n+1} \times [0,1]$. Since the two chains ξ_{+W} and η do not intersect together, then

$$\begin{aligned} \big(A_0 \oplus -A_1\big)(x,y) &= l_{S^{2n+1}}\big((\partial\xi)_+, (\partial\eta)\big), \\ \big(A_0 \oplus -A_1\big)(x,y) &= I_{S^{2n+1}\times[0,1]}(\xi_{+W}, \eta), \\ \big(A_0 \oplus -A_1\big)(x,y) &= 0. \end{aligned}$$

We will now study the manifold W. When the critical points of f are not of index n nor $n+1$ then the associated surgeries on $V_0 \times [0,1]$ do not affect n-homology hence the groups $\mathrm{H}_n(V_0)$ and $\mathrm{H}_n(V_1)$ are isomorphic ; consequently the Seifert forms associated with V_0 and V_1 are the same.

Since the critical points of f are isolated, then it suffices to consider the case where f has only one critical point. Moreover, if f has a critical point of index $n+1$, then it is a critical point of index n for the Morse function $\widetilde{f}(x) = 1 - f(x)$. So we can assume that f has only one critical point of index n. The corresponding surgery means that we attach a n-handle to the upper boundary of a collar neighborhood of V_0. More precisely, we first remove $D^{n+1} \times S^{n-1}$ and then glue $S^n \times D^n$ along the new boundary.

Elementary computations with Mayer-Vietoris sequences give

$$\mathrm{H}_n(W, V_0) \cong \mathrm{H}_{n+1}(W, V_1) \cong \mathbf{Z}$$

and

$$\mathrm{H}_n(W, V_1) \cong \mathrm{H}_{n+1}(W, V_0) = 0.$$

Let a be the image in $\mathrm{H}_n(V_1)$ of the generator of $\mathrm{H}_{n+1}(W, V_1)$, which is given by the homology class of the core the handle we attached.

- If a has a finite order, then Seifert forms associated with V_0 and V_1 are isomorphic since they are defined modulo torsion.
- If a has infinite order, then it is a multiple of a primitive element a_0 of $\mathrm{H}_n(V_1)$. Since the intersection forms of spherical knots are unimodular, then there exists b_0 in $\mathrm{H}_n(V_1)$ such that $\mathfrak{S}_1(a_0, b_0) = 1$. Moreover

  $$\operatorname{rank}\big(\mathrm{H}_n(V_1)\big) = \operatorname{rank}\big(\mathrm{H}_n(V_0)\big) + 2$$

  and we can take $(c_1, \ldots, c_k)$ in $\mathrm{H}_n(V_1)$ such that $(a_0, b_0, c_1, \ldots, c_k)$ is a basis of $\mathrm{H}_n(V_1)$ and $(c_1, \ldots, c_k)$ are homologous to a basis $(d_1, \ldots, d_k)$ of $\mathrm{H}_n(V_0)$.

  There exists a $(n+1)$-chain Γ_i in W such that $\partial\Gamma$ is a n-chain which represent the cycle $d_i - c_i$ for $i = 1, \ldots, k$. Then for all i, j in $\{1, \ldots, k\}$ we have $\mathfrak{A}_0(d_i, d_j) - \mathfrak{A}_1(c_i, c_j)$ is the intersection number of Γ_j and the translate of Γ_i off W in the positive normal direction of W in $S^{2n+1} \times [0,1]$. Since this intersection number is zero then for all i, j in $\{1, \ldots, k\}$ we have

  $$\mathfrak{A}_0(d_i, d_j) = \mathfrak{A}_1(c_i, c_j).$$

  By definition a is null-homologous in W, hence we get $\mathfrak{A}_1(a, c_i) = \mathfrak{A}_1(c_i, a) = \mathfrak{A}_1(a, a) = 0$ for $i = 1, \ldots, k$. Thus we have the following equalities

  $$\mathfrak{A}_1(a_0, c_i) = \mathfrak{A}_1(c_i, a_0) = \mathfrak{A}_1(a_0, a_0) = 0.$$

If A_0 (resp. A_1) is the matrix of $\mathfrak{A}_0$ (resp. $\mathfrak{A}_1$) with respect to the basis $(d_1, \dots, d_k)$ (resp. $(c_1, \dots, c_k, a_0, b_0)$), then

$$A_1 = \begin{pmatrix} A_0 & \mathcal{O} & \nu \\ {}^t\mathcal{O} & 0 & w \\ {}^t\mu & z & v \end{pmatrix},$$

where $\mathcal{O}$ is a column vector whose entries are all 0, and ν, μ are column vector of integers.

Since $\mathfrak{S}_1(a_0, b_0) = 1$ then we have $w + (-1)^n z = 1$. Recall that the Alexander polynomial of K is well defined up to a unit in $\mathbf{Z}[t, t^{-1}]$. If we denote by $\Delta_{A_i}(t)$ the Alexander polynomial associated with $\mathfrak{A}_i$ for $i = 0, 1$, then

$$\Delta_{A_1}(t) = (tw + (-1)^n z)(tz + (-1)^n w)\Delta_{A_0}(t).$$

So w or z must be 0, if $w = 0$ then one can modify the vectors of the basis $(c_1, \dots, c_k, a_0, b_0)$ to get

$$A_1 = \begin{pmatrix} A_0 & \mathcal{O} & \mathcal{O} \\ {}^t\mu' & 0 & 0 \\ {}^t\mathcal{O} & 1 & 0 \end{pmatrix}.$$

Consequently we define the *enlargement* A' of a square integral matrix A as follows

$$A' = \begin{pmatrix} A & \mathcal{O} & \mathcal{O} \\ {}^t\alpha & 0 & 0 \\ {}^t\mathcal{O} & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} A & \beta & \mathcal{O} \\ {}^t\mathcal{O} & 0 & 1 \\ {}^t\mathcal{O} & 0 & 0 \end{pmatrix},$$

where $\mathcal{O}$ is a column vector whose entries are all 0, and, α and β are column vectors of integers. In this case, we also call A a *reduction* of A'.

Definition 3.2. Two square integral matrices are said to be *S-equivalent* if they are related each other by enlargement and reduction operations together with the congruence. We also say that two integral bilinear forms defined on free $\mathbf{Z}$-modules of finite rank are *S-equivalent* if so are their matrix representatives.

This equivalence relation characterize isotopy classes of spherical simple $(2n-1)$-knots with $n \geq 2$ as stated in the following Theorem proved by Levine [91].

Theorem 3.3 ([91]). *For $n \geq 2$, two spherical simple $(2n-1)$-knots are isotopic if and only if they have S-equivalent Seifert forms.*

We will need the two following Lemmas for the proof.

Lemma 3.4. *Let K be a simple spherical $(2n-1)$-knot, and let A be a Seifert matrix associated with a $(n-1)$-connected Seifert manifold for K. If B is an enlargement of A then B is a Seifert matrix associated with a $(n-1)$-connected Seifert manifold for K as well.*

Proof. This result is a direct consequence of Alexander duality (see [16]). □

Lemma 3.5. *If $n \geq 2$, then two simple spherical $(2n-1)$-knots admitting identical Seifert matrices, associated with $(n-1)$-connected Seifert manifolds for K, are isotopic.*

We refer to [91] p.191 for a proof of this Lemma, which is based on handle decompositions for Seifert manifolds. Though the result is valid for all $n \geq 2$, we have to mention that special arguments must be used when $n = 2$.

When K is a spherical $(2n-1)$-knot with Seifert manifold F, then the long exact sequence in homology of (F, K) induces the exact short sequence

$$0 \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F, K) \to 0.$$

Moreover when K is simple then $\mathrm{H}_n(F, K)$ is isomorphic to $\mathrm{Hom}_{\mathbf{Z}}\big(H_n(F), \mathbf{Z}\big)$; and if we equip $\mathrm{H}_n(F, K)$ with the dual basis of the one choosed for $\mathrm{H}_n(F)$ then the matrix of S_* is $A + (-1)^n\, {}^tA$, where A is the Seifert matrix associated with F. So in that case we have $\det\big(A + (-1)^n\, {}^tA\big) = \pm 1$. The converse is also true as stated bellow.

Proposition 3.6 ([91]). *Let n be an integer greater or equal to 2, and let A be an integral square matrix such that $A + (-1)^n\, {}^tA$ is unimodular. If $n \neq 2$, there exists a simple spherical $(2n-1)$-knot with Seifert matrix A; if $n = 2$, there exists a simple spherical 3-knot with Seifert matrix S-equivalent to A.*

Proof of Theorem 3.3. First suppose that two simple spherical $(2n-1)$-knots K_0 and K_1 are isotopic, then using the same argument to compute modifications on Seifert forms corresponding to alterations of Seifert manifolds, we see that their Seifert forms are S-equivalent.

For the converse, start with two simple spherical $(2n-1)$-knots, denoted by K and K', with S-equivalent Seifert forms. Then there exists a finite sequence of matrices $A_1, \ldots, A_k$ such that $A_1 = A$ is a Seifert matrix for K, $A_k = A'$ is a Seifert matrix for K' and for all $i = 1, \ldots, k-1$ the matrix A_{i+1} is an enlargement or a reduction of A_i up to congruence.

Now it is easy to see that K_i and K_{i+1} are isotopic. One can suppose that A_{i+1} is an enlargement of A_i (if necessary we exchange K_i and K_{i+1}), then according to Lemma 3.4 A_{i+1} is a Seifert matrix associated with K_i. But this implies that K_i and K_{i+1} admit the same Seifert matrix associated with simple Seifert manifolds, hence they are isotopic by Lemma 3.5 □

3.2 Cobordism of spherical knots

Let us denote by C_n the set of cobordism classes of spherical n-knots, and by $\widetilde{C}_n$ the set of concordance classes of spherical n-knots. These two sets have a natural group structure. The group operation is given by the connected sum see [72] Chapter III for details.

We say that an n-knot $K \subset S^{n+2}$ is *null-cobordant* if it is cobordant to the trivial knot, i.e., if there exists an $(n+1)$-disk D^{n+1} properly embedded in the $(n+3)$-disk D^{n+3} such that $\partial D^{n+1} = K \subset S^{n+2} = \partial D^{n+3}$. Similarly we define the notion of *null-concordant* knot.

The neutral element of C_n is the class of null-cobordant n-knots, and the neutral element of $\widetilde{C}_n$ is the class of null-concordant n-knot.

To construct the inverse of a n-knot K one can suppose that K is embedded in the upper hemisphere S^{n+2}_+ of the unit $(n+2)$-sphere $\partial D^{n+3} = S^{n+2} \hookrightarrow \mathbf{R}^{n+3}$. Let ρ be the reflection in the equatorial hyperplane $\mathcal{E}$ of D^{n+3}, and $\pi : \mathbf{R}^{n+3} \to \mathcal{E}$ the projection onto $\mathcal{E}$.

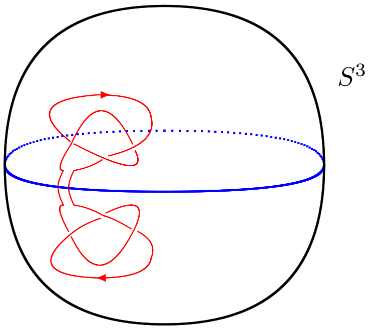


Figure 3.1. The connected sum of trefoil knot and its inverse in C_1

Then we construct the connected sum $K' = K\#\rho(K)$ of K and $\rho(K)$ in S^{2n+1}; we illustrate this construction in Fig. 3.1 when K is the trefoil knot embedded in S^3. Moreover, one can suppose that this connected sum is made in order to have $\pi(K') = \pi(K' \cap S_+^{2n+1})$, where S_+^{2n+1} is the upper hemisphere of S^{2n+1} which contains K.

Then, set $\mathcal{D} = \big(\pi(K') \times [0,1]\big) \cap D^{n+3}$, remark that since $\pi(K')$ is a $(2n-1)$-disk, then $\mathcal{D}$ is homeomorphic to a $(n+1)$-disk ; moreover $\partial\mathcal{D} = K' = K\#\rho(K)$.

Since $K'\#\rho(K')$ bounds a $(n+1)$-disk embedded in D^{n+3} then $K\#\rho(K)$ is null cobordant and $\rho(K)$ is the inverse of K. We have just proved that the inverse of K is given by its mirror image with reversed orientation, which we denote by $-K^!$.

Similarly we can construct the inverse of a knot class in the concordance groups $\widetilde{C}_n$.

First, let us focus on the case of spherical $(2n-1)$-knots. Kervaire and Levine used the notion of *Witt equivalence* for integral bilinear forms.

Witt equivalence of integral bilinear forms

Definition 3.7. Let $\mathfrak{A} : G \times G \to \mathbf{Z}$ be an integral bilinear form defined on a free $\mathbf{Z}$-module G of finite rank. The form $\mathfrak{A}$ is said to be *Witt associated to* $\mathfrak{o}$ if the rank m of G is even and there exists a submodule M of rank $m/2$ in G such that M is a direct summand of G and $\mathfrak{A}$ vanishes on M. Such a submodule M is called a *metabolizer* for $\mathfrak{A}$.

The following theorem was proved by Levine [89] and Kervaire [73].

Theorem 3.8. *For $n \geq 2$, a spherical $(2n-1)$-knot is null-cobordant if and only if its Seifert form is Witt associated to $\mathfrak{o}$.*

We will only give some idea of the proof.

Proof. To prove that the condition on Seifert forms is necessary M. Kervaire constructed a metabolizer associated with Seifert forms of null-cobordant spherical knots. Since we use the same construction in the case on non-spherical knots, we refer to the proof of Theorem 6.5.

To prove that null-cobordant spherical knots have Seifert forms Witt associated to $\mathfrak{o}$, M. Kervaire proved that it is possible to do embedded surgeries on a basis of the metabolizer in order to get an embedded disk with boundary the spherical knot. $\square$

For two spherical $(2n-1)$-knots K_0 and K_1 with Seifert forms $\mathfrak{A}_0$ and $\mathfrak{A}_1$ respectively, the oriented connected sum $K = K_0 \sharp (-K_1^!)$ has $\mathfrak{A} = \mathfrak{A}_0 \oplus (-\mathfrak{A}_1)$ as the Seifert form associated with the oriented connected sum along the boundaries of the Seifert manifolds associated with K_0 and $-K_1^!$, where $-K_1^!$ denotes the mirror image of K_1 with reversed orientation. Hence, as a consequence of Theorem 3.8, we have that two spherical knots K_0 and K_1 are cobordant if and only if the form $\mathfrak{A} = \mathfrak{A}_0 \oplus (-\mathfrak{A}_1)$ is Witt associated to $\mathfrak{o}$. In this case we sometimes say that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are *Witt equivalent*.

Remark 3.9. Witt equivalence is not an equivalence relation on the set of integral bilinear forms of finite rank. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two integral bilinear forms of rank r such that $\mathfrak{A} \oplus -\mathfrak{B}$ is not Witt associated to $\mathfrak{o}$. If we denote by $\mathfrak{O}_r$ the zero form of rank r, then both $\mathfrak{A}$ and $\mathfrak{B}$ are Witt equivalent to $\mathfrak{o}_r$ but $\mathfrak{A}$ and $\mathfrak{B}$ are not Witt equivalent.

For $\varepsilon = \pm 1$, let $C^\varepsilon(\mathbf{Z})$ be the set of all Witt equivalence classes of integral bilinear forms $\mathfrak{A}$ defined on free $\mathbf{Z}$-modules of finite rank such that $\mathfrak{A} + \varepsilon\, {}^t\mathfrak{A}$ is unimodular (for the notation, we follow [73]).

It can be shown that $C^\varepsilon(\mathbf{Z})$ has a natural abelian group structure, where the addition is defined by the direct sum. Then we have the following.

Theorem 3.10 (Levine [89]). *Let $\Phi_n : C_{2n-1} \to C^{(-1)^n}(\mathbf{Z})$ be the (well-defined) homomorphism induced by the Seifert form. Then Φ_n is an isomorphism for $n \geq 3$. But Φ_2 is only a monomorphism whose image $C^{+1}(\mathbf{Z})^0$ is a specified subgroup of $C^{+1}(\mathbf{Z})$ of index 2; and $\Phi_1 : C_1 \to C^{-1}(\mathbf{Z})$ is merely an epimorphism.*

Furthermore, Levine [90] showed the following (see also Remark 6.30).

Theorem 3.11. *For $\varepsilon = \pm 1$, we have*

$$C^\varepsilon(\mathbf{Z}) \cong \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty \oplus \mathbf{Z}^\infty, \tag{3.1}$$

where the right hand side is the direct sum of countably many (but infinite) copies of the cyclic groups $\mathbf{Z}$, $\mathbf{Z}_2$ and $\mathbf{Z}_4$.

Note that the right hand side of (3.1) is *not* an unrestricted direct sum, i.e., each element of the group is a linear combination of *finitely many* elements corresponding to the generators of the factors.

Remark 3.12. Michel [102] showed that for $n \geq 1$, spherical algebraic $(2n-1)$-knots have infinite order in C_{2n-1} as soon as we assume that the associated holomorphic function germ has an isolated singularity at the origin and is not non-singular. Note, however, that they are not independent. See Remark 7.3.

For $n = 1$, $\Phi_1 : C_1 \to C^{-1}(\mathbf{Z})$ is far from being an isomorphism. The non-triviality of the kernel of this epimorphism was first shown by Casson-Gordon [26]. The classification of spherical 1-knots up to cobordism is still an open problem. Moreover, for spherical 1-knots, there is also the important notion of a *ribbon knot* (see, for example, [129]). Ribbon knots are null-cobordant. It is still an open problem whether the converse is true or not.

For even dimensions, we have the following vanishing theorem.

Theorem 3.13 (Kervaire [72]). *For all $n \geq 1$, C_{2n} vanishes.*

Let $\widetilde{C}_n$ be the group of concordance classes of embeddings into S^{n+2} of

1. the n-dimensional standard sphere S^n for $n \leq 4$, or
2. homotopy n-spheres for $n \geq 5$.

In [72] Kervaire showed that the natural surjection $\mathfrak{i} : \widetilde{C}_n \to C_n$ is a group homomorphism.

Let us denote by Θ_n the group of h-cobordism classes of smooth oriented homotopy n-spheres, and by bP_{n+1} the subgroup of Θ_n consisting of the h-cobordism classes represented by homotopy n-spheres which bound compact parallelizable manifolds [74]. Then we have the following

Theorem 3.14 (Kervaire [72]). *For $n \leq 5$ we have $\widetilde{C}_n \cong C_n$, and for $n > 6$ we have the short exact sequence*

$$0 \to \Theta_{n+1}/bP_{n+2} \to \widetilde{C}_n \xrightarrow{\mathfrak{i}} C_n \to 0.$$

Note that for $n \geq 4$, Θ_{n+1}/bP_{n+2} is a finite abelian group. For details, see [74].

Chapter 4

Fibered knots and algebraic knots

"Ce chemin qui débouche sur la route de Chinon, bien au-delà de Ballan, longe une plaine ondulée sans accidents remarquables, jusqu'au pays d'Artanne. Là se découvre une vallée qui commence à Montbazon, finit à la Loire, et semble bondir sous les châteaux posés sur ces doubles collines; une magnifique coupe d'émeraudes au fond de laquelle l'Indre se roule par des mouvements de serpent."
Honoré de Balzac,
- Le lys dans la vallée

In this chapter we will work only with odd dimensional knots. We first define the notion of *fibered knot* and prove that Seifert forms of fibered knots are unimodular, then we define algebraic knots associated with isolated singularities of complex hypersurfaces.

4.1 Fibered knots

As explained in the introduction the set of fibered knots is much more smaller than the set of knots. But using the fibration of the complementary of the knot over S^1 we will be able to define many useful tools for the study of cobordism classes of fibered knots.

Recall (c.f. Definition 1.14) that a $(2n-1)$-knot K is fibered when there exists a trivialization $\tau : N_K \to K \times D^2$ of a closed tubular neighborhood N_K of K in S^{2n+1} and a smooth fibration $\phi : S^{2n+1} \setminus K \to S^1$ such that the following diagram is commutative:

$$\begin{tikzcd} N_K \setminus K \arrow[rr, "\tau"] \arrow[dr, "\phi|_{(N_K \setminus K)}"'] & & K \times (D^2 \setminus \{0\}) \arrow[dl, "p"] \\ & S^1 & \end{tikzcd}$$

where p denotes the obvious projection. In this case, for each $t \in S^1$, we denote by F the closure of $\phi^{-1}(t)$ in S^{2n+1} ; and F is also called a *fiber* of K. Note that $F = \phi^{-1}(t) \cup K$ and is a compact $2n$-dimensional manifold with boundary $\partial F = K$.

4.1.1 Monodromy and variation map

Any C^∞ locally trivial fibration ϕ, as in Definition 1.14, over S^1 with fiber F such that $\partial F \neq \varnothing$, is given up to isomorphism by a map called *geometric monodromy*.

Definition 4.1. The geometric monodromy $\mathfrak{m} : (F, \partial F) \to (F, \partial F)$ is defined up to isotopy such that ϕ identifies with

$$(F, \partial F) \times [0,1]/(x,0) \sim \big(\mathfrak{m}(x), 1\big) \to [0,1]/0 \sim 1,$$

and the restriction $\mathfrak{m}_{|\partial F}$ is the identity.

The geometric monodromy induces two algebraic monodromies.

Definition 4.2. Let K be a fibered knot with fiber F and geometric monodromy $\mathfrak{m} : (F, \partial F) \to (F, \partial F)$.

The *algebraic monodromy* is the homomorphism

$$h : \mathrm{H}_n(F) \to \mathrm{H}_n(F)$$

induced by the geometrical monodromy $\mathfrak{m}$, the characteristic polynomial of the algebraic monodromy is denoted by $\Delta(t)$.

The *relative algebraic monodromy* is the homomorphism in relative homology

$$\widetilde{h} : \mathrm{H}_n(F, \partial F) \to \mathrm{H}_n(F, \partial F)$$

induced by $\mathfrak{m}$.

Using the geometrical monodromy one can define another operator, called *variation map*. More precisely, let K be a fibered $(2n-1)$-knot with fiber F. For any relative n-chain a with $\partial a \in \partial F = K$, we have

$$\partial\big(a - \mathfrak{m}(a)\big) = \partial(a) - \mathfrak{m}(\partial a) = 0.$$

Hence $a - \mathfrak{m}(a)$ is an absolute chain. In the following, if a is a chain, then we denote by $[a]$ its homology class.

Definition 4.3. The following map $\mathcal{V}$ is called *variation map*.

$$\begin{array}{rcl} \mathcal{V} : \mathrm{H}_n(F, \partial F) & \to & \mathrm{H}_n(F) \\ [a] & \mapsto & [a - \mathfrak{m}(a)] \end{array}$$

Let a fibered $(2n-1)$-knot K with fiber F, the Wang exact sequence associated with the fibration $S^{2n+1} \setminus \overset{\circ}{N}(K) \to S^1$ with fiber F provides

$$0 \to \mathrm{H}_{n+1}\big(S^{2n+1} \setminus \overset{\circ}{N}(K)\big) \to \mathrm{H}_n(F) \overset{Id-h}{\to} \mathrm{H}_n(F) \to \mathrm{H}_n\big(S^{2n+1} \setminus \overset{\circ}{N}(K)\big) \to 0$$

by Alexander Duality (see [16]) we get $\mathrm{H}_k\big(S^{2n+1} \setminus \overset{\circ}{N}(K)\ \big) \cong \mathrm{H}^{2n-k}(K)$, and by Poincaré Duality we have $\mathrm{H}^{2n-k}(K) \cong \mathrm{H}_{k-1}(K)$. Hence the previous Wang exact sequence becomes

$$0 \to \mathrm{H}_{n+1}\big(K\big) \to \mathrm{H}_n(F) \overset{Id-h}{\to} \mathrm{H}_n(F) \to \mathrm{H}_n\big(K\big) \to 0 \tag{4.1}$$

Using the variation map, the exact sequences 1.1 and 4.1 can be related together as follows.

First for $k = n, n+1$, let us define Gysin isomorphisms

$$\begin{array}{rcl} g_k : \mathrm{H}_k\big(S^{2n+1} \setminus K\big) & \to & \mathrm{H}_{k-1}(K) \\ [a] & \mapsto & g\big([a]\big) = [b \cap K] \end{array}$$

where b is a boundary chain of dimension $(k+1)$ which meets K transversally in S^{2n+1} and with boundary the k-chain $[a]$.

Then the following diagram is commutative

$$\begin{array}{ccccccccc}
0 \to & \mathrm{H}_{n+1}\big(S^{2n+1}\setminus K\big) & \to & \mathrm{H}_n(F) & \overset{Id-h}{\to} & \mathrm{H}_n(F) & \overset{i}{\to} & \mathrm{H}_n\big(S^{2n+1}\setminus K\big) & \to 0 \\
 & \downarrow g_{n+1} & & || & & \uparrow \mathcal{V} & & \downarrow g_n & \\
0 \to & \mathrm{H}_n(K) & \to & \mathrm{H}_n(F) & \overset{S_*}{\to} & \mathrm{H}_n(F,K) & \to & \mathrm{H}_{n-1}(K) & \to 0
\end{array}$$

The first square is commutative since g_{n+1} is an isomorphism, the second square is commutative because of the definition of $\mathcal{V}$ (recall that S_* is induced by the inclusion). We only have to check the commutativity for the last square.

Start with a relative cycle in $\mathrm{H}_n(F,K)$ given by the homology class $[c]$ of a relative chain c of dimension n. Then $\mathcal{V}([c]) = [c - \mathfrak{m}(c)]$, and if b is a $(n+1)$-chain with boundary $c - \mathfrak{m}(c) = \partial b$ then $g_n\big(i([c-\mathfrak{m}(c)])\big) = [b \cap K] = [\partial c]$. This proves the commutativity, and as a consequence the five Lemma implies that $\mathcal{V}$ is an isomorphism. We proved

Proposition 4.4. *The variation map* $\mathcal{V} : \mathrm{H}_n(F, \partial F) \to \mathrm{H}_n(F)$ *is an isomorphism.*

4.1.2 Seifert form

We already defined Seifert forms associated with simple knots, but in the case of simple fibered knot one can define the Seifert form associated with a fiber using the geometrical monodromy. Let us be more precise, and consider a fibered $(2n-1)$-knot K with fibration ϕ and fiber[1] F. Write $F_\theta = \phi^{-1}(e^{i\theta})$ for any $\theta \in [0, 2\pi]$, then F_θ is homeomorphic to $\overset{\circ}{F}$. Moreover let h be a continuous map

$$h : [0,1] \times F_0 \to S^{2n+1} \setminus K$$

such that h_θ maps F_0 homeomorphically onto F_θ, $\theta \in [0, 2\pi[$, when $\theta = 0$ $h_0 = Id_{F_0}$ and $h_{2\pi}$ is the geometrical monodromy (which is defined up to isotopy).

Since ϕ is a locally trivial fibration, then distinct fibers never meet together. This elementary fact implies that for two cycles $[x]$ and $[y]$ in $\mathrm{H}_n(F)$, and for $\theta \in]0, 2\pi[$ we have

$$l_{S^{2n+1}}\big(i_+(x), y)\big) = l_{S^{2n+1}}\big(h_\theta(x), y\big),$$

where $l_{S^{2n+1}}$ denotes the linking number of chains in S^{2n+1}.

Then the Seifert form $\mathfrak{A}$ is defined as follows

$$\begin{array}{rcl}
\mathfrak{A} : \mathrm{H}_n(F) \times \mathrm{H}_n(F) & \to & \mathbf{Z} \\
([x],[y]) & \mapsto & l_{S^{2n+1}}\big(h_\pi(x), y\big)
\end{array}$$

For ξ in $\mathrm{H}_n(F,K)$ and ζ in $\mathrm{H}_n(F)$ we denote by $< \xi, \zeta >$ the intersection number which is defined by

$$< \xi, \zeta > = \widetilde{\mathfrak{P}}(\xi)(\zeta)$$

where $\widetilde{\mathfrak{P}} : H_n(F,K) \overset{\cong}{\to} \mathrm{Hom}_{\mathbf{Z}}\big(H_n(F), \mathbf{Z}\big)$ is the composite of the Poincaré-Lefschetz duality isomorphism and the universal coefficient isomorphism.

With the last definition of the Seifert form we easily get the following proposition

[1] Recall that we decide to call *fiber* the closure of a preimage of a point.

Proposition 4.5. *Let* $(\alpha, \beta) \in \mathrm{H}_n(F) \times \mathrm{H}_n(F, K)$ *then* $\mathfrak{A}\big(\alpha, \mathcal{V}(\beta)\big) = <\alpha, \beta>$.

Proof. Start with $([a], [b]) \in \mathrm{H}_n(F) \times \mathrm{H}_n(F, K)$ then the following equalities hold

$$\begin{aligned}
\mathfrak{A}\big([a], \mathcal{V}([b])\big) &= l_{S^{2n+1}}\big(h_\pi(a), b - \mathfrak{m}(b)\big) \\
&= l_{S^{2n+1}}\Big(h_\pi(a), \partial\big(\cup_{\theta \in [0,2\pi]} h_\theta(b)\big)\Big) \\
&= I_{D^{2n+2}}\big(h_\pi(a), \cup_{\theta \in [0,2\pi]} h_\theta(b)\big) \\
&= <[h_\pi(a)], [h_\pi(b)]>_{F_\pi} \\
&= <[a], [b]>
\end{aligned}$$

□

As a corollary of the previous proposition we have

Proposition 4.6. *The Seifert form associated with a fibered knot is unimodular.*

Proof. Let K be a fibered knot with fiber F. As before $\mathfrak{A}$ and $\mathcal{V}$ are the Seifert form and the variation map associated with F. We first fix a basis $\mathcal{B} = (\beta_i)_{i \in \mathcal{I}}$ for $\mathrm{H}_n(F)$, and then we take the basis $\mathcal{B}^* = (\beta_i^*)_{i \in \mathcal{I}}$ for $\mathrm{H}_n(F, K)$ which is the dual basis of $\mathcal{B}$. By dual we mean that for all (i, j) in $\mathcal{I}^2$ we have

$$\widetilde{\mathfrak{P}}\Big(\beta_i\Big)(\beta_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. With these choices, when β is a relative chain in $\mathrm{H}_n(F, K)$ and α is a n-chain in $\mathrm{H}_n(F)$ which have the two column vectors b and a respectively as matricial representations, then

$$<\alpha, \beta> = \widetilde{\mathfrak{P}}\big(\alpha\big)(\beta) = {}^t a\, b.$$

Let us denote by A the matrix of the Seifert form $\mathfrak{A}$, and by V the matrix of the variation map $\mathcal{V}$ relatively to the basis $\mathcal{B}$ and $\mathcal{B}^*$. According to Proposition 4.5, for all (α, β) in $\mathrm{H}_n(F, K) \times \mathrm{H}_n(F)$ we have

$$\mathfrak{A}\big(\alpha, \mathcal{V}(\beta)\big) = <\alpha, \beta> .$$

If we denote by a and b the two column vectors which represent α and β relatively to the basis $\mathcal{B}$ and $\mathcal{B}^*$ then the previous equality becomes

$${}^t a (A\, V) b = {}^t a b.$$

Since this equality holds for any column vectors a and b we have $A = V^{-1}$. We already proved that $\mathcal{V}$ is an isomorphism so $\det V = \det A = \pm 1$ and $\mathfrak{A}$ is unimodular. □

Proposition 4.7. *Let* K *be a simple fibered* $(2n-1)$*-knot with fiber* F*. Set* A *be the matrix of the Seifert form,* S *the matrix of the intersection form and* H *be the matrix of the monodromy associated with* F*. If* I *is the matrix of the identity, then the following holds*

$$S = A(I - H), \quad H = (-1)^{n+1} A^{-1}\, {}^t A.$$

Proof. Let α and β be two n-cycles in $\mathrm{H}_n(F)$, set $\alpha = [x]$ and $\beta = [y]$ for two n-chains x and y.

Set $Z = \bigcup_{\theta=0}^{2\pi} h_\theta(y)$ the $(n+1)$-chain in S^{2n+1} with boundary $\partial Z = y - \mathfrak{m}(y)$. And set A and B two $(n+1)$-chains in S^{2n+1} such that $\partial A = y$ and $\partial B = \mathfrak{m}(y)$. Then $Z + B - A$ is a $(n+1)$-chain without boundary which represents the homology class of a $(n+1)$-cycle in S^{2n+1}. Hence the intersection number between $Z + B - A$ and $h_\pi(x)$ in S^{2n+1} must be zero.

If we denote by $\big\langle X, Y \big\rangle$ the intersection number between two chains in S^{2n+1}, then the following equalities hold

$$\begin{aligned}
\big\langle h_\pi(x), Z + B - A \big\rangle &= \big\langle h_\pi(x), Z\big\rangle + \big\langle h_\pi(x), B\big\rangle - \big\langle h_\pi(x), A\big\rangle \\
&= I_{F_\pi}(h_\pi(x), h_\pi(y)) + \big\langle h_\pi(x), \mathfrak{m}(y)\big\rangle + l\big\langle h_\pi(x), y\big\rangle \\
&= \mathfrak{S}(x, y) + \mathfrak{A}\big(x, h(y)\big) - \mathfrak{A}(x, y) \\
&= \mathfrak{S}(x, y) + \mathfrak{A}\big(x, h(y) - y\big).
\end{aligned}$$

The nullity of $\big\langle h_\pi(x), Z + B - A \big\rangle$ gives $S = A(I - H)$.

By Proposition 1.8 we have $S = A + (-1)^n\, {}^t\!A$ and A is invertible, then we get

$$I - H = A^{-1}(A + (-1)^n\, {}^t\!A) = I + (-1)^n A^{-1}\, {}^t\!A.$$

Finally $H = (-1)^{n+1} A^{-1}\, {}^t\!A$ as desired. □

With the unimodularity of Seifert forms associated with fibers of fibered knots Durfee and Kato independently generalized the work of Levine.

Theorem 4.8 ([36],[65]). *Let $n \geq 3$. There is a one-to-one correspondence of isotopy classes of simple fibered knots in S^{2n+1} and equivalence classes of integral unimodular bilinear forms. The correspondence associates to each knot its Seifert form.*

Proof. Let K_0 and K_1 be two simple fibered $(2n-1)$-knots which are isotopic. Using the same proof that we gave for spherical knots, we can see that the Seifert forms associated with the fibers of K_0 and K_1 are S-equivalent. But S-equivalence of unimodular forms reduces to congruence of matrices, hence the associated Seifert forms are equivalent.

Conversely, given an integral matrix A, to realize A as the matrix of an integral bilinear form $\mathfrak{A}$, we can construct a simple knot with Seifert form $\mathfrak{A}$. This is done as Kervaire did in [72] for spherical knots[2], by gluing n-handles on a the boundary of a $(2n-1)$-disk. The knot is the boundary of this handlebody, and the handlebody itself is a Seifert manifold F for this knot K. The core of the handles are the generators of the n^{th} homology group $\mathrm{H}_n(F)$, so we glued such that the linking numbers between the handles correspond to the coefficients of the matrix A. By construction the knot K is simple, we will prove that K is fibered using the h-cobordism Theorem.

First let us fix some notations. Set X be the complementary in S^{2n+1} of an open tubular neighborhood of K in S^{2n+1}, and let $W = F \cap X$. Set N a normal tubular neighborhood of W in X, hence if M is a normal tubular neighborhood ot F in S^{2n+1} then $N = M \cap X$. Moreover, it makes sense to follow notations

[2] The same technic works since Kervaire additional conditions were only used to insure that the knot is spherical.

of Definition 1.5 and set $N \cong W_+ \times [0,1]$ where $W_+ = F_+ \cap X$ correspond to W pushed in the positive normal direction in S^{2n+1}.

Set $Y = X \setminus N$, then the exact long homology sequence of the pair (Y, W_+) gives

$$\ldots \to \mathrm{H}_k(W_+) \to \mathrm{H}_k(Y) \to \mathrm{H}_k(Y, W_+) \to \ldots \tag{4.2}$$

Moreover the manifold W_+ is $(n-1)$-connected; and because of Alexander duality $\mathrm{H}_k(Y) \cong \mathrm{H}^{2n-k}(F)$, so $\mathrm{H}_k(Y) = 0$ for $k \geq n+1$. Hence the relative homology groups $\mathrm{H}_k(Y, W_+)$ vanishe for $k \geq n+1$, by Poincare-Lefshetz duality we also have $\mathrm{H}_k(Y, W_+)$ vanishe for $k \leq n-1$. Then the long exact sequence (4.2) reduces to

$$0 \to \mathrm{H}_n(W_+) \to \mathrm{H}_n(Y) \to \mathrm{H}_n(Y, W_+) \to 0.$$

But since the matrix A is unimodular, then the inclusion $W_+ \hookrightarrow Y$ induces the isomorphism $\mathrm{H}_n(W_+) \overset{\cong}{\to} \mathrm{H}_n(Y)$. Remark that the injectivity also comes from the fact that the image of a non trival homology class x of $\mathrm{H}_n(W_+)$ in $\mathrm{H}_n(Y)$ can't be null homologous otherwise A will be degenerated because $A(x,y) = 0$ for any y in $\mathrm{H}_n(F)$.

The surjectivity is a consequence of the unimodularity of A. To see that, first remark that according to Alexander duality the free $\mathbf{Z}$-modules $\mathrm{H}_n(Y)$ and $\mathrm{H}_n(W_+)$ have same rank. Second, since the inclusion is injective, then if it is not surjective there exists an indivisible element, namely x, in $\mathrm{H}_n(W_+)$ which is homologous to an element $\alpha\, y$ of $\mathrm{H}_n(Y)$ where $\alpha \neq -1, 0, 1$ and y lies in $\mathrm{H}_n(Y)$. But this implies that α divides $\det A$, which contradicts the unimodularity of A.

Finally we get $\mathrm{H}_n(Y, W_+) = 0$ and Y is homeomorphic to $W_+ \times [0,1]$ according to the h-cobordism Theorem.

Now it is not difficult to see that the knot K constructed is fibered. This comes from the decomposition of X in two pieces, namely $N \cong W \times [0,1]$ and $Y \cong W_+ \times [0,1]$. The identification of N and Y along their boundaries induces an homeomorphism $\mathfrak{m} : W \to W$ such that X is homeomorphic to the quotient $W \times [0,1]$ by the equivalence relation $(x, 0) \sim \big(\mathfrak{m}(x), 1\big)$. Since all these maps extend to $S^{2n+1} \setminus K$, then the knot K is simple fibered. $\square$

Remark 4.9. For spherical simple $(2n-1)$-knots, we have another algebraic invariant, called the *Blanchfield pairing*, which is closely related to the Seifert form (see [68, 150]). In fact, it is known that giving an S-equivalence class of a Seifert form is equivalent to giving an isomorphism class of a Blanchfield pairing.

We just saw that fibered knots have unimodular Seifert forms, moreover fibered knots have a nice topological behavior as stated in the following proposition.

Proposition 4.10. *Let $n \geq 1$. Let K be a fiber knot of dimension $2n-1$ and let F be a fiber of the fibration, then we have the following short exact sequence*

$$0 \to H_n(K) \to H_n(F) \overset{S_*}{\to} H_n(F, K) \to H_{n-1}(K) \to 0.$$

Proof. Recall that F is a Seifert surface associated with K. Moreover we know that $S^{2n+1} \setminus K$ is homeomorphic to $\overset{\circ}{F} \times [0,1]_{/(x,0) \sim (\mathfrak{m}(x),1)}$ where $\mathfrak{m}$ is the geometrical monodromy. Hence $S^{2n+1} \setminus F$ has the same homotopy type as F. Now by Alexander duality we have

$$\mathrm{H}_k(F) \cong \mathrm{H}^{2n-k}(S^{2n+1} \setminus F) \cong \mathrm{H}^{2n-k}(F) \quad \text{for} \quad k > 0.$$

Moreover by Poincaré duality we have

$$\mathrm{H}_k(F, K) \cong \mathrm{H}^{2n-k}(F),$$

and this implies

$$\mathrm{H}_k(F, K) \cong \mathrm{H}_k(F) \quad \text{for} \quad k > 0. \tag{4.3}$$

Since K is $(n-2)$-connected, then the long exact sequence

$$\ldots \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F, K) \to \ldots$$

gives the following short exact sequence

$$0 \to \mathrm{H}_{n+1}(F) \xrightarrow{\alpha} \mathrm{H}_{n+1}(F, K) \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \to$$

$$\mathrm{H}_n(F, K) \to \mathrm{H}_{n-1}(K) \to \mathrm{H}_{n-1}(F) \xrightarrow{\beta} \mathrm{H}_{n-1}(F, K) \to 0$$

According to (4.2) the monomorphism α is an isomorphism, and the epimorphism β as well. Finally we get the desired short exact sequence

$$0 \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F, K) \to \mathrm{H}_{n-1}(K) \to 0$$

□

According to this proposition we see that the topological data about the knot K are coming from the Kernel and the Cokernel of the intersection form of F.

Moreover, as a consequence of the short exact sequence of Proposition 4.10 we see that the middle homology group of the fiber is a free abelian group.

4.1.3 Alexander polynomials of fibered knots

Let K be a $(2n-1)$-fibered knot with fiber F. As before, set X be the complementary in S^{2n+1} of an open tubular neighborhood of K in S^{2n+1}, and let $W = F \cap X$ the intersection of the fiber with X.

Then we take the quotient of $W \times \mathbf{R}$ by the equivalence relation $(x, \alpha) \sim (\mathfrak{m}^k, \alpha + k)$ for any $k \in \mathbf{Z}$. This quotient is homeomorphic to X and $W \times \mathbf{R}$ is the infinite cyclic covering of X. Let τ be the generator of the Galois group of the covering $W \times \mathbf{R} \to X$, which is the infinite cyclic covering of X. The action of τ is given by the map which maps (x, α) to $\big(\mathfrak{m}(x), \alpha + 1\big)$. If τ induces an action, denoted by t on $\mathrm{H}_*(W \times \mathbf{R})$ which acts as the monodromy h acts on $\mathrm{H}_*(W)$.

The homology group $\mathrm{H}_n(W \times \mathbf{R})$ is a free abelian group which is finitely generated because it has the homotopy type of a compact CW-complex. The generator of the first elementary ideal of the $\mathbf{Z}[t, t{-}1]$-module $\mathrm{H}_n(W \times \mathbf{R})$, i.e., the ideal generated by minor of maximal rank, is the characteristic polynomial of t. Moreover this polynomial is the Alexander polynomial of $\mathrm{H}_n(W \times \mathbf{R})$. Since the action of t reduce to the action of h on $\mathrm{H}_n(W)$, then we get the folklore Theorem (see also [124])

Theorem 4.11. *Let K be a fiber $(2n-1)$-knot with fiber F. The Alexander polynomial of $\mathrm{H}_n(F \times \mathbf{R})$ is the characteristic polynomial of the algebraic monodromy $h : \mathrm{H}_n(F) \to \mathrm{H}_n(F)$.*

This result is compatible with the previous Definition of the Alexander polynomial of a $(2n-1)$-knot K to be

$$\Delta_K(X) = \det\bigl(tA + (-1)^n\, {}^tA\bigr)$$

since when K is a fibered knot, then the Seifert form A is unimodular and the monodromy has $H = (-1)^{n+1}\, {}^tA\, A^{-1}$ as matrix. This gives

$$\Delta_K(X) = \det\bigl(tA + (-1)^n\, {}^tA\bigr) = \det(tId - H).$$

Since the Alexander Polynomial of a fibered knot K is $\Delta_K(X) = \det(X\,Id - h)$, then as a consequence of the exact sequence (4.1) the fibered knot K is an integral homological sphere if and only if $\Delta_K(1) = \pm 1$. this is also a consequence of the short exact sequence of Proposition 4.10 since the matrix of the intersection form S_* is equal to $A + (-1)^n\, {}^tA$ and $\Delta_K(1) = \det(A + (-1)^n\, {}^tA) = \det S = \pm 1$ if and only if the knot K is an integral homology sphere.

When K is a fibered knot Δ_K is a characteristic polynomial so its leading coefficient must be 1, and its last coefficient is equal to $\pm \det H$ which ± 1, so we get the following Proposition.

Proposition 4.12. *A necessary condition for a knot to fiber is that the extremal coefficients of the Alexander polynomial should be ± 1.*

4.2 Algebraic knots

As said in the introduction, algebraic knots are one motivation to the study of fibered knots. In this section we will review some classical definition and result about algebraic knots, we refer to [33, 109, 119] for details and proofs.

Let $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic function germ with an isolated singularity at the origin. Recall that there exists a positive real number ε_0 such that for all ε in $]0, \varepsilon_0[$ the set

$$K_f = f^{-1}(0) \cap S^{2n+1}_\varepsilon$$

is a $(2n-1)$-dimensional manifold which is naturally oriented, where S^{2n+1}_ε is the sphere in $\mathbf{C}^{n+1}$ of radius ε centered at the origin. Furthermore, its (oriented) isotopy class in $S^{2n+1}_\varepsilon = S^{2n+1}$ does not depend on the choice of ε, and we call it *the algebraic knot* associated with the isolated singularity of f.

Theorem 4.13 ([109]). *Let $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic function germ with an isolated singularity at the origin.*

There exists a positive real number ε_0 such that the following map φ defined on the complement of the algebraic knot K_f

$$\begin{array}{rcl} \varphi : S^{2n+1}_\varepsilon \setminus K_f & \to & S^1 \\ z & \mapsto & \varphi(z) = \frac{f(z)}{|f(z)|} \end{array}$$

is a locally trivial fibration for any $0 < \varepsilon \leq \varepsilon_0$ which is called the Milnor fibration*, its isomorphism class does not depend on the choice of ε.*

The fiber associated with a Milnor fibration is called a Milnor fiber.

Moreover Milnor proved the following

Theorem 4.14 ([109]). *Let $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic function germ with an isolated singularity at the origin.*

1. *The algebraic knot K_f is $(n-2)$-connected,*
2. *The Milnor fiber is $(n-1)$-connected and is homotopic to a bouquet of $(n-1)$-dimensional spheres.*

Since the closure of each fiber is a compact $2n$-dimensional oriented $(n-1)$-connected submanifold of S^{2n+1} which has K_f as boundary, then algebraic knots are simple fibered knots.

Definition 4.15. The n-th Betti number of the Milnor fiber is called the *Milnor number* of f at the origin, we denote it by μ_f (or μ for simplicity).

It is known that

$$\mu = \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^{n+1}} \Big/ \left(\frac{\partial f}{\partial z_i}\right)_{i=1\dots n+1}$$

where $\mathcal{O}_{\mathbf{C}^{n+1}}$ denotes the ring of germs of holomorphic functions at the origin, and $\left(\frac{\partial f}{\partial z_i}\right)_{i=1\dots n+1}$ denote the Jacobian ideal which is generated by the partial derivatives $\partial f/\partial z_i$ for $i = 1, \dots, n+1$.

4.2.1 Functions with independent variables.

Definition 4.16. When $f : \mathbf{C}^{n_1+n_2} \to \mathbf{C}$ is a holomorphic function of the type

$$f(\mathbf{z} + \mathbf{u}) = f_1(\mathbf{z}) + f_2(\mathbf{u}),$$

where $\mathbf{z} \in \mathbf{C}^{n_1}$, $\mathbf{u} \in \mathbf{C}^{n_2}$ and $f_i : \mathbf{C}^{n_i} \to \mathbf{C}$ is a holomorphic function for $i = 1, 2$; then we say f is of *independent variables.*

We will now describe the behavior of the Milnor fiber of holomorphic functions of independent variables.

Theorem 4.17 (Join Theorem). *Let $f = f_1 + f_2$ be a holomorphic function of independent variables, and let F_i be the Milnor fiber of f_i for $i = 1, 2$. Set h_i the algebraic monodromy associated with f_i for $i = 1, 2$.*

Then F the Milnor fiber of f has the same homotopy type of the join $F_1 * F_2$ and the algebraic monodromy associated with f is equal to the join of h_1 and h_2 up to homotopy.

Remark 4.18. The join $F_1 * F_2$ is defined as the quotient space

$$F_1 * F_2 = \Big(F_1 \times F_2 \times [0, 1]\Big)\Big/ \sim$$

of $F_1 \times F_2 \times [0, 1]$ by the identification

$$(\mathbf{z}, \mathbf{u}, 0) \sim (\mathbf{z}', \mathbf{u}, 0)$$

and

$$(\mathbf{z}, \mathbf{u}, 1) \sim (\mathbf{z}, \mathbf{u}', 1)$$

for any $\mathbf{z}, \mathbf{z}' \in F_1$ and $\mathbf{u}, \mathbf{u}' \in F_2$

Moreover in the case of holomorphic functions of independent variables we have some information about the Seifert form as stated below

Theorem 4.19 ([138]). *If* $f_1 : (\mathbf{C}^{k+1}, 0) \to (\mathbf{C}, 0)$ *and* $f_2 : (\mathbf{C}^{\ell+1}, 0) \to (\mathbf{C}, 0)$ *are holomorphic function germs with an isolated critical point at the origin, then the Seifert form associated with the holomorphic function germ*

$$f_1 \oplus f_2 : (\mathbf{C}^{k+\ell+2}, 0) \to (\mathbf{C}, 0)$$

defined by

$$(f_1 \oplus f_2)(z_1, z_2, \dots, z_{k+\ell+2}) = f_1(z_1, z_2, \dots, z_{k+1}) + f_2(z_{k+1}, z_{k+2}, \dots, z_{k+\ell+2})$$

coincides with

$$(-1)^{k\ell} A_{f_1} \otimes A_{f_2},$$

where A_{f_i} *denotes the Seifert form of* f_i *for* $i = 1, 2$.

Since the fiber of a holomorphic function of independent variables is well understood, then we also have the following for the algebraic monodromy.

Theorem 4.20 (Thom-Sebastiani). *Assume that* $f_1 : (\mathbf{C}^{k+1}, 0) \to (\mathbf{C}, 0)$ *and* $f_2 : (\mathbf{C}^{\ell+1}, 0) \to (\mathbf{C}, 0)$ *are holomorphic function germs with an isolated critical point at the origin. Set* F_{f_i} *be the fiber and* h_{f_i} *be the algebraic monodromy for* $i = 1, 2$*, then we have the following commutative diagram*

$$\begin{array}{ccc}
\widetilde{H}_k(F_{f_1}) \otimes \widetilde{H}_l(F_{f_2}) & \stackrel{h_{f_1} \otimes h_{f_2}}{\longrightarrow} & \widetilde{H}_k(F_{f_1}) \otimes \widetilde{H}_l(F_{f_2}) \\
\downarrow\cong & & \downarrow\cong \\
\widetilde{H}_{k+l+1}(F_{f_1+f_2}) & \stackrel{h_{(f_1+f_2)}}{\longrightarrow} & \widetilde{H}_{k+l+1}(F_{f_1 f_2})
\end{array}$$

4.3 Brieskorn knots

Let us consider now some very special functions of independent variables called Brieskorn polynomials[3]

Definition 4.21. A *Brieskorn polynomial* is a polynomial of the form

$$f(z_1, \dots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \dots + z_{n+1}^{a_{n+1}}$$

with $n \geq 0$, the integers $a_j \geq 2$, $j = 1, 2, \dots, n+1$, are called the *exponents.*

The complex hypersurface in $\mathbf{C}^{n+1}$ defined by $f = 0$ has an isolated singularity at the origin, which is called a *Brieskorn singularity.*

According to [36] Proposition 2.1 the Seifert form associated with the one variable Brieskorn polynomial

$$f(z) = z^a$$

has the $(a-1) \times (a-1)$ integral following matrix

$$A(a) = \begin{pmatrix}
1 & 1 & 0 & \dots & \dots & 0 \\
0 & 1 & 1 & 0 & \dots & 0 \\
\vdots & 0 & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \vdots \\
\vdots & \vdots & & & 1 & 1 \\
0 & 0 & \dots & \dots & 0 & 1
\end{pmatrix}$$

[3] Sometimes they are called Brieskorn-Pham polynomials.

Then the Seifert form associated with the Brieskorn polynomial

$$f(z_1, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

has a matrix of the form

$$A_f = A(a_1) \otimes \ldots \otimes A(a_{n+1}).$$

We will now give conditions on exponents of a Brieskorn polynomial to have a Brieskorn knot which is a sphere. But before we have to introduce a graph associated to

$$f(z_1, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}.$$

Let $\mathcal{G}_f$ be the graph which has $n+1$ vertices denoted by the letters $a_1, \ldots, a_{n+1}$, and two vertices a_l and a_k are connected by an edge if $gcd(a_l, a_k)$, the greatest common divisor of a_l and a_k, is strictly greater than 1. We denote by $\mathcal{C}_{ev,f}$ the connected component of $\mathcal{G}_f$ which contains all even exponent a_i. Note that $\mathcal{C}_{ev,f}$ may contain some odd vertices.

Theorem 4.22 ([19]). *Let K_f be the Brieskorn knot associated with*

$$f(z_1, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}.$$

1. *The algebraic knot K_f is a rational homology sphere if and only if $\mathcal{G}_f$ has either*
   - *(a) at least one isolated point,*
   - *(b) the component $\mathcal{C}_{ev,f}$ contains an odd number of vertices and we have* $\gcd(a_i, a_j) = 2$ *for any two vertices a_i and a_j in $\mathcal{C}_{ev,f}$.*
2. *The algebraic knot K_f is a integer homology sphere if and only if $\mathcal{G}_f$ has either*
   - *(a) at least two isolated points,*
   - *(b) an isolated point which is odd and the component $\mathcal{C}_{ev,f}$ contains an odd number of vertices and* $\gcd(a_i, a_j) = 2$ *for any two vertices a_i and a_j in $\mathcal{C}_{ev,f}$.*

We will not prove this Theorem, but we give some important steps in the proof (see [33]).

First it is important to know that K_f is a rational homology sphere if and only if 1 is not a root of the Alexander polynomial Δ_f, and it is equivalent that 1 is not an eigenvalue of the monodromy.

Moreover K_f is an integral homology sphere if and only if $\Delta_f(1) = \pm 1$.

Then according to Thom-Sebastiani Theorem the monodromy h_f is given by the formula

$$h_f = h_{a_1} \otimes \ldots \otimes h_{a_{n+1}}$$

where $h_{a_i} : \widetilde{H}_0(F_{a_i}) \to \widetilde{H}_0(F_{a_i})$ is the monodromy of the zero-dimensional singularity associated with $f_{a_i}(z) = z^{a_i}$ and F_{a_i} is the Milnor fiber

$$F_{a_i} = \{\lambda \in \mathbf{C} | \lambda^{a_i} = 1\}.$$

Set $\xi_{a_i} = exp(2\pi/a_i)$ and $\delta_k = \xi_{a_i}^{k-1} - \xi_{a_i}^{k}$, then the set

$$\big(\delta_k\big)_{k=1,\ldots,a_i-1}$$

is a basis of $\widetilde{H}_0(F_{a_i})$.

Since the geometric monodromy associated with h_{a_i} is in fact the multiplication with ξ_{a_i} then the matrix of h_{a_i} relativley to the basis $\big(\delta_k\big)_{k=1,\ldots,a_i-1}$ is

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & -1 \\ 1 & 0 & 0 & \ldots & -1 \\ 0 & 1 & 0 & \ldots & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 1 & \ldots & -1 \end{pmatrix}$$

then it follows that

$$\Delta_{a_i}(t) = \det(t\,Id - h_{a_i}) = t^{a_i-1} + \ldots + 1$$

and the eigenvalue of the monodomy h_{a_i} are exactly the a_i-roots of unity different from 1.

Finally the eignevalue of the monodromy h_f are exactly the products

$$\xi_{a_0}^{j_0} \ldots \xi_{a_{n+1}}^{j_{n+1}} \quad \text{for} \quad 1 \leq j_k \leq a_k - 1.$$

This shows that K_f is a rational homology sphere if and only if the equation

$$\frac{j_0}{a_0} + \ldots + \frac{j_{n+1}}{a_{n+1}} = m$$

has no solution with $m \in \mathbf{Z}$ and $1 \leq j_k \leq a_k + 1$.

The case of integral homology sphere is slightly more difficult and we refer to [19] for details.

Chapter 5

Algebraic cobordism

" L'algèbre est généreuse, elle donne
souvent plus qu'on ne lui demande."
Jean Le Rond D'Alembert

In this chapter we introduce the notion of *algebraic cobordism* for unimodular integral bilinear forms. We will work only with bilinear forms in a purely algebraic context. Later, we will use algebraic cobordism classes of Seifert forms associated to fibered knots.

5.1 Definitions

First we fix some notations used in this chapter.

Let $\mathcal{A}$ be the set of unimodular bilinear forms defined on free $\mathbf{Z}$-modules G of finite rank.

Let ε be $+1$ or -1.

If A is in $\mathcal{A}$, we denote by

^tA the transpose of A,

S the ε-symmetric form $A + \varepsilon\, {}^tA$ associated to A,

$S^* : G \to G^*$ the adjoint of S (G^* being the dual $\mathrm{Hom}_{\mathbf{Z}}(G; \mathbf{Z})$ of G),

$\overline{S} : \overline{G} \times \overline{G} \to \mathbf{Z}$ the ε-symmetric non degenerated form induced by S on $\overline{G} = G_{/\mathrm{Ker}\, S^*}$.

Recall that a submodule M of G is pure if $G_{/M}$ is torsion free. If M is any submodule of G we will denote by $M^\wedge$ the smallest pure submodule of G which contains M. In fact $M^\wedge$ is equal to $(M \otimes \mathbf{Q}) \cap G$. For a submodule M of G we will denote by $\overline{M}$ the image of M in $\overline{G}$.

Definition 5.1. Let $A : G \times G \to \mathbf{Z}$ be a bilinear form in $\mathcal{A}$. The form A is *Witt associated* to 0 if the rank m of G is even and if there exists a pure submodule M of rank $\frac{m}{2}$ in G such that A vanishes on M ; such a module M is called a *metabolizer* for A.

Remark 5.2. In the case of ε-fomrs, i.e., integral bilinear forms A for which the form $A + \varepsilon\, {}^tA$ is unimodular M. Kervaire [73] and J. Levine [89] said that an ε-form Witt associated to 0 is *null cobordant*.

Definition 5.3. Let $A_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, be two bilinear forms in $\mathcal{A}$. Let G be $G_0 \oplus G_1$ and A be $(A_0 \oplus -A_1)$. The form A_0 is *algebraically cobordant* to A_1 if there exists a metabolizer M for A such that $\overline{M}$ is pure in $\overline{G}$, an isomorphism φ from $\mathrm{Ker}\, S_0^*$ to $\mathrm{Ker}\, S_1^*$ and an isomorphism θ from $\mathrm{Tors}\,(\mathrm{Coker}\, S_0^*)$ to $\mathrm{Tors}\,(\mathrm{Coker}\, S_1^*)$ which satisfy the two following conditions c.1: $M \cap \operatorname{Ker} S^* = \{(x, \varphi(x)); x \in \operatorname{Ker} S_0^*\}$,

c.2: $d(S^*(M)^\wedge) = \{(x, \theta(x)); x \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\}$, where d is the quotient map from G^* to $\operatorname{Coker} S^*$.

Topological meaning of algebraic cobordism. At first reading Definition 5.3 seems very technical. To understand its meaning it is important to consider algebraic cobordism of Seifert forms in the topological context given by fibers of fibered knots.

Recall, cf. Chapter 4, that the Seifert form of a $2n-1$ dimensional fibered knot K, with a Seifert surface F (which is the closure of a fiber), is related to the intersection form of F. More precisely we have $S = A + (-1)^n\, {}^tA$, where S and A are the matrices of the intersection form and the Seifert form respectively. If we choose an integer $n \geq 3$ such that $\varepsilon = (-1)^n$, then we can realize any unimodular bilinear integral form as a Seifert form[1]. Then the long exact sequence of the couple (F, K) gives

$$0 \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \overset{S_*}{\to} \mathrm{H}_n(F, K) \to \mathrm{H}_{n-1}(K) \to 0,$$

where S_* is induced by the intersection form.

So the n^{th}-homology group of K can be identify with the kernel of the intersection form, and the $(n-1)^{th}$-homology group of K with the cokernel of the intersection form. In Definition 5.3 conditions c.1 and c.2 fix the behavior of the elements in the metabolizer which are related to the $(n-1)^{th}$-homology group of K and the $n-^{th}$-homology group of K.

When we consider cobordism classes of knots, then topological data must be related to cobordism classes of Seifert forms, in this sense conditions c.1 and c.2 in Definition 5.3 are natural.

We can also point out that when the knot is a sphere, the homomorphism S^* in the exact sequence just above is an isomorphism. So ε-forms used by M. Kervaire [73] and J. Levine [89] are Seifert forms of spherical knots.

5.2 Examples.

In order to clarify the relation of algebraic cobordism, we present here several examples.

(1) Let us consider any integral bilinear form A in $\mathcal{A}$ such that $A + \varepsilon\, {}^tA$ is unimodular. Then, $A \oplus (-A)$ is always algebraically cobordant to the zero form.

(2) Let us consider the integral bilinear forms A_0 and A_1 represented by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}$$

respectively, which are given in [73, p. 93]. Then it is easy to check that the subgroup of $\mathbf{Z}^4$ generated by ${}^t(3,1,3,0)$ and ${}^t(0,1,2,1)$ is a metabolizer

[1] This is done by adding handles to a $2n$-ball such that the linking numbers of these handles coincide with the entries of the form ; then the resulting manifold is a Seifert surface F for its boundary K with the desired Seifert form.

for $A_0 \oplus (-A_1)$. Since $A_i - {}^tA_i$ are unimodular, $i = 0, 1$, we see that A_0 and A_1 are algebraically cobordant for $\varepsilon = -1$. Note that A_0 and A_1 are not congruent to each other.

(3) This example is a generalization of the examples given in [7]. Let us consider the two matrices

$$A_0 = \begin{pmatrix} p^2 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} q^2 & 1 \\ -1 & 0 \end{pmatrix},$$

which are identified with the corresponding integral bilinear forms, where p and q are odd integers with $1 \leq p < q$. Note that they are both unimodular and

$$S_0 = A_0 + \varepsilon\, {}^tA_0 = S_1 = A_1 + \varepsilon\, {}^tA_1 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

where $\varepsilon = -1$. Let us show that A_0 and A_1 are algebraically cobordant in the sense of Definition 5.3 for $\varepsilon = -1$.

Let r be the greatest common divisor of p and q and set $p = rp'$ and $q = rq'$. Furthermore, set $m = {}^t(q', 0, p', 0)$ and $m' = {}^t(0, p', 0, q')$. Then it is easy to see that the submodule M of $\mathbf{Z}^4$ generated by m and m' constitutes a metabolizer for $A = A_0 \oplus (-A_1)$. Since $S_0 = S_1$ are non-degenerate, we have only to verify condition (c2) of Definition 5.3.

Set $S = S_0 \oplus (-S_1) = A - {}^tA$. Let $S^* : \mathbf{Z}^4 \to \mathbf{Z}^4$, $S_0^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ and $S_1^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the adjoints of S, S_0 and S_1 respectively. It is easy to see that $\operatorname{Coker} S_0^* = \operatorname{Coker} S_1^*$ is naturally identified with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Furthermore, we have

$$S^*(m) = {}^tmS = (0, 2q', 0, -2p') \quad \text{and} \quad S^*(m') = {}^tm'S = (-2p', 0, 2q', 0).$$

Therefore, $S^*(M)^\wedge$, the smallest direct summand of $\mathbf{Z}^4$ containing $S^*(M)$, is the submodule of $\mathbf{Z}^4$ generated by $(0, q', 0, -p')$ and $(-p', 0, q', 0)$. Hence, for the natural quotient map $d : \mathbf{Z}^4 \to \operatorname{Coker} S^* = (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_2)$, we have

$$d\big(S^*(M)^\wedge\big) = \big\{(x, x) \,:\, x \in \operatorname{Coker} S_0^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2\big\},$$

since $\operatorname{Im} S_i^*$ is generated by $(2, 0)$ and $(0, 2)$, $i = 0, 1$, and $\operatorname{Im} S^*$ is generated by $(2, 0, 0, 0)$, $(0, 2, 0, 0)$, $(0, 0, 2, 0)$ and $(0, 0, 0, 2)$. Therefore, we conclude that the unimodular matrices A_0 and A_1 are algebraically cobordant.

Since

$$(0, 1) \begin{pmatrix} p^2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = p^2$$

and

$$(a, b) \begin{pmatrix} q^2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = q^2a^2$$

then there exists an element $x \in \mathbf{Z}^2$ such that ${}^tx\, A_0\, x = p^2$. Moreover such an element does not exist for A_1 because p and q are both odd integers with $1 \leq p < q$. Hence A_0 and A_1 are not congruent,

5.3 Equivalence relation

Let us come back to the algebraic case. We will prove that algebraic cobordism is an equivalence relation on the set $\mathcal{A}$ of unimodular bilinear forms defined on free $\mathbf{Z}$-modules of finite rank, but before we need some preliminary results.

Notations. Let A_0 and A_1 be two algebraically cobordant forms, let A be the form $A_0 \oplus -A_1$ defined on $G = G_0 \oplus G_1$ and S be $A + \varepsilon\, {}^t\!A$.

We will use the following notations, if E is any subset of G we denote by $\langle E \rangle$ the submodule of G, generated by E. If L is any submodule of G then

$$L^{\perp} = \big\{ x \in G \ \mid\ S(x,l) = 0\ \forall l \in L \big\}$$

$$\mathrm{Hom}_{\mathbf{Z}}\big(G_{|L}, \mathbf{Z}\big) = \big\{ f \in G^* \ \mid\ f(l) = 0\ \forall l \in L \big\}$$

Moreover if L_1 and L_2 are two submodules of G, orthogonal for S, we denote by $L_1 \oplus^{\perp} L_2$ their (orthogonal) direct sum.

Variation map. We construct a new map denoted by V and called *variation map*.[2]

Let A, an integral bilinear form defined on a free $\mathbf{Z}$-module of finite rank G, be in $\mathcal{A}$. Since A is unimodular, then for all f in $\mathrm{Hom}(G, \mathbf{Z})$ there exists an unique y_f in G such that $A(., y_f) = f$. We define

$$\begin{array}{rcl} V : \mathrm{Hom}(G, \mathbf{Z}) & \longrightarrow & G \\ f & \mapsto & y_f \end{array}$$

Then for all x in G we have $A(x, y_f) = f(x)$.

If we denote as well by A and V the matrices of the bilinear form and the associated variation map, relatively to a basis for G and its dual basis for $\mathrm{Hom}(G, \mathbf{Z})$, then $A.y_f = f$ and we have the following equality of matrices

$$V = A^{-1}.$$

Moreover since for all f in $\mathrm{Hom}(G, \mathbf{Z})$ and all x in G we have

$$A\big(x, V(f)\big) = f(x),$$

so when $f = S^*(y)$, then for all y in G we have $f(x) = S(x,y)$ and

$$A\Big(x, V\big(S^*(y)\big)\Big) = S(x,y). \tag{5.1}$$

Recall that we have $V = A^{-1}$ and $S^* = A + \varepsilon\, {}^t\!A$. If we denote by I the identity map on G and by I' the identity map on $\mathrm{Hom}(G, \mathbf{Z})$, then the maps

$$V \circ S^* : G \to \mathrm{Hom}(G, \mathbf{Z}) \to G$$

$$S^* \circ V : \mathrm{Hom}(G, \mathbf{Z}) \to G \to \mathrm{Hom}(G, \mathbf{Z})$$

[2] This map is defined as the usual variation map associated with the Seifert form for fibered knots, cf. Definition 4.3.

are well defined. Moreover we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(S^*) \arrow[r] \arrow[dr] & G \arrow[r, "S^*"] \arrow[d, "I-H"'] & \mathrm{Hom}(G,\mathbf{Z}) \arrow[r, "d"] \arrow[dl, "V"'] \arrow[d, "I'-H'"] & \mathrm{Coker}(S^*) \\
 & G \arrow[r, "S^*"] & \mathrm{Hom}(G,\mathbf{Z}) \arrow[ur, "d"'] &
\end{tikzcd} \tag{5.2}$$

Note that the maps S^* and $I - H = V \circ S^*$ have same kernel since V is an isomorphism. Then the previous commutative diagram gives

$$V\big(S^*(G)^\wedge\big) \cap \mathrm{Ker}(S^*) \quad = \quad \{0\}. \tag{5.3}$$

Moreover the morphisms H and H' are fulfilling

$$I - H = A^{-1}\big(A + \varepsilon\, {}^t\!A\big) = I + \varepsilon\, A^{-1}\, {}^t\!A,$$

$$I' - H' = \big(A + \varepsilon\, {}^t\!A\big)A^{-1} = I' + \varepsilon\, {}^t\!AA^{-1}$$

so we have

$$H = -\varepsilon\, A^{-1}\, {}^t\!A,$$
$$H' = -\varepsilon\, {}^t\!AA^{-1}.$$

We will use the variation map V to describe metabolizers associated with algebraically cobordant unimodular integral bilinear forms defined on free modules of finite ranks.

First properties of variation map. From now we suppose that the two elements A_0 and A_1 of $\mathcal{A}$ are algebraically cobordant. And we denote by M the metabolizer in the sense of Definition 5.3. As before set $A = A_0 \oplus -A_1$, $S = A + \varepsilon\, {}^t\!A$ and S^* be the adjoint of S. We denote by V the variation map associated with A.

In the following, when $\varphi : R \to S$ is an isomorphism of $\mathbf{Z}$-modules, then we will denote by $\Delta(\varphi)$ the submodule

$$\Delta(\varphi) = \big\{(x, \varphi(x)); x \in R\big\} \subset R \oplus S.$$

Lemma 5.4. *For all x and y in G we have*

$$A\big(H(x), H(y)\big) = A(x, y).$$

Proof. Let x and y be in G, then

$$\begin{aligned}
A\big(H(x), H(y)\big) &= {}^t(-\varepsilon A^{-1}\, {}^t\!A\, x)\, A\, (-\varepsilon A^{-1}\, {}^t\!A\, y) \\
&= {}^t\!x A\, {}^t\!A^{-1}\, A\, A^{-1}\, {}^t\!A\, y \\
&= {}^t\!x\, A\, y \\
&= A(x, y)
\end{aligned}$$

□

Lemma 5.5. *When M is a metabolizer for A that gives the algebraic cobordism of A_0 and A_1, then the submodule*

$$H(M)^\wedge$$

of G is a metabolizer for A that gives the algebraic cobordism of A_0 and A_1 as well.

Proof. As a direct consequence of the previous Lemma we have that for all x and y in $H(M)^\wedge$

$$A(x, y) = 0.$$

Moreover since H is unimodular then the rank of $H(M)^\wedge$ is half the rank of G.

Since H is the identity on $\operatorname{Ker} S^*$ then $H(M)^\wedge$ fulfills c.1 in Definition 5.3.

With diagram 5.2 we see that $S^* \circ H = H' \circ S^*$ and H' is equal to identity on $d^{-1}\big(\operatorname{Coker} S^*\big)$. Hence $H(M)^\wedge$ fulfills c.2 in Definition 5.3. □

Lemma 5.6. *When M is a metabolizer for A that gives the algebraic cobordism of A_0 and A_1, then we have the following decomposition*

$$M = \Delta(\varphi) \oplus \big(V \circ S^*(M)\big)^\wedge$$

where φ is the isomorphism between $\operatorname{Ker} S_0^*$ and $\operatorname{Ker} S_1^*$ that gives the algebraic cobordism of A_0 and A_1.

Proof. Let m and n be in M, then according to (5.1) we have

$$A\Big(m, V\big(S^*(n)\big)\Big) = S(m, n) = A(m, n) + \varepsilon\, A(n, m) = 0.$$

Hence if $V\big(S^*(n)\big) = \alpha\,\nu$, with ν indivisible, is not in M, then in a basis for G in which ν is an element the matrix of A is of the form

$$\begin{pmatrix} & & & * \\ & \mathcal{O} & & \vdots \\ & & & * \\ 0 & \dots & 0 & * \\ & * & & * \end{pmatrix}$$

where $\mathcal{O}$ is a null square bloc of size half of the matrix corresponding to the metabolizer.

But this imply that the determinant of A must be zero, which is not possible because A is unimodular.

Finally $V\big(S^*(n)\big)$ is in M so

$$V\big(S^*(M)\big) \subset M.$$

Moreover we have

$$M \cap \operatorname{Ker} S^* = \Delta(\varphi),$$

$$V\big(S^*(M)\big) \cap \operatorname{Ker} S^* = \{0\}$$

and

$$\operatorname{rank}\Big(V\big(S^*(M)\big)\Big) = \operatorname{rank} \overline{M},$$

then we have the following decomposition

$$M = \Delta(\varphi) \oplus \big(V \circ S^*(M)\big)^\wedge.$$

□

Preliminary results.

Lemma 5.7. *The following holds* $S^*(G) \cap S^*(M)^\wedge = S^*(M^\perp)$.

Proof. Let r be the rank of $\operatorname{Ker} S_0^*$ and s be the rank of $S^*(M)$. As M is a metabolizer for S which fulfills condition c.1 in Definition 5.3 we have

$$\operatorname{rank}(\operatorname{Ker} S^*) = 2 \operatorname{rank}(M \cap \operatorname{Ker} S^*) = 2 \operatorname{rank}(\operatorname{Ker} S_0^*) = 2\, r,$$

moreover $\operatorname{rank}\big(S^*(G)\big) = 2\, s$ and $\operatorname{rank}(M^\perp) = s + 2\, r$. Hence

$$M^\perp = (M + \operatorname{Ker} S^*)^\wedge$$

and

$$S^*(M^\perp) \subset S^*(G) \cap S^*(M)^\wedge.$$

On the other hand, since $S^*(M)$ is of finite index in $\operatorname{Hom}_{\mathbf{Z}}\big(G_{|M^\perp}; \mathbf{Z}\big)$ and $\operatorname{Hom}_{\mathbf{Z}}\big(G_{|M^\perp}; \mathbf{Z}\big)$ is a pure submodule of G^*, then we have

$$S^*(M)^\wedge = \operatorname{Hom}_{\mathbf{Z}}\big(G_{|M^\perp}; \mathbf{Z}\big).$$

So if $S^*(x) \in S^*(M)^\wedge$, then $S^*(x, l) = 0$ for all l in $M^\perp$ and x is in $M^\perp$. This gives

$$S^*(G) \cap S^*(M)^\wedge \subset S^*(M^\perp),$$

and the Lemma is proved. □

Since $S^*(M)$ is of finite index in $S^*(M)^\wedge$, one can write

$$\big(S^*(M)^\wedge\big)_{/S^*(M)} \cong \bigoplus_{i=1}^{s} \mathbf{Z}_{/a_i\,\mathbf{Z}}$$

where $a_i \in \mathbf{Z} \setminus \{0\}$ and a_i divides a_{i+1} (we do not exclude that there exists an integer l such that $a_i = 1$ for $i = 1, \ldots, l$).

Proposition 5.8. *The submodule* $\overline{M}$ *is pure in* $\overline{G}$ *if and only if*

$$S^*(M^\perp) = S^*(M).$$

Proof.

— First, suppose that $\overline{M}$ is pure in $\overline{G}$. Since we have

$$\operatorname{rank}\big(M \cap \operatorname{Ker} S^*\big) = \operatorname{rank}\big(\Delta(\varphi)\big) = r,$$

then the module $M + \operatorname{Ker} S^*$ has rank $s + 2r$. Then $M + \operatorname{Ker} S^*$ is of finite index in $M^\perp$.

Let x be in $M^\perp$; there exists a positive integer k such that $kx = y + m$, where y is in $\operatorname{Ker} S^*$ and m is in M ; so

$$\overline{m} = k\overline{x}.$$

Since $\overline{M}$ is pure in $\overline{G}$ then $\overline{x}$ is in $\overline{M}$, so there exists y' in $\operatorname{Ker} S^*$ such that $x + y'$ is in M. Finally $S^*(x) = S^*(x + y') \in S^*(M)$, and $S^*(M^\perp) \subset S^*(M)$. But $M \subset M^\perp$ so $S^*(M^\perp) = S^*(M)$.

— Second, suppose that $S^*(M) = S^*(M^\perp)$. We will prove that $\overline{M^\perp}$ is pure in $\overline{G}$.

Let z be in $M^\perp$ with $\overline{z} = k\overline{x}$ where x is in G and k is a positive integer. So there exists y in $\operatorname{Ker} S^*$ such that $kx = z + y$. For all m in M we have $S(kx, m) = S(z + y, m) = 0$, so $S(x, m) = 0$ and x is in $M^\perp$.

Now we prove that $S^*(M^\perp) = S^*(M)$ implies $\overline{M} = \overline{M^\perp}$.

Let z be in $M^\perp$. If $S^*(z) = f$ there exists m in M such that $S^*(m) = f$. So $z - m = y$ is in $\operatorname{Ker} S^*$, and $\overline{z} = \overline{m}$ is in $\overline{M}$. Finally, since $\overline{M^\perp}$ is pure in $\overline{G}$ and $\overline{M^\perp} \subset \overline{M}$ we get $\overline{M^\perp} = \overline{M}$ is pure in $\overline{G}$.

□

By Definition 5.3 $\overline{M}$ is pure in $\overline{G}$, so Lemma 5.7 and Proposition 5.8, and, conditions c.1 and c.2 in Definition 5.3 imply that $\operatorname{Coker} S^*$ is isomorphic to

$$\mathbf{Z}^{2r} \oplus \Big(\bigoplus_{i=1}^{s} \mathbf{Z}/_{a_i}\,\mathbf{Z}\Big)^2.$$

Now we will show how the algebraic cobordism between A_0 and A_1 allows us to describe S. To fix the notation, let M, φ and θ be as in Definition 5.3, m be $\operatorname{rk}(G)$ and r be $\operatorname{rk}(\operatorname{Ker} S_0^*)$. As a consequence of Definition 5.3 we have $s = \operatorname{rk}(S^*(M)) = \frac{1}{2}\operatorname{rk}(S^*(G))$ and $\operatorname{rk}(M) = r + s = \frac{m}{2}$.

Proposition 5.9. *There exists a basis $\mathcal{B} = \{m_i, m_i^*;{\scriptstyle i=1,\dots,s+r}\}$ of G such that:*

1. *$\{m_i;{\scriptstyle i=1,\dots,s+r}\}$ is a basis of M,*
2. *$\{m_i, m_i^*;{\scriptstyle i=s+1,\dots,s+r}\}$ is a basis of* $\operatorname{Ker} S^*$ *and $\{m_i^*;{\scriptstyle i=s+1,\dots,s+r}\}$ is a basis of* $\operatorname{Ker} S_0^*$,
3. *the submodules $\langle m_i, m_i^*\rangle, {\scriptstyle i=1,\dots,s+r}$; are orthogonal for S, and*

$$G = \bigoplus_{1\le i\le s+r}^{\perp} \langle m_i, m_i^*\rangle,$$

4. *when ${\scriptstyle i=1,\dots,s}$, $S(m_i, m_i^*) = a_i$.*

Definition 5.10. Such a basis is called a good basis of G associated to M.

Proof. of Proposition 5.9. We have seen that $S^*(M)^\wedge = \operatorname{Hom}_{\mathbf{Z}^i}(G_{|M^\perp}; \mathbf{Z})$. Let M_0 be any direct summand complement of $(M \cap \operatorname{Ker} S^*)$ in M. There exits a basis $\{m_i;{\scriptstyle i=1,\dots,s}\}$ of M_0 and a basis $\{h_i;{\scriptstyle i=1,\dots,s}\}$ of $\operatorname{Hom}_{\mathbf{Z}^i}(G_{|M^\perp}; \mathbf{Z})$ such that

$$S^*(m_i) = a_i\, h_i$$

where $a_i \in \mathbf{Z} \setminus \{0\}$ and a_i divides a_{i+1}. Let m_1^* be any element in G such that $G = \operatorname{Ker} h_1 \oplus \langle m_1^*\rangle$ and $h_1(m_1^*) = S(m_1, m_1^*).a_1^{-1} = 1$.

We will first prove that for all x in G, a_1 divides $S(x, m_1^*)$.

— If $a_1 = 1$ it is obvious.

— If $a_1 > 1$, condition c.2 in (1.2) implies that $\left(S^*(G)^\wedge\right)_{/S^*(G)}$ is isomorphic to $\left(\left(S^*(M)^\wedge\right)_{/S^*(M)}\right)^2 \cong \left(\bigoplus_{i=1}^{s} \mathbf{Z}_{/a_i\,\mathbf{Z}}\right)^2$ and the rank of $S^*(G)$ is $2\,s$.

So a_1 divides $S^*(x)$ for all x in G.

Now, we will construct an orthogonal complement $(M_1 \oplus R_1)$ for $\langle m_1, m_1^*\rangle$ in G such that

i) $M = \langle m_1\rangle \oplus M_1$,

ii) $\operatorname{Ker} h_1 = M \oplus R_1$.

Let M_1 be the submodule of M generated by $m_i' = m_i - a_1^{-1}\, S(m_i, m_1^*).m_1$, $2 \le i \le s$, and $M \cap \operatorname{Ker} S^*$. By construction M_1 is orthogonal to $\langle m_1, m_1^*\rangle$ and $M = \langle m_1\rangle \oplus M_1$.

By construction $\operatorname{Ker} h_1$ is orthogonal to m_1 and M is in $\operatorname{Ker} h_1$.

If $\{x_{i,\, i=2,\dots,s+r}\}$ is a basis of any direct summand complement of M in $\operatorname{Ker} h_1$, let R_1 be the submodule of $\operatorname{Ker} h_1$ generated by x_i' where

$$x_i' = x_i - a_1^{-1}\, S(x_i, m_1^*).m_1.$$

Then $\operatorname{Ker} h_1 = \langle m_1\rangle \oplus M_1 \oplus R_1$ and R_1 is orthogonal to m_1^*.

Now we have an orthogonal decomposition of G in $\langle m_1, m_1^*\rangle \oplus^\perp (M_1 \oplus R_1)$. By induction on s we obtain an orthogonal decomposition

$$G = (\oplus^\perp \langle m_i, m_i^*\rangle) \oplus^\perp (M_s \oplus R_s) \text{ where } \operatorname{Ker} S^* = M_s \oplus R_s.$$

Let $\{m_{s+1}, \dots, m_{s+r}\}$ be any basis of $\operatorname{Ker} S^* \cap M$. Thanks to condition c.1, $\operatorname{Ker} S^* \cap M = \{(x, \varphi(x)); x \in \operatorname{Ker} S_0^*\}$. So any basis $\{m_{s+1}^*, \dots, m_{s+r}^*\}$ of $\operatorname{Ker} S_0^*$ can be used to build up a basis of G which fulfills the statement of Proposition 5.9. □

The following proposition is sometimes useful since it gives an equivalent definition of algebraic cobordism.

Proposition 5.11. *Let A_0 and A_1 be in $\mathcal{A}$. Then A_0 is algebraically cobordant to A_1 if and only if there exists a pure submodule H of $G = G_0 \oplus G_1$ on which $A = A_0 \oplus -A_1$ vanishes, an isomorphism φ from* $\operatorname{Ker} S_0^*$ *to* $\operatorname{Ker} S_1^*$ *and an isomorphism θ from* $\operatorname{Tors}(\operatorname{Coker} S_0^*)$ *to* $\operatorname{Tors}(\operatorname{Coker} S_1^*)$ *such that:*

c.11: $\Delta(\varphi) \subset H$,

c.12: *the image $\overline{H}$ of H in $\overline{G} = G_{/\operatorname{Ker} S^*}$ is a metabolizer for $\overline{S} = \overline{S_0} \oplus -\overline{S_1}$,*

c.2 : $d(S^*(H)^\wedge) = \Delta(\theta)$.

Proof. Let M, φ, θ be as in Definition 5.3. Then M satisfies conditions c.1 and c.2. The existence of φ shows that $\operatorname{Ker} S_0^*$ and $\operatorname{Ker} S_1^*$ have the same rank, r. So the rank of $\overline{G}$ is $(m_0 + m_1 - 2\,r)$. Since M is a metabolizer for A

$$\operatorname{rk}(M) = \frac{m_0 + m_1}{2},$$

and by c.1 we get
$$M \cap \operatorname{Ker} S^* = \Delta(\varphi).$$
So
$$\operatorname{rk}(\overline{M}) = \frac{m_0 + m_1}{2} - r$$
and $\overline{S}$ vanishes on $\overline{M}$. It implies that $\overline{M}$ is a metabolizer for $\overline{S}$.

Conversely let H, φ and θ be as in the statement of Proposition 5.11. As $\Delta(\varphi)$ is pure in H and in $\operatorname{Ker} S^*$, then there exists a direct sum decomposition
$$H \cap \operatorname{Ker} S^* = \Delta(\varphi) \oplus M_0.$$
Moreover, since $\operatorname{Ker} S^*$ is pure in G, then there exists also a direct sum decomposition
$$H = M_1 \oplus (H \cap \operatorname{Ker} S^*).$$

Let M be $M_1 \oplus \Delta(\varphi)$, by construction A vanishes on M, and
$$M \cap \operatorname{Ker} S^* = \Delta(\varphi), \quad S^*(M) = S^*(H).$$
So M, φ and θ satisfy c.1 and c.2 of Definition 5.3. Furthermore, $\overline{H} = \overline{M_1} = \overline{M}$ and by c.12 the rank of $\overline{H}$ is $\frac{m_0+m_1}{2} - r$. But M_1 being isomorphic to $\overline{M_1}$, the rank of M is $\frac{m_0+m_1}{2}$ and M is a metabolizer for A. $\square$

The following Lemma describes how to construct metabolizers by transitivity.

Lemma 5.12. *Let $B_i : G_i \times G_i \to \mathbf{Z}$ be in $\mathcal{A}$, $i = 0, 1, 2$. Let m_i be the rank of G_i. If there exists a metabolizer H_{01} (resp. H_{12}) for $B_0 \oplus -B_1$ (resp. $B_1 \oplus -B_2$) and if the B_i are non-degenerate, then*

the form $B_0 \oplus -B_2$ vanishes on $H_{02} = \pi(L)$ and,

$\operatorname{rk} H_{02} = \frac{1}{2}\operatorname{rk}(G_0 \oplus G_2)$,

where: $G = G_0 \oplus G_1 \oplus G_1 \oplus G_2$, $H = H_{01} \oplus H_{12}$, $\Delta = \{(y, y) \in G_1 \oplus G_1 \; ; \; y \in G_1\}$, $L = H \cap (G_0 \oplus \Delta \oplus G_2)$ and π is the projection of G on $G_0 \oplus G_2$.

Proof. As $B_0 \oplus -B_2$ vanishes on H_{02} by construction, it is sufficient to prove that the rank of H_{02} is $\frac{m_0+m_1}{2}$. The definition of H_{02} gives the following exact sequence:
$$0 \to L \cap \Delta \xrightarrow{i} L \xrightarrow{\pi} H_{02} \to 0.$$
So we get:
$$(*) \quad \operatorname{rk}(L) = \operatorname{rk}(L \cap \Delta) + \operatorname{rk}(H_{02}).$$

If v is in H, there exists unique x in G_0, y_1 and y_2 in G_1 and z in G_2 such that $v = (x, y_1, y_2, z)$. Let $\rho : H \to G_1 \oplus G_1$ be defined by $\rho(v) = (y_1 - y_2, 0)$. Let us denote by L_1 the image $\rho(H)$. By construction L is the kernel of ρ and we get the exact sequence: $0 \to L \xrightarrow{i} H \xrightarrow{\rho} L_1 \to 0$. Both this sequence and $(*)$ show:
$$(**) \quad \frac{m_0 + m_2 + 2\,m_1}{2} - \operatorname{rk}(L_1) = \operatorname{rk}(L \cap \Delta) + \operatorname{rk}(H_{02}).$$

Relatively to the form $(B_1 \oplus -B_1)$ we have the following decomposition
$$(\Delta \cap L) \oplus^{\perp} (L_1 \oplus \Delta) \quad (***).$$

Indeed, Δ is self-orthogonal ; if (y, y) is in $\Delta \cap L$, then $(0, y)$ is in H_{01} and $(y, 0)$ is in H_{12}. On the other hand, an element of L_1 is of the form $(y_1, -y_2)$ where there exists (x, y_1) in H_{01} and (y_2, z) in H_{12}. So $B_1(y, y_1) = B_1(y_1, y) = 0$ and $-B_1(y, y_2) = -B_1(y_2, y) = 0$.

Since the rank of $L_1 \oplus \Delta$ is equal to $m_1 + \mathrm{rk}(L_1)$, then the property $(***)$ implies that the rank of the restriction of $B_1 \oplus -B_1$ to $(\Delta \cap L) \times (G_1 \oplus G_1)$ is smaller or equal to $m_1 - \mathrm{rk}(L_1)$. But $B_1 \oplus -B_1$ is non-degenerate by hypothesis, so

$$\mathrm{rk}(\Delta \cap L) \leq m_1 - \mathrm{rk}(L_1).$$

With $(**)$ it implies

$$\frac{m_0 + m_2}{2} \leq \mathrm{rk}(H_{02}).$$

But B_0 and B_2 are non-degenerate by hypothesis and as $B_0 \oplus -B_2$ vanishes on H_{02}, so $\mathrm{rk}(H_{02}) \leq \frac{m_0+m_2}{2}$ and

$$\mathrm{rk}(H_{02}) = \frac{m_0 + m_2}{2}.$$

It ends the proof of the lemma. □

Equivalence relation. We are now ready to prove the following Theorem.

Theorem 5.13. *Algebraic cobordism is an equivalence relation on the set $\mathcal{A}$ of unimodular bilinear forms defined on free $\mathbf{Z}$-modules of finite rank.*

Proof. The only non trivial property to check is the transitivity of the relation "algebraic cobordism".

Let A_i be algebraically cobordant to A_{i+1}, $i = 0, 1$. Let $M_{i,i+1}$ be a metabolizer for $A_i \oplus -A_{i+1}$ with the isomorphisms φ_i and θ_i fulfilling conditions c.1 and c.2 in Definition 5.3.

Set

$$G = G_0 \oplus G_1 \oplus G_1 \oplus G_2,$$

$$S_{02} = S_0 \oplus -S_2,$$

$$G_{02} = G_0 \oplus G_2,$$

$$S = S_0 \oplus -S_1 \oplus S_1 \oplus -S_2,$$

$$\Delta = \big\{(x, x)\ ;\ x \in G_1\big\} \subset G_1 \oplus G_1,$$

d be the quotient map from G to $\mathrm{Coker}\, S^*$ and d_{02} be the quotient map from G_{02} to $\mathrm{Coker}\, S_{02}^*$. Let π (resp. $\widetilde{\pi}$) be the obvious projection from G (resp. $\mathrm{Coker}\, S^*$) to $G_0 \oplus G_2$ (resp. $\mathrm{Coker}\, S_{02}^*$).

$$d : G \to \mathrm{Coker} S^*$$

$$d_{02} : G_{02} \to \mathrm{Coker}\ S_{02}^*$$

$$\pi : G \to G_0 \oplus G_2$$

$$\widetilde{\pi} : \mathrm{Coker} S^* \to \mathrm{Coker} S_{02}^*$$

Recall that $\overline{M}_{i,i+1}$ is pure in $\overline{G}_i \oplus \overline{G}_{i+1}$, and with Lemma 5.6 we have the following decompositions

$$G_i \oplus G_j = \operatorname{Ker}(S_i^*) \oplus \Delta(\varphi_i) \oplus V_{i,i+1}\big(S_{i,i+1}^*(M_{i,i+1})^\wedge\big) \oplus R_{i,i+1}.$$

Set $T_{i,i+1} = V_{i,i+1}\big(S_{i,i+1}^*(M_{i,i+1})^\wedge\big) \oplus R_{i,i+1}$ then we have the following decomposition

$$G = \operatorname{Ker} S_{01}^* \oplus \operatorname{Ker} S_{12}^* \oplus T_{01} \oplus T_{12}.$$

Let us denote by T_0 (resp. T_1, T_1', T_2) the projection of T_{01} (resp. T_{01}, T_{12}, T_{12}) to G_0 (resp. G_1, G_1, G_2).

Set $T_{02} = \pi(T_{01} \oplus T_{12}) = T_0 \oplus T_2$, and let

$$M_{02} = \Big(\pi\big(V_{01}(S_{01}^*(M_{01})^\wedge)\big) \oplus V_{12}\big(S_{12}^*(M_{12})^\wedge\big)\big) \cap (G_0 \oplus \Delta \oplus G_2)\Big)^\wedge$$

be the smallest pure submodule of T_{02} which contains the projection of

$$L = \big(V_{01}(S_{01}^*(M_{01})^\wedge)\big) \oplus V_{12}\big(S_{12}^*(M_{12})^\wedge\big)\big) \cap (G_0 \oplus \Delta \oplus G_2)$$

on T_{02}.

Since the forms are unimodular, then they are non-degenerate. Hence, according to Lemma 5.12 the module M_{02} is a metabolizer for $A_0 \oplus -A_1$.

As $A = A_0 \oplus -A_2$, we set $\varphi = \varphi_1 \circ \varphi_0$ and $\theta = -(\theta_1 \circ \theta_0)$.

By Proposition 5.11, to prove that A_0 is algebraically cobordant to A_2 it is sufficient to prove that $M_{02} = \Delta(\varphi) \oplus V_{02}\big(S_{02}^*(M_{02})^\wedge\big)$ is a metabolizer for $A_0 \oplus -A_2$, and, M_{02} fulfill conditions c.11, c.12 and c.2. First we remark that M_{02} fulfills c.11 by definition.

Lemma 5.14. *The submodule M_{02} satisfies $d_{02}\big(S_{02}^*(M_{02})^\wedge\big) = \Delta(-\theta_1 \circ \theta_0)$.*

Proof. By definition we have

$$d\big(S^*(G)^\wedge\big) = \operatorname{Tors}(\operatorname{Coker} S^*)$$

and

$$d_{02}\big(S_{02}^*(M_{02})^\wedge\big) = \widetilde{\pi}\big(d(S^*(L)^\wedge)\big).$$

But c.2 for the two metabolizers M_{01} and M_{12} imply

$$d\big(S^*(L)^\wedge\big) = \big(\Delta(\theta_0) \oplus \Delta(\theta_1)\big) \cap d\big(S^*(G_0 \oplus \Delta \oplus G_2)^\wedge\big),$$

so we get $d(S^*(L)^\wedge) = \Big\{\big(x, \theta_0(x), y, \theta_1(y)\big); x \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\ ,\ y = -\theta_0(x)\Big\}$.

And finally the following holds

$$d_{02}\big(S_{02}^*(M_{02})^\wedge\big) = \Big\{\big(x, -\theta_1 \circ \theta_0(x)\big); x \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\Big\} = \Delta(-\theta_1 \circ \theta_0).$$

$\square$

Lemma 5.15. *The submodule $\overline{M_{02}}$ is a metabolizer for $\overline{S_0} \oplus -\overline{S_2}$.*

Proof. By construction $\overline{S_0} \oplus - \overline{S_2}$ vanish on the submodule $\overline{M_{02}} = \overline{V_{02}\big(S_{02}^*(M_{02})^\wedge\big)}$, and $\mathrm{rk}\, \overline{V_{02}\big(S_{02}^*(M_{02})^\wedge\big)} = \frac{1}{2}\mathrm{rk}\,(\overline{G_0} \oplus \overline{G_2})$.

Suppose that $\overline{M_{02}}$ is not pure, then there exists an indivisible element m in M_{02} and an element x not in M_{02} such that for an integer $\alpha \neq \pm 1$ we have

$$m - \alpha\, x = \beta\, \kappa \in \mathrm{Ker}\, S^*,$$

where κ is indivisible and β is an integer which is relatively prime to α because m is indivisible.

We have $\alpha\, x + \beta\, \kappa$ in M_{02} so if κ is in M_{02} then $\alpha\, x$ is in M_{02}, but M_{02} is a pure submodule of $G_0 \oplus G_2$ so x must be in M_{02}. Hence we can assume that κ is not in M_{02}.

Now for all μ in M_{02} we have

$$A(x, \mu) = \frac{1}{\alpha} A(\alpha\, x, \mu) = \frac{1}{\alpha} A(m - \beta\, \kappa, \mu) = -\frac{1}{\alpha} A(\beta\, \kappa, \mu).$$

Recall that $A(x, \mu)$ is an integer, this implies that $\beta\, A(\kappa, \mu) \in \alpha\, \mathbf{Z}$, but we have α and β relatively prime so

$$A(\kappa, \mu) \in \alpha\, \mathbf{Z}.$$

In a basis for $G_0 \oplus G_2$ beginning with a basis of M_{02} and κ, the matrix of $A_0 \oplus -A_2$ is of the form

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}} & \vdots \\ \star \;\; \dots \;\; \star & \\ \dots & . \end{pmatrix}$$

where the entries $\star$ are all in $\alpha\, \mathbf{Z}$ and the square sub matrix with entries 0 is of size half of the matrix.

But this implies that the determinant of $A_0 \oplus -A_2$ is in $\alpha\, \mathbf{Z}$, and since the forms A_0 and A_2 are unimodular we must have $\alpha = \pm 1$ which is impossible according to our assumption.

Finally $\overline{M_{02}}$ is pure in $\overline{G_0} \oplus G_2$ and it is a metabolizer for $\overline{S_0} \oplus -\overline{S_2}$. $\square$

The above properties of M_{02}, and, Lemmas 5.14-5.15 imply conditions c.11, c.12 and c.2 of Proposition 5.11, and A_0 is algebraically cobordant to A_2. $\square$

Properties of V, H and H'. To finish this chapter we give some useful relations between the forms introduced before.

Set

$$\begin{array}{rccc} <.,.>: & \mathrm{Hom}(G, \mathbf{Z}) \times G & \to & \mathbf{Z} \\ & (\alpha, \beta) & \mapsto & \alpha(\beta) \end{array}$$

be the pairing such that for all x in G and all y in G we have

$$< S^*(x), y >= S(x, y).$$

Lemma 5.16. *For all x in $\mathrm{Hom}(G, \mathbf{Z})$ and all y in $\mathrm{Hom}(G, \mathbf{Z})$ we have*

$$< H'(x), V(y) >= -\varepsilon < y, V(x) > .$$

Proof. Let x be in $\mathrm{Hom}(G,\mathbf{Z})$ and y be in $\mathrm{Hom}(G,\mathbf{Z})$, then

$$\begin{aligned}
< H'(x), V(y) > &= {}^t\!x(-\varepsilon\, {}^t\!A^{-1}A)A^{-1}\,y \\
&= -\varepsilon\, {}^t\!x\, {}^t\!A^{-1}\,y \\
&= -\varepsilon\, {}^t\!(A^{-1}x)\,y \\
&= -\varepsilon\, {}^t\!y(A^{-1}x) \\
&= -\varepsilon < y, V(x) >
\end{aligned}$$

□

Lemma 5.17. *For all x in $\mathrm{Hom}(G,\mathbf{Z})$ and all y in $\mathrm{Hom}(G,\mathbf{Z})$ we have*

$$S\big(V(x),V(y)\big) = \varepsilon < x, V(y) > + < y, V(x) > .$$

Proof. Let x be in $\mathrm{Hom}(G,\mathbf{Z})$ and y be in $\mathrm{Hom}(G,\mathbf{Z})$, then

$$\begin{aligned}
S\big(V(x),V(y)\big) &= {}^t\!x\, {}^t\!A^{-1}(A+\varepsilon\, {}^t\!A)A^{-1}y \\
&= {}^t\!x(\, {}^t\!A^{-1} + \varepsilon\, A^{-1})y \\
&= {}^t\!(A^{-1}x)y + \varepsilon\, {}^t\!xA^{-1}y \\
&= {}^t\!yA^{-1}x + \varepsilon\, {}^t\!xA^{-1}y \\
&= < y, V(x) > + \varepsilon < x, V(y) >
\end{aligned}$$

□

Lemma 5.18. *For all x in $\mathrm{Hom}(G,\mathbf{Z})$ and all y in G we have*

$$< H'(x), H(y) > = < x, y > .$$

Proof. Let x in $\mathrm{Hom}(G,\mathbf{Z})$ and all y in G, then we have

$$\begin{aligned}
< H'(x), H(y) > &= {}^t\!x\, {}^t\!A^{-1}A\,A^{-1}\, {}^t\!Ay \\
&= {}^t\!x\,y \\
&= < x, y >
\end{aligned}$$

□

Chapter 6

Cobordism of simple fibered $(2n-1)$-knots with $n \geq 3$

" L'essence des mathématiques c'est la liberté."
Georg Cantor

In this Chapter, we will give the classification of simple fibered $(2n-1)$-knots up to cobordism for $n \geq 3$.

To begin we have to mention that Durfee [36] and Kato [65] independently proved an analogue of Theorem 3.3 for (not necessarily spherical) simple fibered knots as follows.

Theorem 6.1. *Let $n \geq 3$. There is a one-to-one correspondence of isotopy classes of simple fibered $(2n-1)$-knots in S^{2n+1} and equivalence classes of integral unimodular bilinear forms. the correspondence associates to each knot its Seifert form.*

Where equivalence classes of unimodular bilinear forms are given by isomorphism classes, which correspond to congruence classes of integral square matrices.

We already proved that Seifert forms associated with simple fibered knots are unimodular, hence Theorem 6.1 shows that simple fibered knots can be studied using their Seifert forms.

The classification of odd dimensional simple fibered knots up to cobordism cannot be done by a direct generalization of the results proved by Kervaire and Levine for spherical $(2n-1)$-knots with $n \geq 2$. In fact, we have to consider the topological data contained in the kernel and the cokernel of the intersection form of the fiber (see the exact sequence (1.1)), this is illustrated by the example below.

Example 6.2. Let us consider the two following unimodular matrices

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now one can construct some simple $(2n-1)$-knots K_0 and K_1, with $n \geq 3$ and n odd, which have A_0 and A_1 as Seifert forms. The module M generated by the first elements of the basis in which A_0 is defined, and, the first and the third elements in the basis in which A_1 is defined is a metabolizer for $A_0 \oplus -A_1$.

We have $\operatorname{rank}\big(\mathrm{H}_n(K_0)\big) = 2$ and $\operatorname{rank}\big(\mathrm{H}_n(K_1)\big) = 4$, because the intersections forms associated with A_0 and A_1 are respectively zero forms of rank 2 and 4.

Hence though the knots K_0 and K_1 have Witt-associated Seifert forms they are not cobordant since they are not homeomorphic to each other.

For $n \geq 3$, Du Bois and Michel [35] constructed the first examples of spherical algebraic $(2n-1)$-knots which are cobordant but are not isotopic. So, for algebraic knots of dimension greater than or equal to 5 the notions of cobordism and isotopy are distinct, and they do not have the nice behaviour of algebraic 1-knots.

Moreover, there exist infinitely many examples of knots, not necessary spherical nor algebraic, which are cobordant but are not isotopic in any dimension. For example, for the dimension one, the square knot, which is the connected sum of the right hand and the left hand trefoil knots, is cobordant to the trivial knot, but is not isotopic to it. (For more explicit examples, see Chapter 10.)

Using Seifert forms, we have a complete characterization of cobordism classes of simple fibered knots as follows (see [8, 6]).

Theorem 6.3 ([8]). *For $n \geq 3$, two simple fibered $(2n-1)$-knots are cobordant if and only if their Seifert forms are algebraically cobordant.*

Remark 6.4. Related results had been obtained by Vogt [152, 153], who proved that if two simple (not necessarily fibered) $(2n-1)$-knots, $n \geq 3$, are cobordant, then their Seifert forms are Witt equivalent and satisfy certain properties which are weaker than the algebraic cobordism. Conversely, if two simple $(2n-1)$-knots, $n \geq 3$, with torsion free homologies have such (algebraically) cobordant Seifert forms, then they are cobordant.

In Theorem 6.3 the condition on the integer n is only used to prove the sufficiency, and we have the following theorem which is valid for all odd dimensions.

Theorem 6.5 ([8]). *For $n \geq 1$, two cobordant simple fibered $(2n-1)$-knots have algebraically cobordant Seifert forms.*

Furthermore, the following holds for (not necessarily fibered) simple knots.

Theorem 6.6 ([8]). *For $n \geq 3$, two simple $(2n-1)$-knots are cobordant if their Seifert forms associated with $(n-1)$-connected Seifert manifolds are algebraically cobordant.*

Recall that the knot cobordism is an equivalence relation. Furthermore, any unimodular matrix can be realized as a Seifert matrix associated with a simple fibered $(2n-1)$-knot, $n \geq 3$. Therefore, Theorem 6.3 implies the following

Theorem 6.7. *Algebraic cobordism is an equivalence relation on the set of unimodular forms.*

which gives a topological proof of Theorem 5.13.

We have to mention that we do not know if algebraic cobordism is an equivalence relation on the whole set of integral bilinear forms, the following example illustrate this remark.

Example 6.8. Let us consider the three matrices

$$A_0 = \begin{pmatrix} 0 & 4 & -2 & -3 \\ -4 & 0 & -2 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & -1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 4 & 1 & 2 \\ -4 & 0 & 1 & -2 \\ -1 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 4 & -6 & 1 \\ -4 & 0 & -2 & -1 \\ 6 & 2 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

which are given in [153, p. 45]. We identify A_i with the corresponding bilinear form $A_i : G_i \times G_i \to \mathbf{Z}$ with $G_i \cong \mathbf{Z}^4$, $i = 0, 1, 2$. Set

$$\begin{aligned}
m_1 &= (0,0,1,0,0,0,2,0) \in G_0 \oplus G_1, \\
m_2 &= (0,1,0,2,0,0,0,1) \in G_0 \oplus G_1, \\
m_3 &= (1,0,0,0,1,0,0,0) \in G_0 \oplus G_1, \\
m_4 &= (0,1,0,0,0,1,0,0) \in G_0 \oplus G_1, \\
n_1 &= (0,0,2,0,0,-1,1,0) \in G_1 \oplus G_2, \\
n_2 &= (0,0,0,1,0,0,0,-2) \in G_1 \oplus G_2, \\
n_3 &= (1,0,0,0,1,0,0,0) \in G_1 \oplus G_2, \\
n_4 &= (0,1,0,0,0,1,0,0) \in G_1 \oplus G_2.
\end{aligned}$$

Then we see that the subgroup generated by m_1, m_2, m_3, m_4 of $G_0 \oplus G_1$ gives a metabolizer for $A_0 \oplus (-A_1)$, and that the subgroup generated by n_1, n_2, n_3, n_4 of $G_1 \oplus G_2$ gives a metabolizer for $A_1 \oplus (-A_2)$. Furthermore, it is easy to check that A_i and A_{i+1} are algebraically cobordant for $\varepsilon = +1$ with respect to the identity

$$\mathbf{Z} \oplus \mathbf{Z} \oplus 0 \oplus 0 = \operatorname{Ker} S_i^* \to \operatorname{Ker} S_{i+1}^* = \mathbf{Z} \oplus \mathbf{Z} \oplus 0 \oplus 0,$$

$i = 0, 1$, where $S_i = A_i + {}^tA_i$, $i = 0, 1, 2$.

Using the method described before to prove the transitivity for the algebraic cobordism of unimodular integral bilinear forms, we see that if

$$\begin{aligned}
\nu_1 &= (0,0,1,0,0,-1,-1,0) \in G_0 \oplus G_2, \\
\nu_2 &= (0,1,0,2,0,0,0,-2) \in G_0 \oplus G_2, \\
\nu_3 &= (1,0,0,0,1,0,0,0) \in G_0 \oplus G_2, \\
\nu_4 &= (0,1,0,0,0,1,0,0) \in G_0 \oplus G_2.
\end{aligned}$$

then the submodule M_{02} generated by ν_1, ν_2, ν_3 and ν_4 is a metabolizer for $A_0 \oplus -A_2$.

But since $\overline{\nu_2}$, the image of ν_2 in the quotient $\left(G_0 \oplus G_2\right)/\operatorname{Ker} S_{02}^*$, is not indivisible ; then $\overline{M_{02}}$ is not a pure submodule of $G_0 \oplus G_2$. Hence the metabolizer M_{02} give not the algebraic cobordism of A_0 and A_2.

This shows that to construct a metabolizer as in the proof of the Theorem 5.13 the hypothesis of unimodularity of the forms is necessary.

Presumably, this example would show that the algebraic cobordism is not an equivalence relation on the set of not necessary unimodular integral bilinear forms defined on free $\mathbf{Z}$-modules of finite rank.

Remark 6.9. For general forms which are not necessarily unimodular, we can consider the equivalence relation generated by the algebraic cobordism, called the *weak algebraic cobordism.* Then by using Theorem 6.6,[1] we can show that

[1] Here, we also need the fact that every form in $\mathcal{A}$ can be realized as the Seifert form of a simple $(2n-1)$-knot.

if two simple $(2n-1)$-knots, $n \geq 3$, have weakly algebraically cobordant Seifert forms with respect to $(n-1)$-connected Seifert manifolds, then they are cobordant.

Furthermore, we can prove the following. A simple $(2n-1)$-knot is said to be *C-algebraically fibered* if its Seifert form is algebraically cobordant to a unimodular form (see [9]). Then, two simple C-algebraically fibered $(2n-1)$-knots, $n \geq 3$, are cobordant if and only if their Seifert forms are weakly algebraically cobordant. We do not know if this is true for all simple $(2n-1)$-knots, $n \geq 3$.

Let A_i be Seifert forms associated with $(n-1)$-connected Seifert manifolds V_i of simple $(2n-1)$-knots K_i, $i = 0, 1$, and S_i^* the adjoint of the intersection form of V_i. Since we have the exact sequence

$$\begin{aligned} 0 = H_{n+1}(V_i, K_i) \to H_n(K_i) \to H_n(V_i) \xrightarrow{S_i^*} & H_n(V_i, K_i) \\ & \to H_{n-1}(K_i) \to H_{n-1}(V_i) = 0 \end{aligned}$$

associated with the pair (V_i, K_i), where we identify $H_n(V_i, K_i)$ with the dual of $H_n(V_i)$ (see (1.1)), $\operatorname{Ker} S_i^*$ and $\operatorname{Coker} S_i^*$ are naturally identified with $H_n(K_i)$ and $H_{n-1}(K_i)$ respectively.

As remarked before, in the case of a spherical knot K we have $H_n(K) = H_{n-1}(K) = 0$, and the intersection form is an isomorphism. Hence the algebraic cobordism for Seifert forms associated with *spherical* simple knots is reduced to the Witt equivalence, and Theorem 6.3 follows from the classical result of Kervaire and Levine (see Theorem 3.8 and Proposition 4.10).

6.1 Classification of fibered knots up to cobordism

6.1.1 Algebraic cobordism a necessary condition for knot cobordism

Let $K_0 = \partial F_0$ and $K_1 = \partial F_1$ be two cobordant knots with A_0 and A_1 the Seifert forms associated with F_0 and F_1 respectively. Set $\mathcal{S}$ the product $S^{2n+1} \times [0,1]$ and by Σ its oriented boundary. The definition of cobordism gives a submanifold $C = \Phi(\mathcal{K} \times [0,1])$ of $\mathcal{S}$ such that

$$\Sigma \cap C = K_0 \coprod (-K_1).$$

Let N be $F_0 \cup C \cup (-F_1)$ where F_i is a Seifert manifold for K_i. By construction N is a closed, compact, oriented, $2n$-submanifold of $\mathcal{S}$. Then we have the following Lemma

Lemma 6.10. *There exists a smooth oriented, compact, submanifold W of $\mathcal{S}$ such that N is the boundary of W.*

Proof. When $n \geq 3$ a proof is written in [89] p. 183. As the existence of W is fundamental to our purpose, we give a proof which works in any dimension.

Let C_j for $j = 1, \dots, k$ be the k connected components of C. As C has a trivial normal bundle in $\mathcal{S}$, it is possible to choose disjoint, closed, tubular neighborhoods U_j of C_j and a diffeomorphism

$$\Psi : C \times D^2 \to U = \coprod_{1 \leq j \leq k} U_j.$$

Now we have meridians m_j on ∂U_j defined by

$$m_j = \Psi(P_j \times S^1)$$

where P_j is some point of C_j and m_j is oriented such that the linking number of m_j and C_j (in $\mathcal{S}$) is $+1$. Let $X = \mathcal{S} \setminus \overset{\circ}{U}$, and v be the diffeomorphism induced by the inclusion of ∂X in U. If e is the excision isomorphism and ∂^i (resp. ∂^i_X) is the connectant homomorphism for the pair $(\mathcal{S}, U)$ (resp. $(X, \partial X)$), then we have the following commutative diagram

$$\begin{array}{cccccccccc}
\xrightarrow{\partial^0_X} & H^1(X,\partial X) & \xrightarrow{\rho} & H^1(X) & \xrightarrow{\sigma} & H^1(\partial X) & \xrightarrow{\partial^1_X} & H^2(X,\partial X) & \to & 0 \\
 & \cong\uparrow e & & \uparrow & & v\uparrow & & \cong\uparrow e & & \\
\xrightarrow{\partial^0} & H^1(\mathcal{S},U) & \to & 0 = H^1(\mathcal{S}) & \to & H^1(U) & \xrightarrow{\cong\partial^1} & H^2(\mathcal{S},U) & \to & 0
\end{array}$$

The commutativity of all the squares of the above diagram implies that the homomorphism ρ is zero so σ is injective and ∂^i_X is surjective for $0 \leq i \leq 2n-1$. We have the following direct sum decomposition

$$H^1(\partial X) = \sigma\big(H^1(X)\big) \oplus v\big(H^1(U)\big).$$

Any element of $\sigma\big(H^1(X)\big)$ is represented by a differentiable map from ∂X to S^1, which is, up to homotopy, characterized by its degree on each meridian m_j, and which has a unique extension to X. Let

$$g : X \to S^1$$

be the unique, up to homotopy, differentiable map which has degree $+1$ on each meridian. Thanks to the Thom-Pontriagin construction there exists a differentiable map

$$f : \Sigma \setminus (K_0 \coprod -K_1) \to S^1$$

which has $\overset{\circ}{F_0} \coprod (-\overset{\circ}{F_1})$ as regular fiber and f has degree $+1$ on the meridians of the connected components of $K_0 \coprod (-K_1)$. So f and g have homotopic restrictions on $X \cap \Sigma$ and we can choose g such that its restriction on $X \cap \Sigma$ coincides with f. Then g has a regular fiber $\overline{W}$ such that $\overline{W} \cap \Sigma = (F_0 \coprod -F_1) \cap X$. The union of $\overline{W}$ with a small collar in U is the manifold W such that $N = \partial W$. $\square$

Recall that A_0 (resp. A_1) is the Seifert form associated to a $(n-1)$-connected Seifert surface F_0 (resp. F_1) for K_0 (resp. K_1). Let

$$\begin{array}{rcl}
\tau : K_0 & \to & K_1 \\
P & \mapsto & \Phi\big(\Phi^{-1}(P) \times \{1\}\big)
\end{array}$$

where P is any point of K_0. The diffeomorphism τ induces isomorphisms

$$\theta_j : \mathrm{H}_j(K_0) \to \mathrm{H}_j(K_1)$$

such that for any j-cycle x of K_0, $\big(x, \theta_j(x)\big)$ is a boundary in the manifold $C = \Phi(\mathcal{K} \times [0,1])$. Let $\chi_i : \mathrm{H}_n(K_i) \to \mathrm{H}_n(F_i)$ and $\lambda_i : \mathrm{H}_n(F_i) \to \mathrm{H}_n(N)$, $i = 0, 1$, be the homomorphisms induced by the inclusions $K_i \subset F_i \subset N$. The Mayer-Vietoris exact sequence associated to the decomposition of N in the union of $F_0 \cup C$ and $C \cup (-F_1)$ gives

$$\to \mathrm{H}_n(K_0) \xrightarrow{\chi} \mathrm{H}_n(F_0) \oplus \mathrm{H}_n(F_1) \xrightarrow{\lambda} \mathrm{H}_n(N) \xrightarrow{\delta} \mathrm{H}_{n-1}(K_0) \to \tag{6.1}$$

where $\chi = (\chi_0, \chi_1 \circ \theta_n)$ and $\lambda = (\lambda_0, \lambda_1)$

Remark 6.11. Let m_i be $\mathrm{rk}\big(\mathrm{H}_n(F_i)\big)$, m be $\mathrm{rk}\big(\mathrm{H}_n(N)\big)$ and r be $\mathrm{rk}\Big(\chi\big(\mathrm{H}_n(K_0)\big)\Big)$. By Poincaré duality $m = m_0 + m_1$, $r = \mathrm{rk}\Big(\delta\big(\mathrm{H}_n(N)\big)\Big)$ and $r = \mathrm{rk}(\mathrm{Ker}\, S_i^*)$ where S_i^* is the adjoint of the intersection form S_i on $\mathrm{H}_n(F_i)$.

Now we will construct the isomorphism $\varphi : \mathrm{Ker}\, S_0^* \to \mathrm{Ker}\, S_1^*$ and the isomorphism $\theta : \mathrm{Tors}(\mathrm{Coker}\, S_0^*) \to \mathrm{Tors}(\mathrm{Coker}\, S_1^*)$.

Let

$$S_{i*} : \mathrm{H}_n(F_i) \to \mathrm{H}_n(F_i, K_i)$$

and

$$\partial : \mathrm{H}_n(F_i, K_i) \to \mathrm{H}_{n-1}(K_i)$$

be the homomorphisms given by the long exact sequence for the pair (F_i, K_i). Let

$$U : \mathrm{H}^n(F_i) \to \mathrm{Hom}_{\mathbf{Z}}\big(\mathrm{H}_n(F_i); \mathbf{Z}\big)$$

be the universal coefficient isomorphism (recall that F_i is $(n-1)$-connected) and let

$$P : \mathrm{H}_n(F_i, K_i) \to \mathrm{H}^n(F_i)$$

be the Poincaré duality isomorphism. We have the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \to & \chi_i\big(\mathrm{H}_n(K_i)\big) & \to & \mathrm{H}_n(F_i) & \overset{S_{i*}}{\to} & \mathrm{H}_n(F_i, K_i) & \overset{\partial}{\to} & \partial\big(\mathrm{H}_n(F_i, K_i)\big) & \to & 0 \\
 & & \| & & \| & & \cong\downarrow U \circ P & & \downarrow \Delta_i & & \\
0 & \to & \mathrm{Ker}\, S_i^* & \to & \mathrm{H}_n(F_i) & \overset{S_i^*}{\to} & \mathrm{Hom}_{\mathbf{Z}}\big(\mathrm{H}_n(F_i); \mathbf{Z}\big) & \overset{d}{\to} & \mathrm{Coker}\, S_i^* & \to & 0
\end{array}$$

By definition $\Delta_i : \partial\big(\mathrm{H}_n(F_i, K_i)\big) \to \mathrm{Coker}\, S_i^*$ is the quotient of the isomorphism $U \circ P$, so Δ_i is an isomorphism.

Let us consider again the isomorphism $\theta_j : \mathrm{H}_j(K_0) \to \mathrm{H}_j(K_1)$, which was defined before thanks to the existence of the cobordism. Since F_i is $(n-1)$-connected then $\partial\big(\mathrm{H}_n(F_i, K_i)\big) = \widetilde{\mathrm{H}}_{n-1}(K_i)$ and $\theta_n(\mathrm{Ker}\, \chi_0) = \mathrm{Ker}\, \chi_1$, so

$$\theta_{n-1} \circ \partial\big(\mathrm{H}_n(F_0, K_0)\big) = \partial\big(\mathrm{H}_n(F_1, K_1)\big).$$

Let θ be the restriction of the isomorphism $\Delta_1 \circ \theta_{n-1} \circ \Delta_0^{-1}$ on the $\mathbf{Z}$-torsion of $\mathrm{Coker}\, S_0^*$.

Let φ be the restriction of θ_n on $\chi_0\big(\mathrm{H}_n(K_0)\big)$. As $\chi_i\big(\mathrm{H}_n(K_i)\big) = \mathrm{Ker}\, S_i^*$, then φ is defined on $\mathrm{Ker}\, S_0^*$.

We denote by $\Delta(\varphi)$ the submodule

$$\Delta(\varphi) = \Big\{ \big(x, \varphi(x)\big);\ x \in \operatorname{Ker} S_0^* \Big\}$$

of $\mathrm{H}_n(F_0) \oplus \mathrm{H}_n(F_1)$.

Remark 6.12. By construction φ fulfills $\varphi \circ \chi_0 = \chi_1 \circ \theta_n$ so we have $\Delta(\varphi) = \chi\big(\mathrm{H}_n(K_0)\big)$ where $\chi = (\chi_0, \chi_1 \circ \theta_n)$ as above.

To prove the necessity of algebraic cobordism of Seifert forms associated with cobordant simple fibered knots, we will first construct a submodule M of $\mathrm{H}_n(F_0 \coprod -F_1)$ which will be a metabolizer for $A = A_0 \oplus -A_1$. Then we will prove that this metabolizer M fulfills conditions c.1 and c.2 in Definition 5.3 of the algebraic cobordism, for the isomorphisms φ and θ we have just defined before.

To do that, we have to choose an oriented submanifold W of $\mathcal{S}$ with $\partial(W) = N$ given by Lemma 6.10. Set

$$j : \mathrm{H}_n(N) \to \mathrm{H}_n(W)$$

be the homomorphism induced by the inclusion of N in W.

Lemma 6.13. *The form $A = A_0 \oplus -A_1$ vanishes on $\lambda^{-1}(\operatorname{Ker} j^\wedge)$.*

Proof. It is sufficient to prove that A vanishes on $\lambda^{-1}(\operatorname{Ker} j)$. Let $a = [x]$ and $b = [y]$ be two homology classes in $\lambda^{-1}(\operatorname{Ker} j)$. As λ is induced by the inclusion of $F_0 \coprod -F_1$ in N there exist two $(n+1)$-chains α and β in W such that $\partial\alpha = x$ and $\partial\beta = y$. Let i_+ be the positively oriented normal vector field to W in $\mathcal{S}$. The intersection of α and $i_+(\beta)$ is zero. Hence the linking number in Σ of x and $i_+(y)$ is zero. But this linking number is, by definition, equal to $A(a, b)$, so $A(a, b) = 0$ and the lemma is proved. □

Lemma 6.14. *If m is the rank of $\mathrm{H}_n(N)$, then the rank of $\operatorname{Ker} j$ is $\frac{m}{2}$.*

Proof. The long exact sequence in homology for the pair (W, N) gives the exactness of

$$0 \to \mathrm{H}_{2n+1}(W) \to \mathrm{H}_{2n+1}(W, N) \to \mathrm{H}_{2n}(N) \to \ldots \to \mathrm{H}_{n+1}(W, N) \to \operatorname{Ker} j \to 0$$

The alternating sum of the ranks in this exact sequence together with the Poincaré duality give

$$\operatorname{rk}(\operatorname{Ker} j) = \frac{\operatorname{rk}(\mathrm{H}_n(N))}{2} = \frac{m}{2}.$$

□

Lemma 6.15. *There exists a direct summand decomposition*

$$\lambda^{-1}(\operatorname{Ker} j^\wedge) = \Delta(\varphi) \oplus R_0 \oplus R$$

where $\Delta(\varphi) = \Big\{ \big(x, \varphi(x)\big);\ x \in \operatorname{Ker} S_0^* \Big\}$, $R_0 = \lambda^{-1}(\operatorname{Ker} j^\wedge) \cap \operatorname{Ker} S_0^*$, and R is any direct summand complement of $\lambda^{-1}(\operatorname{Ker} j^\wedge) \cap \operatorname{Ker} S^*$ in $\lambda^{-1}(\operatorname{Ker} j^\wedge)$.

Proof. As the considered submodules of $\lambda^{-1}(\operatorname{Ker} j^\wedge)$ are pure, the decomposition comes from the following equalities proved before

$\chi\big(\mathrm{H}_n(K_0)\big) = \operatorname{Ker}\lambda \subset \lambda^{-1}(\operatorname{Ker} j^\wedge)$,

$\Delta(\varphi) = \chi\big(\mathrm{H}_n(K_0)\big)$,

$\operatorname{Ker} S^* = \chi\big(\mathrm{H}_n(K_0)\big) \oplus \operatorname{Ker} S_0^*$. □

Proposition 6.16. *The submodule $M = \Delta(\varphi) \oplus R$ of $\lambda^{-1}(\operatorname{Ker} j^\wedge)$ is a metabolizer for $A = A_0 \oplus -A_1$, which fulfills $M \cap \operatorname{Ker} S^* = \Delta(\varphi)$.*

Proof. By Lemma 6.15 we have

$$M \cap \operatorname{Ker} S^* = \Delta(\varphi).$$

By Remark 6.12, A vanishes on M, so we have to show that M is of rank $\frac{m}{2}$. As remarked in 6.11, $r = \operatorname{rk}\Big(\delta\big(\mathrm{H}_n(N)\big)\Big)$, so $\operatorname{rk}(\delta(\operatorname{Ker} j^\wedge)) \leq r$.

Let us consider the following exact sequence induced by Equation 6.1

$$0 \to \Delta(\varphi) \xrightarrow{\chi} \lambda^{-1}(\operatorname{Ker} j^\wedge) \xrightarrow{\lambda} \operatorname{Ker} j^\wedge \xrightarrow{\delta} \delta(\operatorname{Ker} j^\wedge) \to 0.$$

This exact sequence and the equalities $\operatorname{rk}(\operatorname{Ker} j^\wedge) = \frac{m}{2}$, and, $\operatorname{rk}\big(\Delta(\varphi)\big) = r$; give

$$\operatorname{rk}\big(\lambda^{-1}(\operatorname{Ker} j^\wedge)\big) = r + \frac{m}{2} - \operatorname{rk}\big(\delta(\operatorname{Ker} j^\wedge)\big).$$

So we get $\operatorname{rk}(\lambda^{-1}\big(\operatorname{Ker} j^\wedge)\big) \geq \frac{m}{2}$.

We can remark that if A is non degenerated then we have $\operatorname{rk}(\lambda^{-1}(\operatorname{Ker} j^\wedge)) \leq \frac{1}{2}\operatorname{rk}(\mathrm{H}_n(F_0) \oplus \mathrm{H}_n(F_1)) = \frac{m}{2}$*, because A vanishes on $\lambda^{-1}(\operatorname{Ker} j^\wedge)$. So, if A is non degenerated,* $\operatorname{rk}(\lambda^{-1}(\operatorname{Ker} j^\wedge)) = \frac{m}{2}$, $\operatorname{rk}(\delta(\operatorname{Ker} j^\wedge)) = r$, $\operatorname{rk}(R_0) = 0$ *and* $M = \lambda^{-1}(\operatorname{Ker} j^\wedge)$ *is a metabolizer for A.*

Come back to the general case. Let r_0 be the rank of R_0. By construction we have

$$\operatorname{rk}(M) = \operatorname{rk}(\lambda^{-1}(\operatorname{Ker} j^\wedge)) - r_0 = r + \frac{m}{2} - \operatorname{rk}(\delta(\operatorname{Ker} j^\wedge)) - r_0.$$

Lemma 6.17. *The rank l of $\delta(\mathrm{H}_n(N))_{/\delta(\operatorname{Ker} j^\wedge)}$ is greater or equal to r_0.*

Proof. Let $\{e_j\}$, $j = 1, \ldots, r_0$ be a basis of R_0. Let $\{e_j^*\}$ be in $\mathrm{H}_n(N) \otimes_{\mathbf{Z}} \mathbf{Q}$ such that $S_N\big(\lambda(e_j), e_j^*\big) = \delta_{ij}$ where S_N is the intersection form defined on $\mathrm{H}_n(N) \otimes_{\mathbf{Z}} \mathbf{Q}$. The e_j^* exists because S_N is unimodular. Let R^* be the submodule of $\mathrm{H}_n(N) \otimes_{\mathbf{Z}} \mathbf{Q}$ generated by $\{e_j^*\}$. Since $R_0 \cap \operatorname{Ker}\lambda = \{0\}$, then $\operatorname{rk}(\lambda(R_0)) = r_0$. As S vanishes on R_0, then S_N vanishes on $\lambda(R_0)$. It implies that $\operatorname{rk}(R^*) = \operatorname{rk}(R_0) = r_0$, and $\operatorname{Ker} j \cap R^* = \{0\}$. Since $R_0 \subset \operatorname{Ker} S_0^*$, we have $S(x, y) = 0$ for all x in R_0 and all y in $\mathrm{H}_n(F_0 \coprod -F_1)$. So $R^* \cap \lambda(\mathrm{H}_n(F_0 \coprod -F_1)) = \{0\}$ and $\operatorname{rk}\Big(\delta\big(\mathrm{H}_n(N)\big)_{/\delta(\operatorname{Ker} j^\wedge)}\Big) = l \geq \operatorname{rk}\big(\delta(R^*)\big) = \operatorname{rk}(R^*) = r_0$. □

In order to end the proof of Proposition 6.16, we only have to show that $\operatorname{rk}(R) = \frac{m}{2} - r$. But $\operatorname{rk}(\delta(\operatorname{Ker} j^\wedge)) = r - l$; so we have

$$\operatorname{rk}(R) = \operatorname{rk}(M) - r = \frac{m}{2} - (r - l) - r_0.$$

By lemma 6.17 we have $l - r_0 \geq 0$, so

$$\mathrm{rk}(R) \geq \frac{m}{2} - r.$$

But $R \cap \operatorname{Ker} S^* = \{0\}$ by construction, and the form $\overline{S}$ induced by S on $\mathrm{H}_n(F_0 \coprod -F_1)_{/\operatorname{Ker} S^*}$ is non-degenerate of rank $m - 2\,r$. So

$$\mathrm{rk}(R) \leq \frac{m}{2} - r$$

because $\overline{S}$ vanishes on $\overline{R} = R_{/(R \,\cap\, \operatorname{Ker} S^*)}$. Finally we have

$$\mathrm{rk}(R) = \frac{m}{2} - r$$

□

Remark 6.18. With the last Proposition, we have found a metabolizer $M = \Delta(\varphi) \oplus R$ for A which fulfills condition c.1 of the algebraic cobordism without any condition on A.

To prove condition c.2 and $\overline{M}$ is pure in $\overline{G}$ in order to prove the algebraic cobordism we must restrict the study to a smaller class of knots. This is why we will have to choose $(n-1)$-connected Seifert surfaces F_i for K_i on which the Seifert forms A_i are unimodular. So the following Proposition together with the previous results will prove that the algebraic cobordism of Seifert forms is necessary for fibered knots.

Let θ_{n-1} be the isomorphism between $\mathrm{H}_{n-1}(K_0)$ and $\mathrm{H}_{n-1}(K_1)$, and let θ the isomorphism between $\mathrm{Tors}(\operatorname{Coker} S_0^*)$ and $\mathrm{Tors}(\operatorname{Coker} S_1^*)$ defined before. According to our previous notation, let $\Delta(\theta_{n-1})$ (resp. $\Delta(\theta)$) be the group $\Big\{\big(x, \theta_{n-1}(x)\big)\ ;\ x \in \mathrm{Tors}(\mathrm{H}_{n-1}(K_0))\Big\}$ (resp. $\Big\{\big(x, \theta(x)\big)\ ;\ x \in \mathrm{Tors}(\operatorname{Coker} S_0^*)\Big\}$).

Proposition 6.19. *If A_0 and A_1 are unimodular the metabolizer $M = \Delta(\varphi) \oplus R$ of $A = A_0 \oplus -A_1$, fulfills $d(S^*(M)^\wedge) = \Delta(\theta)$ and $\overline{M}$ is pure in $\mathrm{H}_n(F)_{/\operatorname{Ker} S^*}$.*

Proof. Let us denote $F_0 \coprod -F_1$ by F, $K_0 \coprod -K_1$ by K, and $S_0^* \oplus -S_1^*$ by S^*. We consider for F the following commutative diagram already constructed for F_i for $i = 0, 1$

$$\begin{array}{ccccccccccc}
0 & \to & \operatorname{Ker} S_* & \hookrightarrow & \mathrm{H}_n(F) & \xrightarrow{S_*} & \mathrm{H}_n(F,K) & \xrightarrow{\partial} & \partial(\mathrm{H}_n(F,K)) & \to 0 \\
 & & \| & & \| & & \cong\downarrow U \circ P & & \cong\downarrow \Delta_0 \oplus \Delta_1 & \\
0 & \to & \operatorname{Ker} S^* & \hookrightarrow & \mathrm{H}_n(F) & \xrightarrow{S^*} & \mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}_n(F); \mathbf{Z}) & \xrightarrow{d} & \operatorname{Coker} S^* & \to 0
\end{array}$$

Lemma 6.20. *The equality $d\big(S^*(M)^\wedge\big) = \Delta(\theta)$ is equivalent to the equality $\partial\big(S_*(M)^\wedge\big) = \Delta(\theta_{n-1})$.*

Proof. The lemma is a consequence of the two following statements

The restriction of $\Delta_0 \oplus \Delta_1$ on $\Delta(\theta_{n-1})$ is an isomorphism to $\Delta(\theta)$ because $\theta \circ \Delta_0 = \Delta_1 \circ \theta_{n-1}$ by construction.

The restriction of $\Delta_0 \oplus \Delta_1$ on $\partial\big(S_*(M)^\wedge\big)$ is an isomorphism to $d\big(S^*(M)^\wedge\big)$ because the commutativity of the above diagram gives $U \circ P\big(S_*(M)^\wedge\big) = S^*(M)^\wedge$.

□

Let

$$\kappa : \mathrm{H}_n(N) \to \mathrm{H}_n(N, C)$$

be the homomorphism which is defined in the long exact sequence for the pair (N, C) and

$$\rho : \mathrm{H}_n(N, C) \to \mathrm{N}_n(F, K)$$

be the inverse of the excision isomorphism induced by the inclusion of the pair $(F, K) \subset (N, C)$. Set

$$\xi = \rho \circ \kappa : \mathrm{H}_n(N) \to \mathrm{H}_n(F, K)$$

and

$$\overline{\theta} = (\mathrm{Id}, \theta_{n-1}) : \mathrm{H}_{n-1}(K_0) \to \mathrm{H}_{n-1}(K).$$

With the previous notations used we have the following commutative diagram

$$(\star) \qquad \begin{array}{ccccccccc}
\to & \mathrm{H}_n(K_0) & \overset{\chi}{\to} & \mathrm{H}_n(F) & \overset{\lambda}{\to} & \mathrm{H}_n(N) & \overset{\delta}{\to} & \mathrm{H}_{n-1}(K_0) & \to \\
 & & & \| & \text{(I)} & \downarrow \xi & \text{(II)} & \downarrow \overline{\theta} & \\
\to & \mathrm{H}_n(K) & \overset{\chi_0 \oplus \chi_1}{\to} & \mathrm{H}_n(F) & \overset{S_*}{\to} & \mathrm{H}_n(F, K) & \overset{\partial}{\to} & \mathrm{H}_{n-1}(K) & \to
\end{array}$$

The square (I) is commutative by functoriality, and (II) is commutative by definition of ξ and $\overline{\theta}$.

Lemma 6.21. *If A_0 and A_1 are unimodular, then we have $\delta(\mathrm{Ker}\, j^\wedge) = \widetilde{\mathrm{H}}_{n-1}(K_0)$.*

Before giving the proof of Lemma 6.21 we finish the proof of Proposition 6.19.

First remark that the module $\overline{M}$ is pure in $\mathrm{H}_n(F)/_{\mathrm{Ker}\, S^*}$ if and only if the quotient $\mathrm{H}_n(F)/_{(\mathrm{Ker}\, S^* + M)}$ is torsion free.

Since $A = A_0 \oplus -A_1$ is non-degenerate, then we have

$$M = \lambda^{-1}(\mathrm{Ker}\, j^\wedge).$$

Furthermore because of the diagram $(\star)$ we get $\lambda(\mathrm{Ker}\, S^*) = \mathrm{Ker}\, \xi$. Let pr be the projection of $\mathrm{H}_n(N)$ on $\mathrm{H}_n(N)/_{(\mathrm{Ker}\, j^\wedge + \mathrm{Ker}\, \xi)}$, so $\mathrm{Ker}\,(\mathrm{pr} \circ \lambda) = M + \mathrm{Ker}\, S^*$. The quotient of $\mathrm{pr} \circ \lambda$ induces an injective map

$$\mathrm{H}_n(F)/_{(\mathrm{Ker}\, S^* + M)} \hookrightarrow \mathrm{H}_n(N)/_{(\mathrm{Ker}\, j^\wedge + \mathrm{Ker}\, \xi)}.$$

Moreover, there exists x_i, $i = 1, \dots, r$, in $\mathrm{Ker}\, j^\wedge$ such that

$$\widetilde{\mathrm{H}}_{n-1}(K_0) = \bigoplus_{i=1}^{r} \langle \delta(x_i) \rangle \oplus \mathrm{Tors}(\widetilde{\mathrm{H}}_{n-1}(K_0)).$$

Let $(y_i)_{i=1,\dots,r}$ a basis of $\mathrm{Ker}\, \xi$ such that $S_N(x_i, y_j) = \delta_{ij}$. By induction on r, we can construct these bases such that $\mathrm{H}_n(N) = T \oplus^\perp T^\perp$ where $T = \bigoplus_{i=1}^{r} \langle x_i, y_i \rangle$. If we denote by D the module $D = T^\perp \cap \mathrm{Ker}\, j^\wedge$ and by D^* any direct summand complement of D in $T^\perp$, then we get $\mathrm{H}_n(N)/_{(\mathrm{Ker}\, \xi + \mathrm{Ker}\, j^\wedge)} \cong D^*$ which is torsion free.

Finaly $\mathrm{H}_n(F)/_{(\mathrm{Ker}\, S^* + M)}$ is torsion free and $\overline{M}$ is pure in $\mathrm{H}_n(F)/_{(\mathrm{Ker}\, S^*)}$.

So if

$n=1$, the knots K_0 and K_1 have torsion free homology groups ($\mathcal{K}$ is a one dimensional compact manifold), so $\mathrm{Tors}(\mathrm{Coker}\, S^*) = \{0\}$ and the Proposition 6.19 is proved.

$n \geq 2$, thanks to Lemma 6.20, the equality $\Delta(\theta_{n-1}) = \partial(S_*(M)^\wedge)$ gives Proposition 6.19. The above diagram $(\star)$ and Lemma 6.21 imply

$$\overline{\theta}(\mathrm{H}_{n-1}(K_0)) = \Delta(\theta_{n-1}) \subset \partial(S_*(M)^\wedge).$$

To show that the inclusion $\Delta(\theta_{n-1}) \subset \partial\big(S_*(M)^\wedge\big)$ is an equality, it is enough to have $\Big(\partial\big(S_*(M)^\wedge\big) \cap \partial(\mathrm{H}_n(F_0, K_0)\big)\Big) = \{0\}$.

Let us denote by L (resp. L_i) the linking form on $\mathrm{Tors}\big(\mathrm{H}_{n-1}(K)\big)$ (resp. $\mathrm{Tors}\big(\mathrm{H}_{n-1}(K_i)\big)$). By definition such a form $L = L_0 \oplus -L_1$ is non degenerated and vanishes on $\partial\big(S_*(M)^\wedge\big)$ because $S_0 \oplus -S_1$ vanishes on M. Let $\big(y, \theta_{n-1}(y)\big)$ be in $\Delta(\theta_{n-1})$. Then $L\Big(x, \big(y, \theta_{n-1}(y)\big)\Big) = L_0(x,y) = 0$ for all $y \in \mathrm{Tors}\big(\mathrm{H}_{n-1}(K_0)\big)$. The non degeneracy of L_0 implies $x = 0$. This ends the proof of Proposition 6.19.

□

Remark 6.22. The linking form L is defined as follows (see [L-L, 75] prop. 2.1): Let x, y be in $\mathrm{Tors}\,(\mathrm{H}_{n-1}(K))$ such that p and q are the smallest positive integers with $p.x = q.y = 0$. Let $\overline{x}$ and $\overline{y}$ be in $\mathrm{H}_n(F)$ such that $\partial(S_*(\overline{x}) \otimes \frac{1}{p}) = x$ and $\partial(S_*(\overline{y}) \otimes \frac{1}{q}) = y$. Then: $L(x,y) \equiv \frac{1}{p.q}\, S(\overline{x}, \overline{y}) \bmod \mathbf{Z}$.

Proof of lemma 6.21. As shown in (3.10), if $A_0 \oplus -A_1$ is non degenerated, $M = \lambda^{-1}(\mathrm{Ker}\, j^\wedge)$ has rank $\frac{m}{2}$ and is the chosen metabolizer. So λ induces a monomorphism $\overline{\lambda}$ on $\mathrm{H}_n(F)_{/M}$ to $\mathrm{H}_n(N)_{/\mathrm{Ker}\, j^\wedge}$ and we get the following exact sequence:

$$0 \to \mathrm{H}_n(F)_{/M} \xrightarrow{\overline{\lambda}} \mathrm{H}_n(N)_{/\mathrm{Ker}\, j^\wedge} \xrightarrow{\overline{\delta}} \widetilde{\mathrm{H}}_{n-1}(K_0)_{/\delta(\mathrm{Ker}\, j^\wedge)} \to 0.$$

As $\overline{\lambda}$ is injective and M is pure in $\mathrm{H}_n(F)$ there exists two $\mathbf{Z}$-bases

$$\{\overline{e}_j;_{\, j=1,\ldots,\frac{m}{2}}\} \text{ of } \mathrm{H}_n(F)_{/M}$$

and

$$\{\overline{k}_j;_{\, j=1,\ldots,\frac{m}{2}}\} \text{ of } \mathrm{H}_n(N)_{/\mathrm{Ker}\, j^\wedge}$$

such that $\overline{\lambda}(\overline{e}_j) = p_j.\overline{k}_j$ with $p_j \in \mathbf{Z} \setminus \{0\}$. Let E (resp. H) be a direct summand complement of M (resp. $\mathrm{Ker}\, j^\wedge$) in $\mathrm{H}_n(F)$ (resp. $\mathrm{H}_n(N)$). Let also $\{e_j;_{\, j=1,\ldots,\frac{m}{2}}\}$ (resp. $\{k_j;_{\, j=1,\ldots,\frac{m}{2}}\}$) be a $\mathbf{Z}$-basis of E (resp. H) such that

$$e_j \equiv \overline{e}_j \bmod M \quad (\text{resp. } k_j \equiv \overline{k}_j \bmod \mathrm{Ker}\, j^\wedge)$$

By construction $\lambda(e_j) - p_j.k_j = x \in \mathrm{Ker}\, j^\wedge$. So there exists a $(n+1)$-chain γ in W and a positive integer a such that: $\partial\gamma = a\,\lambda(e_j) - a\,p_j.k_j$. Let ρ be a $(n+1)$-chain of $S^{2n+1} \times [0,1]$ with $\partial\rho = k_j$. So $a\,e_j$ is the boundary of $\gamma + a\,p_j.\rho$ in $S^{2n+1} \times [0,1]$.

We will now prove that for all m in M, p_j divides $A(e_j, m)$.

Let m be in $M = \lambda^{-1}(\mathrm{Ker} j^{\wedge})$ and Δ be a $(n+1)$-chain in $S^{2n+1} \times [0;1]$ such that $\partial\Delta = i_+(m)$. By definition $A(a\, e_j, m)$ is the intersection in $S^{2n+1} \times [0,1]$ of $\gamma + a\, p_j.\rho$ and Δ. But $\lambda(a\, m) \in \mathrm{Ker}\, j$ so there exists a $(n+1)$-chain μ in W such that $\partial\mu = a\, m$. We have $\partial(i_+(\mu)) = a\, i_+(m)$. Since $\partial(a\,\Delta) = a\, i_+(m)$, we get $\gamma \cap (a\,\Delta) = \gamma \cap (i_+(\mu)) = 0$. But $a > 0$, so $a(\gamma \cap \Delta) = 0$ implies $\gamma \cap \Delta = 0$. Finaly $A(a\, e_j, m) = a\, p_j.(\rho \cap \Delta)$ and p_j divides $A(e_j, m)$.

Since A is unimodular then $p_j = \pm 1$ for all $j = 1, \ldots, \frac{m}{2}$. So $\overline{\lambda}$ is an isomorphism and his cokernel is zero. As asked we have proved

$$\delta(\mathrm{Ker}\, j^{\wedge}) = \widetilde{\mathrm{H}}_{n-1}(K_0).$$

This ends the proof of lemma (3.15). □

Remark 6.23. As above we can also prove that: for all m in M p_j divides $A(m, e_j)$.

6.1.2 Algebraic cobordism a sufficient condition fot knot cobordism

In this Section we will prove that algebraic cobordism of Seifert forms give cobordism of simple fibered knots of dimension $2n-1$ with $n \geq 3$.

Let K_0 and K_1 be two $2n-1$ dimensional simple knots, with $n \geq 3$. We suppose that there exists $(n-1)$-connected Seifert surfaces F_0 and F_1, for K_0 and K_1, such that the associated Seifert forms A_0 and A_1 are algebraically cobordant. We consider K_0 (resp. $-K_1$) as embedded in the sphere $S^{2n+1} \times \{0\}$ (resp. $S^{2n+1} \times \{1\}$) which are oriented as the boundary of $S^{2n+1} \times [0,1]$.

Let x be in $S^{2n+1} \times \{0\}$ such that $(x \times [0,1]) \cap (F_0 \coprod -F_1)$ is empty, and let U be a "small" open ball around x in $S^{2n+1} \times \{0\}$. The boundary S of the disk $D = (S^{2n+1} \times [0,1]) \setminus (U \times [0,1])$ contains $F_0 \coprod -F_1$. Let G be the closure of the connected sum, in S, of the interiors $\mathring{F}_0$ and $-\ \mathring{F}_1$. By construction $A = A_0 \oplus -A_1$ is the Seifert form of $K_0 \coprod -K_1$, associated to G.

We will do in D an embedded surgery on G, the result of which being a manifold $\widetilde{G}$ diffeomorphic to $\mathcal{K} \times [0,1]$.

By Proposition 5.9 we can choose a good basis $\mathcal{B} = \{(m_i, m_i^*);_{i=1,\ldots,s+r}\}$ of $\mathrm{H}_n(G)$. Thanks to J. Milnor ([M1, 61] lemma 6 p. 50), any cycle of G can be represented by the image of an embedding of S^n. Furthermore we have

Proposition 6.24. *There exists $s+r$ disjointed embeddings $\psi_i : D^{n+1} \times D^n \to D$ such that for any $i \in \{1, \ldots, s+r\}$ we have*

1- $[\psi_i(S^n \times \{0\})] = m_i$,

2- $(\psi_i(D^{n+1} \times D^n)) \cap G = \psi_i(D^{n+1} \times D^n) \cap S = \psi_i(S^n \times D^n)$.

Proof. Let $\overline{\psi_i} : S^n \to G$ be an embedding of S^n which represents m_i. Let $i \neq j$, be in $\{1, \ldots, s+r\}$, then m_i and m_j are in the metabolizer M and we have

$$S(m_i, m_j) = A(m_i, m_j) + (-1)^n A(m_j, m_i) = 0.$$

Since $n \geq 3$, thanks to Whitney's procedure [Wh, 44] we can choose the $\overline{\psi_i}$ such that $\overline{\psi_i}(S^n) \cap \overline{\psi_j}(S^n) = \emptyset$. Since $n \geq 2$, the Whitney obstruction to extend $\overline{\psi_i}$ to disjoint embeddings ψ_i of D^{n+1} in the $(2n+2)$-disk D, is the matrix $A(m_i, m_j)$ which is zero. Furthermore, $A(m_i, m_i) = 0$ is the classical obstruction to extend

ψ_i to $\psi_i : D^{n+1} \times D^n \to D$. (see [Br, 72] and for details see [Bl, 94] proposition 5.1.2, p.58). We choose this extension ψ_i such that the restriction to $S^n \times D^n$ is a tubular neighborhood of $\psi_i(S^n)$ in G. $\square$

So thanks to the last Proposition we construct a submanifold $\widetilde{G}$ of D as follows:

$$\widetilde{G} = \Big(G \setminus \big(\coprod_{i=1}^{s+r} \psi_i(S^n \times D^n)\big)\Big) \cup \Big(\coprod_{i=1}^{s+r} \psi_i(D^{n+1} \times S^{n-1})\Big).$$

Lemma 6.25. *The inclusion k_o (resp. k_1) of K_0 (resp. K_1) in $\widetilde{G}$, induces isomorphisms $k_{o,j}$ (resp. $k_{1,j}$) from $\mathrm{H}_j(K_0)$ (resp. $\mathrm{H}_j(K_1)$) to $\mathrm{H}_j(\widetilde{G})$ for all j, and we have*

$$\mathrm{H}_*(\widetilde{G}, K_0) = \mathrm{H}_*(\widetilde{G}, K_1) = 0.$$

Recall the h-cobordism Theorem

Theorem 6.26 (h-cobordism Theorem [107]). *Let $\mathcal{M}$ be a k-dimensional differentiable compact manifold with $\partial\mathcal{M} = \mathcal{M}_0 \coprod \mathcal{M}_1$ such that $\mathcal{M}$, $\mathcal{M}_0$ and $\mathcal{M}_1$ are simply connected. If $\mathrm{H}_*(\mathcal{M}, \mathcal{M}_0) = 0$ and $k \geq 6$ then $\mathcal{M}$ is diffeomorphic to $\mathcal{M}_0 \times [0,1]$.*

Hence with Lemma 6.25 we have that $\widetilde{G}$ is diffeomorphic to $K_0 \times [0,1]$. So to prove that algebraic cobordism is a sufficient condition for knot cobordism it is enough to prove Lemma 6.25.

Proof of Lemma 6.25. According to proposition (2.1), the intersection form on $\mathrm{H}_n(F)$ splits in an orthogonal sum on the submodules $\langle m_i, m_i^* \rangle$, $i = 1, \ldots, s+r$. So the proof when $s + r = 1$ implies the general case.

Let us suppose that $\mathrm{rk}(M) = 1$ and let m be a generator of M, then $\mathrm{H}_n(G) = \langle m, m^* \rangle$. We denote by $\psi : D^{n+1} \times D^n \to D$ an embedding choosen as in Proposition 6.24, by $\eta : S^n \to G$ an embedding such that $[\eta(S^n)] = m^*$, and by G_T the manifold $G_T = G \setminus \psi(S^n \times \overset{\circ}{D^n})$.

The Mayer-Vietoris sequence associated to the following decomposition of the manifold: $G = G_T \cup \psi(S^n \times D^n)$ gives:

$$\begin{aligned} 0 \to \mathrm{H}_n\big(\psi(S^n \times S^{n-1})\big) \to \mathrm{H}_n(G_T) \oplus \mathrm{H}_n\big(\psi(S^n \times D^n)\big) \to \mathrm{H}_n(G) \\ \xrightarrow{\delta} \mathrm{H}_{n-1}\big(\psi(S^n \times S^{n-1})\big) \to \mathrm{H}_{n-1}(G_T) \to 0. \end{aligned} \tag{6.2}$$

where δ is given by the intersection of cycles with m.

The Mayer-Vietoris sequence associated to the following decomposition of the manifold: $\widetilde{G} = G_T \cup \psi(D^{n+1} \times S^{n-1})$ gives:

$$\begin{aligned} 0 \to \mathrm{H}_n(\psi(S^n \times S^{n-1}) \xrightarrow{\alpha} \mathrm{H}_n(G_T) \to \mathrm{H}_n(\widetilde{G}) \xrightarrow{\gamma} \mathrm{H}_{n-1}\big(\psi(S^n \times S^{n-1})\big) \\ \xrightarrow{\beta} \mathrm{H}_{n-1}\big(\psi(D^{n+1} \times S^{n-1})\big) \oplus \mathrm{H}_{n-1}(G_T) \to \mathrm{H}_{n-1}(\widetilde{G}) \to 0. \end{aligned} \tag{6.3}$$

Remark that the homomorphism β is injective into $\mathrm{H}_{n-1}\big(\psi(D^{n+1} \times S^{n-1})\big)$, hence $\gamma = 0$ and the sequence (6.3) splits up into:

$$0 \to \mathrm{H}_n\big(\psi(S^n \times S^{n-1})\big) \xrightarrow{\alpha} \mathrm{H}_n(G_T) \to \mathrm{H}_n(\widetilde{G}) \to 0, \tag{6.4}$$

and

$$0 \to \mathrm{H}_{n-1}\big(\psi(S^n \times S^{n-1})\big) \xrightarrow{\beta} \mathrm{H}_{n-1}\big(\psi(D^{n+1} \times S^{n-1})\big) \oplus \mathrm{H}_{n-1}(G_T) \\ \to \mathrm{H}_{n-1}(\widetilde{G}) \to 0. \qquad (6.5)$$

Since $\mathrm{rk}(M) = 1 = s + r$ we have to consider the two following cases: $s = 0, r = 1$ and $s = 1, r = 0$.

$\star$ 1^{st} case: $s = 0$ and $r = 1$, then $\operatorname{Ker} S^* = \langle m, m^* \rangle$.

In sequence (6.2) we have $\operatorname{Ker} \delta = \langle m, m^* \rangle$, then

$$\mathrm{H}_n(G_T) = \Big\langle \big[\psi(S^n \times \{1\})\big], \big[\eta(S^n)\big] \Big\rangle$$

and

$$\mathrm{H}_{n-1}(G_T) = \Big\langle \big[\psi(\{1\} \times S^{n-1})\big] \Big\rangle.$$

In sequence (6.4) we have $\operatorname{Im} \alpha = \Big\langle \big[\psi(S^n \times \{1\})\big] \Big\rangle$, so

$$\mathrm{H}_n(\widetilde{G}) = \langle [\eta(S^n)] \rangle.$$

By construction of the good basis in Proposition 5.9 $\big[\eta(S^n)\big]$ is a generator of $\operatorname{Im}\big(\mathrm{H}_n(K_0) \to \mathrm{H}_n(G)\big)$. So the inclusion of K_0 in $\widetilde{G}$ induces the isomorphism

$$k_{0,n} : \mathrm{H}_n(K_0) \stackrel{\cong}{\to} \mathrm{H}_n(\widetilde{G}).$$

Since $\mathrm{H}_{n-1}(G_T) = \Big\langle \big[\psi(\{1\} \times S^{n-1})\big] \Big\rangle$ in sequence (6.5), we have

$$\mathrm{H}_{n-1}(\widetilde{G}) = \Big\langle \big[\psi(\{1\} \times S^{n-1})\big] \Big\rangle.$$

Condition c.1 of the algebraic cobordism gives that there exists a in $\operatorname{Ker} S_0^*$ such that $m = \big(a, \varphi(a)\big)$. If we denote by

$$\gamma_0 : \mathrm{H}_n(K_0) \to \mathrm{H}_n(G)$$

the homomorphism induced by the inclusion, then we can choose b in $\mathrm{H}_{n-1}(K_0)$ such that $\mathrm{H}_{n-1}(K_0) = \langle b \rangle$ and b is the dual of $\gamma_0^{-1}(a)$ for the intersection form of K_0. There exists B in $\mathrm{H}_n(G, K_0)$ such that $\partial B = b$ and the intersection between B and m is $+1$. The boundary of the n-chain $\Big(B - \big(B \cap \psi(S^n \times \mathring{D}^n)\big)\Big)$ is homologous to the $(n-1)$-cycle $b - \big(\psi(\{1\} \times S^{n-1})\big)$, hence b and $\big[\psi(\{1\} \times S^{n-1})\big]$ are homologous in $\mathrm{H}_{n-1}(\widetilde{G}) = \Big\langle \big[\psi(\{1\} \times S^{n-1})\big] \Big\rangle$. Thus the inclusion of K_0 in $\widetilde{G}$ induces the isomorphism

$$k_{0,n-1} : \mathrm{H}_{n-1}(K_0) \stackrel{\cong}{\to} \mathrm{H}_{n-1}(\widetilde{G}).$$

⋆ 2^{nd} case: $s = 1$ and $r = 0$, then $\operatorname{Ker} S^* = \{0\}$ and $\mathrm{H}_n(K_0) = 0$.
In sequence (6.2) we have $\operatorname{Ker}\delta = \langle m \rangle$, then

$$\mathrm{H}_n(G_T) = \Big\langle \big[\psi(S^n \times \{1\})\big] \Big\rangle$$

and

$$\mathrm{H}_{n-1}(G_T) = \Big\langle \big[\psi(\{1\} \times S^{n-1})\big] \Big\rangle.$$

In sequence (6.4) we have $\operatorname{Im}\alpha = \Big\langle \big[\psi(S^n \times \{1\})\big] \Big\rangle$. Since $\mathrm{H}_n(G_T) = \Big\langle \big[\psi(S^n \times \{1\})\big] \Big\rangle$ we have $\mathrm{H}_n(\widetilde{G}) = 0 = \mathrm{H}_n(K_0)$.

- if $S_*(m)$ is indivisible (i.e. $\mathrm{H}_{n-1}(K_0) = 0$), then δ in (6.2) is surjective. Thus $\mathrm{H}_{n-1}(\widetilde{G}) = 0 = \mathrm{H}_{n-1}(K_0)$.
- If $a \neq 1$ is the greatest divisor of $S_*(m)$ (i.e. $\mathrm{H}_{n-1}(K_0) \cong \mathbf{Z}_{/a\,\mathbf{Z}}$) then condition c.2 of algebraic cobordism together with Lemma 6.20 give that there exists c in $\mathrm{H}_{n-1}(K_0)$ such that $\partial\big(\frac{1}{a}\, S_*(m)\big) = \big(c, \theta_{n-1}(c)\big)$. Let b in $\mathrm{H}_{n-1}(K_0)$ be the dual of c for the linking form of K_0. There exists B in $\mathrm{H}_n(G, K_0)$ such that $\partial B = b$ and the intersection between B and m is $+1$. As before the boundary of the n-chain

$$B - \big(B \cap \psi(S^n \times \overset{\circ}{D}{}^n)\big)$$

is the n-cycle $b - \psi\big(\{1\} \times S^{n-1}\big)$, hence b and $\big[\psi(\{1\} \times S^{n-1})\big]$ are homologous in $\mathrm{H}_{n-1}(G)$. Since $\mathrm{H}_{n-1}(G_T) = \Big\langle \big[\psi(\{1\} \times S^{n-1})\big] \Big\rangle$ in sequence (6.5) we have $\mathrm{H}_{n-1}(\widetilde{G}) = \Big\langle \big[\psi(\{1\} \times S^{n-1})\big] \Big\rangle$. Thus b and $\big[\psi(\{1\} \times S^{n-1})\big]$ are homologous in $\mathrm{H}_{n-1}(\widetilde{G})$ and the inclusion of K_0 in $\widetilde{G}$ induces the isomorphism: $k_{0,n-1} : \mathrm{H}_{n-1}(K_0) \overset{\cong}{\to} \mathrm{H}_{n-1}(\widetilde{G})$.

Since $\widetilde{G}$ is obtained by surgery on n-cycles, this surgery only modifies homology groups of dimensions n and $n-1$. Hence for $k \neq n, n-1$ we have $\mathrm{H}_k(G) \cong \mathrm{H}_k(K_0) \overset{k_{0,k}}{\cong} \mathrm{H}_k(\widetilde{G})$. By symmetry we also have the same results with K_1. Finally $k_{0,j}$ and $k_{1,j}$ are some isomorphisms for all j. This ends the proof of Lemma 6.25.

□

6.1.3 Comments

For the proof of sufficiency in Theorem 6.3 we have supposed that A_0 and A_1 were algebraically cobordant Seifert forms associated with fibers F_0 and F_1 of two simple fibered knots K_0 and K_1. Then we have considered F_i to be embedded in $S^{2n+1} \times \{i\}$, $i = 0, 1$, and we have denoted by F the connected sum $F = F_0 \sharp F_1$ embedded in $S^{2n+1} \times [0,1]$. Note that in that case we have $H_n(F) = H_n(F_0) \oplus H_n(F_1)$ because $\ngeq 3$. Then we showed that one can perform embedded surgeries on F the connected sum of Seifert manifolds in $S^{2n+1} \times [0,1]$

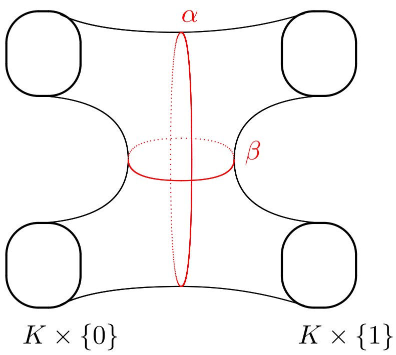


Figure 6.1. The manifold F

so that the result of these surgeries is a simply connected submanifold X of $S^{2n+1} \times [0,1]$ with

$$\partial X = \Big(K_0 \times \{0\}\Big) \coprod \Big(K_1 \times \{0\}\Big)$$

and

$$H_*(X, K_i) = 0 \text{ for } i = 0, 1.$$

According to Smale's h-cobordism Theorem we got $X \cong K_0 \times [0,1]$, and thus X gave a cobordism between K_0 and K_1.

The crucial point in this proof is to see that the technical conditions imposed on the metabolizer in Definition 5.3 give a strategy to perform the right embedded surgeries.

In order to illustrate the idea, let us consider the case of a non-spherical 1-knot K which is the boundary of the disjoint union of two 2-disks embedded in S^3. Note that K is not fibered, nor of dimension ≥ 5. We use this example here just to explain the essential idea for the proof of Theorem 6.3. As a Seifert manifold we can choose an annulus $S^1 \times [0,1]$ trivially embedded in S^3. The knot K is cobordant to itself. Let us try to construct a cobordism by using the same method as described in the proof of Theorem 6.3. First take two copies of K, $K \times \{0\}$ and $K \times \{1\}$, embedded in $S^3 \times [0,1]$, and let A_0 and A_1 be the Seifert forms associated with the Seifert manifolds as above. Let F be the connected sum of the Seifert manifolds associated with the knots.

It is easy to see that $A_0 \oplus (-A_1)$ has a metabolizer M of rank 2, which is generated by the homology classes represented by α and β as in Fig. 6.1. There are two possible surgeries, as shown in Fig. 6.2, and one gives a desired cobordism, while the other does not.

Note that the homology class represented by β belongs to $\operatorname{Ker} S^* \cap M$, while the homology class represented by α does not.

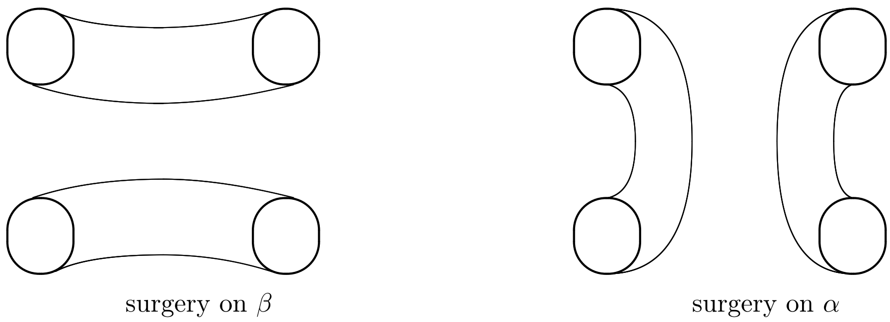


Figure 6.2. The results of the two surgeries on F

6.2 Fox-Milnor type relation

In [44] Fox and Milnor showed that the Alexander polynomials of two cobordant 1-knots should satisfy a certain property. In this section, we explain this property for n-knots and present an application to the cobordism classes of spherical fibered n-knots.

In the following, for a polynomial $f(t) \in \mathbf{Z}[t]$, we set

$$f^*(t) = t^d f(t^{-1}),$$

where d is the degree of $f(t)$. We say that a polynomial $f(t) \in \mathbf{Z}[t]$ is *symmetric* if $f^*(t) = \pm t^a f(t)$ for some $a \in \mathbf{Z}$.

Let K be either a spherical $(2n-1)$-knot or a simple $(2n-1)$-knot with Seifert matrix A. As mentioned before, we still assume that A is associated with an $(n-1)$-connected Seifert manifold when K is simple. Then

$$\Delta_K(t) = \det(tA + (-1)^n\, {}^tA)$$

the *Alexander polynomial* of K (see [2, 88]), is known to be an isotopy invariant of K up to a multiple of $\pm t^a$, $a \in \mathbf{Z}$. For fibered knots, we use (unimodular) Seifert matrices with respect to fibers so that the Alexander polynomial is well-defined up to a multiple of ± 1 and has leading coefficient ± 1.

The following relation is called the *Fox-Milnor type relation* (for proofs, see [89, 8], for example).

Proposition 6.27. *Let K_0 and K_1 be two $(2n-1)$-knots which are both spherical or both simple. If they are cobordant, then there exists a polynomial $f(t) \in \mathbf{Z}[t]$ such that*

$$\Delta_{K_0}(t)\Delta_{K_1}(t) = \pm t^a f(t) f^*(t) \tag{6.6}$$

for some $a \in \mathbf{Z}$.

This result is in fact very powerful, for example, in [35], Du Bois and Michel showed that the algebraic knots constructed in [148] are in fact not cobordant by exploiting the Fox-Milnor type relation.

Let us illustrate again that the above relation, although very simple, gives us a lot of information about knot cobordism.

Let us recall that C_n denotes the cobordism group of spherical n-knots. Let us denote by F_n the subgroup of C_n generated by the cobordism classes of fibered knots. Note that F_n coincides with the set of all cobordism classes which contain a fibered knot.

Then we can prove the following proposition by using the Fox-Milnor type relation, though it might be implicit in the works of Levine [89, 90], Kervaire [73] and Stoltzfus [147] we give here a detailed proof.

Proposition 6.28. *The group C_n/F_n is infinitely generated if n is odd.*

Proof. Set $n = 2k-1$. We only have to prove that $(C_n/F_n) \otimes \mathbf{Z}_2$ contains $\mathbf{Z}_2^\infty$.

First we consider the case where k is odd. For each positive integer p, set $\Delta_p(t) = pt^2 + (1-2p)t + p$. Note that $\Delta_p(t)$ is irreducible over $\mathbf{Z}$. According to Levine (see [89]), there exists a simple spherical $(2k-1)$-knot K_p in S^{2k+1} whose Alexander polynomial $\Delta_{K_p}(t)$ is equal to $\Delta_p(t)$. Let $[K_p]$ denotes the class in

$$(C_n/F_n) \otimes \mathbf{Z}_2 = (C_n/F_n)/2(C_n/F_n) = C_n/(F_n + 2C_n)$$

represented by K_p. In order to show that $(C_n/F_n) \otimes \mathbf{Z}_2$ contains $\mathbf{Z}_2^\infty$, it is sufficient to show that $\{[K_p]\}_{p\geq 2}$ are linearly independent over $\mathbf{Z}_2$.

Suppose that $K_{p_1}\sharp K_{p_2}\sharp\cdots\sharp K_{p_\ell}$ is cobordant to $L\sharp L\sharp L'$, where $p_1, p_2, \ldots, p_\ell$ are distinct positive integers with $p_i \geq 2$, L is a spherical $(2k-1)$-knot, and L' is a spherical fibered $(2k-1)$-knot. Then by Proposition 6.27 we have

$$\Delta_{K_{p_1}}(t)\Delta_{K_{p_2}}(t)\cdots\Delta_{K_{p_\ell}}(t)\Delta_L(t)^2\Delta_{L'}(t) = \pm t^a f(t)f^*(t)$$

for some $a \in \mathbf{Z}$ and $f(t) \in \mathbf{Z}[t]$.

Since $\Delta_{K_{p_i}}(t)$ are irreducible and symmetric, each $\Delta_{K_{p_i}}(t)$ should appear an even number of times in the irreducible decomposition of $f(t)f^*(t)$. Therefore, $\Delta_{K_{p_i}}(t)$ should divide $\Delta_{L'}(t)$, since $\Delta_{K_{p_1}}(t), \Delta_{K_{p_2}}(t), \ldots, \Delta_{K_{p_\ell}}(t)$ are pairwise relatively prime.

On the other hand, since L' is fibered, its Seifert matrix is unimodular and hence $\Delta_{L'}(t)$ has leading coefficient ± 1. This is a contradiction, since the leading coefficient of $\Delta_{K_{p_i}}(t)$ is equal to $p_i \geq 2$.

Therefore, $\{[K_p]\}_{p\geq 2} \subset (C_n/F_n) \otimes \mathbf{Z}_2$ are linearly independent over $\mathbf{Z}_2$.

When k is even, by considering the polynomial $\widetilde{\Delta}_p(t) = pt^4 - (2p-1)t^2 + p$, $p \geq 2$, instead of $\Delta_p(t)$ in the above argument, we get the desired conclusion. This completes the proof. □

Remark 6.29. The above polynomials $\Delta_p(t)$ and $\widetilde{\Delta}_p(t)$ were used by Kervaire in [72, Théorème III.12] for showing that C_{2k-1} is infinitely generated.

Remark 6.30. When k is even, every degree two symmetric polynomial which arises as the Alexander polynomial of a $(2k-1)$-knot is reducible. In fact, in [89], it is mentioned that such a polynomial should be of the form

$$a(a+1)t^2 - (2a(a+1)+1)t + a(a+1) = (at-(a+1))((a+1)t-a).$$

The degree two symmetric polynomial constructed in [90, p. 109] for $\varepsilon = 1$ is also reducible, and it seems that the proof of Theorem 3.11 (or [90, Theorem, p. 108]) given there should appropriately be modified.

6.3 Cobordism of Brieskorn knots

A *Brieskorn polynomial* is a polynomial of the form

$$P(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

with $z = (z_1, z_2, \ldots, z_{n+1})$, $n \geq 1$, where the integers $a_j \geq 2$, $j = 1, 2, \ldots, n+1$, are called the *exponents*. The complex hypersurface in $\mathbf{C}^{n+1}$ defined by $P = 0$ has an isolated singularity at the origin, which is called a *Brieskorn singularity*.

In this section, we will study Brieskorn singularities up to cobordism. We prove that two Brieskorn singularities have cobordant algebraic knots if and only if they have the same set of exponents, provided that no exponent is a multiple of another for each of the two Brieskorn polynomials. Consequently, for such Brieskorn polynomials the multiplicity is an invariant of the cobordism class of the associated algebraic knot.

Definition 6.31. Two bilinear forms $L_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, defined on free abelian groups G_i of finite ranks are said to be *Witt equivalent* if there exists a direct summand M of $G_0 \oplus G_1$ such that $(L_0 \oplus (-L_1))(x, y) = 0$ for all $x, y \in M$ and twice the rank of M is equal to the rank of $G_0 \oplus G_1$. In this case, M is called a *metabolizer*.

Furthermore, we say that L_0 and L_1 are *Witt equivalent over the real numbers* if there exists a vector subspace $M_{\mathbf{R}}$ of $(G_0 \otimes \mathbf{R}) \oplus (G_1 \otimes \mathbf{R})$ such that $(L_0^{\mathbf{R}} \oplus (-L_1^{\mathbf{R}}))(x, y) = 0$ for all $x, y \in M_{\mathbf{R}}$ and $2 \dim_{\mathbf{R}} M_{\mathbf{R}} = \dim_{\mathbf{R}}(G_0 \otimes \mathbf{R}) + \dim_{\mathbf{R}}(G_1 \otimes \mathbf{R})$, where $L_i^{\mathbf{R}} : (G_i \otimes \mathbf{R}) \times (G_i \otimes \mathbf{R}) \to \mathbf{R}$ is the real bilinear form associated with L_i, $i = 0, 1$.

The following lemma is a consequence of Theorem 6.3.

Lemma 6.32. *If two simple fibered $(2n-1)$-knots are cobordant, then their Seifert forms are Witt equivalent. In particular, they are Witt equivalent over the real numbers as well.*

Now, let f be a *weighted homogeneous polynomial* in $\mathbf{C}^{n+1}$, i.e., there exist positive rational numbers $(w_1, w_2, \ldots, w_{n+1})$, called *weights*, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

$$\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1.$$

We say that f is *nondegenerate* if it has an isolated critical point at the origin. Saito [137] has shown that if f is nondegenerate, then by an analytic change of coordinate, f can be transformed to a nondegenerate weighted homogeneous polynomial such that all the weights are greater than or equal to 2. Furthermore, under the assumption that the weights are all greater than or equal to 2, the weights are analytic invariants of the polynomial.

Let f be a nondegenerate weighted homogeneous polynomial in $\mathbf{C}^{n+1}$ with weights $(w_1, w_2, \ldots, w_{n+1})$ such that $w_j \geq 2$ for all j. Set

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1}.$$

Note that $P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbf{Z}$ for some positive integer m. It is known that two nondegenerate weighted homogeneous polynomials f and g in $\mathbf{C}^{n+1}$ have the same weights if and only if $P_f(t) = P_g(t)$ (see [145]).

We start with the following result.

Theorem 6.33. *Let f and g be nondegenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms are Witt equivalent over the real numbers if and only if $P_f(t) \equiv P_g(t) \mod t+1$.*

Proof. Let $h : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a polynomial with an isolated critical point at the origin. It is known that the Seifert form associated with the polynomial

$$\widetilde{h}(z_1, z_2, \ldots, z_{n+2}) = h(z_1, z_2, \ldots, z_{n+1}) + z_{n+2}^2$$

is naturally isomorphic to $(-1)^{n+1}L_h$ (for example, see [138] or [136, Lemma 2.1]). Furthermore, we have $P_{\widetilde{h}}(t) = t^{1/2}P_h(t)$. Hence, by considering $f(z)+z_{n+2}^2$ and $g(z) + z_{n+2}^2$ if necessary, we may assume that n is even.

Recall that

$$H^n(F_h; \mathbf{C}) = \oplus_\lambda H^n(F_h; \mathbf{C})_\lambda,$$

where F_h is the Milnor fiber for h, λ runs over all the roots of the characteristic polynomial $\Delta_h(t)$, and $H^n(F_h; \mathbf{C})_\lambda$ is the eigenspace of the monodromy $H^n(F_h; \mathbf{C}) \to H^n(F_h; \mathbf{C})$ corresponding to the eigenvalue λ ($h = f$ or g). It is easy to see that the intersection form $S_h = L_h + {}^tL_h$ of F_h on $H^n(F_h; \mathbf{C})$ decomposes as the orthogonal direct sum of $(S_h)|_{H^n(F_h;\mathbf{C})_\lambda}$. Let $\mu(h)^+_\lambda$ (resp. $\mu(h)^-_\lambda$) denote the number of positive (resp. negative) eigenvalues of $(S_h)|_{H^n(F_h;\mathbf{C})_\lambda}$. The integer

$$\sigma_\lambda(h) = \mu(h)^+_\lambda - \mu(h)^-_\lambda,$$

is called the *equivariant signature* of h with respect to λ (for details, see [116, 139]). According to Steenbrink [146], putting $P_h(t) = \sum c_\alpha t^\alpha$, we have

$$\sigma_\lambda(h) = \sum_{\substack{\lambda=\exp(-2\pi i\alpha) \\ \lfloor\alpha\rfloor:\ \text{even}}} c_\alpha - \sum_{\substack{\lambda=\exp(-2\pi i\alpha), \\ \lfloor\alpha\rfloor:\ \text{odd}}} c_\alpha$$

for $\lambda \neq 1$, where $i = \sqrt{-1}$, and $\lfloor\alpha\rfloor$ is the largest integer not exceeding α.

Now, suppose that the Seifert forms L_f and L_g are Witt equivalent over the real numbers. Then, the equivariant signatures $\sigma_\lambda(f)$ and $\sigma_\lambda(g)$ coincide for all λ (for example, see [34]. See also [89, 90] for the spherical knot case). Note that by [136, Lemma 2.3], the equivariant signature for $\lambda = 1$ is always equal to zero.

Set $P_f(t) = P_f^0(t) + P_f^1(t)$, where $P_f^0(t)$ (resp. $P_f^1(t)$) is the sum of those terms $c_\alpha t^\alpha$ with $\lfloor\alpha\rfloor \equiv 0 \pmod 2$ (resp. $\lfloor\alpha\rfloor \equiv 1 \pmod 2$). We define $P_g^0(t)$ and $P_g^1(t)$ similarly. Since the equivariant signatures of f and g coincide, we have

$$tP_f^0(t) - P_f^1(t) \equiv tP_g^0(t) - P_g^1(t) \mod t^2 - 1$$

and

$$tP_f^1(t) - P_f^0(t) \equiv tP_g^1(t) - P_g^0(t) \mod t^2 - 1$$

(for details, see [115, 136]). Adding up these two congruences, we have

$$(t-1)P_f(t) \equiv (t-1)P_g(t) \mod t^2 - 1, \tag{6.7}$$

which implies that

$$P_f(t) \equiv P_g(t) \mod t+1. \tag{6.8}$$

Conversely, suppose that (6.8) holds. Then, we have (6.7), which implies that the Seifert forms L_f and L_g have the same equivariant signatures. Then, we see that they are Witt equivalent over the real numbers by virtue of [136, §4]. This completes the proof. □

Remark 6.34. The above theorem should be compared with the result, obtained in [136], which states that the Seifert forms associated with nondegenerate weighted homogeneous polynomials f and g are isomorphic over the real numbers if and only if $P_f(t) \equiv P_g(t) \mod t^2-1$.

Let us now consider the case of Brieskorn polynomials. Note that a Brieskorn polynomial is always a nondegenerate weighted homogeneous polynomial and its weights coincide with its exponents.

Proposition 6.35. *Let*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn polynomials. Then, their Seifert forms are Witt equivalent over the real numbers if and only if

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j} \tag{6.9}$$

holds for all odd integer ℓ.

Proof. Note that $P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some m. Let us put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$. Then, it is easy to see that (6.8) holds if and only if $Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$. Note that ξ is of the form $\exp(\pi\sqrt{-1}\ell/m)$ with ℓ odd and that

$$\frac{-1-\exp(\pi\sqrt{-1}\ell/a_j)}{\exp(\pi\sqrt{-1}\ell/a_j)-1} = \sqrt{-1}\cot\frac{\pi\ell}{2a_j}.$$

Then, we immediately get Proposition 6.35. □

By considering those odd integers ℓ which give zero in (6.9), we get the following.

Proposition 6.36. *Let f and g be the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}.$$

If their Seifert forms are Witt equivalent over the real numbers, then we have

$$\begin{aligned} & \{\ell \in \mathbf{Z} \,|\, \ell \text{ is odd and is a multiple of some } a_j\} \\ = \ & \{\ell \in \mathbf{Z} \,|\, \ell \text{ is odd and is a multiple of some } b_j\}. \end{aligned}$$

In particular, if a_j is odd for some j, then b_k is odd for some k, and the minimal odd exponent for f coincides with that for g.

Remark 6.37. For nondegenerate weighted homogeneous polynomials, we also have results similar to Propositions 6.35 or 6.36. However, the statement becomes complicated, so we omit them here (compare this with [136, Proposition 2.6]).

To each polynomial $Q(t) = \prod_{j=1}^{k}(t - \alpha_j)$, with $\alpha_1, \alpha_2, \ldots, \alpha_k$ in $\mathbf{C}^*$, the multiplicative group of nonzero complex numbers, set

$$\operatorname{divisor} Q(t) = \langle \alpha_1 \rangle + \langle \alpha_2 \rangle + \cdots + \langle \alpha_k \rangle,$$

which is regarded as an element of the integral group ring $\mathbf{ZC}^*$ and is called the *divisor* of Q. For a positive integer a, set $\Lambda_a = \operatorname{divisor}(t^a - 1)$. For the notation and some properties of Λ_a, we refer the reader to [110].

Let f be a nondegenerate weighted homogeneous polynomial in $\mathbf{C}^{n+1}$ with weights $(w_1, w_2, \ldots, w_{n+1})$ such that $w_j \geq 2$ for all j. Let $\Delta_f(t)$ be the characteristic polynomial of the monodromy of f (see [105]). Then, by Milnor–Orlik [110], we have

$$\operatorname{divisor} \Delta_f(t) = \prod_{j=1}^{n+1} \left(\frac{1}{v_j} \Lambda_{u_j} - 1 \right), \tag{6.10}$$

where $w_j = u_j / v_j$, and u_j and v_j are relatively prime positive integers, $j = 1, 2, \ldots, n+1$. In the case of a Brieskorn polynomial, by virtue of the Brieskorn–Pham theorem (for example, see [105]), we have

$$\operatorname{divisor} \Delta_f(t) = \prod_{j=1}^{n+1} (\Lambda_{a_j} - 1),$$

which can also be deduced from the Milnor–Orlik theorem mentioned above.

Proposition 6.38. (1) *Let f and g be nondegenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$ with weights*

$$(u_1/v_1, u_2/v_2, \ldots, u_{n+1}/v_{n+1}) \quad \textit{and} \quad (u'_1/v'_1, u'_2/v'_2, \ldots, u'_{n+1}/v'_{n+1})$$

respectively, where u_j and v_j (resp. u'_j and v'_j) are relatively prime positive integers, $j = 1, 2, \ldots, n+1$. If their Seifert forms are Witt equivalent over the real numbers, then we have

$$\prod_{j=1}^{n+1} \left(\frac{1}{v_j} \Lambda_{u_j} - 1 \right) \equiv \prod_{j=1}^{n+1} \left(\frac{1}{v'_j} \Lambda_{u'_j} - 1 \right) \pmod 2.$$

(2) *Let f and g be Brieskorn polynomials as in Proposition* 6.35*. If their Seifert forms are Witt equivalent over the real numbers, then we have*

$$\prod_{j=1}^{n+1} (\Lambda_{a_j} - 1) \equiv \prod_{j=1}^{n+1} (\Lambda_{b_j} - 1) \pmod 2.$$

Proposition 6.38 is a consequence of the Milnor–Orlik and Brieskorn–Pham theorems on the characteristic polynomials [105, 110] together with the Fox–Milnor type relation. Here, a Fox–Milnor type relation for two polynomials f and g with Witt equivalent Seifert forms means that there exists a polynomial $\gamma(t)$ such that $\Delta_f(t)\,\Delta_g(t) = \pm t^{\deg(\gamma)} \gamma(t)\,\gamma(t^{-1})$ (for details, see [12], for example). Here we give another proof, using Theorem 13.9, as follows.

Proof of Proposition 6.38. Since $P_f(t) \equiv P_g(t) \mod t+1$, there exists a polynomial $R(t) \in \mathbf{Z}[t^{1/m}]$ for some m such that

$$P_f(t) - P_g(t) = (t+1)R(t) = (t-1)R(t) + 2R(t).$$

Therefore, for each $\lambda \in S^1$, the multiplicities of λ in the characteristic polynomials $\Delta_f(t)$ and $\Delta_g(t)$ are congruent modulo 2 to each other (for details, see [115, 136], for example). Then, the result follows in view of the Milnor–Orlik formula (6.10) for the characteristic polynomial. □

Then we have the following Theorem.

Theorem 6.39. *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

no exponent is a multiple of another one. Then, the knots K_f and K_g are cobordant if and only if $a_j = b_j$, $j = 1, 2, \ldots, n+1$, up to order.

For the proof of Theorem 6.39, we need the following.

Lemma 6.40. *For integers $2 \leq a_1 < a_2 < \cdots < a_p$ and $2 \leq b_1 < b_2 < \cdots < b_q$, we have*

$$\sum_{j=1}^{p} \Lambda_{a_j} \equiv \sum_{j=1}^{q} \Lambda_{b_j} \pmod 2 \tag{6.11}$$

if and only if $p = q$ and $a_j = b_j$ for all j.

Proof. Suppose that $a_p < b_q$. Then the coefficient of $\langle \exp(2\pi\sqrt{-1}/b_q) \rangle$ on the right hand side of (6.11) is equal to 1, while the corresponding coefficient on the left hand side is equal to 0. This is a contradiction. So, we must have $a_p = b_q$. Then we have

$$\sum_{j=1}^{p-1} \Lambda_{a_j} \equiv \sum_{j=1}^{q-1} \Lambda_{b_j} \pmod 2.$$

Therefore, by induction, we get the desired conclusion. □

Proof of Theorem 6.39. Suppose that the algebraic knots K_f and K_g are cobordant. We may assume $a_1 < a_2 < \cdots < a_{n+1}$ and $b_1 < b_2 < \cdots < b_{n+1}$. By Proposition 6.38 (2), we have

$$\prod_{j=1}^{n+1} (\Lambda_{a_j} - 1) - (-1)^{n+1} \equiv \prod_{j=1}^{n+1} (\Lambda_{b_j} - 1) - (-1)^{n+1} \pmod 2. \tag{6.12}$$

Recall that for positive integers a and b, we have

$$\Lambda_a \Lambda_b = (a, b)\Lambda_{[a,b]},$$

where (a, b) is the greatest common divisor of a and b, and $[a, b]$ denotes the least common multiple of a and b.

By considering the term of the form Λ_d with the smallest d on both sides of (6.12), we see that $a_1 = b_1$ by Lemma 6.40. By subtracting Λ_{a_1} from the both sides of (6.12), we see $a_2 = b_2$, since a_2 (or b_2) is not a multiple of a_1 (resp. b_1). Then, by further subtracting $\Lambda_{a_2} + (a_1, a_2)\Lambda_{[a_1,a_2]}$ from (6.12), we see $a_3 = b_3$, since a_3 (or b_3) is not a multiple of a_1 or a_2 (resp. b_1 or b_2). Repeating this procedure, we see that $a_j = b_j$ for all j.

Conversely, if f and g have the same set of exponents, then K_f and K_g are isotopic and hence cobordant. This completes the proof. □

Recall that the multiplicity of a Brieskorn polynomial coincides with the smallest exponent, then we have the following Proposition.

Proposition 6.41. *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

the exponents are pairwise distinct. If K_f and K_g are cobordant, then the multiplicities of f and g coincide.

Proof. In the proof of Theorem 6.39, we proved that the smallest exponents of f and g are equal, provided that there is only one smallest exponent for each of f and g. Since we assume that the exponents of f (or g) are pairwise distinct, the same argument works. □

Remark 6.42. Theorem 6.39 implies that two algebraic knots K_f and K_g associated with certain Brieskorn polynomials are isotopic if and only if they are cobordant. Recall that according to Yoshinaga–Suzuki [164], two algebraic knots associated with Brieskorn polynomials in general are isotopic if and only if they have the same set of exponents. In fact, they showed that the characteristic polynomials coincide if and only if the Brieskorn polynomials have the same set of exponents.

Remark 6.43. For the case where $n = 2$ and the knots are homology spheres, Theorem 6.39 has been obtained in [133] by using the Fox–Milnor type relation.

Example 6.44. For all integers $p_1, p_2, \ldots, p_{n-3} \geq 2$, $n \geq 3$, the product of the characteristic polynomials of the algebraic knots associated with

$$f(z) = z_1^{p_1} + z_2^{p_2} + \cdots + z_{n-3}^{p_{n-3}} + z_{n-2}^8 + z_{n-1}^8 + z_n^4 + z_{n+1}^4$$

and

$$g(z) = z_1^{p_1} + z_2^{p_2} + \cdots + z_{n-3}^{p_{n-3}} + z_{n-2}^6 + z_{n-1}^6 + z_n^6 + z_{n+1}^6$$

is a square. This means that the characteristic polynomials $\Delta_f(t)$ and $\Delta_g(t)$ of the algebraic knots K_f and K_g, respectively, satisfy the Fox–Milnor type relation, although their exponents are distinct. Thus the assumptions in Theorem 9.9 and Proposition 6.41 are necessary, as long as the proof depends only on the Fox–Milnor type relation.

6.3.1 Further results

In this section, we give some more precise results for the case of two or three variables. We refer to next Chapters for the study of cobordism of Brieskorn knots of dimension 1 and 3.

Proposition 6.45. *Let f and g be nondegenerate weighted homogeneous polynomials of two variables with weights (w_1, w_2) and (w'_1, w'_2), respectively, with $w_j, w'_j \geq 2$. If their Seifert forms are Witt equivalent over the real numbers, then $w_j = w'_j$, $j = 1, 2$, up to order.*

Proof. Set $w_j = u_j/v_j$ and $w'_j = u'_j/v'_j$, $j = 1, 2$, where u_j and v_j (resp. u'_j and v'_j) are relatively prime positive integers. Let m be a common multiple of u_1, u_2, u'_1 and u'_2. Then, by the same argument as in the proof of [136, Lemma 3.1], we see that the polynomial

$$\begin{aligned} F(\eta) \quad = \quad & -\eta^{m/w_1+m/w_2+m/w'_1} - \eta^{m/w_1+m/w_2+m/w'_2} \\ & +\eta^{m/w_1+m/w'_1+m/w'_2} + \eta^{m/w_2+m/w'_1+m/w'_2} \\ & +\eta^{m/w_1} + \eta^{m/w_2} - \eta^{m/w'_1} - \eta^{m/w'_2} \end{aligned}$$

in η is divisible by $\eta^m + 1$. Note that $F(\eta)$ corresponds to $F(z)$ in the notation of [136].

Since

$$\cot \frac{\pi}{2w_1} \cot \frac{\pi}{2w_2} = \cot \frac{\pi}{2w'_1} \cot \frac{\pi}{2w'_2},$$

we may assume that $w_1 \geq w'_1 \geq w'_2 \geq w_2$. Furthermore, if $w_1 = w'_1$, then we have $w_2 = w'_2$. Therefore, we may assume

$$w_1 > w'_1 \geq w'_2 > w_2 (\geq 2).$$

Note that then the highest degree of F is equal to $m/w_2 + m/w'_1 + m/w'_2$, while the lowest one is equal to m/w_1. Set $V(\eta) = \eta^{-m/w_1} F(\eta)$, which is a polynomial in η of degree

$$\frac{m}{w_2} + \frac{m}{w'_1} + \frac{m}{w'_2} - \frac{m}{w_1},$$

and which is divisible by $\eta^m + 1$. Note that $V(\eta)$ corresponds to $V(z)$ in the notation of [136].

If we have $\deg V < m$, then by the same argument as in the proof of [136, Lemma 3.1], we have the desired conclusion.

If $\deg V \geq m$, then we have the congruence

$$\begin{aligned} V(\eta) \quad \equiv \quad & -\eta^{m/w_2+m/w'_1} - \eta^{m/w_2+m/w'_2} + \eta^{m/w'_1+m/w'_2} \\ & -\eta^{m/w_2+m/w'_1+m/w'_2-m/w_1-m} + 1 + \eta^{m/w_2-m/w_1} \\ & -\eta^{m/w'_1-m/w_1} - \eta^{m/w'_2-m/w_1} \quad \mod \eta^m + 1. \end{aligned} \tag{6.13}$$

Note that all the terms appearing on the right hand side of (6.13) have nonnegative degrees strictly less than m.

Let us consider the monomial $-\eta^{m/w_2+m/w'_1+m/w'_2-m/w_1-m}$ of $V(\eta)$, with negative sign. In order that $V(\eta)$ be divisible by $\eta^m + 1$, a term with positive sign must cancels with $-\eta^{m/w_2+m/w'_1+m/w'_2-m/w_1-m}$. Therefore, three cases arise

1. $1/w_2 + 1/w'_1 + 1/w'_2 - 1/w_1 - 1 = 1/w'_1 + 1/w'_2$,

   this does not occur, since $w_1 > w_2 \geq 2$.

2. $1/w_2 + 1/w_1' + 1/w_2' - 1/w_1 - 1 = 0$,

   then we have

$$\begin{aligned} V(\eta) &\equiv -\eta^{m/w_2+m/w_1'} - \eta^{m/w_2+m/w_2'} + \eta^{m/w_1'+m/w_2'} \\ &\quad +\eta^{m/w_2-m/w_1} - \eta^{m/w_1'-m/w_1} - \eta^{m/w_2'-m/w_1} \quad \bmod \eta^m + 1 \\ &= \eta^{m/w_1'-m/w_1}(-\eta^{m/w_1+m/w_2} - \eta^{m/w_1+m/w_2+m/w_2'-m/w_1'} \\ &\quad +\eta^{m/w_1+m/w_2'} + \eta^{m/w_2-m/w_1'} - 1 - \eta^{m/w_2'-m/w_1'}). \end{aligned}$$

   Note that the difference of the highest and the lowest degrees of the last polynomial is equal to $m/w_1 + m/w_2 + m/w_2' - m/w_1'$, which is strictly positive and is strictly smaller than m, since $1/w_2 + 1/w_2' = 1/w_1 - 1/w_1' + 1$. This means that $V(\eta)$ cannot be divisible by $\eta^m + 1$. This is a contradiction.

3. $1/w_2 + 1/w_1' + 1/w_2' - 1/w_1 - 1 = 1/w_2 - 1/w_1$ In this case we have $1/w_1' + 1/w_2' = 1$, which implies that $w_1' = w_2' = 2$. This is a contradiction, since $w_2' > w_2 \geq 2$.

Therefore, we must have $w_1 = w_1'$ and $w_2 = w_2'$. This completes the proof. $\square$

By using exactly the same argument as in [136, Lemma 3.1], we have the following.

Proposition 6.46. *Let f and g be nondegenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$ with weights $(w_1, w_2, \dots, w_{n+1})$ and $(w_1', w_2', \dots, w_{n+1}')$, respectively, such that $w_j \geq 2$ and $w_j' \geq 2$ for all j. Suppose that the Seifert forms of f and g are Witt equivalent over the real numbers. If*

$$\sum_{j=1}^{n+1} \frac{1}{w_j} + \sum_{j=1}^{n+1} \frac{1}{w_j'} - 2\min\left\{\frac{1}{w_1}, \dots, \frac{1}{w_{n+1}}, \frac{1}{w_1'}, \dots, \frac{1}{w_{n+1}'}\right\} < 1,$$

then we have $w_j = w_j'$, $j = 1, 2, \dots, n+1$, up to order.

Remark 6.47. By Proposition 6.45, we see that if the algebraic knots associated with two weighted homogeneous polynomials of two variables are cobordant, then the polynomials have the same set of weights. In fact, this fact itself is a consequence of already known results as follows.

If two algebraic knots in S^3 are cobordant, then they are in fact isotopic by virtue of the results of Lê [85] and Zariski [166] (for details, see [12, §4]). Then, by Yoshinaga–Suzuki [165] (see also [63, 117]), they have the same set of weights.

Chapter 7

Cobordism of low dimensional knots

"La mathématique est l'art de donner
le même nom à des choses différentes."
Henri Poincaré

7.1 Cobordism of algebraic 1-knots

The classification of 1-knots up to cobordism is still unsolved. However algebraic 1-knots have particular behavior. Let us be more precise.

Consider K, an algebraic 1-knot associated with a holomorphic function germ $f : \mathbf{C}^2, 0 \to \mathbf{C}, 0$ of two variables with an isolated critical point at the origin. Let us further assume that K is spherical. Then it is known that K is an iterated torus knot [17]. Where an *iterated torus knot* is a knot obtained from a torus knot by an iteration of the cabling operation (for example, see [129]), on top of that note that in the case of algebraic 1-knots the cablings have always positive self-linking.

For a knot, the fundamental group of its complement in the ambient sphere is called the *knot group*. In [166] Zariski explicitly gave generators and relations of the knot group of a spherical algebraic 1-knot. When two spherical algebraic 1-knots are isotopic, they have isomorphic knot groups. Although the converse is not true for general spherical (not necessarily algebraic) 1-knots, it was proved that two spherical algebraic 1-knots with isomorphic knot groups are isotopic (see [21, 166, 126, 85]). Furthermore, Burau [21] proved that two spherical algebraic 1-knots with the same Alexander polynomial are isotopic. For a definition of the Alexander polynomial, see §6.2. It is known that the Alexander polynomial of a spherical 1-knot is determined by its knot group (see, for example, [30]).

For general algebraic 1-knots which are not necessarily spherical, the following is known. Let $K = K_1 \cup K_2 \cup \cdots \cup K_s$ and $L = L_1 \cup L_2 \cup \cdots \cup L_t$ be algebraic 1-knots, where K_i, $1 \leq i \leq s$, and L_j, $1 \leq j \leq t$, are components of K and L respectively. Then K and L are isotopic if and only if $s = t$, K_i is isotopic to L_i, $1 \leq i \leq s$, and the linking number of K_i and K_j coincides with that of L_i and L_j for $i \neq j$, after renumbering the indices if necessary (for example, see [126]). It is also known that the multi-variable Alexander polynomial classifies algebraic 1-knots [22, 126, 163].

As to the classification of algebraic 1-knots up to cobordism, we have the following result due to Lê [85]. Let K and L be two cobordant spherical algebraic 1-knots. Let us denote their Alexander polynomials by $\Delta_K(t)$ and $\Delta_L(t)$ respectively, after normalization so that their degree 0 terms are positive. In [44], Fox and Milnor proved that then there exists a polynomial $f(t) \in \mathbf{Z}[t]$ such that $\Delta_K(t)\Delta_L(t) = t^d f(t) f(1/t)$, where d is the degree of $f(t)$ (for details, see §6.2 of the present survey). Using this, one can conclude that the product of the Alexander polynomials of two cobordant spherical algebraic 1-knots is a square in $\mathbf{Z}[t]$. In fact, Lê [85] proved that two cobordant spherical algebraic 1-knots have the same Alexander polynomial, and hence the following holds.

Theorem 7.1 ([85]). *Two cobordant spherical algebraic* 1*-knots are isotopic.*

For general (not necessarily spherical) algebraic 1-knots, since the linking numbers between the components are cobordism invariants, we see that the same conclusion as in Theorem 7.1 holds also for the general case of not necessary spherical algebraic 1-knots.

As mentioned before in the introduction, isotopy of knots implies cobordism. Then we have the following Theorem.

Theorem 7.2. *Two algebraic* 1*-knots are cobordant if and only if they are isotopic.*

Remark 7.3. It has been shown that the images of the cobordism classes of spherical algebraic 1-knots by $\Phi_1 : C_1 \to C^{-1}(\mathbf{Z})$ are not independent. An explicit example is given in [92].

7.2 Brieskorn 1-knots

The following Proposition gives a characterization of cobordism class of Brieskorn 1-knots.

Proposition 7.4. *Let $f(z) = z_1^{a_1} + z_2^{a_2}$ and $g(z) = z_1^{b_1} + z_2^{b_2}$ be Brieskorn polynomials of two variables. If the Seifert forms L_f and L_g are Witt equivalent over the real numbers, then $a_j = b_j$, $j = 1, 2$, up to order.*

Proof. If a_1 or a_2 is odd, then by Proposition 6.36 we may assume that $a_1 = b_1$ is odd. Then by Proposition 6.35, we have

$$\cot \frac{\pi}{2a_2} = \cot \frac{\pi}{2b_2},$$

which implies that $a_2 = b_2$.

Therefore, we may assume that all the exponents for f and g are even. Then by Proposition 6.38 (2), we have

$$(\Lambda_{a_1} - 1)(\Lambda_{a_2} - 1) \equiv (\Lambda_{b_1} - 1)(\Lambda_{b_2} - 1) \pmod 2,$$

which implies that

$$\Lambda_{a_1} + \Lambda_{a_2} \equiv \Lambda_{b_1} + \Lambda_{b_2} \pmod 2.$$

If $a_1 \neq a_2$, then we see that $b_1 \neq b_2$, and $a_j = b_j$, $j = 1, 2$, up to order by Lemma 6.40. If $a_1 = a_2$, then we must have $b_1 = b_2$. In this case, by Proposition 6.35, we have

$$\cot^2 \frac{\pi}{2a_1} = \cot^2 \frac{\pi}{2b_1},$$

which implies that $a_1 = b_1$. This completes the proof. $\square$

Chapter 8

Knots of dimension three

"It would be better for the true physics
if there were no mathematicians on earth."
Daniel Bernoulli

In this Chapter, we deal with 3-dimensional knots and all of them will be oriented. This case is much more difficult than that of higher dimensional knots, since the dimension of the Seifert manifolds associated with a 3-knot is equal to four. The topology of 4-dimensional manifolds is exceptional, and the usual technics like the Whitney trick [162] used in the case of higher dimensional manifolds are not available any more.

The algebraic cobordism of Seifert forms is a necessary condition for the existence of a cobordism between two simple fibered $(2n-1)$-knots for all $n \geq 1$ (see Theorem 6.5). Furthermore, two isotopic simple fibered $(2n-1)$-knots have isomorphic Seifert forms for all $n \geq 1$ (for example, see [36, 65, 131]). However, it is known that there exist 3-dimensional simple fibered knots which are abstractly diffeomorphic and have isomorphic Seifert forms but which are not isotopic (see Example 10.7 below). This shows that the one-to-one correspondence between isotopy classes of knots and isomorphy classes of Seifert forms stated in Theorem 6.1 does not hold for $n = 2$. In fact, these fibered 3-knots are even not cobordant (see Remark 8.16). Hence, for 3-dimensional knots, isotopy classes and cobordism classes must be characterized by new equivalence relations. Isotopy classes of 3-knots were studied in [131, 132, 135] (see also [54]). For cobordism classes we will define a new equivalence relation. For this we need to use Spin structures on manifolds.

Recall that a *Spin structure* on a manifold X means the homotopy class of a trivialization of $TX \oplus \varepsilon^N$ over the 2-skeleton $X^{(2)}$ of X, where TX denotes the tangent bundle and ε^N is a trivial vector bundle of dimension N sufficiently large. Note that X admits a Spin structure if and only if its second Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbf{Z}_2)$ vanishes and that if it admits, then the set of all Spin structures on X is in one-to-one correspondence with $H^1(X; \mathbf{Z}_2)$.

Let K be an oriented 3-knot, with a Seifert manifold V, embedded in S^5. Then K has a natural normal 2-framing $\nu = (\nu_1, \nu_2)$ in S^5 such that the first normal vector field ν_1 is obtained as the inward normal vector field of $K = \partial V$ in V. The homotopy class of this 2-framing does not depend on the choice of the Seifert manifold V. Then K carries a tangent 3-framing on its 2-skeleton $K^{(2)}$ such that the juxtaposition with the above 2-framing gives the standard framing of S^5 restricted to $K^{(2)}$ up to homotopy. This means that K carries a natural Spin structure, which is determined uniquely up to homotopy. Furthermore, this Spin structure coincides with that induced from the Seifert manifold V, which is endowed with the natural Spin structure induced from S^5.

Recall that for high dimensional knots (c.f. [36] and [65]) congruence classes gives isotopy classes of knots. But, in the case of 3-knots, Spin structures must be considered as the following example shows.

Example 8.1. Let K_0 and K_1 be the simple fibered 3-knots which are abstractly diffeomorphic to $S^1 \times \Sigma_g$, constructed in [135, Proposition 3.8], where Σ_g is the closed connected orientable surface of genus $g \geq 2$. They have the property that their Seifert forms are isomorphic, but that there exists no diffeomorphism between K_0 and K_1 which preserves their Spin structures. Consequently they are not isotopic.

In order to study cobordisms of 3-knots, we will use some results valid only for 3-dimensional manifolds without torsion on the first homology group. Hence, we define

Definition 8.2 ([9]). We say that a 3-knot K is *free* if $H_1(K)$ is torsion free over $\mathbf{Z}$.

Moreover, for free knots we do not need to consider condition (c2) in the definition of the algebraic cobordism (see Definition 5.3), which simplifies the argument.

Definition 8.3 ([9]). Consider two simple 3-knots K_0 and K_1. Let A_0 and A_1 be the Seifert forms of K_0 and K_1 respectively with respect to 1-connected Seifert manifolds. We say that the pairs (K_0, A_0) and (K_1, A_1) are Spin *cobordant*, and shorter we also say that the Seifert forms A_0 and A_1 are Spin cobordant, if there exists an orientation preserving diffeomorphism $h : K_0 \to K_1$ such that

(1) h preserves their Spin structures,

(2) A_0 and A_1 are algebraically cobordant with respect to $h_* : H_2(K_0) \to H_2(K_1)$ and $h_*|_{\mathrm{Tors}H_1(K_0)} : \mathrm{Tors}H_1(K_0) \to \mathrm{Tors}H_1(K_1)$, where we identify $H_2(K_i)$ and $H_1(K_i)$ with $\mathrm{Ker}\, S_i^*$ and $\mathrm{Coker}\, S_i^*$ respectively (see the exact sequence (1.1)) and $S_i = A_i + {}^tA_i$, $i = 0, 1$.

Note that if K_0 and K_1 are free 3-knots, then we do not need to consider condition (c2) of Definition 5.3 and hence the isomorphism $h_*|_{\mathrm{Tors}H_1(K_0)}$ in the above definition.

8.1 Spin cobordism as a sufficient condition for knot cobordism

In this section, we shall prove the following, which is valid for simple free 3-knots in general, which may not be fibered.

Theorem 8.4. *Consider two simple free* 3*-knots. If their Seifert forms with respect to* 1*-connected Seifert manifolds are spin cobordant, then the* 3*-knots are cobordant.*

Proof. Let K_0 and K_1 be simple free 3-knots such that the Seifert forms A_0 and A_1 with respect to their 1-connected Seifert manifolds F_0 and F_1, respectively, are spin cobordant. Let M be the metabolizer and $h : K_0 \to K_1$ the diffeomorphism as in Definitions 5.3 and 8.3 respectively. Set $F = F_0 \natural (-F_1)$ and $V = (K_0 \setminus \mathrm{Int}\, D^3) \times [0, 1]$, where the symbol "$\natural$" means a boundary connected sum. Note that $\partial F = K_0 \sharp (-K_1)$ and $\partial V = K_0 \sharp (-K_0)$, where the symbol "$\sharp$" means a usual connected sum (see Fig. 8.1 and Fig. 8.2).

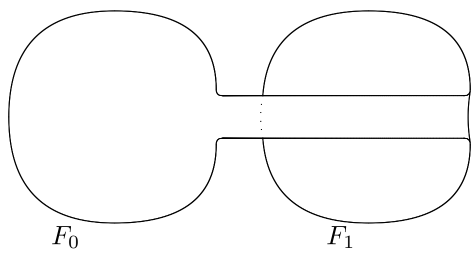


Figure 8.1. $F = F_0\natural(-F_1)$

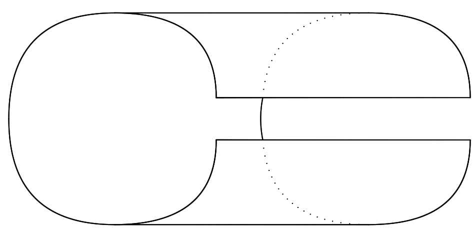


Figure 8.2. $V = (K_0 \setminus \mathrm{Int}D^3) \times [0, 1]$

Note also that the compact 4-manifold V is spin, where the spin structure is induced from K_0. In the following, a *spin surgery* along a simple closed curve c in a spin 4-manifold is a process of taking off the tubular neighborhood $N(c) \cong S^1 \times D^3$ of c and replacing it with $D^2 \times S^2$ by gluing it along the boundary so that the resulting 4-manifold is spin and that the spin structure on the exterior of c coincides with that of the original one.

Lemma 8.5. *For some integer $k \geq 0$, there exists a compact 4-manifold $\widetilde{V}$ and a diffeomorphism $\widetilde{h} : F\sharp k(S^2 \times S^2) \to \widetilde{V}$ such that*

(1) *$\widetilde{V}$ is obtained from V by spin surgeries along simple closed curves, and*

(2) $\widetilde{h}|\partial(F\sharp k(S^2 \times S^2)) = \mathrm{id}_{K_0}\sharp h^{-1} : K_0\sharp(-K_1) \to K_0\sharp(-K_0)$.

Proof. Step 1. Since $H_1(V) \cong H_1(K_0)$ is a finitely generated free abelian group, we can obtain a 4-manifold V_1 with $H_1(V_1) = 0$ from V by spin surgeries along a finite set of simple closed curves c_i, $1 \leq i \leq r = \mathrm{rank}\, H_1(K_0)$, representing a basis of $H_1(V)$.

Step 2. Since $\pi_1(V_1)$ is finitely generated, we can obtain a simply connected 4-manifold V_2 from V_1 by some spin surgeries.

Step 3. Since we have assumed that $H_1(K_i)$ is a free abelian group, the intersection forms of F and V_2 are direct sums of a unimodular form and a zero form, where the dimensions of the null spaces are equal to the rank of $H_1(K_0\sharp(-K_1)) \cong H_1(K_0\sharp(-K_0))$. Furthermore, since they are spin, their intersection forms are of even type. Finally, since the Seifert forms of F_0 and F_1 are algebraically cobordant, the signature of $F = F_0\natural(-F_1)$ vanishes, and that of V_2 is equal to that of V, which is zero. Thus, by the algebraic classification of unimodular forms (see, for example, [111]), by repeating some spin surgeries along trivial simple closed curves in V_2 if necessary, we may assume that there exists an isometry $\Lambda : H_2(F^{(k)}) \to H_2(V_2)$ for some integer $k \geq 0$ such that the diagram

$$
\begin{tikzcd}
0 \arrow[r] & H_2(\partial F^{(k)}) \arrow[r] \arrow[d, "(\mathrm{id}_{K_0}\sharp h^{-1})_*"] & H_2(F^{(k)}) \arrow[r] \arrow[d, "\Lambda"] & H_2(F^{(k)}, \partial F^{(k)}) \arrow[r] & H_1(\partial F^{(k)}) \arrow[r] \arrow[d, "(\mathrm{id}_{K_0}\sharp h^{-1})_*"] & 0 \\
0 \arrow[r] & H_2(\partial V_2) \arrow[r] & H_2(V_2) \arrow[r] & H_2(V_2, \partial V_2) \arrow[r] \arrow[u, "\Lambda^*"'] & H_1(\partial V_2) \arrow[r] & 0
\end{tikzcd}
$$

is commutative, where $F^{(k)} = F\sharp k(S^2 \times S^2)$, we use Poincaré-Lefschetz duality to identify $H_2(F^{(k)}, \partial F^{(k)})$ and $H_2(V_2, \partial V_2)$ with the duals of $H_2(F^{(k)})$ and $H_2(V_2)$ respectively, and Λ^* is the adjoint of Λ.

Step 4. Note that the spin structures of $K_0\sharp(-K_1)$ and $K_0\sharp(-K_0)$ coincide with those induced from $F^{(k)}$ and V_2 respectively. Thus $F^{(k)} \cup_{\mathrm{id}_{K_0}\sharp h^{-1}} (-V_2)$ is a closed spin 4-manifold, since $\mathrm{id}_{K_0}\sharp h^{-1}$ preserves the spin structures by our hypothesis. Furthermore, $K_0\sharp(-K_1)$ and $K_0\sharp(-K_0)$ are connected. Then by an argument of Boyer [15, p. 347], we see that there exists a smooth h-cobordism relative to boundary between $F^{(k)}$ and V_2 such that the induced diffeomorphism between the boundaries of $F^{(k)}$ and V_2 coincides with $\mathrm{id}_{K_0}\sharp h^{-1}$, and that the induced isomorphism between $H_2(F^{(k)})$ and $H_2(V_2)$ coincides with Λ above.

Step 5. Finally, by the 5-dimensional stable h-cobordism theorem due to Lawson [84] and Quinn [123], we see that there exists a diffeomorphism between $F^{(k+k')} = F^{(k)}\sharp k'(S^2 \times S^2)$ and $\widetilde{V} = V_2\sharp k'(S^2 \times S^2)$ extending $\mathrm{id}_{K_0}\sharp h^{-1} : \partial F^{(k)} \to \partial V_2$. Since $\widetilde{V}$ can be obtained from V_2 by repeating k' times the spin surgeries along trivial simple closed curves, we get the result. This completes the proof of Lemma 8.5. □

Remark 8.6. In Step 1, we can choose the curves c_i, $1 \leq i \leq r$, inside $(K_0 \setminus \mathrm{Int}\, D^3) \times \{1/2\}$. After the surgeries, the embedded 2-sphere Σ_i in V_1 corresponding to the center sphere $\{0\} \times S^2$ of the piece $D^2 \times S^2$ replacing $N(c_i)$ is homologous to the boundary of a meridian 3-disk of c_i in V. Let γ_i^*, $1 \leq i \leq r$, be a basis of $H_2(K_0 \setminus \mathrm{Int}\, D^3) \cong H_2(K_0)$ which is Poincaré dual to the basis $[c_i]$, $1 \leq i \leq r$, of $H_1(K_0)$, where $[*]$ denotes the homology class represented by $*$. Then, by the above observation, we have $[\Sigma_i] = i_{0*}\gamma_i^* - i_{1*}\gamma_i^*$, where $i_0 : K_0 \to K_0 \times \{0\} \subset V_1$ and $i_1 : K_0 \to K_0 \times \{1\} \subset V_1$ denote the inclusions.

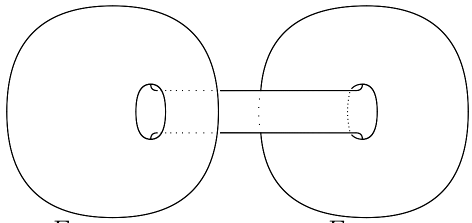


Figure 8.3. $F = F_0\sharp(-F_1)$

Lemma 8.7. *For some integer $k \geq 0$, there exist a compact 4-manifold $\widetilde{V'}$ and a diffeomorphism $\widetilde{h}' : F_0\sharp(-F_1)\sharp k(S^2 \times S^2) \to \widetilde{V'}$ such that*

(1) *$\widetilde{V'}$ is obtained from $V' = K_0 \times [0,1]$ by spin surgeries along simple closed curves, and*

(2) *$\widetilde{h}'|\partial(F_0\sharp(-F_1)\sharp k(S^2 \times S^2)) = \mathrm{id}_{K_0} \coprod h^{-1} : K_0 \coprod(-K_1) \to K_0 \coprod(-K_0)$.*

Proof. Just glue $D^3\times[0,1]$ to $F\sharp k(S^2\times S^2)$ and $\widetilde{V}$ in Lemma 8.5 along $\partial D^3\times[0,1]$ to obtain $F_0\sharp(-F_1)\sharp k(S^2\times S^2)$ and $\widetilde{V'}$ respectively (see Fig. 8.1 and Fig. 8.3). $\square$

Let $\widetilde{V'}$ be as in the above lemma, which is obtained from $K_0 \times [0,1]$ by some spin surgeries. Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$ be the embedded 2-spheres in $\widetilde{V'}$ which have been created in the course of the surgeries in Step 1 of the proof of Lemma 8.5 (for details, see Remark 8.6). Furthermore, let $\Sigma_{r+1}, \Sigma_{r+2}, \ldots, \Sigma_{r+s}$ be the 2-spheres in $\widetilde{V'}$ created in Steps 2–5 in the proof of Lemma 8.5. For the latter spheres, since the surgery curves are all null homologous, we see that there exist homology classes $\sigma^*_{r+1}, \sigma^*_{r+2}, \ldots, \sigma^*_{r+s} \in H_2(\widetilde{V'})$ such that

$$[\Sigma_i] \cdot \sigma^*_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

for $r+1 \leq i, j \leq r+s$. Modifying σ^*_i, $r+1 \leq i \leq r+s$, appropriately, we may further assume that the s submodules $\langle[\Sigma_i], \sigma^*_i\rangle$ are orthogonal to each other with respect to the intersection form $\widetilde{S}$ of $\widetilde{V'}$ and that the intersection matrix of $\langle[\Sigma_i], \sigma^*_i\rangle$ is equal to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where for a subset X of a module, $\langle X\rangle$ denotes the submodule generated by X. Note also that

$$H_2(\widetilde{V'}) = \mathrm{Ker}\, \widetilde{S}^* \oplus \left(\bigoplus_{i=r+1}^{r+s}{}^{\perp} \langle[\Sigma_i], \sigma^*_i\rangle \right),$$

where the symbol "$\oplus^{\perp}$" denotes an orthogonal direct sum.

By taking the connected sum of k copies of $S^2 \times S^2$ with F_0 inside S^5, we may assume that A_0 is the Seifert form with respect to $F_0 \sharp k(S^2 \times S^2)$. Let A_1 be the Seifert form with respect to F_1. Furthermore, let S be the symmetric form associated with $A_0 \oplus (-A_1)$. Note that S can be naturally identified with the intersection form of $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ and hence with that of $\widetilde{V'}$. In the following, we shall identify $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ with $\widetilde{V'}$ by using $\widetilde{h}'$ in Lemma 8.7.

Lemma 8.8. *There exists an isometry Φ of $H_2(F_0 \sharp (-F_1) \sharp k(S^2 \times S^2))$ with respect to S such that*

(1) $\Phi | \operatorname{Ker} S^* = \mathrm{id}$,

(2) $\Phi_*[\Sigma_1], \Phi_*[\Sigma_2], \ldots, \Phi_*[\Sigma_{r+s}]$ *are generators of the metabolizer M.*

Proof. First recall that $[\Sigma_1], [\Sigma_2], \ldots, [\Sigma_r]$ lie in $\operatorname{Ker} S^*$ by Remark 8.6.

As has been shown in [8, Proposition 2.1], there exists a basis

$$\{m_i, m_i^*; i = 1, 2, \ldots, r+s\}$$

of $G = H_2(F_0 \sharp (-F_1) \sharp k(S^2 \times S^2))$ such that

(a) $\{m_i; i = 1, 2, \ldots, r+s\}$ is a basis of M,

(b) $\{m_i, m_i^*; i = 1, 2, \ldots, r\}$ is a basis of $\operatorname{Ker} S^*$ and $\{m_i^*; i = 1, 2, \ldots, r\}$ is a basis of $\operatorname{Ker} S_0^*$, where S_0 is the symmetric form associated with A_0,

(c) the submodules $\langle m_i, m_i^* \rangle$, $i = 1, 2, \ldots, r+s$, are orthogonal for S; i.e.,

$$G = \bigoplus_{i=1}^{r+s}{}^{\perp} \langle m_i, m_i^* \rangle.$$

We may further assume that

$$S(m_i, m_i) = 0, \quad S(m_i, m_i^*) = 1, \quad S(m_i^*, m_i^*) = 0$$

for $r+1 \leq i \leq r+s$, since $\operatorname{Coker} S^*$ is torsion free. Then define the isometry $\Phi : G \to G$ by $\Phi | \operatorname{Ker} S^* = \mathrm{id}$, $\Phi([\Sigma_i]) = m_i$ and $\Phi(\sigma_i^*) = m_i^*$ for $i = r+1, r+2, \ldots, r+s$. This completes the proof of Lemma 8.8. □

Lemma 8.9. *For some integer $k \geq 0$, there exists an orientation preserving self-diffeomorphism φ of the 4-manifold $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ which is the identity on the boundary such that $\varphi_* = \Phi$ on the second homology group.*

Proof. Let J be the submodule of $G = H_2(F_0 \sharp (-F_1) \sharp k(S^2 \times S^2))$ generated by $[\Sigma_i]$ and σ_i^* with $r+1 \leq i \leq r+s$. Note that $G = \operatorname{Ker} S^* \oplus J$ and that the intersection matrix with respect to this decomposition is of the form $0 \oplus Q$, where Q is a unimodular symmetric matrix of even type and zero signature.

Then it is not difficult to see that an arbitrary isometry of $(\operatorname{Ker} S^* \oplus J; 0 \oplus Q)$ which is the identity on $\operatorname{Ker} S^*$ is a composition of the following isometries:

(a) $\mathrm{id} \oplus \Lambda$, where Λ is an isometry of $(J; Q)$,

(b) an isometry represented by the matrix of the form

$$\begin{pmatrix} \mathrm{id} & * \\ 0 & \mathrm{id} \end{pmatrix}$$

with respect to the decomposition $\mathrm{Ker}\, S^* \oplus J$.

We can easily realize isometries of type (a) by diffeomorphisms which are the identity on the boundary, by using Wall's argument [155], since we may assume $k \geq 1$.

In order to realize isometries of type (b), we need the following lemma.

Lemma 8.10. *Increasing k if necessary, we may assume that $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2) \setminus \Sigma_i$ is simply connected for $1 \leq i \leq r+s$.*

Proof. Since $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ is simply connected, $\pi_1(F_0 \sharp (-F_1) \sharp k(S^2 \times S^2) \setminus \Sigma_i)$ is normally generated by a meridian μ_i of Σ_i, where μ_i is the boundary of a fiber of the 2-disk bundle neighborhood of Σ_i. Then, performing the spin surgeries along μ_i, we get a desired situation. □

Now let us go back to the proof of Lemma 8.9. By Lemma 8.10, the spin surgery creating each Σ_i corresponds to the connected sum operation with $S^2 \times S^2$. Thus by [155, Theorem 1], we get a diffeomorphism realizing an isometry of type (b) corresponding to a matrix of the form

$$\begin{pmatrix} \mathrm{id} & E \\ 0 & \mathrm{id} \end{pmatrix},$$

where E is a matrix having one entry equal to 1 and all the others equal to zero. Using this type of diffeomorphisms (sometimes, we have to interchange the two factors of $S^2 \times S^2$, or use the inverse diffeomorphism), we get a desired diffeomorphism. This completes the proof of Lemma 8.9. □

Thus we have proved that the embedded 2-spheres $\varphi(\Sigma_1), \varphi(\Sigma_2), \ldots, \varphi(\Sigma_{r+s})$ in $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ constitute a set of generators for the metabolizer M.

Recall that $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ is embedded in $\mathcal{S} = S^5 \times [0,1]$. Then we can perform appropriate surgeries along these embedded 2-spheres inside $\mathcal{S}$ as in [8, §4]. Since each surgery process is exactly the inverse operation of each spin surgery performed in the construction of $\widetilde{V'}$ (modified by the diffeomorphism φ), the resulting 4-manifold is diffeomorphic to $K_0 \times [0,1]$, which is embedded in $\mathcal{S}$. Thus K_0 and K_1 are cobordant. This completes the proof of Theorem 8.4 and hence Theorem 8.13. □

Remark 8.11. As shown in Example 10.7, algebraic cobordism does not necessarily imply spin cobordism. Hence, Theorem 8.4 does not hold if we replace the spin cobordism with the algebraic cobordism, even if we add the assumption that the 3-knots are abstractly diffeomorphic.

8.2 Cobordism of Brieskorn 3-knots

Proposition 8.12. *Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ be Brieskorn polynomials of three variables. If the Seifert forms L_f and L_g are Witt equivalent over the real numbers, then $a_j = b_j$, $j = 1, 2, 3$, up to order.*

Proof. First suppose that a_1, a_2 and a_3 are all even. Then by Proposition 6.36, b_1, b_2 and b_3 are all even. In this case, by Proposition 6.38 (2), we have

$$\Lambda_{a_1} + \Lambda_{a_2} + \Lambda_{a_3} \equiv \Lambda_{b_1} + \Lambda_{b_2} + \Lambda_{b_3} \pmod 2.$$

Thus, we may assume that $a_1 = b_1$ by Lemma 6.40. Then by Proposition 6.35, we have

$$\cot \frac{\pi\ell}{2a_2} \cot \frac{\pi\ell}{2a_3} = \cot \frac{\pi\ell}{2b_2} \cot \frac{\pi\ell}{2b_3}$$

for all odd integers ℓ. Then, by Proposition 7.4, we see that $a_j = b_j$, $j = 1, 2, 3$, up to order.

Now suppose that a_1, a_2 or a_3 is odd. Then, by Proposition 6.36, we may assume that $a_1 = b_1$ is odd and $a_2 \le a_3$ and $b_2 \le b_3$.

Then by Proposition 6.35, we have

$$\cot \frac{\ell\pi}{2a_2} \cot \frac{\ell\pi}{2a_3} = \cot \frac{\ell\pi}{2b_2} \cot \frac{\ell\pi}{2b_3} \tag{8.1}$$

for all odd integers ℓ that are not a multiple of $a_1 = b_1$. If $a_2 = b_2$, then putting $\ell = 1$, we get $a_3 = b_3$. So, suppose that $a_2 < b_2$. Then by (8.1) with $\ell = 1$, we have $a_2 < b_2 \le b_3 < a_3$.

Let us consider the characteristic polynomials $\Delta_f(t)$ and $\Delta_g(t)$. We have

$$\begin{aligned}
\operatorname{divisor} \Delta_f(t) &= (\Lambda_{a_1} - 1)(\Lambda_{a_2} - 1)(\Lambda_{a_3} - 1) \\
&= (a_1, a_2)([a_1, a_2], a_3)\Lambda_{[a_1,a_2,a_3]} - (a_1, a_2)\Lambda_{[a_1,a_2]} - (a_1, a_3)\Lambda_{[a_1,a_3]} \\
&\quad -(a_2, a_3)\Lambda_{[a_2,a_3]} + \Lambda_{a_1} + \Lambda_{a_2} + \Lambda_{a_3} - 1
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{divisor} \Delta_g(t) &= (b_1, b_2)([b_1, b_2], b_3)\Lambda_{[b_1,b_2,b_3]} - (b_1, b_2)\Lambda_{[b_1,b_2]} - (b_1, b_3)\Lambda_{[b_1,b_3]} \\
&\quad -(b_2, b_3)\Lambda_{[b_2,b_3]} + \Lambda_{b_1} + \Lambda_{b_2} + \Lambda_{b_3} - 1.
\end{aligned}$$

Since $[a_1, a_2, a_3]$, $[a_1, a_3]$, $[a_2, a_3]$, a_3, $[b_1, b_2, b_3]$, $[b_1, b_2]$, $[b_1, b_3]$, $[b_2, b_3]$, b_2 and b_3 are all strictly greater than a_2, by Proposition 6.38 (2) together with $a_1 = b_1$, we must have $[a_1, a_2] = a_2$. Thus a_2 is a multiple of a_1. Then by Proposition 6.38 (2) again, we have

$$\begin{aligned}
\Lambda_{[a_1,a_3]} + \Lambda_{a_3} &\equiv ([b_1, b_2], b_3)\Lambda_{[b_1,b_2,b_3]} + \Lambda_{[b_1,b_2]} + \Lambda_{[b_1,b_3]} \\
&\quad +(b_2, b_3)\Lambda_{[b_2,b_3]} + \Lambda_{b_2} + \Lambda_{b_3} \pmod 2,
\end{aligned}$$

since $a_1 = b_1$ is odd.

If $b_2 < b_3$, then we must have $[b_1, b_2] = b_2$, i.e., b_2 is a multiple of b_1. Then, we see that $[a_1, a_3] = a_3$ and $[b_1, b_3] = b_3$. Therefore, a_2, a_3, b_2 and b_3 are all multiples of $a_1 = b_1$. Since a_1 is odd and $a_1 \ge 3$, there exists an odd integer ℓ ($= a_2 + 1$ or $a_2 + 2$) which is not a multiple of a_1 such that $a_2 < \ell < b_2$. Then for this ℓ, the left hand side of (8.1) is negative, while the right hand side is positive. This is a contradiction.

If $b_2 = b_3$, then we have

$$\Lambda_{[a_1,a_3]} + \Lambda_{a_3} \equiv b_2\Lambda_{[b_1,b_2]} + b_2\Lambda_{b_2} \pmod 2.$$

Thus, $[a_1, a_3] = a_3$, and a_3 is a multiple of a_1. Then, using an odd integer ℓ ($= a_2 + 1$ or $a_2 + 2$) which is not a multiple of a_1 such that $a_2 < \ell < a_3$ in (8.1), we again get a contradiction, since $b_2 = b_3$.

Therefore, we must have $a_2 = b_2$ and $a_3 = b_3$. This completes the proof. □

8.3 Classification

In this section, we give the classification of 3-knots up to cobordism. proved in [9].

Theorem 8.13. *Two simple fibered free* 3*-knots are cobordant if and only if their Seifert forms with respect to* 1*-connected fibers are* Spin *cobordant.*

Remark 8.14. Note that in the case of homology 3-spheres embedded in S^5, the corresponding result had been obtained in [133].

Since the cobordism for knots is an equivalence relation, the Spin cobordism is an equivalence relation on the set of Seifert forms of simple fibered free 3-knots with respect to 1-connected Seifert manifolds.

Let us show that the Spin cobordism is a necessary condition for the existence of a knot cobordism between given two simple fibered 3-knots. Let K_0 and K_1 be two cobordant simple fibered 3-knots with fibers F_0 and F_1 respectively. Denote by $X \cong K_0 \times [0, 1]$ a submanifold of $S^5 \times [0, 1]$ which gives a cobordism between K_0 and K_1, and set $N = F_0 \cup X \cup (-F_1)$. By classical obstruction theory we see that the closed oriented 4-manifold $N \subset S^5 \times [0, 1]$ is the boundary of a compact oriented 5-dimensional submanifold W of $S^5 \times [0, 1]$. Using a normal 2-framing of X in $S^5 \times [0, 1]$ induced from the inward normal vector field along $N = \partial W$ in W, we see that the diffeomorphism h between K_0 and K_1 induced by X preserves their Spin structures.

Moreover, in [8], it has been shown that the two forms A_0 and A_1, associated with the fibers, are algebraically cobordant with respect to

$$h_* : H_2(K_0) \to H_2(K_1)$$

and

$$h_*|_{\mathrm{Tors}H_1(K_0)} : \mathrm{Tors}H_1(K_0) \to \mathrm{Tors}H_1(K_1).$$

Finally we get the following result, in which the knots may not necessarily be free.

Proposition 8.15 ([9]). *If two simple fibered* 3*-knots are cobordant, then their Seifert forms with respect to* 1*-connected fibers are* Spin *cobordant.*

Remark 8.16. In Example 10.7 and 10.8, the Seifert forms of two ”-knots K_0 and K_1 are algebraically cobordant, but are not Spin cobordant. Hence they cannot be cobordant by Proposition 8.15 (or Theorem 8.13). These examples show that Spin structures are essential in the theory of cobordisms of 3-knots as well.

Using the 5-dimensional stable h-cobordism theorem due to Lawson [84] and Quinn [123] together with Boyer's work [15], we also have the following theorem, in which the 3-knots are simple and free, but may not be fibered.

Finally Proposition 8.15 and Theorem 8.4 imply Theorem 8.13.

Chapter 9

Pull back relation for knots

"Nihil est sine ratione."
Gottfried Wilhelm Leibniz

9.1 Pull back relation for knots

For cobordisms of non-spherical knots, Yukio Matsumoto asked the following question.

($\mathcal{Q}$) *If two non-spherical knots (of sufficiently high dimension) are simple homotopy equivalent as abstract manifolds, then are they cobordant after taking connected sums with some spherical knots? In other words, consider the action of the spherical knot cobordism group on the set of cobordism classes of codimension two embeddings of manifolds of a fixed simple homotopy type into a sphere. Then, is the action transitive?*

According to the codimension two surgery theory [96], the answer to the above question is affirmative provided that the material knots satisfy some connectivity conditions and that one of them is obtained as the inverse image of the other one by a certain degree one map between the ambient spheres. This motivates the following definition

Definition 9.1 ([7]). Let K_0 and K_1 be oriented m-knots in S^{m+2}. We say that K_0 is a *pull back* of K_1 if there exists a degree one smooth map $g : S^{m+2} \to S^{m+2}$ with the following properties:

1. g is transverse to K_1,
2. $g^{-1}(K_1) = K_0$,
3. $g|_{K_0} : K_0 \to K_1$ is an orientation preserving simple homotopy equivalence.

In this case, we write $K_0 \succ K_1$. We say that two m-knots are *pull back equivalent* if they are equivalent with respect to the equivalence relation generated by the pull back relation.

Remark 9.2. Here are some direct consequences of the definition.

1. $K \succ K$ for any m-knot K.
2. $K_0 \succ K_1$ and $K_1 \succ K_2$ imply $K_0 \succ K_2$ for any m-knots K_0, K_1 and K_2.
3. $K_0 \succ K_1$ and $K_0' \succ K_1'$ imply $K_0 \sharp K_0' \succ K_1 \sharp K_1'$ for any m-knots K_0, K_0', K_1 and K_1'.

Furthermore, if we restrict ourselves to spherical m-knots, then it is not difficult to show that the *trivial m-knot* (or the *m-dimensional unknot*) K_U is the minimal element, i.e. $K \succ K_U$ for every spherical m-knot K, where K_U is defined to be the isotopy class of the boundary of an $(m+1)$-dimensional disk embedded in S^{m+2}.

Remark 9.3. In the terminology of [93], the map g in Definition 9.1 is *weakly h-regular along* K_1. In fact, the above definition is motivated by the following consequence of the codimension two surgery theory.

For an m-knot K, let $N(K)$ be a tubular neighborhood of K in S^{m+2} and set $E(K) = S^{m+2} \setminus \operatorname{Int} N(K)$. We say that K is *exterior* 2*-connected* if

$$\pi_i(E(K), \partial E(K)) = 0, \quad \forall i \leq 2.$$

(This implies, in particular, that K is simply connected.) The codimension two surgery theory [96] implies that if two exterior 2-connected m-knots K_0 and K_1 with $m \geq 5$ are related by the pull back relation, then they are cobordant after taking connected sums with some spherical knots.

Remark 9.4. In Definition 9.1, if the knots K_0 and K_1 are simply connected, then it is enough that $g|_{K_0} : K_0 \to K_1$ is just an orientation preserving homotopy equivalence for item (3).

Definition 9.5. An m-knot K is *fibered* if there exist a trivialization $\tau : N(K) \to K \times D^2$ of the tubular neighborhood $N(K)$ of K in S^{m+2} and a smooth fibration $\varphi : S^{m+2} \setminus K \to S^1$ such that the following diagram is commutative:

$$\begin{tikzcd} N(K) \setminus K \arrow[rr, "\tau"] \arrow[dr, "\varphi|_{(N(K)\setminus K)}"'] & & K \times (D^2 \setminus \{0\}) \arrow[dl, "p"] \\ & S^1, & \end{tikzcd}$$

where p denotes the obvious projection. In this case, for each $t \in S^1$, the closure F in S^{m+2} of $\varphi^{-1}(t)$ is called a *fiber* of K. Note that $F = \varphi^{-1}(t) \cup K$ is a compact $(m+1)$-dimensional manifold with boundary $\partial F = K$.

We say that a fibered $(2n-1)$-knot K in S^{2n+1} is *simple* if K is $(n-2)$-connected and its fiber is $(n-1)$-connected (see [36]).

Let us first assume that $K_0 \succ K_1$, where K_0 and K_1 are simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. Then, there exists a degree one smooth map $g : S^{2n+1} \to S^{2n+1}$ as in Definition 9.1. By items (1) and (2), we see that there exist trivializations $N(K_i) = K_i \times D^2$ of sufficiently small tubular neighborhoods $N(K_i)$ of K_i in S^{2n+1}, $i = 0, 1$, such that

$$g^{-1}(N(K_1)) = N(K_0)$$

and

$$g|_{N(K_0)} : K_0 \times D^2 = N(K_0) \to N(K_1) = K_1 \times D^2$$

is identified with $(g|_{K_0}) \times \mathrm{id}_{D^2}$. Note that $g|_{K_0} : K_0 \to K_1$ is an orientation preserving homotopy equivalence.

We see that the trivializations $N(K_0) = K_0 \times D^2$ and $N(K_1) = K_1 \times D^2$ are essentially unique, since both $H^1(K_0)$ and $H^1(K_1)$ vanish. Therefore, we may further assume that

$$g|_{(N(K_0)\setminus K_0)} : N(K_0) \setminus K_0 \to N(K_1) \setminus K_1$$

is compatible with the fibrations $S^{2n+1} \setminus K_0 \to S^1$ and $S^{2n+1} \setminus K_1 \to S^1$.

Set $E(K_i) = S^{2n+1} \setminus \operatorname{Int} N(K_i)$, $i = 0, 1$. Note that g induces a smooth map

$$g_E = g|_{E(K_0)} : E(K_0) \to E(K_1) \tag{9.1}$$

whose restriction to $\partial E(K_0)$ is a homotopy equivalence onto $\partial E(K_1)$.

Let $\widetilde{E}(K_i)$ be the universal cover of $E(K_i)$, $i = 0, 1$. Note that $\widetilde{E}(K_i) \cong F_i \times \mathbf{R}$. Since the smooth map g_E in (9.1) induces an isomorphism between the fundamental groups, it lifts to a smooth map $\widetilde{g}_E : \widetilde{E}(K_0) \to \widetilde{E}(K_1)$, whose restriction to the boundary is a homotopy equivalence and respects the product structures $\partial \widetilde{E}(K_i) \cong \partial F_i \times \mathbf{R}$, $i = 0, 1$. Hence, there exists a continuous map $\psi : (F_0, \partial F_0) \to (F_1, \partial F_1)$ such that $\psi|_{\partial F_0} : \partial F_0 \to \partial F_1$ is an orientation preserving homotopy equivalence. (For example, ψ is the composition

$$F_0 = F_0 \times \{0\} \subset F_0 \times \mathbf{R} \cong \widetilde{E}(K_0) \xrightarrow{\widetilde{g}_E} \widetilde{E}(K_1) \cong F_1 \times \mathbf{R} \to F_1,$$

where the last map is the projection to the first factor.)

Note that ψ induces an isomorphism between $H_{2n}(F_0, \partial F_0)$ and $H_{2n}(F_1, \partial F_1)$, since the boundary homomorphism induces an isomorphism

$$H_{2n}(F_i, \partial F_i) \to H_{2n-1}(K_i), \quad i = 0, 1.$$

By the universal coefficient theorem, it also induces an isomorphism between the cohomology groups $H^{2n}(F_1, \partial F_1)$ and $H^{2n}(F_0, \partial F_0)$.

Let $\tau_i : F_i \to F_i$ be the monodromy diffeomorphism of the fibered knot K_i, $i = 0, 1$. Note that $\tau_i|_{\partial F_i}$ is the identity. Since $\widetilde{g}_E$ is compatible with the covering translations, we see that $\psi \circ \tau_0$ and $\tau_1 \circ \psi$ are homotopic relative to boundary.

Lemma 9.6. *The homomorphisms*

$$\psi^* : H^n(F_1) \to H^n(F_0) \quad \textit{and} \quad \psi^* : H^n(F_1, \partial F_1) \to H^n(F_0, \partial F_0)$$

are injective and their images are direct summands of $H^n(F_0)$ and $H^n(F_0, \partial F_0)$ respectively.

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} H^n(F_1) \otimes H^n(F_1, \partial F_1) & \xrightarrow{\smile} & H^{2n}(F_1, \partial F_1) \\ \big\downarrow {\scriptstyle \psi^* \otimes \psi^*} & & \big\downarrow {\scriptstyle \psi^*} \\ H^n(F_0) \otimes H^n(F_0, \partial F_0) & \xrightarrow{\smile} & H^{2n}(F_0, \partial F_0), \end{array}$$

where "$\smile$" denotes the cup product. Let $\xi \in H^n(F_1)$ be an arbitrary primitive element. Then, there exists an element $\zeta \in H^n(F_1, \partial F_1)$ such that $\xi \smile \zeta$ is a generator of $H^{2n}(F_1, \partial F_1) \cong \mathbf{Z}$. Since

$$\psi^* : H^{2n}(F_1, \partial F_1) \to H^{2n}(F_0, \partial F_0)$$

is an isomorphism, we see that $(\psi^* \xi) \smile (\psi^* \zeta)$ is also a generator of $H^{2n}(F_0, \partial F_0)$. This means that $\psi^* \xi$ is a primitive element of $H^n(F_0)$. This shows that

$$\psi^* : H^n(F_1) \to H^n(F_0)$$

is an injection and that its image is a direct summand of $H^n(F_0)$. A similar argument shows the corresponding assertion for $\psi^* : H^n(F_1, \partial F_1) \to H^n(F_0, \partial F_0)$. This completes the proof of Lemma 9.6. $\square$

The above lemma together with the universal coefficient theorem implies that the homomorphisms

$$\psi_* : H_n(F_0, \partial F_0) \to H_n(F_1, \partial F_1) \quad \text{and} \quad \psi_* : H_n(F_0) \to H_n(F_1) \tag{9.2}$$

are surjections.

Let $\Delta_i : H_n(F_i, \partial F_i) \to H_n(F_i)$ be the variation map of the fibered knot K_i, $i = 0, 1$. Recall that for an n-cycle c of $(F_i, \partial F_i)$, $\Delta_i([c])$ is defined to be the homology class represented by $c - \tau_i(c)$, where $[c] \in H_n(F_i, \partial F_i)$ is the homology class represented by c. Note that this is a well-defined homomorphism, since $\tau_i|_{\partial F_i}$ is the identity and the isotopy class of τ_i relative to boundary is uniquely determined. Note also that the variation maps are isomorphisms (see [66]). Then, we see easily that the following diagram is commutative:

$$\begin{array}{ccc} H_n(F_0, \partial F_0) & \xrightarrow{\Delta_0} & H_n(F_0) \\ {\scriptstyle \psi_*}\downarrow & & \downarrow{\scriptstyle \psi_*} \\ H_n(F_1, \partial F_1) & \xrightarrow{\Delta_1} & H_1(F_1), \end{array} \tag{9.3}$$

since $\psi \circ \tau_0$ and $\tau_1 \circ \psi$ are homotopic relative to boundary.

Theorem 9.7. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with fibers F_0 and F_1 respectively, where $n \geq 3$. Suppose $\operatorname{rank} H_n(F_0) = \operatorname{rank} H_n(F_1)$. If $K_0 \succ K_1$, then K_0 and K_1 are orientation preservingly isotopic.*

Proof. If $\operatorname{rank} H_n(F_0) = \operatorname{rank} H_n(F_1)$, then Lemma 9.6 implies that the homomorphisms (9.2) are isomorphisms. Then the commutative diagram (9.3) implies that K_0 and K_1 are orientation preservingly isotopic, since the variation map determines and is determined by the Seifert form, which in turn determines the oriented isotopy class of a simple fibered knot (for details see [66, 36, 65]). □

Corollary 9.8. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. If $K_0 \succ K_1$ and $K_1 \succ K_0$, then K_0 is orientation preservingly isotopic to K_1. In other words, the relation "$\succ$" defines a partial order for simple fibered $(2n-1)$-knots in S^{2n+1} for $n \geq 3$.*

Proof. By Lemma 9.6, we see that $\operatorname{rank} H_n(F_0) = \operatorname{rank} H_n(F_1)$. Then the result follows from Theorem 9.7. □

Theorem 9.9. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. Then $K_0 \succ K_1$ if and only if there exists a spherical simple fibered $(2n-1)$-knot Σ in S^{2n+1} such that K_0 is orientation preservingly isotopic to the connected sum $K_1 \sharp \Sigma$.*

Proof. First, suppose that there exists a spherical simple fibered $(2n-1)$-knot Σ in S^{2n+1} such that K_0 is isotopic to the connected sum $K_1 \sharp \Sigma$. Then by Remark 9.2, we have $\Sigma \succ K_U$ and $K_1 \sharp \Sigma \succ K_1 \sharp K_U$, and hence $K_0 \succ K_1$.

For the converse, let G and G' be the kernels of the homomorphisms $\psi_* : H_n(F_0, \partial F_0) \to H_n(F_1, \partial F_1)$ and $\psi_* : H_n(F_0) \to H_n(F_1)$ respectively. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & G & \to & H_n(F_0, \partial F_0) & \xrightarrow{\psi_*} & H_n(F_1, \partial F_1) & \to & 0 \\ & & \downarrow{\scriptstyle \Delta_0|_G} & & \downarrow{\scriptstyle \Delta_0} & & \downarrow{\scriptstyle \Delta_1} & & \\ 0 & \to & G' & \to & H_n(F_0) & \xrightarrow{\psi_*} & H_n(F_1) & \to & 0. \end{array}$$

Since $H_n(F_1, \partial F_1)$ and $H_n(F_1)$ are free, the exact sequences split. This means that the variation map Δ_0 of K_0 is isomorphic to the direct sum of the variation map Δ_1 of K_1 and the isomorphism $\Delta_0|_G : G \to G'$.

Recall that with respect to certain bases, the matrix associated with the variation map is the inverse of the Seifert matrix (for details see [66]). Since $n \geq 3$, every unimodular matrix is realized as the Seifert matrix of a simple fibered $(2n-1)$-knot (see [36, 65]). So we see that there exists a simple fibered $(2n-1)$-knot Σ which realizes $\Delta_0|_G : G \to G'$ as its variation map.

Then, we see that the Seifert matrices for K_0 and $K_1 \sharp \Sigma$ are congruent. Consequently, they are orientation preservingly isotopic to each other by [36, 65].

Furthermore, since K_0 is homotopy equivalent to both K_1 and $K_1 \sharp \Sigma$, we see that Σ should be homeomorphic to a sphere. This completes the proof. □

Remark 9.10. For $n = 1$, Theorem 9.9 does not hold[1]. Let K_1 be a non-trivial spherical prime fibered 1-knot in S^3 and K_0 a spherical prime satellite fibered 1-knot with companion K_1, where their fibering structures are compatible. Then we can show that $K_0 \succ K_1$. However, K_0 is not isotopic to the connected sum $K_1 \sharp \Sigma$ for any non-trivial 1-knot Σ. Note that such a construction does not give a counter example to Theorem 9.9 for $n \geq 3$, since such a satellite knot in higher dimensions is always a connected sum by virtue of Theorem 6.1.

We do not know if the above results are valid for $n = 2$.

Remark 9.11. Theorem 9.9 implies in particular that the fiber of K_0 is diffeomorphic to the boundary connected sum of the fiber of K_1 and a certain $(n-1)$-connected $2n$-dimensional manifold with spherical boundary. When K_0 and K_1 are spherical, this is also a consequence of [39, Theorem B].

Definition 9.12. Let us consider the equivalence relation generated by the pull back relation defined in Definition 9.1. When two m-knots K_0 and K_1 in S^{m+2} are equivalent with respect to this equivalence relation, we say that K_0 and K_1 are *pull back equivalent*.

The above definition together with Theorem 9.9 implies the following, whose proof is easy and is left to the reader.

Corollary 9.13. *Two simple fibered $(2n-1)$-knots K_0 and K_1 in S^{2n+1} with $n \geq 3$ are pull back equivalent if and only if there exist spherical simple fibered $(2n-1)$-knots Σ_0 and Σ_1 in S^{2n+1} such that $K_0 \sharp \Sigma_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma_1$.*

9.2 Special knots

In this section, we show that for a certain class of simple fibered knots, the pull back equivalence relation is equivalent to the relation generated by connected sums with spherical fibered knots together with the cobordism. For a theory of cobordism of simple fibered knots, refer to [8, 152, 153].

Definition 9.14. Let K be a simple fibered $(2n-1)$-knot with fiber F. Let us denote by $I(K)$ the image of the homomorphism $H_n(K) \to H_n(F)$ induced by the inclusion (or equivalently, the kernel of the homomorphism $H_n(F) \to$

[1] This remark is due to an observation from Shicheng Wang

$H_n(F, \partial F)$). The fibered knot K is said to be *special* if its Seifert form restricted to $I(K)$ is unimodular (for a definition of a Seifert form, see [36]).

Lemma 9.15. *A simple fibered $(2n-1)$-knot K is special if and only if there exist two simple fibered $(2n-1)$-knots K_F and K_T with the following properties:*

1. *K is orientation preservingly isotopic to $K_F \sharp K_T$,*
2. *the intersection form of the fiber of K_F is the zero form,*
3. *$H_{n-1}(K_T)$ is a torsion group (or equivalently, $H_n(K_T) = 0$).*

Proof. If there exist simple fibered $(2n-1)$-knots K_F and K_T with properties (1)–(3), then the Seifert form of K restricted to $I(K)$ coincides with the Seifert form of K_F. Since K_F is fibered, its Seifert form must be unimodular. Hence, K is special.

Conversely, suppose that the simple fibered knot K is special. Let us consider a basis $e_1, \ldots, e_u, e_{u+1}, \ldots, e_{u+v}$ of $H_n(F)$, where $e_1, \ldots, e_u$ constitute a basis of $I(K)$. This is possible, since $I(K)$ is a direct summand of $H_n(F)$. Then, the Seifert matrix L of K with respect to this basis is of the form

$$L = \begin{pmatrix} L_F & A \\ B & C \end{pmatrix}$$

for some $u \times u$ matrix L_F, $u \times v$ matrix A, $v \times u$ matrix B, and $v \times v$ matrix C. Note that $L_F + (-1)^n ({}^tL_F) = 0$ and $A + (-1)^n ({}^tB) = 0$, since the homomorphism $H_n(F) \to H_n(F, \partial F) = \mathrm{Hom}(H_n(F), \mathbf{Z})$ can be identified with the intersection form of F and the intersection matrix of F is given by $L + (-1)^n ({}^tL)$ (for example, see [36]). Since L_F is unimodular by our hypothesis and $L_F = (-1)^{n+1}({}^tL_F)$, we see that L is congruent to a matrix of the form

$$L' = \begin{pmatrix} L_F & 0 \\ 0 & L_T \end{pmatrix}$$

for some $v \times v$ matrix L_T. Since L' is unimodular, so is L_T. Furthermore, $L_T + (-1)^n ({}^tL_T)$ is a nonsingular matrix, since the kernel of the intersection form is generated by $e_1, \ldots, e_u$. Let K_F and K_T be the simple fibered $(2n-1)$-knots realizing L_F and L_T as their Seifert matrices respectively. Then, we can check that conditions (1)–(3) are satisfied. This completes the proof. □

Remark 9.16. In the above lemma, if $H_{n-1}(K)$ is torsion free, then the knot K_T is spherical.

Let us prove the following.

Theorem 9.17. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with $n \geq 3$. Suppose that K_0 is special and that $H_{n-1}(K_0)$ is torsion free. Then the following conditions are all equivalent to each other.*

1. *$K_0 \sharp \Sigma_0$ is cobordant to $K_1 \sharp \Sigma_1$ for some spherical knots Σ_0 and Σ_1.*
2. *$K_0 \sharp \Sigma'_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma'_1$ for some spherical simple fibered knots Σ'_0 and Σ'_1.*
3. *K_0 is pull back equivalent to K_1.*

For the proof, we need the following lemma, which is a direct consequence of [8, Theorem 4] (see also [152, 153]). Recall that a $(2n-1)$-knot is *simple* if it is $(n-2)$-connected and it bounds an $(n-1)$-connected $2n$-dimensional compact manifold in S^{2n+1}.

Lemma 9.18. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with $n \geq 3$. If $K_0 \sharp \Sigma_0$ and $K_1 \sharp \Sigma_1$ are cobordant for some spherical simple knots Σ_0 and Σ_1, then the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.*

Proof of Theorem 9.17. The equivalence of (2) and (3) follows from Corollary 9.13. Condition (2) clearly implies condition (1). Thus, we have only to show that (1) implies (2).

Suppose that (1) holds. Since every spherical $(2n-1)$-knot is cobordant to a spherical simple $(2n-1)$-knot by [91], we may assume that Σ_0 and Σ_1 are simple. Then by Lemma 9.18, the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other. By our assumption, these forms are unimodular, and hence K_1 is also special. Therefore, by Lemma 9.15, there exist simple fibered $(2n-1)$-knots $K_F^{(i)}$, $K_T^{(i)}$, $i = 0, 1$, such that

1. K_i is orientation preservingly isotopic to $K_F^{(i)} \sharp K_T^{(i)}$,
2. the intersection form of the fiber of $K_F^{(i)}$ is the zero form,
3. $H_{n-1}(K_T^{(i)})$ is a torsion group,

for $i = 0, 1$. Note that $K_F^{(0)}$ is orientation preservingly isotopic to $K_F^{(1)}$, since their Seifert forms are isomorphic.

Recall that $H_{n-1}(K_0)$ is torsion free by our assumption. Therefore, $K_T^{(i)}$ are spherical knots for $i = 0, 1$. Since $K_0 \sharp K_T^{(1)}$ is orientation preservingly isotopic to $K_F^{(1)} \sharp K_T^{(0)} \sharp K_T^{(1)}$, it is also orientation preservingly isotopic to $K_1 \sharp K_T^{(0)}$. Hence condition (2) holds. This completes the proof. □

Remark 9.19. Let K_F be the simple fibered $(2n-1)$-knot as in Lemma 9.15. Then its Seifert form is skew-symmetric for n even, and is symmetric for n odd. Note that unimodular skew-symmetric matrices have even ranks and the congruence class of such a matrix is uniquely determined by its rank. Therefore, when n is even, the oriented isotopy class of K_F is determined by its rank, which is even. On the other hand, when n is odd, unimodular symmetric matrices are not determined by its rank. For details, refer to [111], for example.

Proposition 9.20. *For every odd integer $n \geq 3$, there exists a pair (K_0, K_1) of simple fibered $(2n-1)$-knots with the following properties.*

1. *The knots K_0 and K_1 are cobordant.*
2. *The knots K_0 and K_1 are not pull back equivalent.*

Proof. Let us consider the following two matrices:

$$L_0 = \begin{pmatrix} 9 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 25 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that they are both unimodular and that

$$S_0 = L_0 - {}^tL_0 = S_1 = L_1 - {}^tL_1 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Let us show that L_0 and L_1 are algebraically cobordant for $\varepsilon = (-1)^n = -1$.

Set $m = {}^t(5, 0, 3, 0)$ and $m' = {}^t(0, 3, 0, 5)$. Then it is easy to see that the submodule M of $\mathbf{Z}^4$ generated by m and m' constitutes a metabolizer for $L = L_0 \oplus (-L_1)$. Furthermore, M is pure in $\mathbf{Z}^4$: in other words, M is a direct summand of $\mathbf{Z}^4$. Since $S_0 = S_1$ are non-degenerate, we have only to verify the condition c.2 of the definition of algebraic cobordism.

Set $S = S_0 \oplus (-S_1) = L - {}^tL$. Let $S^* : \mathbf{Z}^4 \to \mathbf{Z}^4$, $S_0^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ and $S_1^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the adjoints of S, S_0 and S_1 respectively. It is easy to see that $\operatorname{Coker} S_0^* = \operatorname{Coker} S_1^*$ is naturally identified with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Furthermore, we have

$$S^*(m) = {}^tmS = (0, 10, 0, -6) \quad \text{and} \quad S^*(m') = {}^tm'S = (-6, 0, 10, 0).$$

Therefore, $S^*(M)^\wedge$, the smallest direct summand of $\mathbf{Z}^4$ containing $S^*(M)$, is the submodule of $\mathbf{Z}^4$ generated by $(0, 5, 0, -3)$ and $(-3, 0, 5, 0)$. Hence, for the natural quotient map $d : \mathbf{Z}^4 \to \operatorname{Coker} S^* = (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_2)$, we have

$$d(S^*(M)^\wedge) = \{(x, x) \,:\, x \in \operatorname{Coker} S_0^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2\},$$

since $\operatorname{Im} S_i^*$ is generated by $(2, 0)$ and $(0, 2)$, $i = 0, 1$, and $\operatorname{Im} S^*$ is generated by $(2, 0, 0, 0)$, $(0, 2, 0, 0)$, $(0, 0, 2, 0)$ and $(0, 0, 0, 2)$. Therefore, we conclude that the unimodular matrices L_0 and L_1 are algebraically cobordant.

Now, there exists a simple fibered $(2n-1)$-knot K_i which realizes L_i as its Seifert form with respect to the fiber, $i = 0, 1$ (see [36, 65]). By [8, Theorem 3], K_0 and K_1 are cobordant.

Let us now show that K_0 and K_1 are not pull back equivalent. By Corollary 9.13, we have only to show that for any spherical simple fibered $(2n-1)$-knots Σ_0 and Σ_1 in S^{2n+1}, $K_0 \sharp \Sigma_0$ is never orientation preservingly isotopic to $K_1 \sharp \Sigma_1$.

Since $K_i \sharp \Sigma_i$ is a fibered knot, we can consider the monodromy on the n-th homology group of the fiber, $i = 0, 1$. Let us denote by H_i the monodromy matrix of $K_i \sharp \Sigma_i$ and by $\widetilde{L}_i$ its Seifert matrix with respect to the same basis. Here, we choose a basis which is the union of a basis of the homology of the fiber for K_i and that for Σ_i. It is known that $H_i = (-1)^{n+1} \widetilde{L}_i^{-1}({}^t\widetilde{L}_i)$ (for example, see [36]). Therefore, we have

$$H_0 = \begin{pmatrix} -1 & 0 \\ 18 & -1 \end{pmatrix} \oplus H_0' \quad \text{and} \quad H_1 = \begin{pmatrix} -1 & 0 \\ 50 & -1 \end{pmatrix} \oplus H_1',$$

where H_i' is the monodromy matrix of Σ_i, $i = 0, 1$.

Let us consider $\operatorname{Ker}((I + H_i)^2)$, where I is the unit matrix, $i = 0, 1$. Since Σ_i are spherical knots, the monodromy matrices H_i' cannot have the eigenvalue -1. Therefore, $\operatorname{Ker}((I + H_i)^2)$ corresponds exactly to the homology of the fiber of K_i.

Suppose that $K_0 \sharp \Sigma_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma_1$. Then the Seifert form of $K_0 \sharp \Sigma_0$ restricted to $\operatorname{Ker}((I + H_0)^2)$ should be isomorphic to that of $K_1 \sharp \Sigma_1$ restricted to $\operatorname{Ker}((I + H_1)^2)$. This means that L_0 should be congruent to L_1. However, this is a contradiction, since there exists an element $x \in \mathbf{Z}^2$ such that ${}^txL_0x = 9$, while such an element does not exist for L_1.

Thus, we conclude that K_0 and K_1 are not pull back equivalent. $\square$

Note that the simple fibered knots K_0 and K_1 constructed above are special; however, $H_{n-1}(K_i)$, $i = 0, 1$, are not torsion free.

Remark 9.21. In fact, we can find infinitely many examples as in the above proposition. For example, we could use the matrices

$$\begin{pmatrix} p^2 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^2 & 1 \\ -1 & 0 \end{pmatrix}$$

for arbitrary relatively prime odd integers p and q. Or we could also use $K_0 \sharp K'$ and $K_1 \sharp K'$, instead of K_0 and K_1, for any simple fibered $(2n-1)$-knot K' whose monodromy does not have the eigenvalue -1.

As has been remarked in Remark 9.3, under a certain connectivity condition, if two m-knots K_0 and K_1 with $m \geq 5$ are pull back equivalent, then they are cobordant after taking connected sums with some spherical knots. The above example shows that the converse is not true in general.

Let us now give some examples of pairs of knots which are diffeomorphic but not cobordant even after taking connected sums with (not necessarily simple or fibered) spherical knots. For this, we use the following proposition, which is a slight modification of Lemma 9.18 and is implicitly proved in the proof of Theorem 9.17.

Proposition 9.22. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with $n \geq 3$. If $K_0 \sharp \Sigma_0$ and $K_1 \sharp \Sigma_1$ are cobordant for some spherical knots Σ_0 and Σ_1, then the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.*

Remark 9.23. In fact, the above proposition is implicitly proved also in [153]. Based on this, Vogt proves the following. The usual $(2n-1)$-dimensional spherical knot cobordism group C_{2n-1} acts on the cobordism semi-group of simple $(2n-1)$-knots with torsion free homologies by connected sum. The orbit space of the action inherits a natural semi-group structure. Then this orbit space contains infinitely many free generators as a commutative semi-group for $n \geq 3$.

Vogt [153] also proves that the action of C_{2n-1} on the cobordism semi-group of simple $(2n-1)$-knots is fixed point free for $n \geq 3$. This can also be proved by using [8, (5.1) Proposition]. In fact, for an arbitrary spherical simple $(2n-1)$-knot Σ whose Alexander polynomial is nontrivial and irreducible, $K \sharp \Sigma$ is never cobordant to K for any simple $(2n-1)$-knot K, since the Alexander polynomials of $K \sharp \Sigma$ and K do not satisfy a Fox-Milnor type relation necessary to be cobordant (see [8, (5.1) Proposition]).

Chapter 10

Examples

"An expert is a man who has made all the mistakes,
which can be made, in a very narrow field."
Niels Henrik David Bohr

In this Chapter, we review some examples constructed in [5, 7, 9].

Non-spherical simple fibered $(2n-1)$-knots with $n \geq 3$ which are cobordant but are not isotopic.

Example 10.1 ([5]). Let K_i, with $i = 0, 1$, be the spherical algebraic $(2n-1)$-knots, $n \geq 3$, associated with the isolated singularity at 0 of the polynomial functions $h_i : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ defined by

$$h_i(x_0, x_1, \ldots, x_n) = g_i(x_0, x_1) + x_2^p + x_3^q + \sum_{k=4}^{n} x_k^2$$

with

$$g_0(x_0, x_1) = (x_0 - x_1)\Big((x_1^2 - x_0^3)^2 - x_0^{s+6} - 4x_1 x_0^{(s+9)/2}\Big) \Big((x_0^2 - x_1^5)^2 - x_1^{r+10} - 4x_0 x_1^{(r+15)/2}\Big),$$

and

$$g_1(x_0, x_1) = (x_0 - x_1)\Big((x_1^2 - x_0^3)^2 - x_0^{r+14} - 4x_1 x_0^{(r+17)/2}\Big) \Big((x_0^2 - x_1^5)^2 - x_1^{s+2} - 4x_0 x_1^{(s+7)/2}\Big),$$

where $s \geq 11$, $s \neq r+8$, s and r are odd, and p and q are distinct prime numbers which do not divide the product $330(30+r)(22+s)$ (see [35, p. 166]). Note that the algebraic knots K_i associated with h_i are spherical for $i = 0, 1$. It has been shown in [35] that the algebraic knots K_0 and K_1 are cobordant but are not isotopic.

Now let L be the algebraic $(2n-1)$-knot associated with the isolated singularity at 0 of the polynomial function $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ defined by

$$f(x_0, x_1, \ldots, x_n) = \sum_{k=0}^{n} x_k^2.$$

according to [36] (prop. 2.2 p.50) this algebraic knot has $A = \Big((-1)^{n(n+1)/2}\Big)$, defined on a free $\mathbf{Z}$-module of rank one H, as Seifert matrix. Note that L is not spherical.

Let us consider the connected sums $L_i = K_i \sharp L,\ i = 0,1$, which are simple fibered $(2n-1)$-knots.

We construct L_i the connected sum of L and K_i for $i = 0,1$. The Seifert form for L_i is the integral bilinear form $A \oplus A_i$ defined on a free $\mathbf{Z}$-module $H_i = G \oplus H_i$ of finite rank. The knots L_i are simple fibered since $A \oplus A_i$ is unimodular.

The knots L_0 and L_1 are cobordant by Theorem 6.3 (see [5]), let us see that they cannot be isotopic.

Let $\tau_i,\ i = 0,1$ be the monodromy associated with the fibered knots $L_i,\ i = 0,1$. If there exists an integer e such that $(\tau_i^e - 1)\, G_i = 0$ then e_i is called an *exponent* for L_i.

Recall that for $i = 0,1$ the *ϵ_i-twist group* for K_i is defined as follows: assuming $(t_i^{\epsilon_i} - 1)^2\, H_i = 0$ where t_i is the monodromy associated with K_i, if ϵ_i is an exponent for K_i then the ϵ_i-twist group associated to K_i is the group denoted by $GT^{\epsilon_i}(h_i)$ (or $GT^{\epsilon_i}(K_i)$) which is the $\mathbf{Z}$-torsion subgroup of the quotient $\operatorname{Ker}(t_i^{\epsilon_i} - 1)/(t_i^{\epsilon_i} - 1)H_i$.

According to the monodromy theorem (Brieskorn-Grothendieck), the e-twist group is well defined for one dimensional algebraic knots, and P. Du Bois and F. Michel [35] showed:

1- ϵ is an even exponent for the algebraic knots associated to g_0 and g_1 ; and for all multiples k of ϵ the finite abelian groups $GT^k(g_0)$ and $GT^k(g_1)$ have distinct orders,

2- all multiples k of ϵ are even exponents for the algebraic knots associated with g_0 and g_1, the k-twist group for h_0 and h_1 are well defined and $GT^k(h_i) = \Big(GT^k(g_i)\Big)^{(p-1)(q-1)}$ for $i = 0,1$.

Let k be a multiple of $\epsilon = 330(30+r)(22+s)$. For a fibered knot L, if we denote by A the matrix of the Seifert form and by $\mathcal{T}$ the matrix of the monodromy; then these matrices are related together by $\mathcal{T} = (-1)^n A^{-1}\, {}^t\!A$. Hence for $i = 0,1$ we have $\tau_i = (\pm Id) \oplus t_i$ thus $GT^k(L_i)$ is well defined and we have $GT^k(L_i) = GT^k(h_i)$.

Finally $GT^k(L_0)$ and $GT^k(L_1)$ have distinct order and as $\mathbf{Z}[t,t^{-1}]$-module $\mathrm{H}_n(G_0)$ and $\mathrm{H}_n(G_1)$ are not isomorphic. Hence the knots L_0 and L_1 are not isotopic.

Note that according to [1, Theorem 4, p. 117], the knots L_0 and L_1, which are connected sums of two algebraic knots, are not algebraic.

Let K be a knot. A *stabilization* of K is the operation of taking the connected sum $K \sharp K_S$ for some null-cobordant spherical knot K_S. As the above examples show, stabilization is a natural way to construct knots that are cobordant but are not isotopic. We have other types of constructions as follows.

Example 10.2. The matrices given in Example 5.2 (2) give two spherical simple $(2n-1)$-knots with $n \geq 3$ odd which are cobordant but are not isotopic. Similarly, the matrices given in Example 5.2 (3) give two simple fibered non-spherical $(2n-1)$-knots with $n \geq 3$ odd which are cobordant but are not isotopic.

Non-spherical 3-knots which are cobordant but are not isotopic

Example 10.3 ([9]). A *stabilizer* is a simple fibered spherical 3-knot whose fiber F is diffeomorphic to $(S^2 \times S^2)\sharp(S^2 \times S^2) \setminus \overset{\circ}{D}{}^4$. Such a stabilizer does exist (see [132, §4]). Moreover, we denote by K_S a stabilizer with Seifert matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

with respect to a basis of $H_2(F)$ denoted by a_1, a_2, a_3, a_4 (see [131, p. 600] or [135, §10]).

Since A is not congruent to the zero form, K_S is a non-trivial 3-knot.

Moreover, the submodule generated by a_1 and a_3 is a metabolizer for A, and one can do embedded surgeries on the two cycles a_1 and a_3, represented by two embedded 2-spheres in F. The result of this embedded surgery in D^6 is a 4-dimensional disk properly embedded in D^6 with K_S as boundary. Thus K_S is null-cobordant, i.e., it is cobordant to the trivial spherical 3-knot.

Example 10.4. We can construct cobordant, but not isotopic non-spherical 3-knots as follows. Let K_S be a null cobordant stabilizer as in Example 10.3 Note that K_S is a non-trivial 3-knot which is cobordant to the trivial 3-knot. Then consider any simple fibered 3-knot K which is not spherical. Then the two simple fibered 3-knots $K\sharp K_S$ and K are not isotopic, since the ranks of the second homology groups of their fibers are distinct. However, these knots are cobordant.

Examples related to Pull back In the following two examples we give a pair of diffeomorphic knots for which their connected sums with any spherical knots are never cobordant. This answers question $(\mathcal{Q})$ mentioned at the beginning of this section negatively.

Example 10.5 ([7]). Let us consider the following unimodular matrices:

$$A_0 = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-1)^{n+1} & 0 & 0 & 1 \\ 0 & (-1)^{n+1} & 0 & 0 \end{pmatrix}.$$

Then, for every integer $n \geq 3$, there exist simple fibered $(2n-1)$-knots K_i in S^{2n+1} whose Seifert matrices are given by A_i, $i = 0, 1$. Note that if we denote their fibers by F_i, $i = 0, 1$, then F_1 is orientation preservingly diffeomorphic to $F_0\sharp(S^n \times S^n)$. In particular, K_0 and K_1 are orientation preservingly diffeomorphic to each other.

It is easy to verify that the Seifert form restricted to $I(K_1)$ is the zero form, while it is not zero for K_0. Hence, by Proposition 9.22, $K_0\sharp\Sigma_0$ is never cobordant to $K_1\sharp\Sigma_1$ for any spherical (not necessarily simple or fibered) knots Σ_0, Σ_1.

Note that for this example, we have $H_{n-1}(K_i) \cong \mathbf{Z} \oplus \mathbf{Z}$, $i = 0, 1$.

Let us give another kind of an example together with an argument using the Alexander polynomial.

Example 10.6 ([7]). Let us consider the unimodular matrices

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and their associated simple fibered $(2n-1)$-knots K_i, $i = 0, 1$, with $n \geq 4$ even. As in Example 10.5 we see that K_0 and K_1 are orientation preservingly diffeomorphic to each other.

Now, suppose that for some spherical $(2n-1)$-knots Σ_i, $i = 0, 1$, $K_0 \sharp \Sigma_0$ is cobordant to $K_1 \sharp \Sigma_1$. We may assume that Σ_0 and Σ_1 are simple. The Alexander polynomials of K_0 and K_1 are given by

$$\Delta_{K_0}(t) = \det(tA_0 + {}^tA_0) = t^2 + t + 1$$

and

$$\Delta_{K_1}(t) = \det(tA_1 + {}^tA_1) = -(t^4 + t^3 - t^2 + t + 1)$$

respectively. Both of these polynomials are irreducible over $\mathbf{Z}$. If $K_0 \sharp \Sigma_0$ is cobordant to $K_1 \sharp \Sigma_1$, then by Proposition 6.38, we must have a Fox-Milnor type relation

$$\Delta_{K_0}(t)\Delta_{\Sigma_0}(t)\Delta_{K_1}(t)\Delta_{\Sigma_1}(t) = \pm t^a f(t) f^*(t) \tag{10.1}$$

for some $a \in \mathbf{Z}$ and $f(t) \in \mathbf{Z}[t]$, where $\Delta_{\Sigma_i}(t)$ denotes the Alexander polynomial of Σ_i, $i = 0, 1$.

Note that we have $|\Delta_{K_0}(1)| = |\Delta_{K_1}(1)| = 3$ and $|\Delta_{\Sigma_0}(1)| = |\Delta_{\Sigma_1}(1)| = 1$. Since $\Delta_{K_0}(t)$ is irreducible of degree 2, and $\Delta_{K_1}(t)$ is irreducible of degree 4, the relation (10.1) leads to a contradiction.

Hence, $K_0 \sharp \Sigma_0$ is not cobordant to $K_1 \sharp \Sigma_1$ for any spherical (not necessarily simple or fibered) $(2n-1)$-knots Σ_0, Σ_1. In this example we have $H_{n-1}(K_i) \cong \mathbf{Z}_3$, for $i = 0, 1$.

3-knots and Spin cobordism

Example 10.7. Set $\mathcal{M} = S^1 \times \Sigma_g$, where Σ_g is the closed connected orientable surface of genus $g \geq 2$. Note that $H_1(\mathcal{M})$ is torsion free. Let K_0 and K_1 be the simple fibered $\mathcal{M}$-knots constructed in [135, Proposition 3.8]. They have the property that their Seifert forms are isomorphic, but that there exists no diffeomorphism between K_0 and K_1 which preserves their spin structures. Thus, the Seifert forms of K_0 and K_1 are algebraically cobordant, but are not spin cobordant. Hence they are not cobordant by Proposition 8.15.

This example shows that the spin structure plays an essential role in the theory of cobordism for 3-knots.

Example 10.8. Let $\mathcal{M}$ be a nontrivial orientable S^1-bundle over the closed connected orientable surface of genus $g \geq 2$. Note that $H_1(\mathcal{M})$ is not torsion free in general. Let $K_1, K_2, \ldots, K_n$ be the simple fibered $\mathcal{M}$-knots constructed in [135, Theorem 3.1]. They have the property that their Seifert forms are isomorphic to each other, but that any such isomorphism restricted to $H_2(K_i)$ cannot be realized by a diffeomorphism. Thus, the Seifert forms of K_i are algebraically cobordant to each other, but are not spin cobordant. Hence they are not cobordant by Proposition 8.15, which is valid also for non-free simple fibered 3-knots.

Example 10.9. We can construct simple 3-knots which are C-algebraically fibered, but are not fibered, as follows. Let K be a simple fibered 3-knot. It is easy to see that there are a lot of simple algebraically non-fibered spherical 3-knots K' which are null cobordant. For example, consider the boundary of $S^2 \times S^2 - \operatorname{Int} D^4$ embedded in S^5 so that its Seifert form is isomorphic to

$$\begin{pmatrix} 0 & k \\ 1-k & 0 \end{pmatrix}, \quad k \geq 1.$$

Then the simple 3-knot $K \sharp K'$ is C-algebraically fibered, since its Seifert form is algebraically cobordant to that of K. However, $K \sharp K'$ is not (algebraically) fibered.

Example 10.10. For a simple 3-knot, the algebraic cobordism class of a Seifert form depends on the choice of the Seifert manifold in general. For example, let K be the trivially embedded $S^1 \times S^2$ in S^5; i.e., S^4 is trivially embedded in S^5 and S^4 decomposes into $F_0 = D^2 \times S^2$ and $F_1 = S^1 \times D^3$ along K. Note that F_0 is a 1-connected Seifert manifold of K, while F_1 is a Seifert manifold which is not 1-connected. Then the Seifert forms with respect to F_0 and F_1 are not algebraically cobordant, since the ranks of $H_2(F_i)$, $i = 0, 1$, do not have the same parity.

Example 10.11. Let P be a non-trivial orientable S^1-bundle over the closed connected orientable surface of genus $g \geq 2$. Note that $H_1(P)$ is not torsion free in general. For every positive integer n, let $K_1, K_2, \ldots, K_n$ be the simple fibered 3-knots constructed in [135, Theorem 3.1] which are all abstractly diffeomorphic to P. They have the property that their fibers are all diffeomorphic and their Seifert forms are isomorphic to each other, but any such isomorphism restricted to $H_2(K_i)$ cannot be realized by a diffeomorphism. Thus, the Seifert forms of K_i are algebraically cobordant to each other, but are not Spin cobordant. Hence they are not cobordant by Proposition 8.15, which is valid also for non-free simple fibered 3-knots.

Chapter 11

Cobordism and concordance of surfaces in S^4

Hamlet — ... this was sometime a paradox, but now the time gives it proof.

In [72] Kervaire proved that a $2n$-sphere embedded in $S^{2n+2} = \partial(D^{2n+3})$ is the boundary of a $(2n+1)$-disk properly embedded in D^{2n+3}. This implies that C_{2n} is trivial.

Although there is no group structure on the set of cobordism classes of non-spherical 2-knots, we have a similar result. In fact we show that any connected, closed and orientable surface embedded in S^4 is the boundary of an orientable handlebody properly embedded in the disk D^5. When the surface is non-orientable, it is the boundary of a non-orientable handlebody properly embedded in D^5 if and only if the Euler number of the normal bundle vanishes.

Recall that the normal Euler number of an orientable surface embedded in S^4 always vanishes (see [112]). Let us recall the definition of the normal Euler number of a closed non-orientable surface M embedded in S^4, where S^4 is considered to be oriented. Throughout this section, we use the letter "M" for $2n$-knots rather than "K", since the letter "K" will be used for another purpose. The tubular neighborhood N of M may be regarded as a normal disk bundle over M. Let $p : \widetilde{M} \to M$ be the orientation double cover of M. Consider the induced bundle $\widetilde{N}$ over $\widetilde{M}$ so that we have the commutative diagram

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{p}} & N \\ \downarrow & & \downarrow \\ \widetilde{M} & \xrightarrow{p} & M. \end{array}$$

We orient $\widetilde{N}$ so that the induced map $\widetilde{p} : \widetilde{N} \to N$ preserves the orientations. The *normal Euler number* $e(M)$ of the surface M is then defined by $e(M) = (\widetilde{M} \cdot \widetilde{M})/2$, where $\widetilde{M} \cdot \widetilde{M}$ denotes the self-intersection number of $\widetilde{M}$ in $\widetilde{N}$, which is always even.

Let us denote by N_g the closed connected non-orientable surface of non-orientable genus g. For a closed connected non-orientable surface $M \cong N_g$ embedded in S^4, it is known that $e(M) \in \{-2g, 4-2g, 8-2g, \dots, 2g\}$. Furthermore, all the values in the set can be realized as the normal Euler number of some N_g embedded in S^4 (see [161, 104, 62]).

In [10] we characterized those closed connected surfaces embedded in S^4 which are the boundary of a handlebody properly embedded in D^5. For this purpose, we need to use Pin^- structures on manifolds.

A *Pin^- structure* on a manifold X is the homotopy class of a trivialization of $TX \oplus \det TX \oplus \varepsilon^N$ over the 2-skeleton $X^{(2)}$ of X, where TX denotes the tangent bundle, $\det TX$ denotes the orientation line bundle, and ε^N is a trivial vector bundle of dimension N sufficiently large. A Pin^- structure is equivalent to a Spin structure when X is orientable.

When M is a closed surface embedded in S^4, there is a canonical Pin^- structure defined on M. More precisely, since M is characteristic,i.e., as a submanifold of S^4 it represents the $\mathbf{Z}_2$ homology class dual to the second Stiefel-Whitney class of S^4, there exists a unique Spin structure on $S^4 \setminus M$ which cannot be extended to any normal 2-disk of M. This Spin structure on $S^4 \setminus M$ induces a unique Pin^- structure on M (see [77]).

We denote by H_g the orientable handlebody of dimension three which is obtained by gluing g orientable 1-handles to a 0-handle. The boundary of H_g is the closed connected orientable surface of genus g, denoted by Σ_g. Furthermore, we denote by I_g the non-orientable handlebody of dimension three which is obtained by gluing g non-orientable 1-handles to a 0-handle. Then the boundary of I_g is identified with N_{2g}. In the following we will denote by K_g the handlebody H_g or I_g.

Definition 11.1. Let M be a closed connected surface embedded in S^4. Suppose that M has genus g if M is orientable and $2g$ if M is non-orientable. Let $\psi : \partial K_g \to M$ be a diffeomorphism. We say that ψ is Pin^- *compatible* if the Pin^- structure on ∂K_g induced by ψ extends through K_g.

When M is oriented, there always exists a compact oriented 3-dimensional submanifold V of S^4 such that $\partial V = M$ as oriented manifolds (see, for example, [40]). Such a manifold V is called a *Seifert manifold* associated with M (see §1.2.1). When M is non-orientable, a compact 3-dimensional submanifold V of S^4 with $\partial V = M$ is also called a Seifert manifold. Such a (non-orientable) Seifert manifold exists for M if and only if $e(M) = 0$ (see [46, 62]). When a surface M admits a Seifert manifold V, the unique Spin structure on S^4 induces a Pin^- structure on V and this induces a Pin^- structure on M, which coincides with the Pin^- structure described above (see [41]).

In [10] we proved the following theorem.

Theorem 11.2. *Let M be a closed connected surface embedded in $S^4 = \partial D^5$, and $\psi : \partial K_g \to M$ a diffeomorphism, where K_g denotes the 3-dimensional handlebody with g 1-handles. Then, there exists an embedding $\widetilde{\psi} : K_g \to D^5$ with $\widetilde{\psi}|_{\partial K_g} = \psi$ if and only if $e(M) = 0$ and ψ is* Pin^- *compatible.*

Remark 11.3. Since every closed connected 3-dimensional manifold admits a Heegaard splitting of genus $g \geq 0$, as a consequence of Theorem 11.2 we have a new proof of Rohlin's theorem [128] on the existence of an embedding of an arbitrary closed 3-dimensional manifold into $\mathbf{R}^5$ (see also [157, 159] and [49, p. 90]). For details, see [10].

Let us give a sketch of a proof of Theorem 11.2. First, it is easy to see that the vanishing of $e(M)$ and the Pin^- compatibility of ψ are necessary conditions. The proof of the sufficiency is based on embedded surgeries inside the disk D^5 on a Seifert manifold V of M. To do that we start with the abstract closed 3-manifold $V' = V \cup_\psi K_g$ obtained by attaching V and K_g along their boundaries by using ψ. Since the 3-dimensional cobordism group Ω_3^{Spin} (or $\Omega_3^{\mathrm{Pin}^-}$) of Spin (resp. Pin^-) manifolds is trivial (see [105], [72, Lemme III.7, p. 265], [49, p. 91], [101] or [76] for Ω_3^{Spin}, and [3, 77, 78] for $\Omega_3^{\mathrm{Pin}^-}$), there exists a compact (oriented if so is M) Pin^- 4-manifold W such that $\partial W = V'$ as (oriented) Pin^- manifolds. Let f be a Morse function $f : W \to [0, 1]$ which extends the projection to the second factor $\partial W = (V \times \{0\}) \cup_\psi (\partial K_g \times [0,1]) \cup (K_g \times \{1\}) \to [0,1]$. Note that f can be chosen so that all its critical values lie in the interval $(\varepsilon, 1-\varepsilon)$ for $\varepsilon > 0$ small enough. Moreover, we may assume that the critical points have index $1, 2$ or 3.

Consider the handlebody decomposition of W associated with this Morse function. We can remove handles of index 1 and 3 using modifications described by Wallace in [158], respecting the Pin^- structure. Then we get a new (oriented) Pin^- manifold W' such that $\partial W = \partial W'$. Since the handlebody decomposition of the manifold W' has only handles of index 2, we can attach the handles to $V \times [0,1]$ inside D^5 to get an embedding of W' into D^5. Finally we have a proper embedding of $K_g \cong (\partial K_g \times [0,1]) \cup (K_g \times \{1\}) \subset \partial W'$ into the disk D^5 such that $\partial K_g = M$.

As a corollary to Theorem 11.2 we have

Corollary 11.4 ([10]). *Let M be a closed connected surface embedded in $S^5 = \partial D^5$. Then there exists a 3-dimensional handlebody embedded in D^5 such that its boundary coincides with M if and only if $e(M) = 0$.*

Using Theorem 11.2, we can characterize cobordism classes of closed connected surfaces embedded in S^4 as follows.

Theorem 11.5 ([10]). *Let M_0 and M_1 be two closed connected surfaces embedded in S^4. Then they are cobordant if and only if they are diffeomorphic as abstract manifolds and have the same normal Euler number.*

Remark 11.6. The above theorem in the orientable case is proved by Ogasa [118], although his proof is slightly different from ours explained below.

When two closed connected surfaces embedded in S^4 are cobordant, it is clear that they are diffeomorphic as abstract manifolds and have the same normal Euler number (for details, see [10]). Thus we have the necessity in Theorem 11.5.

For the sufficiency, start with two closed connected surfaces M_0 and M_1 in S^4 which are diffeomorphic as abstract manifolds and have the same normal Euler number. In the following, we consider the case where M_0 and M_1 are non-orientable of non-orientable genus g. (For the orientable case, the proof is similar. For details, see [10].)

By changing M_0 and M_1 by isotopies, we may assume that for a 4-disk D^4 in S^4, we have $M_0 \cap D^4 = M_1 \cap D^4 = D^2$ and (D^4, D^2) is the standard disk pair. Set $\Delta = (S^4 \setminus \mathring{D}^4) \times [0,1] \cong D^5$ and

$$\widetilde{M} = (M_0 \setminus \mathring{D}^2) \cup (\partial D^2 \times [0,1]) \cup (M_1 \setminus \mathring{D}^2) = M_0^! \sharp M_1 \subset \partial\Delta,$$

where $M_0^!$ denotes the mirror image of M_0. Since $e(M_0) = e(M_1)$, we have $e(\widetilde{M}) = 0$. Furthermore, one can prove that there exists a Pin^- compatible diffeomorphism between $\partial((N_g \setminus \mathring{D}^2) \times [0,1]) \cong \partial I_g$ and $\widetilde{M}$ which sends $(N_g \setminus \mathring{D}^2) \times \{i\}$ diffeomorphically onto $M_i \setminus \mathring{D}^2$.

According to Theorem 11.2 we can embed I_g in Δ so that $M_0^! \sharp M_1 = \partial I_g$. The cobordism between M_0 and M_1 is then obtained by gluing back $D^4 \times [0,1]$ to Δ and by replacing $I_g \cong (N_g \setminus \mathring{D}^2) \times [0,1]$ by $N_g \times [0,1]$.

As a consequence of Theorem 11.5 we have that two closed connected orientable surfaces embedded in S^4 are cobordant if and only if they have the same genus. Hence, the monoide of cobordism classes of closed connected orientable surfaces embedded in S^4 is isomorphic to the monoide of non-negative integers $\mathbf{Z}_{\geq 0}$.

Let us consider non-orientable surfaces. First note that by adding the cobordism class of an embedding of S^2 into S^4 to the associative magma[1] of cobordism classes of closed connected non-orientable surfaces embedded in S^4, we get a monoide denoted by $\mathfrak{M}$. We can also describe the monoide structure of $\mathfrak{M}$ as follows. Let $\mathbf{R}P^2_+$ (or $\mathbf{R}P^2_-$) be the projective plane standardly embedded in S^4 with normal Euler number being equal to $+2$ (resp. -2) (see [59]). For a pair of non-negative integers (k,l) such that $k+l \geq 1$, let $M_{k,l}$ be the non-orientable surface embedded in S^4 obtained by taking the connected sum of k copies of $\mathbf{R}P^2_+$ and l copies of $\mathbf{R}P^2_-$. Then we have $e(M_{k,l}) = 2(k-l)$ and the genus of $M_{k,l}$ is equal to $k+l$. Hence, the set of non-orientable surfaces $\{M_{k,l} \, : \, k,l \in \mathbf{Z},\, k,l \geq 0,\, k+l \geq 1\}$ constitutes a complete set of representatives of the cobordism classes of closed connected non-orientable surfaces embedded in S^4. Therefore, $\mathfrak{M}$ is isomorphic to the monoid of pairs of non-negative integers $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$. If we denote by $[M]$ the cobordism class of a closed connected non-orientable surface M embedded in S^4, and by $g(M)$ the genus of M, then the isomorphism $\mathfrak{M} \to \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ is given by

$$\begin{array}{ccccc} \mathfrak{M} & \to & \mathbf{Z}_{\geq 0} & \times & \mathbf{Z}_{\geq 0} \\ {[M]} & \mapsto & \Big(\frac{2g(M)+e(M)}{4} & , & \frac{2g(M)-e(M)}{4} \Big) \end{array}$$

11.0.1 Concordance of embeddings of a surface

In this subsection, we consider the concordance classification of embeddings of closed connected surfaces into S^4. For the definition of the concordance, see Definition 1.11.

Examining the proof of Theorem 11.5 carefully, we see that the following characterization of concordant embeddings of surfaces into S^4 holds.

Theorem 11.7 ([10]). *Let Σ be a closed connected surface. Two embeddings of Σ into S^4 are concordant if and only if the* Pin^- *structures induced by these embeddings coincide and the normal Euler numbers of these embeddings coincide.*

Remark 11.8. When the knots are spherical of dimension two, the notions of cobordism and concordance coincide with each other, since every diffeomorphism of S^2 which preserves the orientation is isotopic to the identity [141]. However, when $g \geq 1$, for an arbitrary embedding $f : \Sigma_g \to S^4$ there exists an orientation preserving diffeomorphism $h : \Sigma_g \to \Sigma_g$ which does not preserve the Pin^- structure induced by f. Therefore, the embeddings $f \circ h$ and f are not concordant. This means that contrary to the spherical case, the notions of cobordism and concordance differ for orientable surfaces of genus $g \geq 1$.

The group of orientation preserving diffeomorphisms of a closed connected oriented surface acts transitively on the set of Pin^- structures with trivial Brown

[1] or semigroup

β	g: odd	g: even
0	0	$2^{(g-2)/2}(2^{(g-2)/2}+1)$
1	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0
2	0	2^{g-2}
3	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
4	0	$2^{(g-2)/2}(2^{(g-2)/2}-1)$
5	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
6	0	2^{g-2}
7	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0

Table 11.1. Number of Pin^-structures on N_g with Brown invariant $\beta \in \mathbf{Z}_8$

invariant (see, for example, [10]). This set is naturally identified with the set of Spin structures with trivial Arf invariant, since the surface is assumed to be orientable. This implies that the number of concordance classes of embeddings of a closed connected oriented surface is equal to the number of Pin^- structures with trivial Brown invariant on this surface. According to [61] this number is equal to $2^{g-1}(2^g+1)$, where g is the genus of the surface. If we denote by ω_g the number of concordance classes of embeddings of Σ_g, then we have $\omega_g = 2^{g-1}(2^g+1)$.

Let us denote by ν_g the number of concordance classes of embeddings of the closed connected non-orientable surface N_g of non-orientable genus g. According to [104, 62], the set of possible normal Euler numbers for such embeddings coincides with $\{-2g, 4-2g, 8-2g, \ldots, 2g\}$. Hence, we have

$$\nu_g = \sum_{i=0}^{g} \nu_{g,-2g+4i},$$

where $\nu_{g,-2g+4i}$ denotes the number of concordance classes of embeddings of N_g into S^4 with normal Euler number equal to $-2g+4i$. Moreover, according to [77, Theorem 6.3], $\nu_{g,-2g+4i}$ is equal to the number of Pin^- structures with Brown invariant equal to $-g+2i$ modulo 8. Such numbers can be calculated as in Table 11.0.1 (see [31]).

Using the values given in Table 11.0.1, we get

$$\nu_g = \begin{cases} 2^{g-2}(g+1) & \text{if } g \text{ is odd,} \\ 2^{g-2}(g+1) + 2^{(g-2)/2} & \text{if } g \text{ is even.} \end{cases}$$

Chapter 12

Cobordism and concordance of 4-knots

"In mathematics you don't understand things.
You just get used to them."
John von Neumann

In the study of cobordism of embeddings of even dimensional manifolds, the only case which remains to be studied is the case of 4-dimensional manifolds embedded in S^6. In [11] we proved the following

Theorem 12.1. *Let M be a closed simply connected 4-dimensional manifold. Then all the embeddings of M into S^6 are concordant.*

In particular, two 4-knots in S^6,i.e., two closed simply connected 4-dimensional manifolds embedded in S^6, are (oriented) cobordant if and only if they are abstractly (orientation preservingly) diffeomorphic to each other.

One will prove Theorem 12.1 by imitating the proofs of Theorems 11.2 and 11.5, and the proof is based essentially on Kervaire's original idea [72].

Let $f_i : M \longrightarrow S^6 \times \{i\}$, $i = 0, 1$, be two embeddings. We denote $K_i = f(M_i)$ the oriented submanifolds of $S^6 \times \{i\}$ such that f_i preserve orientations.

Set V_i some connected, compact and oriented submanifolds of dimension 5 of $S^6 \times \{i\}$ such that $\partial V_i = K_i$ (see § 1 Introduction). Up to ambient isotopy of $S^6 \times \{i\}$ one can assume that there exists a 6-disk Δ in S^6 such that

(i) $\Delta \times \{0\} \cap K_0 = \varnothing = \Delta \times \{1\} \cap K_1$,

(ii) $\Delta \times \{0\} \cap V_0$ coincide to $\Delta \times \{1\} \cap V_1$ if $S^6 \times \{0\}$ is identified with $S^6 \times \{1\}$,

(iii) $\Delta \times \{i\} \cap V_i$ is diffeomorphic to the 5-disk for $i = 0, 1$,

(iv) $V = \big(V_0 \setminus \mathrm{Int}(\Delta \times \{0\})\big) \cup_\partial \big(\partial(\Delta \times \{0\}) \cap V_0\big) \times [0, 1] \cup_\partial \big(V_1 \setminus \mathrm{Int}(\Delta \times \{1\})\big)$ is diffeomorphic to the oriented connected sum of V_0 and V_1,

(v) V is embedded in the boundary of $\big(S^6 \times [0, 1]\big) \setminus \big(\mathrm{Int}(\Delta) \times [0, 1]\big)$ which is diffeomorphic to the 7-disk.

Set W the closed and oriented 5-manifold obtained by gluing the manifolds V and $M \times [0, 1]$ along their boundaries using the maps f_0 and f_1. Moreover M is simply connected and since the natural spin structure of S^6 induces spin structures for V_0 and V_1 then the oriented manifold W admit a spin structure compatible with those of V_0 and V_1.

Recall that the cobordism group[1] of 5-manifolds is trivial, hence there exists a spin compact 6-manifold X such that $\partial X = W$ as spin manifolds. Since $\partial(M \times [0, 1]) = M \times \{0, 1\}$ then we have

$$\partial X = \partial W = \big((V \times \{0\}) \cup_{\partial V \times \{0\}} (\partial V \times [0, 1])\big) \bigcup_{f_0 \coprod f_1} \big((M \times [0, 1]) \times \{1\}\big),$$

[1] In the weak sense, see [105]

where $f_i : (M \times \{i\}) \times \{1\} \to K_i \times \{1\} \subset \partial V \times \{1\}$, $i = 0, 1$, are gluing diffeomorphisms.

Now let

$$\pi : \partial X \longrightarrow [0,1]$$

the projection on the second factor associated to the this decomposition of ∂X. Then there exists a Morse function

$$f : X \longrightarrow [0,1]$$

extending π without critical points of index 0 and 6, and such that all the critical values of f are in the interval $]\varepsilon, 1-\varepsilon[$ for $\varepsilon > 0$ a sufficiently small real.

With the Morse function f one can give a handle decomposition of X with gluing of handles on $V \times [0,1]$ along $V \times \{1\}$ which do not have handles of index 0 or 6.

Now will show that one can modify this decomposition in order to have only handles of index $0, 1$ and 3. First remark that a 5-handle is dual to a 1-handle. Moreover using Wallace's reduction (see [158] § 6) one can replace these dual 1-handles by some 4-handles. Hence one can replace all the 5-handles by some 2-handles, these 2-handles are dual of those 4-handles we just add. Since all the manifolds are orientable and $M \times [0,1]$ is connected, then all these handle modifications can be made by doing spin surgeries on X with no modification on∂X.

Now we have a handle decomposition of X without handle of index 5. Consider a 4-handle in this decomposition, it is dual to a 2-handle and since $M \times [0,1]$ is simply connected this 2-handle is trivially attached. It is the same to do the connected sum, along the boundary, with $S^2 \times D^4$. Then using again Wallace's reduction, we can replace this dual 2-handle by a dual 3-handle wich is a 3-handle.

Finally one can assume that X has a handle decomposition with no handle of index 4 and 5. Moreover since the manifold $V \times [0,1]$ is stably paralelizable and X is a spin manifold, then after the attachment of handles of indexes 1 and 2 we get a stably paralelizable manifold ; and with the nullity of the group $\pi_2(SO)$ implies that after attachment of 3-handles the manifold is still stably paralelizable.

Now to get the result we have to realize this handle decomposition as an embedded manifold in $S^6 \times [0,1]$. But this can easily be done since V is embedded in the boundary of a 7-disk, then we attach handles of indexes $1, 2$ and 3 to $V \times [0,1]$ in the 7-disk. The restriction of this embedding to $S^6 \times [0,1]$ gives the concordance. $\square$

Remark 12.2. It is known that a closed connected orientable 4-dimensional manifold M can be embedded in S^6 if and only if it is Spin and its signature vanishes (see [25]). If in addition M is simply connected, then it can be embedded in S^6 if and only if it is homeomorphic to a connected sum of some copies of $S^2 \times S^2$ by the homeomorphism classification of closed simply connected 4-dimensional manifolds due to Freedman [45].

Remark 12.3. By Park [120], for any sufficiently large odd integer m, there exist infinitely many smooth manifolds which are all homeomorphic to the connected sum of m copies of $S^2 \times S^2$ but which are not diffeomorphic to each other. Let us denote by $\mathfrak{O}_4$ the monoid of (oriented) cobordism classes of closed simply connected 4-manifolds embedded in S^6, and by $\mathbf{Z}_{\geq 0}$ the monoid of non-negative integers. Then the homomorphism $\varphi : \mathfrak{O}_4 \to \mathbf{Z}_{\geq 0}$ which associates to a 4-knot one half of its second Betti number is an epimorphism. The above result of Park shows that this homomorphism is far from being an isomorphism. Compare this with the result of Vogt [152, 153]: the corresponding homomorphism $\mathfrak{O}_{2n} \to \mathbf{Z}_{\geq 0}$ for $n \geq 3$ is an isomorphism, where $\mathfrak{O}_{2n}$ denotes the monoid of (oriented) cobordism classes of $2n$-knots in S^{2n+2}.

Remark 12.4. When $n \neq 2$, for an arbitrary $2n$-knot M, its orientation reversal $-M$ is oriented cobordant to M. For $n = 2$, there exists a closed 4-dimensional manifold N homeomorphic to a connected sum of some copies of $S^2 \times S^2$ such that N is not oriented diffeomorphic to $-N$. In fact, by Kotschick [81], every simply connected compact complex surface of general type which is Spin and has vanishing signature gives such an example. Such a complex surface has been constructed by Moishezon and Teicher [113, 114, 80]. Hence, there exists a closed simply connected oriented 4-dimensional manifold embedded in S^6 which is not oriented cobordant to its orientation reversal.

Chapter 13

Annexe

"En mathématiques, nous sommes davantage
les serviteurs que les maîtres."
Charles Hermite

In this Chapter we first present some results (cf [14]) concerning more general knots than simple fibered knots. In the previous Chapters we strongly used all the good properties of simple fibered knots, here we present generalizations of some results proved before.

Then we extend the result about 3-knots to a larger class.

13.1 Exact knots

Definition 13.1. Suppose $n \geq 2$. A Seifert surface F of a $(2n-1)$-knot K is said to be *exact* if the sequence

$$0 \to H_n(K) \to H_n(F)/\operatorname{Tors} H_n(F) \to H_n(F,K)/\operatorname{Tors} H_n(F,K) \to H_{n-1}(K) \to 0,$$

derived from the homology exact sequence for the pair (F,K), is well defined and exact. Note that the homomorphism

$$H_n(F,K)/\operatorname{Tors} H_n(F,K) \to H_{n-1}(K)$$

may not be well defined in general. Here, we impose the condition that this map should be well defined. A $(2n-1)$-knot is said to be *exact* if it admits an exact Seifert surface.

Example 13.2. Consider $K = S^{n-1} \times S^n$ embedded trivially in $S^{2n} \subset S^{2n+1}$, $n \geq 2$. Then K is a $(2n-1)$-knot and it bounds two Seifert surfaces $F_0 = D^n \times S^n$ and $F_1 = S^{n-1} \times D^{n+1}$, both of which are embedded in S^{2n}. Then F_0 is exact, while F_1 is not, since $H_n(S^{n-1} \times S^n) \to H_n(S^{n-1} \times D^{n+1})$ is not a monomorphism.

Lemma 13.3. *For $n \geq 2$, we have the following.*

(1) *A simple $(2n-1)$-knot is always exact. In fact, every $(n-1)$-connected Seifert surface is exact.*

(2) *A fibered $(2n-1)$-knot is always exact. In fact, every fiber is exact.*

(3) *A $(2n-1)$-knot is always exact. In fact, every Seifert surface is exact.*

Proof. In the following, let K be a $(2n-1)$-knot and F a relevant Seifert surface.

(1) Let us consider the exact sequence

$$H_{n+1}(F,K) \to H_n(K) \to H_n(F) \to H_n(F,K) \to H_{n-1}(K) \to H_{n-1}(F).$$

Then we have the desired result, since $H_{n+1}(F, K) \cong H^{n-1}(F) = 0$, $H_{n-1}(F) = 0$, and $H_n(F)$ and $H_n(F, K)$ are torsion free.

(2) If F is a fiber of a fibered knot, then it is easy to see that $S^{2n+1} \setminus F$ is homotopy equivalent to F. Hence, by Alexander duality, we have

$$\widetilde{H}_i(F) \cong \widetilde{H}^{2n-i}(F)$$

for all i, where $\widetilde{H}_*$ and $\widetilde{H}^*$ denote reduced homology and cohomology groups, respectively. Consider the exact sequence

$$\begin{array}{ccccccc} 0 & \to & \widetilde{H}_{n+1}(F) & \to & H_{n+1}(F, K) & \to \\ \widetilde{H}_n(K) & \to & \widetilde{H}_n(F) & \to & H_n(F, K) & \to \\ \widetilde{H}_{n-1}(K) & \to & \widetilde{H}_{n-1}(F) & \to & H_{n-1}(F, K) & \to 0. \end{array}$$

(Recall that K is $(n-2)$-connected.) Since

$$\widetilde{H}_{n-1}(F) \cong \widetilde{H}^{n+1}(F) \cong H_{n-1}(F, K)$$

and $\widetilde{H}_{n-1}(F) \to H_{n-1}(F, K)$ is an epimorphism, it must be an isomorphism. Hence $H_n(F, K) \to \widetilde{H}_{n-1}(K)$ is an epimorphism. Furthermore, since $\widetilde{H}_{n+1}(F) \cong \widetilde{H}^{n-1}(F) \cong H_{n+1}(F, K)$, $\widetilde{H}_{n+1}(F) \to H_{n+1}(F, K)$ is a monomorphism, and $\widetilde{H}_n(K)$ is torsion free, the homomorphism $\widetilde{H}_{n+1}(F) \to H_{n+1}(F, K)$ must be an isomorphism. Thus $\widetilde{H}_n(K) \to \widetilde{H}_n(F)$ is a monomorphism. Since $\widetilde{H}_n(K)$ is torsion free, the map

$$\widetilde{H}_n(K) \to \widetilde{H}_n(F)/\operatorname{Tors}\widetilde{H}_n(F)$$

is also a monomorphism. Finally, since $\widetilde{H}_n(F) \cong \widetilde{H}^n(F) \cong H_n(F, K)$, we have $\operatorname{Tors}\widetilde{H}_n(F) \cong \operatorname{Tors} H_n(F, K)$. Then we see easily that the sequence

$$0 \to \widetilde{H}_n(K) \to \widetilde{H}_n(F)/\operatorname{Tors}\widetilde{H}_n(F) \to H_n(F, K)/\operatorname{Tors} H_n(F, K) \to \widetilde{H}_{n-1}(K) \to 0$$

is well defined and exact.

(3) If K is a homotopy sphere, then $H_n(K) = 0 = \widetilde{H}_{n-1}(K)$, and hence

$$0 \to \widetilde{H}_n(F) \to H_n(F, K) \to 0$$

is exact. Thus the result is obvious. This completes the proof. □

The following can be regarded as a correction of [5, Proposition 2.1].

Proposition 13.4. *Let K be an exact $(2n-1)$-knot, $n \geq 2$, and A its Seifert form associated with an exact Seifert surface. Then, there exists a simple $(2n-1)$-knot K' cobordant to K such that the Seifert form of K' associated with an $(n-1)$-connected Seifert surface is algebraically cobordant to A.*

Remark 13.5. Note that when $n = 1$, every 1-knot admits a connected Seifert surface, and hence is simple.

Proof of Proposition 13.4. Let F be an exact Seifert surface of K. By exactly the same method as in [5, 89], with the help of an engulfing theorem, we can perform embedded surgeries on F inside the disk D^{2n+2} along spheres a of dimensions $\leq n-1$ embedded in F so that we obtain a simple knot K' cobordant to K and an $(n-1)$-connected Seifert surface F' for K'.

Let us examine the relationship between the Seifert forms with respect to F and F'. If the sphere a along which the surgery is performed is of dimension less than or equal to $n-2$, then it does not affect the n-th homology of F. We again denote by F the result of such surgeries: in particular, F is $(n-2)$-connected. Let us now consider the case where a is of dimension $n-1$. In the following, $[a]$ will denote the homology class in $H_{n-1}(F)$ represented by a, where we fix its orientation once and for all.

Case 1. When $[a]$ has infinite order in $H_{n-1}(F)$.

Since K is exact, the boundary homomorphism $\partial_* : H_n(F,K) \to H_{n-1}(K)$ is surjective. By the exact sequence

$$H_n(F,K) \xrightarrow{\partial_*} H_{n-1}(K) \xrightarrow{i_*} H_{n-1}(F) \xrightarrow{j_*} H_{n-1}(F,K),$$

where $i : K \to F$ and $j : F \to (F,K)$ are the inclusions, we see that j_* is injective and hence $j_*[a]$ has infinite order in $H_{n-1}(F,K) \cong H^{n+1}(F)$. Therefore, there exists an $(n+1)$-cycle $\widetilde{a}$ of F such that the intersection number $a \cdot \widetilde{a}$ does not vanish. We choose $\widetilde{a}$ so that $m = |a \cdot \widetilde{a}| (> 0)$ is the smallest possible.

Let $\psi : D^n \times D^{n+1} \to D^{2n+2}$ be the n-handle used by the surgery in question such that $\psi(S^{n-1} \times \{0\}) = a$. As in [5], let us put

$$F_T = F \setminus \mathrm{Int}(\psi(S^{n-1} \times D^{n+1})), \quad F^\star = F_T \cup \psi(D^n \times S^n).$$

Let us consider the Mayer-Vietoris exact sequence associated with the decomposition $F = F_T \cup \psi(S^{n-1} \times D^{n+1})$:

$$\begin{aligned} &H_{n+1}(F) \xrightarrow{s} H_n(\psi(S^{n-1} \times S^n)) \to H_n(F_T) \to H_n(F) \\ &\xrightarrow{t} H_{n-1}(\psi(S^{n-1} \times S^n)) \xrightarrow{u} H_{n-1}(F_T) \oplus H_{n-1}(\psi(S^{n-1} \times D^{n+1})). \end{aligned}$$

Since the map s is given by the intersection number with a, its image coincides with $m\mathbf{Z} \subset \mathbf{Z} \cong H_n(\psi(S^{n-1} \times S^n))$. Furthermore, since u is an injection, t is the zero map. Therefore, we have the exact sequence

$$0 \to \mathbf{Z}_m \to H_n(F_T) \to H_n(F) \to 0.$$

Therefore, the inclusion $F_T \to F$ induces an isomorphism

$$H_n(F_T)/\operatorname{Tors} H_n(F_T) \to H_n(F)/\operatorname{Tors} H_n(F).$$

Similarly we also have the following exact sequence obtained from the Mayer-Vietoris exact sequence associated with the decomposition

$$F^\star = F_T \cup \psi(D^n \times S^n)$$

$$\begin{aligned} &0 \to H_n(\psi(S^{n-1} \times S^n)) \to H_n(F_T) \oplus H_n(\psi(D^n \times S^n)) \to H_n(F^\star) \\ &\to H_{n-1}(\psi(S^{n-1} \times S^n)) \xrightarrow{u'} H_{n-1}(F_T). \end{aligned}$$

Note that the map u' is injective, since the image of the composition

$$H_{n-1}(\psi(S^{n-1} \times S^n)) \xrightarrow{u'} H_{n-1}(F_T) \xrightarrow{v} H_{n-1}(F)$$

is generated by $[a]$ which is of infinite order, where v is the homomorphism induced by the inclusion. Therefore, we see that the inclusion induces an isomorphism $H_n(F_T) \to H_n(F^\star)$.

Summarizing, we have the isomorphisms

$$H_n(F)/\operatorname{Tors} H_n(F) \xleftarrow{\cong} H_n(F_T)/\operatorname{Tors} H_n(F_T) \xrightarrow{\cong} H_n(F^\star)/\operatorname{Tors} H_n(F^\star)$$

induced by the inclusions.

Case 2. When $[a]$ has finite order in $H_{n-1}(F)$.

Let us denote the order of $[a]$ by $p > 0$. There exists an n-chain σ in F such that $\partial\sigma = pa$. We may assume that σ does not intersect with a outside of its boundary. Then, we have an n-chain σ' in F_T such that $[\partial\sigma'] = p[\psi(S^{n-1} \times \{*\})]$ in $H_{n-1}(\psi(S^{n-1} \times S^n))$.

As before, we have the following exact sequence:

$$H_n(\psi(S^{n-1} \times S^n)) \xrightarrow{w} H_n(F_T) \to H_n(F) \to 0.$$

Since $[\psi(\{*\} \times S^n)] \in H_n(F_T)$ has non-zero intersection number with the homology class in $H_n(F_T, \partial F_T)$ represented by σ', we see that the map w above is injective. Note that then $(\operatorname{Im} w)^\wedge$ is infinite cyclic. Let a generator of $(\operatorname{Im} w)^\wedge$ be denoted by $\ell \in H_n(F_T)$. Then, we have the following exact sequence:

$$0 \to \mathbf{Z}\langle \ell \rangle \to H_n(F_T)/\operatorname{Tors} H_n(F_T) \to H_n(F)/\operatorname{Tors} H_n(F) \to 0,$$

where $\mathbf{Z}\langle \ell \rangle$ denotes the infinite cyclic group generated by ℓ. This implies that

$$H_n(F_T)/\operatorname{Tors} H_n(F_T) \cong (H_n(F)/\operatorname{Tors} H_n(F)) \oplus \mathbf{Z}\langle \ell \rangle.$$

Similarly, we have the exact sequence

$$\begin{aligned} &0 \to H_n(\psi(S^{n-1} \times S^n)) \to H_n(F_T) \oplus H_n(\psi(D^n \times S^n)) \to H_n(F^\star) \\ &\xrightarrow{t'} H_{n-1}(\psi(S^{n-1} \times S^n)) \xrightarrow{u'} H_{n-1}(F_T) \to H_{n-1}(F^\star) \to 0. \end{aligned}$$

The image of p times the generator of $H_{n-1}(\psi(S^{n-1} \times S^n))$ by u' vanishes, since it bounds σ' in F_T. On the other hand, if p' times the generator belongs to $\operatorname{Ker} u'$ for some p' with $0 < p' < p$, then the order of $[a]$ is strictly less than p, which is a contradiction. Therefore, the image of s' is generated by $z = p[\psi(S^{n-1} \times \{*\})]$. Hence, we have the exact sequence

$$0 \to H_n(F_T) \to H_n(F^\star) \xrightarrow{t'} \mathbf{Z}\langle z \rangle \to 0,$$

where $\mathbf{Z}\langle z \rangle$ is the infinite cyclic group generated by $z \in H_{n-1}(\psi(S^{n-1} \times S^n))$. Let $\eta^\star$ be the n-cycle in $F^\star$ obtained by the union of p times $\psi(D^n \times \{*\})$ and σ'. Set $\ell^\star = [\eta^\star] \in H_n(F^\star)$. Then the image of $\ell^\star$ by t' coincides with $\pm z$. Therefore, we see that

$$H_n(F^\star)/\operatorname{Tors} H_n(F^\star) \cong (H_n(F_T)/\operatorname{Tors} H_n(F_T)) \oplus \mathbf{Z}\langle \ell^\star \rangle.$$

Summarizing, we have

$$H_n(F^\star)/\operatorname{Tors} H_n(F^\star) \cong (H_n(F)/\operatorname{Tors} H_n(F)) \oplus \mathbf{Z}\langle \ell \rangle \oplus \mathbf{Z}\langle \ell^\star \rangle.$$

So, in this case, the rank of the n-th homology group increases by two as a result of the surgery.

In the following, we denote by F the original Seifert surface for K and by F' the $(n-1)$-connected Seifert surface for K' obtained as a result of the surgeries. Set $G = H_n(F)/\operatorname{Tors} H_n(F)$. Note that

$$G' = H_n(F')/\operatorname{Tors} H_n(F') \cong G \oplus (\oplus_{i\in\mathcal{I}} (\mathbf{Z}\langle \ell_i\rangle \oplus \mathbf{Z}\langle \ell_i^\star\rangle)), \tag{13.1}$$

where the indices in $\mathcal{I}$ correspond to the surgeries necessary to kill the torsion of the $(n-1)$-th homology, and ℓ_i (or $\ell_i^\star$) corresponds to the generator ℓ (resp. $\ell^\star$) above (see Case 2).

Let A (or A') be the Seifert form for F (resp. F') defined on $H_n(F)/\operatorname{Tors} H_n(F)$ (resp. $H_n(F')/\operatorname{Tors} H_n(F')$). Furthermore, let S (or S') be the intersection form of F (resp. F'). Note that $\operatorname{Ker} S^* \cong H_n(K)$ corresponds to $\operatorname{Ker}(S')^* \cong H_n(K')$ under the isomorphism (13.1).

Set $B = (-A) \oplus A'$ and $S_B = (-S) \oplus S'$, which are bilinear forms defined on $G \oplus G'$. Note that G can be identified with a submodule of G' under the isomorphism (13.1). Let M be the submodule of $G \oplus G'$ generated by the elements of the form (x, x) with $x \in G$ and by ℓ_i, $i \in \mathcal{I}$.

As in [5], we see easily that M is a metabolizer for B. Furthermore, $\overline{M}$ is pure in $\overline{G \oplus G'}$ and we can easily check that

$$M \cap \operatorname{Ker} S_B^* = \{(x, x) \in G \oplus G' \mid x \in \operatorname{Ker} S^*\}.$$

Let y be an arbitrary nonzero element of $\operatorname{Tors}(\operatorname{Coker} S^*)$. We denote the order of y by q. Let

$$\partial'_* : G^* = H_n(F,K)/\operatorname{Tors} H_n(F,K) \to H_{n-1}(K)$$

be the homomorphism induced by the boundary homomorphism, which is well defined and surjective, since F is an exact Seifert surface. Furthermore, the map

$$H_n(F)/\operatorname{Tors} H_n(F) \to H_n(F,K)/\operatorname{Tors} H_n(F,K)$$

induced by the inclusion is identified with S^* by virtue of the Poincaré duality, and its image coincides with $\operatorname{Ker} \partial'_*$. (We also have similar statements for $(S')^*$ as well.)

Thus, there exists a $\widetilde{y} \in G^*$ such that $\partial'_* \widetilde{y} = y$ under the identification $\operatorname{Coker} S^* = H_{n-1}(K)$. Then, $q(\widetilde{y}, \widetilde{y}) \in G^* \oplus (G')^*$ lies in $S_B^*(M)$, which implies that $(\widetilde{y}, \widetilde{y}) \in G^* \oplus (G')^*$ lies in $S_B^*(M)^\wedge$. Therefore, we have

$$d(S_B^*(M)^\wedge) \supset \{(y, y) \mid y \in \operatorname{Tors}(\operatorname{Coker} S^*)\} \tag{13.2}$$

under the natural identification

$$\operatorname{Coker} S^* = H_{n-1}(K) = H_{n-1}(K') = \operatorname{Coker}(S')^*.$$

Lemma 13.6. *The order of* $d(S_B^*(M)^\wedge)$ *coincides with that of* $\operatorname{Tors}(\operatorname{Coker} S^*)$.

Proof. Since $S_B^*(M)$ is of finite index in $S_B^*(M)^\wedge$, we can write

$$S_B^*(M)^\wedge / S_B^*(M) \cong \oplus_{i=1}^k \mathbf{Z}_{a_i},$$

where a_i are positive integers such that a_i divides a_{i+1} for all $i = 1, 2, \ldots, k-1$, and $k = \operatorname{rank} S_B^*(M)^\wedge$. (Here, we do not exclude the case where $a_1 = \cdots = a_r = 1$ for some r with $1 \leq r \leq k$.)

Since $\overline{M}$ is pure in $\overline{G \oplus G'}$, we have $S_B^*(G \oplus G') \cap S_B^*(M)^\wedge = S_B^*(M)$ by [8, §2]. Therefore, the quotient map $d : G^* \oplus (G')^* \to \operatorname{Coker} S_B^*$ restricted to $S_B^*(M)^\wedge$ can be identified with the quotient map $S_B^*(M)^\wedge \to S_B^*(M)^\wedge / S_B^*(M)$.

Let us consider

$$\overline{S_B} : \overline{G \oplus G'} \times \overline{G \oplus G'} \to \mathbf{Z},$$

the ε-symmetric non-degenerate bilinear form induced from S_B on the module $\overline{G \oplus G'} = (G \oplus G') / \operatorname{Ker} S_B^*$. Since $\overline{M}$ is pure in $\overline{G \oplus G'}$, we have a submodule N of $\overline{G \oplus G'}$ such that $\overline{G \oplus G'} = \overline{M} \oplus N$. Note that $S_B^*(M)^\wedge / S_B^*(M)$ is naturally isomorphic to $\overline{S_B}^*(\overline{M})^\wedge / \overline{S_B}^*(\overline{M})$. Therefore, by taking appropriate bases of $\overline{M}$ and N, we may assume that a matrix representative of $\overline{S_B}$ is of the form

$$\begin{pmatrix} 0 & D \\ \varepsilon\, {}^tD & * \end{pmatrix},$$

where D is the $k \times k$ diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_k$. In particular, the order of

$$\operatorname{Tors}(\operatorname{Coker} S_B^*) = \operatorname{Coker} \overline{S_B}^* = \overline{G \oplus G'}^* / \overline{S_B}^*(\overline{G \oplus G'})$$

is equal to $(a_1 a_2 \cdots a_k)^2$.

Note that

$$\overline{S_B}^*(\overline{M})^\wedge / \overline{S_B}^*(\overline{M}) \cong S_B^*(M)^\wedge / S_B^*(M) \cong \oplus_{i=1}^k \mathbf{Z}_{a_i}.$$

Therefore, the order of

$$\operatorname{Coker} \overline{S_B}^* = \operatorname{Coker} \overline{S}^* \oplus \operatorname{Coker} \overline{S'}^*$$

coincides with the square of the order of $\overline{S_B}^*(\overline{M})^\wedge / \overline{S_B}^*(\overline{M})$. Therefore, we have the lemma. □

Combining the above lemma with (13.2), we have

$$d(S_B^*(M)^\wedge) = \{(y, y) \mid y \in \operatorname{Tors}(\operatorname{Coker} S^*)\}.$$

Therefore, we conclude that A and A' are algebraically cobordant. This completes the proof of Proposition 13.4. □

Proposition 13.7. *Let K be an exact $(2n-1)$-knot, $n \geq 3$, and A its Seifert form associated with an exact Seifert surface. Then, there exists a simple $(2n-1)$-knot K' cobordant to K such that the Seifert form of K' associated with an $(n-1)$-connected Seifert surface coincides with A.*

Proof. By Proposition 13.4, there exists a simple $(2n-1)$-knot K'' cobordant to K such that the Seifert form A'' of K'' associated with an $(n-1)$-connected Seifert surface is algebraically cobordant to A. On the other hand, it is known that there exists a simple $(2n-1)$-knot K' whose Seifert form associated with an $(n-1)$-connected Seifert surface coincides with A. Since A and A'' are algebraically cobordant, we see that K' and K'' are cobordant by [8]. Then, K and K' are cobordant, and the desired result follows. □

Remark 13.8. We do not know if the above proposition holds also for $n = 2$ or not.

Theorem 13.9. *Let K and K' be exact $(2n-1)$-knots, $n \geq 3$. If their Seifert forms with respect to exact Seifert surfaces are algebraically cobordant, then K and K' are cobordant.*

Proof. By Proposition 13.7, there exists a simple $(2n-1)$-knot $\widetilde{K}$ (or $\widetilde{K}'$) cobordant to K (resp. K') such that the Seifert form of $\widetilde{K}$ (resp. $\widetilde{K}'$) with respect to an $(n-1)$-connected Seifert surface coincides with the Seifert form of K (resp. K') with respect to an exact Seifert surface. By our assumption, the Seifert forms of $\widetilde{K}$ and $\widetilde{K}'$ are algebraically cobordant. Then, by [8], we see that $\widetilde{K}$ and $\widetilde{K}'$ are cobordant. Therefore, K and K' are cobordant. □

13.2 Cobordism of fibered knots

Theorem 13.10. *Let K and K' be two fibered $(2n-1)$-knots, $n \geq 3$. Then, K and K' are cobordant if and only if their Seifert forms with respect to their fibers are algebraically cobordant.*

Proof. By Lemma 13.3, a fiber of a fibered knot is always exact. Thus, by Theorem 13.9, if the Seifert forms with respect to the fibers are algebraically cobordant, then K and K' are cobordant.

Conversely, suppose that K and K' are cobordant. Let A (or A') be the Seifert form of K (resp. K') with respect to a fiber. By Proposition 13.7 and Lemma 13.3, there exists a simple $(2n-1)$-knot $\widetilde{K}$ (or $\widetilde{K}'$) cobordant to K (resp. K') such that the Seifert form with respect to an $(n-1)$-connected Seifert surface coincides with A (resp. A'). Since A and A' are unimodular, we see that $\widetilde{K}$ and $\widetilde{K'}$ are fibered (for example, see [36, 65]). Since K and K' are cobordant, we see that $\widetilde{K}$ and $\widetilde{K}'$ are also cobordant. Then, by [8], we see that A and A' are algebraically cobordant. This completes the proof. □

13.3 Extension to a larger class of 3-knots

As the arguments of §8.1 show, the sufficiency of Theorem 8.13 holds for simple free 3-knots: i.e., the 3-knots in question may not be fibered. However, for the proof of necessity of the Spin cobordism we have used the hypothesis that the 3-knots are fibered. In this section, we shall try to extend the class of simple fibered free 3-knots in such a way that the necessity continues to hold for a larger class of 3-knots.

First we give a definition which is valid for any dimension.

Definition 13.11. We say that a simple $(2n-1)$-knot K is *C-algebraically fibered*, if the Seifert form of K with respect to an $(n-1)$-connected Seifert manifold is algebraically cobordant to a unimodular form, where the zero form is also considered to be unimodular. In the following, for a C-algebraically fibered $(2n-1)$-knot, we use the Seifert form defined on an $(n-1)$-connected Seifert manifold which is algebraically cobordant to a unimodular form, unless otherwise specified. Note that simple fibered knots are always C-algebraically fibered.

Remark 13.12. A simple $(2n-1)$-knot is said to be *algebraically fibered*, if the Seifert form with respect to an $(n-1)$-connected Seifert manifold is S-equivalent to a unimodular matrix (see [68], [132, §4]). Then we see easily that for a simple $(2n-1)$-knot, we have

$$\text{simple fibered} \Longrightarrow \text{algebraically fibered} \Longrightarrow C\text{-algebraically fibered}.$$

Note that the reverse implications do not hold in general. See [68] and Example 10.9.

Now, let us consider the case of 3-knots. Recall that a *stabilizer* K_S is a simple fibered spherical 3-knot whose fiber is diffeomorphic to $(S^2 \times S^2)\sharp(S^2 \times S^2) \setminus \operatorname{Int} D^4$. Such a stabilizer does exist. For details, see [132, §4]. Furthermore, there also exists a stabilizer which is null cobordant (see [131, p. 600] or [135, §10]). In the following, K_S will denote such a null cobordant stabilizer.

Proposition 13.13. *Let K be a simple free 3-knot. If K is C-algebraically fibered, then there exists a simple fibered free 3-knot K' such that*

(1) *K and K' are cobordant,*

(2) *the Seifert form of K with respect to a 1-connected Seifert manifold and that of K' with respect to a 1-connected fiber are spin cobordant.*

Compare the above proposition with [132, Proposition 4.4].

Proof of Proposition 13.13. Let F be a 1-connected Seifert manifold of K and A the Seifert form for F. Note that A is algebraically cobordant to a unimodular form L by our assumption. Let $\psi : H_2(K) = \operatorname{Ker}(A + {}^tA) \to \operatorname{Ker}(L + {}^tL)$ be the isomorphism with respect to which A and L are algebraically cobordant.

Let us first show that there exists a compact 1-connected oriented spin 4-manifold F' with boundary diffeomorphic to K such that the spin structures induced from F and F' on K coincide with each other and that the intersection form of F' is isomorphic to

$$(L + {}^tL) \oplus 2k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13.3}$$

for some $k \geq 0$. We can construct such a 4-manifold F' as follows.

We first construct a 4-dimensional special handlebody F_1 consisting of one 0-handle and some 2-handles attached to the 0-handle simultaneously such that ∂F_1 is diffeomorphic to K, that F_1 is spin, and that the spin structure induced from F_1 coincides with the given spin structure on K (for details, see [64]). Then by Rohlin's theorem together with Novikov additivity for signature, the difference of the signatures of F and F_1 must be divisible by 16. Hence, by using some copies of a spin 4-dimensional special handlebody with boundary S^3 and with signature ± 16 (see [64]), we may assume that F and F_1 have the same signature. Note that the signature of F is equal to that of $L + {}^tL$. Then by the classification of symmetric unimodular forms, we see that the intersection form of $F' = F_1 \sharp k'(S^2 \times S^2)$ is isomorphic to the form (13.3) for some $k, k' \geq 0$. Here, we need the assumption that $H_1(K)$ is free.

Note that the above isomorphism between the intersection form of F' and the form (13.3) induces an isomorphism $H_2(K) = \operatorname{Ker}(A + {}^tA) \to \operatorname{Ker}(L + {}^tL)$.

Changing the isomorphism between the intersection form of F' and (13.3) if necessary, we may assume that the induced isomorphism coincides with ψ.

Recall that F' has a handlebody decomposition consisting of one 0-handle and some 2-handles. Thus, by using Kervaire's argument [72, pp. 255–257], we can embed F' into S^5 so that its Seifert form is given by $L \oplus kL_S$, where L_S is the Seifert form of a null cobordant stabilizer K_S with respect to the 1-connected fiber. Set $K' = \partial F'$.

Since L is unimodular, by using the stabilization technique developed in [132, §4], we may assume that K' is a simple fibered 3-knot, increasing k if necessary.

Note that the Seifert form $L \oplus kL_S$ for K' is algebraically cobordant to L, which is algebraically cobordant to the Seifert form A for K by our assumption. Furthermore, by the above construction, we see easily that $L \oplus kL_S$ and A are spin cobordant. Thus we have proved the item (2) in the proposition. The item (1) then follows from Theorem 8.4. This completes the proof of Proposition 13.13. □

Corollary 13.14. *If two C-algebraically fibered simple free 3-knots are cobordant, then their Seifert forms with respect to 1-connected Seifert manifolds are spin cobordant.*

Proof. Let K_0 and K_1 be the simple free 3-knots as above. Then by Proposition 13.13, K_0 and K_1 are cobordant to simple fibered free 3-knots K'_0 and K'_1 with spin cobordant Seifert forms respectively. Then, since K'_0 and K'_1 are cobordant, they have spin cobordant Seifert forms by Proposition 8.15. Thus K_0 and K_1 have spin cobordant Seifert forms, since spin cobordism is an equivalence relation. This completes the proof. □

Combining the above corollary with Theorem 8.4, we get the following.

Theorem 13.15. *Two C-algebraically fibered simple free 3-knots are cobordant if and only if their Seifert forms with respect to 1-connected Seifert manifolds are spin cobordant.*

Note that there are a lot of C-algebraically fibered simple free 3-knots which are not fibered (see Example 10.9).

We can prove a similar theorem for higher dimensions as well as follows.

Theorem 13.16. *For $n \geq 3$, two C-algebraically fibered simple $(2n-1)$-knots are cobordant if and only if their Seifert forms with respect to $(n-1)$-connected Seifert manifolds are algebraically cobordant.*

Proof. Replacing Proposition 8.15 in the argument for the 3-dimensional case with [8, Theorem 2'], we see that we have only to show the following: if a simple $(2n-1)$-knot K with $n \geq 3$ is C-algebraically fibered, then K is cobordant to a simple fibered $(2n-1)$-knot K' such that the Seifert form A of an $(n-1)$-connected Seifert manifold for K is algebraically cobordant to the Seifert form of an $(n-1)$-connected fiber of K'.

Since K is C-algebraically fibered, A is algebraically cobordant to a unimodular form L. By Durfee [36], such a form is realized as the Seifert form of a simple fibered $(2n-1)$-knot K'. Then by [8, Theorem 3], K is cobordant to K'. This completes the proof. □

13.4 Special cases

So far, we had to consider spin cobordism of Seifert forms instead of the usual algebraic cobordism for 3-knots. In this section, we shall show that in some special cases, the algebraic cobordism is sufficient.

Let us begin by the following definition.

Definition 13.17. Let $\mathcal{M}$ be a closed connected oriented 3-manifold. A 3-knot K is called an $\mathcal{M}$-knot, if K is abstractly diffeomorphic to $\mathcal{M}$, orientation preservingly.

For a closed connected oriented 3-manifold $\mathcal{M}$, let us consider the following conditions.

(6.1) For any isomorphism $\psi : H_2(\mathcal{M}) \to H_2(\mathcal{M})$, there exists an orientation preserving diffeomorphism $h_1 : \mathcal{M} \to \mathcal{M}$ such that $h_{1*} = \psi$.

(6.2) For any two spin structures of $\mathcal{M}$, there exists an orientation preserving diffeomorphism $h_2 : \mathcal{M} \to \mathcal{M}$ which sends one spin structure to the other such that $h_{2*} : H_2(\mathcal{M}) \to H_2(\mathcal{M})$ is the identity.

Then we have the following.

Proposition 13.18. *Let $\mathcal{M}$ be a closed connected oriented 3-manifold with torsion free first homology group. Suppose that the above conditions (6.1) and (6.2) are satisfied for $\mathcal{M}$. Then two C-algebraically fibered simple $\mathcal{M}$-knots are cobordant if and only if their Seifert forms with respect to 1-connected Seifert manifolds are algebraically cobordant.*

Proof. The necessity follows from Corollary 13.14.

Now, suppose that K_0 and K_1 are C-algebraically fibered simple $\mathcal{M}$-knots whose Seifert forms A_0 and A_1 with respect to 1-connected Seifert manifolds F_0 and F_1, respectively, are algebraically cobordant. We suppose that A_0 and A_1 are algebraically cobordant with respect to the isomorphism $\psi : H_2(K_0) \to H_2(K_1)$. By the conditions (6.1) and (6.2), we see that there exists an orientation preserving diffeomorphism $h : K_0 \to K_1$ such that $h_* = \psi$ and h sends the spin structure of K_0 to that of K_1. Hence, A_0 and A_1 are spin cobordant with respect to h. Thus by Theorem 8.4, K_0 and K_1 are cobordant. This completes the proof. □

For example, if $\mathcal{M}$ is a $\mathbf{Z}$-homology 3-sphere, i.e., if $H_1(\mathcal{M}) = 0$, then $H_2(\mathcal{M}) = 0$ and $\mathcal{M}$ admits a unique spin structure. Thus the conditions (6.1) and (6.2) are automatically satisfied. As another example, consider $\mathcal{M} = \sharp^k(S^1 \times S^2)$, the connected sum of k copies of $S^1 \times S^2$ with $k \geq 1$. Then it is well known that the conditions (6.1) and (6.2) are satisfied also in this case. Thus we have the following.

Corollary 13.19. *Suppose $\mathcal{M}$ is a $\mathbf{Z}$-homology 3-sphere, or $\mathcal{M} = \sharp^k(S^1 \times S^2)$, $k \geq 1$. Then two C-algebraically fibered simple $\mathcal{M}$-knots are cobordant if and only if their Seifert forms with respect to 1-connected Seifert manifolds are algebraically cobordant.*

In fact, when $\mathcal{M}$ is a $\mathbf{Z}$-homology 3-sphere, a stronger result has been known. For details, see [133].

Chapter 14

Open problems

"On résout les problèmes qu'on se pose
et non les problèmes qui se posent."
Henri Poincaré

To conclude this book we list some open problems.

Problem 14.1. *In Definition 1.1, if we remove the connectivity condition on the embedded manifolds, is it possible to characterize isotopy and cobordism classes of such knots?*

Problem 14.2. *Construct efficient invariants of algebraic cobordism.*

Problem 14.3. *Is it true that two simple $(2n-1)$-knots, $n \geq 3$, are cobordant if and only if their Seifert forms associated with $(n-1)$-connected Seifert manifolds are weakly algebraically cobordant? In particular, is there a pair of two simple $(2n-1)$-knots, $n \geq 3$, which are cobordant, but whose Seifert forms are not (weakly) algebraically cobordant?*

Note that for C-algebraically fibered simple knots, the above equivalence is true (see Remark 6.9).

Problem 14.4. *Is the* Spin *cobordism of Seifert forms associated with non-free 3-knots a sufficient condition of cobordism?*

Problem 14.5. *Does Theorem 9.9 (a characterization of the pull back relation for simple fibered $(2n-1)$-knots) hold for $n = 2$?*

As noted in Remark 9.10, the above characterization does not hold for $n = 1$.

Problem 14.6. *Let us fix an oriented simple homotopy type (or an oriented diffeomorphism type) of manifolds, and consider the set of all embeddings of such manifolds into a sphere in codimension two. Then, does there exist a minimal element with respect to the pull back relation?*

As mentioned in §9.1, for spheres, the trivial knot is such a minimal element.

Problem 14.7. *Is C_n/F_n isomorphic to $\mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty \oplus \mathbf{Z}^\infty$ for odd n? Determine the group structure of F_n for odd n. Is F_n a direct summand of C_n?*

Problem 14.8. *Is the multiplicity of a complex holomorphic function germ at an isolated singular point a cobordism invariant of the associated algebraic knot?*

This is known to be true for the case of algebraic 1-knots. See also [168].

Problem 14.9. *Let us consider Brieskorn type polynomials of the form*

$$z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}.$$

If two algebraic knots associated with Brieskorn type polynomials are cobordant, then do their exponents coincide?

A related result is obtained in [133]. Note that the associated Seifert matrix has been explicitly determined (for example, see [138]). It is also known that two algebraic $(2n-1)$-knots associated with Brieskorn polynomials with the same Alexander polynomial have the same exponents [164].

Problem 14.10. *Two fibered n-knots in S^{n+2} are said to be* fibered cobordant *if there exists a cobordism $X \subset S^{n+2} \times [0,1]$ between them whose complement $S^{n+2} \setminus X$ fibers over the circle in a sense similar to Definition 1.14. Is there a pair of two fibered knots which are cobordant but are not fibered cobordant?*

Problem 14.11. *Does there exist a knot which is not exact?*

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