Generalizations of Geodesic Curvature and a Theorem of Gauss Concerning Geodesic Triangles.*

By Gilbert Ames Bliss.

Introduction.

Many of the geometrical invariants of a curve on a surface can be defined in terms of the integral which expresses the length of the curve. Their geometrical invariance corresponds analytically to the fact that they remain invariant under a transformation of the parameters in terms of which the equations of the surface are expressed. In an earlier paper † the author has shown that there exists a function analogous to the angle between two given curves in a plane or on a surface, and related to an integral of the form

$$u = \int_{t_0}^{t_1} f(x, y, \tau) \sqrt{x'^2 + y'^2}\, dt \tag{1}$$

as angle on a surface is related to length. The integral (1) is to be thought of as taken along a curve in the form

$$x = x(t), \quad y = y(t), \qquad (t_0 \leqq t \leqq t_1);$$

x' and y' represent the derivatives of x and y with respect to t; and τ is the angle defined by the equations

$$\cos \tau = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin \tau = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

For the length integral on a surface the function f has the special form

$$f = \sqrt{E \cos^2 \tau + 2 F \cos \tau \sin \tau + G \sin^2 \tau}. \tag{2}$$

In the present paper a similar generalization of the notion of the curvature of a curve in the plane or the geodesic curvature of a curve on a surface is exhibited. This generalization is called *extremal curvature* and is explained in § 1. The possession of invariants corresponding to angle and geodesic

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* The results in § 1 of this paper were presented to the American Mathematical Society, April 28, 1906, under the title, "An Invariant of the Calculus of Variations Corresponding to Geodesic Curvature."

† "A Generalization of the Notion of Angle," *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 184.

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curvature suggests at once the possibility of generalizing in a similar manner the formula of Gauss for the sum of the angles of a geodesic triangle on a surface. The situation is unfortunately not as simple as in the case of the surface theory. In § 2 a notion called the *area of a simply closed curve in a field of extremals* is explained, and in § 3 a generalization of Gauss' theorem is developed in which angle, extremal curvature and the area in a field are involved. In § 4 the invariance under point transformation of the quantities introduced in the preceding sections is discussed, and in § 5 the relation of the results of the paper to the usual formulas of surface theory is elucidated.

§ 1.   *Extremal Curvature.*

The geodesic curvature of a curve on a surface may be defined in a number of different ways, to several of which there correspond by generalization invariants of the integral (1). The most convenient definition for the purposes
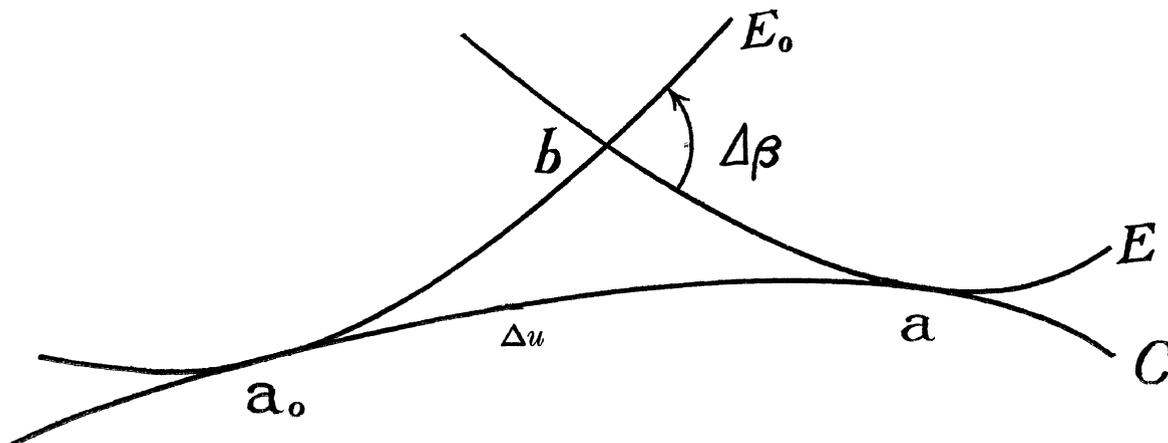
FIG. 1.

of the present paper is the following, which depends upon the notion of angle.* Suppose at a fixed point a_0 of a given curve C the geodesic E_0 tangent to C is drawn. The geodesic tangent E at a movable neighboring point a will in general meet E_0 at a point b. Let the angle between E_0 and E at b be denoted by $\Delta\beta$, and the length of the arc $a_0 a$ by Δu. Then the geodesic curvature of the curve C at the point a_0 is equal to the limit

$$\lim_{\Delta u \doteq 0} \frac{\Delta\beta}{\Delta u}. \tag{3}$$

This definition admits of a ready generalization when the function f in the integral (1) is not restricted to have the special form (2). The geodesics E_0, E

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* See Darboux, *Leçons sur la Théorie Générale des Surfaces*, Vol. III, p. 129.

can be replaced by extremals of the integral (1); the angle $\Delta\beta$ has already been generalized in the paper referred to above, and instead of the length of arc the value of the integral (1) along $a_0\,a$ can be used.

The extremals for the integral (1) are the solutions of the differential equation*

$$T\,(x,y,\tau,\tau_s) = f_x \sin\tau - f_y \cos\tau + f_{x\tau} \cos\tau + f_{y\tau} \sin\tau + (f + f_{\tau\tau})\,\tau_s = 0, \quad (4)$$

which is of the second order. Through each point of the curve C there passes one of these solutions in the direction of the positive tangent to C. The equations of the one-parameter family so determined can be found in the form

$$x = \phi\,(s,\alpha), \quad y = \psi\,(s,\alpha), \tag{5}$$

where s is the length of arc of E measured from the point of tangency with C, and α is the parameter of the family. Then the curve C has the equations

$$x = \phi\,(0,\alpha), \quad y = \psi\,(0,\alpha), \tag{6}$$

with α as the variable parameter. If the function $f\,(x,y,\tau)$ is of class C''' and the expression $f + f_{\tau\tau}$ different from zero in a neighborhood of the values of (x,y,τ) on the curve C, and if the curve C is of class C'', then from the known properties of solutions of the differential equation (4) it follows that the functions $\phi, \psi, \phi_s, \psi_s$ are of class C' for all values of α defining points on the arc C and for $|s| \leqq \delta$, if δ is properly chosen. From the results of a previous paper by the writer † it also follows, when C is not an extremal, that for negative values of s sufficiently near to zero the arcs of the extremals, such as $b\,a$ in the figure, touch the curve C only at a and simply cover a region of the plane which has the curve C as a part of its boundary. An arc of the fixed extremal E_0 extending from a_0 in the direction $s > 0$ can be taken short enough so that it also lies in this region, and consequently so that through each of its points there passes one and but one of the extremals (5). In other words, the equations

$$\phi\,(s,\alpha) = \phi\,(s_0,\alpha_0), \quad \psi\,(s,\alpha) = \psi\,(s_0,\alpha_0), \tag{7}$$

where α_0 is the constant value of α, and s_0 the length of arc for the extremal E_0, have single-valued solutions for $s, \alpha\,(s < 0)$ in terms of s_0. By implicit function theory these solutions $s\,(s_0), \alpha\,(s_0)$ will be of class C' in an interval $0 < s_0 \leqq \varepsilon$ with the value $s_0 = 0$ omitted. But at $s_0 = 0$ they are continuous and approach the values 0 and α_0 respectively.

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* See Bliss, *loc. cit.*, p. 188, where the most general solutions of (5) are given and their properties stated. With the help of these properties the existence and character of the family (5) can be readily derived.

† "Sufficient Conditions for a Minimum with Respect to One-Sided Variations," *Transactions of the American Mathematical Society*, Vol. V (1904), p. 480. See also Bolza, "An Existence Proof for a Field of Extremals Tangent to a Given Curve," the same *Transactions*, Vol. VIII (1907), p. 399.

With the help of these preliminaries the value of the limit (3) can be calculated. The generalized angle $\Delta\beta$ is given by the equation

$$\Delta\beta = \int_\tau^{\tau_0} \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^2}\, d\tau, \tag{8}$$

where, in the functions $f(x, y, \tau)$ and $\bar{f} = f(x, y, \bar{\tau})$, the values x, y are the coordinates of the point b, τ is the variable of integration, and $\bar{\tau}$ is the direction transversal to τ defined by the equations

$$\cos\bar{\tau} = \frac{-f\sin\tau - f_\tau\cos\tau}{\sqrt{f^2 + f_\tau^2}}, \quad \sin\bar{\tau} = \frac{f\cos\tau - f\sin\tau}{\sqrt{f^2 + f_\tau^2}}. \tag{9}$$

The limits τ and τ_0 are the values of τ on the curves E and E_0, respectively, at the point b. The denominator of the fraction (3) of which the limiting value is desired, is

$$\Delta u = \int_{a_0}^a f\, d\sigma,$$

where the arguments of the function f are values of x, y, τ on the curve C taken from equations (5), and σ is the length of arc along C measured from a_0.

From the mean-value theorem for a definite integral it follows that

$$\Delta\beta = \left[\frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^2}\right]_{\tau=\tau'}(\tau - \tau_0), \quad \Delta u = \left[f\sqrt{\phi_\alpha^2 + \psi_\alpha^2}\right]_{\alpha=\alpha'}(\alpha - \alpha_0), \tag{10}$$

where τ' and α' are properly chosen values between τ and τ_0, α_0 and α, respectively. As s_0 approaches zero, the expressions in brackets approach the values they have when x and y are the coordinates of the point a_0, and τ the direction of the curve C at a_0. By Taylor's formula with a remainder term,

$$\tau - \tau_0 = \left[\frac{\partial\tau}{\partial s}\right](s - s_0) + \left[\frac{\partial\tau}{\partial\alpha}\right](\alpha - \alpha_0),$$

where the arguments s', α'' of the expressions in brackets are supposed to be properly chosen mean values between s_0 and s, α_0 and α, respectively. In order to find the limit (3), it is necessary, therefore, to evaluate

$$\lim_{s_0=0}\left\{\left[\frac{\partial\tau}{\partial s}\right]\frac{s - s_0}{\alpha - \alpha_0} + \left[\frac{\partial\tau}{\partial\alpha}\right]\right\}.$$

But the limit of $\left[\dfrac{\partial\tau}{\partial s}\right]$ is the curvature of the extremal E_0 at the point a_0. From the equations

$$\cos\tau = \phi_s(s, \alpha), \quad \sin\tau = \psi_s(s, \alpha), \tag{11}$$

$$\frac{\partial\tau}{\partial\alpha} = \frac{\phi_\alpha\psi_{\alpha\alpha} - \phi_{\alpha\alpha}\psi_\alpha}{\phi_\alpha^2 + \psi_\alpha^2},$$

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* *Transactions*, Vol. VII (1906), p. 192.

it follows that the limit of $\left[\dfrac{\partial \tau}{\partial \alpha}\right]$ is the curvature $\dfrac{d\tau}{d\sigma}$ of the curve C at the point a_0 multiplied by a factor $\sqrt{\phi_a^2 + \psi_a^2}$.

The limit of the fraction $\dfrac{s - s_0}{\alpha - \alpha_0}$ can be evaluated by differentiating numerator and denominator for s_0 and taking the limit of the quotient of the two derivatives. From equations (7),

$$\phi_s \frac{ds}{ds_0} + \phi_a \frac{d\alpha}{ds_0} = \phi_{s_0}(s_0, \alpha_0), \quad \psi_s \frac{ds}{ds_0} + \psi_a \frac{d\alpha}{ds_0} = \psi_s(s_0, \alpha_0). \qquad (12)$$

By multiplying by ϕ_s and ψ_s, respectively, and adding, it follows that

$$\frac{\dfrac{ds}{ds_0} - 1}{\dfrac{d\alpha}{ds_0}} = (\phi_s \psi_a - \phi_a \psi_s) \frac{\phi_s \phi_{s_0} + \psi_s \psi_{s_0} - 1}{\phi_s \psi_{s_0} - \phi_{s_0} \psi_s} - (\phi_s \phi_a + \psi_s \psi_a),$$

since from equations (12)

$$\frac{d\alpha}{ds_0} = \frac{\phi_s \psi_{s_0} - \phi_{s_0} \psi_s}{\phi_s \psi_a - \phi_a \psi_s},$$

and since $\phi_s^2 + \psi_s^2 = 1$. With the help of equations (11),

$$\phi_s \phi_{s_0} + \psi_s \psi_{s_0} = \cos(\tau - \tau_0), \quad \phi \psi_{s_0} - \phi_{s_0} \psi_s = \sin(\tau - \tau_0),$$

and since

$$\lim \frac{\cos(\tau - \tau_0) - 1}{\sin(\tau - \tau_0)} = 0,$$

it follows that

$$\lim_{s_0 \doteq 0} \frac{s - s_0}{\alpha - \alpha_0} = \lim_{s_0 \doteq 0} \frac{\dfrac{ds}{ds_0} - 1}{\dfrac{d\alpha}{ds_0}} = -\left[\phi_s \phi_a + \psi_s \psi_a\right]_{\substack{s_0 = 0 \\ \alpha = \alpha_0}}.$$

The curves E_0 and C are tangent at a_0, and consequently, for $s = 0$, $\alpha = \alpha_0$,

$$\phi_s = \frac{\phi_a}{\sqrt{\phi_a^2 + \psi_a^2}}, \quad \psi_s = \frac{\psi_a}{\sqrt{\phi_a^2 + \psi_a^2}},$$

so that

$$\lim_{s_0 = 0} \frac{s - s_0}{\alpha - \alpha_0} = -\left[\sqrt{\phi_a^2 + \psi_a^2}\right]_{\substack{s = 0 \\ \alpha = \alpha_0}}. \qquad (13)$$

By dividing the expressions (10) and using the limit (13),

$$\lim_{\Delta u \doteq 0} \frac{\Delta \beta}{\Delta u} = \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^3}\left(\frac{d\tau}{d\sigma} - \frac{d\tau}{ds}\right),$$

where the arguments x, y, τ in f and its derivatives are for the curve C at the point a_0, $\dfrac{d\tau}{d\sigma}$ is the curvature of C at a_0, and $\dfrac{d\tau}{ds}$ the curvature of E_0 at the point a_0. From (4) the value of $\dfrac{d\tau}{ds}$ can be found and substituted, giving

$$\frac{d\tau}{d\sigma} - \frac{d\tau}{ds} = \frac{T\left(x,\ y,\ \tau,\ \dfrac{d\tau}{d\sigma}\right)}{f + f_{\tau\tau}}.$$

The extremal curvature of a curve C at a point a_0 (see Fig. 1) is defined as a limit

$$\frac{1}{\rho} = \lim_{\Delta u \doteq 0} \frac{\Delta\beta}{\Delta u}.$$

Here $\Delta\beta$ is the generalized angle (8) between two extremals, E_0 and E, at their point of intersection b; E_0 is supposed to be fixed and tangent to C at a point a_0; while E is a movable extremal tangent to C at a. The quantity Δu is the value of the integral (1) taken along the curve C from a_0 to a.

According to this definition the value of $\dfrac{1}{\rho}$ in terms of the function f and its derivatives turns out to be

$$\frac{1}{\rho} = \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^3(f + f_{\tau\tau})}\, T\,(x, y, \tau, \tau_s), \tag{14}$$

where the arguments of $f, f_\tau, f_{\tau\tau}$ are the coordinates (x, y) and the direction angle τ of the curve C at a_0, while in $\bar{f} = f(x, y, \bar{\tau})$ the angle $\bar{\tau}$ is the angle transversal to τ defined by the equations (9). The expression $T(x, y, \tau, \tau_s)^*$ is the first member of the Euler equation (4) taken at a point a_0 of the curve C.

§ 2. *Area in a Field.*

If a region R on any surface is simply covered by a one-parameter family of curves, the surface area of the region can be calculated by finding the set of curves orthogonal to the original family, and taking the double integral of the product of the elements of length along the curves of the two systems over the entire region. The result is the well-known integral

$$\iint \sqrt{EG - F^2}\, dx\, dy, \tag{15}$$

which is independent of the systems of curves used in the calculation because E, F and G are functions only of the parameters x and y of the surface.

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* Here s is used instead of σ for the length of arc along C.

Thought of in the xy-plane the image of the region R is simply covered by two families of curves one of which is transversal to the other in the sense of the calculus of variations, transversality in the plane being equivalent to orthogonality on the surface.

In a similar manner, for any field of curves a double integral related to the integral (1) can be derived defining what will be called *area in the field*. The result is not independent of the family of curves used in its derivation, for, unlike the integrand of the integral (15), the new integrand contains not only x and y, but the function $\tau(x, y)$ defining the directions of the curves of the family. The area with respect to the integral (1) is dependent, therefore, not only upon the form of the region but upon the field of curves which covers the region.

Let the family of curves E have equations of the form (5). If τ is the direction angle at a point (x, y) of a curve E of the family, then a curve cutting E transversally at (x, y) must have a direction angle $\bar{\tau}$ which satisfies the equations (9). The parameter s can be determined as a function $s(\alpha)$ of α, so that, at any point of the curve

$$x = \phi(s(\alpha), \alpha), \quad y = \psi(s(\alpha), \alpha), \tag{16}$$

the direction angle has the value $\bar{\tau}$ transversal to τ. For, on account of the properties of ϕ, ψ and f, and with the further assumption that f does not vanish in the field, the equations

$$\left.\begin{aligned}
\phi_s s_a + \phi_a &= k(f \sin \tau + f_\tau \cos \tau), \\
\psi_s s_a + \psi_a &= k(-f \cos \tau + f_\tau \sin \tau),
\end{aligned}\right\} \tag{17}$$

are solvable for s_a and k as functions of class C' in s and continuous in α, the arguments x, y and τ being here supposed replaced by their values in terms of s and α in the field. From the existence theorems on differential equations it results from this that through each point of the field there passes one and but one transversal curve (16).

The integrand of the integral (1) taken along a transversal (16) has the form

$$\bar{f}\sqrt{(\phi_s s_a + \phi_a)^2 + (\psi_s s_a + \psi_a)^2}\, d\alpha,$$

where the arguments of $\bar{f}$ are ϕ, ψ and $\bar{\tau}$ for the curve (16). From (17) this expression becomes

$$\bar{f}\sqrt{f^2 + f_\tau^2}\,\frac{\Delta}{f}\, d\alpha, \tag{18}$$

where Δ is the functional determinant of the equations (5). Similarly, along

a curve E the integrand of (1) has the value $f\,ds$, and the area in the field is therefore

$$\iint \bar{f}\,\sqrt{f^2 + f_\tau^2}\,\Delta\,ds\,d\alpha = \iint \bar{f}\,\sqrt{f^2 + f_\tau^2}\,dx\,dy.$$

The area of any closed curve C in a field of curves E may be defined as a double integral over the region bounded by C, the integrand of which is the product of the element $f\,ds$ of the integral (1) taken along one of the curves E at a point (x, y) of the field, by the value of the same element taken along the curve transversal to the field at (x, y). The double integral representing area in the field is then

$$\iint \bar{f}\,\sqrt{f^2 + f_\tau^2}\,dx\,dy,$$

where, in $f(x, y, \tau)$ and $\bar{f} = f(x, y, \bar{\tau})$, the arguments τ and $\bar{\tau}$ are the functions of x and y defining the directions of the curves of the field and the transversals respectively.

§ 3.   *A Generalization of a Theorem of Gauss Concerning Geodesic Triangles.*

One of the most beautiful theorems of the surface theory is that which states that the excess of the sum of the angles of a geodesic triangle over two right angles, is equal to the area of the image of the geodesic triangle on the Gaussian sphere.*   Darboux shows the truth of the theorem by applying Green's theorem to the second member of the formula

$$\int \left(d\omega - \frac{du}{\rho} \right) = \int (r\,dx + r_1\,dy) \tag{19}$$

integrated over the boundary of a region on the surface.†   Here ω is the angle at any point of the bounding curve between the curve and one of the parameter lines; $1/\rho$ and u are the geodesic curvature and length of the bounding curve, respectively; and r, r_1 are two functions of the parameters x, y of the surface. After an application of Green's theorem the second member becomes the double integral of the curvature multiplied by the element of area on the surface, which represents also the area of the image of the triangle on the Gaussian sphere.   If the boundary of the triangle is composed of arcs of geodesics, the geodesic curvature vanishes identically along it, and the first member of equation (19) becomes the sum of the variations of the angle ω along the sides of the

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* Gauss, "General Investigations of Curved Surfaces," translated by J. C. Morehead and A. M. Hiltebeitel, p. 30.

† Darboux, *Leçons*, Vol. III, p. 113; see also Vol. II, p. 354.

geodesic triangle; that is, the difference between the sum of the angles of the triangle and two right angles.

The formula (19) has an analogue in terms of the invariants which have been discussed in the preceding sections. The generalized angle

$$\omega = \int_0^\tau \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^2}\, d\tau \tag{20}$$

between the direction $\tau = 0$ and the tangent to a curve C, is related to the extremal curvature (14) of the curve C by the formula

$$\frac{d\omega}{ds} - \frac{1}{\rho}\frac{du}{ds} = g\,(x, y, \tau), \tag{21}$$

where g has the value

$$g\,(x, y, \tau) = \omega_x \cos\tau + \omega_y \sin\tau - \frac{\omega_\tau}{f + f_{\tau\tau}}\,(f_x \sin\tau - f_y \cos\tau$$
$$+ f_{x\tau}\cos\tau + f_{y\tau}\sin\tau), \tag{22}$$

as is readily seen from the formulas (14) and (20). The last expression is unfortunately not always linear in $\cos\tau$ and $\sin\tau$,* and the sum of the variations of the angle ω along the sides of a triangle whose sides are extremals is therefore not always expressible at once as a double integral over the interior of the triangle. Darboux has remarked, however, for the analogous situation in the surface theory,† that for any triangle whose sides are solutions of the differential equation of the second order

$$\omega_x \cos\tau + \omega_y \sin\tau + \omega_\tau\,\tau_s = M\cos\tau + N\sin\tau, \tag{23}$$

where M and N are arbitrarily chosen functions of x and y, the sum of the variations of the angle on the sides of the triangle is expressible at once by means of Green's theorem as a double integral. For the solutions of the equation (23) the following statements are true:

If a solution of the equation (23) *is an extremal, it must also satisfy the equation*

$$g\,(x, y, \tau) = M\cos\tau + N\sin\tau, \tag{24}$$

and conversely. Further, if the totality of solutions of equation (23) *is identical with the totality of extremals, then the function g must be linear in* $\cos\tau$ *and* $\sin\tau$ *with coefficients functions of x and y, and the functions M and N must be so chosen that equation* (24) *is an identity in* x, y, τ.

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* This is shown by the example $f = x\tau$, or $f = x\cos^2\tau$.

† Darboux, "Sur une série des lignes analogues aux lignes géodésiques," *Annales de l'École Normale,* Vol. VII (1870); see also his *Leçons,* Vol. III, p. 137.

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Since the extremals are the curves along which the curvature $1/\rho$ vanishes, the first part of the theorem is evident from (21) and (23). If f is different from zero for all values of τ, the same will be true of ω_τ, and the equation (23) will have a solution through an arbitrarily chosen element (x, y, τ). Hence, when the totality of extremals is the same as the totality of solutions of (23), the equation (24) must be an identity in x, y, τ. This is the case which occurs in the surface theory when the function f has the form (2), as will be shown later. It is also possible to have an identity of the form (24) for other values of f, for example when f does not contain x and y, but it is difficult to determine the most general function f for which such an identity holds.

If the equation (23) *has among its solutions a one-parameter family of extremals forming a field, the functions M and N are expressible in the form*

$$M = g' \cos\tau' - g'_\tau \sin\tau' - H \sin\tau', \quad N = g' \sin\tau' + g'_\tau \cos\tau' + H \cos\tau'. \quad (25)$$

Here H is a suitably chosen function of x and y, and the arguments of g' and g'_\tau are x, y and the function $\tau'(x, y)$ which represents the angle at the point (x, y) between the x-axis and the tangent to an extremal of the field.

For the equation (24) must be an identity in the field when $\tau'(x, y)$ is substituted for τ, and the expressions (25) with $H = 0$ are solutions of this equation. It follows readily that any functions M and N for which the equation is satisfied are expressible in the form (25). When H is identically zero, the following theorem holds:

The sum of the variations of the angle ω along the sides of a triangle Δ whose sides are solutions of the equation

$$\frac{d\omega}{ds} = (g' \cos\tau' - g'_\tau \sin\tau') \cos\tau + (g' \sin\tau' + g'_\tau \cos\tau') \sin\tau, \quad (26)$$

where $\tau'(x, y)$ is the angle function for an arbitrarily chosen field of curves, is expressible in the form

$$\iint_\Delta T(g')\, dx\, dy.$$

The expression in the second member of (26) *is exactly $g(x, y, \tau)$ when g is linear in $\cos\tau$ and $\sin\tau$ with coefficients functions of x and y.*

For, if both sides of equation (26) are integrated around the boundary of Δ, the result on the left is the sum of the variations of the angle ω on the sides of the triangle, while on the right Green's theorem gives

$$\iint_\Delta \left\{ \frac{\partial}{\partial x}(g' \sin\tau' + g'_\tau \cos\tau') - \frac{\partial}{\partial y}(g' \cos\tau' - g'_\tau \sin\tau') \right\} dx\, dy,$$

and it is readily seen that the expression under the integral sign is exactly $T(g')$.

If the angle function $\tau(x, y)$ (used for convenience instead of the $\tau'(x, y)$ in the last theorem) *belongs to a field of extremals, the value of* $T(g)$ *is*

$$T(g) = \left(K + \frac{1}{V} \frac{d^2 V}{d u^2} \right) \bar{f} \sqrt{f^2 + f_\tau^2}, \tag{27}$$

where K and V are given by formulas (29) *and* (30) *below.*

The function K is an invariant associated with Jacobi's equation which reduces to the curvature of a surface when f is the element of length on a surface, in which case also the value of V is unity. It has been shown in the preceding section that the expression $\bar{f} \sqrt{f^2 + f_\tau^2}\, dx\, dy$ may be regarded as an element of area in the field. For the case of the surface theory it reduces to the element of area on the surface.

When the function f involves the curvature τ_s as well as the variables x, y, τ, the value of $T(f)$ will be defined to be

$$T(f) = f_x \sin \tau - f_y \cos \tau + \tau_s (f - \tau_s f_{\tau_s}) + \frac{d}{d s} \left(f_\tau - \frac{d}{d s} f_{\tau_s} \right).^*$$

The operator T has the following properties for arbitrary functions f and g:

$$\left. \begin{aligned} T \left[\frac{d}{d s} f(x, y, \tau) \right] &\equiv 0, \quad T(f + g) = T(f) + T(g), \\ T(fg) = f\, T(g) + g\, T(f) &- f g \tau_s \\ + \left(f_\tau - \frac{d}{d s} f_{\tau_s} \right) \frac{dg}{ds} &+ \left(g_\tau - \frac{d}{d s} g_{\tau_s} \right) \frac{df}{ds} - \frac{d}{ds} \left(f_{\tau_s} \frac{dg}{ds} + g_{\tau_s} \frac{df}{ds} \right). \end{aligned} \right\} \tag{28}$$

The relation (21) may be written

$$g(x, y, \tau) = \frac{d \omega}{d s} - W\, T(f),$$

where

$$W = \frac{\omega_\tau}{f_1} = \frac{\bar{f} \sqrt{f^2 + f_\tau^2}}{f^2 f_1}.$$

On account of the relations (28) it follows, for the function g just discussed, that

$$T(g) = - W\, T[T(f)] + \frac{d}{d s} \left(f_1 \frac{d W}{d s} \right) - [T(W) - W \tau_s]\, T(f) - W_\tau \frac{d T(f)}{d s}.$$

In a field of extremals of the function f the last two terms vanish, while

$$T[T(f)] = \frac{\partial T(f)}{\partial x} \sin \tau - \frac{\partial T(f)}{\partial y} \cos \tau - f_1 \tau_s^2 = f_2.^\dagger$$

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* See Radon, "Über das Minimum des Integrals $\int F(x, y, \theta, \kappa)\, d s$," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften zu Wien*, Vol. CXIX (1910), Abtheilung II a, p. 1257.

† See Bliss, "A New Form of the Simplest Problem of the Calculus of Variations," *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 405.

Let K and V be defined by the formulas *

$$K = -\frac{f_2}{f_1 f^2} - \frac{1}{\sqrt{f f_1}} \frac{d^2}{d u^2} \sqrt{f f_1}, \qquad (29)$$

$$V = \sqrt{\pm f f_1}\, W = \frac{\overline{f}\,\sqrt{f^2 + f_\tau^2}}{f\,\sqrt{\pm f f_1}}, \qquad (30)$$

in which the sign under the radical may be so chosen that the radical is real Furthermore, let the differentiations with respect to s be transformed into differentiations with respect to u, where

$$\frac{d u}{d s} = f(x, y, \tau).$$

Then, after some calculation, it is found that the value of $T(g)$ is exactly that given in the formula (27). For convenience in notation the prime has been dropped throughout these calculations.

In any field of extremals there exists a two-parameter family of curves, defined by the equation (26), to which the extremals of the field themselves belong, and which have the property that in any polygon whose sides consist of these curves the sum of the variations of the generalized angle ω along the sides of the polygon is equal to the double integral

$$\iint \left(K + \frac{1}{V} \frac{d^2 V}{d u^2} \right) d A$$

taken over the interior of the polygon. Here K is a known invariant (29) connected with Jacobi's equation, dA is the element of area in the field, and the function V is given by the equation (30). The derivative $d^2 V/d u^2$ is taken with respect to the variable

$$u = \int f(x, y, \tau)\, d s,$$

in which the integral is taken along an extremal of the field.

§ 5.   *Invariantive Properties.*

The expressions which enter into the results of the preceding sections are either relative or absolute invariants under point transformations. Let the variables x, y, τ be transformed by the transformation

$$x = X(\xi, \eta) \qquad y = Y(\xi, \eta) \qquad (31)$$

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* Compare the formula given for K with that given by Underhill for K_0 in his paper, "Invariants of the Function $F(x, y, x', y')$ in the Calculus of Variations," *Transactions of the American Mathematical Society*, Vol, IX (1908), p. 334, formula (35). For the relation of K_0 to the second variation, see p. 336. The value of the function K in the text above is the negative of Underhill's K_0.

into a new set ξ, η, θ. The relation between τ and θ is determined by the equations

$$R \cos \tau = X_\xi \cos \theta + X_\eta \sin \theta, \qquad R \sin \tau = Y_\xi \cos \theta + Y_\eta \sin \theta, \qquad (32)$$

where

$$R = \sqrt{(X_\xi \cos \theta + X_\eta \sin \theta)^2 + Y_\xi \cos \theta + Y_\eta \sin \theta)^2}$$
$$= \cos \tau \, (X_\xi \cos \theta + X_\eta \sin \theta) + \sin \tau \, (Y_\xi \cos \theta + Y_\eta \sin \theta). \qquad (33)$$

By differentiating (32), it is found that

$$\left.\begin{aligned}
R_\theta &= \cos \tau \, (-X_\xi \sin \theta + X_\eta \cos \theta) + \sin \tau \, (-Y_\xi \sin \theta + Y_\eta \cos \theta), \\
R\,\tau_\theta &= -\sin \tau \, (-X_\xi \sin \theta + X_\eta \cos \theta) + \cos \tau \, (-Y_\xi \sin \theta + Y_\eta \cos \theta) = \frac{D}{R}, \\
R_\xi &= \cos \tau \, \frac{d X_\xi}{d \sigma} + \sin \tau \, \frac{d Y_\xi}{d \sigma}, \\
R\,\tau_\xi &= -\sin \tau \, \frac{d X_\xi}{d \sigma} + \cos \tau \, \frac{d Y_\xi}{d \sigma},
\end{aligned}\right\} \quad (34)$$

with similar expressions for the derivatives with respect to η. The symbol D stands for the functional determinant of the transformation (31), and σ is the length of arc along a curve in the $\xi\eta$-plane.

The integral (1) becomes

$$\int h\,(x, y, \theta) \sqrt{\xi'^2 + \eta'^2} \, d t, \qquad (35)$$

where

$$\left.\begin{aligned}
h &= f R \\
&= (f\cos\tau - f_\tau\sin\tau)(X_\xi\cos\theta + X_\eta\sin\theta) + (f\sin\tau + f_\tau\cos\tau)(Y_\xi\cos\theta + Y_\eta\sin\theta), \\
h_\theta &= f R_\theta + f_\tau R\,\tau_\theta \\
&= (f\cos\tau - f_\tau\sin\tau)(-X_\xi\sin\theta + X_\eta\cos\theta) + (f\sin\tau + f_\tau\cos\tau)(-Y_\xi\sin\theta + Y_\eta\cos\theta), \\
h_\xi &= R\,(f_x X_\xi + f_y Y_\xi) f R_\xi + f_\tau R\,\tau_\xi \\
&= R\,(f_x X_\xi + f_y Y_\xi) + (f\cos\tau - f_\tau\sin\tau)\frac{d X_\xi}{d\sigma} + (f\sin\tau + f_\tau\cos\tau)\frac{d Y_\xi}{d\sigma},
\end{aligned}\right\} \quad (36)$$

with a similar expression for the derivative of h with respect to η.

It follows readily that

$$\left.\begin{aligned}
h \cos \theta - h_\theta \sin \theta &= X_\xi\,(f\cos\tau - f_\tau\sin\tau) + Y_\xi\,(f\sin\tau + f_\tau\cos\tau), \\
h \sin \theta + h_\theta \cos \theta &= X_\eta\,(f\cos\tau - f_\tau\sin\tau) + Y_\eta\,(f\sin\tau + f_\tau\cos\tau).
\end{aligned}\right\} \quad (37)$$

From equations (9), therefore,

$$\left.\begin{aligned}
X_\xi \cos \bar{\theta} + X_\eta \sin \bar{\theta} &= D \sqrt{\frac{f^2 + f_\tau^2}{h^2 + h_\theta^2}} \, \cos \bar{\tau}, \\
Y_\xi \cos \bar{\theta} + Y_\eta \sin \bar{\theta} &= D \sqrt{\frac{f^2 + f_\tau^2}{h^2 + h_\theta^2}} \, \sin \bar{\tau},
\end{aligned}\right\} \quad (38)$$

showing that *the direction $\bar{\tau}$ transversal to τ at a point in the xy-plane is transformed by equations* (32) *into the direction $\bar{\theta}$ transversal to θ at the corresponding point of the $\xi\eta$-plane.*

The relation between the Euler expressions, $T(f)$ and $T(g)$, for the integrals (1) and (35) can be calculated from the relations (36) and (37). From (31),

$$\frac{ds}{dt} = R\frac{d\sigma}{dt}, \quad \frac{d}{d\sigma} = \frac{d}{dt} \Big/ \frac{d\sigma}{dt} = R\frac{d}{ds}.$$

It follows then that

$$\sin\theta \, T(h) = h_\xi - \frac{d}{d\sigma}(h\cos\theta - h_\theta\sin\theta) = \quad D\sin\theta \, T(f),$$

$$-\cos\theta \, T(h) = h_\eta - \frac{d}{d\sigma}(h\sin\theta + h_\theta\cos\theta) = -D\cos\theta \, T(f).^*$$

Hence, the Euler expression $T(f)$ for the integral (1) *and the corresponding quantity $T(h)$ for* (35) *are related by the formula*

$$T(h) = D \, T(f). \tag{39}$$

From equations (33) and (38),

$$R(\xi, \eta, \bar{\theta}) = D\sqrt{\frac{f^2 + f_\tau^2}{h^2 + h_\theta^2}}, \tag{40}$$

so that, with the help of the second of equations (34) and the first of (36),

$$\omega = \int_\theta^{\theta'} \frac{\bar{h}\sqrt{h^2 + h_\theta^2}}{h^2} \, d\theta = \int_\tau^{\tau'} \frac{f\sqrt{f^2 + f_\tau^2}}{f^2} \, d\tau, \tag{41}$$

where τ' corresponds to θ', and τ to θ, by means of the transformation (32).

The generalized angle with respect to the integral (1) *between two directions τ and τ' at a point of the xy-plane, is equal to the generalized angle with respect to the integral* (35) *between the corresponding directions at the corresponding point of the $\xi\eta$-plane.*

By differentiating equations (37) for θ and making use of (32) and the value of τ_θ from the second of equations (34), it follows that

$$h + h_{\theta\theta} = \frac{D^2}{R^3}(f + f_{\tau\tau}). \tag{42}$$

From the first of the relations (36), with (40), (42) and (39), it can be shown that *the extremal curvature*

$$\frac{1}{\rho} = \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^3(f + f_{\tau\tau})} \, T(f)$$

---

with respect to the integral (1), *is the same as the extremal curvature with respect to the integral* (35) *at corresponding points of curves which are equivalent under the transformation* (31).

Similarly, from the first of the relations (36), with (40),

$$\iint \bar{h} \sqrt{h^2 + h_\theta^2}\, d\xi\, d\eta = \iint \bar{f} \sqrt{f^2 + f_\tau^2}\, D\, d\xi\, d\eta = \iint \bar{f} \sqrt{f^2 + f_\tau^2}\, dx\, dy,$$

so that *the area enclosed by a curve C in a field of curves in the xy-plane, is equal to the area enclosed by the image of C in the ξη-plane taken with respect to the image of the original field defined by the transformation* (31).

It remains to show that the two integrals which occur in the generalization of Gauss' theorem have also invariantive properties. From the behavior of ω, ρ and h it follows that

$$g(h) = R\, g(f),$$

where $g(h)$ and $g(f)$ are the values of the function (22) formed for h and f, respectively. From this it follows also that the integral

$$\int g(x, y, \tau)\, ds$$

is invariant under the transformation (31). Equations (37) show that the expression

$$(g\cos\tau - g_\tau \sin\tau)\, dx + (g\sin\tau + g_\tau \cos\tau)\, dy$$

is equal to the expression formed in a similar way for the integral (35). The Euler expression $T(g)$ is multiplied by the factor D when the transformation (31) is applied. Consequently, the function $T(g)/(\bar{f}\sqrt{f^2 + f_\tau^2})$ is an absolute invariant. From equation (27), the value of this fraction in a field consisting of extremals is

$$\frac{T(g)}{\bar{f}\sqrt{f^2 + f_\tau^2}} = K + \frac{1}{V}\frac{d^2 V}{du^2}, \tag{43}$$

and it follows at once that the value of the second member of the equation, taken in a field of extremals, is an absolute invariant. But from equations (30), (36), (40) and (42) it is seen that V has the same property. Furthermore, the derivative of any invariant with respect to t is always invariant, and the same is therefore true of the derivatives of V with respect to u, since u is an invariant and

$$\frac{dV}{du} = \frac{dV}{dt} \bigg/ \frac{du}{dt}.$$

It follows then that *the three integrals*

$$\int g(x, y, \tau)\, ds, \qquad \iint \left(K + \frac{1}{V}\frac{d^2 V}{du^2}\right) dA,$$

$$\int [(g\cos\tau - g_\tau \sin\tau)\, dx + (g\sin\tau + g_\tau \cos\tau)\, dy],$$

where g is the function defined by equation (22), *are all invariant under the transformation* (31). *In the integrand of the second of these the function V and its derivative d^2V/du^2 are invariants for all values of the arguments x, y, τ, τ_s, τ_{ss}, while K is invariant at least when the arguments τ, τ_s, τ_{ss} are the functions of x and y defining the direction, curvature and derivative of curvature for an extremal of a field.*

§ 6. *Application to the Case of the Surface Theory.*

It is interesting to note how the invariants found above are related to the well-known invariants of the surface theory when the function f has the value (2). With the help of the notations

$$\lambda = E \cos \tau + F \sin \tau, \qquad \mu = F \cos \tau + G \sin \tau,$$

it follows, as in a preceding paper,[*] that

$$f = \frac{\lambda \cos \tau + \mu \sin \tau}{f}, \qquad\qquad f_\tau = \frac{-\lambda \sin \tau + \mu \cos \tau}{f},$$

$$\bar{f} = \sqrt{E\,G - F^2}\, \frac{f}{\sqrt{\lambda^2 + \mu^2}}, \quad f^2 + f_\tau^2 = \frac{\lambda^2 + \mu^2}{f^2}, \quad f + f_{\tau\tau} = \frac{E\,G - F^2}{f^3},$$

so that the integral (41) giving the generalized angle takes the form

$$\omega = \int_\tau^{\tau'} \frac{\sqrt{E\,G - F^2}}{E \cos^2 \tau + 2\,F \cos \tau \sin \tau + G \sin^2 \tau}\, d\tau.$$

The Euler expression $T(f)$ has often been calculated. It has the value

$$f^3\, T(f) = (E\,G - F^2)\, \tau_s$$
$$+ (E \cos \tau + F \sin \tau)\, [\, (F_x - \tfrac{1}{2} E_y) \cos^2 \tau + G_x \cos \tau \sin \tau + \tfrac{1}{2} G_y \sin^2 \tau\,]$$
$$- (F \cos \tau + G \sin \tau)\, [\, \tfrac{1}{2} E_x \cos^2 \tau + E_y \cos \tau \sin \tau + (F - \tfrac{1}{2} G_x) \sin^2 \tau\,],$$

which, substituted in (14), gives the well-known formula for the geodesic curvature[†]

$$\frac{1}{\rho} = \frac{T(f)}{\sqrt{E\,G - F^2}}.$$

The integral of $g(x, y, \tau)$ from equation (22), which occurs in the proof of Gauss' theorem, can be found after some calculation. It has the value

$$g(x, y, \tau) = \frac{-2\,E\,F_x + E\,E_y + E_x\,F}{2\,E \sqrt{E\,G - F^2}} \cos \tau + \frac{E_y\,F - E\,G_x}{2\,E \sqrt{E\,G - F^2}} \sin \tau. \quad (44)$$

Since g is linear in $\cos \tau$ and $\sin \tau$, the value of $T(g)$ is independent of the

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* *Transactions*, Vol. VII (1906), p. 193.

† See, for example. Bolza. "Vorlesungen über Variationsrechnung," p. 210.

direction angle τ and consequently entirely independent of the particular field in which it is taken.

The value of V is found to be unity, so that the invariant K in a field of extremals is exactly the expression $T(g)/(\bar{f}\sqrt{f^2+f_\tau^2})$ in equation (43). The numerator of the latter, calculated directly from (44), is given by the equation

$$
\begin{aligned}
4\,(E\,G-F^2)^{3/2}\,T\,(g) = {} & E\,(G_x^2 + E_y\,G_y - 2\,F_x\,G_y) \\
& + F\,(E_x\,G_y + 4\,F_x\,F_y - 2\,E_y\,F_y - 2\,F_x\,G_x - E_y\,G_x) \\
& + G\,(E_y^2 - 2\,E_x\,F_y + E_x\,G_x) \\
& - 2\,(E\,G-F^2)\,(E_{yy} - 2\,F_{xy} + G_{xx}),
\end{aligned}
$$

which, taken with the expression

$$
\bar{f}\sqrt{f^2+f_\tau^2} = \sqrt{E\,G-F^2},
$$

shows that K is the Gaussian curvature.

These results were found by direct calculations which were somewhat long. It would be still more difficult to identify K with the curvature by substituting in the expression (43), without a special choice of coordinates, the value of f from (2) and the values of τ_s and τ_{ss} derived from Euler's equation. But the calculation becomes very simple when a transformation has been made which takes the transversals of the field into the lines $x = a$, and the extremals of the field into the lines $y = b$, and which furthermore makes the new x-coordinate of any point equal to the value of the integral (1) taken along the extremal arc joining the point in question to some fixed initial transversal. When such a transformation has been made, the function $f(x, y, \tau)$ will have special values for $\tau = 0$. Since

$$
x \equiv \int_0^x f(x, y, 0)\,dx,
$$

it follows that

$$
f(x, y, 0) \equiv 1. \tag{45}
$$

The direction $\bar{\tau} = \pi/2$ is everywhere transversal to $\tau = 0$, and consequently, from equations (9),

$$
f_\tau(x, y, 0) = 0. \tag{46}
$$

In the field the values of τ and τ_s are zero, and it is easy to see from the formula just preceding (29), with (45) and (46), that $f_2 = 0$. Consequently, from (29),

$$
K = -\frac{1}{\sqrt{f f_1}} \frac{d^2}{d x^2} \sqrt{f f_1}. \tag{47}
$$

3

For the length integral on a surface the conditions (45) and (46) mean that

$$f(x, y, \tau) = \sqrt{\cos^2 \tau + m^2 \sin^2 \tau},$$

where G has been put equal to m^2. Then, in the field, $f = 1$ and $f_1 = m^2$; and it follows, by substituting in equation (47), that

$$K = -\frac{1}{m} \frac{d^2 m}{d x^2}.$$

This is the well-known formula for the curvature of a surface when geodesic coordinates are used.*

THE UNIVERSITY OF CHICAGO, *Chicago, Ill.*

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* Gauss, *loc. cit.*, p. 28.