Cobordism and Concordance of Knots

Vincent Blanlœil

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to V., V. & A.

孔子说: "温习过去所学的知识,能有新体会,新发展,这样就可以当老师了."

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Chapter 1

Introduction

"... the theory of "Cobordisme" which has, within the few years of its existence, led to the most penetrating insights into the topology of differentiable manifolds." H. Hopf, International Congress of Mathematics, 1958.

1.1 History

In the early fifties Rohlin [113] and Thom [129] studied the *cobordism groups of* manifolds. At the 1958 International Congress of Mathematicians in Edinburgh, René Thom received a Fields Medal for his development of cobordism theory.

Then, Fox and Milnor [37, 38] were the first to study cobordism of knots, i.e., cobordism of embeddings of the circle S^1 into the 3-sphere S^3 . Knot cobordism is slightly different from the general cobordism, since its definition is more restrictive. After Fox and Milnor, Kervaire [65] and Levine [80] studied embeddings of the *n*-sphere S^n (or homotopy *n*-spheres) into the (n + 2)-sphere S^{n+2} , and gave classifications of such embeddings up to cobordism for $n \ge 2$. Moreover, Kervaire defined group structures on the set of cobordism classes of *n*-spheres embedded in S^{n+2} , and on the set of concordance classes of embeddings of S^n into S^{n+2} . The structures of these groups for $n \ge 2$ were clarified by Kervaire [65], Levine [80, 81] and Stoltzfus [127].

Note that embeddings of spheres were studied only in the codimension two case, since in the PL category Zeeman [146] proved that all such embeddings in codimension greater than or equal to three are unknotted, and Stallings [126] proved that it is also true in the topological category (here, one needs to assume the locally flatness condition), provided that the ambient sphere has dimension greater than or equal to five. In the smooth category Haefliger [46] proved that a cobordism of spherical knots in codimension greater than or equal to three implies isotopy.

Later people studied embeddings of manifolds, which are not necessary homeomorphic to spheres, into codimension two spheres. One motivation comes from the topology of complex hupersurfaces near isolated singular points. More precisely, Milnor [100] showed that, in a neighborhood of an isolated singular point, a complex hypersurface is homeomorphic to the cone over the *algebraic* knot associated with the singularity. Hence, the embedded topology of a complex hypersurface around an isolated singular point is given by the algebraic knot, which is a special case of a *fibered knot*. After Milnor's work, the class of fibered knots has been recognized as an important class of knots to study. Usually algebraic knots are not homeomorphic to spheres, and this motivated the study of embeddings of general manifolds (not necessarily homeomorphic to spheres) into spheres in codimension two. Moreover, in the beginning of the seventies, $L\hat{e}$ [76] proved that isotopy and cobordism are equivalent for 1dimensional algebraic knots. Lê proved this for the case of *connected* (or spherical) algebraic 1-knots, and the generalization to arbitrary algebraic 1-knots follows easily (for details, see \S ??).

1 Introduction

During Arcata's symposium of pure mathematics in 1974, Durfee [33] listed several unsolved problems about algebraic knots. After Le's previous result, the following question seems natural

Problem 5([33]): Are cobordant algebraic knots (with K homeomorphic to a sphere) isotopic?

About twenty years later, Du Bois and Michel [30] gave the first examples of algebraic spherical knots that are cobordant but are not isotopic. These examples motivated the classification of fibered knots up to cobordism.

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1.1.1 Contents

This book is organized as follows. In Chapter 1 we give several apropos definitions to the cobordism theory of knots. The Seifert form associted with a knot is also introduced. In §?? we review the classifications of (simple) spherical (2n-1)-knots with $n \ge 2$ up to isotopy and up to cobordism. In §?? we review the properties of algebraic 1-knots and present the classification theorem of algebraic 1-knots up to cobordism due to Lê [76]. In §?? we present the classifications of simple fibered (2n-1)-knots with $n \ge 3$ up to isotopy and up to cobordism. The classification up to cobordism is based on the notion of the algebraic cobordism. In order to clarify the definition of algebraic cobordism, we give several explicit examples. We also explain why this relation may not be an equivalence relation on the set bilinear forms defined on free Z-modules of finitge rank. The classification of 3-dimensional simple fibered knots up to cobordism is given in §??. In §?? we recall the Fox-Milnor type relation on the Alexander polynomials of cobordant knots. As an application, we show that the usual spherical knot cobordism group modulo the subgroup generated by the cobordism classes of fibered knots is infinitely generated for odd dimensions. In \S ?? we present several examples of knots with interesting properties in view of the cobordism theory of knots. In §?? we define the pull back relation for knots which naturally arises from the viewpoint of the codimension two surgery theory. We illustrate several results on pull back of fibered knot with examples. Some results for even dimensional knots are given in \S ?, where we explain recent results about embedded surfaces in S^4 and embedded 4-manifolds in S^6 . Finally in §??, we give several open problems related to the cobordism theory of non-spherical knots, where a "non-spherical manifold" refers to a general manifold which may not necessarily be a homotopy sphere.

With all the results collected in this paper, we have classifications of knots up to cobordism in any dimensions. Only the classical case of one dimensional knots, and the case of three dimensional knots remain not to have complete classifications.

This book is made of a serie of lectures for graduate students in Louis Pasteur university of Strasbourg during the academic year 2006-2007. The purpose of these lectures was to give the opportunity to students to learn topology of high dimensional manifolds while studying knot cobordism.

Many proofs and results in this book are comming from papers writen before on the subject, and published in different journals. I want to thank here all my

1.2 Definitions

co-authors.

1.1.2 Notations

We will work in the smooth category, but sometimes manifolds might have corners. When a manifold M has boundary we denote it by ∂M . Moreover, if M is an oriented manifold with boundary we use the outward first convention to orient its boundary ∂M . All the homology and cohomology theory used have integer coefficients. The symbol \cong denotes a diffeomorphism between manifolds or an isomorphism between algebraic objects. An embedding of a manifold Kin a manifold M is denoted by $K \hookrightarrow M$. The closure of X is denoted by \overline{X} , and its interior is denote by $\overset{\circ}{X}$ or by IntX. We denote by ${}^{t}A$ the transpose of a matrix A.

1.2 Definitions

In this section we introduce knot cobordism. We also present some dedtailled constructions in order to give to the reader a precise idea of the subject.

Since our aim is to study cobordism and concordance of codimension two embeddings of manifolds which are not necessarily homeomorphic to spheres, we define knots as follows.

Definition 1.1. Let K be a closed n-dimensional manifold embedded in the (n+2)-dimensional sphere S^{n+2} . We suppose that K is

(k-2)-connected if n = 2k - 1 and $k \ge 2$, or

(k-1)-connected if n = 2k and $k \ge 1$.

When K is orientable, we further assume that it is oriented. Then we call K or its (oriented) isotopy class an *n*-knot, or simply a knot.

An *n*-knot K is *spherical* if K is

- 1. diffeomorphic to the *n*-dimensional standard sphere S^n for $n \leq 4$, or
- 2. a homotopy *n*-sphere for $n \ge 5$.

Remark 1.2. We use the above definition of a spherical knot for $n \leq 4$ in order to avoid the difficulty related to the smooth Poincaré conjecture in dimensions three and four.

Remark 1.3. With our definition one dimensional knots may have several connected components. But spherical 1-knots are connected and diffeomorphic to S^1 , see Figure 1.1 and Figure 1.2.

We impose a connectivity condition in Definition 1.1, this is first motivated by the usual definition of *algebraic knot* (see Definition ??), and second because later we will need connectivity conditions to perform embedded surgeries.

In order to define, and compute, invariants of isotopy and cobordism classes of knots, we will need some algebraic data associated with knots like *Seifert forms* and *Alexander polynomials*. In the classical knot theory, i.e., the case of spherical 1-knots, it is usual to make combinatorial computations associated 1 Introduction



Figure 1.1. The trefoil knot is a spherical 1-knot



Figure 1.2. The Hopf link is not a spherical 1-knot

with crossing of planar representations. We will have another approach, in a sense may be more algebraic, since we will do computations using integral bilinear forms.

The first step is to define *Seifert manifolds* associated with knots.

1.2.1 Seifert manifolds associated with knots

Proposition 1.4. For every oriented n-knot K with $n \ge 1$, there exists a compact oriented (n+1)-dimensional submanifold V of S^{n+2} having K as boundary. Such a manifold V is called a Seifert manifold associated with K. When K is a one dimensional knot, the manifold V is usually called a Seifert surface.

Remark 1.5. Seifert manifolds are not unique. For a given Seifert manifold of dimension k, one can construct a new one by doing its connected sum with a compact closed k-manifold embedded in S^{k+1} .

Proof. The construction of Seifert surfaces associated with 1-knots is elementary.

Start by assigning an orientation to each component of the knot, and then choose a regular projection into the plane. Around each crossing do the following modification:



Then the regular projection of K has become a disjoint collection of oriented S^1 embedded in the plane. Each one bounds a disk, and by pushing the interor of these disks off the plane in the three sphere they can be made disjoint. The orientations of the S^1 induce orientations of disks. Hence we we can connect these oriented disks at each crossing with half twisted strips in order to form an embedded, 2-manifold in S^3 , whose boundary is K as depicted bellow:



This construction gives the desired surface, embedded in S^3 , which has the knot as boundary.

When K is not spherical it is moreover necessary to connect the components with oriented connected sums.

For general dimensions, the existence of a Seifert manifold associated with a n-knot K can be proved by using the obstruction theory as follows.

Let $p: \tau_K \to K$ be the normal bundle of $K \hookrightarrow S^{n+2}$, and let $p_0: \tau_K^0 \to K$ be the bundle p without the zero section, i.e., for all $x \in K$ the fibers satisfy $p_0^{-1}(x) = p^{-1}(x) \setminus \{0\}$. A global orientation for τ_K means that we choosed a prefered generator μ of $\mathrm{H}^2(\tau_K, \tau_K^0)$.

The zero section of the bundle τ_K is an embedding of K in τ_K , moreover K is a deformation retract of τ_K and $p^* : \mathrm{H}^2(K) \xrightarrow{\cong} \mathrm{H}^2(\tau_K)$ is an isomorphism.

Let us denote *i* the inclusion map of (τ_K, \emptyset) into $(\tau_K, \tau_K \setminus K)$, which induces the morphism $i^* : \mathrm{H}^2(\tau_K, \tau_K \setminus K) \to \mathrm{H}(\tau_K)$ in cohomology.

Recall that the Euler class $e(\tau_K) = p^{*^{-1}} \circ i^*(\mu)$ of the normal bundle is an obstruction to having a nonzero normal section.¹

Let $T_K \cong K \times D^2$ be an open tubular neighbourhood of K in S^{n+2} . The 2-disk bundle T_K is diffeomorphic to τ_K and we have the following commutative diagram

¹Since K is a *n*-knot then we have $e(\tau_K) \in H^2(K) = 0$ as soon as K is 2-connected. Then we already have that τ_K is trivial for $n \ge 5$.

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$$\begin{array}{cccc} \mathrm{H}^{2}(S^{n+2}, S^{n+2} \setminus K) & \xrightarrow{\epsilon^{*}} & \mathrm{H}^{2}(T_{K}, T_{K} \setminus K) & \xrightarrow{\varphi^{*}} & \mathrm{H}^{2}(\tau_{K}, \tau_{K}^{0}) \\ & & & \downarrow & & \downarrow^{i^{*}} \\ & & & \downarrow & & \downarrow^{i^{*}} \\ 0 = \mathrm{H}^{2}(S^{n+2}) & \xrightarrow{\nu^{*}} & \mathrm{H}^{2}(K) & \xrightarrow{p^{*}} & \mathrm{H}^{2}(\tau_{K}) \end{array}$$

Where $\mathrm{H}^{2}(S^{n+2}, S^{n+2} \setminus K) \stackrel{\epsilon^{*}}{\cong} \mathrm{H}^{2}(T_{K}, T_{K} \setminus K)$ is given by the excision, and the morphisms j^{*} and ν^{*} are induced by inclusions. Since $e(\tau_{K}) = p^{*^{-1}} \circ i^{*}(\mu)$, the commutativity of the diagram gives $e(\tau_{K}) = p^{*^{-1}} \circ i^{*}(\mu) = \nu^{*} \circ j^{*} \circ \epsilon^{*^{-1}} \circ \varphi^{*^{-1}}(\mu) = 0$. So the normal bundle of $K \hookrightarrow S^{n+2}$ is trivial.

Let $N_K \cong^{\tau} K \times D^2$, the closure of T_K in S^{n+2} , be a closed tubular neighborhood of K in S^{n+2} , and $\Phi : \partial N_K \xrightarrow{\cong} K \times S^1 \xrightarrow{pr_2} S^1$ the composite of the restriction of $\overline{\tau}$ to the boundary of N_K and the projection pr_2 to the second factor. Using the exact sequence

$$H^1(S^{n+2} \setminus T_K) \to H^1(\partial N_K) \to H^2(S^{n+2} \setminus T_K, \partial N_K)$$

associated with the pair $(S^{n+2} \setminus T_K, \partial N_K)$, we see that the obstruction to ex-tending Φ to $\tilde{\Phi} : S^{n+2} \setminus T_K \to S^1$ lies in the cohomology group

$$H^2(S^{n+2} \setminus T_K, \partial N_K) \cong H_n(S^{n+2} \setminus T_K).$$

By Alexander duality we have

$$H_n(S^{n+2} \setminus T_K) \cong H^1(K),$$

which vanishes if $n \ge 4$, since K is simply connected for $n \ge 4$. When $n \le 3$, we can show that by choosing the trivialization τ appropriately, the obstruction in question vanishes. Therefore, a desired extension Φ always exists. Now, for a regular value y of $\tilde{\Phi}$, the manifold $\tilde{\Phi}^{-1}(y)$ is a submanifold of S^{n+2} with boundary being identified with $K \times \{y\}$ in $K \times S^1$. The desired Seifert manifold associated with K is obtained by gluing a small collar $K \times [0,1]$ to $\tilde{\Phi}^{-1}(y)$. \Box

Let us now recall the classical definition of Seifert forms of odd dimensional oriented knots, which were first introduced in [122] and play an important role in the study of knots cobordism.

Definition 1.6. Suppose that V is a compact oriented 2n-dimensional submanifold of S^{2n+1} , and let G be the quotient of $H_n(V)$ by its **Z**-torsion. The Seifert form associated with V is the bilinear form $\mathfrak{A}: G \times G \to \mathbf{Z}$ defined as follows

$$\begin{array}{rcl} \mathfrak{A}:G\times G&\longrightarrow&\mathbf{Z}\\ (x,y)&\mapsto&\mathfrak{A}(x,y)=l_{S^{2n+1}}(\xi_+,\eta). \end{array}$$

where $l_{S^{2n+1}}(.,.)$ denotes the linking number of chains in S^{2n+1} , the two nchains ξ and η are representing the cycles x and y respectively, and ξ_{+} is the *n*-chain η pushed off V into the positive normal direction to V in S^{2n+1} .

Recall that the linking number of two *n*-chains ξ and η in S^{2n+1} is given by the algebraic intersection number in S^{2n+1} of a (n+1)-chain Θ , which bounds ξ in S^{2n+1} , and η ; or by the algebraic intersection number in D^{2n+2} of a (n+1)chain Θ , which bounds $\xi \ D^{2n+2}$, and a (n+1)-chain Ω , which bounds η in D^{2n+2} .

By definition a Seifert form associated with an oriented (2n - 1)-knot K is the Seifert form associated with V, where V is a Seifert manifold associated with K. A matrix representative of a Seifert form with respect to a basis of G is called a Seifert matrix.

Remark 1.7. One can as well define the Seifert form $\mathfrak{A}'(x, y)$ to be the linking number of ξ and η_+ instead of ξ_+ and η , where ξ_+ is the *n*-cycle ξ pushed off V into the positive normal direction to V in S^{2n+1} . There is no essential difference between the two forms \mathfrak{A} and \mathfrak{A}' . However some formulas may take different forms.

More precisely, for a given *n*-chain ξ in F we denote by ξ_{-} the *n*-chain ξ pushed off V into the negative normal direction to V in S^{2n+1} . Then we have

$$l_{S^{2n+1}}(\xi,\eta_+) = l_{S^{2n+1}}(\xi_-,\eta),$$

and recall

$$l_{S^{2n+1}}(\xi,\eta) = (-1)^{n+1} l_{S^{2n+1}}(\eta,\xi).$$

According to these formulas we get

$$\begin{aligned} \mathfrak{A}(x,y) &= l_{S^{2n+1}}(\xi_+,\eta) \\ \mathfrak{A}(x,y) &= (-1)^{n+1} l_{S^{2n+1}}(\eta,\xi_+) \\ \mathfrak{A}(x,y) &= (-1)^{n+1} \mathfrak{A}'(y,x) \end{aligned}$$

So if A a the Seifert matrix associated with \mathfrak{A} and A' is the Seifert matrix associated with \mathfrak{A}' we have $A' = (-1)^{n+1} A^T$

Let us illustrate the above definition in the case of the trefoil knot. First consider the Seifert manifold F associated with the trefoil knot as depicted in Fig. 1.3, where "+" indicates the positive normal direction. Note that rank $H_1(V) = 2$. We denote by ξ and η the 1-cycles which represent the generators of $H_1(F)$. Then, with the aid of Fig. 1.3, we see that the Seifert matrix for the trefoil knot is given by

$$A = \begin{pmatrix} -1 & 1\\ 0 & -1 \end{pmatrix}.$$

Definition 1.8. Let $n \ge 1$. We say that an (2n-1)-knot is *simple* if it admits an (n-1)-connected Seifert manifold.

Let K be a simple knot with an (n-1)-connected Seifert manifold F. The Universal coefficient Theorem states that the following short exact sequence is exact

$$0 \to \operatorname{Ext}(\operatorname{H}_{k-1}(F,K)) \to \operatorname{H}^{k}(F,K) \to \operatorname{Hom}(\operatorname{H}_{k}(F,K)) \to 0.$$

Since F is an (n-1)-connected Seifert manifold, then $\text{Ext}(\text{H}_{n-1}(F,K)) = 0$ and the group $\text{H}^n(F,K)$ is torsion free. But by Poincaré-Lefschetz duality we have $\text{H}^n(F,K) \cong \text{H}_n(F)$. Hence $\text{H}_n(F)$ is trosion free. 1 Introduction + ξ ξ η η F

Figure 1.3. Computing a Seifert matrix for the trefoil knot

In the following, when a (2n - 1)-knot is simple, we consider an (n - 1)connected Seifert manifold associated with this knot unless otherwise specified.

When $n \ge 2$, the long exact sequence associated with a simple (2n-1)-knot K and its (n-1)-connected Seifert manifold F, induces the following short exact sequence

$$0 \to H_n(K) \to H_n(F) \xrightarrow{S_*} H_n(F, K) \to H_{n-1}(K) \to 0$$
(1.1)

where the homomorphism S_* is induced by the inclusion. Let $\tilde{\mathfrak{P}} : H_n(F, K) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{Z}}(H_n(F), \mathbf{Z})$ be the composite of the Poincaré-Lefschetz duality isomorphism and the universal coefficient isomorphism.

If we denote by \mathfrak{S} the intersection pairing $\mathfrak{S} : \mathrm{H}_n(F) \times \mathrm{H}_n(F) \to \mathbf{Z}$, then for all $(a,b) \in \mathrm{H}_n(F) \times \mathrm{H}_n(F)$ we have $\mathfrak{S}(a,b) = (\tilde{\mathfrak{P}} \circ S_*(b))(a)$.

Proposition 1.9. Let K be a simple (2n - 1)-knot with an (n - 1)-connected Seifert manifold F. Let \mathfrak{A} be the Seifert form associated with F and \mathfrak{S} the intersection pairing. If we denote by A the Seifert matrix and by S the matrix representative of \mathfrak{S} , then $S = A + (-1)^n A^T$.

Proof. Let $0 < \varepsilon << 1$. For $t \in [-\varepsilon, \varepsilon]$ we define diffeomorphisms $i_t : F \to S^{2n+1}$ which is a translation, in the positive normal direction when t is positive and in the negative normal direction when t is negative. Remark that for a *n*-chain γ we have

$$\gamma_{+} = i_{\varepsilon}(\gamma). \tag{1.2}$$

Let x and y be two n-cycle in $H_n(F)$, set $x = [\eta]$ and $y = [\xi]$ for two n-chains $\eta \xi$.

As consequence of Equation 1.2 we get

$$l_{S^{2n+1}}(\xi, i_{\varepsilon}(\eta)) = l_{S^{2n+1}}(i_{-\varepsilon}(\xi), \eta).$$

Let $\Lambda = \bigcup_{t \in [-\varepsilon,\varepsilon]} i_t(\eta) \cong \eta \times [-\varepsilon,\varepsilon]$ the oriented (n+1)-chain in S^{2n+1} with $\partial \Lambda = (i_\varepsilon(\eta) - i_{-\varepsilon}(\eta))$ since we use the outward first convention for the orientation of the boundary of an oriented manifold. The intersection of ξ and η in F is equal to the intersection of ξ and Λ in S^{2n+1} , this implies the following equalities

$$\begin{split} \mathfrak{S}(x,y) &= l_{S^{2n+1}}(\partial\Lambda,\xi) \\ \mathfrak{S}(x,y) &= l_{S^{2n+1}}(i_{\varepsilon}(\eta),\xi) - l_{S^{2n+1}}(i_{-\varepsilon}(\eta),\xi) \\ \mathfrak{S}(x,y) &= \mathfrak{A}(x,y) - (-1)^{n+1}l(i_{\varepsilon}(\xi),\eta) \\ \mathfrak{S}(x,y) &= \mathfrak{A}(x,y) + (-1)^{n}\mathfrak{A}(y,x) \end{split}$$



Figure 1.4. A cobordism between K_0 and K_1



Figure 1.5. A cobordism which is not an isotopy

This implies the desired relation between matrices.

Remark 1.10. Intersection forms \mathfrak{S} are $(-1)^n$ -symmetrical, contrary to Seifert forms, which are not generally symmetrical. For example see the matrix of the trefoil knot we computed with the aid of Fig. 1.3.

Let us now focus on cobordism and concordance classes of knots.

Definition 1.11. Two *n*-knots K_0 and K_1 in S^{n+2} are said to be *cobordant* if there exists a properly embedded (n+1)-dimensional manifold X of $S^{n+2} \times [0, 1]$ such that

- 1. X is diffeomorphic to $K_0 \times [0, 1]$, and
- 2. $\partial X = (K_0 \times \{0\}) \cup (K_1 \times \{1\}).$

The manifold X is called a *cobordism* between K_0 and K_1 . When the knots are oriented, we say that K_0 and K_1 are *oriented cobordant* (or simply *cobordant*) if there exists an oriented cobordism X between them such that $\partial X = (-K_0 \times \{0\}) \cup (K_1 \times \{1\})$, where $-K_0$ is obtained from K_0 by reversing the orientation.

Recal that a manifold with boundary Y embedded in a manifold X with boundary is said to be *properly embedded* if $\partial Y = \partial X \cap Y$ and Y is transverse to ∂X .

It is clear that isotopic knots are always cobordant. However, the converse is not true in general (see Fig. 1.5). For explicit examples, see §??.

We also introduce the notion of *concordance* for embedding maps as follows.

Definition 1.12. Let K be a closed n-dimensional manifold. We say that two embeddings $f_i : K \to S^{n+2}$, i = 0, 1, are *concordant* if there exists a proper embedding $\Phi : K \times [0, 1] \to S^{n+2} \times [0, 1]$ such that $\Phi|_{K \times \{i\}} = f_i : K \times \{i\} \to S^{n+2} \times \{i\}, i = 0, 1$.

Where an embedding map $\varphi: Y \to X$ between manifolds with boundary is said to be *proper* if $\partial Y = \varphi^{-1}(\partial X)$ and Y is transverse to ∂X .

Remark 1.13. Concordant knots are cobordant, but the converse is not true in general. See Theorem **??** for the spherical case and Remark **??** for non spherical examples.

Cobordant knots are diffeomorphic. Hence, to have a cobordism between two given knots, we need to have topological informations about the knots. Since a simple fibered (2n - 1)-knot is the boundary of the closure of a fiber, which is an (n - 1)-connected Seifert manifold associated with the knot, by considering the above exact sequence (1.1) we can use the kernel and the cokernel of the homomorphism S^* to get topological data of the knot. Note that in the case of spherical knots, these considerations are not necessary since S_* and S^* are isomorphisms.

1.3 Fibered knots

Definition 1.14. We say that an oriented *n*-knot *K* is *fibered* if there exists a smooth fibration $\phi : S^{n+2} \setminus K \to S^1$ and a trivialization $\tau : N_K \to K \times D^2$ of a closed tubular neighborhood N_K of *K* in S^{n+2} such that $\phi|_{N_K \setminus K}$ coincides with $\pi \circ \tau|_{N_K \setminus K}$, where $\pi : K \times (D^2 \setminus \{0\}) \to S^1$ is the composition of the projection to the second factor and the obvious projection $D^2 \setminus \{0\} \to S^1$. Note that then the closure of each fiber of ϕ in S^{n+2} is a compact (n+1)-dimensional oriented manifold whose boundary coincides with *K*. We shall often call the closure of each fiber.

Furthermore, for $n \ge 1$ we say that a fibered (2n-1)-knot K is simple if each fiber of ϕ is (n-1)-connected.

Though the notion of fibered knot it is much more restrictive, this definition will give additional data, like *monodromy* and *variation map* see Chapter 4, which are very usefull.

When K is a fibered knot, the closure of a fiber is always a Seifert manifold associated with K.

In the following, for a fibered (2n-1)-knot, we use the Seifert form associated with a fiber unless otherwise specified.

Moreover the definition of fibered knots gives a topological framework for algebraic knots associated with isolated singularities.

1.4 Complex hypersurfaces isolated singularities

One motivation for Definition 1.1 is the study of the topology of isolated singularities of complex hypersurfaces.

Let $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic function germ with an isolated singularity at the origin. If $\varepsilon > 0$ is sufficiently small, then $K_f = f^{-1}(0) \cap S_{\varepsilon}^{2n+1}$

is a (2n-1)-dimensional manifold which is naturally oriented, where S_{ε}^{2n+1} is the sphere in \mathbf{C}^{n+1} of radius ε centered at the origin. Furthermore, its (oriented) isotopy class in $S_{\varepsilon}^{2n+1} = S^{2n+1}$ does not depend on the choice of ε (see [100]).

Definition 1.15. We call K_f the algebraic knot associated with f.

Since the pair $(D_{\varepsilon}^{2n+2}, f^{-1}(0) \cap D_{\varepsilon}^{2n+2})$ is homeomorphic to the cone over the pair $(S_{\varepsilon}^{2n+1}, K_f)$, the algebraic knot completely determines the local embedded topological type of $f^{-1}(0)$ near the origin, where D_{ε}^{2n+2} is the disk in \mathbf{C}^{n+1} of radius ε centered at the origin.

Milnor [100] considers only polynomial functions. However, it is known that a holomorphic function germ with an isolated critical point is topologically equivalent to a polynomial function germ.

In [100], Milnor proved that, when $n \geq 2$, algebraic knots associated with isolated singularities of holomorphic function germs $f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$ are (2n-1)-dimensional closed, oriented and (n-2)-connected submanifolds of the sphere S^{2n+1} . This means that algebraic knots are some knots in the sense of Definition 1.1. Moreover, the complement of an algebraic knot K_f in the sphere S^{2n+1} admits a fibration over the circle S^1 , and the closure of each fiber is a compact 2n-dimensional oriented (n-1)-connected submanifold of S^{2n+1} which has K_f as boundary. Note that an algebraic knot is always a simple fibered knot.

1.5 Alexander polynomial

The Alexander polynomial associated with a knot K was initially defined for spherical 1-knots, and was computed with a combinatorial presentation of 1-knots, i.e., crossings. But, with the aid of a Seifert form associated with a knot it is possible to define Alexander polynomials for knots of every dimension.

Let K a (2n-1)-knot, with $n \ge 1$. Set A be a Seifert form for K associated with a Seifert manifold F. The polynomial

$$\Delta_A(t) = \det(tA + (-1)^n {}^tA)$$

of $\mathbf{Z}[t, t^{-1}]$, is well defined up to units of $\mathbf{Z}[t, t^{-1}]$, and is called the *Alexander* polynomial of K.

Remark that the Alexander polynomial is defined up to units of $\mathbf{Z}[t, t^{-1}]$ since the Seifert manifold associated with the knot is not unique.

Chapter 2

h-cobordism Theorem and surgeries on manifolds

Macbeth ...— What is the night? Lady Macbeth Almost at odds with morning, which is which. MACBETH ACT III, SC IV

The goal of this Chapter is to prove the *h*-cobordism Theorem. In fact we will prove a slightly more general theorem, which is called *s*-cobordism Theorem. We choose to give the proof of the *s*-cobordism theorem because of the similarity of the proofs, though we need to consider Whitehead torsions to prove the *s*-cobordism Theorem. The first step is to introduce Morse theory and handlebody decomposition for manifolds. In conclusion of this Chapter we will describe modifications of manifolds called *surgeries*.

2.1 Morse functions and handle decompositions of manifolds

In this section we recall briefly some classical results on Morse theory, we refer to [97] and [89] for detailed proofs.

We will consider functions defined on manifolds. Let M^n be a *n*-dimensionnal manifold with $n \in \mathbf{N}^*$, recall that we only consider smooth manifolds. A function $f: M \to \mathbf{R}$ is smooth if there exists a local coordinate system (x_1, \ldots, x_n) around each point p of M in which f is C^{∞} . By opposition we define

Definition 2.1. A point $p_0 \in M$ is a *critical* point of the function $f: M \to \mathbf{R}$ if $\frac{\partial f}{\partial x_i}(p_0) = 0, \ i = 1, \dots, n$.

It is easy to check that this definition does not depend on the choice of a coordinate system.

Definition 2.2. We say that a critical point p_0 of f is *non-degenerate* if the determinant

$$H_f(p_0) = \det \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(p_0) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(p_0) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(p_0) & \dots & \frac{\partial^2 f}{\partial x_n^2}(p_0) \end{pmatrix}$$

is not zero, and it is degenerate if $H_f(p_0) = 0$. We call $H_f(p_0)$ the Hessian of f at the critical point p_0 .

Let (x_1, \ldots, x_n) and (y_1, \ldots, y_n) be two coordinate systems, and set

$$J(p_0) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1}(p_0) & \dots & \frac{\partial x_1}{\partial y_n}(p_0) \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1}(p_0) & \dots & \frac{\partial x_n}{\partial y_n}(p_0) \end{pmatrix}$$

which is usually called the *Jacobian matrix* of the coordinate transformation evaluated at p_0 .

If we denote by $H_f^x(p_0)$ the Hessian of f in the coordinate system $x = (x_1, \ldots, x_n)$, then by direct computation we get

$$H_f^y(p_0) = {}^t J(p_0) H_f^x(p_0) J(p_0).$$

Definition 2.3. A real number c is called a *critical value* of a $f: M \to \mathbf{R}$ if there exists a critical point $p_0 \in M$ such that $f(p_0) = c$.

Since the Jacobian of the coordinate transformation at a point p_0 has a non-zero determinant, then we have

$$\det H_f^y(p_0) = \det \left({}^t J(p_0) \right) \det \left(H_f^x(p_0) \right) \det \left(J(p_0) \right).$$

But the determinant of the Jacobian of any coordinate transformation at a point p_0 has a non-zero determinant. Hence det $H_f^y(p_0) \neq 0$ if and only if det $H_f^x(p_0) \neq 0$, and the property of a critical point of a function being non-degenerate or degenerate does not depend on the choice of a coordinate system at p_0 .

Definition 2.4. A function $f : M \to \mathbf{R}$ is called a *Morse function* if every critical point of f is non-degenerate.

Theorem 2.5 (Morse Lemma). Let p_0 be a non-degenerate critical point of $f: M \to \mathbf{R}$. Then there exists a local coordinate system (x_1, \ldots, x_n) at p_0 such that with respect to these coordinates f has the form

$$-x_1^2 - \dots - x_{\lambda}^2 + x_{\lambda+1}^2 + \dots + x_n^2 + f(p_0)$$

Sylvester's law implies that $0 \leq \lambda \leq n$ is well defined and do not depend on the choice of the coordinate system. Since λ depends only on the function fand the critical point p_0 , then we define

Definition 2.6. The integer λ is called the *index* of the non-degenerate critical point p_0 of the function f.

Proof of Morse Lemma. Without loss of generality one can assume that $f(p_0) = 0$, and let (x_1, \ldots, x_n) be a local coordinate system around the origin p_0 . Since $f(p_0) = 0$, then according to the fundamental Theorem of calculus one can find n smooth functions $h_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt$, $i = 1, \ldots, n$ such that

$$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i h_i(x_1, \dots, x_n).$$

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With this decomposition we get $\frac{\partial f}{\partial x_i}(0,\ldots,0) = h_i(0,\ldots,0)$ for $i = 1,\ldots,n$. Now, since the origin p_0 in the local coordinate system (x_1,\ldots,x_n) is a

Now, since the origin p_0 in the local coordinate system (x_1, \ldots, x_n) is a critical point for the function f, then we have $h_i(0, \ldots, 0) = 0$ for $i = 1, \ldots, n$. As made before for f, for each $h_i, i = 1, \ldots, n$ one can find n smooth functions $h_{i,j}, j = 1, \ldots, n$ such that

$$h_i(x_1,...,x_n) = \sum_{j=1}^n x_j h_{i,j}(x_1,...,x_n).$$

Putting these decompositions all together, we get

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^n x_i x_j h_{i,j}(x_1, \dots, x_n),$$

setting $H_{i,j}=\frac{h_{i,j}+h_{j,i}}{2}$ gives $H_{i,j}=H_{j,i}$ and the following quadratic representation of f

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^n x_i x_j H_{i,j}(x_1, \dots, x_n).$$
(2.1)

We will now reduce this representation to the wanted one using the Gauss algorithm on quadratic forms.

The computation of the second order partial derivative of 2.1 gives

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0,\ldots,0) = 2 H_{i,j}(0,\ldots,0).$$

Since p_0 is a non-degenerate critical point of the function f, then we have $\det H_f(p_0) = \det H_f^x(0,\ldots,0) = \det (H_{i,j}(0,\ldots,0))_{i,j} \neq 0$. Moreover, up to a change of local coordinates, we can assume that

$$\frac{\partial^2 f}{\partial x_1^2}(0,\dots,0) \neq 0,$$

hence since the functions $H_{i,j}$ are continuous this gives $H_{1,1} \neq 0$ (eventually on a smaller neighborhood of p_0 than the one of the local coordinate system).

Now for an appropriate choice of local coordinate (X_1, x_2, \ldots, x_n) the function f is of the form

$$f(X_1, x_2, \dots, x_n) = \pm X_1^2 + \varphi(x_2, \dots, x_n)$$
(2.2)

with $\varphi(x_2, \ldots, x_n)$ a quadratic form with n-1 variables x_2, \ldots, x_n . By induction on the number of variables one can reduce the function f to the desired form. \Box

Corollary 2.7. Let $f : M \to \mathbf{R}$ be a Morse function. Any non-degenerate critical point of f is isolated, and when M is a compact n-manifold f admits finitely many critical points.

Proof. According to Morse Lemma, in a small coordinate neighbourhood of a critical point p_0 , the function f is of the form $-x_1^2 - \ldots - x_{\lambda}^2 + x_{\lambda+1}^2 + \ldots + x_n^2 + f(p_0)$. So the origin, i.e., the point p_0 , is the only critical point in the

coordinate neighbourhood of p_0 . Recall that for a Morse function any critical point is non-degenerate.

Assume that the Morse function f admits infinitely many distinct critical points $(p_i)_{i \in \mathcal{I}}$ where \mathcal{I} is an infinite set. Since non-degenerated critical points are isolated there exists disjoint open sets $(U_i)_{i \in \mathcal{I}}$ such that $U_i \subset M$ contains only one critical point p_i . First construct $U \subset M$ an open set such that for all i in \mathcal{I} the point p_i is not in U, then the infinite cover

$$M \subset U \bigcup_{i \in \mathcal{I}} U_i$$

can't be reduced to a finite one. This is in contradiction with the hypothesis of compactness for M.

Finally the Morse function f admits only finitely many critical points. \Box

Now we will see that every function $f: M \to \mathbf{R}$ on a compact manifold can be approximate by a Morse function.

Definition 2.8. Let M be a compact manifold, and let $\varepsilon > 0$ be a real. A function $f: M \to \mathbf{R}$ is a C_{ε}^2 -approximation of a function $\varphi: M \to \mathbf{R}$ if there exists a compact covering $M \subset \bigcup_{i=1,\dots,m} Y_i$ and on each compact $Y_i \subset M$, $i = 1, \dots, m$ the following hold

- 1. $\forall y \in Y_i |f(y) g(y)| < \varepsilon$,
- 2. $\forall y \in Y_i | \frac{\partial f(y)}{\partial x_j} \frac{\partial g(y)}{\partial x_j} | < \varepsilon, \ j = 1, \dots, n,$

3.
$$\forall y \in Y_i | \frac{\partial^2 f(y)}{\partial x_j \partial x_k} - \frac{\partial^2 g(y)}{\partial x_j \partial x_k} | < \varepsilon, \ j, k = 1, \dots, n.$$

Theorem 2.9 (Existence of Morse functions). Let M be a compact manifold without boundary, and $f: M \to \mathbf{R}$ a smooth function. Then for each real $\varepsilon > 0$ there exists a Morse function ψ on M which is a C_{ε}^2 -approximation of f. Moreover one can assume that the critical values associated with distincts critical points of ψ are distincts.

We refer to [89] for a detailed proof of this Theorem.

Using Morse functions defined on a manifold M, we will explain now how to construct some particular tangent vector fields on M. These vector fields make easier to understand the behaviour of the manifold around the critical points of the Morse functions.

Before, recall, that for a given vector $v \in T_p M$ the directional derivative of a function $f: M \to \mathbf{R}$ can be defined as follows. Let $c(\tau) = (x_1(\tau), \ldots, x_n(\tau))$ be a curve in M such that c(0) = p and $\frac{\mathrm{d}c}{\mathrm{d}t}(0) = v$. Then the directional derivative of f in the direction v at p is the real function defined on M

$$v.f = \sum_{i=1}^{n} \frac{\mathrm{d}x_i}{\mathrm{d}t}(0) \frac{\partial f}{\partial x_i}.$$

When X is a tangent vector field on M, i.e., to each point p in M we associate a tangent vector X(p) in $T_p(M)$, we extend this definition. We compute the 2 h-cobordism Theorem and surgeries on manifolds



Figure 2.1. The gradient vector field of $x_1^2 - \ldots - x_{\lambda}^2 + x_{\lambda+1}^2 + \ldots + x_n^2$

directional derivative of f in the direction X(p) at p. Then we can differentiate f with respect to X as well. A tangent vector field is defined by

$$X(p) = \sum_{i=1}^{n} \xi_i(p) \left(\frac{\partial}{\partial x_i}\right)_p,$$

where $\xi_i(p)$ are smooth functions defined on a coordinate system at p for i = 1, ..., n. Then set

$$(X.f)(p) = \left(\sum_{i=1}^{n} \xi_i(p) \left(\frac{\partial}{\partial x_i}\right)_p f\right)(p)$$

Now let us consider the gradient vector field of a Morse function $f: M \to \mathbf{R}$ in a small neighborhood of a critical point for f. We saw that in an appropriate local coordinate system (x_1, \ldots, x_n) the function f has the form

$$-x_1^2 - \ldots - x_{\lambda}^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

Its gradient vector field is

$$\nabla_f = -2x_1 \frac{\partial}{\partial x_1} - \ldots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + \ldots + 2x_n \frac{\partial}{\partial x_n}$$

Remark that $\nabla_f f = \sum_{i=1}^n (\frac{\partial f}{\partial x_i})^2 \ge 0$, and $(\nabla_f f)(p) > 0$ when p is not a

critical point of the Morse function f. This inequality means that locally the gradient vector field of f follows a direction into which f is increasing.

This induces the following definition.

Definition 2.10. We say that a vector field X on M is a gradient like vector field for the Morse function $f: M \to \mathbf{R}$ if

- 1. (X.f)(p) > 0 for any non-critical point $p \in M$,
- 2. around any critical point of f there exists an appropriate coordinate system such that $X = \nabla_f$.

Theorem 2.11. Let $f : M \to \mathbf{R}$ be a Morse function on a compact manifold. Then there exists a gradient like vector field on M.

A way to prove this Theorem is to glue all together gradient vector fields of f defined on a finite number of coordinate neighbourhoods. We refer to [89] for a detailed proof.

We illustrate the utility of gradient like vector fields with the two following Propositions.

Proposition 2.12. Let $f : M \to \mathbf{R}$ be a Morse function. If the function f has no critical value in a real interval $[\alpha, \beta]$, then the manifold $M_{[\alpha,\beta]} = \{p \in M | \alpha \leq f(p) \leq \beta\}$ is diffeomorphic to the product $f^{-1}(\alpha) \times [\alpha, \beta]$, and M_{α} is diffeomorphic to M_{β} .

Proof. Let X be a gradient like vector field of f. Since f has no critical point on $M_{[\alpha,\beta]}$, then (X.f)(p) > 0 for all $p \in M_{[\alpha,\beta]}$. Set $Y = \frac{1}{X.f}X$ a vector field on $M_{[\alpha,\beta]}$, and let $\gamma_x(\tau)$ the integral curve of Y which start at $x \in f^{-1}(\alpha)$.

Since $\frac{d}{dt}f(\gamma_x(\tau)) = Y \cdot f = 1$, then the integral curve $\gamma_x(\tau)$ starts at $x \in M_\alpha$ when $\tau = 0$ and is reaching M_β when $\tau = \beta - \alpha$. We know that the integral curves $\gamma_x(\tau)$ depend smoothly on both x and τ and two distinct integral curves never meet, hence the map

$$\begin{array}{rccc} h: & M_{\alpha} \times [0, \beta - \alpha] & \to & M_{[\alpha, \beta]} \\ & (x, \tau) & \mapsto & h(x, \tau) = \gamma_x(\tau) \end{array}$$

is a diffeomorphism.



Proposition 2.13 (Existence of collar neighbourhood). Let M be a manifold with compact boundary ∂M . Then there exists a neighbourhood V of ∂M in M, which is diffeomorphic to $\partial M \times [0, 1)$.

Proof. First glue two copies of M along their boundary ∂M to get a smooth closed manifold $W = M \cup_{\partial} M$. Then if $f: W \to \mathbf{R}$ is a Morse function on W, up to change one can suppose that f has no critical value in a neighbourhood of 0 and $f(\partial M) = 0$. Then we have $M = W_{f\geq 0} = \{p \in W | 0 \leq f(p)\}$. Hence we may assume that there exists a Morse function $f: M \to \mathbf{R}^+$ on M such that $f^{-1}(0) = \partial M$ and 0 is not a critical value. As in the previous Proposition, one can construct a gradient like vector field, for which integral curves give the desired diffeomorphism.



2.1.1 Handle decompositions of manifolds

In this subsection we will use Morse functions to describe handle decompositions of compact manifolds.

Let $f: M \to \mathbf{R}$ be a Morse function on a compact *n*-manifold M with a critical point at $p_0 \in M$ of index λ , and set $M_{\leq \tau} = \{p \in M | f(p) \leq \tau\}$. We will describe the changes of $M_{\leq \tau}$ when $\tau \in]c - \varepsilon, c + \varepsilon[$ where $\varepsilon > 0$ is a real such that $c = f(p_0)$ is the only critical value of f in $|c - \varepsilon, c + \varepsilon[$.

As seen before, in a local coordinate system around p_0 , the function f is of the form $-x_1^2 - \ldots - x_{\lambda}^2 + x_{\lambda+1}^2 + \ldots + x_n^2$. In the following picture we illustrated the behaviour of f on M in a small coordinate neighbourhood of the critical point p_0 , we made a normal projection of a small neighbourhood of the critical point p_0 of the manifold M onto \mathbb{R}^n . The shaded area correspond to the set points of M for which the value of f is greater or equal to $\tau + \varepsilon$, the doted area correspond to the set of points of M for which the value of f is lower or equal to $\tau - \varepsilon$.



Definition 2.14. The product manifold $D^{\lambda} \times D^{n-\lambda}$ is called a λ -handle, and the λ -disk $D^{\lambda} \times \{0\} \subset D^{\lambda} \times D^{n-\lambda}$ is called the *core* of the handle.

In the following picture we glued a λ -handle $D^{\lambda} \times D^{n-\lambda}$, along $D^{\lambda-1} \times D^{n-\lambda}$, to the boundary of the set of points of M for which f takes value lower or equal to $\tau - \varepsilon$.



With the gradient like vector field depicted by the arrows on the picture, one can see that, after smoothing, the manifold $M_{\leq \tau-\varepsilon} \cup D^{\lambda} \times D^{n-\lambda}$ is diffeomorphic to $M_{\leq \tau+\varepsilon}$.

Remark 2.15. Let c_1, \ldots, c_k the distinct critical values of a Morse function $f: M \to \mathbf{R}$ defined on a compact manifold M without boundary. Let $\varepsilon > 0$ a real small enough, then the following hold

- 1. $M_{\leq c_1-\varepsilon} = \emptyset$,
- 2. $M_{\leq c_1+\varepsilon} = D^n$, is a 0-handle,
- 3. $M_{\leq c_k+\varepsilon} = M$.

Let X be a n-manifold with non-empty boundary, and let $\varphi : S^{\lambda-1} \times D^{n-\lambda} \to \partial M$ be an embedding. Using φ we can attach a λ -handle to X. Set $Y = X \cup_{\varphi} (D^{\lambda} \times D^{n-\lambda})$, which is the manifold obtained from X by gluing the λ -handle $D^{\lambda} \times D^{n-\lambda}$ to ∂X along $\varphi(S^{\lambda-1} \times D^{n-\lambda})$. After smoothing corners if necessary we can assume that Y is smooth.

Definition 2.16. We say that Y is obtained by attaching a λ -handle to X, and φ is called the *attaching map* of the λ -handle. We will use the notation $Y = X \cup (\varphi^{\lambda})$. The disk $D^{\lambda} \times \{0\}$ is called the *core* of the λ -handle, and the sphere $\{0\} \times S^{n-\lambda-1}$ is called the *transverse* sphere of the λ -handle.



Remark 2.17. Sometimes, the transverse sphere to a handle is called a *belt sphere*.

When we attach several handles to X, we use the same notation, e.g. $Y = X \cup (\varphi^{\lambda}) \cup (\psi^{\mu})$. But beware of the meaning of this description. When we write $Y = X \cup (\varphi^{\lambda}) \cup (\psi^{\mu})$, this means that the λ -handle is attached to ∂X and then the μ -handle is attached to $\partial (X \cup (\varphi^{\lambda}))$.

Definition 2.18. A manifold obtained from D^n by attaching handles of various indices is called a *handlebody*.

When the boundary of a compact manifold X is of the form $X_0 \coprod X_1$, then it is sometimes more convenient to give a handle decomposition in which we attach the first handles to a collar neighbourhood of the component $X_0 \subset \partial X$ of the boundary.

To do that, it is enough to start with a Morse function $f: X \to \mathbf{R}$ which maps X_0 to $f(X_0) = 0$, X_1 to $f(X_1) = 1$ and such that all the critical values $\lambda_1, \ldots, \lambda_k$ of $f \circ$ are in]0,1[. Then the first handle, corresponding to the first X

critical value λ_1 of f, must be attach to a collar neighbourhood of X_0 (see the following picture).



Then using this Morse function we have a handle decomposition for X as stated in the following Proposition

Proposition 2.19 (Handle decomposition of boundary manifolds). Let X be a compact manifold with boundary $\partial X = X_0 \coprod X_1$. Then X possesses a handle-body decomposition up to diffeomorphism

$$X = X_0 \times [0, 1] \bigcup_{i=1,\dots,m} (\varphi_i^{\lambda_i}).$$

Remark 2.20. When $\partial X = \emptyset$ the statement remains valid since in that case the first handle must be of index 0 and the last one must be of index *n*. The process strat with a collection of *n*-disks, the 0-handles, then handles of index greater or equal to one are glued on these disks.

The decomposition given in Theorem 2.19 is not unique. So we will try to find good decompositions for our purpose. First we have to describe modifications of handlebody decompositions which do not change the diffeomorphism type. The goal is to find decompositions with less handles, and as few as possible of distinct indexes of handles. Note that all the following lemmas are due to Smale [124], see [64] and [85] as well for proofs.

Lemma 2.21 (Isotopy lemma). Let X be a manifold of dimension n such that its boundary ∂X is $X_0 \coprod X_1$. Let $\varphi, \psi : S^{\lambda-1} \times D^{n-\lambda} \to X_1$ be two isotopic embbedings. Then there exists a diffeomorphism between $X \cup (\varphi)$ and $X \cup (\psi)$ which is the identity on X_0 .

Proof. The idea of the proof is to find an ambiant isotopy on X which is identity on X_0 . It induces a diffeomorphism h on X with $h \circ \varphi = \psi$, and then a diffeomorphism between $X \cup (\varphi)$ and $X \cup (\psi)$.

Remark 2.22. We sometimes call isotopy between attaching map of handles *sliding* of handles. This terminology comes from the fact that we can illustrate this isotopy by the moving of one handle to the other by the sliding of the gluing set.

In the following, for two handle decompositions

$$X = X_0 \times [0, 1] \bigcup_{i=1, \dots, m} (\varphi_i^{\lambda_i}),$$
$$X = X_0 \times [0, 1] \bigcup_{i=1, \dots, m} (\psi_i^{\lambda_i}),$$

of X, we will construct diffeormorphism of X which is the identity on $X_0 \times \{0\}$.

Definition 2.23. We say that the two handle decompositions

$$X = X_0 \times [0, 1] \bigcup_{i=1,...,m} (\varphi_i^{\lambda_i}),$$
$$X = X_0 \times [0, 1] \bigcup_{i=1,...,m} (\psi_i^{\lambda_i}),$$

of X, are diffeormorphic together *relatively* to X_0 when the diffeomorphism is the identity on $X_0 \times \{0\}$.

Lemma 2.24. Let X be a manifold of dimension n such that its boundary ∂X is $X_0 \coprod X_1$. If $\lambda \leq \mu$ are some positive integers, then $X_0 \times [0,1] \cup (\psi^{\mu}) \cup (\varphi^{\lambda})$ is diffeomorphic to $X_0 \times [0,1] \cup (\varphi^{\lambda}_{\star}) \cup (\psi^{\mu})$ relatively to X_0 for an appropriate attaching map φ_{\star} .

Proof. The inequality of dimensions $(\lambda - 1) + (n - \mu - 1) < n - 1$ holds, so up to an isotopy $\varphi(S^{\lambda-1} \times \{0\})$ does not meet the transverse sphere of the μ -handle. Hence one can find an embedding $\varphi_{\star} : S^{\lambda-1} \times D^{n-\lambda} \to \partial(X \cup (\psi^{\mu}))$ which does not meet the image of ψ in ∂X , namely $\psi(S^{\mu-1} \times D^{n-\mu})$. By Lemma 2.21 $X_0 \times [0,1] \cup (\psi^{\mu}) \cup (\varphi^{\lambda})$ is diffeomorphic to $X_0 \times [0,1] \cup (\varphi^{\lambda}) \cup (\psi^{\mu})$. \Box

Remark 2.25. Let $\lambda \leq \mu$, and let $X_0 \times [0,1] \cup (\varphi^{\lambda}) \cup (\psi^{\mu})$ the manifold obtained by attaching two handles. Note that the attaching map of the μ handle $\psi : S^{\mu-1} \times D^{n-\mu} \to \partial (X \cup (\varphi^{\lambda}))$ may not be isotopic to an embedding $\psi_* : S^{\mu-1} \times D^{n-\mu} \to \partial (X \setminus (\varphi(S^{\lambda-1} \times D^{n-\lambda}))))$. This means that the formula $X_0 \times [0,1] \cup (\psi^{\mu}) \cup (\varphi^{\lambda})$ may be meaningless (up to diffeomorphism as well) in this situation, since the attaching map ψ may not be defined (up to isotopy) on $X_0 \times \{1\}$. Hence the order in which handles appear is very important and changing this order must be done carefully.

Let us consider the manifold Y obtained from $X_0 \times [0,1]$ by adding two handles of consecutive index, say λ and $\lambda + 1$. If φ and ψ are the attaching maps one can write $Y = X_0 \times [0,1] \cup (\varphi^{\lambda}) \cup (\psi^{\lambda+1})$. Assume that $\psi(S^{\lambda} \times \{0\})$ meets the transverse sphere of the λ -handle, namely $\{0\} \times S^{n-\lambda-1}$, transversaly in exactly one point \varkappa . Let \mathcal{U} be a small neighbourhood of the transverse sphere $\{0\} \times S^{n-\lambda-1}$ in the λ -handle (φ^{λ}) . Then one can find an isotopy between $D^{n-\lambda} \times \mathcal{U}$ and the λ -handle. Then we have $\psi(S^{\lambda} \times \{0\}) \cap (\varphi^{\lambda}) = D^{\lambda} \times \{\varkappa\}$.

Then it is technical, but not difficult, to check that the *n*-manifold $D^{\lambda} \times D^{n-\lambda} \cup_{\psi} D^{\lambda+1} \times D^{n-\lambda-1}$, which is the gluing of the λ -handle and the $(\lambda + 1)$ -handle along $\psi(S^{\lambda} \times D^{n-\lambda}) \cap D^{\lambda} \times D^{n-\lambda}$, is homeomorphic to the contractible manifold D^{n} . This implies that X and Y are diffeomorphic. The following picture illustrate this cancellation phenomenon.



We proved

Lemma 2.26 (Cancellation Lemma). Let X_0 be a manifold without boundary, and let $Y = X_0 \times [0,1] \cup (\varphi^{\lambda}) \cup (\psi^{\lambda+1})$ such that $\psi(S^{\lambda} \times \{0\})$ meets the transverse sphere of the λ -handle transversaly in exactly one point. Then $X_0 \times [0,1]$ and Y are diffeomorphic.

Using this Lemma, if needed, one can change a handle decomposition and add two handles with consecutive indexes. First choose an embedded n-disk D in $X_0 \times \{0\}$. Then construct an embedding $\varphi: S^\lambda \times D^{n-\lambda} \to D$ and an embedding $\psi: S^{\lambda+1} \times D^{n-\lambda-1} \to \partial(X \cup (\varphi^{\lambda}))$ such that $\psi(S^{\lambda} \times \{0\})$ meets the transverse sphere of the λ -handle transversally in exactly one point. According to the Cancellation Lemma 2.26 the manifolds $X_0 \times [0,1]$ and $X_0 \times [0,1] \cup (\varphi^{\lambda}) \cup (\psi^{\lambda+1})$ are diffeomorphic relatively to X_0 .

Let us to describe how to remove a λ -handle. The first step is to construct a $(\lambda + 1)$ -handle with a transversality condition with the λ -handle which allows cancellation. Then construct a handle of indexe $\lambda + 2$ such that the two handles of indexes $\lambda + 1$ and $\lambda 2$ are cancelling together.

Now, up to technical assumptions we are ready to eliminate a λ -handle and replace it by a $(\lambda + 2)$ -handle as stated in the following Lemma.

First we have to fix some notations. Suppose that we have a handle decom-

position of a manifold $Y = X_0 \times [0,1] \bigcup_{i=1}^{p_1} (\varphi_i^1) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$, then we denote

• $Y^q = X_0 \times [0,1] \bigcup_{i=1}^{p_1} (\varphi_i^1) \dots \bigcup_{i=1}^{p_q} (\varphi_i^q)$, the manifold obtained from $X_0 \times [0,1]$ after the gluing of handles of index less or equal to q,

•
$$\hat{\partial}Y^q = \partial Y^q \setminus \prod_{i=1}^{p_{q+1}} \varphi_i^{q+1}(S^q \times \overset{\circ}{D}^{n-1-q})$$

Lemma 2.27. Let X_0 be a (n-1)-manifold without boundary and $1 \le \lambda \le n-3$. Fix a handle decomposition of $Y = X_0 \times [0,1] \bigcup_{i=1}^{p_{\lambda}} (\varphi_i^{\lambda}) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n),$

with no handle of index strictly less than λ .

Let $1 \leq k \leq p_{\lambda}$ be a fixed integer. Suppose that there exists an embedding $\psi^{\lambda+1}: S^{\lambda} \times D^{n-1-\lambda} \to \hat{\partial} Y^{\lambda}$ such that

- 1. $\psi^{\lambda+1}|S^{\lambda} \times \{0\}$ is isotopic in ∂Y^{λ} to an embedding $\xi^{\lambda+1}: S^{\lambda} \times \{0\} \to \partial Y^{\lambda}$ which meets the transverse sphere of the handle (φ_k^{λ}) and is disjoint from the transverse spheres of the handles (φ_i^{λ}) $i = 1, ..., p_{\lambda}$ $i \neq k$
- 2. $\psi^{\lambda+1}|_{S^{\lambda} \times \{0\}}$ is isotopic in $\partial Y^{\lambda+1}$ to an embedding of S^{λ} into a (n-1)-disk $D^{n-1} \subset \partial Y^{\lambda+1}$.

Then Y is diffeomorphic, relatively to X_0 , to a manifold of the shape

$$X_0 \times [0,1] \bigcup_{i = 1, \dots, p_{\lambda} \atop i \neq k} (\varphi_i^{\lambda}) \bigcup_{i=1}^{p_{\lambda+1}} (\overline{\varphi}_i^{\lambda+1}) \cup (\psi^{\lambda+2}) \bigcup_{i=1}^{p_{\lambda+2}} (\overline{\varphi}_i^{\lambda+2}) \dots \bigcup_{i=1}^{p_n} (\overline{\varphi}_i^n)$$

Proof. All the technical asymptions made in this Lemma allow to add first a new $(\lambda + 1)$ -handle $(\psi^{\lambda+1})$ which cancel with the handle (φ_k^{λ}) , second to glue a new $(\lambda + 2)$ -handle $(\psi^{\lambda+2})$ which cancel with $(\psi^{\lambda+1})$. With the second assumption made in the statement, the gluing of the two handles $(\psi^{\lambda+1})$ and $(\psi^{\lambda+2})$ can be made in a (n-1)-disk embedded in $\partial Y^{\lambda+2}$.

Then according to the Isotopy and Cancellation Lemmas (2.21 and 2.26) we can find the appropriate embeddings $\{\overline{\varphi}i_{i_k}^k | k = \lambda + 1, \ldots, n; i_k = 1, \ldots, p_k\}$ to give the desired handle decomposition of a manifold which is diffeomorphic to Y relatively to X_0 .

This Lemma will be very useful to prove the h-cobordism Theorem. But first we have to introduce a CW-complex associated with handle decompositions of manifolds. This CW-complex will allow us to compute the Whitehead torsion that appears in the *s*-cobordism Theorem.

2.1.2 CW-complex and handlebodies

In this subsection, we briefly recall some elementary properties of relative CWcomplexes, and then we will construct a CW-complex which is associated with the handlebody decomposition of a manifold.

Let us denote by $X^{(0)}$ a set of discrete points. Let $n \ge 1$ be an integer. If the set $X^{(n-1)}$ has been defined, then consider $\{\psi_{\alpha}\}_{\alpha\in\mathcal{A}_n}$ a set of maps $f_{\alpha}: S^{n-1} \to X^{(n-1)}$. Set $X^{(n)} = X^{(n-1)} \cup \left(\bigcup_{\psi_{\alpha}} D^n_{\alpha}\right)_{\alpha\in\mathcal{A}_n}$ be the gluing of $X^{(n-1)}$ and some *n*-dimensional disks along their boundaries $\partial D^n_{\alpha} \cong S^{n-1}$ with the maps ψ_{α} .

This induces a filtration

$$X^{(0)} \subset X^{(1)} \subset \ldots \subset X^{(n)} \subset \ldots,$$

the path components of $X^{(n)} \setminus X^{(n-1)}$ are called *open n-cells*, the maps ψ_{α} are called *attaching maps*, and the maps $\Psi_{\alpha} : D_n \to X^{(n)}$ induced by ψ_{α} are called *characteristic maps*.

The set $X = \bigcup_{n \in \mathcal{N}} X^{(n)}$ is called a CW-complex. When \mathcal{N} is not finite, then a set is open in X if its intersection with each $X^{(n)}$ is open in $X^{(n)}$. The letter "C" stands for "closure finite" and the letter "W" stands for "weak topology". A set is *open* if its intersection with each $X^{(n)}$ is open in $X^{(n)}$.

Remark 2.28. An open *n*-cell is open in $X^{(n)}$, but usually is not an open set in X.

The image of a characteristic map is a compact subset of X, which is sometimes called a *closed cell*, but usually is not homeomorphic to D^n .

A relative CW-complex (X, A) consists of a pair of topological spaces $A \subset X$, such that X is obtained from A by gluing λ -cells, with $\lambda \geq 1$, as we did for CWcomplexes. The associated filtration is $A = X^{(\lambda-1)} \subset X^{(\lambda)} \subset \ldots \subset X^{(n)} \subset \ldots$

Let (X, A) be a relative CW-complex. Assume that X is arcwise connected¹ and set $\pi = \pi_1(X)$. Let $\rho : \tilde{X} \to X$ be the universal covering of X, and set

 $^{^1\}mathrm{This}$ assumption is only made in order to avoid considerations about base points and simplify the argument.

 $\tilde{X}^{(q)} = \rho^{-1}(X^{(q)})$ and $\tilde{A} = \rho^{-1}(A)$. Then (\tilde{X}, \tilde{A}) is a relative CW-complex with the filtration $\tilde{A} \subset \tilde{X}^{(1)} \subset \ldots \subset \tilde{X}^{(n)} \subset \ldots$

Recall that the homology of the relative CW-complex (\tilde{X}, \tilde{A}) can be computed using a $\mathbb{Z}[\pi]$ -chain complex $C_*(\tilde{X}, \tilde{A})$. The $q^{\text{th}} \mathbb{Z}[\pi]$ -chain module is the singular homology $H_q(\tilde{X}^{(q)}, \tilde{X}^{(q-1)})$ and the π -action is coming from the covering transformations, the q^{th} differential is then given by the composite map

$$\mathbf{H}_{q}(\tilde{X}^{(q)}, \tilde{X}^{(q-1)}) \xrightarrow{\partial_{q}} \mathbf{H}_{q-1}(\tilde{X}^{(q-1)}) \xrightarrow{i_{q}} \mathbf{H}_{q-1}(\tilde{X}^{(q-1)}, \tilde{X}^{(q-2)}),$$

where ∂_q is the q^{th} boundary map associate with the homology long exact sequence of the pair $(\tilde{X}^{(q)}, \tilde{X}^{(q-1)})$ and i_q is induced by the inclusion.

One can see that if we denote by β_i the image of a generator of $\operatorname{H}_q(D^q, S^{q-1}) \cong$ **Z** under the map $(\Psi_i^q, \psi_i^q)_q : \operatorname{H}_q(D^q, S^{q-1}) \to C_q(\tilde{X}, \tilde{A}) = \operatorname{H}_q(\tilde{X}^{(q)}, \tilde{X}^{(q-1)}),$ then the set $\{\beta_i\}_{i \in \mathcal{A}_q}$ is a $\mathbf{Z}[\pi]$ -basis for $C_q(\tilde{X}, \tilde{A})$. We call this basis the *cellular* basis.

Recal that the homology of a relative CW-complex is given by the homology of the $\mathbf{Z}[\pi]$ -chain complex we just defined, i.e.,

$$\mathcal{H}_*(X, A) \cong \mathcal{H}_*(C_*(X, A)).$$

Let M be a closed (n-1)-manifold. Now suppose we have a handle decomposition of a manifold

$$Y = M \times [0,1] \bigcup_{i=1}^{p_{\lambda}} (\varphi_i^{\lambda}) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n),$$

where the λ -handles are attached on $M \times \{0\}$. We denote by Y^q the manifold

$$Y^{q} = M \times [0,1] \bigcup_{i=1}^{p_{\lambda}} (\varphi_{i}^{\lambda}) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_{i}^{\lambda+1}) \dots \bigcup_{i=1}^{p_{q}} (\varphi_{i}^{q})$$

obtained from $M \times [0, 1]$ by adding handles of index less or equal to q.

Let us denote $M \times \{0\}$ by M_0 . Then we construct by induction over $q = \lambda, \ldots, n$ a sequence of spaces $X^{(q)}$ with a filtration $M_0 \subset X^{(\lambda)} \subset \ldots \subset X^{(n)} = \mathfrak{X}$ such that (\mathfrak{X}, M_0) is a relative CW-complex. We define the attaching maps of the relative CW-complex $(\mathfrak{X}, M \times \{0\})$ using the attaching maps of the handlebody decomposition of Y.

More precisely set $f_{\lambda-1}: Y^{\lambda-1} = M \times [0,1] \to X^{(\lambda-1)} = M_0$ the projection, which is a homotopy equivalence.

Assume that, for $q \geq \lambda$, the set $X^{(q-1)}$ is constructed and there exists a homotopy equivalence $f_{q-1}: Y^{q-1} \to X^{(q-1)}$. Then define the attaching maps $f_{q-1} \circ \varphi_i^q |_{S^{q-1} \times \{0\}}$ for $i = 1, \ldots, p_q$ to construct Y^q . Now consider the relative CW-complex (\mathcal{Y}^q, Y^q) , where \mathcal{Y}^q is constructed from Y^q by adding qcells with the attaching maps $\{\varphi_i^q |_{S^{q-1} \times \{0\}}\}_{i=1,\ldots,p_q}$. One can see that both $X^{(q)}$ and Y^q are homotopically equivalent to \mathcal{Y}^q , hence there exists a homotopy equivalence $f_q: Y^q \to X^{(q)}$ such that $f_q|_{Y^{q-1}} = f_{q-1}$.

Denote by $\rho : \tilde{Y} \to Y$ the universal covering of Y with $\pi = \pi_1(Y)$ as covering transformations group. Set $\tilde{Y}^q = \rho^{-1}(Y^q)$. As done before in the general context of relative CW-complexex, one can associate a $\mathbf{Z}[\pi]$ -chain complex $C_*(\tilde{Y}, \tilde{M}_0)$. The $q^{\text{th}} \mathbf{Z}[\pi]$ -chain module is the singular homology $H_q(\tilde{Y}^{(q)}, \tilde{Y}^{(q-1)})$ and the q^{th} differential is then given by the composite map

$$\mathbf{H}_{q}(\tilde{Y}^{(q)}, \tilde{Y}^{(q-1)}) \xrightarrow{\partial_{q}} \mathbf{H}_{q-1}(\tilde{Y}^{(q-1)}) \xrightarrow{i_{q}} \mathbf{H}_{q-1}(\tilde{Y}^{(q-1)}, \tilde{Y}^{(q-2)}),$$

where ∂_q is the q^{th} boundary map associated with the homology long exact sequence of the pair $(\tilde{Y}^{(q)}, \tilde{Y}^{(q-1)})$ and i_q is induced by the inclusion.

Since the maps $f_q: Y^q \to X^{(q)}$ constructed before are homotopy equivalcences, then we get an isomorphism of $\mathbb{Z}[\pi]$ -chain complexes

$$C_*(\tilde{Y}, \tilde{M}_0) \stackrel{\Theta}{\cong} C_*(\tilde{\mathfrak{X}}, \tilde{M}_0).$$

Moreover each handle of index q with attaching map φ_i^q for $i = 1, \ldots, p_q$ determines an element $[\varphi_i^q] \in C_q(\tilde{Y}, \tilde{M}_0)$. And the basis $\{[\varphi_i^q]\}_{i=1,\ldots,p_q}$ of $C_q(\tilde{Y}, \tilde{M}_0)$ maps to the cellular basis of $C_*(\tilde{\mathfrak{X}}, \tilde{M}_0)$ under Θ .

Now we are ready to prove the h-cobordism Theorem.

2.2 *h*-cobordism Theorem

First let us state the h-cobordism Theorem due to Smale.

Theorem 2.29 (h-cobordism [124]). Let M_1 and M_2 be two closed oriented and simply connected manifolds of dimension $n \ge 5$. If there exists an oriented compact manifold W with ∂W diffeomorphic to the disjoint union of M_1 and $-M_2$, and each component of ∂W is a deformation retract of W then W is diffeomorphic to $M_1 \times [0, 1]$.

The manifold $-M_2$ is the manifold M_2 with the reversed orientation.

Remark 2.30. As an important consequence we have that the two manifolds M_1 and M_2 are diffeomorphic to each other.

Remark that the inclusions $M_i \hookrightarrow W$, for i = 1, 2, are homotopy equivalences. And the letter "h" in "h-cobordism" is for homotopy equivalence.

The *h*-cobordism Theorem can be reformulated a follows.

Theorem 2.31 (h-cobordism). Let M_1 and M_2 be two closed oriented and simply connected manifolds of dimension $n \ge 5$. If there exists an oriented compact manifold W with ∂W diffeomorphic to the disjoint union of M_1 and $-M_2$, and $H_*(W, M_1) = 0$ then W is diffeomorphic to $M_1 \times [0, 1]$.

Remark 2.32. In the second statement of the *h*-cobordism Theorem it is equivalent to replace $H_*(W, M_1) = 0$ by $H_*(W, M_2) = 0$.

More precisely, when $H_*(W, M_1) = 0$ the universal coefficient Theorem implies $H^*(W, M_1) \cong Hom(H_*(W, M_1)) = 0$, and by Poincaré duality we get $H_*(W, M_2) = 0$. Similarly $H_*(W, M_2) = 0$ implies $H_*(W, M_1) = 0$

Proof of Theorem 2.31. First remark that if M_1 and M_2 are both deformation retracts of W then we have $H_*(W, M_1) = 0$, and $H_*(W, M_2) = 0$ as well.

Second when $\pi_1(M_1) = 0$, $\pi_1(W, M_1) = 0$ and $H_*(W, M_1) = 0$ then, according to the relative Hurewicz isomorphism Theorem (see [13]), we have

 $\pi_i(W, M_1) = 0$ for $i \in \mathbf{N}$. Then one can construct a deformation retraction from W to M_1 . As explained in Remark 2.31 the nullity of $\mathrm{H}_*(W, M_1)$ implies $\mathrm{H}_*(W, M_2) = 0$, and M_2 is, by the same argument, a deformation retract of W.

The *h*-cobordism Theorem is crucial for the study of cobordism classes of high dimensional knots. It concerns simply connected manifolds, but this connectivity condition is automatic for knots of dimension greater or equal to 2.

In the subsection 2.2.1 we will prove an extension to non-simply connected manifolds called *s*-cobordism theorem. Though we will not need this extension for the study of knot cobordism, we choose to give this proof since the core of the proof is the same of the proof of the *h*-cobordism Theorem and is essentially made of Smale's lemmas .

The *s*-cobordism Theorem was proved by Barden in [4], by Mazur in [90] and by Stallings (who never published his proof). For additional details we refer to Kervaire's paper [64] devoted to a detailed proof of this Theorem.

2.2.1 s-cobordism Theorem

Theorem 2.33 (s-cobordism Theorem). Let M_1 and M_2 be two closed oriented and connected manifolds of dimension $n \ge 5$, and let $\pi = \pi_1(M_1)$ the fundamental group of M_1 . If there exists an oriented compact manifold W with ∂W diffeomorphic to the disjoint union of M_1 and $-M_2$, and each component of ∂W is a deformation retract of W then W is diffeomorphic to $M_1 \times [0, 1]$ if and only if the Whitehead torsion $\tau(W, M_1) \in Wh(\pi)$ vanishes.

To make this statement understandable we have to define briefly Whitehead groups and Whitehead torsion, see [131] for details.

Whitehead groups. Let π be a group, and let $GL(n, \mathbb{Z}[\pi])$ the group of invertible matrices of order n on the group ring $\mathbb{Z}[\pi]$. We denote by $GL(\mathbb{Z}[\pi])$ the set of disjoint union $\bigcup_{n \in \mathbb{Z}} GL(n, \mathbb{Z}[\pi])$, it is the set of invertible matrices of

arbitrary size with entries in $\mathbf{Z}[\pi]$.

Let us denote by $E_{i,j}^n$ a $n \times n$ matrix with all entries 0 except for a 1 in the (i, j) spot; and by $\Delta_i^n(\gamma)$ a $n \times n$ diagonal matrix with entries on the diagonal equal to 1 except for γ in the (i, i) spot. If I_n denotes the identity matrix of rank n, then an *elementary matrix* is a matrix of the form $(I_n + aE_{i,j}^n)$, with $a \in \mathbf{Z}[\pi]$; and let $E(\mathbf{Z}[\pi])$ be the subgroup of $GL(\mathbf{Z}[\pi])$ generated by the elementary matrices.

It is not difficult to show that $E(\mathbf{Z}[\pi])$ is the commutator subgroup of $GL(\mathbf{Z}[\pi])$, and any subgroup of $GL(\mathbf{Z}[\pi])$ which contains $E(\mathbf{Z}[\pi])$ is a normal subgroup of $GL(\mathbf{Z}[\pi])$.

Let us consider the subgroup $\pm \pi$ of $\mathbf{Z}[\pi]$ of trivial units, namely

$$\{p|p \in \pi\} \cup \{-p|p \in \pi\} = \pm \pi < \mathbf{Z}[\pi].$$

Then we define $I_{\pm\pi}$ to be the set

$$I_{\pm\pi} = \left\{ M \in GL(\mathbf{Z}[\pi]) \mid M = \Delta_i^n(\gamma) \text{ with } \gamma \in \pm\pi, \text{ or } M \in E(\mathbf{Z}[\pi]) \right\}.$$

In $I_{\pm\pi}$ we collected the matrices of $E(\mathbf{Z}[\pi])$ and the matrices of the form

$$\begin{pmatrix} I_i & 0 & 0\\ 0 & \gamma & 0\\ 0 & 0 & I_j \end{pmatrix}$$

with $\gamma \in \pm \pi$.

Hence the group E_{π} , which is generated by the matrices of $I_{\pm\pi}$, is a normal subgroup of $GL(\mathbf{Z}[\pi])$.

Definition 2.34. The whitehead group $Wh(\pi)$ is the abelian quotient group $GL(\mathbf{Z}[\pi])_{/E_{\pi}}$.

In the following we will use another definition of $Wh(\pi)$, which is more complicated but more convenient for our purpose. On $GL(\mathbf{Z}[\pi])$ we define an equivalence relation, denoted by \mathcal{R} , generated by the elementary operations listed below.

Let A be a matrix in $GL(\mathbf{Z}[\pi])$,

- 1. multiply the *i*-th row of A from left by $\pm \gamma$ with $\gamma \in \pi$;
- 2. add the *i*-th row to *j*-th row of A;
- 3. change the matrix $A \in GL(n, \mathbf{Z}[\pi])$ to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;
- 4. change the matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1, \mathbf{Z}[\pi])$ to A (this is the inverse of the previous item).

Remark 2.35. We do not use column operations in our definition, i.e., right product with elementary matrices. Because if two matrices A and B are related together with column and row operations, then there exist two matrices E_1 and E_2 , which are product of elementary matrices, such that $I_n = E_1 \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix} B^{-1} E_2$. But this means that $E_2^{-1} = E_1 \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix} B^{-1}$, and then $I_n = E_2 E_1 \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix} B^{-1}$. This implies that A and B are related together only using row operations.

One can define a product on classes of matrices in $GL(\mathbf{Z}[\pi])_{\mathcal{R}}$. We denote by $[A] \in GL(\mathbf{Z}[\pi])_{\mathcal{R}}$ the class of a matrix $A \in GL(\mathbf{Z}[\pi])$. Let [A] and [B] be in $GL(\mathbf{Z}[\pi])_{\mathcal{R}}$, then there exist two integers *i* and *j* (may be equal to 0) such that the two matrices $A \oplus I_i$ and $B \oplus I_j$ are invertible matrices of same rank. We define $[A].[B] = [(A \oplus I_i).(B \oplus I_j)]$. The neutral element is given by $[I_n]$ for any positive integer *n*. The inverse of [A] is given by $[A^{-1}]$. One can prove that $(GL(\mathbf{Z}[\pi])_{\mathcal{R}}, .)$ is an abelian group, and Wh(π) is the quotient $GL(\mathbf{Z}[\pi])_{\mathcal{R}}$.

Proposition 2.36. These two definitions of Whitehead groups are equivalent together.

See [25] for this equivalence.

In the following we will denote by A both a matrix in $GL(\mathbf{Z}[\pi])$ and its class in Wh(π).

Whitehead torsion. We will define the Whitehead torsion of a pair (X, Y) when both X and Y are CW-complexes such that Y is a deformation retract of X. But Whitehead torsion may be defined algebraically for acyclic chain complexes over a ring R under some assumptions for R, we refer to [131] and [99] for detailed expositions on Whitehead torsion.

Since the inclusion $Y \hookrightarrow X$ is a homotopy equivalence, then it induces an isomorphism of fundamental groups $\pi_1(Y) \cong \pi_1(X) = \pi$, provided we choose a base point in Y. Let us consider again the universal covering $\tilde{\rho} : \tilde{X} \to X$, it induces the covering $\tilde{\rho}_{|\tilde{Y}} : \tilde{Y} \to Y$ and the subcomplex \tilde{Y} is a deformation

retract of \tilde{X} . Therefore the $\mathbb{Z}[\pi]$ -chain complex $C_*(\tilde{X}, \tilde{Y})$ of length n is acyclic. Recall that π acts on $C_*(\tilde{X}, \tilde{Y})$, and this makes it a free chain complex over $\mathbb{Z}[\pi]$; each $\mathbb{Z}[\pi]$ -module $C_q(\tilde{X}, \tilde{Y})$ equiped with the cellular basis $\mathcal{B}_q = \{\beta_i\}_{i \in \mathcal{A}_q}$ see § 2.1.2.

1. First assume that for all integer $0 \le q \le n$ the $\mathbb{Z}[\pi]$ -module $\operatorname{Im} d_q$ is free. Since the complex is acyclic, then we have the short exact sequences

$$0 \to \operatorname{Im} d_q \to C_q(\tilde{X}, \tilde{Y}) \stackrel{d_q}{\to} \operatorname{Im} d_{q-1} \to 0.$$

By exactness of the last short sequences we get sections s_q of d_q , then set $\mathcal{I}_{q-1}^* = s_q(\mathcal{I}_q)$ the image of the basis \mathcal{I}_{q-1} of $\operatorname{Im} d_{q-1}$ under s_q . Note that, since for any distinct integers i and j the two $\mathbf{Z}[\pi]$ -modules $\mathbf{Z}[\pi]^i$ and $\mathbf{Z}[\pi]^j$ are not isomorphic, then the juxtaposition of the two basis \mathcal{I}_q and \mathcal{I}_{q-1}^* is a basis of $C_q(\tilde{X}, \tilde{Y})$. Set $T_{\mathcal{I}_q \mathcal{I}_{q-1}^*} \to \mathcal{B}_q$ the transition matrix from $\mathcal{I}_q \mathcal{I}_{q-1}^*$ to \mathcal{B}_q .

The following product matrix

$$\tau = \prod_{i=0}^{n} T_{\mathcal{I}_q \mathcal{I}_{q-1}^{\star} \to \mathcal{B}_q}^{(-1)^{i+1}}$$

is invertible.

Moreover one can prove that its class in $Wh(\pi)$ does not depend on the choices of the basis and is invariant under cellular subdivisions. According to these facts when for all integer $0 \leq q \leq n$ the $\mathbf{Z}[\pi]$ -module $\operatorname{Im} d_q$ are free, then we define the *torsion* of the complex $C_*(\tilde{X}, \tilde{Y})$ to be the class of τ in $Wh(\pi)$.

2. When the $\mathbf{Z}[\pi]$ -module Im d_q are not free we have the following Lemma

Lemma 2.37. For all integers $0 \le q \le n$ there exists a free $\mathbb{Z}[\pi]$ -module F_q such that the $\mathbb{Z}[\pi]$ -module $\operatorname{Im} d_q \oplus F_q$ is free.

Proof. Note that $\operatorname{Im} d_0 = C_0(\tilde{X}, \tilde{Y})$ is free.

We will prove the property by induction on q. Assume there exists an integer $k \geq 0$ for which there exists a free $\mathbb{Z}[\pi]$ -module F_k such that the $\mathbb{Z}[\pi]$ -module $\operatorname{Im} d_k \oplus F_k$ is free.

Since the $\mathbf{Z}[\pi]$ -chain complex $C_*(\tilde{X}, \tilde{Y})$ of length n is acyclic, then we have the following short exact sequence

$$0 \to \operatorname{Im} d_{k+1} \to C_k(\tilde{X}, \tilde{Y}) \oplus F_k \xrightarrow{d_k \oplus Id} \operatorname{Im} d_k \oplus F_k \to 0.$$

2 h-cobordism Theorem and surgeries on manifolds

The last $\mathbf{Z}[\pi]$ -module is free, hence there exists a section σ_k for $d_k \oplus Id$. The $\mathbf{Z}[\pi]$ -module $\sigma_q(\operatorname{Im} d_k \oplus F_k)$ is free, and $\operatorname{Im} d_{k+1} \oplus \sigma_q(\operatorname{Im} d_k \oplus F_k) = C_k(\tilde{X}, \tilde{Y}) \oplus F_k$ as well.

Let us denote by $C^q_*(F)$ the free based acyclic $\mathbf{Z}[\pi]$ -chain complex associated with a free based $\mathbf{Z}[\pi]$ -module F, which has $d_q: F \to F$ as the only non-trivial differential

$$\ldots \to 0 \to F \xrightarrow{a_q} F \to 0 \to \ldots$$

Define a new $\mathbf{Z}[\pi]$ -chain complex $C_*(\tilde{X}, \tilde{Y}) \bigoplus_{k=0}^n C_*^k(F_k)$. Since in this free acyclic $\mathbf{Z}[\pi]$ -chain complex the image of the differential are some free $\mathbf{Z}[\pi]$ -modules, then we can compute its torsion as just made before. One can prove that the torsion of this complex does not depend on the choices made on the free $\mathbf{Z}[\pi]$ -modules F_q for $q = 0, \ldots, n$.

We define the torsion $\tau(X, Y)$ to be the torsion of the $\mathbf{Z}[\pi]$ -chain complex $C_*(\tilde{X}, \tilde{Y}) \bigoplus_{k=0}^n C^k_*(F_k)$.

Come back to the statement of the *s*-cobordism Theorem. Assume that W is an oriented compact manifold with boundary $\partial W \cong M_1 \coprod -M_2$, such that both M_1 and M_2 are deformation retracts of W. To a handlebody decomposition

$$W = M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \dots \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n),$$

one can associate first a $\mathbf{Z}[\pi]$ -chain complex $C_*(\tilde{W}, \tilde{M}_1)$ and second a relative CW-complex $(\tilde{\mathfrak{X}}, \tilde{M}_1)$ such that the $\mathbf{Z}[\pi]$ -chain complex $C_*(\tilde{\mathfrak{X}}, \tilde{M}_1)$ is isomorphic to $C_*(\tilde{W}, \tilde{M}_1)$.

Since M_1 is a deformation retract of W, in the relative CW complex $(\tilde{\mathfrak{X}}, \tilde{M}_1)$ we have that \tilde{M}_1 is a deformation retract of $\tilde{\mathfrak{X}}$ as well. Hence $\tau(\tilde{\mathfrak{X}}, \tilde{M}_1)$ is well defined, and the torsion $\tau(W, M_1)$ is by definition equal to the torsion $\tau(\tilde{\mathfrak{X}}, \tilde{M}_1)$.

Simple homotopy equivalences. When the map $f: E \to F$ is a homotopy equivalence between CW-complexes, then F is a deformation retract of the mapping cylinder

$$M_f = (X \times [0,1]) \prod Y_{/(x,1)} \sim f(x)$$

of f.

We define the Whitehead torsion of f, denoted by $\tau(f) \in Wh(\pi_1(Y))$, to be the image of the torsion $\tau(M_f, Y) \in Wh(\pi_1(M_f))$ in $Wh(\pi_1(Y))$ under the isomorphism between $Wh(\pi_1(M_f)) \xrightarrow{\cong} Wh(\pi_1(Y))$ induced by the isomorphism $\pi_1(M_f) \xrightarrow{\cong} \pi_1(Y)$.

This torsion is well defined, and when two cellular homotopy equivalences between two CW-complexes are homotopic the torsion are equal.

Definition 2.38. We say that a homotopy equivalence $f : X \to Y$ of finite CW-complexes is *simple* if the torsion $\tau(f)$ vanishes in Wh $(\pi_1(Y))$.

This definition extend to homotopy equivalences between smooth manifolds.

Remark 2.39. In the statement of the *s*-cobordism Theorem the inclusions $M_i \hookrightarrow W$ are simple homotopy equivalences. The letter "s" in "s-cobordism" refer to simple homotopy equivalence.

Proof of the *s***-cobordism Theorem.** To prove the *s*-cobordism Theorem we need some technical Lemmas. There exists many written proofs of these crucial Lemmas in the literature, see Lück [85] and Kervaire [64].

Let us fix some notations. In the following we will consider handle decompositions of a manifold W which has $M_1 \coprod M_2$ has boundary.

$$W = M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \dots \bigcup_{i=1}^{p_\lambda} (\varphi_i^\lambda) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n).$$

Then we will denote

$$W^{\lambda} = M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \dots \bigcup_{i=1}^{p_{\lambda}} (\varphi_i^{\lambda})$$

the manifold obtained from $M_1 \times [0, 1]$ after the gluing of handles of indexes less or equal to λ , and

$$\hat{\partial} W^{\lambda}_{+} = \partial W^{\lambda} \setminus \Big(\prod_{i=1}^{p_{\lambda+1}} \varphi_{i}^{\lambda+1} (S^{\lambda} \times \overset{\circ}{D^{n-1-\lambda}}) \coprod M \times \{0\} \Big)$$

the upper boundary of W^{λ} without the gluing sets of handles of index $\lambda + 1$.

Lemma 2.40. Let W be an oriented compact n-manifold with $n \ge 6$ and ∂W is diffeomorphic to the disjoint union of two compact (n-1)-manifolds M_1 and M_2 . Suppose that each component of ∂W is a deformation retract of W, then W is diffeomorphic to $M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\varphi_i^2) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$ relatively to M_1 .

Proof. Let $M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\varphi_i^0) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$ be a handle decomposition of W. To prove this Lemma we have to show that we can remove the handle of indexes 0 and 1.

Recall that to add a 0-handle we make the disjoint union with a *n*-disk. But since W is connected there exists almost one 1-handle joining $M_1 \times [0, 1]$ to this *n*-disk. Up to isotopy all the gluing sets of 1-handles, which are not in the 0-handles, are in $M_1 \times \{1\}$, hence the order of attaching 1-handles is not important. So if (φ_1^0) is the first 0-handle, one can assume that the 1-handle (φ_1^1) is joining $M_1 \times [0, 1]$ to (φ_1^0) . But the gluing of (φ_1^1) with (φ_1^0) is homeomorphic to a *n*-disk since we only attach one connected component of the boundary of the 1-handle to the 0-handle. These two handles (φ_1^1) and (φ_1^0) are cancelling together, so we can remove the 0-handle (φ_1^0) and the 1-handle (φ_1^1) . Finally one may assume that there is no 0-handle.

The handle decomposition of W became $M_1 \times [0,1] \bigcup_{i=1}^{p_1-p_0} (\varphi_i^1) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$.

Since $\hat{\partial}W^0_+$ consists only in $M \times \{1\}$ with $2p_1$ disks of dimension (n-1) removed, then $\pi_1(\hat{\partial}W^0_+) = \pi_1(M \times \{1\})$. Moreover $M_1 \times \{1\}$ is a deformation retract of W, so $\pi_1(\hat{\partial}W^0_+)$ maps surjectively onto $\pi_1(W)$. Let $\phi^1_1 : D^1 \times D^{n-1} \to W^1$ the embedding of the 1-handle (φ^1_1) . Consider now $[\sigma] \in \pi_1(W^1)$ given by the homotopy class of $\sigma = \phi^1_1(D^1 \times \{0\}) \cup \gamma_{\varphi^1_1(S^0 \times \{0\})}$ the gluing , along their

boundary of the core the 1-handle and a path γ , which join in ∂W^0_+ the two points of $\varphi_1^1(S^0 \times \{0\})$. By construction $[\sigma]$ is in not equal to 0 in $\pi_1(W^1)$; but since $\pi_1(W) \cong \pi_1(M_1)$, then $[\sigma]$ must be 0 in $\pi_1(W)$. This means that σ is nullhomotopic in W. Because of the dimensions, one can find some attaching maps $\{\varphi_i'^2\}_{i=1,\dots,p_2}$ isotopic to $\{\varphi_i^2\}_{i=1,\dots,p_2}$ such that for all $i = 1,\dots,p_2$ the images of $\varphi_i'^2$ do not meet the loop σ . Hence one can construct an embedding $\phi: S^1 \to \partial W^1$ such that $[\phi(S^1)] = [\sigma]$ and $\phi(S^1)$ meets the transverse sphere of (φ_1^1) transversally in exactly one point. Since σ is nullhomotopic in W, then ϕ is nullhomotopic in W and in ∂W^2 as well. This means that the image of ϕ bounds an immersed 2-disk, and twice the dimension of this disk is strictly less than the dimension of ∂W^2 , which is 5. According to Whintney's embedding Theorem, this homotopy can be realized with an embedding $\phi: S^1 \times D^{n-1} \to \partial W^2$ which is isotopic to a trivial embedding in ∂W^2 . By construction Φ fulfils the hypothesis of Lemma 2.27, so we can eliminate the first 1-handle in the decomposition of W. By induction we get the desired decomposition

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\overline{\varphi}_i^2) \bigcup_{i=1}^{p_3} (\overline{\varphi}_i^3) \dots \bigcup_{i=1}^{p_n} (\overline{\varphi}_i^n)$$

Remark 2.41. In the proof we strongly used the assumption $n \ge 6$ to smooth immersed disks to embedded disks.

As a consequence of this Lemma one can give a description of the $\mathbb{Z}[\pi]$ -chain complex $C_*(\tilde{W}, \tilde{M}_1)$ in term of homotopy groups, see § 2.1.2 for the definition of this complex, where we have identified $M_1 \times \{0\}$ to M_1 , the manifold \tilde{W} is the universal covering of W and $\pi = \pi_1(W)$.

First we fix a base point in $M_1 \times \{0\}$ and a lift of that point in $\rho^{-1}(W)$, all the homotopy groups will be considered with respect to these base points. Now we define the $\mathbf{Z}[\pi]$ -chain complex

$$\pi_*(W^*, W^{*-1}) = \begin{cases} 0 & \text{if } q \le 1, \\ \pi_q(W^q, W^{q-1}) & \text{if } q \ge 2. \end{cases}$$

The differentials are given by the composite maps

$$\pi_q(W^q, W^{q-1}) \xrightarrow{\partial_q} \pi_{q-1}(W^{q-1}) \xrightarrow{i_{q-1}} \pi_{q-1}(W^{q-1}, W^{q-2})$$

where ∂_q is a boundary operator, and i_{q-1} is induced by the inclusion.

For all $q \geq 1$ the group $\pi_1(\tilde{W}^{q-1})$ is trivial, then the relative Hurewicz homeomorphism $\pi_q(\tilde{W}^q, \tilde{W}^{q-1}) \to H_q(\tilde{W}^q, \tilde{W}^{q-1})$ is an isomorphism. Moreover the covering maps $\tilde{\rho}_q : \tilde{W}^q \to W^q$ induce the isomorphisms $\pi_q(\tilde{W}^q, \tilde{W}^{q-1}) \cong$ $\pi_q(W^q, W^{q-1})$. Finally we get an isomorphism of $\mathbf{Z}[\pi]$ -chain complexes

$$C_*(W, \tilde{M}_1) \cong \pi_*(W^*, W^{*-1}).$$

Each basis element $[\varphi_i^q] \in C_q(\tilde{W}, \tilde{M}_1)$, associate with the attaching maps of the handles, can be considered as an element of $\pi_q(W^q, W^{q-1})$ with this isomorphism. It corresponds to the element given by the homotopy class of the mapping $(D^q \times \{0\}, \varphi^q(S^{q-1} \times \{0\}) \hookrightarrow (W^q, W^{q-1}).$

In the following Lemma we give conditions which ensure that the embedding of a sphere meets suitably the transverse spheres of a handle decomposition.

Lemma 2.42. Let W be a compact n-manifold with $n \ge 6$ and ∂W is diffeomorphic to the disjoint union of two compact (n-1)-manifolds M_1 and M_2 . Suppose that W is diffeomorphic to $M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\varphi_i^2) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$ relatively M_1 .

Fix $\lambda \in \{1, \ldots, n-3\}$ and $k \in \{1, \ldots, p_{\lambda}\}$. Let $f : S^{\lambda} \to \hat{\partial} W^{\lambda}_{+}$ be an embedding. Then the following are equivalent

- 1. There exists an embedding $g: S^{\lambda} \to \hat{\partial} W^{\lambda}_{+}$ isotopic to f which meets the transverse spheres of the λ -handle (φ_{k}^{λ}) transversally in exactly one point and is disjoint from the transverse spheres of the λ -handles $\{(\varphi_{i}^{\lambda})\}_{i\neq k}$,
- 2. For any lift $\tilde{f}: S^{\lambda} \to \tilde{W}^{\lambda}$ of f under $\tilde{\rho}_{|\tilde{W}^{\lambda}}$; if $[\tilde{f}]$ denotes the image of funder the composite map $\pi_{\lambda}(\tilde{W}^{\lambda}) \to \pi_{\lambda}(\tilde{W}^{\lambda}, \tilde{W}^{\lambda-1}) \to H_{\lambda}(\tilde{W}^{\lambda}, \tilde{W}^{\lambda-1})$, then there exists $\gamma \in \pi$ such that $[\tilde{f}] = \pm \gamma[\varphi_{k}^{\lambda}]$.

Proof. When the transversality conditions of the first statement are fulfilled, the second follows easily.

Let us explain the converse. Because of dimensions the image of f meets the set of transverse spheres of the λ -handles only in a finite number of points, set

$$\operatorname{Im} f \bigcap \left\{ \{0\} \times S_i^{n-\lambda-1} \right\}_{i=1,\dots,p_{\lambda}} = \left\{ x_{i,1},\dots,x_{i,n_i} \right\}_{i=1,\dots,p_{\lambda}}.$$

Fix $* \in \text{Im } f$ a base point in W, and in each transverse sphere $\{0\} \times S_i^{n-\lambda-1}$ fix a base point $*_i$, for $i = 1, \ldots, p_\lambda$, such that $*_i \notin \{x_{i,1}, \ldots, x_{i,n_i}\}_{i=1,\ldots,p_\lambda}$.

Now let $c_{i,j}:[0,1] \to S^{\lambda}$ be a path such that for all $(i,j) \in \{1,\ldots,p_{\lambda}\} \times \{1,\ldots,n_i\}$ we have $f \circ c_{i,j}(0) = *$ and $f \circ c_{i,j}(1) = x_{i,j}$. Let $b_{i,j}:[0,1] \to W^{\lambda}$ be a path such that for all $(i,j) \in \{1,\ldots,p_{\lambda}\} \times \{1,\ldots,n_i\}$ we have $b_{i,j}(0) = x_{i,j}$ and $b_{i,j}(1) = *_i$. And let $a_i:[0,1] \to W^{\lambda}$ be a path such that for all $i \in \{1,\ldots,p_{\lambda}\}$ we have $a_i(0) = *_i$ and $a_i(1) = *$.

Now let $l_{i,j}$ a loop base in *, which is the composite path of $f(c_{i,j})$, $b_{i,j}$ and a_i . if we denote by $\gamma_{i,j}$ the homotopy class of $l_{i,j}$ in $\pi = \pi_1(W, *)$, then we have

$$[\tilde{f}] = \sum_{i=1}^{p_{\lambda}} \sum_{j=1}^{n_i} \epsilon_{i,j} \gamma_{i,j} [\varphi_i^{\lambda}]$$

where $\epsilon_{i,j} = \pm 1$.

We assume that there exists $\gamma \in \mathbf{Z}[\pi]$ such that $[\tilde{f}] = \pm \gamma[\varphi_k^{\lambda}]$, but since the set $\{[\varphi_i^{\lambda}]\}_{i=1,...,p_{\lambda}}$ is a basis of $\mathcal{H}_{\lambda}(\tilde{W}^{\lambda}, \tilde{W}^{\lambda-1})$ then, for $i \neq k$, we can associate the elements of $\{x_{i,1}, \ldots, x_{i,n_i}\}_{i=1,...,p_{\lambda}}$ by pairs such that for each pair, say (x_{i,j_1}, x_{i,j_2}) , we have $\epsilon_{i,j_1} \epsilon_{i,j_2} = -1$. This means that the loop, which is the composite path of $f(c_{i,j_1}), b_{i,j_1}$, the inverse of b_{i,j_2} and the inverse of $f(c_{i,j_2})$ is nullhomotopic in $\partial W_{\lambda}^{\lambda}$.

Now, since $n \ge 6$, then one can apply the Whitney trick (see [141]) to modify f with an isotopy, and get new embedding of S^{λ} in ∂W^{λ}_{+} with the two intersection points x_{i,j_1} and x_{i,j_2} removed and no change to the other intersection points with the transverse spheres.

2 h-cobordism Theorem and surgeries on manifolds

By induction we get the first statement with
$$\gamma = \pm \sum_{j=1}^{n_k} \epsilon_{k,j} \gamma_{k,j}$$
.

Lemma 2.43. Let $f: S^{\lambda} \to \hat{\partial} W^{\lambda}_{+}$ be an embedding, and let $\{x_j\}_{j=1,\ldots,p_{\lambda+1}}$ be a set of elements of $\mathbf{Z}[\pi]$.

An embedding $g: S^{\lambda} \to \hat{\partial} W^{\lambda}_{+}$ is isotopic to f if and only if to each lift $\tilde{f}: S^{\lambda} \to \tilde{W}^{\lambda}$ of f under $\tilde{\rho}_{|\tilde{W}^{\lambda}}$ one can find a lift $\tilde{g}: S^{\lambda} \to \tilde{W}^{\lambda}$ of g such that in $\mathcal{H}_{\lambda}(\tilde{W}^{\lambda}, W^{\lambda-1})$ we have

$$[\tilde{g}] = [\tilde{f}] + \sum_{j=1}^{p_{\lambda+1}} x_j \, d_{\lambda+1}[\varphi_j^{\lambda+1}]$$

where $d_{\lambda+1}$ is the $(\lambda+1)$ -differential of the complex $C_*(\tilde{W}, \tilde{M}_1)$.

This Lemma is more or less proved in Smale's work [124], for a proof see [64] or [85].

Lemma 2.44. Let W be an oriented compact n-manifold with $n \ge 6$ and ∂W is diffeomorphic to the disjoint union of two compact (n-1)-manifolds M_1 and M_2 . Suppose that each component of ∂W is a deformation retract of W, then for any $\lambda \in \{2, \ldots, n-3\}$ there exists a handlebody decomposition of W of the form

$$M_1 \times [0,1] \bigcup_{i=1}^{p_{\lambda}} (\varphi_i^{\lambda}) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}).$$

Proof. We saw that handles of indexes 0 and 1 can be removed so we start with a handle decomposition for W of the form

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_2} (\varphi_i^2) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n).$$

Now we will show that we can decrease p_q by one provided that $p_r = 0$ for $r \le q - 1$ and $q \le n - 3$.

Start with a decomposition $W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_q} (\varphi_i^q) \dots \bigcup_{i=1}^{p_n} (\varphi_i^n)$. As done before the trick is to attach a new (q+1)-handle, which cancel with (φ_1^q) , and a new (q+2)-handle such that the two new handles cancel together. To do that we will use Lemma 2.27.

Let $\Psi^{q+1}: S^{q+1} \times D^{n-q-1} \to \hat{\partial} W^q_+$ be an embedding such that its image is included in a *n*-disk $D^n \subset \hat{\partial} W^q_+$.

included in a *n*-disk $D^n \subset \hat{\partial} W^q_+$. Since the inclusion $M_1 \hookrightarrow W$ is a homotopy equivalence, then the $\mathbb{Z}[\pi]$ chain complex $C_*(\tilde{W}, \tilde{M}_1)$ is acyclic. But we assume that there is no *k*-handle with $k \leq q-1$ in the handle decomposition for W, hence the the $\mathbb{Z}[\pi]$ -module $C_{q-1}(\tilde{W}, \tilde{M}_1) = \mathrm{H}_{q-1}(\tilde{W}^{q-1}, \tilde{W}^{q-2})$ is trivial. So the (q+1)-differential of the complex $C_*(\tilde{W}, \tilde{M}_1)$, namely $d_{q+1} : C_{q+1}(\tilde{W}, \tilde{M}_1) \to C_q(\tilde{W}, \tilde{M}_1)$, is surjective. This implies that there exists a set $\{x_k\}_{i=1,\dots,p_{q+1}}$ of elements in $\mathbb{Z}[\pi]$, such that

$$\mathbf{H}_{q}(\tilde{W}^{q}, \tilde{W}^{q-1}) \ni [\varphi_{1}^{q}] = \sum_{i=1}^{p_{q+1}} x_{i} \, d_{q+1} \big([\varphi_{i}^{q+1}] \big).$$

According to Lemma 2.43, one can find an embedding $\psi^{q+1}: S^{q+1} \times D^{n-q-1} \to \hat{\partial}W^{q+1}_+$, which is isotopic to Ψ^{q+1} in $\hat{\partial}W^{q+1}_+$, such that

$$[\psi_{|S^q \times \{0\}}^{q+1}] = [\Psi_{|S^q \times \{0\}}^{q+1}] + \sum_{i=1}^{p_{q+1}} x_i \, d_{q+1} \big([\varphi_i^{q+1}] \big).$$

But $[\psi_{|S^q \times \{0\}}^{q+1}] = [\varphi_1^q]$ since $[\Psi_{|S^q \times \{0\}}^{q+1}]$ is nullhomotopic in $\hat{\partial}W_+^{q+1}$. Moreover, according to Lemma 2.42 the embedding $\psi_{|S^q \times \{0\}}^{q+1}$ is isotopic in $\hat{\partial}W_+^{q+1}$ to an embbeding $S^q \to \hat{\partial}W_+^{q+1}$ which meets the transverse sphere of (φ_1^q) transversally exactly in one point and do not meet the transverse spheres of the other *q*-handles.

We can apply Lemma 2.27 to find a new handle decomposition

$$W \cong M_1 \times [0,1] \bigcup_{i=2}^{p_q} (\varphi_i^q) \bigcup_{i=2}^{p_{q+1}} (\overline{\varphi}_i^{q+1}) \cup (\psi^{q+2}) \bigcup_{i=2}^{p_{q+2}} (\overline{\varphi}_i^{q+2}) \dots \bigcup_{i=1}^{p_n} (\overline{\varphi}_i^n),$$

and the number of q-handle decreased by one. By induction we can remove all q-handles.

Now using the dual handle decomposition for W, i.e., the handle decomposition associated with the Morse function -f instead of f which start with $M_2 \times [0,1]$; we have the following decomposition

$$W \cong M_2 \times [0,1] \bigcup_{i=1}^{p_0} (\phi_i^n) \dots \bigcup_{i=1}^{p_\lambda} (\phi_i^{n-\lambda}) \dots \bigcup_{i=1}^{p_n} (\phi_i^0).$$

As just explained before one can remove handles of indexes less or equal to $n - \lambda - 2$ in this decomposition, and

$$W \cong M_2 \times [0,1] \bigcup_{i=1}^{p_0} (\phi_i^n) \dots \bigcup_{i=1}^{p_{\lambda+1}} (\phi_i^{n-\lambda-1}).$$

If we take again the dual handle decomposition of the last one, then one can find a handle decomposition for W of the form

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_0} (\Phi_i^0) \dots \bigcup_{i=1}^{p_{\lambda+1}} (\Phi_i^{\lambda+1}).$$

Now we remove all handles of indexes less or equal to $\lambda - 1$ in the last decomposition and we get the desired result

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^{p_{\lambda}} (\varphi_i^{\lambda}) \bigcup_{i=1}^{p_{\lambda+1}} (\varphi_i^{\lambda+1}).$$

We are ready to finish the proof of the s-cobordism Theorem.

Proof of s-cobordism Theorem. With the previous Lemma 2.44 we can assume that the manifold W admits a handle decomposition of the form

$$W \cong M_1 \times [0,1] \bigcup_{i=1}^p (\varphi_i^{\lambda}) \bigcup_{i=1}^p (\varphi_i^{\lambda+1}).$$

The number of handle is the same since we assume that both M_1 and M_2 are deformation retracts of W.

The acyclic $\mathbf{Z}[\pi]$ -chain complex $C_*(\tilde{W}, \tilde{M}_1)$ has only one differential which is non zero, namely $d_{\lambda+1} : \mathrm{H}_{\lambda+1}(\tilde{W}_{\lambda+1}, \tilde{W}_{\lambda}) \to \mathrm{H}_{\lambda}(\tilde{W}_{\lambda}, \tilde{W}_{\lambda-1})$. Let D be the matrix of the isomorphism $d_{\lambda+1}$ with respect to the basis $\{[\varphi_i^{\lambda+1}]\}_{i=1,\ldots,p}$ of $C_{\lambda+1}(\tilde{W}, \tilde{M}_1) = \mathrm{H}_{\lambda+1}(\tilde{W}_{\lambda+1}, \tilde{W}_{\lambda})$ and the basis $\{[\varphi_i^{\lambda}]\}_{i=1,\ldots,p}$ of $C_{\lambda}(\tilde{W}, \tilde{M}_1) =$ $\mathrm{H}_{\lambda}(\tilde{W}_{\lambda}, \tilde{W}_{\lambda-1})$. The entries $d_{i,j} \in \mathbf{Z}[\pi]$ of the matrix D, for $i, j = 1, \ldots, p$, are defined by the equations

$$d_{\lambda+1}\left(\left[\varphi_i^{\lambda+1}\right]\right) = \sum_{j=1}^p d_{i,j}\left[\varphi_j^{\lambda}\right].$$

By definition, the Whitehead torsion $\tau(W, M_1)$ is given by the class of the matrix D in Wh(π).

Let us give the geometrical interpretation of the four elementary operations which generate the Whitehead group described in Definition 2.34 and Proposition 2.36, when these operations are made on the matrix D we just defined.

- 1. The multiplication of the k-th row of D by $\pm \gamma$ with $\gamma \in \mathbb{Z}[\pi]$ correspond to the modification of the lift in $\tilde{W}^{\lambda+1}$ of φ_k^{λ} . But according to Lemma 2.43 this corresponds to the gluing of a new λ -handle $(\varphi_k^{\prime\lambda})$ instead of (φ_k^{λ}) . The resulting manifold is diffeomorphic to W.
- 2. Similarly to the previous item, the addition to the k-th row of the j-th row of D can be realized by the gluing of a new λ -handle which is isotopic to (φ_k^{λ}) .
- 3. This operation corresponds to the gluing of a new λ -handle (ψ^{λ}) and a new $(\lambda + 1)$ -handle $(\psi^{\lambda+1})$ in a *n*-disk of $\partial W_{+}^{\lambda+1}$, such that these handles are cancelling together.
- 4. This operation is the converse of the previous one, when we do it we just remove to handles, which are cancelling together, from the handle decomposition of W.

Since all of the modifications on the matrix D correspond to modifications of the handle decomposition of W which do not change W up to diffeomorphism, then we see that the Whitehead torsion $\tau(W, M_1)$ vanishes if and only if W admits a handle decomposition in which all handles can be removed, and then $W \cong M_1 \times [0, 1]$.

Proposition 2.45. The s-cobordism Theorem implies the h-cobordism Theorem.

Proof. Recall that any invertible matrix over the integers can be reduced by elementary operations to the identity matrix. So all the matrices in $GL(\mathbf{Z})$ are equivalent in Wh(\mathbf{Z}) which is trivial. When the manifolds M_1 and M_2 are simply connected, then Wh(π) = {0} and the s-cobordism Theorem implies the h-cobordism Theorem.

2.3 Surgery on manifolds

2.2.2 The relative case

The notion of *relative h-cobordism* was introduced by Heaffiger [45].

Definition 2.46. Two pairs (M_1, V_1) and (M_2, V_2) of manifolds with $V_i \subset M_i$ for i = 1, 2 are *h*-cobordant if there exist a pair of manifold (M, V) with $V \subset M$ such that $\partial M = M_1 - M_2$, $\partial V = V_1 - V_2$ and the inclusion $M_i \hookrightarrow M$, $V_i \hookrightarrow V$ are homotopy equivalences for i = 1, 2.

Then the h-cobordism and s-cobordism theorems can be extended to the relative case.

2.3 Surgery on manifolds

In this section we describe modifications on manifolds called *surgeries*. We introduce them now since they are very similar to handle gluing. In handlebody decompositions of manifolds, we attach handles in order to give some descriptions of the manifolds.

Start with a *n*-manifold X, and let $\psi : S^k \times D^{n-k} \to X$ be an embedding for 0 < k < n. Set Y be the manifold obtained from X as follows

$$Y = X \setminus \left(\psi(S^k \times D^{n-k}) \cup_{\partial} \left(D^{k+1} \times S^{n-k-1} \right), \right.$$

where the gluing is given by the identification of boundaries.

Definition 2.47. We say that Y is obtained from X after a surgery on $\psi(S^k)$. When the manifold X is embedded in a manifold W, if there exist an embed-

ding $\phi: D^{k+1} \times S^{n-k-1} \to W \setminus (\psi(S^k \times D^{n-k}))$ such that $\psi(S^k \times S^{n-k-1}) = \psi(S^k \times S^{n-k-1})$, then we say that the manifold

$$Y = X \setminus \left(\psi(S^k \times D^{n-k}) \cup_{\partial} \phi(D^{k+1} \times S^{n-k-1}), \right.$$

is obtained from X after an embedded surgery on $\psi(S^k)$.

In fact surgeries can be described with handles gluing. The manifold Y constructed by surgery on $\psi(S^k)$ can be viewed as the upper boundary of

 $X\times [0,1]\cup (\psi^{k+1}),$

as depicted bellow



Modifications of manifolds with surgeries change homology groups. More precisely a suregry on $\psi(S^k)$ in a manifold X gives a manifold Y in which the homology class of $\psi(S^k)$ is zero. So if $\psi(S^k)$ is a *n*-dimensional chain wich represent a non trivial homology class in X, then the rank of the k^{th} homology group of Y may not be equal to those of X. Moreover if $\psi(S^k)$ is a *n*-dimensional chain wich represent a trivial homology class in X then a surgery on $\psi(S^k)$ must add some homology class of dimension not equal to n.

Anyway, using Mayer-Vietoris exact sequences associated with decomposition of manifolds like

$$X \setminus \left(\psi(S^k \times D^{n-k}) \cup_{\partial} \left(D^{k+1} \times S^{n-k-1} \right), \right.$$

one can compute exactly how a surgery modifies the homology of X.

We will combine surgeries and *h*-cobordism Theorem to construct cobordism of knots. More precisely, to prove that two knots K_0 and K_1 are cobordant, we need to find a manifold X such that $\partial X = K_0 \coprod K_1$ and $X \cong K_0 \times [0, 1]$. A way to do that is to start with a manifold Z such that $\partial Z = K_0 \coprod K_1$ and do some surgeries on Z to get a manifold X with $H_*(X, K_0) = 0$ and then apply the *h*-cobordism Theorem to get $X \cong K_0 \times [0, 1]$.

Chapter 3

Spherical knots

"Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so bersetzen sie es in ihre Sprache, und dann ist es alsbald etwas anderes." J.W. von Goethe, - Maximen und Reflexionen

In this chapter, we consider the case of spherical knots. In the sixties, Kervaire and Levine gave classifications of spherical knots up to cobordism, we will recall some of their results in the following.

Unless specified all knots in this chapter are simple spherical (2n-1)-knots.

3.1 Alexander polynomial

3.2 S-equivalence

The Seifert form is the main tool to study cobordisms of odd dimensional spherical knots. Since spherical knots are not in general fibered, then there exists many distinct Seifert manifolds for a given spherical knot. Before going futher, the first step is to know what happen on Seifert forms when we change the Seifert manifolds associated with a spherical knot. In [82] Levine described the possible modifications on Seifert forms of a spherical simple knot corresponding to alterations of Seifert manifolds.

For a given (2n-1)-knot K embedded in S^{2n+1} let us consider two Seifert manifolds V_1 and V_2 associated with K. One can suppose that $V_i \times \{i\}$ is embedded in $S^{2n+1} \times \{i\} \hookrightarrow S^{2n+1} \times [0,1]$ for i = 0, 1. We denote by \mathfrak{A}_i the Seifert form associated with V_i , and by $\mathfrak{S}_i = \mathfrak{A}_i + (-1)^n \mathfrak{A}_i$ the intersection form associated with V_i for i = 0, 1.

With similar arguments as those used to proved that every knot bounds an embedded Seifert manifold, one can see that there is no obstruction to construct an embedded submanifold W of $S^{2n+1} \times [0,1]$ such that $\partial W = V_0 \cup K \times [0,1] \cup V_1$. Then the handle decomposition associated with a Morse function $f: W \to [0,1]$ show that V_0 and V_1 are related each other by embedded surgeries.

Remark 3.1. The manifold W is very usefull to construct submodules on which the Seiferts forms vanishe. More precisely, when two *n*-cycle α and β in $H(V_0) \oplus H(V_1)$ are null-homologous in $H_n(W)$ then $A_0 \oplus -A_1(x, y) = 0$.

To prove the last equality, remark that the positive direction of the normal bundle of $V_0 \coprod V_1$ in S^{2n+1} extend to a positive direction of the normal bundle of W in $S^{2n+1} \times [0,1]$. Set ξ and η some (n+1)-chains in W such that $[\partial \xi] = x$ and $[\partial \eta] = y$ and ξ_{+_W} the chain ξ pushed out W in the positive normal direction in $S^{2n+1} \times [0,1]$. Since the two chains ξ_{+_W} and η do not intersect together, then

$$\begin{array}{rcl} A_0 \oplus -A_1(x,y) &=& l_{S^{2n+1}} \big((\partial \xi)_+, (\partial \eta) \big), \\ A_0 \oplus -A_1(x,y) &=& I_{S^{2n+1} \times [0,1]} (\xi_{+_W}, \eta), \\ A_0 \oplus -A_1(x,y) &=& 0. \end{array}$$

3 Spherical knots

When the critical points of f are not of index n nor n+1 then the associated surgeries do not modify the homolgy groups $H_n(V_0)$ and $H_n(V_1)$; hence these modifications do not affect the Seifert forms associated with V_0 and V_1 .

Since the critical points of f are isolated, then it suffices to consider the case where f has only one critical point. First, suppose that the critical point is of index n. The corresponding surgery means that we attach a n-handle to V_0 . More precisely, remove $D^{n+1} \times S^{n-1}$ and glue $S^n \times D^n$ along the new boundary. Elementary computation with Mayer-Vietoris sequences give $H_n(W, V_0) \cong H_{n+1}(W, V_1) \cong \mathbb{Z}$ and $H_n(W, V_1) \cong H(W, V_0) = 0$. Let a be the image in $H(V_1)$ of the generator of $H(W, V_1)$. If a has a finite order, then Seifert forms associated with V_0 and V_1 are isomorphic since they are defined modulo torsion.

If a has infinite order, then it is a multiple of a primitive element a_0 of $H(V_1)$. Moreover there exists b_0 in $H(V_1)$ such that $\mathfrak{S}_1(a_0, b_0) = 1$. Since $rk(H(V_1)) = rk(H(V_0)) + 2$ one can take (c_1, \ldots, c_k) in $H(V_1)$ such that $(a_0, b_0, c_1, \ldots, c_k)$ is a basis of $H(V_1)$ and (c_1, \ldots, c_k) are homologous to a basis (d_1, \ldots, d_k) of $H(V_0)$.

There exist a (n+1)-chain Γ_i in W such that $\partial\Gamma$ is a n-chain which represent the cycle $d_i - c_i$ for $i = 1, \ldots, k$. Then for all i, j in $\{1, \ldots, k\}$ we have $\mathfrak{A}_0(d_i, d_j) - \mathfrak{A}_1(c_i, c_j)$ is the intersection number of Γ_j and the translate of Γ_i off W in the positive normal direction of W in $S^{2n+1} \times [0, 1]$. Since this intersection number is zero then for all i, j in $\{1, \ldots, k\}$ we have $\mathfrak{A}_0(d_i, d_j) = \mathfrak{A}_1(c_i, c_j)$. By definition a is null-homologous in W, hence we get $\mathfrak{A}_1(a, c_i) = \mathfrak{A}_1(c_i, a) = \mathfrak{A}_1(a, a) = 0$ for $i = 1, \ldots, k$. Thus we have $\mathfrak{A}_1(a_0, c_i) = \mathfrak{A}_1(c_i, a_0) = \mathfrak{A}_1(a_0, a_0) = 0$. If A_0 (resp. A_1) is the matrix of \mathfrak{A}_0 (resp. \mathfrak{A}_1) with respect to the basis (d_1, \ldots, d_k) (resp. $(c_1, \ldots, c_k, a_0, b_0)$), then

$$A_1 = \begin{pmatrix} A_0 & \mathcal{O} & \nu \\ \mathcal{O}^T & 0 & w \\ \mu^T & z & v \end{pmatrix},$$

where \mathcal{O} is a column vector whose entries are all 0, and ν, μ are column vector of integers.

Since $\mathfrak{S}_1(a_0, b_0) = 1$ then we have $w + (-1)^n z = 1$. Recall that the Alexander polynomial of K is well defined up to a unit in $\mathbb{Z}[t, t^{-1}]$. If we denote by $\Delta_{A_i}(t)$ the Alexander polynomial associated with \mathfrak{A}_i for i = 0, 1, then $\Delta_{A_1}(t) = (tw + (-1)^n z)(tz + (-1)^n w)\Delta_{A_0}(t)$. So w or z must be 0, if w = 0 then one can modify the vectors of the basis $(c_1, \ldots, c_k, a_0, b_0)$ to get

$$A_1 = \begin{pmatrix} A_0 & \mathcal{O} & \mathcal{O} \\ {\mu'}^T & 0 & 0 \\ \mathcal{O}^T & 1 & 0 \end{pmatrix}.$$

Hence we define the *enlargement* A' of a square integral matrix A as follows:

$$A' = \begin{pmatrix} A & \mathcal{O} & \mathcal{O} \\ \alpha^T & 0 & 0 \\ \mathcal{O}^T & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} A & \beta & \mathcal{O} \\ \mathcal{O}^T & 0 & 1 \\ \mathcal{O}^T & 0 & 0 \end{pmatrix},$$

where \mathcal{O} is a column vector whose entries are all 0, and, α and β are column vectors of integers. In this case, we also call A a *reduction* of A'.

Definition 3.2. Two square integral matrices are said to be *S*-equivalent if they are related each other by enlargement and reduction operations together with the congruence. We also say that two integral bilinear forms defined on free \mathbf{Z} -modules of finite rank are *S*-equivalent if so are their matrix representatives.

This equivalence relation characterize isotopy classes of spherical simple (2n - 1)-knots with $n \geq 2$ as stated in the following Theorem proved by Levine [82].

Theorem 3.3 ([82]). For $n \ge 2$, two spherical simple (2n-1)-knots are isotopic if and only if they have S-equivalent Seifert forms.

We will need the two following Lemmas for the proof.

Lemma 3.4. Let K be a simple spherical (2n-1)-knot, and let A be a Seifert matrix associated with a (n-1)-connected Seifert manifold for K. If B is an enlargement of A then B is a Seifert matrix associated with a (n-1)-connected Seifert manifold for K as well.

Proof. By Alexander duality

Lemma 3.5. If $n \ge 2$, then two simple spherical (2n - 1)-knots admitting identical Seifert matrices, associated with (n - 1)-connected Seifert manifolds for K, are isotopic.

We refer to [82] p.191 for a proof of this Lemma, which is based on handle decompositions for Seifert manifolds. Though the result is valid for all $n \ge 2$, we have to mention that special arguments must be used when n = 2.

When K is a spherical (2n-1)-knot with Seifert manifold F, then the long exact sequence in homology of (F, K) induces the exact short sequence

$$0 \to \operatorname{H}_n(F) \xrightarrow{S_*} \operatorname{H}_n(F, K) \to 0.$$

Morever when K is simple then $H_n(F, K)$ is isomorphic to $Hom_{\mathbf{Z}}(H_n(F), \mathbf{Z})$; and if we equip $H_n(F, K)$ with the dual basis of the one choosed for $H_n(F)$ then the matrix of S_* is $A + (-1)^n {}^tA$, where A is the Seifert matrix associated with F. So in that case we have $\det(A + (-1)^n {}^tA) = \pm 1$. The converse is also true as stated below.

Proposition 3.6 ([82]). Let n be an integer greater or equal to 2, and let A be an integral square matrix such that $A + (-1)^{n} A$ is unimodular. If $n \neq 2$, there exists a simple spherical (2n - 1)-knot with Seifert matrix A; if n = 2, there exists a simple spherical 3-knot with Seifert matrix S-equivalent to A.

Proof of Theorem 3.3. First suppose that two simple spherical (2n - 1)-knots K_0 and K_1 are isotopic, then using the same argument to compute modifications on Seifert forms corresponding to alterations of Seifert manifolds, we see that their Seifert forms are S-equivalent.

For the converse, star with two simple spherical (2n-1)-knots, denoted by K and K', with S-equivalent Seifert forms. Then there exist a finite sequence of matrices A_1, \ldots, A_k such that $A_1 = A$ is a Seifert matrix for K, $A_k = A'$ is a Seifert matrix for K' and for all $i = 1, \ldots, k-1$ the matrix A_{i+1} is an enlargement or a reduction of A_i up to congruence.

3 Spherical knots



Figure 3.1. The connected sum of trefoil knot and its inverse in C_1

3.3 Cobordism of spherical knots

Let us denote by C_n the set of cobordism classes of spherical *n*-knots, and by \tilde{C}_n the set of concordance classes of spherical *n*-knots. These two sets have a natural group structure. The group operation is given by the connected sum see [65] Chapter III for details.

We say that an n-knot $K \subset S^{n+2}$ is null-cobordant if it is cobordant to the trivial knot, i.e., if there exists an (n + 1)-disk D^{n+1} properly embedded in the (n+3)-disk D^{n+3} such that $\partial D^{n+1} = K \subset S^{n+2} = \partial D^{n+3}$. Similarly we define the notion of null-concordant knot.

The neutral element of C_n is the class of null-cobordant *n*-knots, and the neutral element of \tilde{C}_n is the class of null-concordant *n*-knot.

To construct the inverse of a *n*-knot *K* one can suppose that *K* is embedded in the upper hemisphere S^{n+2}_+ of the unit (n+2)-sphere $\partial D^{n+3} = S^{n+2} \hookrightarrow \mathbf{R}^{n+3}$. Let ρ be the reflection in the equatorial hyperplane \mathcal{E} of D^{n+3} , and $\pi: \mathbf{R}^{n+3} \to \mathcal{E}$ the projection onto \mathcal{E} .

Then we construct the connected sum $K' = K \# \rho(K)$ of K and $\rho(K)$ in S^{2n+1} ; we illustrate this construction in Fig. 3.1 when K is the trefoil knot embedded in S^3 . Moreover, one can suppose that this connected sum is made in order to have $\pi(K') = \pi(K' \cap S^{2n+1}_+)$, where S^{2n+1}_+ is the upper hemisphere of S^{2n+1} which contains K.

Then, set $\mathcal{D} = (\pi(K') \times [0,1]) \cap D^{n+3}$, remark that since $\pi(K')$ is a (2n-1)-disk, then \mathcal{D} is homeomorphic to a (n+1)-disk; moreover $\partial \mathcal{D} = K' = K \# \rho(K)$.

Since $K' \# \rho(K')$ bounds a (n + 1)-disk embedded in D^{n+3} then $K \# \rho(K)$ is null cobordant and $\rho(K)$ is the inverse of K. We have just proved that the inverse of K is given by its mirror image with reversed orientation, which we denote by $-K^!$.

Similarly we can construct the inverse of a knot class in the concordance groups \tilde{C}_n .

First, let us focus on the case of spherical (2n - 1)-knots. Kervaire and Levine used the notion of *Witt equivalence* for integral bilinear forms.

Witt equivalence of integral bilinear forms

Definition 3.7. Let $\mathfrak{A} : G \times G \to \mathbb{Z}$ be an integral bilinear form defined on a free Z-module G of finite rank. The form \mathfrak{A} is said to be *Witt associated to* \mathfrak{o} if the rank m of G is even and there exists a submodule M of rank m/2 in G such

that M is a direct summand of G and \mathfrak{A} vanishes on M. Such a submodule M is called a *metabolizer* for \mathfrak{A} .

The following theorem was proved by Levine [80] and Kervaire [66].

Theorem 3.8. For $n \ge 2$, a spherical (2n - 1)-knot is null-cobordant if and only if its Seifert form is Witt associated to \mathfrak{o} .

Proof.

For two spherical (2n-1)-knots K_0 and K_1 with Seifert forms \mathfrak{A}_0 and \mathfrak{A}_1 respectively, the oriented connected sum $K = K_0 \sharp (-K_1^!)$ has $\mathfrak{A} = \mathfrak{A}_0 \oplus (-\mathfrak{A}_1)$ as the Seifert form associated with the oriented connected sum along the boundaries of the Seifert manifolds associated with K_0 and $-K_1^!$, where $-K_1^!$ denotes the mirror image of K_1 with reversed orientation. Hence, as a consequence of Theorem 3.8, we have that two spherical knots K_0 and K_1 are cobordant if and only if the form $\mathfrak{A} = \mathfrak{A}_0 \oplus (-\mathfrak{A}_1)$ is Witt associated to \mathfrak{o} . In this case we sometimes say that \mathfrak{A}_0 and \mathfrak{A}_1 are *Witt equivalent*.

Proposition 3.9.

Remark 3.10. Witt equivalence is not an equivalence relation on the set of integral bilinear forms of finite rank. Let \mathfrak{A} and \mathfrak{B} be two integral bilinear forms of rank r such that $\mathfrak{A} \oplus -\mathfrak{B}$ is not Witt associated to \mathfrak{o} . If we denote by \mathfrak{O}_r the zero form of rank r, then both \mathfrak{A} and \mathfrak{B} are Witt equivalent to \mathfrak{o}_r but \mathfrak{A} and \mathfrak{B} are not Witt equivalent.

For $\varepsilon = \pm 1$, let $C^{\varepsilon}(\mathbf{Z})$ be the set of all Witt equivalence classes of integral bilinear forms \mathfrak{A} defined on free **Z**-modules of finite rank such that $\mathfrak{A} + \varepsilon \mathfrak{A}^T$ is unimodular (for the notation, we follow [66]).

Proposition 3.11.

It can be shown that $C^{\varepsilon}(\mathbf{Z})$ has a natural abelian group structure, where the addition is defined by the direct sum. Then we have the following.

Theorem 3.12 (Levine [80]). Let $\Phi_n : C_{2n-1} \to C^{(-1)^n}(\mathbf{Z})$ be the (welldefined) homomorphism induced by the Seifert form. Then Φ_n is an isomorphism for $n \geq 3$. But Φ_2 is only a monomorphism whose image $C^{+1}(\mathbf{Z})^0$ is a specified subgroup of $C^{+1}(\mathbf{Z})$ of index 2; and $\Phi_1 : C_1 \to C^{-1}(\mathbf{Z})$ is merely an epimorphism.

Furthermore, Levine [81] showed the following (see also Remark ??).

Theorem 3.13. For $\varepsilon = \pm 1$, we have

$$C^{\varepsilon}(\mathbf{Z}) \cong \mathbf{Z}_{2}^{\infty} \oplus \mathbf{Z}_{4}^{\infty} \oplus \mathbf{Z}^{\infty}, \qquad (3.1)$$

where the right hand side is the direct sum of countably many (but infinite) copies of the cyclic groups \mathbf{Z} , \mathbf{Z}_2 and \mathbf{Z}_4 .

Note that the right hand side of (3.1) is *not* an unrestricted direct sum, i.e., each element of the group is a linear combination of *finitely many* elements corresponding to the generators of the factors.

3 Spherical knots

Remark 3.14. Michel [93] showed that for $n \ge 1$, spherical algebraic (2n-1)-knots have infinite order in C_{2n-1} as soon as we assume that the associated holomorphic function germ has an isolated singularity at the origin and is not non-singular. Note, however, that they are not independent. See Remark ??.

For n = 1, $\Phi_1 : C_1 \to C^{-1}(\mathbf{Z})$ is far from being an isomorphism. The nontriviality of the kernel of this epimorphism was first shown by Casson-Gordon [22]. The classification of spherical 1-knots up to cobordism is still an open problem. Moreover, for spherical 1-knots, there is also the important notion of a *ribbon knot* (see, for example, [115]). Ribbon knots are null-cobordant. It is still an open problem whether the converse is true or not.

For even dimensions, we have the following vanishing theorem.

Theorem 3.15 (Kervaire [65]). For all $n \ge 1$, C_{2n} vanishes.

Let \tilde{C}_n be the group of concordance classes of embeddings into S^{n+2} of

1. the *n*-dimensional standard sphere S^n for $n \leq 4$, or

2. homotopy *n*-spheres for $n \ge 5$.

In [65] Kervaire showed that the natural surjection $i : \tilde{C}_n \to C_n$ is a group homomorphism.

Let us denote by Θ_n the group of *h*-cobordism classes of smooth oriented homotopy *n*-spheres, and by bP_{n+1} the subgroup of Θ_n consisting of the *h*cobordism classes represented by homotopy *n*-spheres which bound compact parallelizable manifolds [67]. Then we have the following

Theorem 3.16 (Kervaire [65]). For $n \leq 5$ we have $\tilde{C}_n \cong C_n$, and for n > 6 we have the short exact sequence

$$0 \to \Theta_{n+1}/bP_{n+2} \to \tilde{C}_n \xrightarrow{i} C_n \to 0.$$

Note that for $n \ge 4$, Θ_{n+1}/bP_{n+2} is a finite abelian group. For details, see [67].

Chapter 4

Fibered knots and algebraic knots

"Ce chemin qui débouche sur la route de Chinon, bien au-delà de Ballan, longe une plaine ondulée sans accidents remarquables, jusqu'au pays d'Artanne. Là se découvre une vallée qui commence à Montbazon, finit à la Loire, et semble bondir sous les châteaux posés sur ces doubles collines; une magnifique coupe d'émeraudes au fond de laquelle l'Indre se roule par des mouvements de serpent." Honoré de Balzac, - Le lys dans la vallée

In this chapter we will work only with odd dimensional knots. We first define the notion of *fibered knot* and prove that Seifert forms of fibered knots are unimodular, then we define algebraic knots associated with isolated singularities of complex hypersurfaces.

4.1 Fibered knots

As explained in the introduction the set of fibered knots is much more smaller than the set of knots. But using the fibration of the complementary of the knot over S^1 we will be able to construct many usefull data for the study of cobordsim classes of fibered knots.

Recall (c.f. Definition 1.14) that a (2n-1)-knot K is fibered when there exist a trivialization $\tau : N_K \to K \times D^2$ of a closed tubular neighborhood N_K of K in S^{2n+1} and a smooth fibration $\phi : S^{2n+1} \setminus K \to S^1$ such that the following diagram is commutative:

$$\begin{array}{ccc} N_K \setminus K & & \xrightarrow{\tau} & K \times \left(D^2 \setminus \{0\} \right) \\ \phi|_{(N_K \setminus K)} & & \swarrow & p \\ & & & \swarrow & p \end{array}$$

where p denotes the obvious projection. In this case, for each $t \in S^1$, the closure F in $S^{(2n+1)}$ of $\phi^{-1}(t)$ is also called a *fiber* of K. Note that $F = \phi^{-1}(t) \cup K$ is a compact 2n-dimensional manifold with boundary $\partial F = K$.

4.1.1 Monodromy and variation map

Any C^{∞} locally trivial fibration ϕ , as in Definition 1.14, over S^1 with fiber F such that $\partial F \neq \emptyset$, is given up to isomorphism by a map called *geometric* monodromy.

Definition 4.1. The geometric monodromy $\mathfrak{m} : (F, \partial F) \to (F, \partial F)$ is defined up to isotopy such that ϕ identifies with

$$(F,\partial F) \times [0,1]/(x,0) \sim (\mathfrak{m}(x),1) \xrightarrow{\to} [0,1]/0 \sim 1,$$

and the restriction $\mathfrak{m}_{|\partial F}$ is the identity.

The geometric monodromy induces two algebraic monodromies.

Definition 4.2. Let *K* be a fibered knot with fiber *F* and geometric monodromy $\mathfrak{m} : (F, \partial F) \to (F, \partial F).$

The algebraic monodromy is the homomorphism $h : H_n(F) \to H_n(F)$ induced by the geometrical monodromy \mathfrak{m} , and the characteristic polynomial of the algebraic monodromy is denoted by $\Delta(t)$.

The relative algebraic monodromy is the homomorphism in relative homology $\tilde{h}: H_n(F, \partial F) \to H_n(F, \partial F)$ induced by \mathfrak{m} .

Using the geometrical monodromy one can define another operator, called variation map. More precisely, let K be a fibered (2n - 1)-knot with fiber F. For any relative *n*-chain a with $\partial a \in \partial F = K$, we have $\partial (a - \mathfrak{m}(a)) = \partial (a) - \mathfrak{m}(\partial a) = 0$. Hence $a - \mathfrak{m}(a)$ is an absolute chain. In the following, if a is a chain, then we denote by [a] its homology class.

Definition 4.3. The following map \mathcal{V} is called *variation map*.

$$\begin{array}{rccc} \mathcal{V}: \mathrm{H}_n(F, \partial F) & \to & \mathrm{H}_n(F) \\ [a] & \mapsto & [a - \mathfrak{m}(a)] \end{array}$$

Let a fibered (2n-1)-knot K with fiber F, the Wang exact sequence associated with the fibration $S^{2n+1} \setminus N(K) \to S^1$ with fiber F provides

$$0 \to \mathrm{H}_{n+1}(S^{2n+1} \setminus \overset{\circ}{N(K)}) \to \mathrm{H}_n(F) \xrightarrow{Id-h} \mathrm{H}_n(F) \to \mathrm{H}_n(S^{2n+1} \setminus \overset{\circ}{N(K)}) \to 0$$

by Alexander Duality (see [13]) we get $\operatorname{H}_k(S^{2n+1} \setminus N(K)) \cong \operatorname{H}^{2n-k}(K)$, and by Poincaré Duality we have $\operatorname{H}^{2n-k}(K) \cong \operatorname{H}_{k-1}(K)$. Hence the previous Wang exact sequence becomes

$$0 \to \mathcal{H}_{n+1}(K) \to \mathcal{H}_n(F) \xrightarrow{Id-h} \mathcal{H}_n(F) \to \mathcal{H}_n(K) \to 0$$
(4.1)

Using the variation map, the exact sequences 1.1 and 4.1 can be related together as follows.

First for k = n, n+1, let us define Gysin isomorphisms $g_k : H_k(S^{2n+1} \setminus K) \to H_{k-1}(K)$ by $g([a]) = [b \cap K]$ where b is a boundary chain of dimension (k+1) which meets K transversaly in S^{2n+1} and with boundary the k-chain [a].

Then the following diagram is commutative

$$0 \to \operatorname{H}_{n+1}\left(S^{2n+1} \setminus K\right) \to \operatorname{H}_{n}(F) \xrightarrow{Id-h} \operatorname{H}_{n}(F) \xrightarrow{i} \operatorname{H}_{n}\left(S^{2n+1} \setminus K\right) \to 0$$
$$\downarrow g_{n+1} \qquad || \qquad \uparrow_{\mathcal{V}} \qquad \downarrow g_{n}$$

$$0 \to \qquad \mathbf{H}_n(K) \qquad \to \quad \mathbf{H}_n(F) \qquad \stackrel{S_*}{\to} \qquad \mathbf{H}_n(F,K) \quad \to \qquad \mathbf{H}_{n-1}(K) \qquad \to 0$$

The first square is commutative since g_{n+1} is an isomorphism, the second square is commutative because of the definition of \mathcal{V} (recall that S_* is induced by the inclusion). We only have to check the commutativity for the last square. Start with a relative cycle in $H_n(F, K)$ given by the homology class [c] of a relative chain c of dimension n. Then $\mathcal{V}([c]) = [c - \mathfrak{m}(c)]$, and if b is a (n + 1)chain with boundary $c - \mathfrak{m}(c) = \partial b$ then $g_n(i([c - \mathfrak{m}(c)])) = [b \cap K] = [\partial c]$. This proves the commutativity, and as a consequence the five Lemma implies that \mathcal{V} is an isomorphism. We proved

Proposition 4.4. The variation map $\mathcal{V} : \mathrm{H}_n(F, \partial F) \to \mathrm{H}_n(F)$ is an isomorphism.

4.1 Fibered knots

4.1.2 Seifert form

We already defined Seifert forms associated with simple knots, but in the case of simple fibered knot one can define the Seifert form associated with a fiber using the geometrical monodromy. Let us be more precise, and consider a fibered (2n-1)-knot K with fibration ϕ and fiber F. Write $F_{\theta} = \phi^{-1}(e^{i\theta})$ for any $\theta \in [0, 2\pi]$, then F_{θ} is homeomorphic to $\overset{\circ}{F}$. Moreover let h be a continuous map $h: [0, 1] \times F_0 \to S^{2n+1} \setminus K$ such that h_{θ} maps F_0 homeomorphically onto $F_{\theta}, \ \theta \in [0, 2\pi[$, when $\theta = 0$ $h_0 = Id_{F_0}$ and $h_{2\pi}$ is the geometrical monodromy (which is defined up to isotopy).

Since ϕ is a locally trivial fibration, then distinct fibers never meet together. This elementary fact implies that for two cycles [x] and [y] in $\mathcal{H}_n(F)$, and for $\theta \in]0, 2\pi[$ we have

$$l_{S^{2n+1}}(i_+(x),y)) = l_{S^{2n+1}}(h_\theta(x),y),$$

where $l_{S^{2n+1}}$ denotes the linking number of chains in S^{2n+1} . Then the Seifert form \mathfrak{A} is defined as follows

$$\begin{aligned} \mathfrak{A}: \mathrm{H}_n(F) \times \mathrm{H}_n(F) & \to & \mathbf{Z} \\ ([x], [y]) & \mapsto & l_{S^{2n+1}} \left(h_{\pi}(x), y \right) \end{aligned}$$

For ξ in $H_n(F, K)$ and ζ in $H_n(F)$ we denote by $\langle \xi, \zeta \rangle$ the intersection number which is defined by

$$\langle \xi, \zeta \rangle = \mathfrak{P}(\xi)(\zeta)$$

where $\tilde{\mathfrak{P}} : H_n(F, K) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{Z}}(H_n(F), \mathbf{Z})$ is the composite of the Poincaré-Lefschetz duality isomorphism and the universal coefficient isomorphism.

With the last definition of the Seifert form we easily get the following proposition

Proposition 4.5. Let $(\alpha, \beta) \in H_n(F) \times H_n(F, K)$ then $\mathfrak{A}(\alpha, \mathcal{V}(\beta)) = \langle \beta, \alpha \rangle$. *Proof.* Start with $([a], [b]) \in H_n(F) \times H_n(F, K)$ then the following equalities hold

$$\begin{aligned} \mathfrak{A}([a], \mathcal{V}([b])) &= l_{S^{2n+1}}(h_{\pi}(a), b - \mathfrak{m}(b)) \\ &= l_{S^{2n+1}}(h_{\pi}(a), \partial(\cup_{\theta \in [0, 2\pi]} h_{\theta}(b))) \\ &= I_{D^{2n+2}}(h_{\pi}(a), \cup_{\theta \in [0, 2\pi]} h_{\theta}(b)) \\ &= < [h_{\pi}(b)], [h_{\pi}(a)] >_{F_{\pi}} \\ &= < [b], [a] > \end{aligned}$$

As a corollary of the previous proposition we have

Proposition 4.6. The Seifert form associated with a fibered knot is unimodular.

Proof. Let K be a fibered knot with fiber F. As before \mathfrak{A} and \mathcal{V} are the Seifert form and the variation map associated with F. We first fix a basis $\mathcal{B} = (\beta_i)_{i \in \mathcal{I}}$ for $\mathrm{H}_n(F)$, and then we take the basis $\mathcal{B}^* = (\beta_i^*)_{i \in \mathcal{I}}$ for $\mathrm{H}_n(F, K)$ which is the dual basis of \mathcal{B} . By dual we mean that for all (i, j) in \mathcal{I}^2 we have

$$\tilde{\mathfrak{P}}(\beta_i)(\beta_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if i = j and $\delta_{ij} = 0$ if $i \neq j$. With these choices, when β is a relative chain in $H_n(F, K)$ and α is a *n*-chain in $H_n(F)$ have the two column vectors b and a respectively as matricial representations, then $\langle \beta, \alpha \rangle = \tilde{\mathfrak{P}}(\beta)(\alpha) = {}^{t}ba$.

Let us denote by A the matrix of the Seifert form \mathfrak{A} , and by V the matrix of the variation map \mathcal{V} relatively to the basis \mathcal{B} and \mathcal{B}^* . According to Proposition 4.5, for all (β, α) in $\mathrm{H}_n(F, K) \times \mathrm{H}_n(F)$ we have

$$\mathfrak{A}(\alpha, \mathcal{V}(\beta)) = <\beta, \alpha > .$$

If we denote by a and b the two column vectors which represent α and β relatively to the basis \mathcal{B} and \mathcal{B}^* then the previous equality becomes ${}^ta AV b = {}^tb a$. Hence we have ${}^t\!\!b({}^t\!V {}^t\!A)a = {}^t\!\!ba$. Since this equality holds for any column vectors aand b we have $A = V^{-1}$. We already proved that \mathcal{V} is an isomorphism so det $V = \det A = \pm 1$ and \mathfrak{A} is unimodular. \Box

Proposition 4.7. Let K be a simple fibered (2n - 1)-knot with fiber F. Set A be the matrix of the Seifert form, S the matrix of the intersection form and H be the matrix of the monodromy associated with F. If I if the matrix of the identity, then the following holds

$$S = A(I - H), \quad H = (-1)^{n+1} {}^{t}A A^{-1}.$$

Proof. Let α and β be two *n*-cycles in $H_n(F)$, set $\alpha = [x]$ and $\beta = [y]$ for two *n*-chains x and y.

Set $Z = \bigcup_{\theta=0}^{2\pi} h_{\theta}(y)$ the (n+1)-chain in S^{2n+1} with boundary $\partial Z = y - \mathfrak{m}(y)$.

And set A and B two (n + 1)-chains in S^{2n+1} such that $\partial A = y$ and $\partial B = \mathfrak{m}(y)$. Then Z + B - A is a (n + 1)-chain without boundary wich represent the homology class of a (n + 1)-cycle in S^{2n+1} . Hence the intersection number between Z + B - A and $h_{\pi}(x)$ in S^{2n+1} must be zero.

If we denote by $\langle X, Y \rangle$ the intersection number between two chains in S^{2n+1} , then the following equalities hold

$$\begin{aligned} \left\langle h_{\pi}(x), Z + B - A \right\rangle &= \left\langle h_{\pi}(x), Z \right\rangle + \left\langle h_{\pi}(x), B \right\rangle - \left\langle h_{\pi}(x), A \right\rangle \\ &= I_{F_{\pi}}(h_{\pi}(x), h_{\pi}(y)) + \left\langle h_{\pi}(x), \mathfrak{m}(y) \right\rangle + l \left\langle h_{\pi}(x), y \right\rangle \\ &= \mathfrak{S}(x, y) + \mathfrak{A}(x, h(y)) - \mathfrak{A}(x, y) \\ &= \mathfrak{S}(x, y) + \mathfrak{A}(x, h(y) - y). \end{aligned}$$

The nullity of $\langle h_{\pi}(x), Z + B - A \rangle$ gives S = A(I - H). Since $S = A + (-1)^n {}^t A$ and A is invertible, then we have

$$I - H = A^{-1}(A + (-1)^{n} A) = I + (-1)^{n} A^{-1} A.$$

Finally $H = (-1)^{n+1} {}^{t}A A^{-1}$ as desired.

With the unimodularity of Seifert forms associated with fibers of fibered knots Durfee and Kato independently generalized the work of Levine.

Theorem 4.8 ([31],[58]). Let $n \geq 3$. There is a one-to-one correspondence of isotopy classes of simple fibered knots in S^{2n+1} and equivalence classes of integral unimodular bilinear forms. The correspondence associates to each knot its Seifert form.

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Proof. Let K_0 and K_1 be two simple fibered (2n-1)-knots which are isotopic. Using the same proof that we give for spherical knots, we can see that the Seifert forms associated with the fibers of K_0 and K_1 are S-equivalent. But Sequivalence of unimodular forms reduces to congruence of matrices, hence the associated Seifert forms are equivalent.

Conversely, given an integral matrix A, to realize A as the matrix of an integral bilinear form \mathfrak{A} , we can construct a simple knot with Seifert form \mathfrak{A} . This is done as Kervaire did in [65] for spherical knots¹, by gluing *n*-handles on a the boundary of a (2n-1)-disk. The knot is the boundary of this handlebody, and the handlebody itself is a Seifert manifold F for this knot K. The core of the handles are the generators of the n^{th} homology group $H_n(F)$, so we glued such that the linking numbers between the handles correspond to the coefficients of the matrix A. By construction the knot K is simple, we will prove that K is fibered using the h-cobordism Theorem.

First let us fix some notations. Set X be the complementary in S^{2n+1} of an open tubular neighbourhood of K in S^{2n+1} , and let $W = F \cap X$. Set N a normal tubular neighbourhood of W in X, hence if M is a normal tubular neighbourhood ot F in S^{2n+1} then $N = M \cap X$. Moreover, it makes sense to follow notations of Definition 1.6 and set $N \cong W_+ \times [0,1]$ where $W_+ = F_+ \cap X$ correspond to W pushed in the positive normal direction in S^{2n+1} .

Set $Y = X \setminus N$, then the exact long homology sequence of the pair (Y, W_+) gives

$$\dots \to \operatorname{H}_k(W_+) \to \operatorname{H}_k(Y) \to \operatorname{H}_k(Y, W_+) \to \dots$$
 (4.2)

Moreover the manifold W_+ is (n-1)-connected; and because of Alexander duality $H_k(Y) \cong H^{2n-k}(F)$, so $H_k(Y) = 0$ for $k \ge n+1$. Hence the relative homology groups $H_k(Y, W_+)$ vanishe for $k \ge n+1$, by Poincare-Lefshetz duality we also have $H_k(Y, W_+)$ vanishe for $k \le n-1$. Then the long exact sequence (4.2) reduces to

$$0 \to \mathrm{H}_n(W_+) \to \mathrm{H}_n(Y) \to \mathrm{H}_n(Y, W_+) \to 0.$$

But since the matrix A is unimodular, then the inclusion $W_+ \hookrightarrow Y$ induces the isomorphism $\operatorname{H}_n(W_+) \xrightarrow{\cong} \operatorname{H}_n(Y)$. Remark that the injectivity also comes from the fact that the image of a non trivial homology class x of $\operatorname{H}_n(W_+)$ in $\operatorname{H}_n(Y)$ can't be null homologous otherwise A will be degenerated because A(x, y) = 0 for any y in $\operatorname{H}_n(F)$.

The surjectivity is a consequence of the unimodularity of A. To see that, first remark that according to Alexander duality the free **Z**-modules $H_n(Y)$ and $H_n(W_+)$ have same rank. Second, since the inclusion is injective, then if it is not surjective there exists an indivisible element, namely x, in $H_n(W_+)$ wich is homologous to an element αy of $H_n(Y)$ where $\alpha \neq -1, 0, 1$ and y lies in $H_n(Y)$. But this implies that α divides det A, which contradicts the unimodularity of A.

Finally we get $H_n(Y, W_+) = 0$ and Y is homeomorphic to $W_+ \times [0, 1]$ according to the *h*-cobordism Theorem.

Now it not difficult to see that the knot K constructed is fibered. This comes from the decomposition of X in two pieces, namely $N \cong W \times [0, 1]$ and

 $^{^1{\}rm The}$ same technic works since Kervaire additional conditions were only used to insure that the knot is spherical.

 $Y \cong W_+ \times [0,1]$. The identification of N and Y along their boundaries induces an homeomorphism $\mathfrak{m} : W \to W$ such that X is homeomorphic to the quotient $W \times [0,1]$ by the equivalence relation $(x,0) \sim (\mathfrak{m}(x),1)$. Since all these maps extend to $S^{2n+1} \setminus K$, then the knot K is simple fibered. \Box

Remark 4.9. For spherical simple (2n - 1)-knots, we have another algebraic invariant, called the *Blanchfield pairing*, which is closely related to the Seifert form (see [62, 130]). In fact, it is known that giving an S-equivalence class of a Seifert form is equivalent to giving an isomorphism class of a Blanchfield pairing.

We just saw that fibered knots have unimodular Seifert forms, moreover fibered knots have a nice topological behaviour as stated in the following proposition.

Proposition 4.10. Let $n \ge 1$. Let K be a fiber knot of dimension 2n - 1 and let F be a fiber of the fibration, then we have the following short exact sequence

$$0 \to H_n(K) \to H_n(F) \xrightarrow{S_*} H_n(F, K) \to H_{n-1}(K) \to 0.$$

Proof. Recall that F is a Seifert surface associated with K. Moreover we know that $S^{2n+1} \setminus K$ is homeomorphic to $\overset{\circ}{F} \times [0,1]_{/(x,0)} \sim (\mathfrak{m}(x),1)$ where \mathfrak{m} is the geometrical monodromy. Hence $S^{2n+1} \setminus F$ has the same homotopy type as F. Now by Alexander duality we have

$$\mathrm{H}_{k}(F) \cong \mathrm{H}^{2n-k}(S^{2n+1} \setminus F) \cong \mathrm{H}^{2n-k}(F) \text{ for } k > 0.$$

Moreover by Poincaré duality we have

$$\operatorname{H}_k(F,K) \cong \operatorname{H}^{2n-k}(F),$$

and this implies

$$\mathbf{H}_k(F,K) \cong \mathbf{H}_k(F) \quad \text{for} \quad k > 0. \tag{4.3}$$

Since K is (n-2)-connected, then the long exact sequence

$$\dots \to \operatorname{H}_n(K) \to \operatorname{H}_n(F) \xrightarrow{S_*} \operatorname{H}_n(F, K) \to \dots$$

gives the following short exact sequence

$$0 \to \mathrm{H}_{n+1}(F) \xrightarrow{\alpha} \mathrm{H}_{n+1}(F, K) \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \to$$

$$\operatorname{H}_{n}(F,K) \to \operatorname{H}_{n-1}(K) \to \operatorname{H}_{n-1}(F) \xrightarrow{\rho} \operatorname{H}_{n-1}(F,K) \to 0$$

According to (4.2) the monomorphism α is an isomorphism, and the epimorphism β as well. Finally we get the desired short exact sequence

$$0 \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F, K) \to \mathrm{H}_{n-1}(K) \to 0$$

According to this proposition we see that the topological data about the knot K are coming from the Kernel and the Cokernel of the intersection form of F.

Moreover, as a consequence of the short exact sequence of Proposition 4.10 we see that the middle homology group of the fiber is a free abelian group.

4.1.3 Alexander polynomials of fibered knots

Let K be a (2n - 1)-fibered knot with fiber F. As before, set X be the complementary in S^{2n+1} of an open tubular neighbourhood of K in S^{2n+1} , and let $W = F \cap X$ the intersection of the fiber with X.

Then we take the quotient of $W \times \mathbf{R}$ by the equivalence ralation $(x, \alpha) \sim (\mathfrak{m}^k, \alpha + k)$ for any $k \in \mathbf{Z}$. This quotient is homeomorphic to X and $W \times \mathbf{R}$ is the infinite cyclic covering of X. Let τ be the generator of the Galois group of the covering $W \times \mathbf{R} \to X$, which is the infinite cyclic covering of X. The action of τ is given by the map which maps (x, α) to $(\mathfrak{m}(x), \alpha + 1)$. If τ induces an action, denoted by t on $\mathrm{H}_*(W \times \mathbf{R})$ which acts as the monodromy h acts on $\mathrm{H}_*(W)$.

The homology group $H_n(W \times \mathbf{R})$ is a free abelian group which is finitely generated because it has the homotopy type of a compact CW-complex. The generator of the first elementary ideal of the $\mathbf{Z}[t, t-1]$ -module $H_n(W \times \mathbf{R})$, i.e., the ideal generated by minor of maximal rank, is the caracteristic polynomial of t. Moreover this polynomial is the Alexander polynomial of $H_n(W \times \mathbf{R})$. Since the action of t reduce to the action of h on $H_n(W)$, then we get the folklore Theorem

Theorem 4.11. Let K be a fiber (2n - 1)-knot with fiber F. The Alexander polynomial of $H_n(F \times \mathbf{R})$ is the caracteristic polynomial of the algebraic monodromy $h : H_n(F) \to H_n(F)$.

This result is compatible with the definition the previous Definition of the Alexander polynomial of a (2n - 1)-knot K to be

$$\Delta_K(X) = \det(tA + (-1)^n {}^tA)$$

since when K is a fibered knot, then the Seifert form A is unimodular and the monodromy has $H = (-1)^{n+1} {}^{t}A A^{-1}$ as matrix. This gives

$$\Delta_K(X) = \det(tA + (-1)^n {}^tA) = \det(tId - H).$$

Since the Alexander Polynomial of a fibered knot K is $\Delta_K(X) = \det(X \operatorname{Id} - h)$, then as a consequence of the exact sequence (4.1) the fibered knot K is an integer homological sphere if and only if $\Delta_K(1) = \pm 1$. this is also a consequence of the short exact sequence of Proposition 4.10 since the matrix of the intersection form S_* is equal to $A + (-1)^n A$ and $\Delta_K(1) = \det(A + (-1)^n A) = \det S = \pm 1$ if and only if the knot K is an integral homology sphere.

When K is a fibered knot Δ_K is a characteristic polynomial so its leading coefficient must be 1, and its last coefficient is equal to $\pm \det H$ which ± 1 , so we get the following Proposition.

Proposition 4.12. A necessary condition for a knot to fiber is that the extremal coefficients of the Alexander polynomial should be ± 1 .

4.2 Algebraic knots

As said in the introduction, algebraic knots are one motivation to the study of fibered knots. In this section we will review some classical definition and result about algebraic knots, we refer to [29, 100, 106] for details and proofs.

Let $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic function germ with an isolated singularity at the origin. Recall that for $\varepsilon > 0$ sufficiently small the set $K_f = f^{-1}(0) \cap S_{\varepsilon}^{2n+1}$ is a (2n-1)-dimensional manifold which is naturally oriented, where S_{ε}^{2n+1} is the sphere in \mathbf{C}^{n+1} of radius ε centered at the origin. Furthermore, its (oriented) isotopy class in $S_{\varepsilon}^{2n+1} = S^{2n+1}$ does not depend on the choice of ε , and we call it *the algebraic knot* associated with the isolated singularity of f.

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