On the Farrell–Jones Conjecture and its applications

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Abstract

We present the status of the Farrell–Jones Conjecture for algebraic K-theory for a group G and arbitrary coefficient rings R. We add new groups for which the conjecture is known to be true, and we study inheritance properties. We discuss new applications, focussing on the Bass Conjecture, the Kaplansky Conjecture, and conjectures generalizing Moody's Induction Theorem. Thus, we considerably extend the class of groups for which these conjectures are known.

1. *Introduction and statements of results*

1.1. *Background*

The *Farrell–Jones Conjecture for algebraic K-theory* predicts the structure of $K_n(RG)$ for a group G and a ring R. There is also an L-theory version. For applications in topology and geometry, the case $R = \mathbb{Z}$ is the most important one, since many topological invariants of manifolds and CW-complexes such as the finiteness obstruction, the Whitehead torsion and the surgery obstruction take values in the algebraic K- or L-theory of the integral group ring $\mathbb{Z}\pi$ of the fundamental group π. The Farrell–Jones Conjecture for $R = \mathbb{Z}$ implies several famous conjectures, for example, the *Novikov Conjecture*, (in high dimensions) the *Borel Conjecture*, and the triviality of compact h-cobordisms with torsionfree fundamental group. On the other hand, proofs of the Farrell–Jones Conjecture for certain groups often rely on working with integral coefficients since they are based on these geometric connections. This is the reason why more is known about the algebraic K- and L-theory of $\mathbb{Z}G$ than of $\mathbb{C}G$ which is in some sense surprising, since $\mathbb{C}G$ has better ring-theoretic properties than $\mathbb{Z}G$. For the status of the Farrell–Jones Conjecture with coefficient in $\mathbb{Z}$, we refer for instance to [**45**, Sections 5.2 and 5.3].

Recently, the geometric approaches have been generalized so far that they also apply to other coefficient rings than $\mathbb{Z}$ (see, for instance, [**6**, **7**, **32**, **54**]). This is interesting for algebraic and ring-theoretic applications, where one would like to consider, for example, fields, rings of integers in algebraic number fields, and integral domains. The purpose of this article is to describe the status of the Farrell–Jones Conjecture for algebraic K-theory for arbitrary coefficient rings and to discuss applications, for instance to the *Bass Conjectures*, the *Kaplansky Conjecture*, generalizations of *Moody's Induction Theorem*, *Nil-groups*, and *Fuglede–Kadison determinants*.

1.2. *Status of the Farrell–Jones Conjecture for algebraic K-theory*

There is a stronger version of the Farrell–Jones Conjecture, the so-called *Fibered Farrell–Jones Conjecture*. The Fibered Farrell–Jones Conjecture does imply the Farrell–Jones Conjecture, and has better inheritance properties than the Farrell–Jones Conjecture. We will give the precise technical formulations of these conjectures in Section 3. The original source for

the (Fibered) Farrell–Jones Conjecture is the paper [**28**] from 1993; see in particular [**28**, 1.6, p. 257 and 1.7, p. 262].

'Ring' will always mean an associative ring with unit. It is not necessarily commutative. Fields are understood to be commutative unless they are called skew-fields.

One of the main results of this article is the next theorem, the proof of which will be given in Subsection 3.1.

THEOREM 1.1. *Let R be a ring. Let $\mathcal{FJ}(R)$ be the class of groups which satisfy the Fibered Farrell–Jones Conjecture for algebraic K-theory with coefficients in R. Then:*

(i) *Every word-hyperbolic group and every virtually nilpotent group belongs to $\mathcal{FJ}(R)$.*

(ii) *If G_1 and G_2 belong to $\mathcal{FJ}(R)$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}(R)$.*

(iii) *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps), such that $G_i \in \mathcal{FJ}(R)$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}(R)$.*

(iv) *If H is a subgroup of G and $G \in \mathcal{FJ}(R)$, then $H \in \mathcal{FJ}(R)$.*

If one restricts to lower and middle K-theory for torsionfree groups and regular rings R, the Farrell–Jones Conjecture for algebraic K-theory reduces to the following, easier to understand, conclusions, which are already very interesting in their own right.

Let R be a ring and let G be a group. Denote by $i\colon R \to RG$ the obvious inclusion. Sending $(g, [P]) \in G \times K_0(R)$ to the class of the RG-automorphism

$$R[G] \otimes_R P \to R[G] \otimes_R P, \quad u \otimes x \mapsto ug^{-1} \otimes x$$

defines a map $\Phi\colon G/[G,G] \otimes_{\mathbb{Z}} K_0(R) \to K_1(RG)$. Define the homomorphism

$$A := \Phi \oplus K_1(i)\colon (G/[G,G] \otimes_{\mathbb{Z}} K_0(R)) \oplus K_1(R) \to K_1(RG). \tag{1.1}$$

Define $\operatorname{Wh}^R(G)$ to be the cokernel of A. If $\widetilde{K}_0(R) = 0$ and the obvious map $R^{\times} \to K_1(R)$ is surjective, then $\operatorname{Wh}^R(G)$ coincides with $K_1(RG)/\langle (r \cdot g) \mid r \in R^{\times}, g \in G\rangle$. If $R = \mathbb{Z}$, then $\operatorname{Wh}^{\mathbb{Z}}(G)$ is the classical *Whitehead group* $\operatorname{Wh}(G)$ which appears for instance in the *s*-cobordism theorem.

THEOREM 1.2. *Let R be a regular ring. Suppose that G is torsionfree and that the Farrell–Jones Conjecture for algebraic K-theory with coefficients in R holds for G. Then:*

(i) *$K_n(RG) = 0$ for $n \leqslant -1$.*

(ii) *The change of rings map $K_0(R) \to K_0(RG)$ is bijective. In particular, $\widetilde{K}_0(RG)$ is trivial if and only if $\widetilde{K}_0(R)$ is trivial.*

(iii) *The Whitehead group $\operatorname{Wh}^R(G)$ is trivial.*

The proof of Theorem 1.2 can be found in [**45**, Conjecture 1.1, p. 652, Conjecture 1.1, p. 657 and Corollary 2.3, p. 685].

In particular, we see for $R = \mathbb{Z}$ that $\operatorname{Wh}(G) = 0$, $\widetilde{K}_0(\mathbb{Z}G) = 0$ and $K_n(\mathbb{Z}G) = 0$ for $n \leqslant -1$ hold if the torsionfree group G satisfies the Farrell–Jones Conjecture with coefficients in $\mathbb{Z}$.

These vanishing results have important geometric consequences. Namely, let G be a finitely presented group. Then $\operatorname{Wh}(G)$ vanishes if and only if every compact h-cobordism of dimension at least 6 with G as fundamental group is trivial, and $\widetilde{K}_0(\mathbb{Z}G)$ vanishes if and only if every finitely dominated CW-complex with G as fundamental group is homotopy equivalent to a compact CW-complex. The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ implies that the group G is already of type FF (which is sometimes also called 'type FL' in the literature) if it is of type FP (see [**16**, Chapter 8; Section 6]).

The conclusions appearing in Theorem 1.2 are known to be true for a torsionfree group G, provided that G belongs to the class $\mathcal{FJ}(R)$ appearing in Theorem 1.1. Examples are torsionfree

subgroups of a finite product $\prod_{i=1}^{r} G_i$, where each group G_i is word-hyperbolic. Subgroups of products of word-hyperbolic groups are studied, for instance, in [**15**].

More information about torsionfree groups G for which Theorem 1.2 is true in the case $R = \mathbb{Z}$ can be found in [**45**, Theorem 5.5, p. 722]. There, results due to Aravinda, Farrell, Hu, Jones, Linnell, and Roushon are listed.

1.3. *Homotopy K-theory and rings with finite characteristic*

Sometimes one wants to consider special rings, or one is interested in rational information only. Then the Fibered Farrell–Jones Conjecture is known to be true for more groups. This is illustrated by Theorem 1.3.

For the definition and basic properties of homotopy K-theory, we refer to Weibel [**67**]. For a positive integer N, let $\mathbb{Z}[1/N]$ be the subring of $\mathbb{Q}$ consisting of rational numbers m/n with $m, n \in \mathbb{Z}$ for which each prime dividing n divides N. For an abelian group A, let $A[1/N]$ be $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$. A ring R is said to be *of finite characteristic* if there is an integer $N \geqslant 2$ such that $N \cdot 1_R = 0$. In this case, the minimal positive integer with this property is called the *characteristic of R*.

The proof of the next theorem will be given in Subsection 3.3. The differences between the conclusions appearing in Theorem 1.1 and the following theorem are that now 'virtually nilpotent' is replaced by 'elementary amenable' in assertion (i), and that there are new assertions about extensions and actions on trees, namely assertions (v) and (vi). The extension result is intriguing since elementary amenable groups and word-hyperbolic groups form separate branches in Bridson's universe of groups (see [**14**]).

THEOREM 1.3. *Let R be a ring. Consider the following assertions for a group G.*
- *(KH) The group G satisfies the Fibered Farrell–Jones Conjecture for homotopy K-theory with coefficients in R.*
- *(FC) The ring R has finite characteristic N. The Fibered Farrell–Jones Conjecture for algebraic K-theory for G with coefficients in R for both the families $\mathcal{F}in$ and $\mathcal{V}\mathcal{C}yc$ is true after applying $-\otimes_{\mathbb{Z}}\mathbb{Z}[1/N]$ to the assembly map.*

Let $\mathcal{F}\mathcal{J}_{KH}(R)$ be the class of groups for which assertion (KH) holds. If R has finite characteristic, then let $\mathcal{F}\mathcal{J}_{FC}(R)$ be the class of groups for which assertion (FC) is true. Let $\mathcal{F}$ be $\mathcal{F}\mathcal{J}_{FC}(R)$ or $\mathcal{F}\mathcal{J}_{KH}(R)$. Then:
- *(i) Every word-hyperbolic and every elementary amenable group belongs to $\mathcal{F}$.*
- *(ii) If G_1 and G_2 belong to $\mathcal{F}$, then $G_1 \times G_2$ belongs to $\mathcal{F}$.*
- *(iii) Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps), such that $G_i \in \mathcal{F}$ for $i \in I$. Then $\mathrm{colim}_{i \in I} G_i$ belongs to $\mathcal{F}$.*
- *(iv) If H is a subgroup of G and $G \in \mathcal{F}$, then $H \in \mathcal{F}$.*
- *(v) Let $1 \to H \to G \to Q \to 1$ be an extension of groups, such that H is either elementary amenable or word-hyperbolic and Q belongs to $\mathcal{F}$. Then G belongs to $\mathcal{F}$.*
- *(vi) Suppose that G acts on a tree T. Assume that for each $x \in T$ the isotropy group G_x belongs to $\mathcal{F}$. Then G belongs to $\mathcal{F}$.*

Moreover, if R has finite characteristic, then we have $\mathcal{F}\mathcal{J}_{KH}(R) \subseteq \mathcal{F}\mathcal{J}_{FC}(R)$.

COROLLARY 1.4. *Let R be a regular ring of finite characteristic N. Let G be torsionfree. Suppose that G belongs to the class $\mathcal{F}\mathcal{J}_{FC}(R)$ defined in Theorem 1.3. Then:*
- *(i) $K_n(RG)[1/N] = 0$ for $n \leqslant -1$.*
- *(ii) The change of rings map induces a bijection $K_0(R)[1/N] \to K_0(RG)[1/N]$. In particular, $\widetilde{K}_0(RG)[1/N]$ is trivial if and only if $\widetilde{K}_0(R)[1/N]$ is trivial.*
- *(iii) $\mathrm{Wh}^R(G)[1/N]$ is trivial.*

The proof of Corollary 1.4 is analogous to that of Theorem 1.2. Corollary 1.4, together with Theorem 1.3, substantially extends Theorem 1.1 of Farrell–Linnell [**32**], where $\mathrm{Wh}^F(G) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ is proven for G a torsionfree elementary amenable group and F, a field of prime characteristic.

The vanishing of $\mathrm{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is needed in the definition of a p-adic logarithmic Fuglede–Kadison determinant for G, due to Deninger [**22**].

1.4. Induction from finite subgroups

The next result will be explained and proven in Section 4.

THEOREM 1.5. *Let G be a group. Then:*

(i) *Let R be a regular ring such that the order of any finite subgroup of G is invertible in R, for example, a field of characteristic zero. Suppose that G satisfies the Farrell–Jones Conjecture with coefficients in R. Then the map given by induction from finite subgroups of G (see equation (4.1)),*

$$I(G, R) \colon \underset{\mathrm{Or}_{\mathcal{F}in}(G)}{\mathrm{colim}}\, K_0(RH) \to K_0(RG),$$

is an isomorphism.

(ii) *Let D be a skew-field of characteristic p for a prime number p. Suppose that G satisfies the Farrell–Jones Conjecture with coefficients in D after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$.*

Then the map

$$I(G, D)[1/p] \colon \underset{\mathrm{Or}_{\mathcal{F}in}(G)}{\mathrm{colim}}\, K_0(DH)[1/p] \to K_0(DG)[1/p]$$

is an isomorphism.

Theorem 1.5 is an example of a result of the type that certain K-groups of a group ring are given by induction over finite subgroups. A prominent example is *Moody's Induction Theorem* (see [**19**, **49**, **50**]) which deals with the surjectivity of a corresponding map to $G_0(RG)$ instead of $K_0(RG)$ for virtually poly-$\mathbb{Z}$ groups G and Noetherian rings R. If R is regular and the order of any finite subgroup in the virtually poly-$\mathbb{Z}$ group G is invertible in R, then RG is regular and there is no difference between the G-theoretic and the K-theoretic statement. Thus, Theorem 1.5(i) is linked to Moody's Induction Theorem.

Every group in the family $\mathcal{FJ}(R)$ appearing in Theorem 1.1 satisfies the assumptions of Theorem 1.5(i).

Theorem 1.5(ii) applies to every group G which belongs to the family $\mathcal{FJ}_{FC}(D)$ appearing in Theorem 1.3. Thus, we have substantially generalized Theorem 1.2 of Farrell–Linnell [**32**], where the surjectivity of the map

$$I(G, F) \otimes_{\mathbb{Z}} \mathbb{Q} \colon \underset{\mathrm{Or}_{\mathcal{F}in}(G)}{\mathrm{colim}}\, K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is proven for elementary amenable groups G and fields F of prime characteristic.

1.5. Bass conjectures

The following conjecture is due to Bass [**11**, 4.5].

CONJECTURE 1.6 (Bass Conjecture for commutative integral domains).   Let R be a commutative integral domain, and let G be a group. Let $g \in G$. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.

Then for every finitely generated projective RG-module, the value of its Hattori–Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) (see equation (5.3)) is zero.

The Bass Conjecture 1.6 can be interpreted topologically. Namely, the Bass Conjecture 1.6 is true for a finitely presented group G in the case $R = \mathbb{Z}$, if and only if every homotopy idempotent selfmap of an oriented smooth closed manifold whose dimension is greater than 2 and whose fundamental group is isomorphic to G is homotopic to one that has precisely one fixed point (see [13]). The Bass Conjecture 1.6 for G in the case $R = \mathbb{Z}$ (or $R = \mathbb{C}$) also implies, for a finitely dominated CW-complex with fundamental group G, that its Euler characteristic agrees with the L^2-Euler characteristic of its universal covering (see [24]).

The next results follows from the argument in [32, Section 5].

THEOREM 1.7. *Let G be a group. Suppose that*

$$I(G, F) \otimes_{\mathbb{Z}} \mathbb{Q} : \underset{\mathrm{Or}_{\mathcal{F}in}(G)}{\mathrm{colim}} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic.
Then the Bass Conjecture 1.6 is satisfied for every commutative integral domain R.

Hence, by Theorem 1.5, the Bass Conjecture for commutative integral domains (Conjecture 1.6) is true for every group G which lies in the class $\mathcal{FJ}_{FC}(F)$ for all fields F of prime characteristic. The case of elementary amenable groups has already been treated in [32, Theorem 1.6]. The Bass Conjecture 1.6 has been proved by Bass [11, Proposition 6.2 and Theorem 6.3] for $R = \mathbb{C}$ and G a linear group, by Linnell [38, Lemma 4.1] for $|g| < \infty$ and $R = \mathbb{Z}$, and by Eckmann [23, Theorem 3.3] for $R = \mathbb{Q}$, provided that G has at most cohomological dimension 2 over $\mathbb{Q}$. Further results have been proved by Emmanouil [27].

Here is another version of the Bass Conjecture.

CONJECTURE 1.8 (Bass Conjecture for fields of characteristic zero as coefficients). Let F be a field of characteristic zero, and let G be a group. The Hattori–Stallings homomorphism (see equation (5.3)) induces an isomorphism

$$\mathrm{HS}_{FG} : K_0(FG) \otimes_{\mathbb{Z}} F \to \mathrm{class}_F(G)_f.$$

For a field F of characteristic zero, Conjecture 1.8 obviously implies Conjecture 1.6. The proof of the next result will be given in Section 5.

THEOREM 1.9. *Let F be a field of characteristic zero, and let G be a group. If G satisfies the Farrell–Jones Conjecture with coefficients in F, then G satisfies the Bass Conjecture 1.8 for F.*

In particular, the Bass Conjecture for a field F of characteristic zero as coefficients (Conjecture 1.8) is true for all groups in the class $\mathcal{FJ}(F)$ by Theorem 1.1.

Berrick–Chatterji–Mislin [12] prove that a group G satisfies the Bass Conjecture 1.8 for $F = \mathbb{C}$ and the Bass Conjecture 1.6 for integral domains for $R = \mathbb{Z}$, if G satisfies the Bost Conjecture. Because the Bost Conjecture is known for many groups, this is also true for the Bass Conjecture for $F = \mathbb{C}$. Since the Bost Conjecture deals with l^2-spaces, this strategy can only work for subrings of $\mathbb{C}$.

The following result describes another conclusion of the Farrell–Jones Conjecture, which is in the spirit of the Bass Conjecture. It is hence true for all groups in the family $\mathcal{FJ}(R)$.

THEOREM 1.10. *Let G be a group. Let R be a commutative integral domain with field of fractions F such that no prime divisor of the order of a finite subgroup of G is invertible in R. (An example is $R = \mathbb{Z}$ and $F = \mathbb{Q}$.) Suppose that G satisfies the Farrell–Jones Conjecture for algebraic K-theory with coefficients in R.*

Then the change of rings homomorphisms

$$K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

agrees with the composite

$$K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where the three maps come from the change of ring homomorphisms associated to the augmentation $RG \to R$, the inclusion $R \to F$, and the inclusion $F \to FG$. In particular, the homomorphism

$$\widetilde{K}_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q} \to \widetilde{K}_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is trivial.

(We remind the reader that $\widetilde{K}_0(RG)$ and $\widetilde{K}_0(FG)$ are, respectively, the cokernel of the canonical map $K_0(\mathbb{Z}) \to K_0(RG)$ and that of the canonical map $K_0(\mathbb{Z}) \to K_0(FG)$.)

If G is finite, G satisfies the Farrell–Jones Conjecture for algebraic K-theory with coefficients in R for trivial reasons; hence Theorem 1.10 reduces to a Theorem of Swan (see [**63**, Theorem 8.1; **11**, Corollary 4.2]).

The conclusion of Theorem 1.10 is related to the theorem that for every group G, the change of rings homomorphism $\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathcal{N}(G))$ is trivial, where $\mathcal{N}(G)$ is the group von Neumann algebra (see [**41**, Theorem 9.62, p. 363; **60**]).

1.6. The Kaplansky Conjecture

CONJECTURE 1.11 (Kaplansky Conjecture). Let R be an integral domain, and let G be a torsionfree group. Then all idempotents of RG are trivial, that is, equal to 0 or 1.

In the next theorem, we will use the notion of a *sofic group* that was introduced by Gromov and originally called a *subamenable group*. Every residually amenable group is sofic but the converse is not true. The class of sofic groups is closed under taking subgroups, direct products, free amalgamated products, colimits, and inverse limits, and, if H is a sofic normal subgroup of G with amenable quotient G/H, then G is sofic. For more information about the notion of a sofic group, we refer to [**26**].

The proof and further explanations of the next theorem will be given in Section 6.

THEOREM 1.12. *Let G be a group. Let R be a ring whose idempotents are all trivial. Suppose that*

$$K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Then the Kaplansky Conjecture holds for R and G if one of the following conditions is satisfied:
  (i) *RG is stably finite.*
 (ii) *R is a field of characteristic zero.*
(iii) *R is a skew-field and G is sofic.*

Next, we discuss some special cases of Theorem 1.12. Notice that we get assertions also for skew-fields, and not only for fields as coefficients.

Theorems 1.2 and 1.12 imply that for a skew-field D of characteristic zero and a torsionfree group G belonging to the class of groups $\mathcal{FJ}(D)$ defined in Theorem 1.1, the Kaplansky Conjecture 1.11 is true for DG, provided that D is commutative or that G is sofic.

Suppose that D is a skew-field of prime characteristic p, all finite subgroups of G are p-groups and G belongs to the class $\mathcal{FJ}_{FC}(D)$ defined in Theorem 1.3. Then $K_0(D) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_0(DG) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism by Theorem 1.5(ii), since for a finite p-group H, the group ring DH is a local ring and hence $\widetilde{K}_0(DH) = 0$. If we furthermore assume that G is sofic, then Theorem 1.12 implies that all idempotents in DG are trivial. This has already been proved by Farrell–Linnell [**32**, Theorem 1.7] in the case where G is elementary amenable and D is commutative.

To the authors' knowledge, there is no example in the literature of a group which is not sofic and of a group which is word-hyperbolic and not residually finite. So it is conceivable that all word-hyperbolic groups are sofic.

Next we mention some results of others.

Let F be a field of characteristic zero, and let $u = \sum_{g \in G} x_g \cdot g \in KG$ be an idempotent. Let K be the finitely generated field extension of $\mathbb{Q}$ given by $K = \mathbb{Q}(x_g \mid g \in G)$. Obviously, $u \in KG$ is an idempotent. There exists an embedding of K in $\mathbb{C}$. Hence all idempotents in FG are trivial if all idempotents in $\mathbb{C}G$ are trivial.

The *Kadison Conjecture* says that all idempotents in the reduced group C*-algebra $C_r^*(G)$ of a torsionfree group are trivial. Hence, the Kadison Conjecture implies the Kaplansky Conjecture for all fields of characteristic zero. The Kadison Conjecture follows from the *Baum–Connes Conjecture* (as explained, for instance, in [**42**, **45**, 1.8.1 and 1.8.2]). Hence, a torsionfree group G satisfies the Kaplansky Conjecture for all fields of characteristic zero, if it satisfies the Baum–Connes Conjecture. For a survey of groups satisfying the Baum–Connes Conjecture, we refer to [**45**, Sections 5.1 and 5.3]. We mention that subgroups of word-hyperbolic groups satisfy the Baum–Connes Conjecture by a result of Mineyev–Yu [**48**, Theorem 20] based on the work of Lafforgue [**37**] (see also [**61**]). A proof of the Kadison Conjecture for a torsionfree word-hyperbolic group using cyclic homology is given by Puschnigg [**52**]. Notice that all these analytic methods work only for fields of characteristic zero and cannot be extended to skew-fields or fields of prime characteristic.

Formanek [**30**] (see also [**17**, Lemma 4.1 and Proposition 4.2]) has shown that all idempotents of FG are trivial, provided that F is a field of prime characteristic p, the group G contains no p-torsion and there do not exist an element $g \in G$, $g \neq 1$, and an integer $k \geqslant 1$ such that g and g^{p^k} are conjugate, or provided that F is a field of characteristic zero and there are infinitely many primes p for which there do not exist an element $g \in G$, $g \neq 1$, and an integer $k \geqslant 1$ such that g and g^{p^k} are conjugate. Torsionfree word-hyperbolic groups satisfy these conditions. Hence, Formanek's results imply that all idempotents in FG are trivial if G is torsionfree word-hyperbolic and F is a field.

Delzant [**21**] has proven the Kaplansky Conjecture 1.11 for all integral domains R for a torsionfree word-hyperbolic group G, provided that G admits an appropriate action with large enough injectivity radius. Delzant actually deals with zero-divisors and units as well.

1.7. Homotopy invariance of L^2-torsion

The following conjecture for a group G is stated and explained in Lück [**41**, Conjecture 3.94 (1), p. 163].

CONJECTURE 1.13. Define the homomorphism

$$\Phi = \Phi^G \colon \mathrm{Wh}(G) \to \mathbb{R}$$

by sending the class $[A]$ of an invertible matrix $A \in \mathrm{GL}_n(\mathbb{Z}G)$ to $\ln(\det(r_A^{(2)}))$, where $\det(r_A^{(2)})$ is the Fuglede–Kadison determinant of the G-equivariant bounded operator $l^2(G)^n \to l^2(G)^n$ given by right multiplication with A.

Then Φ is trivial.

It is important because of the following conclusion, explained in [**41**, Theorem 3.94 (1), p. 161]: If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsions agree:

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(Y; \mathcal{N}(G)).$$

THEOREM 1.14. *Suppose that G satisfies the Farrell–Jones Conjecture for algebraic K-theory with coefficients in $\mathbb{Z}$. Then G satisfies Conjecture* 1.13.

We will omit the proof of Theorem 1.14 since it is similar to that of Theorem 1.10 using the fact that for a finite group H, we have $\widetilde{K}_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and Conjecture 1.13 is true for finite groups for elementary reasons.

Let G be a torsionfree word-hyperbolic group. Suppose that its L^2-Betti numbers $b_p^{(2)}(G)$ are trivial for all $p \geqslant 0$ and that G is of det $\geqslant$ 1-class. (If G is residually finite, it is of det $\geqslant$ 1-class.) Choose a cocompact model for EG. Then we can define the L^2-torsion of G:

$$\rho^{(2)}(G) := \rho^{(2)}(EG; \mathcal{N}(G)) \in \mathbb{R}.$$

This is indeed independent of the choice of a cocompact model for EG, and hence depends only on G, by Theorems 1.1 and 1.14. If M is a compact hyperbolic manifold of dimension $2n + 1$ whose interior admits a complete hyperbolic Riemannian metric of finite volume, then $\pi = \pi_1(M)$ satisfies all these assumptions and there exists a number $C_n > 0$ depending only on n such that $C_n \cdot (-1)^n \cdot \rho^{(2)}(\pi)$ is the volume of the interior of M [**47**, Theorem 0.5]. Hence, $\rho^{(2)}(G)$ can be viewed as a kind of volume of a word-hyperbolic group G, provided that G satisfies the above assumptions.

1.8. *Searching for counterexamples*

There is no group known for which the Farrell–Jones Conjecture, the Fibered Farrell–Jones Conjecture or the Baum–Connes Conjecture is false. However, Higson, Lafforgue, and Skandalis [**35**, Section 7] construct counterexamples to the *Baum–Connes Conjecture with coefficients*, actually with a commutative C*-algebra as coefficients. They describe precisely what properties a group Γ must have so that it does *not* satisfy the Baum–Connes Conjecture with coefficients. Gromov [**33**] describes the construction of such a group Γ as a colimit over a directed system of groups $\{G_i \mid i \in I\}$ such that each G_i is word-hyperbolic. We conclude from Bartels–Echterhoff–Lück [**3**] and Theorem 1.1 that the Fibered Farrell–Jones Conjecture and the Bost Conjecture do hold for Γ.

1.9. *Nil-groups*

In Section 7, we discuss some consequences for Nil-groups in the sense of Bass and Waldhausen, and for the passage from algebraic K-theory to homotopy K-theory. There we will prove the following application of Theorem 1.1(i) to Waldhausen's Nil-groups.

THEOREM 1.15. *Let G, H, and K be finite groups. Let $C := \mathbb{Z}[K]$.*
  (i) *Let $\alpha \colon K \to G$ and $\beta \colon K \to H$ be injective group homeomorphisms. Consider the C bimodules $A' := \mathbb{Z}[G - \alpha(K)]$ and $B' := \mathbb{Z}[H - \beta(K)]$. Then*

$$\mathrm{Nil}_n(C; A', B') \otimes_{\mathbb{Z}} \mathbb{Q} = 0;$$

  (ii) *Let $\alpha \colon K \to G$ and $\beta \colon K \to G$ be injective group homeomorphisms. Let $A' := \mathbb{Z}[G - \alpha(K)]$, $A'' := \mathbb{Z}[H - K]$ and $A := \mathbb{Z}[G]$. Then*

$$\mathrm{Nil}_n(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta) \otimes_{\mathbb{Z}} \mathbb{Q} = 0,$$

where we use the lower indices to indicate the relevant C-bimodule structures.

1.10. *The Farrell–Jones Conjecture for L-theory*

In Section 8, we briefly explain some results about the *L*-theoretic version of the Farrell–Jones Conjecture.

2. *Inheritance properties of the (Fibered) Isomorphism Conjecture*

In this section, we formulate a (Fibered) Isomorphism Conjecture for a given equivariant homology theory and a family of groups. In this general setting, we study the behavior of this Fibered Isomorphism Conjecture under directed colimits and extensions. The Farrell–Jones Conjecture is a special case; one has to choose a specific equivariant homology theory and a specific family of subgroups. The payoff of this general setting is that some of the proofs become easier and more transparent, and that it applies to other related Isomorphism Conjectures such as the *Farrell–Jones Conjecture for L-theory*, the *Baum–Connes Conjecture*, the *Bost Conjecture*, and other types of Isomorphism Conjectures predicting the bijectivity of certain assembly maps.

For this section, we fix the following data:
(1) a discrete group G;
(2) an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules;
(3) a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, for example, the family $\mathcal{F}\text{in}$ of finite groups and the family $\mathcal{VC}\text{yc}$ of virtually cyclic groups. For a group G, we denote by $\mathcal{C}(G)$ the family of subgroups of G which belong to $\mathcal{C}$.

Here, *equivariant homology theory* with values in Λ-modules for a commutative ring Λ satisfying the disjoint union axiom is understood in the sense of [40, Section 1] with one important modification: we require that for every group homomorphism $\alpha \colon H \to K$, we get a natural transformation

$$\text{ind}_\alpha \colon \mathcal{H}_*^H(X) \to \mathcal{H}_n^K(\text{ind}_\alpha X)$$

satisfying the obvious variations of the axioms.
(1) Compatibility with the boundary operator.
(2) Naturality.
(3) Compatibility with conjugation, but the map ind_α is only required to be an isomorphism in the case, where $X = \text{pt}$ and α is injective. This implies that ind_α is bijective for a *G-CW*-complex X, if the kernel of α acts freely on X (see [3, Lemma 1.5]). Every Groupoids-spectrum gives an equivariant homology theory with values in $\mathbb{Z}$-modules in the sense above (see [45, 6.5]). In particular, we get an equivariant homology theory with values in $\mathbb{Z}$-modules in the sense above for algebraic *K*-theory (see [20, Section 2; 45, Theorem 6.1]). If $\mathcal{H}_*^?$ is an equivariant homology theory with values in $\mathbb{Z}$-modules, then $\mathcal{H}_*^? \otimes_\mathbb{Z} \Lambda$ is an equivariant homology theory with values in Λ-modules for $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$.

Notice that this is one of the key differences between the Farrell–Jones Conjecture for algebraic *K*- and *L*-theory and the Baum–Connes Conjecture for topological *K*-theory of reduced group C*-algebras. In the latter case, induction is only defined if the kernel of the group homomorphism acts freely because the corresponding spectrum lives over Groupoids$^{\text{inj}}$ and not over Groupoids as in the Farrell–Jones setting (see [45, 6.5]).

2.1. *The Fibered Isomorphism Conjecture for equivariant homology theories*

A *family of subgroups of G* is a collection of subgroups of G which is closed under conjugation and taking subgroups. Let $E_\mathcal{F}(G)$ be the *classifying space associated to $\mathcal{F}$*. It is uniquely characterized up to *G*-homotopy by the properties that it is a *G-CW*-complex, and that $E_\mathcal{F}(G)^H$ is contractible if $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$. For more information about these

spaces $E_{\mathcal{F}}(G)$, we refer to the survey article [**44**]. Given a group homomorphism $\phi\colon K \to G$ and a family $\mathcal{F}$ of subgroups of G, define the family $\phi^*\mathcal{F}$ of subgroups of K by

$$\phi^*\mathcal{F} = \{H \subseteq K \mid \phi(H) \in \mathcal{F}\}. \tag{2.1}$$

If ϕ is an inclusion of a subgroup, we also write $\mathcal{F}|_K$ instead of $\phi^*\mathcal{F}$.

DEFINITION 2.1 ((Fibered) Isomorphism Conjecture for $\mathcal{H}_*^?$). A group G together with a family of subgroups $\mathcal{F}$ satisfies the *Isomorphism Conjecture for* $\mathcal{H}_*^?$ if the projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to \mathrm{pt}$ to the one-point-space pt induces an isomorphism

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \xrightarrow{\cong} \mathcal{H}_n^G(\mathrm{pt})$$

for $n \in \mathbb{Z}$.

The pair $(G, \mathcal{F})$ satisfies the *Fibered Isomorphism Conjecture for* $\mathcal{H}_*^?$ if for every group homomorphism $\phi\colon K \to G$, the pair $(K, \phi^*\mathcal{F})$ satisfies the Isomorphism Conjecture.

We mostly work with a fixed equivariant homology theory $\mathcal{H}_*^?$ and hence we will often omit it in the statements.

The following results are proven in [**4**, Lemma 1.6; **5**, Lemma 1.2 and Theorem 2.4].

LEMMA 2.2. *Let G be a group, and let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of G. Suppose that $(G, \mathcal{F})$ satisfies the Fibered Isomorphism Conjecture.*

Then $(G, \mathcal{G})$ satisfies the Fibered Isomorphism Conjecture.

LEMMA 2.3. *Let $\phi\colon K \to G$ be a group homomorphism, and let $\mathcal{F}$ be a family of subgroups. If $(G, \mathcal{F})$ satisfies the Fibered Isomorphism Conjecture, then $(K, \phi^*\mathcal{F})$ satisfies the Fibered Isomorphism Conjecture.*

THEOREM 2.4 (Transitivity Principle). *Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of G. Assume that for every element $H \in \mathcal{G}$, the group H satisfies the (Fibered) Isomorphism Conjecture for $\mathcal{F}|_H$.*

Then $(G, \mathcal{G})$ satisfies the (Fibered) Isomorphism Conjecture if and only if $(G, \mathcal{F})$ satisfies the (Fibered) Isomorphism Conjecture.

The next lemma follows from Lemma 2.3 applied to the inclusion $H \to G$, since $\mathcal{C}(H) = \mathcal{C}(G)|_H$.

LEMMA 2.5. *Suppose that the Fibered Isomorphism Conjecture holds for $(G, \mathcal{C}(G))$. Let $H \subseteq G$ be a subgroup.*

Then the Fibered Isomorphism Conjecture holds for $(H, \mathcal{C}(H))$.

2.2. Colimits over directed systems of groups

We collect some basic facts about the behavior of the Fibered Isomorphism Conjecture under directed colimits.

We consider a *directed* set I and a *directed system of groups* $\{G_i \mid i \in I\}$. The *structure maps* $\phi_{i,j}\colon G_i \to G_j$ for $i,j \in I$ with $i \leqslant j$ are *not* required to be injective. Let $\mathrm{colim}_{i \in I} G_i$ be the *colimit*. Denote by $\psi_i\colon G_i \to G$ the *structure maps of the colimit* for $i \in I$.

We say that G is the *directed union* of the subgroups $\{G_i \mid i \in I\}$ if I is a directed set and $\{G_i \mid i \in I\}$ is a directed system of subgroups, directed by inclusion, such that $G = \bigcup_{i \in I} G_i$. This is essentially the same as a directed system of groups such that all structure maps $\phi_{i,j}$ are inclusions of groups and $G = \mathrm{colim}_{i \in I} G_i$. For a group homomorphism $\psi\colon G' \to G$, define

the Λ-map

$$\alpha_n(\psi)\colon \mathcal{H}_n^{G'}(\mathrm{pt}) \to \mathcal{H}_n^{G}(\mathrm{pt})$$

as the composition of ind_ψ with the map induced by the projection $\psi^*\mathrm{pt} \to \mathrm{pt}$ of G-spaces.

The next definition is an extension of [**4**, Definition 3.1].

DEFINITION 2.6 ((Strongly) Continuous equivariant homology theory). An equivariant homology theory $\mathcal{H}_*^?$ is called *continuous* if for each group G which is the directed union of subgroups $\{G_i \,|\, i \in I\}$, the Λ-map

$$\operatorname*{colim}_{i \in I} \alpha_n(G_i \to G)\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\mathrm{pt}) \to \mathcal{H}_n^{G}(\mathrm{pt})$$

is an isomorphism for every $n \in \mathbb{Z}$.

An equivariant homology theory $\mathcal{H}_*^?$ is called *strongly continuous* if for each directed system of groups $\{G_i \,|\, i \in I\}$ with $G = \operatorname*{colim}_{i \in I} G_i$, the Λ-map

$$\operatorname*{colim}_{i \in I} \alpha_n(\psi_i)\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\mathrm{pt}) \to \mathcal{H}_n^{G}(\mathrm{pt})$$

is an isomorphism for every $n \in \mathbb{Z}$.

The next theorem generalizes the result of Farrell–Linnell [**31**, Theorem 7.1] to a more general setting about equivariant homology theories as developed in Bartels–Lück [**4**]. Its proof can be found in [**3**, Theorems 3.4 and 4.6].

THEOREM 2.7. (i) *Let G be the directed union $G = \bigcup_{i \in I} G_i$ of subgroups G_i Suppose that $\mathcal{H}_*^?$ is continuous and that the (Fibered) Isomorphism Conjecture is true for $(G_i, \mathcal{C}(G_i))$ for all $i \in I$. Then the (Fibered) Isomorphism Conjecture is true for $(G, \mathcal{C}(G))$;*

(ii) *Let $\{G_i \,|\, i \in I\}$ be a directed system of groups. Put $G = \operatorname*{colim}_{i \in I} G_i$. Suppose that $\mathcal{H}_*^?$ is strongly continuous and that the Fibered Isomorphism Conjecture is true for $(G_i, \mathcal{C}(G_i))$ for all $i \in I$. Then the Fibered Isomorphism Conjecture is true for $(G, \mathcal{C}(G))$.*

2.3. *Extensions*

For the remainder of this section, fix the following data:

(1) a discrete group G;
(2) an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules;
(3) a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, for example, $\mathcal{F}\mathrm{in}$ or $\mathcal{VC}\mathrm{yc}$;
(4) an exact sequence of groups $1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1$.

We want to investigate the inheritance properties of the (Fibered) Isomorphism Conjecture 2.1 under exact sequences.

LEMMA 2.8. *Suppose that the Fibered Isomorphism Conjecture holds for $(Q, \mathcal{C}(Q))$, and for every $H \in p^*\mathcal{C}(Q)$ the Isomorphism Conjecture is true for $(H, \mathcal{C}(H))$.*

Then the Isomorphism Conjecture is true for $(G, \mathcal{C}(G))$.

Proof. Since the Fibered Isomorphism Conjecture holds for $(Q, \mathcal{C}(Q))$ by assumption, the Isomorphism Conjecture holds for $(G, p^*\mathcal{C}(Q))$. We have to show that the Isomorphism Conjecture holds for $(G, \mathcal{C}(G))$. But this follows from the Transitivity Principle (Theorem 2.4). $\square$

LEMMA 2.9.   *Suppose that the Fibered Isomorphism Conjecture holds for $(Q, \mathcal{C}(Q))$. Then the following assertions are equivalent.*

(i) *The Fibered Isomorphism Conjecture is true for $(p^{-1}(H), \mathcal{C}(p^{-1}(H)))$ for every $H \in \mathcal{C}(Q)$.*

(ii) *The Fibered Isomorphism Conjecture is true for $(G, \mathcal{C}(G))$.*

Proof.   (ii) ⇒(i) This follows from Lemma 2.3 applied to the inclusion $p^{-1}(H) \to G$.

(i) ⇒(ii) Let $q \colon L \to G$ be a group homomorphism. We have to show that $(L, q^*\mathcal{C}(G))$ satisfies the Isomorphism Conjecture. Since $(Q, \mathcal{C}(Q))$ satisfies the Fibered Isomorphism Conjecture, we conclude that $(L, q^*p^*\mathcal{C}(Q))$ satisfies the Isomorphism Conjecture. Because of the Transitivity Principle 2.4, it remains to show, for any $H \subseteq L$ for which there exists $V \in \mathcal{C}(Q)$ with $q(H) \subseteq p^{-1}(V)$, that $(H, (q^*\mathcal{C}(G))|_H)$ satisfies the Isomorphism Conjecture. This follows from the assumption that $(p^{-1}(V), \mathcal{C}(p^{-1}(V)))$ satisfies the Fibered Isomorphism Conjecture since the families $(q^*\mathcal{C}(G))|_H$ and $(q|_H)^*\mathcal{C}(p^{-1}(V))$ coincide.   □

LEMMA 2.10.   *Suppose that $p^{-1}(H)$ belongs to $\mathcal{C}(G)$ if $H \in \mathcal{C}(Q)$. Then $(G, \mathcal{C}(G))$ satisfies the Fibered Isomorphism Conjecture, if $(Q, \mathcal{C}(Q))$ satisfies the Fibered Isomorphism Conjecture.*

Proof.   This follows from Lemma 2.3, since $p^*\mathcal{C}(Q) = \mathcal{C}(G)$.   □

Lemma 2.10 is interesting in the case where $\mathcal{C}$ is $\mathcal{F}$in or $\mathcal{VC}$yc and K is finite.

LEMMA 2.11.   (i) *Suppose that $H_1 \times H_2$ belongs to $\mathcal{C}$ if H_1 and H_2 belong to $\mathcal{C}$. Then $(G_1 \times G_2, \mathcal{C}(G_1 \times G_2))$ satisfies the Fibered Isomorphism Conjecture if and only if both $(G_1, \mathcal{C}(G_1))$ and $(G_2, \mathcal{C}(G_2))$ satisfy the Fibered Isomorphism Conjecture.*

(ii) *Suppose that $(D_\infty \times D_\infty, \mathcal{VC}yc(D_\infty \times D_\infty))$ satisfies the Fibered Isomorphism Conjecture, where $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2$ is the infinite dihedral group. Then $(G_1 \times G_2, \mathcal{VC}yc(G_1 \times G_2))$ satisfies the Fibered Isomorphism Conjecture if and only if both $(G_1, \mathcal{VC}yc(G_1))$ and $(G_1, \mathcal{VC}yc(G_1))$ satisfy the Fibered Isomorphism Conjecture.*

Proof.   (i) If the Fibered Isomorphism Conjecture holds for $(G_1 \times G_2, \mathcal{C}(G_1) \times \mathcal{C}(G_2))$, it also holds, by Lemma 2.5, for $(G_i, \mathcal{C}(G_i))$ since $\mathcal{C}(G_1 \times G_2)|_{G_i} = \mathcal{C}(G_i)$.

Suppose that the Fibered Isomorphism Conjecture holds for both $(G_1, \mathcal{C}(G_1))$ and $(G_2, \mathcal{C}(G_2))$. In view of Lemma 2.9, we can assume without loss of generality that G_2 belongs to $\mathcal{C}$. Applying this argument again, we can assume without loss of generality that G_1 and G_2 belong to $\mathcal{C}$. This case is obviously true since $G_1 \times G_2 \in \mathcal{C}$. (cf. [58, Lemma 5.1] for a similar argument.)

(ii) Analogously to the proof of assertion 2.11, one reduces the claim to the assertion that $(G_1 \times G_2, \mathcal{VC}yc(G_1 \times G_2))$ satisfy the Fibered Isomorphism Conjecture if G_1 and G_2 are virtually cyclic. Since every virtually cyclic group admits an epimorphism to $\mathbb{Z}$ or D_∞ with finite kernel and $\mathbb{Z} \subseteq D_\infty$, the product $G_1 \times G_2$ admits a group homomorphism to $D_\infty \times D_\infty$ with finite kernel. Now apply Lemma 2.5 and Lemma 2.10.
□

Assertion 2.11 appearing in Lemma 2.11 is interesting in the case $\mathcal{C} = \mathcal{F}$in.

LEMMA 2.12.   *Suppose that $\mathcal{H}^?_*$ is continuous. Suppose that any virtually finitely generated abelian group satisfies the Fibered Isomorphism Conjecture for $\mathcal{F}$in. Then every elementary amenable group satisfies the Fibered Isomorphism Conjecture for $\mathcal{F}$in.*

Proof. Using the same transfinite induction strategy and the same notation as in the proof of [**32**, Corollary 3.9], one reduces the claim to the following assertion. The group G satisfies the Fibered Isomorphism Conjecture for $\mathcal{F}$in provided that there exists an extension $1 \to H \to G \xrightarrow{p} A \to 1$ such that A is virtually finitely generated abelian, H belongs to $L\mathcal{X}_{\alpha-1}$ and the Fibered Isomorphism Conjecture for $\mathcal{F}$in holds for every group in the class of groups $\mathcal{X}_{\alpha-1}$. Here $L\mathcal{X}_{\alpha-1}$ is the class of groups for which every finitely generated subgroup occurs as a subgroup of some group in $\mathcal{X}_{\alpha-1}$. Theorem 2.72.7 and Lemma 2.5 imply that the Fibered Isomorphism Conjecture holds for every group in $L\mathcal{X}_{\alpha-1}$. Because of Lemma 2.9, it remains to prove, for any finite subgroup $K \subseteq A$, that the Fibered Isomorphism Conjecture for $\mathcal{F}$in holds for $p^{-1}(K)$. We conclude from the short exact sequence $1 \to H \to p^{-1}(K) \to K \to 1$ and [**32**, Lemma 2.1(iii)] that $p^{-1}(K)$ is a member of $L\mathcal{X}_{\alpha-1}$, and therefore satisfies the Fibered Isomorphism Conjecture for $\mathcal{F}$in. $\square$

LEMMA 2.13. *Suppose that $\mathcal{H}_*^?$ is continuous. Suppose that any virtually finitely generated abelian group satisfies the Fibered Isomorphism Conjecture for $\mathcal{VC}yc$.*

Then every virtually nilpotent group satisfies the Fibered Isomorphism Conjecture for $\mathcal{VC}yc$.

Proof. Any finitely generated subgroup of a virtually abelian group is virtually finitely generated abelian. The assumptions and Theorem 2.72.7 imply that any virtually abelian group satisfies the Fibered Isomorphism Conjecture for $\mathcal{VC}yc$.

For a group H, we denote by $\mathrm{cent}(H)$ its center. Recall that a group N is called *nilpotent* if we can find a sequence of epimorphisms $N = N_0 \xrightarrow{p_0} N_1 \xrightarrow{p_1} \cdots \xrightarrow{p_r} N_r$ such that $\ker(p_i) = \mathrm{cent}(N_{i-1})$ for $i = 1, 2, \dots, r$ and $N_r = \{1\}$. The class of N is the smallest nonnegative integer r for which such a sequence of epimorphisms exists. Let G be virtually nilpotent. Hence, we can find a normal subgroup $N \subseteq G$ such that N is nilpotent and G/N is finite. We show by induction over the class of N that G satisfies the Fibered Isomorphism Conjecture for $(G, \mathcal{VC}yc)$. The induction beginning where the class is at most 1, and hence G is virtually abelian, has already been taken care of.

We can arrange that r is the class of N. Since N is normal in G and $\mathrm{cent}(N)$ is a characteristic subgroup of N, $\mathrm{cent}(N)$ is a normal subgroup of G. We obtain the exact sequence $1 \to N/\mathrm{cent}(N) \to G/\mathrm{cent}(N) \to G/N \to 1$. Since the class of $N/\mathrm{cent}(N)$ is smaller than the class of N, the Fibered Isomorphism Conjecture holds for $(G/\mathrm{cent}(N), \mathcal{VC}yc)$ by the induction hypothesis. Because of Theorem 2.9, it remains to show for any virtually cyclic subgroup $V \subseteq G/\mathrm{cent}(N)$ that $(q^{-1}(V), \mathcal{VC}yc)$ satisfies the Fibered Isomorphism Conjecture, where $q \colon G \to G/\mathrm{cent}(N)$ is the canonical projection.

Let $\phi \colon G \to \mathrm{aut}(N)$ be the group homomorphism sending $g \in G$ to the automorphism of N given by conjugation with g. Since $\mathrm{cent}(N)$ is a characteristic subgroup of N, it induces a homomorphism $\phi' \colon G \to \mathrm{aut}(\mathrm{cent}(N))$. Since the conjugation action of N on itself is the identity on $\mathrm{cent}(N)$, the homomorphism ϕ' factorizes through the finite group G/N and hence has finite image. Hence, we can find $g \in G$ such that $q(g)$ generates an infinite cyclic subgroup C in V whose index in V is finite and $\phi'(g) = \mathrm{id}_{\mathrm{cent}(N)}$. Hence $q^{-1}(C)$ is isomorphic to $\mathrm{cent}(N) \times C$ and has finite index in $q^{-1}(V)$. Therefore, $q^{-1}(V)$ is virtually abelian and $(q^{-1}(V), \mathcal{VC}yc)$ satisfies the Fibered Isomorphism Conjecture. $\square$

3. *The Farrell–Jones Conjecture for algebraic K-theory*

Recall that the (*Fibered*) *Farrell–Jones Conjecture for algebraic K-theory with coefficients in R for the group G* is the (Fibered) Isomorphism Conjecture 2.1 in the special case where the family $\mathcal{F}$ consists of all virtually cyclic subgroups of G and $\mathcal{H}_*^?$ is the equivariant homology theory $\mathcal{H}_*^?(-; \mathbf{K}_R)$ associated to the Groupoids-spectrum given by algebraic K-theory and R

as coefficient ring (see [**45**, Section 6]). So the Farrell–Jones Conjecture for algebraic K-theory with coefficients in R for the group G predicts that the map

$$H_n^G(E_{\mathcal{VCyc}}(G), \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$. The original source for the (Fibered) Farrell–Jones Conjecture is [**28**, 1.6, p. 257 and 1.7, p. 262].

Recall that the (*Fibered*) *Farrell–Jones Conjecture for homotopy K-theory with coefficients in R for the group G* is the (Fibered) Isomorphism Conjecture 2.1 in the special case where the family $\mathcal{F}$ consists of all finite subgroups of G and $\mathcal{H}_*^?$ is the equivariant homology theory $\mathcal{H}_*^?(-;\mathbf{KH}_R)$ associated to the Groupoids-spectrum given by homotopy K-theory and R as coefficient ring (see [**5**, Section 7]). So the Farrell–Jones Conjecture for homotopy K-theory with coefficients in R for the group G predicts that the map

$$H_n^G(E_{\mathcal{F}\text{in}}(G), \mathbf{KH}_R) \to KH_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

The following theorem follows from the main result of Bartels–Lück–Reich [**6**] together with [**8**, Corollary 4.3].

THEOREM 3.1. *The Fibered Farrell–Jones Conjecture for algebraic K-theory is true for every word-hyperbolic group and every coefficient ring.*

Next, we extend a result of Quinn [**54**, Theorem 1.2.2] for virtually abelian groups to virtually nilpotent groups.

THEOREM 3.2. *The Fibered Farrell–Jones Conjecture for algebraic K-theory is true for every virtually nilpotent group and every coefficient ring.*

Proof. The Fibered Farrell–Jones Conjecture for algebraic K-theory is true for every virtually abelian group and every coefficient ring, by a result of Quinn [**54**, Theorem 1.2.2]. (Quinn deals only with commutative coefficient rings but this assumption is not needed in his argument.) Now apply Lemma 2.13. $\square$

For more information about groups satisfying the Farrell–Jones Conjecture for algebraic K-theory with coefficients in $\mathbb{Z}$ and the Farrell–Jones Conjecture for homotopy K-theory $KH_*(RG)$, we refer to [**45**, Sections 5.2 and 5.3; **5**, Theorem 0.5].

LEMMA 3.3. *The equivariant homology theories $\mathcal{H}_*^?(-;\mathbf{K}_R)$ and $\mathcal{H}_*^?(-;\mathbf{KH}_R)$ are strongly continuous.*

Proof. We have to show for every directed system of groups $\{G_i \mid i \in I\}$ with $G = \text{colim}_{i \in I} G_i$ that the canonical maps

$$\text{colim}_{i \in I} K_n(RG_i) \to K_n(RG),$$
$$\text{colim}_{i \in I} KH_n(RG_i) \to KH_n(RG),$$

are bijective for all $n \in \mathbb{Z}$. Obviously, RG is the colimit of rings $\text{colim}_{i \in I} RG_i$. Now the claim follows for $K_n(RG)$ for $n \geqslant 0$ from [**53**, (12), p. 20]. Using the Bass–Heller–Swan decomposition, one gets the results for $K_n(RG)$ and also for the Nil-groups $N^p K_n(RG)$ defined by Bass [**10**, XII] for all $n \in \mathbb{Z}$ and $p \geqslant 1$. The claim for $KH_n(RG)$ follows from the spectral sequence [**67**, Theorem 1.3]. $\square$

We conclude from Theorem 2.7 and Lemma 3.3 that the (Fibered) Farrell–Jones Conjecture is inherited under directed colimits.

3.1. *Extensions*

LEMMA 3.4.   *Let* $1 \to K \to G \to Q \to 1$ *be an extension of groups. Suppose that* K *is virtually cyclic and* Q *satisfies the Fibered Farrell–Jones Conjecture with coefficients in* R.
   Then G *satisfies the Fibered Farrell–Jones Conjecture with coefficients in* R.

Proof.   Because of Lemma 2.9, it suffices to prove that $(G; \mathcal{VC}\mathrm{yc}(G))$ satisfies the Fibered Farrell–Jones Conjecture in the case that Q is virtually cyclic. Choose an infinite cyclic subgroup C of Q. Let $\phi \colon K \to K$ be the automorphism given by conjugation with an element in G which is mapped to a generator of C under the epimorphism $G \to Q$. Then $p^{-1}(C)$ is a subgroup of G which has finite index and is isomorphic to the semidirect product $K \rtimes_\phi C$. Since K is virtually cyclic, its automorphism group has finite order. Hence, by replacing C by a subgroup of the order of this automorphism group as index, we can arrange that $p^{-1}(C)$ is a subgroup of finite index in G and $p^{-1}(C) \cong K \times C$. Since K is virtually cyclic, we conclude that G contains a subgroup of finite index which is isomorphic to $\mathbb{Z}^2$. In particular, G is virtually abelian. Hence, $(G, \mathcal{VC}\mathrm{yc}(G))$ satisfies the Fibered Farrell–Jones Conjecture, by Theorem 3.2. $\square$

LEMMA 3.5.   *Let* G_1 *and* G_2 *be groups. Then* $G_1 \times G_2$ *satisfies the Fibered Farrell–Jones Conjecture with coefficients in* R *if and only if both* G_1 *and* G_2 *satisfy the Fibered Farrell–Jones Conjecture with coefficients in* R.

Proof.   Because of Lemma 2.11(ii), it suffices to show that $D_\infty \times D_\infty$ satisfies the Fibered Farrell–Jones Conjecture. This follows from Theorem 3.2. $\square$

Next we give the proof of Theorem 1.1.

Proof.   (i) This follows from Theorems 3.1 and 3.2.
 (ii) This follows from Lemma 3.5.
 (iii) This follows from Theorem 2.7(ii) and Lemma 3.3.
 (iv) This follows from Lemma 2.5. $\square$

A ring R is called *regular coherent* if every finitely presented R-module possesses a finite resolution by finitely generated projective R-modules. A ring R is *regular* if and only if it is regular coherent and Noetherian. A group G is called *regular* or *regular coherent*, respectively, if for any regular ring R, the group ring RG is regular or, respectively, regular coherent. Poly-$\mathbb{Z}$-groups and free groups are regular coherent (see [**64**, p. 247]. For more information about these notions, we refer to [**64**, Theorem 19.1].

THEOREM 3.6.   *Suppose that* Q *is torsionfree and that the Fibered Farrell–Jones Conjecture holds for* Q. *Suppose that* K *is a regular coherent group. Suppose that* R *is regular. Let* $1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1$ *be an extension of groups.*
   Then the assembly map

$$H_n(BG; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Proof. Let Γ be a torsionfree group. The relative assembly map $H_n(B\Gamma; \mathbf{K}_R) \xrightarrow{\cong} H_n^\Gamma(E_{\mathcal{VC}yc}(\Gamma); \mathbf{K}_R)$ is bijective (see [**45**, Proposition 2.6, p. 686]) since R is regular and Γ is torsionfree. Hence, the Farrell–Jones Conjecture for Γ and R boils down to the claim that the assembly map $H_n(B\Gamma; \mathbf{K}_R) \xrightarrow{\cong} K_n(R\Gamma)$ is bijective for all $n \in \mathbb{Z}$. This implies that Theorem 3.6 follows directly from Lemma 2.8 if we can show that the second assumption appearing in Lemma 2.8 is satisfied.

Let $\mathcal{CL}$ be the class of groups introduced in [**64**, Definion 19.2, p. 248] or [**5**, Definition 0.10]. Let $V \subseteq Q$ be virtually cyclic. Since K is regular coherent and Q is torsionfree, and hence V is isomorphic to $\mathbb{Z}$, $p^{-1}(V)$ belongs to the class $\mathcal{CL}$. Since by [**64**, Proposition 19.3, p. 249] $\mathcal{CL}$ is closed under taking subgroups, every element in $p^*\mathcal{VC}yc(Q)$ belongs to $\mathcal{CL}$. One of the main results in Waldhausen's article [**64**] is that for a regular ring R the K-theoretic assembly map $H_n(BG'; \mathbf{K}_R) \to K_n(RG')$ is an isomorphism for $G' \in \mathcal{CL}$. Hence, the second assumption appearing in Lemma 2.8 is satisfied. $\square$

3.2. *Passage from $\mathcal{F}in$ to $\mathcal{VC}yc$*

LEMMA 3.7. *Let G be a group. Let R be a regular ring such that the order of any finite subgroup of G is invertible in R.*

Then the relative assembly map

$$H_n^G(E_{\mathcal{F}in}(G); \mathbf{K}_R) \to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R)$$

(induced by the up to G-homotopy unique G-map $E_{\mathcal{F}in}(G) \to E_{\mathcal{VC}yc}(G)$) is an isomorphism for all $n \in \mathbb{Z}$.

Proof. This is proved, for instance, in [**45**, Proposition 2.6, p. 686]. $\square$

LEMMA 3.8. *Let G be a group. Then the relative assembly map induces, for all $n \in \mathbb{Z}$, isomorphisms*

$$H_n^G(E_{\mathcal{F}in}(G); \mathbf{K}_\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q} \to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q}.$$

Proof. This is proved in [**34**, Theorem 5.6]. $\square$

Notice that the two Lemmas 3.7 and 3.8 above do *not* deal with the Fibered version. A discussion of the Fibered version of Lemma 3.7 can be found in Subsection 7.3. The fibered version of Lemma 3.8 is definitely false. For homotopy K-theory, one can prove in the fibered situation that the passage from $\mathcal{F}in$ to $\mathcal{VC}yc$ does not matter. For $\Lambda = \mathbb{Z}$, the next lemma is proven in [**5**, Remark 7.4]. The following, more general, statement follows by the same argument.

LEMMA 3.9. *Let R be a ring. A group G satisfies the (Fibered) Isomorphism Conjecture for homotopy K-theory with coefficients in R for the family $\mathcal{F}in$ after applying $- \otimes_\mathbb{Z} \Lambda$ for $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ to the assembly map, if and only if G satisfies the (Fibered) Isomorphism Conjecture for homotopy K-theory with coefficients in R for the family $\mathcal{VC}yc$ after applying $- \otimes_\mathbb{Z} \Lambda$ for $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ to the assembly map.*

3.3. *Homotopy K-theory and rings with finite characteristic*

LEMMA 3.10. *Let R be a ring of finite characteristic N. Let $NK_n(R)$ be the Nil-group of Bass. Then we get, for $n \in \mathbb{Z}$,*

$$NK_n(R)[1/N] = 0.$$

Proof. The proof can be found in [**66**, Corollary 3.2]. We give a brief outline for the reader's convenience.

Put $\Lambda = \mathbb{Z}/N$. Then R is a Λ-algebra. Let $W(\Lambda)$ be the ring of big Witt vectors over Λ. The underlying additive group is the multiplicative group $1 + t\Lambda[[t]]$ of formal power series with leading term 1. We do not need the explicit multiplicative structure, but do need to know that the identity element is $1 - t$. Let $\mathrm{End}(\Lambda)$ be the Grothendieck group of endomorphisms $f\colon P \to P$ of finitely generated projective Λ-modules. We get an injective homomorphism $K_0(\Lambda) \to \mathrm{End}(\Lambda)$ by sending $[P]$ to $[0\colon P \to P]$. Its cokernel is denoted by $\mathrm{End}_0(\Lambda)$. The tensor product induces the structure of a commutative ring on $\mathrm{End}(\Lambda)$ for which $K_0(\Lambda)$ becomes an ideal. Hence, $\mathrm{End}_0(\Lambda)$ is a commutative ring. There is an $\mathrm{End}_0(\Lambda)$-module structure on $NK_n(R)$ for all $n \in \mathbb{Z}$. Almkvist [**1**] shows that the characteristic polynomial defines an injective ring homomorphism

$$\chi\colon \mathrm{End}_0(\Lambda) \to W(\Lambda), \quad [f\colon P \to P] \mapsto \det(\mathrm{id}_P - t \cdot f).$$

For a positive integer N, let I_N be the ideal in $\mathrm{End}_0(\Lambda)$

$$I_N = \{[f\colon P \to P] \mid \chi([f]) \equiv 1 \mod t^N\}.$$

Stienstra [**62**] (see also Theorem 1.3 and the following paragraph in [**66**]) has proven that for every element $x \in NK_n(R)$, there exists $N(x)$ such that the ideal $I_{N(x)}$ annihilates x. Now choose a sufficiently large positive integer k such that $\binom{N^k}{j}$ is a multiple of N for $1 \leqslant j < N(x)$. Then we conclude that

$$\chi(N^k \cdot [\mathrm{id}_\Lambda]) = (1 - t)^{N^k} = \sum_{j=0}^{N^k} \binom{N^k}{j}(-t)^j = 1 + \sum_{j=N(x)}^{N^k} \binom{N^k}{j}(-t)^j \equiv 1 \mod t^{N(x)}.$$

This shows that $N^k \cdot [\mathrm{id}_\Lambda] \in I_{N(x)}$. We compute

$$N^k \cdot x = N^k \cdot ([\mathrm{id}_\Lambda] \cdot x) = (N^k \cdot [\mathrm{id}_\Lambda]) \cdot x = 0.$$

This implies that $NK_n(R)[1/N] = 0$. $\qquad\square$

LEMMA 3.11. (i) *Let R be a ring of finite characteristic N. Then the canonical map from algebraic K-theory to homotopy K-theory induces an isomorphism*

$$K_n(R)[1/N] \xrightarrow{\cong} KH_n(R)[1/N]$$

for all $n \in \mathbb{Z}$.

(ii) *Let H be a finite group. Then the canonical map from algebraic K-theory to homotopy K-theory induces an isomorphism*

$$K_n(\mathbb{Z}[H]) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} KH_n(\mathbb{Z}[H]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for all $n \in \mathbb{Z}$.

Proof. (i) We conclude from Lemma 3.10 that $NK_n(R)[1/N] = 0$. This implies that $N^p K_n(R)[1/N] = 0$ for $n \in \mathbb{Z}$ and $p \geqslant 1$. Now apply the spectral sequence from [**67**, Theorem 1.3].

(ii) By [**66**, Corollary] and [**5**, Remark 8.3] $NK_n(\mathbb{Z}[H]) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Now proceed as in 3.11. $\qquad\square$

LEMMA 3.12. *Let (X, A) be a pair of G-CW-complexes.*

(i) *Let R be a ring of finite characteristic N. Then the natural map*

$$H_n^G(X, A; \mathbf{K}_R)[1/N] \to H_n^G(X, A; \mathbf{KH}_R)[1/N]$$

is bijective for every $n \in \mathbb{Z}$.

(ii) *Assume that X is a G-$\mathcal{F}$in-CW-complex; that is, the isotropy groups of X are finite. Then the natural map*

$$H_n^G(X, A; \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(X, A; \mathbf{KH}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is bijective for every $n \in \mathbb{Z}$.

Proof. This follows from Lemma 3.11 and a spectral sequence argument based on the equivariant Atiyah–Hirzebruch spectral sequence (see for instance, [**20**, Theorem 4.7]).   □

LEMMA 3.13.   *Let R be a ring, and let G be a group. Let $N \geqslant 2$ be an integer.*

(i) *If G satisfies the (Fibered) Farrell–Jones Conjecture for algebraic K-theory with coefficients in $R[x_1, x_2, \ldots, x_k]$ for all $k \geqslant 0$, then G satisfies the (Fibered) Farrell–Jones Conjecture for homotopy K-theory with coefficients in R.*

(ii) *Suppose that $N \cdot 1_R = 0$ and that G satisfies the (Fibered) Farrell–Jones Conjecture for homotopy K-theory with coefficients in R after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$. Then G satisfies the (Fibered) Farrell–Jones Conjecture for algebraic K-theory with coefficients in R after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ for both the family $\mathcal{F}$in and $\mathcal{VC}$yc.*

Proof. (i) This is proven in [**5**, Theorem 8.4].

(ii) Consider for any family $\mathcal{F}$ of subgroups of G, the following commutative diagram

$$\begin{array}{ccc} H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R)[1/N] & \longrightarrow & H_n(\mathrm{pt}; \mathbf{K}_R)[1/N] = K_n(RG))[1/N] \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ H_n^G(E_{\mathcal{F}}(G); \mathbf{KH}_R)[1/N] & \longrightarrow & H_n(\mathrm{pt}; \mathbf{KH}_R)[1/N] = KH_n(RG)[1/N] \end{array}$$

where the horizontal maps are the assembly maps induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$ and the vertical maps are induced by the passage from algebraic K-theory to homotopy K-theory. Lemma 3.12(i) implies that the vertical maps are bijective. Now apply Lemma 3.9.   □

Next we can give the proof of Theorem 1.3.

Proof. (i) Word-hyperbolic groups and virtually abelian groups satisfy the Farrell–Jones Conjecture for algebraic K-theory with coefficients in any ring R by Theorem 3.1 and Theorem 3.2. We conclude from Lemma 3.13(i) that word-hyperbolic groups and virtually abelian groups belong to $\mathcal{FJ}_{KH}(R)$ for all rings R. We conclude from Lemma 2.12 that all elementary amenable groups belong to $\mathcal{FJ}_{KH}(R)$ for all rings R. Lemma 3.13(ii) implies that all word-hyperbolic groups and all elementary amenable groups belong to $\mathcal{FJ}_{FC}(R)$.

In particular, every virtually cyclic group belongs to $\mathcal{FJ}_{FC}(R)$. We conclude from Theorem 2.4 that for a ring R of characteristic N, a group satisfies the Fibered Farrell–Jones Conjecture for algebraic K-theory for G with coefficients in R after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ to the assembly map for the family $\mathcal{F}$in if and only if it does for the family $\mathcal{VC}$yc. Hence in the sequel we only have to consider the family $\mathcal{F}$in when dealing with $\mathcal{FJ}_{FC}(R)$.

(ii) This follows from Lemma 2.11(i).

(iii) This follows from Theorem 2.7(ii) and Lemma 3.3.

(iv) This follows from Lemma 2.5.

(v) This follows from Lemma 2.9 since a group which is commensurable to a word-hyperbolic group is again word-hyperbolic, and the same is true for elementary amenable groups.

(vi) This is proven for $\mathcal{FJ}_{KH}$ in Bartels–Lück [**4**, Theorem 0.5]. The same proof applies to $\mathcal{FJ}_{FC}$ if we can show that $\mathcal{H}_*^?(-; \mathbf{K}_R)[1/N]$ has the tree property (see [**4**, Definition 4.1

and Theorem 4.2]). This follows from Lemma 3.12(i), since the equivariant homology theory $\mathcal{H}^?_*(-;\mathbf{KH}_R)$ has the tree property by [**4**, Theorem 11.1].

The claim $\mathcal{FJ}_{KH}(R) \subseteq \mathcal{FJ}_{FC}(R)$ follows from Lemma 3.13(ii). $\qquad\square$

3.4. *Coefficient rings with operation*

In the setup developed so far, we have not dealt with the more general version developed in Bartels–Reich [**8**], where one fixes a group and considers an additive category with G-action. This setup can deal with crossed products $R \rtimes G$ and not only with group rings RG. However, a slight modification of the proofs above allows us to carry over the result above to this setting. This is explained in Bartels–Echterhoff–Lück [**3**].

4. *The projective class group and induction from finite subgroups*

Let $\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)$ be the category whose objects are homogeneous spaces G/H with finite H and whose morphisms are G-maps. We obtain a functor from $\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)$ to abelian groups by sending G/H to $K_0(RH)$. It sends a morphism $G/H \to G/K$, $gH \mapsto gg_0 K$ to the map $K_0(RH) \to K_0(RK)$ coming from the group homomorphism $H \to K$, $h \mapsto g_0 h g_0^{-1}$. This is well defined since inner automorphisms of H induce the identity on $K_0(RH)$. The various inclusions of finite subgroups of G yield a homomorphism

$$I(G;R) : \operatorname*{colim}_{\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(RH) \to K_0(RG). \qquad (4.1)$$

Notice for the sequel that the canonical map of Λ-modules

$$\left(\operatorname*{colim}_{\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(RH) \right) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\cong} \operatorname*{colim}_{\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)} (K_0(RH) \otimes_{\mathbb{Z}} \Lambda)$$

is bijective for every ring Λ.

The next lemma is proven in [**32**, Lemma 2.9] for fields, and carries over directly to skew-fields.

LEMMA 4.1. *Let D be a skew-field (of arbitrary characteristic), and let H be a finite group. Then $K_n(DH) = 0$ for $n \leqslant -1$.*

Now we can give the proof of Theorem 1.5.

Proof. (i) This is proved in [**45**, p. 691]:
(ii) This follows analogously to the proof of assertion (i) using Lemma 3.10 and Lemma 4.1. $\qquad\square$

If G satisfies the Farrell–Jones Conjecture for algebraic K-theory with coefficients in $\mathbb{Z}$, then the following maps are injective (see [**45**, p. 692]):

$$\operatorname*{colim}_{\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)} K_n(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{for } n \in \mathbb{Z};$$

$$\operatorname*{colim}_{\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)} \mathrm{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The injectivity for the map involving the Whitehead group is proven for groups G satisfying mild homological finiteness conditions in a forthcoming paper by Lück, Reich, Rognes and Varisco. In general, these maps are not surjective. In particular,

$$\operatorname*{colim}_{\mathsf{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is in general not surjective.

5. *Bass Conjectures*

In this section, we explain the relationship between the Farrell–Jones Conjecture for algebraic K-theory and the Bass Conjecture.

Let G be a group. Let $\mathrm{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$. Denote by $\mathrm{con}(G)_f$ the subset of $\mathrm{con}(G)$ consisting of those conjugacy classes (g) for which each representative g has finite order. Let R be a commutative ring. Let $\mathrm{class}_R(G)$ and $\mathrm{class}_R(G)_f$ be the free R-module with the set $\mathrm{con}(G)$ and $\mathrm{con}(G)_f$ as basis. This is the same as the R-module of R-valued functions on $\mathrm{con}(G)$ and $\mathrm{con}(G)_f$ with finite support. Define the *universal R-trace*

$$\mathrm{tr}^u_{RG} \colon RG \to \mathrm{class}_R(G), \quad \sum_{g \in G} r_g \cdot g \mapsto \sum_{g \in G} r_g \cdot (g). \tag{5.1}$$

It extends to a function $\mathrm{tr}^u_{RG} \colon M_n(RG) \to \mathrm{class}_R(G)$ on (n,n)-matrices over RG by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective RG-module. Choose a matrix $A \in M_n(RG)$ such that $A^2 = A$ and the image of the RG-map $r_A \colon RG^n \to RG^n$ given by right multiplication with A is RG-isomorphic to P. Define the *Hattori–Stallings rank* of P as

$$\mathrm{HS}_{RG}(P) = \mathrm{tr}^u_{RG}(A) \in \mathrm{class}_R(G). \tag{5.2}$$

The Hattori–Stallings rank depends only on the isomorphism class of the RG-module P. It induces an R-homomorphism, the *Hattori–Stallings homomorphism*,

$$\mathrm{HS}_{RG} \colon K_0(RG) \otimes_{\mathbb{Z}} R \to \mathrm{class}_R(G), \quad [P] \otimes r \mapsto r \cdot \mathrm{HS}_{RG}(P). \tag{5.3}$$

Let F be a field of characteristic zero. Fix an integer $m \geqslant 1$. Let $F(\zeta_m) \supset F$ be the Galois extension given by adjoining the primitive mth root of unity ζ_m to F. Denote by $\Gamma(m, F)$ the Galois group of this extension of fields, that is, the group of automorphisms $\sigma \colon F(\zeta_m) \to F(\zeta_m)$ which induce the identity on F. It can be identified with a subgroup of $\mathbb{Z}/m^*$ by sending σ to the unique element $u(\sigma) \in \mathbb{Z}/m^*$ for which $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$ holds. Let g_1 and g_2 be two elements of G of finite order. We call them *F-conjugate* if for some (and hence all) positive integers m with $g_1^m = g_2^m = 1$, there exists an element σ in the Galois group $\Gamma(m, F)$ with the property that $g_1^{u(\sigma)}$ and g_2 are conjugate. Two elements g_1 and g_2 are F-conjugate for F equal to $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ respectively, if the cyclic subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are conjugate, if g_1 and g_2 or g_1 and g_2^{-1} are conjugate, or if g_1 and g_2 are conjugate, respectively.

Denote by $\mathrm{con}_F(G)_f$ the set of F-conjugacy classes $(g)_F$ of elements $g \in G$ of finite order. Let $\mathrm{class}_F(G)_f$ be the F-vector space with the set $\mathrm{con}_F(G)_f$ as basis, or, equivalently, the F-vector space of functions $\mathrm{con}_F(G)_f \to F$ with finite support. There are obvious inclusions of F-modules $\mathrm{class}_F(G)_f \subseteq \mathrm{class}_F(G)$.

LEMMA 5.1. *Suppose that F is a field of characteristic zero and H is a finite group. Then the Hattori–Stallings homomorphism (see equation (5.3)) induces an isomorphism*

$$\mathrm{HS}_{FH} \colon K_0(FH) \otimes_{\mathbb{Z}} F \xrightarrow{\cong} \mathrm{class}_F(H) = \mathrm{class}_F(H)_f.$$

Proof. Since H is finite, an FH-module is a finitely generated projective FH-module if and only if it is a (finite-dimensional) H-representation with coefficients in F and $K_0(FH)$ is the same as the representation ring $R_F(H)$. The Hattori–Stallings rank $\mathrm{HS}_{FH}(V)$ and the character χ_V of a G-representation V with coefficients in F are related by the formula

$$\chi_V(h) = |Z_G\langle h\rangle| \cdot \mathrm{HS}_{FH}(V)(h)$$

for $h \in H$, where $Z_G \langle h \rangle$ is the centralizer of h in G. Hence, Lemma 5.1 follows from representation theory; see for instance [**60**, Corollary 1, p. 96]. $\square$

Notice that the Bass Conjecture for fields of characteristic zero as coefficients (Conjecture 1.8) is the obvious generalization of Lemma 5.1 to infinite groups.

LEMMA 5.2. *Suppose that F is a field of characteristic zero and G is a group. Then the composite*

$$\operatorname*{colim}_{G/H \in \operatorname{Or}_{\mathcal{F}in}(G)} K_0(FH) \otimes_{\mathbb{Z}} F \xrightarrow{I(G,F) \otimes_{\mathbb{Z}} \operatorname{id}_F} K_0(FG) \otimes_{\mathbb{Z}} F \xrightarrow{\operatorname{HS}_{FG}} \operatorname{class}_F(G)$$

is injective and has as image $\operatorname{class}_F(G)_f$.

Proof. This follows from the following commutative diagram; compare [**39**, Lemma 2.15, p. 220].

$$
\begin{array}{ccc}
\operatorname*{colim}_{H \in \operatorname{Or}_{\mathcal{F}in}(G)} K_0(FH) \otimes_{\mathbb{Z}} F & \xrightarrow{\quad I(G,F) \otimes_{\mathbb{Z}} \operatorname{id}_F \quad} & K_0(FG) \otimes_{\mathbb{Z}} F \\
{\scriptstyle \operatorname*{colim}_{H \in \operatorname{Or}_{\mathcal{F}in}(G)} \operatorname{HS}_{FH}} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \operatorname{HS}_{FG}} \\
\operatorname*{colim}_{H \in \operatorname{Or}_{\mathcal{F}in}(G)} \operatorname{class}_F(H)_f & \xrightarrow[\cong]{\quad j \quad} \operatorname{class}_F(G)_f \xrightarrow{\quad i \quad} & \operatorname{class}_F(G)
\end{array}
$$

Here the isomorphism j is the colimit over the obvious maps $\operatorname{class}_F(H)_f \to \operatorname{class}_F(G)_f$ given by extending a class function in the trivial way, and the map i is the natural inclusion and in particular injective. $\square$

LEMMA 5.3. *Let F be a field (of arbitrary characteristic). Then there is a natural commutative diagram*

$$
\begin{array}{ccc}
\operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(FH) & \xrightarrow[\cong]{\operatorname{edge}} & H_0(E_{\mathcal{F}in}(G); K_F) \\
{\scriptstyle I(G,F)} \downarrow & & \downarrow \\
K_0(FG) & \xrightarrow{\quad \cong \quad} & H_0(\operatorname{pt}; K_F)
\end{array}
$$

whose horizontal maps are bijective and whose right vertical arrow is the assembly map, that is, the map induced by the projection $E_{\mathcal{F}in}(G) \to \operatorname{pt}$.

Proof. This follows from Lemma 4.1, the equivariant Atiyah–Hirzebruch spectral sequence, which converges to $H_{p+q}(E_{\mathcal{F}in}(G); K_F)$ in the strong sense and whose E_2-term $E^2_{p,q}$ is given by the Bredon homology

$$H_p^{\operatorname{Or}_{\mathcal{F}in}(G)}(E_{\mathcal{F}in}(G), K_q(F?)),$$

and the natural identification

$$\operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(FH) \cong H_0^{\operatorname{Or}_{\mathcal{F}in}(G)}(E_{\mathcal{F}in}(G), K_0(F?)). \qquad \square$$

Theorem 1.9 follows now from Lemma 3.7, Lemma 5.2, and Lemma 5.3.

LEMMA 5.4. *Let R be a commutative integral domain. Let F be its field of fractions. Let G be a group.*

(i) *Then we obtain a commutative diagram*

$$
\begin{array}{ccc}
\operatorname{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(RH) & \longrightarrow & \operatorname{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(FH) \\
\big\downarrow {\scriptstyle \mathrm{edge}(G,R)} & & \big\downarrow {\scriptstyle \mathrm{edge}(G,F)} \\
H_0(E_{\mathcal{F}in}(G); K_R) & \longrightarrow & H_0(E_{\mathcal{F}in}(G); K_F) \\
\big\downarrow & & \big\downarrow \\
H_0(E_{\mathcal{VC}yc}(G); K_R) & \longrightarrow & H_0(E_{\mathcal{VC}yc}(G); K_F) \\
\big\downarrow & & \big\downarrow \\
K_0(RG) & \longrightarrow & K_0(FG)
\end{array}
$$

where all horizontal maps are change of rings homomorphisms for the inclusion $R \to F$, the maps $\mathrm{edge}(G,R)$ and $\mathrm{edge}(G,F)$ are edge homomorphisms appearing in the equivariant Atiyah–Hirzebruch spectral sequence and the other vertical maps come from the obvious maps $E_{\mathcal{F}in}(G) \to E_{\mathcal{VC}yc}(G)$ and $E_{\mathcal{VC}yc}(G) \to \mathrm{pt}$.

(ii) *The image of the composite*

$$
\alpha : H_0(E_{\mathcal{VC}yc}(G); K_R) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_0(E_{\mathcal{VC}yc}(G); K_F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

is the same as the image of the composite

$$
\beta : \bigoplus_{C \in \mathcal{FC}yc(G)} K_0(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \to \bigoplus_{C \in \mathcal{FC}yc(G)} K_0(FC) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q},
$$

where $\mathcal{FC}yc$ is the family all finite cyclic subgroups of G.

Proof. (i) This follows from the naturality of the constructions.

(ii) For every group G, every ring R, and every $n \in \mathbb{Z}$, the relative assembly map

$$
H_n^G(E_{\mathcal{F}in}(G); \mathbf{K}_R) \to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R)
$$

is split-injective [**9**]. This map and the splitting are natural with respect to change of rings homomorphisms. Hence, Lemma 3.7 implies that the image of

$$
H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_F) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

and the image of the composite

$$
H_n^G(E_{\mathcal{F}in}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{F}in}(G); \mathbf{K}_F) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_F) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

agree. By [**46**, Theorem 2.13], a variation of the Chern character in [**40**], we have, for every ring R and group G, an isomorphism, natural in R,

$$
\mathrm{chern}_n^{G,R} : \bigoplus_{p+q=n} \bigoplus_{\substack{(C) \\ C \in \mathcal{FC}yc(G)}} H_p(BZ_GC; \mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot (K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q})
$$

$$
\to H_n^G(E_{\mathcal{F}in}(G); K_R) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad (5.4)
$$

where W_GC is the quotient of the normalizer by the centralizer Z_GC of the finite cyclic subgroup C in G, and θ_C is an idempotent in $A(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $A(C)$ the Burnside ring of C which singles out a direct summand $\theta_C \cdot (K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q})$ in $K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the $A(C) \otimes_{\mathbb{Z}} \mathbb{Q}$-module structure on $K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$. This isomorphism is natural under change of rings homomorphisms. Recall from Lemma 4.1 that $K_q(FC)$ is trivial for $q \leqslant -1$. Hence the image of

$$
H_0^G(E_{\mathcal{F}in}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_0^G(E_{\mathcal{F}in}(G); \mathbf{K}_F) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

agrees with the image of the composition

$$
\bigoplus_{(C)\in\mathcal{FC}\mathrm{yc}(G)} H_0(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot (K_0(RC)\otimes_\mathbb{Z}\mathbb{Q})
$$

$$
\rightarrow \bigoplus_{(C)\in\mathcal{FC}\mathrm{yc}(G)} H_0(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot (K_0(FC)\otimes_\mathbb{Z}\mathbb{Q})
$$

$$
= \bigoplus_{p+q=0} \bigoplus_{(C)\in\mathcal{FC}\mathrm{yc}(G)} H_p(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot (K_q(FC)\otimes_\mathbb{Z}\mathbb{Q})
$$

$$
\xrightarrow{\mathrm{chern}_0^{G,F}} H_0^G(E_{\mathcal{F}\mathrm{in}}(G);K_F) \otimes_\mathbb{Z} \mathbb{Q}.
$$

Since $H_0(BZ_GC;\mathbb{Q})$ is $\mathbb{Q}$ with the trivial W_GC-action, $H_0(BZ_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C \cdot (K_0(RC)\otimes_\mathbb{Z}\mathbb{Q})$ is a quotient of $\theta_C\cdot(K_0(RC)\otimes_\mathbb{Z}\mathbb{Q})$, which is a direct summand in $K_0(RC)\otimes_\mathbb{Z}\mathbb{Q}$. Hence, $\mathrm{im}(\alpha)\subseteq\mathrm{im}(\beta)$. We get $\mathrm{im}(\beta)\subseteq\mathrm{im}(\alpha)$ from the commutative diagram appearing in assertion (i). This finishes the proof of Lemma 5.4 □

Finally, we can give the proof of Theorem 1.10.

Proof. For any finite group of $H\subseteq G$ such that its order is not invertible in R, the map $\widetilde{K}_0(RH)\to\widetilde{K}_0(FH)$ is trivial by a result of Swan [**63**, Theorem 8.1] (see also [**11**, Corollary 4.2]). Lemma 5.4(ii) implies that the composite

$$
H_0(E_{\mathcal{VC}\mathrm{yc}}(G);K_R)\otimes_\mathbb{Z}\mathbb{Q} \to H_0(E_{\mathcal{VC}\mathrm{yc}}(G);K_F)\otimes_\mathbb{Z}\mathbb{Q} \to K_0(FG)\otimes_\mathbb{Z}\mathbb{Q} \to \widetilde{K}_0(FG)\otimes_\mathbb{Z}\mathbb{Q}
$$

is trivial. Since G satisfies the Farrell–Jones Conjecture for algebraic K-theory with coefficients in R by assumption, the map $H_0(E_{\mathcal{VC}\mathrm{yc}}(G);K_R)\otimes_\mathbb{Z}\mathbb{Q}\to K_0(RG)\otimes_\mathbb{Z}\mathbb{Q}$ is surjective. Because of the commutative diagram appearing in Lemma 5.4(i), the map

$$
K_0(RG)\otimes_\mathbb{Z}\mathbb{Q} \to K_0(FG)\otimes_\mathbb{Z}\mathbb{Q} \to \widetilde{K}_0(FG)\otimes_\mathbb{Z}\mathbb{Q}
$$

is trivial. Now Theorem 1.10 follows. □

6. *The Kaplansky Conjecture*

We need the following definition.

DEFINITION 6.1. An R-module M is called *directly finite* if every R-module N satisfying $M\cong_R M\oplus N$ is trivial. A ring R is called *directly finite* (or *von Neumann finite*) if it is directly finite as a module over itself; that is, if $r,s\in R$ satisfy $rs=1$, then $sr=1$. A ring is called *stably finite* if the matrix algebra $M(n,n,R)$ is directly finite for all $n\geqslant 1$.

REMARK 6.2. Stable finiteness for a ring R is equivalent to the following statement. Every finitely generated projective R-module P whose class in $K_0(R)$ is zero is already the trivial module; that is, $0=[P]\in K_0(R)$ implies $P\cong 0$.

THEOREM 6.3. (i) *If F is a field of characteristic zero, then FG is stably finite for every group G.*
(ii) *If R is a skew-field and G is a sofic group, then RG is stably finite.*

Proof. (i) This is proved by Kaplansky [**36**] (see also Passman [**51**, Corollary 1.9, p. 38]).
(ii) This is proved for free-by-amenable groups by Ara–Meara–Perera [**2**] and extended to sofic groups by Elek–Szabo [**25**, Corollary 4.7]. □

LEMMA 6.4. *Let R be a ring whose idempotents are all trivial. Let G be a group such that the map induced by the inclusion $i\colon R \to RG$*

$$i_*\colon K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is bijective and RG is stably finite.
Then all idempotents in RG are trivial.

Proof.   Let p be an idempotent in RG. We want to show that $p \in \{0, 1\}$. Denote by $\epsilon\colon RG \to R$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Since $\epsilon(p) \in R$ is 0 or 1 by assumption, it suffices to show that $p = 0$ under the assumption that $\epsilon(p) = 0$. Let $(p) \subseteq RG$ be the ideal generated by p which is a finitely generated projective RG-module. Since $i_*\colon K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective by assumption, we can find a finitely generated projective R-module P and integers $k, m, n \geqslant 0$ satisfying

$$(p)^k \oplus RG^m \cong_{RG} i_*(P) \oplus RG^n.$$

If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \mathrm{id}$, $i_* \circ \epsilon_*(RG^l) \cong RG^l$ and $\epsilon(p) = 0$, we obtain

$$RG^m \cong i_*(P) \oplus RG^n.$$

Inserting this in the first equation yields

$$(p)^k \oplus i_*(P) \oplus RG^n \cong i_*(P) \oplus RG^n$$

and therefore, $0 = [(p)^k] \in K_0(RG)$. Using Remark 6.2, we conclude that $(p)^k = 0$ and hence that $p = 0$. □

Now Theorem 1.12 follows from Theorem 6.3 and Lemma 6.4.

7. *Nil-groups*

In this section, we give a brief discussion about Waldhausen's Nil-groups.

7.1. *Applications of homotopy K-theory to Waldhausen's Nil-groups*

The Nil-groups due to Bass $NK_n(R)$ have been generalized to *Waldhausen's Nil-groups* as follows. These groups were defined by Waldhausen (see [**64**, **65**]) for $n \geqslant 0$, and their extension to $n \in \mathbb{Z}$ is explained in [**4**, Section 9]. For more details, we refer to these papers. There are two kinds of Nil-groups, the first one taking care of amalgamated products, and the second of generalized Laurent extensions.

We begin with a discussion of the one for amalgamated products. Let $\alpha\colon C \to A$ and $\beta\colon C \to B$ be pure and free, and write $A = \alpha(C) \oplus A'$ and $B = \beta(C) \oplus B'$ and put $R = A *_C B$. Then the Nil-groups of the first kind are denoted by $\mathrm{Nil}_n(C; A', B')$. The group $\mathrm{Nil}_n(C; A', B')$ is a direct summand in $K_{n+1}(R)$ and there is a long exact sequence

$$\cdots \to K_{n+1}(RC) \to K_{n+1}(A) \oplus K_{n+1}(B) \to K_{n+1}(R)/\mathrm{Nil}_n(C; A', B')$$
$$\to K_n(C) \to K_n(A) \oplus K_n(B) \to \cdots. \qquad (7.1)$$

There is an analogous long exact sequence for homotopy K-theory which does not involve Nil-terms (see [**4**, Section 9]):

$$\cdots \to KH_{n+1}(RC) \to KH_{n+1}(A) \oplus KH_{n+1}(B)$$
$$\to KH_{n+1}(R) \to KH_n(C) \to KH_n(A) \oplus KH_n(B) \to \cdots. \qquad (7.2)$$

The natural transformation from algebraic K-theory to homotopy K-theory induces a map between the long exact sequences given in equations (7.1) and (7.2). Now suppose that for

$\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ the maps

$$K_n(A) \otimes_{\mathbb{Z}} \Lambda \to KH_n(A) \otimes_{\mathbb{Z}} \Lambda,$$
$$K_n(B) \otimes_{\mathbb{Z}} \Lambda \to KH_n(B) \otimes_{\mathbb{Z}} \Lambda,$$
$$K_n(C) \otimes_{\mathbb{Z}} \Lambda \to KH_n(C) \otimes_{\mathbb{Z}} \Lambda,$$
$$K_n(R) \otimes_{\mathbb{Z}} \Lambda \to KH_n(R) \otimes_{\mathbb{Z}} \Lambda,$$

are bijective for all $n \in \mathbb{Z}$. Then a Five-Lemma argument implies that

$$\mathrm{Nil}(A; B', C') \otimes_{\mathbb{Z}} \Lambda = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Recall from the spectral sequence of [**67**, Theorem 1.3] that $K_n(A) \otimes_{\mathbb{Z}} \Lambda \to KH_n(A) \otimes_{\mathbb{Z}} \Lambda$ is bijective if we have $NK_n(A[x_1, x_2, \dots, x_k]) \otimes_{\mathbb{Z}} \Lambda = 0$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$, $k \geqslant 0$, and analogously for B, C, and R instead of A.

Next we deal with generalized Laurent extensions. Let R be the generalized Laurent extension of pure and free ring maps $\alpha, \beta \colon C \to A$. Associated to it is the Nil-term $\mathrm{Nil}_n(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta)$ which is a direct summand in $K_{n+1}(R)$. We obtain long exact sequences

$$\cdots \to K_{n+1}(C) \xrightarrow{\alpha_* - \beta_*} K_{n+1}(A) \to K_{n+1}(R)/\mathrm{Nil}_n(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta)$$
$$\to K_n(C) \xrightarrow{\alpha_* - \beta_*} K_n(A) \to \cdots \qquad (7.3)$$

and

$$\cdots \to KH_{n+1}(C) \xrightarrow{\alpha_* - \beta_*} KH_{n+1}(A) \to KH_{n+1}(R)$$
$$\to KH_n(C) \xrightarrow{\alpha_* - \beta_*} KH_n(A) \to \cdots . \qquad (7.4)$$

If, for $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$, the maps

$$K_n(A) \otimes_{\mathbb{Z}} \Lambda \to KH_n(A) \otimes_{\mathbb{Z}} \Lambda,$$
$$K_n(C) \otimes_{\mathbb{Z}} \Lambda \to KH_n(C) \otimes_{\mathbb{Z}} \Lambda,$$
$$K_n(R) \otimes_{\mathbb{Z}} \Lambda \to KH_n(R) \otimes_{\mathbb{Z}} \Lambda,$$

are bijective for all $n \in \mathbb{Z}$, then we conclude as above that

$$\mathrm{Nil}_n(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta) \otimes_{\mathbb{Z}} \Lambda = 0 \quad \text{for all } n \in \mathbb{Z}$$

holds.

We can now prove Theorem 1.15.

Proof. (i) Because G, H, and K are finite, we conclude from Lemma 3.11(ii) that the maps

$$K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \to KH_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$K_n(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} \to KH_n(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$K_n(\mathbb{Z}K) \otimes_{\mathbb{Z}} \mathbb{Q} \to KH_n(\mathbb{Z}K) \otimes_{\mathbb{Z}} \mathbb{Q},$$

are bijective. Let $\Gamma := G *_K H$ be the amalgamated product. Then Γ acts properly and cocompactly on a tree and is therefore, word-hyperbolic. Theorems 1.1(i) and 1.3(i) imply $\Gamma \in \mathcal{FJ}(\mathbb{Z})$ and $\Gamma \in \mathcal{FJ}_{KH}(\mathbb{Z})$. Consider the following commutative diagram

$$\begin{array}{ccc}
H_n^\Gamma(E_{\mathcal{F}\mathrm{in}}(\Gamma); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & H_n(\mathrm{pt}; \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} = K_n(R\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\downarrow & & \downarrow \\
H_n^\Gamma(E_{\mathcal{F}\mathrm{in}}(\Gamma); \mathbf{KH}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & H_n(\mathrm{pt}; \mathbf{KH}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} = KH_n(R\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{array}$$

where the horizontal maps are the assembly maps induced by the projection $E_{\mathcal{F}\text{in}}(\Gamma) \to \text{pt}$ and the vertical maps are induced by the passage from algebraic K-theory to homotopy K-theory. The lower horizontal map is an isomorphism since $\Gamma \in \mathcal{FJ}_{KH}(\mathbb{Z})$, and the upper horizontal map is an isomorphism by Lemma 3.8 since $\Gamma \in \mathcal{FJ}(\mathbb{Z})$. The left vertical map is an isomorphism by Lemma 3.12(ii). Therefore, $K_n(\mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \cong KH_n(\mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Five-Lemma argument from above now implies the rational vanishing of the Nil-groups, as claimed.

(ii) Here we can argue as in (i), but now using the HNN-extension associated to the inclusions α and β instead of the amalgamated product. □

7.2. Nil-groups and rings with finite characteristic

For this subsection, fix an integer $N \geqslant 2$ and a ring A of characteristic N. In Lemma 3.10, we have shown that we get $NK_n(A)[1/N] = 0$ for every $n \in \mathbb{Z}$. This implies, by the above Five-Lemma argument for all $n \in \mathbb{Z}$, and all pure and free maps $\alpha\colon C \to A$ and $\beta\colon C \to B$, and all pure and free maps $\alpha, \beta\colon C \to A$, respectively

$$\text{Nil}_n(C; A', B') \otimes_{\mathbb{Z}} [1/N] = 0;$$
$$\text{Nil}_n(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta) \otimes_{\mathbb{Z}} [1/N] = 0.$$

7.3. Nil-groups and regular rings R with $\mathbb{Q} \subset R$

CONJECTURE 7.1. Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then we get, for all groups G and all $n \in \mathbb{Z}$,

$$NK_n(RG) = 0,$$

and the canonical map

$$K_n(RG) \to KH_n(RG)$$

is bijective.

By the following discussion, Conjecture 7.1 is true if G belongs to the class $\mathcal{FJ}(R)$ appearing in Theorem 1.1, but we do not know it for all groups G.

If R is regular, then the polynomial ring $R[t]$ is regular as well. Thus, if $NK_n(RG) = 0$ holds for a fixed group G and all regular rings R with $\mathbb{Q} \subseteq R$, then $K_n(RG) \to KH_n(RG)$ is bijective for this fixed group G and all regular rings R with $\mathbb{Q} \subseteq R$. This follows from the spectral sequence appearing in [67, Theorem 1.3]. Using the Bass–Heller–Swan decomposition, it is not hard to see that, similarly, if $NK_n(RG) = 0$ holds for all groups G and a fixed regular ring R with $\mathbb{Q} \subseteq R$, then $K_n(RG) \to KH_n(RG)$ is bijective for all groups G and this fixed regular ring R with $\mathbb{Q} \subseteq R$.

If Conjecture 7.1 is true, then also Waldhausen's Nil-groups associated to a free amalgamated product of groups $G = G_1 *_{G_0} G_1$ or to an HNN-extension with coefficients in a regular ring R with $\mathbb{Q} \subseteq R$ vanish by the general argument from Section 7.1.

Let $\mathcal{FJ}_{Fin}(R)$ be the class of those groups for which the Fibered Farrell–Jones Conjecture for algebraic K-theory with coefficients in R is true for the family $\mathcal{F}$in. We get $\mathcal{FJ}_{Fin}(R) \subseteq \mathcal{FJ}(R)$ from Lemma 2.2 for all rings R but $\mathcal{FJ}_{Fin}(\mathbb{Z}) \neq \mathcal{FJ}(\mathbb{Z})$.

If Conjecture 7.1 holds and R is a regular ring with $\mathbb{Q} \subseteq R$, then the map $H_n^G(X; \mathbf{K}_R) \to H_n^G(X; \mathbf{KH}_R)$ induced by the transition from K-theory to homotopy K-theory is an isomorphism for any G-CW-complex X and the assembly maps for K-theory and homotopy K-theory can be identified. Thus, Conjecture 7.1 implies that $\mathcal{FJ}_{Fin}(R) = \mathcal{FJ}_{KH}(R)$. In particular, if Conjecture 7.1 holds, then by Theorem 1.3 (vi), every virtually cyclic group satisfies the Fibered Farrell–Jones Conjecture for algebraic K-theory with coefficients in R for the family $\mathcal{F}$in. We omit the proof (based on the Bass–Heller–Swan formula) that Conjecture 7.1 is in fact, equivalent to the statement that $\mathbb{Z} \in \mathcal{FJ}_{Fin}(R)$ for any regular ring R with $\mathbb{Q} \subseteq R$.

If Conjecture 7.1 is true and R is a regular ring with $\mathbb{Q} \subseteq R$, then the Transitivity Principle (see Theorem 2.4) implies that $\mathcal{FJ}(R) = \mathcal{FJ}_{Fin}(R)$, the conclusions appearing in Theorem 1.3 are true for each group in $\mathcal{FJ}(R) = \mathcal{FJ}_{Fin}(R)$, and assertion (i) of Theorem 1.5 is true for all groups in $\mathcal{FJ}_{Fin}(R)$, and in particular for all elementary amenable groups.

Let R be any ring. Recall that $\mathcal{FJ}(R)$ contains, for instance, word-hyperbolic groups and virtually nilpotent groups by Theorem 1.1, but we do not know whether elementary amenable groups belong to $\mathcal{FJ}(R)$ and whether they satisfy assertion (i) of Theorem 1.5. Our methods do not give this conclusion since we know only that $G \in \mathcal{FJ}(R)$ implies $G \times \mathbb{Z} \in \mathcal{FJ}(R)$, but we would at least need $G \rtimes_\alpha \mathbb{Z} \in \mathcal{FJ}(R)$ for any automorphism $\alpha \colon G \to G$. More generally, it would be interesting to solve the question of whether, for any extensions $1 \to G \to \Gamma \to V \to 1$ for virtually cyclic V and $G \in \mathcal{FJ}(R)$, we have $\Gamma \in \mathcal{FJ}(R)$ because a positive answer would imply that $\mathcal{FJ}(R)$ is closed under extensions.

8. The Farrell–Jones Conjecture for L-theory

In this section, 'ring R' will always mean an associative ring with unit and involution.

Recall that the (*Fibered*) *Farrell–Jones Conjecture for algebraic L-theory with coefficients in R for the group G* is the (Fibered) Isomorphism Conjecture 2.1 in the special case, where the family $\mathcal{F}$ consists of all virtually cyclic subgroups of G, and $\mathcal{H}_*^?$ is the equivariant homology theory $\mathcal{H}_*^?(-; \mathbf{L}_R^{\langle -\infty \rangle})$ associated to the Groupoids-spectrum given by algebraic L-theory and R as coefficient ring (see [**45**, Section 6]). So the Farrell–Jones Conjecture for algebraic L-theory with coefficients in R for the group G predicts that the map

$$H_n^G \left(E_{\mathcal{VC}yc}(G), \mathbf{L}_R^{\langle -\infty \rangle} \right) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$. The original source for (Fibered) Farrell–Jones Conjecture is [**28**]. The corresponding conjecture is false if one replaces the decoration $\langle -\infty \rangle$ with the decoration p, h or s (see [**29**]). For the status of the Farrell–Jones Conjecture with coefficients in $\mathbb{Z}$, we refer for instance to [**45**, Sections 5.2 and 5.3].

The next result is proved in [**3**, Lemma 5.2].

LEMMA 8.1. *Let R be a ring. Then the equivariant homology theory $\mathcal{H}_*^?(-; \mathbf{L}_R^{\langle -\infty \rangle})$ is strongly continuous.*

Theorem 2.7 2.7 and Lemma 8.2 imply that for any ring R and any direct system of groups $\{G_i \mid i \in I\}$ (with not necessarily injective structure maps), $G = \operatorname{colim}_{i \in I} G_i$ satisfies the Fibered Farrell–Jones Conjecture for algebraic L-theory with coefficients in R if each group G_i does.

In algebraic L-theory, the situation simplifies if one inverts 2.

LEMMA 8.2. *Let R be a ring, and let G be a group. Then the Fibered Farrell–Jones Conjecture with coefficients in R after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ to the assembly map holds for $(G, \mathcal{VC}yc)$, if and only if the same holds for $(G, \mathcal{F}in)$.*

Proof. Because of Lemma 2.10 and the fact that every virtually cyclic group maps surjectively onto $\mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$, it suffices to show the claim for $G = \mathbb{Z}$ and $G = \mathbb{Z}/2 * \mathbb{Z}/2$. These cases follow from the exact sequence involving UNil-terms and the proof that UNil-groups are 2-torsion, due to Cappell [**18**] in the case $\mathbb{Z}/2 * \mathbb{Z}/2$. In the case $\mathbb{Z}$, there are no UNil-terms for infinite virtually cyclic groups of the first kind. This follows essentially from [**55**] and [**56**] as carried out in [**43**, Lemma 4.2]. $\square$

The Transitivity Principle (see Theorem 2.4) implies that for a ring R and a group G, the Farrell–Jones Conjecture for algebraic L-theory with coefficients in R is true after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ to the assembly map for the family $\mathcal{VC}\mathrm{yc}$ if and only if this is true for the family $\mathcal{F}\mathrm{in}$. This implies that assertions (ii), (iii), (iv), and (vi) appearing in Theorem 1.3 are true for the family $\mathcal{FJ}_{L[1/2]}(R)$ of those groups for which the Farrell–Jones Conjecture for algebraic L-theory with coefficients in R is true after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$. We do not know whether all word-hyperbolic groups belong to $\mathcal{FJ}_{L[1/2]}(R)$. Farrell and Jones claim in [**28**, Remark 2.1.3] without giving details that the L-theory version of their celebrated [**28**, Theorem 1.2] is true. This, together with [**57**, 3.6.4], implies that all virtually finitely generated abelian groups belong to $\mathcal{FJ}_{L[1/2]}(R)$ for $R = \mathbb{Z}, \mathbb{Q}$. Hence, all elementary amenable groups belong to $\mathcal{FJ}_{L[1/2]}(R)$ for $R = \mathbb{Z}, \mathbb{Q}$, by Lemma 2.12. This has already been proved in [**31**, Section 5].

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