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The K-theoretic Farrell–Jones conjecture for hyperbolic groups

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Abstract. We prove the K-theoretic Farrell–Jones conjecture for hyperbolic groups with (twisted) coefficients in any associative ring with unit.

Introduction

The main result of this paper is the following theorem.

Main theorem. *Let G be a hyperbolic group. Then G satisfies the K*-theoretic Farrell–Jones conjecture with coefficients*, i.e., if $\mathcal{A}$ is an additive category with right G-action, then for every $n \in \mathbb{Z}$ the assembly map*

(0.1) $$H_n^G(E_{\mathcal{VC}\text{yc}}G; \mathbf{K}_{\mathcal{A}}) \to H_n^G(\text{pt}; \mathbf{K}_{\mathcal{A}}) \cong K_n(\mathcal{A} *_G \text{pt})$$

is an isomorphism. This implies in particular that G satisfies the ordinary Farrell–Jones conjecture with coefficients in an arbitrary coefficient ring R.

Some explanations are in order.

Basic notations and conventions. *Hyperbolic group* is to be understood in the sense of Gromov (see for instance [12, 14, 33, 34]).

K-theory is always *non-connective K-theory*, i.e., $K_n(\mathcal{B}) = \pi_n(\mathbb{K}^{-\infty}\mathcal{B})$ for an additive category $\mathcal{B}$ and the associated non-connective K-theory spectrum as constructed for instance in [49].

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We denote by $\mathcal{VC}\mathrm{yc}$ the family of virtually cyclic subgroups of G. A *family* $\mathcal{F}$ of subgroups of G is a non-empty collection of subgroups closed under conjugation and taking subgroups. We denote by $E_{\mathcal{F}}G$ the associated *classifying space of the family* $\mathcal{F}$ (see for instance [45]).

A ring is always understood to be a (not necessarily commutative) associative ring with unit.

The K-theoretic Farrell–Jones conjecture with coefficients. Given an additive category $\mathcal{A}$ with right G-action, a covariant functor

$$\mathbf{K}_{\mathcal{A}}\colon \mathsf{Or}\,G \to \mathsf{Spectra}, \quad T \mapsto \mathbb{K}^{-\infty}(\mathcal{A} *_G T)$$

is defined in [7, Sect. 2], where $\mathsf{Or}\,G$ is the orbit category of G and $\mathsf{Spectra}$ is the category of spectra with (strict) maps of spectra as morphisms. To any such functor one can associate a G-homology theory $H_n^G(-;\mathbf{K}_{\mathcal{A}})$ (see [19, Sects. 4 and 7]). The *assembly map* for a family $\mathcal{F}$ and an additive category $\mathcal{A}$ with right G-action

$$H_n^G(E_{\mathcal{F}}G;\mathbf{K}_{\mathcal{A}}) \to H_n^G(\mathrm{pt};\mathbf{K}_{\mathcal{A}}) \cong K_n(\mathcal{A} *_G \mathrm{pt}) \tag{0.2}$$

is induced by the projection $E_{\mathcal{F}}G \to \mathrm{pt}$ onto the space pt consisting of one point. The right hand side of the assembly map $H_n^G(\mathrm{pt};\mathbf{K}_{\mathcal{A}})$ can be identified with $K_n(\mathcal{A} *_G \mathrm{pt})$, the K-theory of a certain additive category $\mathcal{A} *_G \mathrm{pt}$. We say that the K-theoretic Farrell–Jones conjecture with coefficients for a group G holds if the map (0.2) is bijective for $\mathcal{F} = \mathcal{VC}\mathrm{yc}$, every $n \in \mathbb{Z}$ and every additive category $\mathcal{A}$ with right G-action.

The original K-theoretic Farrell–Jones conjecture. If $\mathcal{A}$ is the category of finitely generated free R-modules and is equipped with the trivial G-action, then $\pi_n(\mathbf{K}_{\mathcal{A}}(G/G)) \cong K_n(RG)$ and the assembly map becomes

$$H_n^G(E_{\mathcal{VC}\mathrm{yc}}G;\mathbf{K}R) \to H_n^G(\mathrm{pt};\mathbf{K}R) \cong K_n(RG). \tag{0.3}$$

This map can be identified with the one that appears in the original formulation of the *Farrell–Jones conjecture* [28, 1.6 on p. 257], compare [37]. So the main theorem implies that the K-theoretic version of the Farrell–Jones conjecture is true for hyperbolic groups and any coefficient ring R.

The benefit of the K-theoretic Farrell–Jones conjecture is that it computes $K_n(RG)$ by a G-homology group which is given in terms of $K_n(RV)$ for all $V \in \mathcal{VC}\mathrm{yc}$. So it reduces the computation of $K_n(RG)$ to the one of $K_n(RV)$ for all $V \in \mathcal{VC}\mathrm{yc}$ together with all functoriality properties coming from inclusion and conjugation.

Let $\alpha\colon G \to \mathrm{aut}(R)$ be a homomorphism with the group of ring automorphisms of R as target. Let $R_\alpha G$ be the associated twisted group ring. Then one can define an additive category $\mathcal{A}(R,\alpha)$ such that $K_n(\mathcal{A}(R,\alpha) *_G G/H) \cong K_n(R_{\alpha|_H}H)$, see [7, Example 2.6]. The assembly map in the K-theoretic Farrell–Jones conjecture with coefficients in $\mathcal{A}(R,\alpha)$ has as target $K_n(R_\alpha G)$.

Farrell–Jones [28] formulate a *fibered version* of their conjecture which has much better inheritance properties. It turns out that the version of the *Farrell–Jones conjecture with coefficients* as formulated in the main theorem is stronger than the fibered version and has even better inheritance properties (see [7, Sect. 4]).

The case of a torsionfree hyperbolic group. Suppose that G is a subgroup of a torsionfree hyperbolic group and R is a ring. Then the main theorem implies for all $n \in \mathbb{Z}$ the existence of an isomorphism, natural in R,

$$H_n(BG; \mathbf{K}R) \oplus \bigoplus_{(C)} (NK_n(R) \oplus NK_n(R)) \xrightarrow{\cong} K_n(RG),$$

where $H_n(BG; \mathbf{K}R)$ is the homology theory associated to the (non-connective) K-theory spectrum $\mathbf{K}R$ of R evaluated at the classifying space BG of G, (C) runs through the conjugacy classes of maximal infinite cyclic subgroups of G and $NK_n(R)$ denotes the nth Bass–Nil-group of R. This follows from [9, Theorem 1.3] and [45, Theorem 8.11]. If R is regular, then $NK_n(R) = 0$ for $n \in \mathbb{Z}$ and $K_n(R) = \pi_n(\mathbf{K}R) = 0$ for $n \leq -1$.

Previous results. A lot of work about the Farrell–Jones conjecture has been done during the last decade. Its original formulation is due to Farrell–Jones [28, 1.6 on p. 257]. Celebrated results of Farrell and Jones prove the pseudo-isotopy version of their conjecture for certain classes of groups, e.g., for any subgroup G of a group Γ such that Γ is a cocompact discrete subgroup of a Lie group with finitely many path components (see [28, Theorem 2.1]). The pseudo-isotopy version implies the K-theoretic Farrell–Jones conjecture for $R = \mathbb{Z}$ and $n \leq 1$ and the rational K-theoretic version for $R = \mathbb{Z}$ and all $n \in \mathbb{Z}$. For more explanations, information about the status and references concerning the Farrell–Jones conjecture we refer to the survey article [46].

Most of the results about the K-theoretic version of the Farrell–Jones conjecture deal with dimensions $n \leq 1$ and $R = \mathbb{Z}$. The first result dealing with arbitrary coefficient rings R appear in Bartels–Farrell–Jones–Reich [3], where the K-theoretic Farrell–Jones conjecture was proven in dimension ≤ 1 for G the fundamental group of a negatively curved closed Riemannian manifold. In Bartels–Reich [8] this result was extended to all $n \in \mathbb{Z}$. In this paper we replace the condition that G is the fundamental group of a negatively curved closed Riemannian manifold by the much weaker condition that G is hyperbolic in the sense of Gromov, and also allow twisted coefficients.

Further results. We mention that the main theorem implies that the K-theoretic Farrell–Jones conjecture with coefficients in any ring R holds not only for hyperbolic groups but for instance for any group which occurs as a subgroup of a finite product of hyperbolic groups and for any directed colimit of hyperbolic groups (with not necessarily injective structure maps). Such groups can be very wild and can have exotic properties (see Bridson [13] and Gromov [36]). This follows from some general inheritance properties. All this will be explained in Bartels–Lück–Reich [6] and Bartels–Echterhoff–Lück [4], where further classes of groups are discussed, for which certain versions or special cases of the K-theoretic Farrell–Jones conjecture hold.

Applications. In order to illustrate the potential of the K-theoretic Farrell–Jones conjecture we mention some conclusions. We will not try to state the most general versions. For explanations, proofs and further applications in a more general context we refer to [6].

In the sequel we suppose that G satisfies the K-theoretic Farrell–Jones conjecture for any ring R, i.e., the assembly map (0.3) is bijective for every $n \in \mathbb{Z}$ and every ring R. Examples for G are subgroups of finite products of hyperbolic groups. Then the following conclusions hold:

- *Induction from finite subgroups for the projective class group*
  If R is a regular ring and the order of any finite subgroup of G is invertible in R, then the canonical map
  $$\operatorname{colim}_{H \subseteq G, |H| < \infty} K_0(RH) \to K_0(RG)$$
  is bijective.
  If R is a skew-field of prime characteristic p, then the canonical map
  $$\operatorname{colim}_{H \subseteq G, |H| < \infty} K_0(RH)[1/p] \to K_0(RG)[1/p]$$
  is bijective.
- *Bass conjectures*
  The *Bass conjecture for commutative integral domains* holds for G, i.e., for a commutative integral domain R and a finitely generated projective RG-module P its *Hattori–Stallings rank* $\mathrm{HS}(P)(g)$ evaluated at $g \in G$ is trivial if g has infinite order or the order of g is finite and not invertible in R.
  The *Bass conjecture for fields of characteristic zero* holds for G, i.e., for any field F of characteristic zero the Hattori-Stallings rank induces an isomorphism
  $$K_0(FG) \otimes_{\mathbb{Z}} F \xrightarrow{\cong} \operatorname{class}_F(G)_f$$
  to the F-vector space of functions $G \to F$ which vanish on elements of infinite order, are constant on F-conjugacy classes and are non-trivial only for finitely many F-conjugacy classes.
- *Bass–Nil-groups and homotopy K-theory*
  If R is a regular ring and the order of any finite subgroup of G is invertible in R, then the *Bass–Nil-groups* $NK_n(RG)$ are trivial and the canonical map
  $$K_n(RG) \xrightarrow{\cong} KH_n(RG)$$
  to the homotopy K-theory of RG in the sense of Weibel [58] is bijective for every $n \in \mathbb{Z}$.

- *Kaplansky conjecture for prime characteristic*
  Suppose that R is a field of prime characteristic p or suppose that R is a skew-field of prime characteristic p and G is sofic. (For the notion of a sofic group we refer for instance to [21]. Every residually amenable group is sofic.) Moreover, assume that every finite subgroup of G is a p-group. Then RG satisfies the *Kaplansky conjecture*, i.e., 0 and 1 are the only idempotents in RG.

Now suppose additionally that G is torsionfree. Then:

- *Negative K-groups*
  $K_n(RG) = 0$ for any regular ring R and $n \leq -1$.
- *Projective class group*
  The change of rings map $K_0(R) \to K_0(RG)$ is bijective for a regular ring R. In particular $\widetilde{K}_0(\mathbb{Z}G) = 0$. Hence any finitely dominated connected CW-complex with G as fundamental group is homotopy equivalent to a finite CW-complex.
- *Whitehead group*
  The Whitehead group $\mathrm{Wh}(G)$ is trivial. Hence any compact h-cobordism of dimension ≥ 6 with G as fundamental group is trivial.
- *Kaplansky conjecture for characteristic zero*
  If R is a field of characteristic zero or if R is a skew-field of characteristic zero and G is sofic, then RG satisfies the Kaplansky conjecture.

Searching for counterexamples. There is no group known for which the Farrell–Jones conjecture, the Farrell–Jones with coefficients or the Baum–Connes conjecture is false. However, Higson, Lafforgue and Skandalis [39, Sect. 7] construct counterexamples to the *Baum–Connes conjecture with coefficients*, actually with a commutative C^*-algebra as coefficients. They describe precisely what properties a group Γ must have so that it does *not* satisfy the Baum–Connes conjecture with coefficients. Gromov [36] constructs such a group Γ as a colimit over a directed system of groups $\{G_i | i \in I\}$ for which each G_i is hyperbolic. It will be shown in [4] that the main theorem implies that the Farrell–Jones conjecture with coefficients in any ring holds for Γ. It will also be shown that the Bost conjecture with coefficients in a C^*-algebra holds for Γ.

Controlled topology. A prototype of a result involving controlled topology and showing its potential is the *α-approximation theorem* of Chapman–Ferry (see [18,31]). It says, roughly speaking, that a homotopy equivalence $f\colon M \to N$ between closed manifolds is homotopic to a homeomorphism if it is controlled enough over N, i.e., there is a homotopy inverse $g\colon N \to M$ such that the compositions $f \circ g$ and $g \circ f$ are close to the identity and homotopic to the identity via homotopies whose tracks are small. Here “close” and “small” are understood to be measured in N considered as a metric space. In particular it says that a homotopy equivalence which is controlled enough represents the trivial element in the Whitehead group.

Controlled topology and its variations have been important for a number of further celebrated results in geometric topology. Some of these are concerned with the Novikov conjecture [17,22,32,41,59], ends of maps [52,53], controlled h-cobordisms [2,54], Whitehead groups and lower K-theory [23–25,30,42], topological rigidity [26,27,29,30], homology manifolds [15], parametrized Euler characteristics and higher torsion [20] and topological similarity [38]. Of course this list is not complete.

A key theme in controlled topology is to associate a size to geometric objects and then prove that objects of small size are trivial in an appropriate sense. Such a result is sometimes called a *stability* or *squeezing* result. A good example is the α-approximation theorem mentioned above. Related is the reformulation of the assembly maps into a *"forget control"* version, i.e., the domain of the assembly map is described by objects whose size is very small while the target is described by bounded objects. This formulation of forget-control is often referred to as the ε-version. Now it is clear what one has to do to prove for instance surjectivity, one must be able to manipulate a representative of an element in K-theory so that it becomes better and better controlled without changing its K-theory class. This opens the door to apply geometric methods. In their celebrated work Farrell–Jones used three decisive ideas to carry out such manipulations: transfers, geodesic flows and foliated control theory.

There is also a somewhat different approach to the assembly map as a forget-control map, sometimes called the bounded or categorical version. Here the emphasis is not on single objects and their sizes but on (the category of) all bounded objects. Then the way boundedness is measured can be varied, for instance on non-compact spaces very different metrics can be considered. A good example is the description of the homology theory associated to the K-theory spectrum of a ring in [50]. This formulation is very elegant, but less concrete (and involves usually a dimension shift).

Controlled topology is the main ingredient in proofs of the Farrell–Jones conjecture, whereas for the Baum–Connes conjecture the main strategy is the Dirac-Dual-Dirac-method.

A rough outline of the proof. We will use the bounded (more precisely, the continuous controlled) version of the forget-control assembly map. This quickly leads to a description of the homotopy fiber of the assembly map as the K-theory of a certain additive category, see Proposition 3.8. We call this category the obstruction category. A somewhat artificial construction makes the obstruction category a functor of metric spaces with G-action, see Subsect. 3.4. In the simplest case the metric space in question is the group G equipped with a word metric, but it will be important to vary the metric space. This will be done in two steps. Firstly, we use a transfer to replace G by $G \times \overline{X}$, where $\overline{X}$ is a compactification of the Rips complex for G, see Theorem 6.1. The benefit of the G-space $\overline{X}$ is to have place for certain equivariant constructions which cannot be carried out in G itself. In particular, in [5] we constructed certain G-invariant open covers, see Assumption 1.4. The existence of these covers can be viewed as an equivariant version of the fact that hyperbolic groups have finite asymptotic dimension. Secondly, we apply contracting maps associated to open covers of $G \times \overline{X}$, see Proposition 5.3. This map will only be contracting with respect to the G-coordinate and will expand in the $\overline{X}$ coordinate. This defect can be compensated, because the transfer produces arbitrary small control with respect to the $\overline{X}$-coordinate. Improving on an idea from [10] we formulate and prove a kind of stability result for the obstruction category in Theorem 7.2. This result is not formulated in terms of single elements, but as a K-theory equivalence of certain categories. (However, for K_1 it is not hard to extract a more concrete statement along the lines of the above stability statements, see [10, Corollary 4.6].) The general strategy of the proof is worked out in Sect. 4, see in particular Diagram (4.4).

Our approach is very much influenced by the general strategy of Farrell–Jones. However, our more general setting involves new ideas and techniques. We prove the K-theoretic Farrell–Jones conjecture for arbitrary coefficient rings and also for higher K-theory. We also would like to mention that our proof unlike many other proofs treats the surjectivity and injectivity part simultaneously. One main difficulty is that we cannot work with manifolds and simplicial complexes anymore and do not have transversality or general position arguments at hand, since in the world of hyperbolic groups we can at best get metric spaces with very complicated compactifications. This forces us to use open covers. A benefit of our approach is that we avoid the hard foliated control theory. Other ingredients of the Farrell–Jones strategy are still used. Namely, in order to show that hyperbolic groups fulfill Assumption 1.4 we build in [5] on Mineyevs [48] replacement of the geodesic flow and generalize the long and thin cells of Farrell–Jones for manifolds to certain covers of metric spaces.

Open problems. There is an L-theoretic version of the Farrell–Jones conjecture. An obvious problem is to extend our methods for K-theory to L-theory. The main difficulties concern the transfer and the fact that in L-theory one needs to control the signature of the fiber and not – as in K-theory – the Euler characteristic.

If both the K-theoretic and the L-theoretic Farrell–Jones conjecture hold for $R = \mathbb{Z}$ as coefficients for a group G, then the *Borel conjecture* is true for G, i.e., if M and N are closed aspherical topological manifolds of dimension ≥ 5 whose fundamental groups are isomorphic to G, then M and N are homeomorphic and every homotopy equivalence $M \to N$ is homotopic to a homeomorphism.

Another problem is to prove the Farrell–Jones conjecture with coefficients for groups which act proper and cocompactly on a CAT(0)-space.

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1. Axiomatic formulation

Theorem 1.1 (Axiomatic formulation). *Let G be a finitely generated group. Let $\mathcal{F}$ be a family of subgroups of G. Let $\mathcal{A}$ be an additive category with right G-action. Suppose*

(i) *There exists a G-space X such that the underlying space X is the realization of an abstract simplicial complex;*
(ii) *There exists a G-space $\overline{X}$ which contains X as an open G-subspace such that the underlying space of $\overline{X}$ is compact, metrizable and contractible;*
(iii) *Assumption 1.2 holds;*
(iv) *Assumption 1.4 holds for $\mathcal{F}$.*

Then for every $m \in \mathbb{Z}$ the assembly map

$$H_m^G(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \to K_m(\mathcal{A} *_G \mathrm{pt})$$

is an isomorphism.

Sections 3 to 7 are devoted to the proof of Theorem 1.1. The general structure of the argument is described in Subsect. 4.4. We now formulate the two assumptions that appear in Theorem 1.1.

Assumption 1.2 (Weak Z-set condition). *There exists a homotopy $H \colon \overline{X} \times [0, 1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$.*

In order to state the second assumption we introduce the notion of an open $\mathcal{F}$-cover.

Definition 1.3. Let Y be a G-space. Let $\mathcal{F}$ be a family of subgroups of G. An *open $\mathcal{F}$-cover* of Y is a collection $\mathcal{U}$ of open subsets of Y such that the following conditions are satisfied:

(i) $Y = \bigcup_{U \in \mathcal{U}} U$;
(ii) For $g \in G$, $U \in \mathcal{U}$ the set $g(U) := \{gx \mid x \in U\}$ belongs to $\mathcal{U}$;
(iii) For $g \in G$ and $U \in \mathcal{U}$ we have $g(U) = U$ or $U \cap g(U) = \emptyset$;
(iv) For every $U \in \mathcal{U}$, the subgroup $\{g \in G \mid g(U) = U\}$ lies in $\mathcal{F}$.

Suppose $\mathcal{U}$ is an open $\mathcal{F}$-cover. Then $|\mathcal{U}|$, the realization of the nerve, is a simplicial complex with cell preserving simplicial G-action and hence a G-CW complex. (A G-action on a simplicial complex is called *cell preserving* if for every simplex σ and element $g \in G$ such that the intersection of the interior σ° of σ with $g\sigma^\circ$ is non-empty we have $gx = x$ for every $x \in \sigma$. Notice that a simplicial action is not necessarily cell preserving, but the induced simplicial action on the barycentric subdivision is cell preserving.) Moreover all its isotropy groups lie in $\mathcal{F}$. Recall that by definition the dimension $\dim \mathcal{U}$ of an open cover is the dimension of the CW-complex $|\mathcal{U}|$.

If G is a finitely generated discrete group, then d_G denotes the word metric with respect to some chosen finite set of generators. Recall that d_G

depends on the choice of the set of generators but its quasi-isometry class is independent of it.

Assumption 1.4 (Wide open $\mathcal{F}$-covers). *There exists $N \in \mathbb{N}$, which only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists an open $\mathcal{F}$-cover $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:*

(*i*) *For every $g \in G$ and $x \in \overline{X}$ there exists $U \in \mathcal{U}(\beta)$ such that*

$$\{g\}^{\beta} \times \{x\} \subset U.$$

Here $\{g\}^{\beta}$ denotes the open β-ball around g in G with respect to the word metric d_G, i.e., the set $\{h \in G \mid d_G(g,h) < \beta\}$;

(*ii*) *The dimension of the open cover $\mathcal{U}(\beta)$ is smaller than or equal to N.*

We remark that if Assumption 1.4 holds, then it is possible to massage the covers $\mathcal{U}(\beta)$ (using for example Lemma 5.1) in order to additionally obtain the property that each $\mathcal{U}(\beta)$ is locally finite, i.e., every point in $G \times \overline{X}$ has a neighborhood U that intersects only a finite number of members of $\mathcal{U}$. We will however not use this fact.

2. The case of a hyperbolic group

Lemma 2.1. *Let G be a word-hyperbolic group. Then the assumptions appearing in Theorem 1.1 are satisfied for the family $\mathcal{F} = \mathcal{VC}\mathrm{yc}$ of virtually cyclic subgroups of G.*

Proof. (i) Fix a set of generators S. Equip G with the corresponding word metric. Choose δ such that G becomes a δ-hyperbolic space. Choose an integer $d > 4\delta + 6$. Let $P_d(G)$ be the associated *Rips complex*. It is a finite-dimensional contractible locally finite simplicial complex. The obvious simplicial G-action on $P_d(G)$ is proper and cocompact. In particular $P_d(G)$ is uniformly locally finite and connected. Its 1-skeleton is the Cayley graph of G with respect to the set of generators consisting of non-trivial elements in the ball of radius d about the identity in G. All these claims are proven for instance in [14, pp. 468ff]. Since the quasi-isometry type of the Cayley graph of a group is independent of the choice of the finite set of generators, the 1-skeleton of $P_d(G)$ with the word metric is a hyperbolic metric space. Hence $P_d(G)$ equipped with the word metric is a hyperbolic complex in the sense of Mineyev [48, p. 438]. We take $X = P_d(G)$.

We mention that $P_d(G)$ is quasi-isometric to the Cayley graph of the group. Moreover, the barycentric subdivision of $P_d(G)$ is a G-CW-complex which is for large enough d a model for the classifying space for proper G-actions (see [47]), but we will not use this fact.

(ii) We take $\overline{X} = X \cup \partial X$ to be the compactification of X in the sense of Gromov (see [34], [14, III.H.3]).

(iii) According to [11, Theorem 1.2] the subspace $\partial X \subseteq \overline{X}$ satisfies the Z-set condition. This implies our (weaker) Assumption 1.2 which is a consequence of Part (2) of the characterization of Z-sets before [11, Theorem 1.2].

(iv) This assumption is proved in [5, Theorem 1.2]. □

Because of Lemma 2.1 the main theorem follows from Theorem 1.1. The remainder of this paper is devoted to the proof of Theorem 1.1.

3. Controlled algebra and the fiber of the assembly map

3.1. A quick review of controlled algebra. Let Y be a space and let $\mathcal{A}$ be a small additive category. Define the additive category

$$\mathcal{C}(Y;\mathcal{A})$$

as follows. An object is a collection $A = (A_x)_{x\in Y}$ of objects in $\mathcal{A}$ which is locally finite, i.e., its *support* $\operatorname{supp}(A) := \{x \in Y \mid A_x \neq 0\}$ is a locally finite subset of Y. Recall that a subset $S \subseteq Y$ is called *locally finite* if each point in Y has an open neighborhood U whose intersection with S is a finite set. A morphism $\phi = (\phi_{x,y})_{x,y\in Y} \colon A = (A_y)_{y\in Y} \to B = (B_x)_{x\in Y}$ consists of a collection of morphisms $\phi_{x,y} \colon A_y \to B_x$ in $\mathcal{A}$ for $x, y \in Y$ such that the set $\{x \mid \phi_{x,y} \neq 0\}$ is finite for every $y \in Y$ and the set $\{y \mid \phi_{x,y} \neq 0\}$ is finite for every $x \in Y$. Composition is given by matrix multiplication, i.e.,

$$(\psi \circ \phi)_{x,z} := \sum_{y\in Y} \psi_{x,y} \circ \phi_{y,z}.$$

The category $\mathcal{C}(Y;\mathcal{A})$ inherits in the obvious way the structure of an additive category from $\mathcal{A}$. We will often drop $\mathcal{A}$ from the notation.

If Y and $\mathcal{A}$ come with a G-action, we get a G-action on $\mathcal{C}(Y;\mathcal{A})$ by $(g^*A)_x := g^*(A_{gx})$ and $(g^*\phi)_{x,y} := g^*(\phi_{gx,gy})$. Here the action on Y is a left action, and the action on $\mathcal{A}$ is a right action, i.e., $(g^* \circ h^*)(A) = (hg)^*A$. The action on $\mathcal{C}(Y;\mathcal{A})$ is again a right action.

Denote by

$$\mathcal{C}^G(Y;\mathcal{A})$$

the fixed point category. This is an additive subcategory of $\mathcal{C}(Y;\mathcal{A})$. An object in $\mathcal{C}^G(Y;\mathcal{A})$ is given by a locally finite collection $(A_x)_{x\in Y}$ of objects in $\mathcal{A}$ such that $A_x = g^*(A_{gx})$ holds for all $g \in G$ and $x \in Y$. A morphism $(\phi_{x,y})_{x,y\in Y}$ in $\mathcal{C}(Y;\mathcal{A})$ between two objects which belong to $\mathcal{C}^G(Y;\mathcal{A})$ is a morphism in $\mathcal{C}^G(Y;\mathcal{A})$ if and only if $g^*(\phi_{gx,gy}) = \phi_{x,y}$ holds for all $g \in G$ and $x, y \in Y$.

We are seeking certain additive subcategories of $\mathcal{C}^G(Y,\mathcal{A})$, where support conditions are imposed on the objects and morphisms. This is formalized by the notion of a *coarse structure* following [40]. For us it consists of a set $\mathcal{E}$ of subsets of $Y \times Y$ and a set $\mathcal{F}$ of subsets of Y fulfilling certain axioms stated as (i) to (iv) in [3, p. 167]. An object is called *admissible* if there exists $F \in \mathcal{F}$ which contains its support. A morphisms $(\phi_{x,y})$ in $\mathcal{C}^G(Y; \mathcal{A})$ is called *admissible* if there exists $J \in \mathcal{E}$ which contains its support $\operatorname{supp}(\phi) := \{(x, y) \mid x, y \in Y, \phi_{x,y} \neq 0\}$. The axioms are designed such that the admissible objects together with the admissible morphisms form an additive subcategory of $\mathcal{C}^G(Y; \mathcal{A})$ which we will denote by

$$\mathcal{C}^G(Y, \mathcal{E}, \mathcal{F}; \mathcal{A}).$$

Let $f\colon Y \to Z$ be a G-equivariant map. The formula $(f_*(A))_z := \bigoplus_{y \in f^{-1}(z)} A_y$ defines a functor $\mathcal{C}^G(Y, \mathcal{E}^Y, \mathcal{F}^Y; \mathcal{A}) \to \mathcal{C}^G(Z, \mathcal{E}^Z, \mathcal{F}^Z; \mathcal{A})$ if f maps locally finite sets to locally finite sets and takes $\mathcal{E}^Y$ to $\mathcal{E}^Z$ and $\mathcal{F}^Y$ to $\mathcal{F}^Z$, see [3, Subsect. 3.3]. If $g\colon Y \to Z$ is a second G-equivariant map that induces a functor, then there is always a candidate for a natural equivalence between the two functors, namely we can use the identity on each A_y. Viewed over Z this candidate for a morphism will have a nontrivial support. This yields indeed a natural equivalence if the following holds.

(3.1) For each object $A \in \mathcal{C}^G(Y, \mathcal{E}^Y, \mathcal{F}^Y; \mathcal{A})$ there is an element $J_A \in \mathcal{E}^Z$ such that $(f(y), g(y)) \in J_A$ for all $y \in \operatorname{supp} A$.

3.2. Some control condition. Let Z be a space equipped with a quasi-metric d. (We remind the reader that the difference between a metric and a quasi-metric is that in the later case the distance ∞ is allowed.) Then we define $\mathcal{E}_d^Z$ to be the collection of all subsets J of $Z \times Z$ of the form $J_\alpha = \{(z, z') \mid d(z, z') \leq \alpha\}$ with $\alpha < \infty$. A morphism $\varphi \in \mathcal{C}(Z, \mathcal{E}_d^Z)$ is said to be δ-controlled if $\operatorname{supp} \varphi \subseteq J_\delta$. This terminology will be used in Subsect. 6.3 and we will often be interested in small δ.

Let Y be a G-space. A subset $C \subset Y$ is called *G-compact* if there exists a compact subset $C' \subseteq Y$ satisfying $C = G \cdot C'$. For a G-CW-complex Y a subset $C \subseteq Y$ is G-compact if and only if its image under the projection $Y \to G\backslash Y$ is a compact subset of the quotient $G\backslash Y$. Denote by $\mathcal{F}_{Gc}^Y$ the set which consist of all G-compact subsets of Y.

Let Y be a G-space. We denote by G_x the isotropy group of a point $x \in Y$. Equip $Y \times [1, \infty)$ with the G-action given by $g(y, t) := (gy, t)$. As in [3, Definition 2.7] we define $\mathcal{E}_{Gcc}^Y$ to be the collection of subsets $J \subseteq (Y \times [1, \infty)) \times (Y \times [1, \infty))$ satisfying

(3.2) For every $x \in Y$, every G_x-invariant open neighborhood U of (x, ∞) in $Y \times [1, \infty]$ there exists a G_x-invariant open neighborhood $V \subseteq U$ of (x, ∞) in $Y \times [1, \infty]$ such that

$$((Y \times [1, \infty] - U) \times V) \cap J = \emptyset;$$

(3.3) The image of J under the projection $(Y \times [1, \infty))^{\times 2} \to [1, \infty)^{\times 2}$ sends J to a member of $\mathcal{E}_d^{[1,\infty)}$ where $d(t, s) = |t - s|$;

(3.4) J is symmetric and invariant under the diagonal G-action.

$\mathcal{E}_{Gcc}^Y$ is called the equivariant continuous control condition.

3.3. Controlled algebra and the assembly map. Let G be finitely generated group equipped with a word-metric d_G. For a G-space Y let $p\colon G \times Y \times [1,\infty) \to Y \times [1,\infty)$, $q\colon G \times Y \times [1,\infty) \to G \times Y$ and $r\colon G \times Y \times [1,\infty) \to G$ be the canonical projections. We will abuse notation and set

$$p^{-1}\mathcal{E}^Y_{Gcc} \cap r^{-1}\mathcal{E}^G_{d_G} := \left\{ (p \times p)^{-1}(J) \cap (r \times r)^{-1}(J') \,\middle|\, J \in \mathcal{E}^Y_{Gcc},\ J' \in \mathcal{E}^G_{d_G} \right\};$$
$$q^{-1}\mathcal{F}^{G\times Y}_{Gc} := \left\{ q^{-1}(F) \,\middle|\, F \in \mathcal{F}^{G\times Y}_{Gc} \right\}.$$

We define

$$\begin{aligned}
\mathcal{T}^G(Y;\mathcal{A}) &:= \mathcal{C}^G\left(G \times Y, \mathcal{F}^{G\times Y}_{Gc}; \mathcal{A}\right);\\
\mathcal{O}^G(Y;\mathcal{A}) &:= \mathcal{C}^G\left(G \times Y \times [1,\infty), p^{-1}\mathcal{E}^Y_{Gcc} \cap r^{-1}\mathcal{E}^G_{d_G}, q^{-1}\mathcal{F}^{Y\times G}_{Gc}; \mathcal{A}\right);\\
\mathcal{D}^G(Y;\mathcal{A}) &:= \mathcal{C}^G\left(G \times Y \times [1,\infty), p^{-1}\mathcal{E}^Y_{Gcc} \cap r^{-1}\mathcal{E}^G_{d_G}, q^{-1}\mathcal{F}^{Y\times G}_{Gc}; \mathcal{A}\right)^\infty.
\end{aligned}$$

We will often drop the $\mathcal{A}$ from the notation. Here the upper index ∞ in the third line denotes germs at infinity. This means that the objects of $\mathcal{D}^G(Y)$ are the objects of $\mathcal{O}^G(Y)$ but morphisms are identified if their difference can be factored over an object whose support is contained in $G \times Y \times [1,t]$ for some $t \in [1,\infty)$, compare [3, Subsect. 2.4].

We remark that in [3, Subsect. 3.2] a slightly different definition of $\mathcal{D}^G(Y)$ is given, where the metric control condition $\mathcal{E}^G_{d_G}$ does not appear. Using Theorem 3.7 below it can be shown that this does not change the K-theory of these categories. The metric control condition on G will be important in the construction of the transfer, see in particular Proposition 6.13. The interested reader may compare this difference to different possible definitions of cone and suspension rings. Often it is convenient to add finiteness condition to obtain formulas such as $(\Lambda R)G = \Lambda(RG)$, compare [3, Remark 7.2].

The following is the so-called germs at infinity sequence.

$$\mathcal{T}^G(Y) \to \mathcal{O}^G(Y) \to \mathcal{D}^G(Y). \tag{3.5}$$

Here the first map is induced by $\{1\} \subset [1,\infty)$ and the second is the quotient map. We will need the following facts.

Lemma 3.6

(*i*) *The sequence* (3.5) *induces a long exact sequence in K-theory;*
(*ii*) *The K-theory of $\mathcal{O}^G(\mathrm{pt})$ is trivial.*

Proof. We can replace $\mathcal{T}^G(Y)$ by an equivalent category, namely by the full subcategory of $\mathcal{O}^G(Y)$ on all objects that are isomorphic to an object in $\mathcal{T}^G(Y)$. These are precisely the objects in $\mathcal{O}^G(Y)$ whose support is contained in $G \times Y \times [1,r]$ for some $r \geq 0$. Then the first map in (3.5) becomes a Karoubi filtration and $\mathcal{D}^G(Y)$ is its quotient. Now (i) follows because Karoubi filtrations induce long exact sequences in K-theory, see for example [16].

To prove (ii) it suffices to observe that there is an Eilenberg-swindle on $\mathcal{D}^G(\mathrm{pt})$ induced by the map $(g,t) \mapsto (g,t+1)$, compare for example [3, Proposition 4.4]. □

Theorem 3.7. *The assignment $Y \mapsto K_*(\mathcal{D}^G(Y))$ is a G-equivariant homology theory on G-CW-complexes. The projection $E_{\mathcal{F}}G \to \mathrm{pt}$ induces the assembly map* (0.2).

Proof. This is proven in [3, Sect. 5, Corrollary 6.3], see also [7, Theorem 7.3]. As mentioned above a slightly different definition is used in these references, but this does not affect the proof and the arguments can be carried over word for word. □

The following is now an easy consequence, compare [37, Theorem 7.4].

Proposition 3.8. *Suppose there exists an $m_0 \in \mathbb{Z}$ such that for all $\mathcal{A}$ and all $m \geq m_0$ we have*

$$K_m\big(\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A})\big) = 0.$$

Then the assembly map (0.2) *is an isomorphism for all $n \in \mathbb{Z}$ and all $\mathcal{A}$.*

Proof. If the assembly map is an isomorphism for all $m \geq m_0$ and all $\mathcal{A}$, then it is an isomorphism for all $n \in \mathbb{Z}$ and all $\mathcal{A}$ by [7, Corollary 4.7]. If we apply Lemma 3.6 (i) to the map $E_{\mathcal{F}}G \to \mathrm{pt}$ we obtain a map between homotopy fibration sequences

$$\begin{array}{ccccc}
\mathbb{K}^{-\infty}\mathcal{T}^G(E_{\mathcal{F}}G) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{O}^G(E_{\mathcal{F}}G) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{D}^G(E_{\mathcal{F}}G) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{K}^{-\infty}\mathcal{T}^G(\mathrm{pt}) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{O}^G(\mathrm{pt}) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{D}^G(\mathrm{pt}).
\end{array}$$

It is not hard to check that the left vertical map is induced by an equivalence of categories and is therefore an equivalence of spectra. Because the homotopy groups of the lower middle spectrum vanish by Lemma 3.6 (ii) the claim follows by considering the long exact ladder of homotopy groups associated to the diagram above. □

3.4. The obstruction category as a functor of metric spaces. We will now allow for (G, d_G) to be replaced by a metric space (Z, d) with a free G-action by isometries in the definition of $\mathcal{O}^G(Y;\mathcal{A})^G$. We define

$$\mathcal{O}^G(Y,Z,d;\mathcal{A}) := \mathcal{C}^G\big(Z \times Y \times [1,\infty),\, p^{-1}\mathcal{E}^Y_{Gcc} \cap r^{-1}\mathcal{E}^Z_d,\, q^{-1}\mathcal{F}^{Z\times Y}_{Gc};\mathcal{A}\big),$$

where p, q, r the same projections as before, but with G replaced by the free G-space Z. As before we will often drop the $\mathcal{A}$ from the notation. The construction is functorial for G-equivariant maps $f\colon Z \to Z'$ that satisfy the following condition.

(3.9) For every $\alpha > 0$ there exists a $\beta > 0$ such that $d(x, y) \leq \alpha$ implies $d'(f(x), f(y)) \leq \beta$.

Let (Z_n, d_n) be a sequence of metric spaces with free isometric G-action. We define

$$\mathcal{O}^G(Y, (Z_n, d_n)_{n\in\mathbb{N}}) \subseteq \prod_{n\in\mathbb{N}} \mathcal{O}^G(Y, Z_n, d_n)$$

as a subcategory of the indicated product category by requiring additional conditions on the morphisms. A morphism $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is allowed if it is bounded with respect to the sequence of metrics, i.e., if there exists a constant $\alpha = \alpha(\varphi)$, such that for every $n \in \mathbb{N}$ and for every $((y, z, t), (y', z', t')) \in \operatorname{supp} \varphi_n \subset (Y \times Z_n \times [1, \infty))^{\times 2}$ one has $d_n(z, z') \leq \alpha$. The sum $\bigoplus_{n\in\mathbb{N}} \mathcal{O}^G(Y, Z_n, d_n)$ is in a canonical way a full subcategory of $\mathcal{O}^G(Y, (Z_n, d_n)_{n\in\mathbb{N}})$.

Later on, in Sect. 7, we will allow the d_n to be quasi-metrics rather than metrics. The definitions are clearly meaningful in this case as well.

These constructions are functorial for sequences of G-equivariant maps $f_n : Z_n \to Z'_n$ that satisfy the following uniform growth condition.

(3.10) For every $\alpha > 0$ there is $\beta > 0$ such that for all $n \in \mathbb{N}$

$$d_n(x, y) \leq \alpha \implies d'_n(f_n(x), f_n(y)) \leq \beta.$$

4. The core of the proof

4.1. The map to the realization of the nerve. Let (Z, d) be a metric space. Let $\mathcal{U}$ be a finite dimensional cover of Z by open sets. Recall that points in the realization of the nerve $|\mathcal{U}|$ are formal sums $x = \sum_{U\in\mathcal{U}} x_U U$, with $x_U \in [0, 1]$ such that $\sum_{U\in\mathcal{U}} x_U = 1$ and such that the intersection of all the U with $x_U \neq 0$ is non-empty, i.e., $\{U \mid x_U \neq 0\}$ is a simplex in the nerve of $\mathcal{U}$. There is a map

$$f = f^{\mathcal{U}} : Z \to |\mathcal{U}|, \quad x \mapsto \sum_{U\in\mathcal{U}} f_U(x)U, \tag{4.1}$$

where

$$f_U(x) = \frac{a_U(x)}{\sum_{V\in\mathcal{U}} a_V(x)} \quad \text{with}$$
$$a_U(x) = d(x, Z - U) = \inf\{d(x, u) \mid u \notin U\}.$$

It is well-defined since $\mathcal{U}$ is finite dimensional. If Z is a G-space, d is G-invariant and $\mathcal{U}$ is an open $\mathcal{F}$-cover, compare Definition 1.3, then the map $f = f^{\mathcal{U}}$ is G-equivariant. In our application $f^{\mathcal{U}}$ will be strongly contracting with respect to the l^1-metric on $|\mathcal{U}|$, see Proposition 5.3.

4.2. The l^1-metric on a simplicial complex. Every simplicial complex and in particular the realization of the nerve of an open cover can be equipped with the l^1-metric, i.e., the metric where the distance between points $x = \sum_U x_U U$ and $y = \sum_U y_U U$ is given by $d^1(x, y) = \sum_U |x_U - y_U|$. We remark that this metric does not generate the weak topology, unless the simplical complex is locally finite. We will never consider the weak topology and only be interested in the l^1-metric.

4.3. The metric d_C on $G \times \overline{X}$. Let $\overline{X}$ be as in Theorem 1.1. We will now define a G-invariant metric d_C depending on a constant $C > 0$ on the G-space $G \times \overline{X}$. Recall that $\overline{X}$ is assumed to be metrizable. We choose some (not necessarily G-invariant) metric $d_{\overline{X}}$ on $\overline{X}$ which generates the topology. We fix now for the rest of this paper some choice of a word-metric d_G on G.

Definition 4.2. Let $C > 0$. For $(g, x), (h, y) \in G \times \overline{X}$ set

$$d_C((g, x), (h, y)) = \inf \sum_{i=1}^{n} C d_{\overline{X}}\big(g_i^{-1} x_{i-1}, g_i^{-1} x_i\big) + d_G(g_{i-1}, g_i),$$

where the infimum is taken over all finite sequences $(g_0, x_0), (g_1, x_1), \ldots,$ (g_n, x_n) with $(g_0, x_0) = (g, x)$ and $(g_n, x_n) = (h, y)$.

Proposition 4.3

(*i*) *d_C defines a G-invariant metric on $G \times \overline{X}$, with respect to the diagonal action;*
(*ii*) *$d_G(g, h) \le d_C((g, x), (h, y))$ for all $g, h \in G$ and $x, y \in \overline{X}$;*
(*iii*) *$d_G(g, h) = d_C((g, x), (h, x))$ for all $g, h \in G$ and $x \in \overline{X}$;*

Proof. (i) It is immediate from the definition that d_C is G-invariant, and satisfies the triangle inequality. Because $d_G(g, h) \ge 1$ for all $g \ne h$ we have $d_C((g, x), (h, y)) \ge C d_{\overline{X}}(g^{-1}x, g^{-1}y)\}$ if $g = h$, and $d_C((g, x), (h, y)) \ge 1$ if $g \ne h$, for all $(g, x), (h, y) \in G \times \overline{X}$. Hence d_C is a metric.

(ii) and (iii) are obvious. □

For $C = 1$ we will denote the restriction of d_1 to $\{g\} \times \overline{X} = \overline{X}$ by d_g. Note that considered as a metric on $\overline{X}$ this metric varies with g. Often we will be interested in d_e, where e denotes the unit element in G. (If the diameter of $d_{\overline{X}}$ is less than 2, then d_e will in fact coincide with $d_{\overline{X}}$, but this will not be important for us.) Proposition 4.3 (i) implies that $d_g(x, y) = d_e(g^{-1}x, g^{-1}y)$ for $g \in G$ and $x, y \in \overline{X}$.

4.4. The diagram. Let $\overline{X}$ be the G-space appearing in Theorem 1.1. Choose a G-CW complex E which is a model for $E_{\mathcal{F}}G$, the classifying space for the family $\mathcal{F}$. Fix an $N \in \mathbb{N}$ as it appears in Assumption 1.4 and for every

$n \in \mathbb{N}$ choose an open $\mathcal{F}$-cover $\mathcal{U}(n)$ of $G \times \overline{X}$ satisfying the conditions in Assumption 1.4 with $\beta = n$, i.e., the dimension of $\mathcal{U}(n)$ is smaller than N and for every $(g, x) \in G \times \overline{X}$ we can find $U \in \mathcal{U}(n)$ such that $\{g\}^n \times \{x\} \subset U$. Here $\{g\}^n$ denotes the open ball with respect to the word-metric d_G in G of radius n around g. According to Lemma 5.1 below we can choose for every $n \in \mathbb{N}$ a constant $C(n)$ such that the open $\mathcal{F}$-cover $\mathcal{U}(n)$ satisfies the following condition:

> For every $(g, x) \in G \times \overline{X}$ there exists a $U \in \mathcal{U}(n)$ such that the open ball of radius n with respect to the metric $d_{C(n)}$ around the point (g, x) lies in U.

We will use the following sequences of metric spaces with free isometric G-action

$$(G \times \overline{X}, d_{C(n)})_{n \in \mathbb{N}}, \quad \big(G \times |\mathcal{U}(n)|, d_n^1\big)_{n \in \mathbb{N}}.$$

Here the metric d_n^1 is a product metric of the l^1-metric on the simplicial complex $|\mathcal{U}(n)|$ scaled by the factor n and the word-metric d_G on G, i.e.,

$$d_n^1((g, x), (h, y)) = d_G(g, h) + n d^1(x, y).$$

The map $G \times \overline{X} \to G \times |\mathcal{U}(n)|$ defined by $(g, x) \mapsto (g, f^{\mathcal{U}(n)}(g, x))$ satisfies Condition (3.9) and yields the functor

$$F^{\mathcal{U}(n)} \colon \mathcal{O}^G(E, G \times \overline{X}, d_{C(n)}) \to \mathcal{O}^G\big(E, G \times |\mathcal{U}(n)|, d_n^1\big).$$

We will construct the following diagram of additive categories around which the proof is organized. Here the arrows labelled inc are the obvious inclusions. The functors p_k and q_k are defined by first projecting onto the k-th factor and then applying the projection map $G \times \overline{X} \to G$ and $G \times |\mathcal{U}(k)| \to G$ respectively. Both projections clearly satisfy Condition (3.9).

(4.4)

$$\begin{tikzcd}
 & \bigoplus_{n \in \mathbb{N}} \mathcal{O}^G\big(E, G \times |\mathcal{U}(n)|, d_n^1\big) \arrow[d, "(3)"] \\
\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n \in \mathbb{N}}) \arrow[r, dashed, "(2)"] \arrow[d, "\mathrm{inc}"] & \mathcal{O}^G\big(E, \big(G \times |\mathcal{U}(n)|, d_n^1\big)_{n \in \mathbb{N}}\big) \arrow[d, "\mathrm{inc}"] \\
\prod_{n \in \mathbb{N}} \mathcal{O}^G(E, G \times \overline{X}, d_{C(n)}) \arrow[r, "\prod_{n \in \mathbb{N}} F^{\mathcal{U}(n)}"] \arrow[d, "p_k"] & \prod_{n \in \mathbb{N}} \mathcal{O}^G\big(E, G \times |\mathcal{U}(n)|, d_n^1\big) \arrow[d, "q_k"] \\
\mathcal{O}^G(E) \arrow[r, "\mathrm{id}"] \arrow[uu, dashed, bend left, "(1)"] & \mathcal{O}^G(E)
\end{tikzcd}$$

The lower square commutes. In the remaining sections we will establish the following facts.

(4.5) After applying $K_m(-)$ for $m \geq 1$ to the diagram the dotted arrow (1) exists and has the property that $K_m(p_k \circ \mathrm{inc}) \circ (1)$ is the identity on $K_m(\mathcal{O}^G(E))$ for all $k \in \mathbb{N}$. This will be proven in Theorem 6.1;

(4.6) The dotted horizontal functor (2) defined as the restriction of $\prod_{n\in\mathbb{N}} F^{\mathcal{U}(n)}$ to the indicated subcategories is well defined. This is the content of Corollary 5.6;

(4.7) The inclusion (3) from Subsect. 3.4 gives an isomorphism on K-theory. This follows from Theorem 7.2.

Proof of Theorem 1.1. According to Proposition 3.8 it suffices to show that the group $K_m(\mathcal{O}^G(E))$ vanishes for all $m \geq 1$. So for $m \geq 1$ apply K_m to Diagram (4.4). Pick an element

$$\xi \in K_m(\mathcal{O}^G(E))$$

at the lower left corner of the diagram. A quick diagram chase following the arrows (1), (2) and (3) and using Properties (4.5), (4.6) and (4.7) shows that there is

$$\eta \in K_m\Big(\bigoplus_{n\in\mathbb{N}} \mathcal{O}^G\big(E, G \times |\mathcal{U}(n)|, d_n^1\big)\Big)$$

whose image under the map induced by $q_k \circ \mathrm{inc} \circ (3)$ is ξ for all $k \in \mathbb{N}$. Since K-theory commutes with colimits (see Quillen [51, (12) on p. 20]) we have the canonical isomorphism

$$\bigoplus_{n\in\mathbb{N}} K_m\big(\mathcal{O}^G\big(E, G \times |\mathcal{U}(n)|, d_n^1\big)\big) \xrightarrow{\cong} K_m\Big(\bigoplus_{n\in\mathbb{N}} \mathcal{O}^G\big(E, G \times |\mathcal{U}(n)|, d_n^1\big)\Big).$$

Hence there exists a $k = k(\eta) \in \mathbb{N}$ such that for the projection pr_k onto the k-th factor we get $K_m(\mathrm{pr}_k)(\eta) = 0$. This implies that the image of η under the map induced by $q_k \circ \mathrm{inc} \circ (3)$ is trivial as well. This implies $\xi = 0$. □

5. Contracting maps induced by wide covers

In this section we will use Assumption 1.4 to prove (4.6).

Lemma 5.1. *Let $\beta \geq 1$. Suppose that $\mathcal{U}(\beta)$ is an open $\mathcal{F}$-cover of $G \times \overline{X}$ as it appears in Assumption 1.4, i.e., for every $(g, x) \in G \times \overline{X}$ there exists $U \in \mathcal{U}(\beta)$ such that $\{g\}^\beta \times \{x\} \subset U$. Then there exists a constant $C = C(\mathcal{U}(\beta)) > 1$ such that the following holds:*

> *For every $(g, x) \in G \times \overline{X}$ there exists $U \in \mathcal{U}(\beta)$ such that the open β-ball with respect to the metric d_C around (g, x) lies in U.*

Proof. For every $z \in \overline{X}$ we can find by assumption $U_z \in \mathcal{U}(\beta)$ with $\{e\}^\beta \times \{z\} \subseteq U_z$, where $e \in G$ is the unit element. Choose for $g \in \{e\}^\beta$

an open neighborhood $V_{g,z}$ of $z \in \overline{X}$ such that $\{g\} \times V_{g,z} \subseteq U_z$. Put $V_z := \bigcap_{g \in \{e\}^\beta} V_{g,z}$. Then $\{V_z \mid z \in \overline{X}\}$ is an open cover of the compact metric space $(\overline{X}, d_{\overline{X}})$. Let $\varepsilon > 0$ be a Lebesgue number for this open cover, i.e., for $x \in \overline{X}$ the ball x^ε lies in $V_{z(x)}$ for an appropriate $z(x) \in \overline{X}$.

Since $\overline{X}$ is compact, the map $\overline{X} \to \overline{X}$, $x \mapsto gx$ is uniformly continuous. Hence we can find $\delta(\varepsilon, g) > 0$ such that $d_{\overline{X}}(gx, gy) < \frac{\varepsilon}{\beta}$ holds for all $x, y \in \overline{X}$ with $d_{\overline{X}}(x, y) < \delta(\varepsilon, g)$. Since there are only finitely many elements in $\{e\}^\beta$, we can choose a constant C such that $\frac{\beta}{C} < \delta(\varepsilon, g)$ holds for all $g \in \{e\}^\beta$. Thus we get

$$(5.2) \quad d_{\overline{X}}(gx, gy) < \frac{\varepsilon}{\beta} \quad \text{for } x, y \in \overline{X} \text{ with } d_{\overline{X}}(x, y) < \frac{\beta}{C} \text{ and } g \in \{e\}^\beta.$$

Because d_C and the cover $\mathcal{U}$ are G-invariant, it suffices to prove the claim for an element of the shape $(e, x) \in G \times \overline{X}$. Let (h, y) be an element in the ball of radius β around (e, x) with respect to the metric d_C. We want to show $(h, y) \in U_{z(x)}$. By definition of d_C we can find a sequence of elements $(e, x) = (g_0, x_0), (g_1, x_1), \ldots, (g_{n-1}, x_{n-1}), (g_n, x_n) = (h, y)$ in $G \times \overline{X}$ such that

$$\sum_{i=1}^{n} d_G(g_{i-1}, g_i) + \sum_{i=1}^{n} C \cdot d_{\overline{X}}\big(g_i^{-1} x_{i-1}, g_i^{-1} x_i\big) < \beta.$$

We can arrange $g_{i-1} \neq g_i$, otherwise delete the element (g_i, x_i) from the sequence, the inequality above remains true because of the triangle inequality for $d_{\overline{X}}$. Since $d_G(g_{i-1}, g_i) \geq 1$, we conclude

$$n \leq \beta.$$

By the triangle inequality $d_G(e, g_i) \leq \beta$ for $i = 1, 2, \ldots, n$. In other words $g_i \in \{e\}^\beta$ for $i = 1, 2, \ldots, n$.

We have $d_{\overline{X}}(g_i^{-1} x_{i-1}, g_i^{-1} x_i) < \frac{\beta}{C}$ for $i = 1, 2, \ldots, n$. We conclude from (5.2) that

$$d_{\overline{X}}(x_{i-1}, x_i) < \frac{\varepsilon}{\beta}$$

holds for $i = 1, 2, \ldots, n$. The triangle inequality implies together with $n \leq \beta$

$$d_{\overline{X}}(x, y) < \varepsilon.$$

Hence $y \in V_{z(x)}$. Since $h \in \{e\}^\beta$ holds, we conclude $y \in V_{z(x)} \subseteq V_{h,z(x)}$. This implies $(h, y) \in U_{z(x)}$. □

The following proposition yields contracting properties of the map from a metric space to the nerve of an open cover of the space. Similar ideas appear already in [35, Sect. 1].

Proposition 5.3. *Let $X = (X, d)$ be a metric space and let $\beta \geq 1$. Suppose $\mathcal{U}$ is an open cover of X of dimension less than or equal to N with the following property:*

> *For every $x \in X$ there exists $U \in \mathcal{U}$ such that the β-ball around x lies in U.*

Then the map $f^{\mathcal{U}}\colon X \to |\mathcal{U}|$ (defined in Subsect. 4.1) has the following contracting property. If $d(x, y) \leq \frac{\beta}{4N}$ then

$$d^1(f^{\mathcal{U}}(x), f^{\mathcal{U}}(y)) \leq \frac{16N^2}{\beta} d(x, y).$$

Note that if β gets bigger, the estimate applies more often and $f^{\mathcal{U}}$ contracts stronger.

Proof. Recall that $f^{\mathcal{U}}(x) = \sum_U f_U(x)U$, where $f_U(x) = \frac{a_U(x)}{\sum_V a_V(x)}$ with $a_U(x) = d(x, X - U) = \inf\{d(x, u) \mid u \notin U\}$. For every $V \in \mathcal{U}$ we set $b_V(x, y) = a_V(x) - a_V(y)$. Since d is a metric we have $|b_V(x, y)| \leq d(x, y)$. Since the covering dimension is smaller than N there are at most $2N$ covering sets V for which $b_V(x, y) \neq 0$. Hence we have

$$\sum_V |b_V(x, y)| \leq 2Nd(x, y) \leq \frac{\beta}{2}. \tag{5.4}$$

For every x there exists by assumption $U \in \mathcal{U}$ such that the β-ball around x is contained in U. For this U we have

$$\sum_V a_V(x) \geq a_U(x) \geq \beta. \tag{5.5}$$

We compute

$$f_U(y) - f_U(x) = \frac{a_U(x) \sum_V b_V(x, y) - b_U(x, y) \sum_V a_V(x)}{\left(\sum_V a_V(x)\right)\left(\sum_V a_V(x) - b_V(x, y)\right)}.$$

Now one can estimate using (5.4) for the third, (5.4) and (5.5) for the fourth inequality and (5.4) for the last inequality.

$$\begin{aligned}
\sum_U |f_U(x) - f_U(y)| &\leq \sum_U \left| \frac{\sum_V b_V(x, y)}{\sum_V a_V(x) - b_V(x, y)} \right| \\
&\quad + \sum_U \left| \frac{b_U(x, y)}{\sum_V a_V(x) - b_V(x, y)} \right| \\
&\leq 4N \frac{\sum_V |b_V(x, y)|}{\left|\sum_V a_V(x) - b_V(x, y)\right|} \\
&\leq 4N \frac{2Nd(x, y)}{\left|\sum_V a_V(x) - b_V(x, y)\right|}
\end{aligned}$$

$$\leq \frac{8N^2 d(x,y)}{\sum_V a_V(x) - \sum |b_V(x,y)|}$$
$$\leq \frac{8N^2 d(x,y)}{\beta - 2Nd(x,y)} \leq \frac{8N^2 d(x,y)}{\beta - \frac{\beta}{2}} = \frac{16N^2 d(x,y)}{\beta}. \qquad \square$$

Combining these two statements we can now establish (4.6).

Corollary 5.6. *The map* (2) *in diagram* (4.4) *is well defined.*

Proof. Let $\varphi = (\varphi_n)$ be a morphism in the source, then there exists a constant $K = K(\varphi)$ such that for every $n \in \mathbb{N}$ we have that $((g, x, e, t), (g', x', e', t')) \in \operatorname{supp} \varphi_n \subset (G \times \overline{X} \times E \times [1, \infty))^{\times 2}$ implies $d_{C(n)}((g, x), (g', x')) \leq K$. By Proposition 4.3(ii) it suffices to show that there exists a constant L such that

$$nd^1(f^{\mathcal{U}(n)}((g,x)), f^{\mathcal{U}(n)}((g',x'))) \leq L,$$

compare (3.10). By the construction of the sequence $(C(n))_{n\in\mathbb{N}}$ the assumptions in Proposition 5.3 are satisfied for the cover $\mathcal{U}(n)$ of $G \times \overline{X}$ with $\beta = n$ for every $n \in \mathbb{N}$. We conclude for $n \geq 4KN$ and $((g, x, e, t), (g', x', e', t')) \in \operatorname{supp} \varphi_n$ that

$$\begin{aligned} nd^1(f^{\mathcal{U}(n)}((g,x)), f^{\mathcal{U}(n)}((g',x'))) &\leq 16N^2 d_{C(n)}((g,x),(g',x')) \\ &\leq 16N^2 K =: L. \end{aligned}$$

The distance of two points of a simplicial complex with respect to the l^1-metric is at most 2. Because $(4KN) \cdot 2 \leq L$ this implies that the above estimate holds in fact for all n. $\square$

6. The transfer

In this section we will use Assumption 1.2 to deal with the dotted arrow (1) in Diagram (4.4). The following result establishes (4.5).

Theorem 6.1. *Let* $m \geq 1$. *There exists a map*

$$\operatorname{trans}\colon K_m \mathcal{O}^G(E) \to K_m \mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n\in\mathbb{N}})$$

that is for all $k \in \mathbb{N}$ *a right inverse for the map induced by* $\operatorname{pr}_k := p_k \circ \operatorname{inc}$, *compare* (4.4).

This will be proven as follows. For an additive category $\mathcal{O}$ of the type appearing above we define Waldhausen categories $\operatorname{ch}_{\mathrm{hfd}} \mathcal{O}$ and $\widetilde{\operatorname{ch}}_{\mathrm{hfd}} \mathcal{O}$ together with natural inclusions

$$\mathcal{O} \xrightarrow{\operatorname{inc}} \operatorname{ch}_{\mathrm{hfd}} \mathcal{O} \xrightarrow{\operatorname{inc}} \widetilde{\operatorname{ch}}_{\mathrm{hfd}} \mathcal{O}$$

that induce isomorphisms on K_m for every $m \geq 1$, compare Lemma 6.5. We then construct the functor trans in Subsect. 6.4 (see in particular Proposition 6.13) in order to obtain for every $k \in \mathbb{N}$ the following diagram of Waldhausen categories and exact functors

$$\begin{tikzcd} & \widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E, (G \times \overline{X}, d_C(n))_{n\in\mathbb{N}}) \arrow[d, "\widetilde{\mathrm{ch}}_{\mathrm{hfd}}(\mathrm{pr}_k)"] \\ \mathcal{O}^G(E) \arrow[ur, "\mathrm{trans}"] \arrow[r, "\mathrm{inc}"] & \widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E). \end{tikzcd} \tag{6.2}$$

In Lemma 6.16 we show that this diagram commutes after applying K-theory. From this Theorem 6.1 follows.

6.1. Review of the classical transfer. As a motivation for the forthcoming construction we briefly review the transfer for the Whitehead group associated to a fibration $F \to E \xrightarrow{p} B$ of connected finite CW-complexes. Recall that the fiber transport gives a homomorphism $t\colon \pi_1(B) \to [F, F]$. Under mild conditions on t one can define geometrically a transfer homomorphism $\mathrm{trans}\colon \mathrm{Wh}(B) \to \mathrm{Wh}(E)$ by sending the Whitehead torsion of a homotopy equivalence $f\colon B' \to B$ of finite CW-complexes to the Whitehead torsion of the pull back $\overline{f}\colon p^*E \to E$ (see [1], [43, Sect. 5]). An algebraic description in terms of chain complexes is given in [43, Sect. 4] and is identified with the geometric construction. It involves the chain complex of an appropriate cover of F and the action up to homotopy of $\pi_1(B)$ coming from the fiber transport. The map $\mathrm{trans}\colon \mathrm{Wh}(B) \to \mathrm{Wh}(E)$ is bijective if F is contractible.

The desired transfer

$$\mathrm{trans}\colon \mathcal{O}^G(E) \to \widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n\in\mathbb{N}})$$

is a controlled version on the category level of the algebraic description of the classical transfer above in the situation $G \times E \times \overline{X} \to G \times E$ which one may consider after dividing out the diagonal G-action as a flat bundle with the contractible space $\overline{X}$ as fiber and $(G \times E)/G \cong E$ as base space. The group G plays the role of $\pi_1(B)$ and the fiber transport comes from the honest G-action on $\overline{X}$. Having this in mind it becomes clear why in the sequel we will have to deal with categories of chain complexes.

6.2. Some preparations. Fix an infinite cardinal κ. Let $\mathcal{F}^\kappa(\mathbb{Z})$ be a small model for the category of all free $\mathbb{Z}$-modules which admit a basis B with $\mathrm{card}(B) \leq \kappa$. Let $\mathcal{F}^f(\mathbb{Z})$ be the full subcategory of $\mathcal{F}^\kappa(\mathbb{Z})$ that consists of finitely generated free $\mathbb{Z}$-modules. These categories will always be equipped with the trivial G-action. Let $\mathcal{A}$ be an additive category with G-action. According to [7, Lemma 9.2] there exist additive categories with G-action $\mathcal{A}^f$ and $\mathcal{A}^\kappa$ together with G-equivariant inclusion functors

$$\mathcal{A} \to \mathcal{A}^f \to \mathcal{A}^\kappa.$$

In $\mathcal{A}^\kappa$ there exist direct sums with indexing sets of cardinality than or equal to κ and $\mathcal{A} \to \mathcal{A}^f$ is an equivalence of categories. There exists a "tensor product", i.e., a bilinear functor

$$- \otimes -\colon \mathcal{A}^\kappa \times \mathcal{F}^\kappa(\mathbb{Z}) \to \mathcal{A}^\kappa \tag{6.3}$$

which is compatible with the G-action on $\mathcal{A}^\kappa$, i.e., $g^*(A \otimes M) = g^*A \otimes M$ and restricts to

$$- \otimes -\colon \mathcal{A}^f \times \mathcal{F}^f(\mathbb{Z}) \to \mathcal{A}^f.$$

For all practical purposes we can and will identify $\mathcal{A}$ with $\mathcal{A}^f$.

For a G-space Y and a metric space (Z, d) with a free action of G by isometries we define the category

$$\overline{\mathcal{O}}^G(Y, Z, d; \mathcal{A}^\kappa)$$

in the same way as before but we replace $\mathcal{A}$ by $\mathcal{A}^\kappa$ and drop the assumption that the support of objects is locally finite. Moreover, instead of defining a morphism φ to be a family of morphisms $\varphi_{y,x}$ and requiring that for fixed x, respectively y, the sets $\{y \mid \varphi_{y,x} \neq 0\}$, respectively $\{x \mid \varphi_{y,x} \neq 0\}$, are finite we define a morphism $\varphi\colon A = (A_x) \to B = (B_y)$ to be a morphism $\bigoplus_x A_x \to \bigoplus_y B_y$ in the category $\mathcal{A}^\kappa$. Note that for suitably chosen κ these direct sums exist in $\mathcal{A}^\kappa$, compare [7, Lemma 9.2]. For a sequence $(Z_n, d_n)_{n\in\mathbb{N}}$ of metric spaces with G-action by isometries we define

$$\overline{\mathcal{O}}^G\big(Y, (Z_n, d_n)_{n\in\mathbb{N}}; \mathcal{A}^\kappa\big) \subset \prod_{n\in\mathbb{N}} \overline{\mathcal{O}}^G\big(Y, Z_n, d_n; \mathcal{A}^\kappa\big)$$

by requiring conditions on the morphisms precisely as in Subsect. 3.4. One should think of the inclusions

$$\begin{aligned} \mathcal{O}^G(Y; \mathcal{A}) &\subset \overline{\mathcal{O}}^G(Y; \mathcal{A}^\kappa), \\ \mathcal{O}^G(Y, (Z_n, d_n)_{n\in\mathbb{N}}; \mathcal{A}) &\subset \overline{\mathcal{O}}^G\big(Y, (Z_n, d_n)_{n\in\mathbb{N}}; \mathcal{A}^\kappa\big), \quad \text{etc.} \end{aligned} \tag{6.4}$$

as inclusions of full subcategories on objects satisfying finiteness conditions into large categories which give room for constructions. The prototype of such a situation is the inclusion $\mathcal{F}^f(\mathbb{Z}) \subset \mathcal{F}^\kappa(\mathbb{Z})$.

Let $\mathcal{O}$ be an additive category. We write $\mathrm{Idem}(\mathcal{O})$ for its idempotent completion. We define $\mathrm{ch}_\mathrm{f}\,\mathcal{O}$ to be the category of chain complexes in $\mathcal{O}$ that are bounded above and below and $\mathrm{ch}^{\geq}\,\mathcal{O}$ to be the category of chain complexes that are bounded below. For these categories the notion of chain homotopy leads to a notion of weak equivalence, and we define cofibrations to be those chain maps which are degreewise the inclusion of a direct summand.

Now let $\overline{\mathcal{O}}$ be an additive category and let $\mathcal{O} \subset \overline{\mathcal{O}}$ be the inclusion of a full additive subcategory. We write

$$\mathrm{ch}_{\mathrm{hf}}(\mathrm{Idem}(\mathcal{O}) \subset \mathrm{Idem}(\overline{\mathcal{O}}))$$

for the full subcategory of $\mathrm{ch}^{\geq} \operatorname{Idem}(\overline{\mathcal{O}})$ consisting of chain complexes that are chain homotopy equivalent to a chain complex in $\mathrm{ch}_{\mathrm{f}} \operatorname{Idem}(\mathcal{O})$. We write

$$\mathrm{ch}_{\mathrm{hfd}}(\mathcal{O} \subset \operatorname{Idem}(\overline{\mathcal{O}}))$$

for the full subcategory of $\mathrm{ch}^{\geq} \operatorname{Idem}(\overline{\mathcal{O}})$ consisting of objects C which are homotopy retracts of objects in $\mathrm{ch}_{\mathrm{f}} \mathcal{O}$, i.e., there exists a diagram $C \xrightarrow{i} D \xrightarrow{r} C$ with D in $\mathrm{ch}_{\mathrm{f}} \mathcal{O}$ such that the composition $r \circ i$ is chain homotopic to the identity on C.

Lemma 6.5. *We have an equality*

$$\mathrm{ch}_{\mathrm{hf}}(\operatorname{Idem}(\mathcal{O}) \subset \operatorname{Idem}(\overline{\mathcal{O}})) = \mathrm{ch}_{\mathrm{hfd}}(\mathcal{O} \subset \operatorname{Idem}(\overline{\mathcal{O}}))$$

and the inclusions

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[d] & \mathrm{ch}_{\mathrm{f}} \mathcal{O} \arrow[r] \arrow[d] & \mathrm{ch}_{\mathrm{hfd}}(\mathcal{O} \subset \operatorname{Idem}(\overline{\mathcal{O}})) \arrow[d, equal] \\ \operatorname{Idem}(\mathcal{O}) \arrow[r] & \mathrm{ch}_{\mathrm{f}} \operatorname{Idem}(\mathcal{O}) \arrow[r] & \mathrm{ch}_{\mathrm{hf}}(\operatorname{Idem}(\mathcal{O}) \subset \operatorname{Idem}(\overline{\mathcal{O}})) \end{tikzcd}$$

induce equivalences on K_m for all $m \geq 1$.

Proof. Suppose C is a chain complex in $\mathrm{ch}_{\mathrm{f}} \operatorname{Idem}(\mathcal{O})$. Then by adding elementary chain complexes of the form

$$\cdots \to 0 \to 0 \to P \xrightarrow{\mathrm{id}} P \to 0 \to 0 \to \ldots$$

one can produce a chain homotopy equivalent chain complex C' in $\operatorname{Idem}(\mathcal{O})$ such that all C'_i except the one in the top-non-vanishing dimension n lie in $\mathcal{O}$ instead of $\operatorname{Idem}(\mathcal{O})$. By adding a complement to C'_n one can easily produce a chain complex in $\mathrm{ch}_{\mathrm{f}} \mathcal{O}$ which contains C' as an (honest) retract. Since the homotopy relation is transitive this proves the inclusion "$\subset$".

Suppose we have $C \xrightarrow{i} D \xrightarrow{r} C$ with $r \circ i \simeq \mathrm{id}$, where C belongs to $\mathrm{ch}^{\geq} \operatorname{Idem}(\overline{\mathcal{O}})$ and D to $\mathrm{ch}_{\mathrm{f}} \mathcal{O}$. Then the proof of Proposition 11.11 in [44] yields a chain complex D' in $\mathrm{ch}^{\geq} \mathcal{O}$ which is chain homotopic to C and of a special form. Namely, there exists an $n \in \mathbb{Z}$ and an object D'_∞ such that $D'_m = D'_\infty$ for all $m \geq n$. Moreover there exists a map $p\colon D'_\infty \to D'_\infty$ with $p \circ p = p$ such that the chain complex is 2-periodic above n and of the form

$$\ldots \xrightarrow{1-p} D'_\infty \xrightarrow{p} D'_\infty \xrightarrow{1-p} D'_\infty \xrightarrow{p} D'_\infty = D'_n \to D'_{n-1} \to D'_{n-2} \to \ldots .$$

In $\mathrm{ch}^{\geq} \operatorname{Idem}(\mathcal{O})$ this chain complex is homotopic to

$$\ldots \to 0 \to 0 \to (D'_\infty, p) \xrightarrow{p} (D'_n, \mathrm{id}) \to (D'_{n-1}, \mathrm{id}) \to (D'_{n-2}, \mathrm{id}) \to \ldots .$$

This proves the other inclusion. The two horizontal inclusions on the left are well known to induce isomorphisms on K_m, for $m \geq 0$, compare [56, 16].

The lower horizontal inclusion on the right induces an isomorphism on K_m, for $m \geq 0$ by an application of the Approximation Theorem 1.6.7 in [57]. The vertical inclusion on the left induces an isomorphism on K_m for $m \geq 1$ by the cofinality theorem, compare Theorem 2.1 in [55]. □

Notation 6.6. In the following we will use the abbreviation

$$\mathrm{ch}_{\mathrm{hfd}}\, \mathcal{O} = \mathrm{ch}_{\mathrm{hfd}}(\mathcal{O} \subset \mathrm{Idem}(\overline{\mathcal{O}}))$$

because $\overline{\mathcal{O}}$ will be clear from the context. In fact we will always be in the situation described in (6.4).

We recall from [8, Subsect. 8.2] that for a given Waldhausen category $\mathcal{W}$ there exists a Waldhausen category $\widetilde{\mathcal{W}}$ whose objects are sequences

$$C_0 \xrightarrow{c_0} C_1 \xrightarrow{c_1} C_2 \xrightarrow{c_2} \ldots,$$

where the c_n are morphisms in $\mathcal{W}$ that are simultaneously cofibrations and weak equivalences. A morphism f in $\widetilde{\mathcal{W}}$ is represented by a sequence of morphisms $(f_m, f_{m+1}, f_{m+2}, \ldots)$ which makes the diagram

$$\begin{tikzcd} C_m \arrow[r, "c_m"] \arrow[d, "f_m"] & C_{m+1} \arrow[r, "c_{m+1}"] \arrow[d, "f_{m+1}"] & C_{m+2} \arrow[r, "c_{m+2}"] \arrow[d, "f_{m+2}"] & \cdots \\ D_{m+k} \arrow[r, "d_{m+k}"] & D_{m+k+1} \arrow[r, "d_{m+k+1}"] & D_{m+k+2} \arrow[r, "d_{m+k+2}"] & \cdots \end{tikzcd}$$

commutative. Here m and k are non-negative integers. If we enlarge m or k the resulting diagrams represent the same morphism, i.e., we identify $(f_m, f_{m+1}, f_{m+2}, \ldots)$ with $(f_{m+1}, f_{m+2}, f_{m+3}, \ldots)$ but also with $(d_{m+k} \circ f_m, d_{m+k+1} \circ f_{m+1}, d_{m+k+2} \circ f_{m+2}, \ldots)$. Sending an object to the constant sequence defines an inclusion

$$\mathcal{W} \to \widetilde{\mathcal{W}}.$$

According to [8, Proposition 8.2] the inclusion induces an isomorphism on K_m for $m \geq 0$ under some mild conditions about $\mathcal{W}$. These conditions will be satisfied in all our examples.

6.3. The singular chain complex of $\overline{X}$. In the next subsection we will construct the functor denoted trans in Diagram (6.2). It will essentially replace objects $A \in \mathcal{A}$ by $A \otimes C_*^{\mathrm{sing},\delta}(\overline{X}, d_e)$. Here we use a chain subcomplex of the singular chain complex and consider it as a chain complex over $\overline{X}$. We now collect some facts about the singular chain complex of a metric space that will be needed in the construction of the transfer.

Let $X = (X, d)$ be a metric space. As before we denote the singular chain complex of X by $C_*^{\mathrm{sing}}(X)$. For $\delta > 0$ we define

$$C_*^{\mathrm{sing},\delta}(X, d) \subset C_*^{\mathrm{sing}}(X)$$

as the chain subcomplex generated by all singular simplices $\sigma\colon \Delta \to X$ for which the diameter of $\sigma(\Delta)$ is less or equal to δ, i.e., for all $y, z \in \Delta$ we have $d(\sigma(y), \sigma(z)) \leq \delta$.

This chain complex can be considered as a chain complex over X via the barycenter map, i.e., for $x \in X$ the module $C_n^{\mathrm{sing},\delta}(X, d)_x$ is generated by all singular n-simplices which satisfy the condition above and map the barycenter to x. A map $f\colon C_* \to D_*$ of chain complexes over X is called a δ-controlled homotopy equivalence if there exists a chain homotopy inverse g and chain homotopies $h\colon f \circ g \simeq \mathrm{id}$ and $h'\colon g \circ f \simeq \mathrm{id}$ such that f, g, h and h' are all δ-controlled when considered as morphisms over X, see Subsect. 3.2.

Lemma 6.7. *Let $X = (X, d)$ be a metric space.*

(*i*) *For $\delta' > \delta > 0$ the inclusion*

$$i\colon C_*^{\mathrm{sing},\delta}(X, d) \to C_*^{\mathrm{sing},\delta'}(X, d)$$

is a δ'-controlled chain homotopy equivalence;

(*ii*) *For every $\delta > 0$ the inclusion*

$$i\colon C_*^{\mathrm{sing},\delta}(X, d) \to C_*^{\mathrm{sing}}(X)$$

is a chain homotopy equivalence;

(*iii*) *Suppose $X = |T|$ is a simplicial complex, i.e., the realization of an abstract simplicial complex T. Let $C_*(T)$ denote the simplicial chain complex considered as a chain complex over $X = |T|$ via the barycenters. Suppose all simplices of $|T|$ have diameter smaller than δ. Then realization defines a map*

$$C_*(T) \to C_*^{\mathrm{sing},\delta}(X, d)$$

which is a δ-controlled chain homotopy equivalence.

Proof. (i) Let $\mathcal{C}$ denote the category whose objects are the closed subsets of X and whose morphisms are the inclusions. We can consider

$$X \supset A \mapsto C_*^{\mathrm{sing},\delta}(A) = C_*^{\mathrm{sing},\delta}(A, d|_A)$$

as a functor from $\mathcal{C}$ to the category of chain complexes, i.e., as a $\mathbb{Z}\mathcal{C}$-chain complex. (For basic definitions and facts of $\mathbb{Z}\mathcal{C}$-modules we refer to [44, Sect. 9].) We claim that the inclusion $C_*^{\mathrm{sing},\delta}(?) \to C_*^{\mathrm{sing},\delta'}(?)$ is a chain homotopy equivalence of $\mathbb{Z}\mathcal{C}$-chain complexes. Note that for every $n \in \mathbb{Z}$

$$C_n^{\mathrm{sing},\delta}(?) = \bigoplus_{\sigma} \mathbb{Z}\,\mathrm{mor}_{\mathcal{C}}(\sigma(\Delta), ?)$$

is a free $\mathbb{Z}\mathcal{C}$-chain complex, here the sum runs over all singular simplices in X whose image have a diameter less or equal to δ. Because of the fundamental theorem for homological algebra in the setting of $R\mathcal{C}$-chain complexes (see [44, Lemma 11.7 on p. 220]), it suffices to prove that for every closed subset $A \subset X$ the inclusion

$$C_*^{\mathrm{sing},\delta}(A) \to C_*^{\mathrm{sing},\delta'}(A) \tag{6.8}$$

induces a homology isomorphism. In order to see this one uses that the usual subdivision chain selfmap sd of the singular chain complex restricts to a selfmap of $C^{\mathrm{sing},\delta'}(A)$ and so does the chain homotopy proving that sd is homotopic to the identity. Moreover for each individual singular simplex σ in $C_*^{\mathrm{sing},\delta'}(A)$ there exists an N, such that $\mathrm{sd}^N \sigma$ lies in $C^{\mathrm{sing},\delta}(A)$ by a Lebesgue-number argument.

We now have a homotopy inverse p and homotopies h and h' as maps of $\mathbb{Z}\mathcal{C}$-chain complexes. Evaluating p at X yields a chain homotopy inverse p_X of ordinary chain complexes that restricts to every closed subset of X. In particular for every singular simplex $\sigma\colon \Delta \to X$ in $C_*^{\mathrm{sing},\delta'}(X,d)$ the image under p_X lies in $C_*^{\mathrm{sing},\delta}(\sigma(\Delta),d)$. Hence p_X considered as a morphism over X is bounded by δ' because $\sigma(\Delta)$ lies within a δ'-ball around $\sigma(\mathrm{bary}(\Delta))$ and the same argument works for the homotopies h_X and h'_X.

(ii) It suffices to prove that the inclusion induces a homology isomorphism. This is a less careful version of the argument used above for the map (6.8).

(iii) The proof starts similar to the proof of Assertion (i). Instead of the category $\mathcal{C}$ of closed subsets and inclusions one works with the category of simplicial subcomplexes of T and inclusions. Let $S \subset T$ be a simplicial subcomplex then the composition in the sequence

$$C_*(S) \to C_*^{\mathrm{sing},\delta}(|S|) \to C_*^{\mathrm{sing}}(|S|)$$

is well known to be a homology isomorphism and the second map is a homology isomorphism by Assertion (ii). If we evaluate at $S = T$ we see that the map in question is a homotopy equivalence and that the homotopy inverse and the homotopies can be chosen in such a way that they restrict to every simplex. Since the simplices have diameter at most δ we see that these maps are δ-controlled. $\square$

Next we prove for $\overline{X}$ as in Theorem 1.1 and the metric d_e from Subsect. 4.3 that $C_*^{\mathrm{sing},\delta}(\overline{X},d_e)$ is homotopy finitely dominated in a controlled sense. We will make use of Assumption 1.2, i.e., we assume the following.

There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$.

Lemma 6.9. *Let $\delta > 0$ be given. There exists a finite chain complex D_*^δ in $\mathrm{ch}_{\mathrm{f}}\, \mathcal{C}(\overline{X})$ all whose differentials are δ-controlled with respect to d_e together with maps*

$$C_*^{\mathrm{sing},\delta}(\overline{X}) \xrightarrow{i} D_*^\delta \xrightarrow{r} C_*^{\mathrm{sing},\delta}(\overline{X})$$

and a chain homotopy $h\colon r \circ i \simeq \mathrm{id}$ *such that* i, r *and* h *are bounded by* 6δ *as morphisms over* $\overline{X} = (\overline{X}, d_e)$.

Proof. Let H be a homotopy as in Assumption 1.2. For $t > 0$ let K_t be the union of all simplices of X that meet $H_t(\overline{X}) \subset X$. Since $H_t(\overline{X})$ is compact this is a finite simplicial subcomplex of X. Since $\overline{X} \times I$ is compact H is uniformly continuous. Since H_0 is the identity, we can find for a given $\delta > 0$ an $\varepsilon = \varepsilon(\delta) > 0$ such that $H(\{x\}^{\varepsilon} \times [0, \varepsilon]) \subset \{x\}^{\delta/2}$ holds for all $x \in \overline{X}$. We conclude that $H(\{x\}^{\delta} \times [0, \varepsilon]) \subseteq \{x\}^{2\delta}$ holds for $x \in \overline{X}$. In particular H_ε maps δ-balls to 2δ-balls. Let $\mathrm{inc}\colon K_\varepsilon \to \overline{X}$ be the inclusion. Then

$$C_*^{\mathrm{sing},\delta}(\overline{X}) \xrightarrow{(H_\varepsilon)_*} C_*^{\mathrm{sing},2\delta}(K_\varepsilon) \xrightarrow{\mathrm{inc}_*} C_*^{\mathrm{sing},2\delta}(\overline{X})$$

is well defined and the composition is homotopic to the inclusion

$$C_*^{\mathrm{sing},\delta}(\overline{X}) \xrightarrow{\mathrm{inc}_*} C_*^{\mathrm{sing},2\delta}(\overline{X})$$

via a homotopy that is 2δ-controlled. The latter inclusion is a 2δ-controlled homotopy equivalence by Lemma 6.7(i). After a suitable subdivision we can assume that in the simplicial complex $K_\varepsilon = |T_\varepsilon|$ all simplices have diameter smaller than δ. Lemma 6.7(iii) says that there exists a 2δ-controlled homotopy equivalence $C(T_\varepsilon) \to C_*^{\mathrm{sing},2\delta}(K_\varepsilon)$. Now set $D_*^{\delta} = C(T_\varepsilon)$. □

6.4. The controlled transfer. Fix an infinite cardinal κ large enough such that the following constructions make sense. For $\delta > 0$ we define a chain complex over $G \times \overline{X}$, more precisely a chain complex

$$C_*(\delta) \in \mathrm{ch}^{\geq} \overline{\mathcal{C}}^G(G \times \overline{X}; \mathcal{F}^{\kappa}(\mathbb{Z}))$$

as follows. The n-th module $C_n(\delta)$ is as a module over $G \times \overline{X}$ given by

$$C_n(\delta)_{(g,x)} = C_n^{\mathrm{sing},\delta}(\overline{X}, d_e)_{g^{-1}x}. \tag{6.10}$$

(Note that $C_n(\delta)$ is indeed an object in the subcategory that is fixed under G.) Here d_e is the (non-invariant) metric on $\overline{X}$ from Subsect. 4.3. The differential $\partial\colon C_n(\delta) \to C_{n-1}(\delta)$ is given by

$$\partial_{(g',x'),(g,x)} = \begin{cases} \partial_{g^{-1}x',g^{-1}x} & \text{if } g' = g \\ 0 & \text{otherwise} \end{cases},$$

where $\partial_{g^{-1}x',g^{-1}x}$ are the components of the differential

$$\partial\colon C_n^{\mathrm{sing},\delta}(\overline{X}, d_e) \to C_{n-1}^{\mathrm{sing},\delta}(\overline{X}, d_e),$$

considered as a map over $\overline{X}$. Note that differentials have non-diagonal support only in the $\overline{X}$-direction.

Similarly using the chain complexes D_*^δ appearing in Lemma 6.9 we define a chain complex $D_*(\delta)$ over $G \times \overline{X}$ by

$$D_*(\delta)_{(g,x)} = (D_*^\delta)_{g^{-1}x}.$$

Lemma 6.11. *Let $\delta > 0$ and $C > 1$. The chain complex $C_*(\delta)$ is a homotopy retract of the chain complex $D_*(\delta)$. The differentials of $C_*(\delta)$ and $D_*(\delta)$ and the maps and homotopies proving that $C_*(\delta)$ is a homotopy retract satisfy the following control condition. If $((g', x'), (g, x))$ lies in the support of one of these maps, then $g' = g$ and $d_C((g, x'), (g, x)) \leq 6C\delta$.*

Proof. Note that $C_*(\delta)$ is the unique G-invariant chain complex whose restriction to $\{e\} \times \overline{X} \subset G \times \overline{X}$ coincides with $C_*^{\mathrm{sing},\delta}(\overline{X}, d_e)$ considered as a chain complex over $\{e\} \times \overline{X}$ via the identification $\{e\} \times \overline{X} \cong \overline{X}$. Similarly one can extend all the maps and homotopies from Lemma 6.9 to maps over $G \times \overline{X}$. The statement about the support of these maps follows immediately from the definitions. □

Now let $(C(n))_{n\in\mathbb{N}}$ be the sequence of numbers $C(n) > 1$ that we have chosen towards the beginning of Subsect. 4.4. Assume that $(\delta(n))_{n\in\mathbb{N}}$ is a sequence of positive numbers which satisfies the following condition.

(6.12) There exists a constant $\alpha > 1$ such that $\delta(n) \leq \frac{\alpha}{C(n)}$ for all $n \in \mathbb{N}$.

Depending on the sequence $(\delta(n))_{n\in\mathbb{N}}$ we now would like to define the transfer functor

$$\mathrm{trans}\colon \mathcal{O}^G(E) \to \mathrm{ch}_{\mathrm{hfd}}\, \mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n\in\mathbb{N}}).$$

However, we will see soon that we have to modify the target category in order to get a well defined functor. In order to motivate this modification we describe the naive attempt to define the functor and explain where the problem occurs. On objects the functor should be given by

$$A \mapsto (A \otimes C_*(\delta(n)))_{n\in\mathbb{N}},$$

where $A \otimes C_k(\delta(n))$ is an object over $G \times \overline{X} \times E \times [1, \infty)$ via

$$\begin{aligned}(A \otimes C_k(\delta(n)))_{(g,x,e,t)} &= A_{(g,e,t)} \otimes C_k(\delta(n))_{(g,x,t)} \\ &= A_{(g,e,t)} \otimes C_k^{\mathrm{sing},\delta(n)}(\overline{X}, d_e)_{g^{-1}x},\end{aligned}$$

and the differentials are given by

$$(\mathrm{id} \otimes \partial(n))_{(g',x',e',t'),(g,x,e,t)} = \begin{cases} \mathrm{id} \otimes \partial_{(g',x'),(g,x)} & \text{if } (g', e', t') = (g, e, t); \\ 0 & \text{otherwise.} \end{cases}$$

Again off-diagonal support for the differentials occurs only in the $\overline{X}$-direction. Lemma 6.11 and (6.12) imply that $(A \otimes C_*(\delta(n)))_{n\in\mathbb{N}}$ is a well defined object in $\mathrm{ch}_{\mathrm{hfd}}\,\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n\in\mathbb{N}})$.

On morphisms a problem occurs. We would like to map the morphism $\varphi\colon A \to B$ with components $\varphi_{(g',e',t'),(g,e,t)}\colon A_{(g,e,t)} \to B_{(g',e',t')}$ to the morphism $(\varphi \otimes l(n))_{n\in\mathbb{N}}$ whose components are given by

$$(\varphi \otimes l(n))_{(g',x',e',t'),(g,x,e,t)} = \varphi_{(g',e',t'),(g,e,t)} \otimes l(n)_{(g',x'),(g,x)},$$

with

$$l(n)_{(g',x'),(g,x)} = \begin{cases} l(n)_{g'^{-1}g} & \text{if } x' = x; \\ 0 & \text{if } x' \neq x, \end{cases}$$

where $l(n)_{g'^{-1}g}$ is the map

$$(l_{g'^{-1}g})_*\colon C_k^{\mathrm{sing},\delta(n)}(\overline{X}, d_e)_{g^{-1}x} \to C_k^{\mathrm{sing},\delta(n)}(\overline{X}, d_e)_{g'^{-1}x}$$

which is induced by left multiplication with $g'^{-1}g$, i.e., a singular simplex $\sigma\colon \Delta \to \overline{X}$ is mapped to $l_{g'^{-1}g} \circ \sigma$, where $l_{g'^{-1}g}\colon \overline{X} \to \overline{X}$ is the map $x \mapsto g'^{-1}gx$. However the map $l(n)_{g'^{-1}g}$ is not well defined, its target is not as stated. One only has a well defined map (in fact an isomorphism)

$$(l_{g'^{-1}g})_*\colon C_k^{\mathrm{sing},\delta(n)}(\overline{X}, d_e)_{g^{-1}x} \to C_k^{\mathrm{sing},\delta(n)}(\overline{X}, d_{g'^{-1}g})_{g'^{-1}x},$$

where in the target we work with the metric $d_{g'^{-1}g}$ instead of d_e. We will compose this map with the inclusion

$$C_k^{\mathrm{sing},\delta(n)}(\overline{X}, d_{g'^{-1}g})_{g'^{-1}x} \subset C_k^{\mathrm{sing},\widetilde{\delta}(n)}(\overline{X}, d_e)_{g'^{-1}x}$$

for a suitable chosen $(\widetilde{\delta}(n))_{n\in\mathbb{N}}$ with $\widetilde{\delta}(n) \geq \delta(n)$ in order to at least obtain a well defined morphism

$$(\varphi \otimes l(n))_{n\in\mathbb{N}}\colon (A \otimes C_*(\delta(n)))_{n\in\mathbb{N}} \to (B \otimes C_*(\widetilde{\delta}(n)))_{n\in\mathbb{N}}.$$

Now the $\widetilde{\ }$-construction that was reviewed at the end of Subsect. 6.2 comes into play.

Proposition 6.13. *Choose a collection of numbers $\delta^k(n)$, $k \in \mathbb{N}$, $n \in \mathbb{N}$ as in Lemma 6.14. Then there exists a functor depending on that choice*

$$\mathrm{trans}\colon \mathcal{O}^G(E) \to \widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n\in\mathbb{N}})$$

which sends a morphism $\varphi\colon A \to B$ to the morphism whose n-th component is represented by

$$\begin{array}{ccccccc}
A \otimes C_*(\delta^{\alpha}(n)) & \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} & A \otimes C_*(\delta^{\alpha+1}(n)) & \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} & A \otimes C_*(\delta^{\alpha+2}(n)) & \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} & \cdots \\
\downarrow{\scriptstyle \varphi\otimes l(n)} & & \downarrow{\scriptstyle \varphi\otimes l(n)} & & \downarrow{\scriptstyle \varphi\otimes l(n)} & & \\
B \otimes C_*(\delta^{\alpha+1}(n)) & \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} & B \otimes C_*(\delta^{\alpha+2}(n)) & \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} & B \otimes C_*(\delta^{\alpha+3}(n)) & \xrightarrow{\mathrm{id}\otimes\mathrm{inc}} & \cdots .
\end{array}$$

Here $\alpha = \alpha(\varphi) \in \mathbb{N}$ *is chosen such that for every* $((g', e', t'), (g, e, t)) \subset \operatorname{supp} \varphi$ *we have* $d_G(g, g') \leq \alpha$.

It is here that we use the metric control condition on G in the definition of $\mathcal{O}^G(E)$: it ensures the existence of α in the above statement.

Proof. The boundedness condition with respect to the sequence of metrics $(d_{C(n)})_{n\in\mathbb{N}}$ for the differentials follows because of Lemma 6.14(ii). That we have a homotopy finitely dominated object follows from Lemma 6.11. Hence $(A \otimes C_*(\delta^k(n)))_{n\in\mathbb{N}}$ is indeed a well defined object. Lemma 6.14(i) assures that one has the horizontal inclusions. The vertical maps exist because of Lemma 6.14(iii). Because the E-coordinate is left unchanged in this construction, the equivariant continuous control condition is preserved. □

Lemma 6.14. *Let* $(C(n))_{n\in\mathbb{N}}$ *be a monotone increasing sequence of numbers. There exists a collection of numbers* $\delta^k(n) > 0$ *with* n, $k \in \mathbb{N}$, *such that the following conditions are satisfied.*

(i) *For every fixed* $n \in \mathbb{N}$ *the sequence* $(\delta^k(n))_{k\in\mathbb{N}}$ *is increasing, i.e.,*

$$\delta^1(n) \leq \delta^2(n) \leq \delta^3(n) \leq \ldots;$$

(ii) *For every* $k \in \mathbb{N}$ *there exists* $\alpha(k) \geq 0$ *such that*

$$\delta^k(n) \leq \frac{\alpha(k)}{C(n)}$$

for all $n \in \mathbb{N}$;

(iii) *Consider* g, $h \in G$, x, $y \in \overline{X}$ *and* k, $n \in \mathbb{N}$. *If* $d_G(g, h) \leq k$ *and* $d_g(x, y) \leq \delta^k(n)$, *then*

$$d_h(x, y) \leq \delta^{k+1}(n).$$

Proof. For $L \in \mathbb{N}$ and $\delta \geq 0$ put

$$\begin{aligned} \widetilde{R}_L(\delta) &:= \sup\{d_g(x, y) \mid (g, x, y) \in G \times \overline{X} \times \overline{X} \\ &\qquad\qquad \text{with } d_G(g, e) \leq L, d_e(x, y) \leq \delta\}; \\ R_L(\delta) &:= \max\{\delta, \widetilde{R}_L(\delta)\}. \end{aligned}$$

Since $\overline{X}$ is compact and there are only finitely many g with $d_G(g, e) \leq L$, this defines a monotone increasing map $R_L \colon [0, \infty) \to [0, \infty)$ with $R_L(\delta) \geq \delta$. In particular $R_L(\delta) > 0$ for $\delta > 0$. Moreover R_L is continous at 0 because the identity yields a uniformly continous map $(\overline{X}, d_e) \to (\overline{X}, d_g)$. Note that $R_L(\delta) \leq R_{L+1}(\delta)$. Using $d_g(x, y) = d_e(g^{-1}x, g^{-1}y)$ we can conclude that

(6.15) $\quad d_G(g, h) \leq L$ and $d_g(x, y) \leq \delta$ implies $d_h(x, y) \leq R_L(\delta)$.

Define $R_L^{-1}(\delta) = \min\{\delta, \sup\{\alpha \in [0, \infty) | R_L(2\alpha) \leq \delta\}\}$. Here by abuse of notation R_L^{-1} stands for some function but need not be the inverse of R_L.

The function R_L^{-1} is monotone increasing and satisfies $0 < R_L^{-1}(\delta) \leq \delta$ for $\delta > 0$. We claim that $R_L(R_L^{-1}(\delta)) \leq \delta$. In fact for $s < R_L^{-1}(\delta)$ we have $R_L(2s) \leq \delta$. Hence for $s = \frac{3}{4}R_L^{-1}(\delta)$ we have by monotony $R_L(R_L^{-1}(\delta)) \leq R_L(2\frac{3}{4}R_L^{-1}(\delta)) \leq \delta$. For $n \in \mathbb{N}$ define

$$\delta^n(n) = \frac{1}{C(n)}.$$

For $k = n + l$ with $l \geq 1$ put

$$\delta^k(n) = R_{n+l-1} \circ \cdots \circ R_{n+1} \circ R_n(\delta^n(n))$$

and for $k = n - l$, with $l = 1, 2, \ldots, n - 1$ set

$$\delta^k(n) = R_{n-l}^{-1} \circ \cdots \circ R_{n-2}^{-1} \circ R_{n-1}^{-1}(\delta^n(n)).$$

It remains to check that the numbers $\delta^k(n)$ have the desired properties.

(i) This follows since $R_L(\delta) \geq \delta$ and $R_L^{-1}(\delta) \leq \delta$.
(ii) For $n \geq k$ we have $\delta^k(n) \leq \delta^n(n) = \frac{1}{C(n)}$ by (i) Now we can choose $\alpha(k)$ to be $\max\{1, C(n)\delta^k(1), \ldots, C(n)\delta^k(k-1)\}$.
(iii) Since $R_k(R_k^{-1}(\delta)) \leq \delta$ we conclude

$$R_k(\delta^k(n)) \leq \delta^{k+1}(n).$$

We derive from (6.15)

$$d_h(x, y) \leq R_k(\delta^k(n)) \leq \delta^{k+1}(n).$$

□

Lemma 6.16. *After applying K-theory Diagram* (6.2) *is commutative.*

Proof. Because of [57, Proposition 1.3.1] it suffices to construct a natural transformation T of functors $\mathcal{O}^G(E) \to \widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E)$ between $\widetilde{\mathrm{ch}}_{\mathrm{hfd}}(\mathrm{pr}_k) \circ$ trans and the obvious inclusion such that $T(A)$ is a weak homotopy equivalence in $\widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E)$ for every object A in $\mathcal{O}^G(E)$.

Consider a $\mathbb{Z}G$-chain complex C_* such that after forgetting the G-action

$$C_* \in \mathrm{ch}_{\mathrm{hfd}}\,\mathcal{F}^f(\mathbb{Z}) = \mathrm{ch}_{\mathrm{hfd}}(\mathcal{F}^{\mathrm{f}}(\mathbb{Z}) \subset \mathcal{F}^{\kappa}(\mathbb{Z})).$$

Examples are $C_*^{\mathrm{sing}}(\overline{X})$ and $C_*^{\mathrm{sing},\delta}(\overline{X})$ by Lemma 6.7(ii) and the (easier) uncontrolled version of Lemma 6.9. We define a functor

$$l_{C_*} \colon \mathcal{O}^G(E) \to \mathrm{ch}_{\mathrm{hfd}}\,\mathcal{O}^G(E)$$

as follows. Let

$$A = (A_{(g,y,t)})_{(g,y,t)\in G\times E\times[1,\infty)} \quad \text{and} \quad B = (B_{(g',y',t')})_{(g',y',t')\in G\times E\times[1,\infty)}$$

be objects in $\mathcal{O}^G(E)$ and let $\varphi\colon A \to B$ be a morphism in $\mathcal{O}^G(E)$ with components $\varphi_{(g',y',t'),(g,y,t)}\colon A_{(g,y,t)} \to B_{(g',y',t')}$. Define $A \mapsto A \otimes C_*$ and $\varphi \mapsto \varphi \otimes l$, where for (g, y, t), $(g', y', t') \in G \times E \times [1, \infty)$ we put

$$(A \otimes C_*)_{(g,y,t)} = A_{(g,y,t)} \otimes C_*$$

with differential

$$\partial_{(g',y',t'),(g,y,t)} = \mathrm{id}_{(g',y',t'),(g,y,t)} \otimes \partial$$

and

$$(\varphi \otimes l)_{(g',y',t'),(g,y,t)} = \varphi_{(g',y',t'),(g,y,t)} \otimes l_{g'^{-1}g},$$

where $l_{g'^{-1}g}$ is left multiplication with $g'^{-1}g$.

Let C_* and D_* be $\mathbb{Z}G$-chain complexes belonging to $\mathrm{ch}_{\mathrm{hfd}}\,\mathcal{F}^f(\mathbb{Z})$ and $f_*\colon C_* \to D_*$ be a $\mathbb{Z}$-chain map. Then for every object A in $\mathcal{O}^G(E)$, we have an induced chain map $\mathrm{id}_A \otimes f_*\colon l_{C_*}(A) \to l_{D_*}(A)$. If f_* is moreover a $\mathbb{Z}G$-chain map, then this construction is compatible with l_{C_*} and l_{D_*} on morphisms and defines a natural transformation $l_{f_*}\colon l_{C_*} \to l_{D_*}$ of functors. If f_* is a chain homology equivalence (after forgetting the group action), then $l_{f_*}(A)\colon l_{C_*}(A) \to l_{D_*}(A)$ is a chain homotopy equivalence in $\mathrm{ch}_{\mathrm{hfd}}\,\mathcal{O}^G(E)$: since C_* and D_* are free as $\mathbb{Z}$-chain complexes, we can choose a (not necessarily G-equivariant) $\mathbb{Z}$-chain homotopy inverse $u_*\colon D_* \to C_*$ for f_*. Then $\mathrm{id}_A \otimes u_*$ is a homotopy inverse for $\mathrm{id}_A \otimes f_* = l_{f_*}(A)$.

Let $0(\mathbb{Z})_*$ be the $\mathbb{Z}G$-chain complex concentrated in dimension zero and given there by $\mathbb{Z}$ with the trivial G-operation. Let $\varepsilon_*\colon C_*^{\mathrm{sing}}(\overline{X}) \to 0(\mathbb{Z})_*$ be the $\mathbb{Z}G$-chain complex map given by augmentation. Denote by $\varepsilon(\delta_k^n)_*\colon C_*^{\mathrm{sing},\delta^n(k)}(\overline{X}) \to 0(\mathbb{Z})_*$ its composition with the inclusion $C_*^{\mathrm{sing},\delta^n(k)}(\overline{X}) \to C_*^{\mathrm{sing}}(\overline{X})$. We obtain the following commutative diagram in $\mathrm{ch}_{\mathrm{hfd}}\,\mathcal{O}^G(E)$.

$$\begin{tikzcd}
l_{C_*^{\mathrm{sing},\delta^1(k)}(\overline{X})}(A) \arrow[r, "l_{\mathrm{inc}}"] \arrow[d, "l_{\varepsilon(\delta_k^1)_*}"] & l_{C_*^{\mathrm{sing},\delta^2(k)}(\overline{X})}(A) \arrow[r, "l_{\mathrm{inc}}"] \arrow[d, "l_{\varepsilon(\delta_k^2)_*}"] & l_{C_*^{\mathrm{sing},\delta^3(k)}(\overline{X})}(A) \arrow[r, "l_{\mathrm{inc}}"] \arrow[d, "l_{\varepsilon(\delta_k^3)_*}"] & \cdots \\
l_{0(\mathbb{Z})_*}(A) \arrow[r, "\mathrm{id}"] & l_{0(\mathbb{Z})_*}(A) \arrow[r, "\mathrm{id}"] & l_{0(\mathbb{Z})_*}(A) \arrow[r, "\mathrm{id}"] & {}
\end{tikzcd}$$

Here inc denotes the obvious inclusions. All arrows are homotopy equivalences in $\mathrm{ch}_{\mathrm{hfd}}\,\mathcal{O}^G(E)$ by the argument above since $\varepsilon_*\colon C_*^{\mathrm{sing}}(\overline{X}) \to 0(\mathbb{Z})_*$ and each inclusion $C_*^{\mathrm{sing},\delta^n(k)}(\overline{X}) \to C_*^{\mathrm{sing}}(\overline{X})$ are chain homology equivalences by the contractibility of $\overline{X}$ and Lemma 6.7(ii). One easily checks that the upper row is an element in $\widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E)$, namely $\widetilde{\mathrm{ch}}_{\mathrm{hfd}}(\mathrm{pr}_k) \circ \mathrm{trans}(A)$, and that the lower row is an element in $\widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E)$, namely, the one given by $A \xrightarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} \dots$. Hence we obtain the desired natural transformation T whose evaluation at an object A is a weak equivalence in $\widetilde{\mathrm{ch}}_{\mathrm{hfd}}\mathcal{O}^G(E)$. $\square$

7. Stability

In this section we will prove a stability result that implies (4.7).

Notation 7.1. Let E be a model for the classifying space $E_{\mathcal{F}}G$. Let $(X_n, d_n)_{n\in\mathbb{N}}$ be a sequence of quasi-metric spaces equipped with an isometric G-action. Denote by $\widetilde{d}_n$ the product quasi-metric on $G \times X_n$ defined by $\widetilde{d}_n((g,x),(g',x')) = d_G(g,g') + d_n(x,x')$. We abbreviate

$$\mathcal{L}^G_{\oplus}((X_n,d_n)_{n\in\mathbb{N}}) = \bigoplus_{n\in\mathbb{N}} \mathcal{O}^G(E,(G\times X_n,\widetilde{d}_n));$$
$$\mathcal{L}^G((X_n,d_n)_{n\in\mathbb{N}}) = \mathcal{O}^G(E,(G\times X_n,\widetilde{d}_n)_{n\in\mathbb{N}}).$$

The inclusion $\mathcal{L}^G_{\oplus}((X_n,d_n)_{n\in\mathbb{N}}) \to \mathcal{L}^G((X_n,d_n)_{n\in\mathbb{N}})$ is a Karoubi filtration and we denote the quotient by $\mathcal{L}^G_{\oplus}((X_n,d_n)_{n\in\mathbb{N}})^{>\oplus}$, its objects are the same as the objects of $\mathcal{L}^G((X_n,d_n)_{n\in\mathbb{N}})$ but morphism are identified if they factor over an object in $\mathcal{L}^G_{\oplus}((X_n,d_n)_{n\in\mathbb{N}})$.

Theorem 7.2. *Let X_n, $n\in\mathbb{N}$ be a sequence of simplicial complexes with a cell preserving simplicial G-action. Suppose that there exists an $N\in\mathbb{N}$ such that* $\dim X_n \le N$ *for all $n\in\mathbb{N}$. For every $n\in\mathbb{N}$ let d_n be a quasi-metric on X_n satisfying*

$$d_n(x,y) \ge nd^1(x,y) \quad \forall x,y\in X_n$$

with equality if x and y are contained in a common simplex of X_n. (Recall that d^1 denotes the l^1-metric on simplicial complexes.) Assume that all isotropy groups of the action of G on X_n are contained in $\mathcal{F}$. Then the inclusion

$$\mathcal{L}^G_{\oplus}((X_n,d_n)_{n\in\mathbb{N}}) \to \mathcal{L}^G((X_n,d_n)_{n\in\mathbb{N}})$$

induces an equivalence on the level of K-theory.

Note that (4.7) is a consequence of this Theorem. In this application to the inclusion (3) in Diagram (4.4) the quasi-metrics d_n are equal to nd^1, the l^1-metrics scaled by n, but for the proof it will be convenient to also allow disjoint unions of simplicial complexes which carry a scaled l^1-metric, but where different components are infinitely far apart. We start by introducing some notation. Next we will state a special case and an excision result. The proof of Theorem 7.2 will then be an easy induction.

Notation 7.3. Retain the assumptions of Theorem 7.2. Recall that $\dim X_n \le N$. Let $Y_n = \coprod \Delta^N$ be the disjoint union of the N-simplices of X_n. Equip Y_n with the quasi-metric d_n^∞ for which $d_n^\infty(x,y) = nd^1(x,y)$ if x and y are contained in a common N-simplex and $d_n^\infty(x,y)=\infty$ otherwise. Let $\partial Y_n = \coprod \partial\Delta^N$ be the disjoint union of the boundaries of the N-simplices of Y_n and equip ∂Y_n with the subspace quasi-metric. Let $X_n^{(N-1)}$ be the $(N-1)$-skeleton of X_n equipped with the subspace quasi-metric.

Proposition 7.4. *Retain Notation 7.3. Then the K-theory of $\mathcal{L}^G((Y_n, d_n^\infty)_{n\in\mathbb{N}})^{>\oplus}$ is trivial.*

Proposition 7.5. *Retain Notation 7.3. Then diagram induced from the pushout-diagram that describes the attaching of the N-simplices*

$$\begin{tikzcd}
\mathcal{L}^G\big(\big(\partial Y_n, d_n^\infty\big)_{n\in\mathbb{N}}\big)^{>\oplus} \arrow[r] \arrow[d] & \mathcal{L}^G\big(\big(Y_n, d_n^\infty\big)_{n\in\mathbb{N}}\big)^{>\oplus} \arrow[d] \\
\mathcal{L}^G\big(\big(X_n^{(N-1)}, d_n\big)_{n\in\mathbb{N}}\big)^{>\oplus} \arrow[r] & \mathcal{L}^G((X_n, d_n)_{n\in\mathbb{N}})^{>\oplus}
\end{tikzcd} \tag{7.6}$$

becomes homotopy cartesian after applying K-theory.

Proof of Theorem 7.2. Karoubi filtrations induce fibration sequences in K-theory, [16]. Therefore

$$\mathcal{L}^G_\oplus((X_n, d_n)_{n\in\mathbb{N}}) \to \mathcal{L}^G((X_n, d_n)_{n\in\mathbb{N}}) \to \mathcal{L}^G((X_n, d_n)_{n\in\mathbb{N}})^{>\oplus}$$

induces a fibration sequence in K-theory. Hence the statement of the theorem is equivalent to showing that the K-theory of $\mathcal{L}^G((X_n, d_n)_{n\in\mathbb{N}})^{>\oplus}$ vanishes. We proceed by induction on N. If $N = -1$, then there is nothing to prove. Consider (7.6). By the induction hypothesis the K-theory of the categories on the left both are trivial. By Proposition 7.4 the K-theory of the upper right category of (7.6) vanishes. Proposition 7.5 implies now that the K-theory of $\mathcal{L}^G((X_n, d_n)_{n\in\mathbb{N}})^{>\oplus}$ has to vanish as well. □

It remains to prove Propositions 7.4 and 7.5.

Proof of Proposition 7.4. This proof will be similar to the proof that homology theories constructed using controlled algebra satisfy homotopy invariance and uses an Eilenberg swindle.

We will construct for each n an Eilenberg-swindle on $\mathcal{O}^G(E, G \times Y_n, \widetilde{d}_n^\infty)$. Since the construction leaves the $G \times Y_n$-direction untouched it will be clear that these Eilenberg-swindles can be combined to an Eilenberg-swindle on $\mathcal{L}^G((Y_n, d_n^\infty)_{n\in\mathbb{N}})^{>\oplus}$. If $E = \mathrm{pt}$, then we can define this swindle by pushing along the $[1, \infty)$-direction, compare Lemma 3.6(ii). In the general case, we will also need to use contractions in E towards fixed points for isotropy groups.

Fix $n \in \mathbb{N}$.

Let R_n be the set of N-simplices of Y_n. The isotropy groups of R_n agree with the isotropy groups of Y_n and are thus all contained in $\mathcal{F}$. By the universal property of E there exists a G-equivariant map $\iota\colon R_n \to E$ and a G-equivariant homotopy $h\colon R_n \times E \times [0, 1] \to E$ with $h_0(r, e) = e$ and

$h_1(r, e) = \iota(r)$. Denote by $p\colon Y_n \to R_n$ the canonical projection map that collapses each N-simplex to a point. For $k \in \mathbb{N}_0$ the map

$$(g, y, e, t) \mapsto (g, y, h_{k/(k+t)}(p(y), e), t + k)$$

where $g \in G$, $y \in Y_n$, $e \in E$, $t \in [1, \infty)$ induces a functor S_k from $\mathcal{O}^G(E, G \times Y_n, \widetilde{d}_n^\infty)$ to itself. Our first claim is that $\bigoplus_{k=0}^{\infty} S_k$ also yields a well defined functor. There is a canonical natural transformation τ_k between S_k and S_{k+1}, see (3.1). Our second claim is that $\bigoplus_{k=0}^{\infty} \tau_k$ yields a natural equivalence from $\bigoplus_{k=0}^{\infty} S_k$ to $\bigoplus_{k=1}^{\infty} S_k$. Since $S_0 = \mathrm{id}$ this gives the desired Eilenberg-swindle.

It remains to prove the two claims above. In both cases the only nontrivial claim is that the continuous control condition (3.2) is preserved.

We start with the first claim. Let φ be a morphism in $\mathcal{O}^G(E, G \times Y_n, \widetilde{d}_n^\infty)$. The support of $(\bigoplus_{k=0}^{\infty} S_k)(\varphi)$ is given by all pairs of points in $G \times Y_n \times E \times [1, \infty)$ of the form

$$((g, y, h_{k/(k+t)}(p(y), e), t + k), (g', y', h_{k/(k+t)}(p(y'), e'), t' + k)),$$

where $k \in \mathbb{N}_0$ and $((g, y, e, t), (g', y', e', t')) \in \operatorname{supp} \varphi$. Let U be an $G_{\bar{e}}$-invariant open neighborhood of $\bar{e} \in E$ and $\kappa > 0$. We need to show that there is an $G_{\bar{e}}$-invariant neighborhood V of $\bar{e}$ and $\sigma > \kappa$ such that if $((g, y, e, t), (g', y', e', t')) \in \operatorname{supp} \varphi$, $k \in \mathbb{N}_0$, $h_{k/(k+t)}(p(y), e) \in V$ and $t > \sigma$, then $h_{k/(k+t')}(p(y'), e') \in U$ and $t' > \kappa$. By [3, Proposition 3.4] there is a sequence $V^1 \supset V^2 \supset \cdots$ of open $G_{\bar{e}}$-invariant neighborhoods of $\bar{e} \in E$ such that $\bigcap_{l=1}^{\infty} \overline{G V^l} = G\bar{e}$ and $gV^l \cap V^l = \emptyset$ if $g \in G - G_{\bar{e}}$. We proceed now by contradiction and assume that for every l there is $P_l = ((g_l, y_l, e_l, t_l), (g_l', y_l', e_l', t_l')) \in \operatorname{supp} \varphi$ and $k_l \in \mathbb{N}_0$ such that

$$\begin{aligned} &(h_{k_l/(k_l+t_l)}(p(y_l), e_l), t_l + k_l) \in V^l \times (\kappa + l, \infty) \\ \text{but}\ &(h_{k_l/(k_l+t_l')}(p(y_l'), e_l'), t_l' + k_l) \notin U \times (\kappa, \infty). \end{aligned}$$

The metric control with respect to $\widetilde{d}_n^\infty$ for the morphism φ implies that $p(y_l) = p(y_l')$ for all sufficiently large l. From the metric control condition with respect to the projection to $[1, \infty)$, see (3.3), we conclude that $|t_l - t_l'| \leq \alpha$ for some $\alpha > 0$ independent of l. Since $t_l + k_l > \kappa + l$ this implies $t_l' + k_l > \kappa$ for sufficiently large l. Therefore we may assume that

$$h_{k_l/(k_l+t_l')}(p(y_l'), e_l') \notin U \quad \forall l. \tag{7.7}$$

Passing to a subsequence, if necessary, we can assume that $k_l/(k_l + t_l)$ and $k_l/(k_l + t_l')$ both converge for $l \to \infty$. Since $t_l + k_l > \kappa + l$ we conclude from

$$\left| \frac{k_l}{k_l + t_l} - \frac{k_l}{k_l + t_l'} \right| = \left| \frac{k_l \cdot (t_l' - t_l)}{(k_l + t_l) \cdot (k_l + t_l')} \right| \leq \frac{\alpha}{k_l + t_l} \leq \frac{\alpha}{\kappa + l}$$

that both sequences have the same limit $s \in [0, 1]$. Because the morphism φ has G-compact support with respect to the projection to $G \times Y_n \times E$, we can assume that there are $a_l \in G$ and $(\tilde{g}, \tilde{y}, \tilde{e})$ such that

$$a_l(g_l, y_l, e_l) \to (\tilde{g}, \tilde{y}, \tilde{e}) \quad \text{as } l \to \infty.$$

Now $a_l h_{k_l/(k_l+t_l)}(p(y_l), e_l) = h_{k_l/(k_l+t_l)}(p(a_l y_l), a_l e_l)$ converges to $h_s(p(\tilde{y}), \tilde{e})$. Since $h_{k_l/(k_l+t_l)}(p(y_l), e_l) \in V^l$ we conclude $h_s(p(\tilde{y}), \tilde{e}) = a\tilde{e}$ for some $a \in G$ and $a^{-1}a_l \in G_{\bar{e}}$ for sufficiently large l from the properties of the V^l. Because U and the V^l are $G_{\bar{e}}$ invariant and $\operatorname{supp}\varphi$ is G-invariant, we can replace P_l by $a^{-1}a_l P_l$. Therefore we may now assume that

$$(g_l, y_l, e_l) \to (\tilde{g}, \tilde{y}, \tilde{e}) \quad \text{as } l \to \infty.$$

Because R_l is discrete we can also assume that $p(y_l) = p(y_l') = p(\tilde{y})$ for all l. Let $\tilde{U} \subset E$ be the preimage of $U \subset E$ under the G-equivariant map $e \mapsto h_s(p(y), e)$. Now we use that $\operatorname{supp}\varphi$ satisfies the continuous control condition (3.2) to conclude that there exists an open G_e-invariant neighborhood W of $e \in E$ and $\sigma > 0$ such that if $((g, y, e, t), (g', y', e', t')) \in \operatorname{supp}\varphi$, $e \in W$, $t > \sigma$ and $t' > \kappa$ then $e' \in \tilde{U}$. Since $e_l \to \tilde{e}$ we can apply this to $P_l \in \operatorname{supp}\varphi$ and conclude that $e_l' \in \tilde{U}$ for sufficiently large l. Thus $h_s(p(y), e_l') \in U$ for sufficiently large l. But this contradicts (7.7) since $k_l/(k_l + t_l) \to s$ as $l \to \infty$ and $p(y) = p(y_l')$ for all l. This finishes the proof of the first claim.

For the second claim will use a similar argument. Let A be an object of $\mathcal{O}^G(E, G \times Y_n, \widetilde{d_n^\infty})$. Then the support of the isomorphism

$$\Big(\bigoplus_{k=0}^{\infty} \tau_k\Big)(A)\colon \bigoplus_{k=0}^{\infty} S_k(A) \to \bigoplus_{k=1}^{\infty} S_k(A)$$

is the set of all pairs of points in $G \times Y_n \times E \times [1, \infty)$ of the form

$$((g, y, h_{k/(k+t)}(p(y), e), t + k), (g, y, h_{k+1/(k+1+t)}(p(y), e), t + k + 1))$$

where $k \in \mathbb{N}_0$ and $(g, y, e, t) \in \operatorname{supp} A$, compare (3.1). We need to show that this set satisfies the continuous control condition (3.2). Let U be an $G_{\bar{e}}$-invariant open neighborhood of $\bar{e} \in E$ and $\kappa > 0$. Let V^l be a sequence of open neighborhoods as used in the proof of the first claim. We proceed as before by contradiction and assume that for every l there is $(g_l, y_l, e_l, t_l) \in \operatorname{supp} A$ and $k_l \in \mathbb{N}_0$ such that

$$\begin{aligned} &(h_{k_l/(k_l+t_l)}(p(y_l), e_l), t_l + k_l) \in V^l \times (\kappa + l, \infty) \\ \text{but } &(h_{k_l+1/(k_l+1+t_l)}(p(y_l), e_l), t_l + k_l + 1) \notin U \times (\kappa, \infty). \end{aligned}$$

(Strictly speaking we also need to consider the case where we interchange k_l and $k_l + 1$ because of (3.4), but this case can be treated by essentially the same argument.) From $t_l + k_l + 1 > \kappa$ we conclude

$$h_{k_l+1/(k_l+1+t_l)}(p(y_l), e_l) \notin U \quad \forall l. \tag{7.8}$$

Passing to a subsequence, if necessary, we can assume that $k_l/(k_l + t_l)$ and $k_l + 1/(k_l + 1 + t_l)$ both converge. As above we conclude that both sequences have the same limit $s \in [0, 1]$. Because the object A has G-compact support with respect to the projection to $G \times Y_n \times E$, we can assume that there are $a_l \in G$ such that

$$a_l(g_l, y_l, e_l) \to (\tilde{g}, \tilde{y}, \tilde{e}) \quad \text{as} \quad l \to \infty.$$

By a similar argument as above we can in fact assume that a_l is trivial. Since

$$\bar{e} = \lim_{l\to\infty} h_{k_l/(k_l+t_l)}(p(y_l), e_l) = \lim_{l\to\infty} h_{k_l+1/(k_l+1+t_l)}(p(y_l), e_l)$$

we obtain a contradiction to (7.8). $\square$

Before we can prove Proposition 7.5 we will need to unravel the definition of the categories appearing in (7.6) and introduce some more notation.

Notation 7.9. Retain Notation 7.3. Let

$$B := \coprod_{n\in\mathbb{N}} G \times \partial Y_n \times E \times [1, \infty), \qquad Y := \coprod_{n\in\mathbb{N}} G \times Y_n \times E \times [1, \infty),$$
$$A := \coprod_{n\in\mathbb{N}} G \times X_n^{(N-1)} \times E \times [1, \infty), \quad X := \coprod_{n\in\mathbb{N}} G \times X_n \times E \times [1, \infty).$$

Let $\mathcal{F}_{\oplus}^{B}$ be the collection of subsets of B of the form $\coprod_{n=1}^{\bar{n}} G \times \partial Y_n \times E \times [1, \infty)$ for some $\bar{n} \in \mathbb{N}$. Similar we have collections $\mathcal{F}_{\oplus}^{A}$, $\mathcal{F}_{\oplus}^{Y}$ and $\mathcal{F}_{\oplus}^{X}$ respectively of subsets of A, Y and X respectively. Let $\mathcal{F}_{cs}^{B}$ be the collection of subsets of B of the form $\coprod_{n\in\mathbb{N}} K_n$, where each K_n is the preimage of a G-compact subset under the projection $G \times \partial Y_n \times E \times [1, \infty) \to G \times \partial Y_n \times E$. Similar we have collections $\mathcal{F}_{cs}^{A}$, $\mathcal{F}_{cs}^{Y}$ and $\mathcal{F}_{cs}^{X}$ respectively of subsets of A, Y and X respectively. Let $\mathcal{E}^{B}$ be the collection of subsets $J \subset B \times B$ satisfying the following conditions:

(7.10) $J \subseteq \coprod_{n\in\mathbb{N}} J_n$ with respect to the canonical inclusion

$$\coprod_{n\in\mathbb{N}} \left((G \times \partial Y_n \times E \times [1, \infty))^{\times 2}\right) \to B \times B,$$

where for every $n \in \mathbb{N}$ the set $J_n \subset (G \times \partial Y_n \times E \times [1, \infty))^{\times 2}$ is such that J_n is the preimage of some $J_n' \in \mathcal{E}_{Gcc}^{E}$ with respect to the canonical projection $G \times \partial Y_n \times E \times [1, \infty) \to E \times [1, \infty)$, compare Sect. 3.2;

(7.11) There is $\alpha > 0$ such that $((g, y, e, t), (g', y', e', t')) \in J$ with $g, g' \in G$, $y, y' \in \partial Y_n$, $e, e' \in E$ and $t, t' \in [0, \infty)$ implies $d_n^{\infty}(y, y') \le \alpha$ and $d_G(g, g') \le \alpha$.

Similar we have collections $\mathcal{E}^A$, $\mathcal{E}^Y$ and $\mathcal{E}^X$ respectively of subsets of $A^{\times 2}$, $Y^{\times 2}$ and $X^{\times 2}$ respectively. Of course we use the quasi-metrics d_n in the definition of $\mathcal{E}^A$ and $\mathcal{E}^X$.

With this notation Diagram (7.6) becomes

$$\begin{tikzcd} \mathcal{C}^G\big(B, \mathcal{E}^B, \mathcal{F}_{cs}^B; \mathcal{A}\big)^{>\mathcal{F}_\oplus^B} \arrow[r] \arrow[d] & \mathcal{C}^G\big(Y, \mathcal{E}^Y, \mathcal{F}_{cs}^Y; \mathcal{A}\big)^{>\mathcal{F}_\oplus^Y} \arrow[d] \\ \mathcal{C}^G\big(A, \mathcal{E}^A, \mathcal{F}_{cs}^A; \mathcal{A}\big)^{>\mathcal{F}_\oplus^A} \arrow[r] & \mathcal{C}^G\big(X, \mathcal{E}^X, \mathcal{F}_{cs}^X; \mathcal{A}\big)^{>\mathcal{F}_\oplus^X} \end{tikzcd} \tag{7.12}$$

where we used a more general germ notation to denote Karoubi quotients, see for instance [8, Sect. 2.1.6]. For example, for the upper left category this just means that morphisms are identified if their difference factors over an object whose support lies in some $F \in \mathcal{F}_\oplus^B$. We will drop $\mathcal{A}$ from the notation.

For $\alpha > 0$ and $F \in \mathcal{F}_{cs}^A$ let F^α be the subset of X consisting of all points $(g, x, e, t) \in X$ with the property that if $x \in X_n$ then there is $x' \in X_n^{(N-1)}$ with $(g, x', e, t) \in F$ and $d_n(x, x') \leq \alpha$. Define $\mathcal{F}_A^X$ as the collection of all subsets of X of the form F^α for all $\alpha > 0$, $F \in \mathcal{F}_{cs}^A$ and $F' \in \mathcal{F}_{cs}^X$. We will follow [8, Sect. 8.4] and abuse notation to denote by $\mathcal{F}_{cs}^X \cap \mathcal{F}_A^X$ the collection of all subsets of the form $F \cap F'$ with $F \in \mathcal{F}_{cs}^X$ $F' \in \mathcal{F}_A^X$ Similar definitions yield $\mathcal{F}_B^Y$, a collection of subsets of Y and $\mathcal{F}_{cs}^Y \cap \mathcal{F}_B^Y$.

Lemma 7.13. *Retain Notation 7.9. The inclusions*

$$\mathcal{C}^G\big(B, \mathcal{E}^B, \mathcal{F}_{cs}^B\big)^{>\mathcal{F}_\oplus^B} \to \mathcal{C}^G\big(Y, \mathcal{E}^Y, \mathcal{F}_{cs}^Y \cap \mathcal{F}_B^Y\big)^{>\mathcal{F}_\oplus^Y},$$
$$\mathcal{C}^G\big(A, \mathcal{E}^A, \mathcal{F}_{cs}^A\big)^{>\mathcal{F}_\oplus^A} \to \mathcal{C}^G\big(X, \mathcal{E}^X, \mathcal{F}_{cs}^X \cap \mathcal{F}_A^X\big)^{>\mathcal{F}_\oplus^X}$$

are equivalences of categories.

Proof. It is a formal consequence of the definitions that both functors yield isomorphisms on morphism groups. It remains to show that every object in the target category is isomorphic to an object in the image of the functor. We consider the second functor. Let M be an object in $\mathcal{C}^G(X, \mathcal{E}^X, \mathcal{F}_{cs}^X \cap \mathcal{F}_A^X)^{>\mathcal{F}_\oplus^X}$. By definition $\operatorname{supp} M$ is a locally finite subset of $F^\alpha \cap F'$ for some $\alpha > 0$, $F \in \mathcal{F}_{cs}^A$, $F' \in \mathcal{F}_{cs}^X$. Therefore there is a G-equivariant map $f\colon \operatorname{supp} M \to F$ with the property that if $f(g, x, e, t) = (g', x', e', t')$ with $x \in X_n$ then $g' = g$, $e' = e$, $t' = t$, $x' \in X_n$ and $d_n(x, x') \leq \alpha$. (The map is not canonical; we have to choose x' for every x in a G-equivariant way.) It is not hard to see that f is finite-to-one and has a locally finite image. Thus we can apply f to M to obtain an object $f_*(M)$ of $\mathcal{C}^G(A, \mathcal{E}^A, \mathcal{F}_{cs}^A)^{>\mathcal{F}_\oplus^A}$. Clearly, $\{(s, f(s)) \mid s \in \operatorname{supp} M\} \in \mathcal{E}^X$. Thus M and $f_*(M)$ are isomorphic.

The first functor can be treated similarly. $\square$

Proof of Proposition 7.5. Retain Notation 7.9. Because of (7.12) and Lemma 7.13 it suffices to prove that

$$\begin{tikzcd} \mathcal{C}^G\big(Y, \mathcal{E}^Y, \mathcal{F}^Y_{cs} \cap \mathcal{F}^Y_B\big)^{>\mathcal{F}^Y_\oplus} \arrow[r] \arrow[d] & \mathcal{C}^G\big(Y, \mathcal{E}^Y, \mathcal{F}^Y_{cs}\big)^{>\mathcal{F}^Y_\oplus} \arrow[d] \\ \mathcal{C}^G\big(X, \mathcal{E}^X, \mathcal{F}^X_{cs} \cap \mathcal{F}^X_A\big)^{>\mathcal{F}^X_\oplus} \arrow[r] & \mathcal{C}^G\big(X, \mathcal{E}^X, \mathcal{F}^X_{cs}\big)^{>\mathcal{F}^X_\oplus} \end{tikzcd}$$

yields a homotopy cartesian diagram in K-theory.

The two rows of this diagram are now Karoubi filtrations and on the quotients we obtain an induced functor

$$\mathcal{C}^G\big(Y, \mathcal{E}^Y, \mathcal{F}^Y_{cs}\big)^{>\mathcal{F}^Y_\oplus \cup \mathcal{F}^Y_B} \to \mathcal{C}^G\big(X, \mathcal{E}^X, \mathcal{F}^X_{cs}\big)^{>\mathcal{F}^X_\oplus \cup \mathcal{F}^X_A}. \tag{7.14}$$

(Here we are again abusing notation following [8, Sect. 8.4]: $\mathcal{F}^Y_\oplus \cup \mathcal{F}^Y_B$ is the collection of all sets of the form $F \cup F'$ with $F \in \mathcal{F}^Y_\oplus$ and $F' \in \mathcal{F}^Y_B$ and the definition of $\mathcal{F}^X_\oplus \cup \mathcal{F}^X_A$ is similar.)

Because Karoubi filtrations induce fibration sequences in K-theory [16], it suffices to show that (7.14) is an equivalence of categories. Because the canonical map $Y_n \to X_n$ induces a homeomorphism $Y_n - \partial Y_n \to X_n - X_n^{(N-1)}$ every object in the target category is isomorphic to an object in the image. Hence it suffices to show that (7.14) is full and faithful.

Every morphism in the category $\mathcal{C}^G(X, \mathcal{E}^X, \mathcal{F}^X_{cs}; \mathcal{A})$ can be written as the sum of two morphisms $\varphi + \psi$, where φ does not connect different k-simplices of $\coprod_{n\in\mathbb{N}} X(k)_n$ and ψ has no component that connects two points on the same simplex. Clearly, φ can be lifted to $\mathcal{C}^G(Y, \mathcal{E}^Y, \mathcal{F}^Y_{cs})$. It follows from Lemma 7.15 below that ψ can be factored over an object whose support is contained in some $F \in \mathcal{F}^X_A$. The definition of the Karoubi quotient implies that ψ is trivial in $\mathcal{C}^G(X, \mathcal{E}^X, \mathcal{F}^X_{cs})^{>\mathcal{F}^X_\oplus \cup \mathcal{F}^X_A}$. Therefore (7.14) is surjective on morphism sets. The injectivity on morphism sets follows from the fact that the preimage of an $F \in \mathcal{F}^X_A$ is contained in some $F' \in \mathcal{F}^Y_B$. □

Lemma 7.15. *Let Z be an n-dimensional simplicial complex. If Δ is an n-simplex in Z, $x \in \Delta$, $y \in Z - \Delta$ then there is $z \in \partial\Delta$ such that $d^1(x, z) \leq 2d^1(x, y)$. (Here d^1 denotes the l^1-metric on Z).*

Proof. Let Δ' be the simplex uniquely determined by the property that y lies in its interior. Then $\Delta \cap \Delta' \neq \Delta'$. Let x_i, $i \in I$ be the barycentric coordinates of x and $y_{i'}$, $i' \in I'$ be the barycentric coordinates of y, where I and I' respectively are the vertices of Δ and Δ'. We can assume $x \notin \partial\Delta$, because otherwise we simply set $z = x$. Therefore $x_i \neq 0$ for all $i \in I$. Since $\Delta \cap \Delta' \neq \Delta'$ there exists an $i_0 \in I$ with $i_0 \notin I'$. We have $x_{i_0} \neq 0$ and

$x_{i_0} \leq d^1(x, y)$. Now let $z \in \partial\Delta$ be the point with coordinates $z_i = \frac{x_i}{1-x_{i_0}}$ if $i \neq i_0$ and $z_{i_0} = 0$. Then $d^1(x, z) = 2x_{i_0}$ and hence $d^1(x, z) \leq 2d^1(x, y)$. $\square$

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