# On Hopf Invariants

J. M. Boardman and B. Steer [1])

## Introduction

There are various operations in homotopy theory all called generalized Hopf invariants. In § 2 we give axioms for Hopf invariants, resembling those for characteristic classes. We prove that these axioms define a unique sequence of homotopy operations

$$\lambda_n : [EA, EB] \to [E^n A, EB \wedge EB \wedge \cdots \wedge EB].$$

From the axioms we can find the behaviour of $\lambda_n$ on composites and Whitehead products. This enables us to express $\lambda_n$ in terms of the Hilton-Hopf invariants [12], in § 4. On the other hand, we show in § 3 that $\lambda_n$ is the $(n-1)$-th suspension of the $n$-th James-Hopf invariant [16]. We deduce that the James-Hopf invariants and the Hilton-Hopf invariants determine one another, apart from a few suspensions.

In § 5 we construct a sequence of homotopy operations by writing down explicit maps. Since the axioms hold, these operations coincide with the operations $\lambda_n$. We know after Pontrjagin [23] and Thom [30] that one may regard an element of $\pi_r(S^k)$ as a framed-cobordism class of framed, compact, smooth submanifolds of $\mathbf{R}^r$ with codimension $k$. We interpret $\lambda_n$ in this context as a geometric construction on framed submanifolds of a smooth manifold, which generalizes the geometric interpretation given in [10] of the suspension of the Hopf invariant due to G. W. Whitehead [35] and Hilton [12]. (Notice that with our conventions there is a difference of sign between our invariant $\lambda_2$ and the invariant $h'$ of [10].) Kervaire [18] has given another geometric construction for the suspended Hopf invariant, which is easily seen to be a stable suspension (up to sign) of $\lambda_2$. One may regard our geometric invariant, or that of [10], as superseding it.

Perhaps the most interesting section is § 6, where we illustrate how the geometric invariants occur in differential topology. Many of their properties were initially proved (in the case of spheres) by direct geometric methods.

Particular attention has been paid throughout to signs. We include an Appendix, § 7, comparing the signs of the various Hopf invariants on homotopy groups of spheres.

---

[1]) Pendant une partie de ce travail le deuxième auteur a bénéficié d'un bourse du Fonds National Suisse.

## 1. Notation

The only spaces we shall consider are *connected* CW-complexes and their loop spaces. All our spaces are assumed to be equipped with a basepoint $o$, whenever one is needed; it is to be respected by maps and homotopies. In the case of a CW-complex the basepoint must be a vertex. The *support* of a map $f: A \to B$ is the closure of the inverse image $f^{-1}(B-o)$. We say f is *zero* at $a \in A$ if $fa = o$. We denote the composite of $f: A \to B$ and $g: B \to C$ by $g \circ f$.

The *wedge* $A \vee B$ of spaces $A$ and $B$ is their union with the two basepoints identified together to form the new basepoint. If $B$ is a subspace of $A$, $A/B$ will denote the identification space formed from $A$ by identifying $B$ to a new basepoint $o$. We may include $A \vee B$ in $A \times B$ as $(A \times o) \cup (o \times B)$. The *smash product* (or *reduced join*) $A \wedge B$ of two CW-complexes is the identification space $(A \times B)/(A \vee B)$, where we give $A \times B$, and hence $A \wedge B$, the obvious CW-topology. We shall write $\Lambda^n B$ for the smash product of $n$ copies of $B$.

We shall write $\mathbf{R}^n$ for euclidean $n$-space, $D^n$ for the unit disk in $\mathbf{R}^n$, and $S^{n-1}$ for its boundary, the unit sphere. In homotopy theory it is useful to have also the sphere $\Sigma^n \equiv D^n/S^{n-1}$, which is homeomorphic to $S^n$, but not canonically. (The choice of homeomorphism $S^n \cong \Sigma^n$ is discussed in more detail in the Appendix.) The *suspension* $EA$ of $A$ is defined as $A \wedge (I/\partial I)$ (which is canonically homeomorphic to $A \wedge \Sigma^1$), where $I$ is the closed unit interval [0, 1], with endpoints $\partial I = 0 \cup 1$, and basepoint 0; $E$ is a functor. The $n$-fold suspension $E^n$ is the functor $E$ iterated $n$ times. We shall denote by $s$ any identification map. It will be convenient always to regard $EA$ as being obtained from $A \times \mathbf{R}$ by the map $s: A \times \mathbf{R} \to EA$ identifying $o \times \mathbf{R}$, $A \times (-\infty, 0]$ and $A \times [1, \infty)$, to the basepoint. Similarly for $s: A \times \mathbf{R}^n \to E^n A$.

We denote by $[A, B]$ the set of homotopy classes of maps from $A$ to $B$, where the maps and homotopies must, of course, respect the basepoints. By [4], track addition, which we write as $+$, makes $[EA, B]$ into a group, and $[E^2 A, B]$ into an abelian group. Then the class $-\iota \in [EB, EB]$ is defined, where $\iota$ is the class of the identity map of $EB$. The involution $U$ on $[A, EB]$ is induced by composition with $-\iota$. We stress that, even when $\alpha \in \pi_r(S^n)$, the elements $\alpha$, $U\alpha$, $-\alpha$, and $-U\alpha$ are in general all distinct. However, we do have, trivially, $UE\beta = EU\beta = -E\beta$ for any $\beta \in [A, EB]$.

Associativity and commutativity of the smash product define *shuffles* $E^m A \wedge E^n B \cong E^{m+n}(A \wedge B)$ uniquely, provided the copies of $I/\partial I$ remain in the same order. These shuffles are frequently used and suppressed from the notation.

Given any wedge $B_1 \vee B_2 \vee \cdots \vee B_n$ or product $B_1 \times B_2 \times \cdots \times B_n$, we shall write $p_r: B_1 \vee B_2 \vee \cdots \vee B_n \to B_r$ or $p_r: B_1 \times B_2 \times \cdots \times B_n \to B_r$ for the projection to the $r$-th factor, and $i_r: B_r \subset B_1 \vee B_2 \vee \cdots \vee B_n$ or $i_r: B_r \subset B_1 \times B_2 \times \cdots \times B_n$ for the inclusion of the $r$-th factor (making use of basepoints). We write $\iota_r$ for the homotopy class of $i_r$, and

$\pi_r$ for the homotopy class of $p_r$. (We shall normally use Greek letters for homotopy classes, and Roman letters for maps.)

The *pinch map* $r_n: EA \to EA \vee EA \vee \cdots \vee EA$ ($n$ factors) is given by

$$r_n s(a, t) = i_k s(a, nt - k + 1) \quad \text{for} \quad k - 1 \le nt \le k, \tag{1.1}$$

where $s: A \times \mathbf{R} \to EA$ is the identification map, and $a \in A$, $t \in \mathbf{R}$. We write $\rho_n$ for the homotopy class of $r_n$. Then $\rho_n = \iota_1 + \iota_2 + \cdots + \iota_n$. We also need the *backward pinch map* $\bar{r}_n: EA \to EA \vee EA \vee \cdots \vee EA$ defined by

$$\bar{r}_n s(a, t) = i_k s(a, nt - n + k) \quad \text{for} \quad n - k \le nt \le n - k + 1, \tag{1.2}$$

and its homotopy class $\bar{\rho}_n = \iota_n + \iota_{n-1} + \cdots + \iota_1$. It is clear that $\bar{\rho}_n = \rho_n$ when $A$ is a suspension. In general we have $\bar{\rho}_n = -U\rho_n$.

For any space $A$, the *reduced diagonal* $\Delta: A \to A \wedge A$ is the map given by $\Delta a = a \wedge a (a \in A)$. It is nullhomotopic when $A$ is a suspension. The smash product functor defines a pairing

$$[E^m A, B] \times [E^n A, C] \to [E^m A \wedge E^n A, B \wedge C].$$

The reduced diagonal in $A$ yields the map

$$E^{m+n} A \xrightarrow[E^{m+n}\Delta]{} E^{m+n}(A \wedge A) \cong E^m A \wedge E^n A,$$

which induces the operation

$$[E^m A \wedge E^n A, B \wedge C] \to [E^{m+n} A, B \wedge C].$$

DEFINITION 1.3. The *cup product* pairing

$$[E^m A, B] \times [E^n A, C] \to [E^{m+n} A, B \wedge C]$$

is the composite of these two operations. We write $\alpha \cdot \beta$ for the cup product of $\alpha$ and $\beta$.

These products are relative to $A$. If $A$ is itself a suspension $ED$, we have two cup products, defined with respect to $A$ or $D$. Those defined with respect to $A$ vanish, because $\Delta: ED \to ED \wedge ED$ is nullhomotopic. We summarize the elementary properties of the cup product.

LEMMA 1.4. *The cup product is bilinear and associative. It vanishes when $A$ is a suspension.*

As a particular case of the cup product pairing, we have

$$[EA, EB] \times [EA, EC] \to [E^2 A, EB \wedge EC].$$

From [28] we know that this can be desuspended, as a pairing

$$[EA, EB] \times [EA, EC] \to [EA, E(B \wedge C)], \tag{1.5}$$

which is induced from a map $\natural: \Omega EB \times \Omega EC \to \Omega E(B \wedge C)$.

*Loop spaces.* The usual loop space on a space $A$ is a $H$-space which is not associative. Therefore we shall use MOORE's loop space [22], which is.

DEFINITION 1.6. Given a space $A$, the *Moore loop space* $\Omega A$ is the set of pairs $(f, k)$, where $k \in \mathbf{R}$, $k \geqslant 0$, and $f: \mathbf{R} \to A$ is a map such that $ft = o$ unless $0 < t < k$. We topologize $\Omega A$ as a subspace of $A^{\mathbf{R}} \times \mathbf{R}$, where $A^{\mathbf{R}}$ is given the compact-open topology. The basepoint of $\Omega A$ is the *zero loop* $(o, 0)$. We call $k$ the *length* of the loop $(f, k)$.

The multiplication (or 'addition') of loops in $\Omega A$ is defined by $(f, k) + (g, m) = (h, k+m)$, where

$$ht = \begin{cases} f\,t & \text{if } 0 \leq t \leq k\,, \\ g(t-k) & \text{if } k \leq t \leq k+m\,, \\ o & \text{otherwise.} \end{cases}$$

This makes $\Omega A$ into an associative $H$-space, having the zero loop as identity element.

The ordinary loop space $\Omega_1 A$ is the subspace of $\Omega A$ consisting of loops of length 1.

LEMMA 1.7. *$\Omega_1 A$ is naturally a deformation retract of $\Omega A$.*

*Proof.* Define the retraction $q: \Omega A \to \Omega_1 A$ by $q(f, k) = (f_1, 1)$, where $f_1 t = f(kt)$. Then $q$ is continuous and is a deformation retraction. A deforming homotopy is easily constructed.

Given a map $f: EA \to B$, we have $f \circ s: A \times \mathbf{R} \to B$, and therefore $f': A \to \Omega B$, defined by $(f'a)t = fs(a, t)$ $(a \in A,\ t \in \mathbf{R})$. We have a loop $(f'a, 1)$ of length 1. This defines the natural *adjoint* isomorphism of groups

$$[EA, B] \cong [A, \Omega B]\,,$$

where the multiplication in $\Omega B$ is used to make $[A, \Omega B]$ into a group. The functors $E$ and $\Omega$ are adjoint functors (on homotopy classes). The above isomorphism may also be regarded as the transgression of the fibration $LB \to B$, where $LB$ is the space of Moore paths on $B$.

## 2. Axioms for Hopf invariants

In the following definition $B$ runs through all connected based CW-complexes, and $A$ runs through all finite connected based CW-complexes. We use the cup product 1.3.

DEFINITION 2.1. A *Hopf ladder* is a sequence of natural transformations (operations)

$$\lambda_n: [EA, EB] \to [E^n A, \Lambda^n EB]\,, \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

such that:

(a) (identity) $\lambda_1$ is the identity operation,

(b) (normalization) $\lambda_n E\alpha = 0$ if $\alpha \in [A, B]$ and $n > 1$,

(c) (Cartan formula)

$$\lambda_n(\alpha+\beta)=\lambda_n\alpha+\lambda_{n-1}\alpha\cdot\lambda_1\beta+\lambda_{n-2}\alpha\cdot\lambda_2\beta+\cdots+\lambda_1\alpha\cdot\lambda_{n-1}\beta+\lambda_n\beta$$
$$\text{whenever}\quad \alpha,\beta\in[EA,EB].$$

(If $n>1$, the order of the terms in (c) is irrelevant, because then $[E^nA, \Lambda^nEB]$ is abelian.)

THEOREM 2.2. *There exists precisely one Hopf ladder.*

The proof of this main theorem is deferred to § 3, where it will be included in 3.15.

In the subsequent sections we shall use these axioms to express $\lambda_n$ in terms of various kinds of previously defined Hopf invariants. This will justify the name 'Hopf ladder'.

A particularly useful special case is when $A$ is a suspension. Then the cup products vanish, by 1.4.

COROLLARY 2.3. *$\lambda_n:[EA, EB]\to[E^nA, \Lambda^nEB]$ is a homomorphism whenever $A$ is a suspension.*

The Cartan formula (c) suggests the usual formalism.

DEFINITION 2.4. We define formally the *exponential Hopf invariant*

$$e^\alpha=1+\alpha+\lambda_2\alpha+\lambda_3\alpha+\lambda_4\alpha+\cdots$$

The terms lie in different groups, except that 1 is purely formal. With this definition we can rewrite the axioms succinctly as

$$e^{E\alpha}=1+E\alpha,\quad\text{and}\quad e^{\alpha+\beta}=e^\alpha\cdot e^\beta. \tag{2.5}$$

Further support for the name 'exponential' will be given by 3.17, when we show that in certain special cases $n!\ \lambda_n\alpha=\alpha^n$ (the cup power): so that, very formally, $e^\alpha=\Sigma\alpha^n/n!$.

Various extensions of Theorem 2.2 are possible.

REMARK. Our proof of 2.2 will show uniqueness of *truncated* Hopf ladders, in which we are given $\lambda_n$ only for $n\leqslant n_0$, satisfying the relevant axioms. Further, we use only the naturality in $A$, not that in $B$. Again, we can allow $A$ to run through finite-dimensional CW-complexes.

REMARK. We can give a desuspended form of 2.1. Instead of the cup product, we use the $\natural$-pairing 1.5. We postulate operations

$$[EA, EB]\to[EA, E\Lambda^nB]$$

satisfying (a) and (b) as before, but in (c) we demand equality only modulo an 'ideal' generated by certain Whitehead products. Uniqueness is thus modulo this 'ideal', which is killed by one suspension.

## 3. The invariants of James

In this section we introduce JAMES's theory [15] of reduced product spaces, and the resulting James-Hopf invariants [16]

$$\gamma_n : [EA, EB] \to [EA, EA^n B].$$

This theory enables us to prove our main theorem 2.2, and to show that the suspended operations $E^{n-1}\gamma_n$ form a Hopf ladder.

As a by-product, we deduce the formula for $\lambda_n(\beta \circ \alpha)$.

As always, $B$ is to be a connected CW-complex with basepoint $o$. We collect from [15] the salient facts about the *reduced product space* $B_\infty$.

LEMMA 3.1. *$B_\infty$ is the free monoid on the points of $B-o$, with $o$ as identity, topologized as a* CW*-complex. It contains $B$ as a subcomplex. Given any associative $H$-space $X$, with basepoint as the identity, any map $f: B \to X$ (or homotopy $f_t: B \to X$) extends uniquely to a continuous homomorphism $g: B_\infty \to X$ (or homotopy of homomorphisms $g_t: B_\infty \to X$).*

*Proof.* This is essentially Theorem 1.11 of [15].

If $B$ has countably many cells, $B_\infty$ is an $H$-space. In any case, the multiplication $B_\infty \times B_\infty \to B_\infty$ (which we write as $+$, even though it is obviously not commutative) is continuous if we use the CW-topology on the product $B_\infty \times B_\infty$. The subcomplex $B_n$ of $B_\infty$ is the subspace consisting of all $n$-fold products of points of $B$. Thus $B_\infty$ is the union of the sequence of subcomplexes $B_n$, and $B_n / B_{n-1} \cong A^n B$.

A *distance* $d$ on $B$ is a real-valued continuous function defined on $B$, such that $do = 0$, and $db > 0$ for all $b \neq o$. Such functions always exist on a CW-complex $B$, and any two are homotopic through distances, because they form a convex subset of $\mathbf{R}^B$.

Suppose given a distance $d$ on $B$. Given $b \in B$ and $k \in \mathbf{R} (k > 0)$, let $w(b, k)$ be the particular loop in $\Omega EB$ with length $k$ defined by

$$w(b, k)(t) = s(b, t/k),$$

where $s: B \times \mathbf{R} \to EB$. Also, define $w(o, 0)$ to be the zero loop.

DEFINITION 3.2. The *canonical homomorphism* (relative to $d$)

$$u: B_\infty \to \Omega EB$$

is the homomorphism extending the map $B \to \Omega EB$ given by

$$ub = w(b, db) \quad \text{(which is continuous).}$$

We can now state the main theorem on $B_\infty$.

THEOREM 3.3. *Any canonical homomorphism $u: B_\infty \to \Omega EB$ is a homotopy equivalence. Any two are homotopic.*

*Proof.* In Theorem 5.6 of [15], JAMES proved that the composite $q \circ u : B_\infty \to \Omega_1 EB$ is a singular homotopy equivalence, where $q : \Omega EB \to \Omega_1 EB$ is the deformation retraction we used in the proof of 1.7. For examination of the formula (7.1) of [15] reveals that $q \circ u$ is precisely the canonical map as defined by JAMES. Finally, we may omit the word 'singular', because MILNOR has proved [21] that $\Omega_1 EB$ has the homotopy type of a CW-complex.

We shall use an adjoint form of this theorem.

DEFINITION 3.4. Define the homotopy class $\omega \in [EB_\infty, EB]$ as the adjoint to the homotopy class of any canonical map $u : B_\infty \to \Omega EB$. Write $\omega_n = i^* \omega \in [EB_n, EB]$, where $i : B_n \subset B_\infty$.

LEMMA 3.5. *Let $A$ be a finite* CW-*complex, and $\alpha \in [EA, EB]$. Then there exists an integer $n$, and $\beta \in [A, B_n]$, such that $\alpha = \omega_n \circ E\beta$.*

*Proof.* The class in $[A, \Omega EB]$ adjoint to $\alpha$ can be factored through some $B_n$, by 3.3 and the finiteness of $A$.

Let $B^n$ denote the product of $n$ copies of $B$. It will play the same rôle in this section as the maximal torus in the theory of Lie groups.

The following lemma is well known.

LEMMA 3.6. *The identification (i.e. multiplication) map $s : B^n \to B_n$ induces an injection*

$$s^* : [EB_n, X] \to [EB^n, X]$$

*for any space $X$.*

*Proof.* We know from Theorem 8.2 of [4] that

$$s^* : [E\Lambda^n B, X] \to [EB^n, X]$$

is injective. We consider the commutative diagram

$$\begin{array}{ccccc} [E\Lambda^n B, X] & \to & [EB_n, X] & \to & [EB_{n-1}, X] \\ & & \downarrow & & \downarrow \\ & & [EB^n, X] & \to & [EB^{n-1}, X], \end{array}$$

in which the top row is exact because $\Lambda^n B \cong B_n / B_{n-1}$ (see [4]). By induction on $n$ assume that $s^* : [EB_{n-1}, X] \to [EB^{n-1}, X]$ is injective. Then the diagram shows that $s^* : [EB_n, X] \to [EB^n, X]$ is injective.

The induction starts trivially with $n = 1$. Hence the result holds generally.

We combine 3.5 and 3.6 to prove an important lemma, which may be viewed as a splitting principle. (It has an interpretation for certain types of quasifibrations; see [17].)

LEMMA 3.7. *Let $A$ be a variable finite* CW-*complex, and $X$ and $Y$ be fixed spaces. Suppose we have two operations*

$$\Phi, \Psi: [EA, EX] \to [EA, Y],$$

*natural in A, which agree on all elements of the form* $E\alpha_1 + E\alpha_2 + \cdots + E\alpha_n$, *where* $\alpha_1, \alpha_2, \ldots, \alpha_n \in [A, X]$, *for all A and all n. Then* $\Phi = \Psi$.

*Similarly for operations*

$$[E(A \vee A), EX] \to [EA, Y].$$

*Proof.* Take $\alpha \in [EA, EX]$; we have to show that $\Phi\alpha = \Psi\alpha$. Since $A$ is finite, there exists a finite CW-complex $B$ and homotopy classes $\beta \in [EA, EB]$, $\gamma \in [B, X]$, such that $\alpha = E\gamma \circ \beta$. By 3.5, there exists $n$ such that $\beta$ factors as $\omega_n \circ E\eta$, where $\eta \in [A, B_n]$. Naturality in $A$ yields the commutative diagram

$$\begin{array}{ccccc} [EA, EX] & \xleftarrow{\eta^*} & [EB_n, EX] & \xrightarrow{s^*} & [EB^n, EX] \\ \downarrow{\scriptstyle \Phi, \Psi} & & \downarrow{\scriptstyle \Phi, \Psi} & & \downarrow{\scriptstyle \Phi, \Psi} \\ [EA, Y] & \xleftarrow{\eta^*} & [EB_n, Y] & \xrightarrow{s^*} & [EB^n, Y]. \end{array}$$

We have lifted $\alpha \in [EA, EX]$ to $E\gamma \circ \omega_n \in [EB_n, EX]$, which gives $s^*(E\gamma \circ \omega_n) \in [EB^n, EX]$. The crucial observation is that from the definition of $\omega_n$, we have

$$s^*\omega_n = E\pi_1 + E\pi_2 + \cdots + E\pi_n \in [EB^n, EB].$$

Hence

$$s^*(\gamma \circ \omega_n) = E(\gamma \circ \pi_1) + E(\gamma \circ \pi_2) + \cdots + E(\gamma \circ \pi_n),$$

on which $\Phi$ and $\Psi$ agree by hypothesis. Since $s^*$ is injective by 3.6, it follows that $\Phi\alpha = \Psi\alpha$.

In the second case, $\alpha \in [E(A \vee A), EX]$ has two components, $\alpha_j \in [EA, EX]$ ($j = 1, 2$). We factor each $\alpha_j$ as $E\gamma_j \circ \beta_j$, where $\gamma_j \in [B_j, X]$ and $\beta_j \in [EA, EB_j]$. Then we put $B = B_1 \vee B_2$, $\beta = \iota_1 \circ \beta_1 + \iota_2 \circ \beta_2 \in [EA, EB]$, and define $\gamma \in [B \vee B, X]$ as the class including $\gamma_1$ on $B_1 \vee o$, $\gamma_2$ on $o \vee B_2$, and zero on $B_2 \vee o$ and $o \vee B_1$. The proof can now be completed much as before.

It is time to introduce the James-Hopf invariants [16].

Definition 3.8. We define, for each $n \geq 1$,

$$g_n: B_\infty \to (\Lambda^n B)_\infty \quad \text{by}$$
$$g_n(b_1 + b_2 + \cdots + b_m) = \Sigma_\sigma b_{\sigma 1} \wedge b_{\sigma 2} \wedge \cdots \wedge b_{\sigma n}, \quad (b_i \in B) \tag{3.9}$$

summing over all strictly increasing functions

$$\sigma: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\},$$

where the terms are to be ordered lexicographically from the left. (Compare Lemma 2.5 of [15].)

This map is not a homomorphism.

DEFINITION 3.10. The *James-Hopf invariant*

$$\gamma_n : [EA, EB] \to [EA, E\Lambda^n B]$$

is obtained from $g_n^* : [A, B_\infty] \to [A, (\Lambda^n B)_\infty]$ by taking adjoints and using Theorem 3.3.

In order to compare $\gamma_n$ with other operations by 3.7, we need to compute its value on elements of the form $E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m$.

LEMMA 3.11. *Take any elements* $\alpha_j \in [A, B]$, $(1 \leqslant j \leqslant m)$. *Then*

$$\gamma_n(E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m) = \sum_\sigma E(\alpha_{\sigma 1} \cdot \alpha_{\sigma 2} \cdot \cdots \cdot \alpha_{\sigma n})$$

(*the cup product*), *where we sum over* $\sigma$ *as in* 3.8.

*Proof.* For each $j$, choose a representative $f_j : A \to B$ of $\alpha_j$. A convenient representative of the adjoint to the track sum $E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m$ is the map $f : A \to B_\infty$ given by

$$f\,a = f_1 a + f_2 a + \cdots + f_m a . \quad (a \in A)$$

Then

$$\begin{aligned} g_n f\,a &= \sum_\sigma f_{\sigma 1} a \wedge f_{\sigma 2} a \wedge \cdots \wedge f_{\sigma n} a \\ &= \sum_\sigma (f_{\sigma 1} \wedge f_{\sigma 2} \wedge \cdots \wedge f_{\sigma n})(a \wedge a \wedge \cdots \wedge a), \end{aligned}$$

from which the result is clear.

REMARK 3.12. The terms in 3.9 may be ordered in different ways. JAMES, in [15], orders them lexicographically from the right. TODA [32] orders them lexicographically from the left, as we do. In fact, one could use any system for ordering the terms, provided it gives rise to a continuous map $g_n$. One can show that there are just $2^n \cdot n!$ such systems of ordering, all of them essentially lexicographic. From 3.11 one can deduce that in general (e.g. when $B$ is the $n$-fold wedge $P_{2n}(\mathbf{R}) \vee P_{2n}(\mathbf{R}) \vee \cdots \vee P_{2n}(\mathbf{R})$) no two of the resulting maps $g_n$ are homotopic. On the other hand, we see from 3.7 and 3.11 that the corresponding suspended operations

$$E\gamma_n : [EA, EB] \to [E^2 A, E^2 \Lambda^n B]$$

are independent of this choice.

We consider the suspended James-Hopf invariants

$$E^{n-1}\gamma_n : [EA, EB] \to [E^n A, E^n \Lambda^n B] \cong [E^n A, \Lambda^n EB], \tag{3.13}$$

which we now know are canonical (in the sense that they do not depend on any arbitrary choices). For these, 3.11 yields

$$(E^{n-1}\gamma_n)(E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m) = \sum_\sigma E\alpha_{\sigma 1} \cdot E\alpha_{\sigma 2} \cdot \cdots \cdot E\alpha_{\sigma n} . \tag{3.14}$$

THEOREM 3.15. *The suspended James-Hopf invariants* 3.13

$$E^{n-1}\gamma_n\colon [EA, EB] \to [E^n A, E^n \Lambda^n B] \cong [E^n A, \Lambda^n EB]$$

*form a Hopf ladder. For any Hopf ladder* $(\lambda_n)$ *we have* $\lambda_n = E^{n-1}\gamma_n$.

*Proof.* Write $\lambda_n' = E^{n-1}\gamma_n$ for the operation 3.13. We must verify the axioms 2.1. Trivially, (a) and (b) hold. To prove the Cartan formula, we compare the two operations given on $(\alpha, \beta) \in [EA, EB] \times [EA, EB] \cong [E(A \vee A), EB]$ by $\lambda_n'(\alpha+\beta)$ and

$$\lambda_n'\alpha + \lambda_{n-1}'\alpha \cdot \lambda_1'\beta + \lambda_{n-2}'\alpha \cdot \lambda_2'\beta + \cdots + \lambda_1'\alpha \cdot \lambda_{n-1}'\beta + \lambda_n'\beta.$$

That they agree on $(\alpha, \beta)$ when $\alpha = E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m$ and $\beta = E\beta_1 + E\beta_2 + \cdots + E\beta_r$ is clear from 3.14. By the second part of 3.7 (taking $X = B$ and $Y = \Omega^{n-1}\Lambda^n EB$), these two operations must agree generally. Thus (c) holds, and we have a Hopf ladder.

For any Hopf ladder $(\lambda_n)$, the axioms 2.1 determine the value of $\lambda_n$ on all elements of the form $E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m$. Then 3.7 shows that $\lambda_n' = \lambda_n$ generally. This completes the proof.

This theorem includes Theorem 2.2, which is therefore now proved.

We can also deduce from 3.7 the expansion of $\lambda_n(\beta \circ \alpha)$. Given strictly positive integers $j_r$ $(1 \le r \le q)$ such that $j_1 + j_2 + \cdots + j_q = n$, define the permutation map

$$T(j_1, j_2, \ldots, j_q)\colon (EB)^q \times \mathbf{R}^{n-q} \to (EB \times \mathbf{R}^{j_1-1}) \times (EB \times \mathbf{R}^{j_2-1}) \times \cdots \times (EB \times \mathbf{R}^{j_q-1})$$

by grouping the factors of $\mathbf{R}^{n-q}$ as $\mathbf{R}^{j_1-1} \times \mathbf{R}^{j_2-1} \times \cdots \times \mathbf{R}^{j_q-1}$ and rearranging the factors of $(EB)^q \times \mathbf{R}^{n-q}$, taking care to keep the $q$ copies of $EB$ in the same order. That is,

$$\begin{aligned} &T(j_1, j_2, \ldots, j_q)(b_1, b_2, \ldots, b_q, t_1, t_2, \ldots, t_{n-q}) \\ &\quad = ((b_1, t_1, t_2, \ldots, t_{j_1-1}), (b_2, t_{j_1}, t_{j_1+1}, \ldots, t_{j_2-2}), \ldots, (b_q, t_{n-q-j_q+1}, \ldots, t_{n-q})), \end{aligned}$$

where $b_i \in EB$ and $t_k \in \mathbf{R}$. Let $\eta(j_1, j_2, \ldots, j_q)$ be the class of the map

$$E^{n-q}\Lambda^q EB \cong E^{j_1}B \wedge E^{j_2}B \wedge \cdots \wedge E^{j_q}B$$

induced from $T(j_1, j_2, \ldots, j_q)$ by identification. This is *not* a shuffle in the sense of § 1.

THEOREM 3.16. *Let* $\alpha \in [EA, EB]$ *and* $\beta \in [EB, EC]$. *Then*

$$\lambda_n(\beta \circ \alpha) = \sum (\lambda_{j_1}\beta \wedge \lambda_{j_2}\beta \wedge \cdots \wedge \lambda_{j_q}\beta) \circ \eta(j_1, j_2, \ldots, j_q) \circ E^{n-q}\lambda_q\alpha,$$

*where we sum over all sequences* $(j_1, j_2, \ldots, j_q)$ *of strictly positive integers satisfying* $j_1 + j_2 + \cdots + j_q = n$.

*Proof.* We regard both sides of the formula as operations on $\alpha \in [EA, EB]$, and in 3.7 take $X = B$, $Y = \Omega^{n-1}\Lambda^n EC$. By 3.7 we need verify equality only when $\alpha$ has the form $E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m$.

Suppose $\alpha = E\alpha_1 + E\alpha_2 + \cdots + E\alpha_m$. Then

$$\beta \circ \alpha = \beta \circ E\alpha_1 + \beta \circ E\alpha_2 + \cdots + \beta \circ E\alpha_m.$$

By 2.1, we obtain the formulae, in which $(j_1, j_2, \ldots, j_q)$ ranges over all sets of integers satisfying $j_1+j_2+\cdots+j_q=n$, as in the statement of the theorem, and $\sigma$ runs through all strictly increasing functions $\{1, 2, \ldots, q\}\to\{1, 2, \ldots, m\}$.

$$\begin{aligned}\lambda_n(\beta\circ\alpha) &= \sum\sum\lambda_{j_1}(\beta\circ E\alpha_{\sigma 1})\cdot\lambda_{j_2}(\beta\circ E\alpha_{\sigma 2})\cdot\cdots\cdot\lambda_{j_q}(\beta\circ E\alpha_{\sigma q})\\ &= \sum\sum(\lambda_{j_1}\beta\circ E^{j_1}\alpha_{\sigma 1})\cdot(\lambda_{j_2}\beta\circ E^{j_2}\alpha_{\sigma 2})\cdot\cdots\cdot(\lambda_{j_q}\beta\circ E^{j_q}\alpha_{\sigma q})\\ &= \sum\sum(\lambda_{j_1}\beta\wedge\lambda_{j_2}\beta\wedge\cdots\wedge\lambda_{j_q}\beta)\circ(E^{j_1}\alpha_{\sigma 1}\cdot E^{j_2}\alpha_{\sigma 2}\cdot\cdots\cdot E^{j_q}\alpha_{\sigma q})\\ &= \sum\sum(\lambda_{j_1}\beta\wedge\lambda_{j_2}\beta\wedge\cdots\wedge\lambda_{j_q}\beta)\circ\eta(j_1, j_2, \ldots, j_q)\\ &\qquad\qquad\circ E^{n-q}(E\alpha_{\sigma 1}\cdot E\alpha_{\sigma 2}\cdot\cdots\cdot E\alpha_{\sigma q})\\ &= \sum(\lambda_{j_1}\beta\wedge\lambda_{j_2}\beta\wedge\cdots\wedge\lambda_{j_q}\beta)\circ\eta(j_1, j_2, \ldots, j_q)\circ E^{n-q}\lambda_q\alpha.\end{aligned}$$

Thus the formula holds for $\alpha$, and therefore generally, by 3.7.

We next investigate in what sense the elements $\lambda_n\alpha$ are divided powers of $\alpha$. If $\alpha\in[EA, EB]$, we wish to compare its $n$-th cup power $\alpha^n$ with $\lambda_n\alpha$, which both lie in $[E^nA, \Lambda^nEB]$.

THEOREM 3.17. *If $B$ is a suspension, and $\alpha\in[EA, EB]$, then*

$$\alpha^n = \sum_\pi(-1)^{\varepsilon(\pi)}\pi\circ\lambda_n\alpha,$$

*where we sum over all permutations $\pi$ of the factors $EB$ of $\Lambda^nEB$, and $\varepsilon(\pi)$ denotes the sign of the permutation $\pi$.*

*Proof.* Both sides are natural in $A$; therefore by 3.7 we need verify the formula only when $\alpha$ has the form $\Sigma\, E\alpha_j$.

Suppose $\alpha=E\alpha_1+E\alpha_2+\cdots+E\alpha_m$, where each $\alpha_j\in[A, B]$. Then by 2.1

$$\lambda_n\alpha = \sum_\sigma E\alpha_{\sigma 1}\cdot E\alpha_{\sigma 2}\cdot\cdots\cdot E\alpha_{\sigma n},$$

where $\sigma$ runs through all functions $\sigma:\{1, 2, \ldots, n\}\to\{1, 2, \ldots, m\}$ satisfying $\sigma 1<\sigma 2<\cdots<\sigma n$. (The order of the terms is irrelevant.) Hence, by composing with a permutation $\pi$,

$$(-1)^{\varepsilon(\pi)}\pi\circ\lambda_n\alpha = \sum_\sigma E\alpha_{\sigma 1}\cdot E\alpha_{\sigma 2}\cdot\cdots\cdot E\alpha_{\sigma n},$$

where this time we sum over those functions $\sigma$ satisfying $\sigma(\pi 1)<\sigma(\pi 2)<\cdots<\sigma(\pi n)$. Summing over $\pi$ yields

$$\sum_\pi(-1)^{\varepsilon(\pi)}\pi\circ\lambda_n\alpha = \sum_\sigma E\alpha_{\sigma 1}\cdot E\alpha_{\sigma 2}\cdot\cdots\cdot E\alpha_{\sigma n},$$

where we sum on the right over all functions $\sigma$ such that $\sigma 1$, $\sigma 2$, ..., and $\sigma n$, are all distinct. But

$$\alpha^n = \sum_\sigma E\alpha_{\sigma 1}\cdot E\alpha_{\sigma 2}\cdot\cdots\cdot E\alpha_{\sigma n},$$

with no condition on $\sigma$. The extra terms all contain a repeated factor. Now for any $\gamma \in [A, B]$, we may use the naturality of the reduced diagonal to rewrite $E\gamma \cdot E\gamma$ as the composite

$$E^2 A \xrightarrow[E^2\gamma]{} E^2 B \xrightarrow[E^2\Delta]{} E^2(B \wedge B) \cong EB \wedge EB.$$

Since by hypothesis $B$ is a suspension, the diagonal $\Delta: B \to B \wedge B$ is nullhomotopic, and therefore $E\gamma \cdot E\gamma = 0$. Thus the unwanted terms in the expansion of $\alpha^n$ are all zero, and we have proved the theorem.

As an example, taken $A$ and $B$ to be spheres.

COROLLARY 3.18. *Suppose $\alpha \in \pi_r(S^k)$, where $k$ is odd. Then $\lambda_n \alpha \in \pi_{r+n-1}(S^{nk})$ satisfies $n!\,\lambda_n \alpha = 0$.*

For the case $n=2$, compare Theorem 5.42 of [35], apart from the question of identifying our Hopf invariant $\lambda_2$ with the usual one (up to sign; see Appendix).

For the sake of completeness, let us note the behaviour of $\lambda_n$ on smash products.

THEOREM 3.19. *Given $\alpha \in [EA, EB]$, and $\beta \in [C, D]$, let $\Delta: D \to A^n D$ be the n-fold reduced diagonal, Then*

$$\lambda_n(\alpha \wedge \beta) = \lambda_n \alpha \wedge (\Delta \circ \beta),$$

*apart from some shuffles.*

*Proof.* This is trivial for the James-Hopf invariant 3.10, which are we entitled to use (after suspension) for $\lambda_n$, by 3.15.

## 4. The invariants of Hilton

In this section we introduce the Hilton-Hopf invariants [12], [20],

$$H_c: [EA, EC] \to [EA, EA^r C],$$

which are defined by the identity

$$(\iota_1 + \iota_2) \circ \alpha = \sum_c \iota_c \circ H_c \alpha \in [EA, EC \vee EC]. \tag{4.1}$$

Here, $\iota_c$ runs through certain iterated Whitehead products called *basic* products; there are many choices of such a system, and consequently many choices for $H_c$.

We compare these invariants with our axiomatic invariants $\lambda_n$, and hence indirectly with the James-Hopf invariants, by applying $\lambda_n$ to each side of 4.1; this was the method used in [10] to compare $h'$ with $h$. We evaluate $\lambda_n$ on a sum by the axioms 2.1, and on a composite by 3.16. We need to evaluate $\lambda_n$ also on Whitehead products.

For this purpose we use BARRATT's definition of the Whitehead product, as given in § 3 of [5]. It is observed there that for any spaces $A$ and $B$, the Barratt-Puppe exact sequence (see [4] or [24]) for the inclusion map $i: A \vee B \subset A \times B$ breaks up into short exact sequences, in particular

$$0 \to [E(A \wedge B), X] \underset{s^*}{\to} [E(A \times B), X] \underset{i^*}{\to} [E(A \vee B), X] \to 0.$$

The projection maps $p_1: A \times B \to A$ and $p_2: A \times B \to B$ embed $[EA, X]$ and $[EB, X]$ (though not, of course, their direct sum!) in $[E(A \times B), X]$. Given $\alpha \in [EA, X]$ and $\beta \in [EB, X]$, we can form the commutator $\xi = \alpha' + \beta' - \alpha' - \beta' \in [E(A \times B), X]$, where $\alpha' = p_1^* \alpha$ and $\beta' = p_2^* \beta$. Evidently $i^* \xi = 0$, because inclusion induces $[E(A \vee B), X] \cong [EA, X] \times [EB, X]$; and therefore $\xi$ lifts uniquely to $[E(A \wedge B), X]$.

DEFINITION 4.2. Given $\alpha \in [EA, X]$ and $\beta \in [EB, X]$, we define their *Whitehead product* $[\alpha, \beta] \in [E(A \wedge B), X]$ by

$$s^*[\alpha, \beta] = p_1^* \alpha + p_2^* \beta - p_1^* \alpha - p_2^* \beta.$$

In case $A$ and $B$ are spheres, this differs by a sign from J. H. C. WHITEHEAD'S classical definition [36] (see Appendix). It has the property that if $\delta: [EA, X] \cong [A, \Omega X]$ is the adjoint isomorphism, then $\delta[\alpha, \beta]$ is the Samelson product [25] of $\delta\alpha$ and $\delta\beta$.

Standard identities for commutators in groups yield corresponding formulae for Whitehead products. We clearly have, always,

$$[\beta, \alpha] = -[\alpha, \beta] \circ E\eta(A, B), \tag{4.3}$$

where $\eta(A, B)$ denotes the class of the map $B \wedge A \cong A \wedge B$ interchanging the factors. It will be quite safe to ignore natural isomorphisms arising from the associativity of the smash product, but not in general those from commutativity, except for shuffles $E^m A \wedge E^n B \cong E^{m+n}(A \wedge B)$.

Let us write $(x, y)$ for the commutator $xyx^{-1}y^{-1}$ in a (multiplicative) group. From the identity

$$(x, yz) = (x, y)(y, (x, z))(x, z)$$

and the fact that the reduced diagonal $\Delta: B \to B \wedge B$ is nullhomotopic when $B$ is a suspension, we deduce that the Whitehead product

$$[EA, X] \times [EB, X] \to [E(A \wedge B), X]$$

is linear in the second factor, when $B$ is a suspension. From this and 4.3, it is linear in the first factor when $A$ is a suspension. Take also $\gamma \in [EC, X]$. From the Witt identity

$$(x, ((x^{-1}, z), y))((x^{-1}, z), y)(y, ((y^{-1}, x), z))((y^{-1}, x), z)$$
$$(z, ((z^{-1}, y), x))((z^{-1}, y), x) = 1$$

we deduce the Jacobi identity for the Whitehead product

$$[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] \circ E\eta(B \wedge C, A) + [[\gamma, \alpha], \beta] \circ E\eta(C, A \wedge B) = 0, \tag{4.4}$$

again provided that $A$, $B$, and $C$ are suspensions.

Given elements $\alpha \in [E^k A, \Lambda^m X]$ and $\beta \in [E^l B, \Lambda^n X]$, we can form by using shuffles the elements $\alpha \wedge \beta$ and $(\beta \wedge \alpha) \circ E^{k+l} \eta(B, A)$ in $[E^{k+l}(A \wedge B), \Lambda^{m+n} X]$. We call the *smash commutator* $\langle \alpha, \beta \rangle$ of $\alpha$ and $\beta$ the element

$$\langle \alpha, \beta \rangle = \alpha \wedge \beta - (\beta \wedge \alpha) \circ E^{k+l} \eta(B, A) \in [E^{k+l}(A \wedge B), \Lambda^{m+n} X]. \tag{4.5}$$

It is defined if $k+l>0$, and is bilinear. This commutator extends by linearity to formal sums.

THEOREM 4.6. *Suppose $\alpha \in [EA, EX]$ and $\beta \in [EB, EX]$, where $A$ and $B$ are suspensions. Then we have, for the exponential Hopf invariants* 2.4,

$$e^{[\alpha, \beta]} = 1 + [\alpha, \beta] + \langle e^\alpha, e^\beta \rangle.$$

*Explicitly, for $n \geq 2$,*

$$\lambda_n [\alpha, \beta] = \sum_{i=1}^{i=n-1} \lambda_i \alpha \wedge \lambda_{n-i} \beta - (\lambda_i \beta \wedge \lambda_{n-i} \alpha) \circ E^n \eta(B, A).$$

*Proof.* Write as before $\alpha' = p_1^* \alpha$ and $\beta' = p_2^* \beta$ in $[E(A \times B), EX]$. Then by definition $s^*[\alpha, \beta] = \alpha' + \beta' - \alpha' - \beta'$. Naturality and the Cartan formula 2.5 yield

$$\begin{aligned} s^* e[\alpha, \beta] &= e^{\alpha'} \cdot e^{\beta'} \cdot e^{-\alpha'} \cdot e^{-\beta'} \\ &= (e^{\alpha'} \cdot e^{\beta'} - e^{\beta'} \cdot e^{\alpha'}) \cdot e^{-\alpha'} \cdot e^{-\beta'} + 1. \end{aligned}$$

We see that the cup product $(e^{\alpha'} - 1) \cdot (e^{\beta'} - 1)$ can be written $s^* \{(e^\alpha - 1) \wedge (e^\beta - 1)\}$, and similarly for $(e^{\beta'} - 1) \cdot (e^{\alpha'} - 1)$. Hence we have

$$s^* e^{[\alpha, \beta]} = s^* \{\langle e^\alpha, e^\beta \rangle + [\alpha, \beta]\} \cdot e^{-\alpha'} \cdot e^{-\beta'} + 1.$$

The hypothesis that $A$ and $B$ are suspensions implies that all the cup products except $s^* \{\langle e^\alpha, e^\beta \rangle + [\alpha, \beta]\} \cdot 1 \cdot 1$ vanish, since they involve the diagonal in $A$ or in $B$. The remaining terms are those we need.

*The Hilton-Milnor theorem.* To state the Hilton-Milnor theorem precisely, we need a certain amount of formal algebra. We shall consider from now on the wedge $B = B_1 \vee B_2 \vee \cdots \vee B_k$ of connected CW-complexes, and a finite CW-complex $A$. We shall eventually assume that each $B_r$ is a suspension, in order to simplify the theorems and the proofs.

Take abstract symbols $z_1, z_2, \ldots, z_k$, and let

$L$ be the free Lie algebra (over $\mathbf{Z}$) generated by the letters $z_1, z_2, \ldots, z_k$;

$U$ be the free associative algebra on $z_1, z_2, \ldots, z_k$;

$M$ be the set of monomials in $U$, which is the free monoid on the letters $z_1, z_2, \ldots, \ldots, z_k$; and

$F$ be the free non-associative algebraic object generated by $z_1, z_2, \ldots, z_k$, with one binary operation. The *weight* $\mathrm{wt}(a)$ of an element $a$ in $M$ or $F$ is the number of factors in it. $F$ is often called the set of *formal commutators* in the letters $z_1, z_2, \ldots, z_k$. There are obvious homomorphisms $F \to L$ and $F \to M \subset U$, which we suppress from our notation, obtained by taking the binary operation in $F$ as [, ] or as multiplication.

It is customary to make $U$ into a Lie algebra by setting $[x, y] = xy - yx$; then there is a homomorphism $\kappa: L \to U$ of Lie algebras sending each $z_j$ to $z_j$. The Poincaré-Birkhoff-Witt theorem asserts in this case that $U$ is the universal enveloping algebra of $L$, and that $\kappa$ embeds $L$ as a direct summand (considered as additive groups) of $U$. Hence $L$ is free abelian, and recipes for a base are available (e.g. [12]).

By induction on weight, we define for each $a \in M (a \neq 1)$,

$$\Lambda^c B = \begin{cases} B_r & \text{if} \quad c = z_r, \\ \Lambda^a B \wedge \Lambda^b B & \text{if} \quad c = a\,b; \end{cases}$$

and iterated Whitehead products $\iota_c \in [E\Lambda^c B, EB]$, for each $c \in F$, by

$$\iota_c = \begin{cases} \iota_r, & \text{the class of the inclusion} \quad EB_r \subset EB, \quad \text{if} \quad c = z_r, \\ [\iota_a, \iota_b] & \text{if} \quad c = a\,b. \end{cases}$$

Given any family $(P_\alpha)$ of spaces with basepoint, we denote by $\prod_\alpha P_\alpha$ the *restricted product* of the $P_\alpha$, which is the union of all the finite subproducts of the cartesian product. We give this space the direct limit topology (rather than the cartesian product topology), in which a function in $\prod_\alpha P_\alpha$ is continuous if and only if it is continuous on every finite subproduct. We can at last state the Hilton-Milnor theorem in a suitable form (compare [12], [20], [5], [28]). (The methods of [21] show that each space involved has the homotopy type of a CW-complex; so that a singular homotopy equivalence is a homotopy equivalence.)

THEOREM 4.7. (HILTON-MILNOR). *Suppose the subset $Q$ of $F$ yields a base of $L$, and give $Q$ any total ordering. Then the map*

$$\prod_{c \in Q} \Omega\, i_c : \prod_{c \in Q} \Omega E \Lambda^c B \to \Omega E B, \qquad (i_c \in \iota_c)$$

*defined by using the multiplication in $\Omega EB$ in the order indicated by $Q$, is a homotopy equivalence.*

If $c$ has weight $n$, the space $E\Lambda^c B$ is $n$-connected, because $B$ is connected. It is possible to deduce for any CW-complex $Y$ an isomorphism of sets

$$[EY, EB] \cong \prod_{c \in Q} [EY, E\Lambda^c B],$$

which becomes an isomorphism of groups when $Y$ is a suspension. We have the projection to the $c$-th factor

$$h_c : [EY, EB] \to [EY, E\Lambda^c B], \tag{4.8}$$

which is a homomorphism when $Y$ is a suspension. Suppose now that $A$ is a finite

CW-complex (or even finite-dimensional). Then the operations $h_c$ can be described more simply. Take $\beta\in[EA, EB]$; then

$$\beta=\sum_{c\in Q}\iota_c\circ h_c\beta. \tag{4.9}$$

The terms must remain in the correct order, that given by the ordering on $Q$. Once $Q$ has been chosen, the particular Whitehead products $\iota_c$ for which $c\in Q$ are called the *basic* Whitehead products.

Now suppose that $B_1=B_2=\cdots=B_k=C$. Then $B=C\vee C\vee\cdots\vee C$.

DEFINITION 4.10. Given $\alpha\in[EA, EC]$, the *Hilton-Hopf invariants* $H_c\alpha\in[EA, E\Lambda^m C]$ $(m=\mathrm{wt}(c))$ are defined by $H_c\alpha=h_c(\rho_k\circ\alpha)$, where

$$\rho_k=\iota_1+\iota_2+\cdots+\iota_k\in[EC, E(C\vee C\vee\cdots\vee C)]$$

is the class of the pinch map 1.1.

For these operations we have the defining relation

$$(\iota_1+\iota_2+\cdots+\iota_k)\circ\alpha=\sum_{c\in Q}H_c\alpha\circ\iota_c. \tag{4.11}$$

Of course, the element $H_c\alpha$ depends in general on the choice of the ordered base $Q$, and not merely on $c$.

We propose to apply $\lambda_n$ to each side of 4.11, with the help of 3.16 and 4.6. *For the rest of this section we assume that A and C are suspensions.*

We first compute $\lambda_n\iota_c$. Define, by induction on weight, iterated smash commutators 4.5 $\omega_c\in[E^n\Lambda^c B, \Lambda^n EB]$, where $c\in F$ and $n=\mathrm{wt}(c)$, by

$$\omega_c=\begin{cases}\iota_r & \text{if} \quad c=z_r,\\ \langle\omega_a,\omega_b\rangle & \text{if} \quad c=ab.\end{cases}$$

We shall also write $\lambda\alpha$ rather than $e^\alpha$ for the exponential Hopf invariant 2.4 of $\alpha$.

LEMMA 4.12. *If $c=z_r$, then $\lambda\iota_c=1+\iota_c$. If $\mathrm{wt}(c)\geq 2$, then $\lambda\iota_c=1+\iota_c+\omega_c$.*

*Proof.* We proceed by induction on weight. We have $\lambda\iota_c=1+\iota_c$ if $c$ has weight 1, by 2.5. Assume the result for $\iota_a$ and $\iota_b$. Then by 4.6 $\lambda\iota_{ab}$ is given by one of the four formulae,

$$\lambda\iota_{ab}=\begin{cases}1+\iota_{ab}+\langle 1+\iota_a, 1+\iota_b\rangle & (\text{if } \mathrm{wt}(a)=\mathrm{wt}(b)=1)\\ 1+\iota_{ab}+\langle 1+\iota_a, 1+\iota_b+\omega_b\rangle & (\text{if } \mathrm{wt}(a)=1, \mathrm{wt}(b)>1)\\ 1+\iota_{ab}+\langle 1+\iota_a+\omega_a, 1+\iota_b\rangle & (\text{if } \mathrm{wt}(a)>1, \mathrm{wt}(b)=1)\\ 1+\iota_{ab}+\langle 1+\iota_a+\omega_a, 1+\iota_b+\omega_b\rangle & (\text{if } \mathrm{wt}(a)>1, \mathrm{wt}(b)>1).\end{cases}$$

Any smash commutator of the form $\langle 1,\alpha\rangle$ or $\langle\alpha,[\beta,\gamma]\rangle$, where $\alpha\in[EX, EY]$, vanishes (since $E[\beta,\gamma]=0$); hence in all cases the third term reduces to $\omega_{ab}$, which proves the lemma.

We need to rewrite $\omega_c$ in a more algebraic form. Grade the algebra $U$ by writing $U_n$ for the subgroup of homogeneous elements of weight $n$. Let $\mathscr{S}_n$ be the permutation group on $n$ symbols, with $\circ$ as multiplication, which will act on $\Lambda^n EB$ by permuting the factors, and $G_n = \mathbf{Z}[\mathscr{S}_n]$ its integral group-ring. Then the identity $\Lambda^{m+n}EB \cong \Lambda^m EB \wedge \Lambda^n EB$ induces a homomorphism of groups $\mathscr{S}_m \times \mathscr{S}_n \to \mathscr{S}_{m+n}$, and hence $G_m \otimes G_n \to G_{m+n}$. These maps make the additive groups $G_n$ into a graded ring $G_*$, quite apart from the composition products $\circ$ in each $G_n$. Construct a new graded ring $V_*$ by defining $V_n = U_n \otimes G_n$ for each $n \geq 0$; $V_n$ is also a $(G_n, \circ)$-module. If $a, b \in F$ have weights $m$ and $n$ respectively, define $\eta(a, b) \in \mathscr{S}_{m+n}$ as the permutation sending $j$ to $j+n$ (if $j \leq m$) or $j-m$ (if $j > m$). Then we define, by induction on weight, elements $u_c \in U$ and $v_c \in V$ for each $c \in F$, by

$$\begin{cases} u_c = z_r, & v_c = z_r, \qquad \text{if} \quad c = z_r, \\ u_c = u_a u_b - u_b u_a, & v_c = v_a v_b - \eta(b, a) \circ v_b v_a \quad \text{if} \quad c = ab. \end{cases}$$

The augmentations $\varepsilon_n : G_n = \mathbf{Z}[\mathscr{S}_n] \to \mathbf{Z}$ induce the augmentation $\varepsilon : V_* \to U_*$ of graded rings. Clearly $\varepsilon v_c = u_c$ for all $c \in F$, by induction on weight.

Denote by $v_c(\iota)$ the element of $[E^n \Lambda^c B, \Lambda^n EB]$ obtained by replacing $z_r$ by $\iota_r$ for each $r$, and multiplication by smash product, where $n = \mathrm{wt}(c)$. Then our observation is that by induction on weight we have

$$\omega_c = v_c(\iota),$$

for all $c \in F$.

The map sending $c$ to $u_c$ extends to the additive map $\kappa : L \to U$ that embeds $L$ in its universal enveloping algebra $U$. The Poincaré-Birkhoff-Witt theorem asserts that if the ordered subset $Q$ of $F$ yields a $\mathbf{Z}$-base of $L$, then the elements $u_{q_1} u_{q_2} \dots u_{q_m}$, where $q_1 \leq q_2 \leq \cdots \leq q_m$ in $Q$ and $m \geq 0$, form a base of $U$. An analogous proof (formally similar to that of Theorem 3.2 of [12]) shows that a corresponding result holds for $V$, as follows.

LEMMA 4.13. *Suppose the ordered subset $Q \subset F$ yields a base of $L$. Then the elements $v_{q_1} v_{q_2} \dots v_{q_m}$ of weight $n$, such that $q_1 \leq q_2 \leq \cdots \leq q_m$ in $Q$, form a $\mathbf{Z}[\mathscr{S}_n]$-base of the $(\mathbf{Z}[\mathscr{S}_n], \circ)$-module $V_n$.*

We are assuming that $B_1 = B_2 = \cdots = B_k = C$, so that $B = C \vee C \vee \cdots \vee C$. By 2.3, since $A$ is a suspension, $\lambda_n : [EA, EB] \to [E^n A, \Lambda^n EB]$ is a homomorphism. Also, since $C$ is a suspension, the pinch class $\rho_k \in [EC, EB]$ is a suspension. Thus $\lambda_n$, applied to 4.11, gives

$$(\rho_k \wedge \rho_k \wedge \cdots \wedge \rho_k) \circ \lambda_n \alpha = \sum_{c \in Q} \lambda_n (\iota_c \circ H_c \alpha).$$

For each monomial $a \in M$ of weight $n$, we have the obvious projection map

$$p_a : \Lambda^n EB \to \Lambda^a EB \cong \Lambda^n EC,$$

where $\Lambda^a EB$ is defined in a similar way to $\Lambda^a B$. Then composition with $\pi_a$, the class of $p_a$, yields

$$\lambda_n \alpha = \sum_{c \in Q} \pi_a \circ \lambda_n (\iota_c \circ H_c \alpha). \tag{4.14}$$

Assuming that $n \geq 2$, we see from 4.12 and the composition formula 3.16 that $\lambda_n(\iota_c \circ H_c \alpha) = 0$ unless the weight of $c$ divides $n$. If $\mathrm{wt}(c) = m$ and $n = rm$, we find

$$\lambda_n(\iota_c \circ H_c \alpha) = (-1)^e \Lambda^r(v_c(\iota)) \circ E^{n-r} \lambda_r H_c \alpha, \tag{4.15}$$

where $e = \frac{1}{2}(r-1)(n-r)$, and the sign comes from 3.16. (We have concealed a shuffle.) Now for each $c \in Q$ of weight $n$, let $v_c: V_n \to \mathbf{Z}[\mathscr{S}_n]$ be the $(\mathbf{Z}[\mathscr{S}_n]\circ,)$-module homomorphism that picks out the coefficient of $v_c$ in the $\mathbf{Z}[\mathscr{S}_n]$-base of $V_n$ given by 4.13. We deduce

$$E^{n-1} H_c \alpha = \sum_a v_c(a \otimes 1) \circ \lambda_q \alpha, \tag{4.16}$$

where $\mathscr{S}_n$ also acts on $\Lambda^n EC$ by permuting the factors, and we sum over the monomials $a \in M$ of weight $n$.

In [3], Barcus and Barratt pick out the particular commutators $\sigma_n = [[\ldots [[z_2, z_1], z_1], \ldots], z_1]$ of weight $n \geq 1$, with $n-1$ entries $z_1$. They suppose that $k=2$, and that the ordered base $Q$ contains all the $\sigma_n$. Write $H_n$ for the corresponding Hilton-Hopf invariant $H_{\sigma_n}$, which still depends on $Q$. (To some extent the elements $\sigma_n$ are canonical: if one orders $F$ arbitrarily subject only to the conditions (i) $\mathrm{wt}(a) < \mathrm{wt}(b)$ implies $a < b$, and (ii) $z_1 < z_2$, and picks out the corresponding set of basic commutators as in [12] or [28], then each $\sigma_n$ will be contained in every such set.) In 4.14 take $a = z_2 z_1^{n-1}$. Since $Q$ can obtain only one element with $n-1$ factors $z_1$ and one factor $z_2$, only one term of 4.15 survives substitution into 4.14, and we find $\lambda_n \alpha = E^{n-1} H_n \alpha$.

Let us summarize.

Theorem 4.17. *If $A$ and $C$ are suspensions, and $\alpha \in [EA, EC]$, then*

(a) *$\lambda_n \alpha = E^{n-1} H_n \alpha$ for $n \geq 2$, where $H_n$ is the Hilton invariant corresponding to $\sigma_n$ as above.*

(b) *For every basic commutator $c$ of weight $n$, $E^{n-1} H_c \alpha$ is expressed in terms of $\lambda_n \alpha$ and permutations by the formula* 4.16.

Now we know from 3.15 that $\lambda_n \alpha = E^{n-1} \gamma_n \alpha$, where $\gamma_n$ is the James-Hopf invariant. Thus we can relate the James-Hopf invariants $\gamma_n$ to the Hilton-Hopf invariants $H_c$.

Theorem 4.18. *If $A$ and $C$ are suspensions, and $\alpha \in [EA, EC]$, then*

(a) *If the ordered base $Q$ contains the commutators $\sigma_n$, then*

$$E^{n-1} \gamma_n \alpha = E^{n-1} H_n \alpha \quad \textit{for} \quad n \geq 2,$$

(b) *For every basic commutator $c$ of weight $n$, $E^{n-1} H_c \alpha$ can be expressed in terms of $E^{n-1} \gamma_n \alpha$ and permutations, by the formula* 4.16.

This theorem is well known, and can be desuspended, as the assertion that the Hilton-Hopf invariants and the James-Hopf invariants determine each other. The first proof, of the desuspended theorem, was given by BARRATT [6]. The desuspension of (a) is Lemma 3.12 of [29].

When $A$ and $C$ are not suspensions, a similar result, with numerous extra terms involving cup products of terms of lower weight, can be proved in exactly the same way. The cup products vanish in the case of 4.18, by 1.4.

## 5. A geometric invariant

In this section we construct a sequence of homotopy operations by writing down explicit maps. We prove that they form a Hopf ladder, and hence provide a geometric interpretation of the suspended James-Hopf invariants. The second, $\lambda_2$, is closely related to the generalized Hopf invariant $H^*$ given by HILTON [11]. We show that $\lambda_2$ also includes the functional cup products, and is therefore related to STEENROD's cohomology definition of the classical Hopf invariant [27].

Before we construct the operations $\lambda_n$, we construct a sequence of operations $\mu_n$, which is slightly more general, and is technically more convenient in certain proofs, but seems to lack independent interest.

We consider $n$ spaces $B_1, B_2, \ldots, B_n$, instead of a single space $B$. We recall from § 1 the identification maps $s: A \times \mathbf{R}^m \to E^m A$.

DEFINITION 5.1. Given any map (based, of course)

$$f: EA \to B_1 \vee B_2 \vee \cdots \vee B_n \quad (n \geq 0)$$

we define a new map

$$\mu_n f: E^n A \to B_1 \wedge B_2 \wedge \cdots \wedge B_n$$

as follows. Put $f_j = p_j \circ f \circ s: A \times \mathbf{R} \to B_j$. Define

$$q: A \times \mathbf{R}^n \to B_1 \wedge B_2 \wedge \cdots \wedge B_n$$

by the formula

$$q(a, t_1, t_2, \ldots, t_n) = \begin{cases} f_1(a, t_1) \wedge f_2(a, t_2) \wedge \cdots \wedge f_n(a, t_n) \\ \qquad \text{if} \quad t_1 \leq t_2 \leq \cdots \leq t_n, \\ o \quad \text{otherwise.} \end{cases} \tag{5.2}$$

Then $q$ is continuous, for if $t_j = t_{j+1}$, one at least of $f_j$ and $f_{j+1}$ is zero at $(a, t_j)$, and hence $q$ is zero there. We also observe that $q$ vanishes whenever any $t_j \leq 0$ or $t_j \geq 1$; therefore $q$ factors through the identification map $s: A \times \mathbf{R}^n \to E^n A$, to yield the required map $\mu_n f$.

LEMMA 5.3. *The homotopy class of $\mu_n f$ depends only on the homotopy class of $f$, and we therefore have a homotopy operation*

$$\mu_n\colon [EA, B_1 \vee B_2 \vee \cdots \vee B_n] \to [E^n A, B_1 \wedge B_2 \wedge \cdots \wedge B_n],$$

*natural in* $A$, $B_1$, $B_2$, ..., $B_n$.

*Proof.* We have merely to apply the formula 5.2 to a homotopy $f_t$ of $f$ to obtain a homotopy $\mu_n f_t$ of $\mu_n f$. Naturality is clear.

DEFINITION 5.4. Given a map $f\colon EA \to EB$ and $n>0$, define the map $\lambda_n f\colon E^n A \to \Lambda^n EB$ by $\lambda_n f = \mu_n(\bar{r}_n \circ f)$, where $\bar{r}_n\colon EB \to EB \vee EB \vee \cdots \vee EB$ is the *backward* pinch map 1.2. Hence we have the operation

$$\lambda_n\colon [EA, EB] \to [E^n A, \Lambda^n EB]$$

on homotopy classes, which may be regarded as the composite operation

$$[EA, EB] \xrightarrow[\bar{r}_{n*}]{} [EA, EB \vee EB \vee \cdots \vee EB] \xrightarrow[\mu_n]{} [E^n A, \Lambda^n EB].$$

REMARK 5.5. We note that from $\lambda_n$ we can recover a particular case of the operation $\mu_n$, namely

$$\mu_n\colon [EA, EB_1 \vee EB_2 \vee \cdots \vee EB_n] \to [E^n A, EB_1 \wedge EB_2 \wedge \cdots \wedge EB_n].$$

For put $B = B_1 \vee B_2 \vee \ldots \vee B_n$, and take $\alpha \in [EA, EB]$. Then by naturality

$$\begin{aligned}\mu_n \alpha &= \mu_n \{(E\pi_1 \vee E\pi_2 \vee \cdots \vee E\pi_n) \circ \bar{\rho}_n \circ \alpha\} \\ &= (E\pi_1 \wedge E\pi_2 \wedge \cdots \wedge E\pi_n) \circ \mu_n(\bar{\rho}_n \circ \alpha) \\ &= (E\pi_1 \wedge E\pi_2 \wedge \cdots \wedge E\pi_n) \circ \lambda_n \alpha,\end{aligned}$$

where $\pi_j$ is the class of the projection $p_j\colon B \to B_j$.

THEOREM 5.6. *These operations $\lambda_n$ form a Hopf ladder.*

Hence by 2.2 and 3.15 we have an interpretation of the suspended James-Hopf invariant $E^{n-1}\gamma_n$, when $A$ is a finite-dimensional CW-complex. (Actually, the dimensional restriction is unnecessary, but we shall not prove this here.)

The main work in proving 5.6 is in establishing the Cartan formula. We shall deduce it from Cartan formula for $\mu_n$.

We need a well-known lemma for adding homotopy classes, which we state without proof.

LEMMA 5.7. *Suppose the classes $\alpha_i \in [E^n A, X]$ are represented by maps $g_i\colon E^n A \to X$, for $1 \le i \le k$, where $n>1$. Suppose the support* (see § 1) *of $g_i \circ s\colon A \times \mathbf{R}^n \to X$ is contained in $A \times D_i$, where the sets $D_i$ $(1 \le i \le k)$ are convex subsets of $\mathbf{R}^n$ whose interiors are disjoint. The the track sum $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is represented by the map $g\colon E^n A \to X$ defined as follows: $ga = g_i a$ if $g_i a \ne o$, and $ga = o$ if $g_i a = o$ for all $i$ ($\alpha \in E^n A$). The result holds even for $n=1$, provided the sets $D_i \subset \mathbf{R}$ occur in the correct increasing order.*

This lemma is quite false if $D_i$ is not required to be convex, for then linking can occur (see § 6).

There are obvious projection maps ($1 \leq j \leq n-1$)

$$B_1 \vee B_2 \vee \cdots \vee B_n \to B_1 \vee B_2 \vee \cdots \vee B_j, \; B_1 \vee B_2 \vee \cdots \vee B_n \to B_{j+1} \vee B_{j+2} \vee \cdots \vee B_n,$$

which induce homomorphisms of track groups

$$L_j : [EA, B_1 \vee B_2 \vee \cdots \vee B_n] \to [EA, B_1 \vee B_2 \vee \cdots \vee B_j]$$

and

$$R_{n-j} : [EA, B_1 \vee B_2 \vee \cdots \vee B_n] \to [EA, B_{j+1} \vee B_{j+2} \vee \cdots \vee B_n].$$

LEMMA 5.8. *Suppose* $\alpha, \beta \in [EA, B_1 \vee B_2 \vee \cdots \vee B_n]$. *Then*

$$\mu_n(\alpha + \beta) = \mu_n \alpha + \mu_{n-1} L_{n-1} \alpha \cdot \mu_1 R_1 \beta + \mu_{n-2} L_{n-2} \alpha \cdot \mu_2 R_2 \beta + \cdots$$
$$\cdots + \mu_1 L_1 \alpha \cdot \mu_{n-1} R_{n-1} \beta + \mu_n \beta.$$

*Proof.* We may choose $f \in \alpha$ and $g \in \beta$ such that $f \circ s$ has support in $A \times [0, \frac{1}{2}]$ and $g \circ s$ has support in $A \times [\frac{1}{2}, 1]$. Then by 5.7 we may represent $\alpha + \beta$ by the map $k$, where $k \circ s$ agrees with $f \circ s$ on $A \times [0, \frac{1}{2}]$ and with $g \circ s$ on $A \times [\frac{1}{2}, 1]$.

We have to consider the map $q : A \times \mathbf{R}^n \to B_1 \wedge B_2 \wedge \cdots \wedge B_n$ defined by

$$q(a, t_1, t_2, \ldots, t_n) = \begin{cases} k_1(a, t_1) \wedge k_2(a, t_2) \wedge \cdots \wedge k_n(a, t_n) \\ \qquad\qquad \text{when} \quad t_1 < t_2 < \cdots < t_n \\ o \quad \text{otherwise}, \end{cases} \tag{5.9}$$

where $k_i = p_i \circ k \circ s$. This map represents $\mu_n(\alpha + \beta)$ apart from identification. We see that $q$ is zero on each of the hyperplanes $t_j = \frac{1}{2}$. These hyperplanes divide the region in $\mathbf{R}^n$ satisfying 5.9 into various convex subsets. Consider that on which

$$t_1 < t_2 < \cdots < t_j < \tfrac{1}{2} < t_{j+1} < \cdots < t_n.$$

Let $q_j$ agree with $q$ on this set, and be zero outside. Then

$$q_j(a, t_1, t_2, \ldots, t_n) = f_1(a, t_1) \wedge f_2(a, t_2) \wedge \cdots \wedge f_j(a, t_j)$$
$$\wedge g_{j+1}(a, t_{j+1}) \wedge \cdots \wedge g_n(a, t_n),$$

in which $f_i = p_i \circ f \circ s$ and $g_i = p_i \circ g \circ s$, subject to certain inequalities which, owing to the special form of $f$ and $g$, we may write as

$$t_1 < t_2 < \cdots < t_j \quad \text{and} \quad t_{j+1} < t_{j+2} < \cdots < t_n.$$

Thus $q_j$, after identification, represents $\mu_j L_j \alpha \cdot \mu_{n-j} R_{n-j} \beta$.

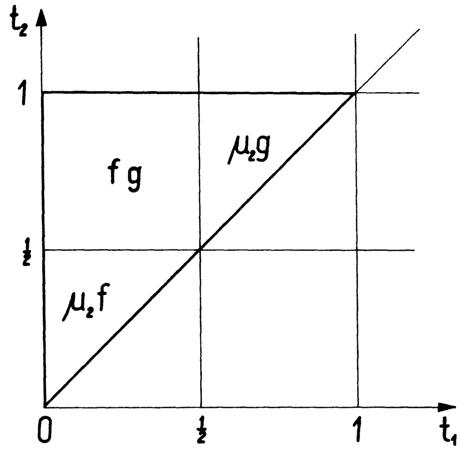
The lemma now follows by applying 5.7 to the maps $q_j$, for $0 \leq j \leq n$.

The case $n = 2$ is illustrated in the figure overleaf.

*Proof of* 5.6. We must verify the axioms 2.1 for the operations $\lambda_n$.

The identity axiom (a) holds, trivially.

The Cartan formula (c) follows from 5.8, when we observe that $L_j \bar{\rho}_n = \bar{\rho}_j$ and $R_j \rho_n = \bar{\rho}_j$.




We verify the normalization axiom (b) by showing that we actually have $\lambda_n(Ef)=0$ for any map $f:A\to B$ when $n\geq 2$. For $\lambda_n(Ef)$ is obtained by identification from the map $q:A\times\mathbf{R}^n\to\Lambda^n EB$ given by 5.2, namely

$$q(a,t_1,t_2,\dots,t_n)=\begin{cases} f_1(a,t_1)\wedge f_2(a,t_2)\wedge\cdots\wedge f_n(a,t_n) \\ \qquad\qquad \text{when} \quad t_1\leq t_2\leq\cdots\leq t_n, \\ o \quad \text{otherwise}, \end{cases}$$

where $f_j=p_j\circ\bar{r}_n\circ Ef\circ s:A\times\mathbf{R}\to EB$. In this case the support of $f_j$ is contained in $A\times[(n-j)/n,\ (n-j+1)/n]$, by 1.2. Hence $q(a,t_1,t_2,\dots,t_n)\neq o$ only if $n-j<nt_j<n-j+1$ for all $j$, which contradicts $t_1\leq t_2\leq\cdots\leq t_n$ if $n\geq 2$. Therefore $q$, and $\lambda_n(Ef)$, are zero if $n\geq 2$.

This completes the proof of Theorem 5.6.

Remark. The normalization axiom would not hold if we had used the ordinary pinch map instead of the backward pinch map in 5.4.

We next recall another generalized Hopf invariant, and show that it is included in $\lambda_2$.

Take a CW-complex $B$. For $k\geq 1$ the homotopy exact sequence of the pair $(EB\times EB,\ EB\vee EB)$ splits, to yield the short exact sequence

$$0\to\pi_{k+1}(EB\times EB,\ EB\vee EB)\xrightarrow[\partial]{}\pi_k(EB\vee EB)\xrightarrow[j_*]{}\pi_k(EB\times EB)\to 0. \tag{5.10}$$

Given an element $\alpha\in\pi_k(EB)$, we can form

$$\bar{p}_2\circ\alpha-\iota_1\circ\alpha-\iota_2\circ\alpha\in\pi_k(EB\vee EB),$$

which evidently lies in the kernel of $j_*$, since $\pi_k(EB\times EB)\cong\pi_k(EB)\oplus\pi_k(EB)$. It therefore lifts uniquely to $\pi_{k+1}(EB\times EB, EB\vee EB)$. We have also, by identification, a map $\pi_{k+1}(EB\times EB, EB\vee EB)\to\pi_{k+1}(EB\wedge EB)$. These yield the operation

$$H^*:\pi_k(EB)\to\pi_{k+1}(EB\wedge EB). \tag{5.11}$$

This map is called a generalized Hopf invariant by HILTON [11] in the case when $B$ is a sphere. (When $B$ is a suspension, it does not matter whether we use the usual pinch class $\rho_2$ or the backward pinch class $\bar{\rho}_2$, because they coincide.)

THEOREM 5.12. *We have*

$$\lambda_2=-H^*:\pi_k(EB)\to\pi_{k+1}(EB\wedge EB).$$

*Proof.* We put $A=\Sigma^{k-1}=D^{k-1}/S^{k-2}$ in the definition 5.4 of $\lambda_2$. Given $f:EA\to EB$, we constructed $\lambda_2 f$ by means of a map

$$q:(A\times T, A\times\partial T)\to(EB\times EB, EB\vee EB),$$

followed by identification, where $T$ is the triangle in $\mathbf{R}^2$ given by $0\le t_1\le t_2\le 1$, $\partial T$ is its boundary, and

$$q(a,t_1,t_2)=\big(f_1(a,t_1), f_2(a,t_2)\big).$$

The three sides of the triangle yield $\bar{r}_2\circ f$, $i_1\circ f_1$, and $i_2\circ f_2$. Hence we have here the construction for $H^*$, and the theorem is established, apart from the sign.

If we use the homotopy boundary convention (see Appendix), the three sides of $\Sigma^{k-1}\times\partial T$ become oriented so as to represent $\iota_1\circ\alpha$, $\iota_2\circ\alpha$, and $-\bar{\rho}_2\circ\alpha$, where $f\in\alpha\in[EA, EB]$. We therefore have $\lambda_2=-H^*$. (Use of a different boundary convention would result in a different sign.)

Finally we show that the Hopf invariant $\lambda_2$ induces important cases of the functional cup product described by STEENROD [27]. Take any map $f:EA\to EB$ of spaces with basepoint. We can form the *reduced mapping cone* $X=EB\bigcup_f TEA$ of $f$, where $TEA$ denotes the reduced cone obtained from $EA\times[0,1]$ by identifying $EA\times 1$ and $o\times[0,1]$ to the basepoint $o$, and we attach $TEA$ to $EB$ along $EA\times 0$ by $f$.

We know that the reduced diagonal of a suspension is nullhomotopic. This fact enables us to simplify the reduced diagonal $\Delta:X\to X\wedge X$ of $X$ by a homotopy.

More specifically, let us define explicit homotopies $g_u:EC\to EC$ and $k_u:EC\to EC$ by the formulae

$$\begin{cases} g_u s(c,t)=s(c,t+tu) \\ k_u s(c,t)=s(c,t+tu-u) \end{cases} \quad (0\le u\le 1;\, c\in C) \tag{5.13}$$

where $s:C\times\mathbf{R}\to EC$ stands for the usual identification map. Then $(g_u\wedge k_u)\circ\Delta:EC\to EC\wedge EC$ is a nullhomotopy of $\Delta$.

From these, we construct a homotopy $F_u: X \to X \wedge X$ in three stages, starting from $F_0 = \Delta$.

*First stage*: $0 \leq u \leq 1$.

On $EB$ we take the constant homotopy,

$$F_u = \Delta_{EB}: EB \to EB \wedge EB \subset X \wedge X.$$

On $s(EA \times [0, \frac{1}{2}])$ we take

$$F_u s(z, t) = s(k_{2tu} z, t) \wedge s(g_{2tu} z, t) \in X \wedge X \qquad (z \in EA).$$

On $s(EA \times [\frac{1}{2}, 1])$ we take

$$F_u s(z, t) = s(k_u z, t) \wedge s(g_u z, t) \in X \wedge X \qquad (z \in EA).$$

These fit together at $t=0$ and $t=\frac{1}{2}$ to define $F_u: X \to X \wedge X$ for $0 \leq u \leq 1$. We see from 5.13 that $F_1$ is zero on $s(EA \times [\frac{1}{2}, 1])$.

*Second stage*: $1 \leq u \leq 2$.

We note that the image of $F_1$ lies in $M \wedge M$, where $M = EB \cup s(EA \times [0, \frac{1}{2}])$. We may regard $M$ as the mapping cylinder of $f$. It contains $EB$ as a canonical deformation retract. For $F_u (1 \leq u \leq 2)$ we compose $F_1: X \to M \wedge M$ with a homotopy which starts with the identity map of $M \wedge M$ and ends with the canonical projection $M \wedge M \to EB \wedge EB$. We shall therefore find:

On $EB$, $F_2 = \Delta_{EB}$, still.

On $s(EA \times [0, \frac{1}{2}])$,

$$F_2 s(z, t) = f\, k_{2t} z \wedge f\, g_{2t} z \in EB \wedge EB \subset X \wedge X \qquad (z \in EA).$$

On $s(EA \times [\frac{1}{2}, 1])$, $F_2$ is zero.

*Third stage*: $2 \leq u \leq 3$.

We compose the factored map $F_2: X \to EB \wedge EB$ with the homotopy $k_{u-2} \wedge g_{u-2}$: $EB \wedge EB \to EB \wedge EB$, Thus $F_3$ is zero except on $s(EA \times [0, \frac{1}{2}])$, on which we have

$$F_3(z, t) = k_1 f\, k_{2t} z \wedge g_1 f\, g_{2t} z \qquad (z \in EA).$$

If we now compare $F_3$ with $\lambda_2 f$ by 5.2 and 5.4, we see that $F_3$ factors as $F_3 = \lambda_2 f \circ j$, where $j: E^2 A \to E^2 A / Y$ is the map given by

$$j\, s(a, v, t) = s(a, v + 2tv - 2t, v + 2tv), \qquad (a \in A, t \geq 0)$$

and $Y$ is the subset of $E^2 A$ given by $t_1 \geq t_2$ (on which $\lambda_2 f$ vanishes by 5.2). But it is easily seen that $j$ is homotopic to the identification map $E^2 A \to E^2 A / Y$. It follows that $F_3$ and $\lambda_2 f$ are homotopic. Let us state what this proves.

THEOREM 5.14. *Let $X = EB \bigcup_f TEA$ be the reduced mapping cone of a map $f: EA \to EB$. Then the reduced diagonal $\Delta: X \to X \wedge X$ is homotopic to the composite map*

$$X \to X/EB \cong E^2 A \xrightarrow[\lambda_2 f]{} EB \wedge EB \subset X \wedge X,$$

*where $X \to X/EB$ is the identification map.*

COROLLARY 5.15. *The map $\lambda_2 f$ induces the functional cup product in cohomology* (*up to sign*)

$$H^p(EB) \otimes H^q(EB) \to H^{p+q-1}(EA),$$

*with zero indeterminacy*!

*Proof.* We have essentially the definition in § 5 of [27] of the functional cup product, apart from the lack of indeterminacy.

We do not yet have a corresponding interpretation of $\lambda_n$ for $n>2$.

*Example.* Let us take $A=S^{2n-2}$, and $B=S^{n-1}$, and $f: S^{2n-1} \to S^n$. Then $\lambda_2 f: S^{2n} \to S^{2n}$ is a map, of degree $k$, say. $A$ and $B$ give rise to cohomology classes $x \in H^{2n}(X;\mathbf{Z})$ and $y \in H^n(X;\mathbf{Z})$. By 5.15 we have $x^2 = \pm ky$, which is one of the well-known definitions [27] of the Hopf invariant $k$ of $f$.

## 6. A geometric construction in framed cobordism

We know after PONTRJAGIN [23], KERVAIRE [18], etc. how to interpret the homotopy groups of spheres as framed-cobordism classes of framed smooth manifolds. KERVAIRE [18] and HAEFLIGER and STEER [10] gave geometric interpretations of the generalized Hopf invariant in this language. We show in this section that the geometric Hopf invariant $\lambda_n$ described in § 5 gives rise to a construction on framed submanifolds. In particular, for $n=2$, we find the construction of [10]. Again just as in [18], we can easily evaluate $\lambda_n$ on the image of the $J$-homomorphism. Finally we show how the geometric construction has already arisen in differential topology, together with several of its elementary properties.

All manifolds in this section will be smooth (in the sense $C^\infty$) and paracompact. Given a $m$-manifold $M$, and a $v$-submanifold $V$ of $M$ having codimension $k=m-v$, a *framing* of $V$ in $M$ is a sequence $\mathscr{F}=(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$ of sections of the normal bundle of $V$ in $M$ which are everywhere linearly independent. We then say that $V$ is a *framed submanifold* of $M$.

Now suppose that $V$ is compact, that its boundary $\partial V$ (if any) is $V \cap \partial M$, and that $V$ meets $\partial M$ transversely (see e.g. [30]). A suitable chosen tubular neighbourhood $N$ of $V$ in $M$ is diffeomorphic to $V \times D^k$ ($D^k$ being the standard closed $k$-disk) by means of the framing sections. The Pontrjagin-Thom construction [23], [30], associates to this tubular neighbourhood of $V$ the *Thom map* $M \to \Sigma^k = D^k/\partial D^k$ as follows: on $N$ we use the composite

$$N \cong V \times D^k \to D^k \to \Sigma^k,$$

which maps $\partial N$ to the basepoint $o$, and outside $N$ we take the zero map. This map has compact support, and therefore extends to a based map $M_c \to \Sigma^k$, where $M_c$ denotes the one-point compactification of $M$. The extra point in $M_c$, which we call $\infty$, is taken as the basepoint of $M_c$. (If $M$ is already compact, $M_c$ must be taken as the disjoint union of $M$ and a point $\infty$, for consistency.)

Suppose $M$ is without boundary. Two compact framed submanifolds $V_0$ and $V_1$ of $M$ are said to be *framed-cobordant* if there exists a compact framed submanifold $W$ of $M \times [0, 1]$ such that $V_i = W \cap (M \times i)$ $(i = 0, 1)$, and the framing of $V_i$ in $M$ is obtained by restriction from the framing of $W$ in $M \times [0, 1]$. This is an equivalence relation. The equivalence classes are called *framed-cobordism classes*. The fundamental result of THOM [30] implies the following as a special case.

THEOREM 6.1. *Let $M$ be a smooth manifold without boundary. The Pontrjagin-Thom construction induces an isomorphism between the set of framed-cobordism classes of compact framed submanifolds of $M$ with codimension $k$, and $[M_c, \Sigma^k]$.*

As an alternative notation to $[M_c, \Sigma^k]$, we write $\pi^k(M, \infty)$, and call it the $k$-th *compact cohomotopy set* of $M$. More generally, if $M$ has a boundary $\partial M$, we define $\pi^k(M, \partial M, \infty) = [M_c/(\partial M)_c, \Sigma^k]$, and the theorem extends in a suitable sense to $\pi^k(M, \infty)$ and $\pi^k(M, \partial M, \infty)$ (see below).

As observed above, this theorem enables us to translate results about framed submanifolds of a given manifold $M$ into results about the compact cohomotopy sets of $M$, and vice versa. We give a glossary of the commonest terms.

*Suspension.* Since $(A \times B)_c \cong A_c \wedge B_c$, and we can identify $\mathbf{R}_c$ with $\Sigma^1$ canonically up to homotopy, we have the Freudenthal suspension map

$$E: \pi^k(M, \infty) \to \pi^{k+1}(M \times \mathbf{R}, \infty).$$

If $\alpha \in \pi^k(M, \infty)$ is represented by the framed submanifold $V$ of $M$, $E\alpha$ is represented by the submanifold $V \subset M \subset M \times \mathbf{R}$. To frame $V$ in $M \times \mathbf{R}$, we take the framing of $V$ in $M$, *followed* by the positive unit section of the normal bundle of $M$ in $M \times \mathbf{R}$ (as in [18]).

*Isotopy.* If the framed compact submanifolds $V_1$ and $V_2$ of $M$, together with their framings, are isotopic, they represent the same element of $\pi^k(M, \infty)$, since isotopy may be regarded as a special kind of cobordism. Hence we may move submanifolds around in $M$ to suit our purposes.

*Track addition.* Now $\pi^k(M \times \mathbf{R}, \infty)$ is a group, by track addition. Suppose $\alpha, \beta \in \pi^k(M \times \mathbf{R}, \infty)$ are represented by the framed submanifolds $V$ and $W$ of $M \times \mathbf{R}$. Since $V$ and $W$ are compact, we can move them by isotopies until $V \subset M \times (-\infty, 0)$ and $W \subset M \times (0, \infty)$. Then $V \cup W$ is another compact framed submanifold of $M \times \mathbf{R}$ and represents $\alpha + \beta$ (compare 5.7). (It is easy to see geometrically that $\pi^k(M \times \mathbf{R}^2, \infty)$ is abelian.)

*Induced homomorphisms.* Let $f: N \to M$ be a proper map (i.e. $f^{-1}(K)$ is compact whenever the subset $K$ of $M$ is compact), so that $f$ extends to $f_c: N_c \to M_c$. Suppose $V$

represents $\alpha \in \pi^k(M,\infty)$. If $f$ is transverse to $V$ (which can be arranged without changing its homotopy class), $f^{-1}(V)$ is a compact submanifold of $N$, with framing induced from that of $V$ in $M$, and represents $f^*\alpha$.

*Reflection of the framing.* Suppose the framed submanifold $V$ of $M$ represents $\alpha \in \pi^k(M,\infty)$. If we change the framing $\mathscr{F}$ of $V$ in $M$ at each point $v \in V$ by a linear transformation $\sigma$ independent of $v$, then $V$, with the altered framing $\sigma\mathscr{F}$, represents $\alpha$ if the determinant of $\sigma$ is positive, or $U\alpha$ if the determinant of $\sigma$ is negative. (Here, $U$ is the operation on $\pi^k(M,\infty)$ obtained by composing with a map from $\Sigma^k$ to itself of degree $-1$; see § 1.)

*Products.* If the framed submanifolds $V \subset M$ and $W \subset N$, with framings $\mathscr{F} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$ and $\mathscr{G} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_l)$ respectively, represent $\alpha \in \pi^k(M,\infty)$ and $\beta \in \pi^l(N,\infty)$, their product $V \times W \subset M \times N$ represents $\alpha \wedge \beta \in \pi^{k+l}(M \times N, \infty)$, if we endow $V \times W$ with the *product framing*

$$\mathscr{F} \oplus \mathscr{G} = \left(p_1^* \mathbf{x}_1, p_1^* \mathbf{x}_2, \ldots, p_1^* \mathbf{x}_k, p_2^* \mathbf{y}_1, \ldots, p_2^* \mathbf{y}_l\right).$$

*Manifolds with boundary.* Suppose $M$ has boundary $\partial M$. An element of $\pi^k(M,\infty)$ is represented by a framed submanifold $V$ with boundary $\partial V = V \cap \partial M$. An element of $\pi^k(M, \partial M, \infty)$ is represented by a compact framed submanifold $V$ without boundary. (We can always move $V$ away from $\partial M$ if desired.)

*Exact sequences of a pair.* We have the exact sequence of the pair $(M, \partial M)$ (see [4] or [24])

$$\cdots \to \pi^k(\partial M \times \mathbf{R}, \infty) \xrightarrow[\partial]{} \pi^k(M, \partial M, \infty) \xrightarrow{i^*} \pi^k(M, \infty) \xrightarrow[j^*]{} \pi^k(\partial M, \infty).$$

The interpretation of $i^*$ is obvious. For $j^*$, we take the boundary $\partial V$ of a framed submanifold $V$ of $M$; by restriction $\partial V$ is framed in $\partial M$. Now $\partial M$ is collared in $M$, i.e. has a tubular neighbourhood $\partial M \times [0, 1]$, with $\partial M = \partial M \times 0$. Then $\partial M \times \mathbf{R} \cong \partial M \times (0, 1) \subset \partial M \times [0, 1] \subset M$, by means of an order-preserving diffeomorphism $\mathbf{R} \cong (0, 1)$. Hence a compact framed submanifold of $\partial M \times \mathbf{R}$ yields by inclusion a framed submanifold of $M$, not meeting $\partial M$. This interprets $\partial$.

*Cup product.* Let $V_i \subset M \times \mathbf{R}^{r_i}$, with framing $\mathscr{F}_i$, represent $\alpha_i \in \pi^{k_i}(M \times \mathbf{R}^{r_i}, \infty)$, for $1 \le i \le n$. In 1.3 we defined the cup product $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n \in \pi^k(M \times \mathbf{R}^r, \infty)$, where $r = r_1 + r_2 + \cdots + r_n$ and $k = k_1 + k_2 + \cdots + k_n$. We seek a framed compact submanifold that represents it.

Let $f_i : M \times \mathbf{R}^{r_i} \to \Sigma^{k_i} = D^{k_i}/\partial D^{k_i}$ be the Thom map of $V_i$. Let $u_i \in \mathbf{R}^{r_i}$ be a parameter. Then the map defining the cup product may be written as

$$M \times \mathbf{R}^r \xrightarrow[l]{} \Sigma^{k_1} \times \Sigma^{k_2} \times \cdots \times \Sigma^{k_n} \xrightarrow[s]{} \Sigma^{k_1} \wedge \Sigma^{k_2} \wedge \cdots \wedge \Sigma^{k_n} \cong \Sigma^k,$$

where

$$l(m, u_1, u_2, \ldots, u_n) = \left(f_1(m, u_1), f_2(m, u_2), \ldots, f_n(m, u_n)\right). \tag{6.2}$$

Let $b_i \in \Sigma^{k_i}$ be the image of the centre of $D^{k_i}$, so that $V_i = f_i^{-1}(b_i)$. Then *if* the map $l$ is transverse to $(b_1, b_2, \ldots, b_n)$, the theory of THOM [30] shows that $V = l^{-1}(b_1, b_2, \ldots, b_n)$, with the framing induced from that of $(b_1, b_2, \ldots, b_n)$ in $\Sigma^{k_1} \times \Sigma^{k_2} \times \cdots \times \Sigma^{k_n}$, will be a framed submanifold of $M \times \mathbf{R}^r$ representing $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$. Hence we need a condition to ensure the transversality of $l$.

DEFINITION 6.3. Let $g_i: W_i \to N$ be a smooth map of manifolds $(1 \leq i \leq n)$. We say these maps are *mutually transverse* if

$$g_1 \times g_2 \times \cdots \times g_n: W_1 \times W_2 \times \cdots \times W_n \to N \times N \times \cdots \times N$$

is transverse to the diagonal $\Delta N$ of $N \times N \times \cdots \times N$, where $\Delta N$ is the set of all points $(x, x, \ldots, x)$ for $x \in N$.

Let us write $q_i: M \times \mathbf{R}^{r_i} \to M$ for the projection.

LEMMA 6.4. *Suppose the maps $q_i | V_i: V_i \to M (1 \leq i \leq n)$ are mutually transverse. Then the map $l$ (see 6.2) is transverse to $(b_1, b_2, \ldots, b_n)$, and $l^{-1}(b_1, b_2, \ldots, b_n)$ is a smooth compact framed submanifold of $M \times \mathbf{R}^r$ representing the cup product $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$.*

*Proof.* We may express the definition of $l$ as a commutative diagram

$$\begin{array}{ccc} M \times \mathbf{R}^{r_1} \times \mathbf{R}^{r_2} \times \cdots \times \mathbf{R}^{r_n} & \xrightarrow[l]{} & \Sigma^{k_1} \times \Sigma^{k_2} \times \cdots \times \Sigma^{k_n} \\ \downarrow{\scriptstyle \Delta \times 1 \times 1 \times \cdots \times 1} & & \uparrow{\scriptstyle f_1 \times f_2 \times \cdots \times f_n} \\ M \times M \times \cdots \times M \times \mathbf{R}^{r_1} \times \mathbf{R}^{r_2} \times \cdots \times \mathbf{R}^{r_n} & \cong & (M \times \mathbf{R}^{r_1}) \times (M \times \mathbf{R}^{r_2}) \times \cdots \times (M \times \mathbf{R}^{r_n}). \end{array}$$

By construction, $f_1 \times f_2 \times \cdots \times f_n$ is transverse to $(b_1, b_2, \ldots, b_n)$, and

$$V_1 \times V_2 \times \cdots \times V_n = (f_1 \times f_2 \times \cdots \times f_n)^{-1}(b_1, b_2, \ldots, b_n).$$

In $M \times M \times \cdots \times M \times \mathbf{R}^r$, we need to have $\Delta M \times \mathbf{R}^r$ transverse to $V_1 \times V_2 \times \cdots \times V_n$, or equivalently, $V_1 \times V_2 \times \cdots \times V_n$ transverse to $\Delta M \times \mathbf{R}^r$. This in turn is equivalent to having the projection map $V_1 \times V_2 \times \cdots \times V_n \to M \times M \times \cdots \times M$ transverse to $\Delta M$, which is precisely what we have assumed. Thus we have the required framed submanifold.

It is also necessary to know that there are enough sets of maps realizing the condition of 6.4.

LEMMA 6.5. *Given any smooth maps $g_i: W_i \to N (1 \leq i \leq n)$, there exist new maps $g_i': W_i \to N$ which are arbitrarily close (in the $C^p$ sense, for any integer $p$) to $g_i$, and mutually transverse.*

*Proof.* This is an easy consequence of work of THOM [31].

It follows that we can always move the manifolds $V_i$ by small isotopies so as to make the projections $q_i | V_i: V_i \to M$ mutually transverse. In this case we can construct the submanifold $W$ of $M \times \mathbf{R}^r$ representing $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$ directly, by the condition: $(m, u_1, u_2, \ldots, u_n) \in W$ if and only if

$$(m, u_i) \in V_i \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

We can frame $W$ canonically, by referring back to $l$. This framing is the restriction to $W$ of the product framing $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \cdots \oplus \mathscr{F}_n$ of $V_1 \times V_2 \times \cdots \times V_n$ in

$$(M \times \mathbf{R}^{r_1}) \times (M \times \mathbf{R}^{r_2}) \times \cdots \times (M \times \mathbf{R}^{r_n}).$$

*Hopf invariants.* Our main object in this section is to interpret the Hopf invariant

$$\lambda_n : \pi^k(M \times \mathbf{R}, \infty) \to \pi^{nk}(M \times \mathbf{R}^n, \infty),$$

in the geometric form given in § 5. To do this, we shall first interpret the operation

$$\mu_n : [(M \times \mathbf{R})_c, \Sigma^{k_1} \vee \Sigma^{k_2} \vee \cdots \vee \Sigma^{k_n}] \to \pi^k(M \times \mathbf{R}^n, \infty)$$

defined in 5.1, where $k = k_1 + k_2 + \cdots + k_n$. Let $b_i \in \Sigma^{k_i}$ be the image of the centre of $D^{k_i}$. Without loss of generality take a map $f: M \times \mathbf{R} \to \Sigma^{k_1} \vee \Sigma^{k_2} \vee \cdots \vee \Sigma^{k_n}$ transverse to all the points $b_i$, representing $\alpha$. Then $V_i = f^{-1}(b_i)$ is a framed submanifold of $M \times \mathbf{R}$, and these submanifolds are disjoint. We may suppose, from 6.5, that the projections $V_i \to M$ are mutually transverse. Then as above we define a compact framed submanifold $W'$ of $M \times \mathbf{R}^n$ representing $\mu_n \alpha$ by: $(m, t_1, t_2, \ldots, t_n) \in W'$ if and only if

$$(m, t_i) \in V_i \quad \text{for all} \quad i = 1, 2, \ldots, n, (m \in M, t_i \in \mathbf{R}) \tag{6.6}$$

and

$$t_1 < t_2 < \cdots < t_n, \quad (t_i \in \mathbf{R}) \tag{6.7}$$

For the same reasons as in the discussion of the cup product, we would find a compact framed submanifold $W$ if we had omitted the condition 6.7. Now $W$ avoids the hyperplanes $t_i = t_j$; therefore $W'$ is the union of certain components of $W$. It follows that $W'$ is a framed compact submanifold of $M \times \mathbf{R}^n$ representing $\mu_n \alpha$. Its framing is obtained from the product framing of $V_1 \times V_2 \times \cdots \times V_n$.

In order to deduce the interpretation of $\lambda_n$, it remains to evaluate the effect of the backward pinch map $\bar{r}_n$. Take a framed submanifold $V$ of $M \times \mathbf{R}$ representing $\alpha \in \pi^k(M \times \mathbf{R}, \infty)$, with Thom map $f: M \times \mathbf{R} \to \Sigma^k$. Then we see that we have to choose points $b_i (1 \leq i \leq n)$ in $\Sigma^k$, distinct from each other and from the basepoint $o$. If $k \geq 2$, it does not matter how we choose the points, since all choices are isotopic; if $k = 1$, we must choose them in reverse order round the circle $\Sigma^1$, to comply with the definition 1.2 of $\bar{r}_n$. Set $V_i = f^{-1}(b_i)$; thus each $V_i$ is obtained from $V$ by 'pushing $V$ off itself along one of its framing sections'. We obtain a framed submanifold $W'$ representing $\lambda_n \alpha$ by applying the geometric construction for $\mu_n$ just described to the disjoint framed submanifolds $V_1, V_2, \ldots. V_n$, from the definition 5.4 of $\lambda_n$. The framing of $W'$ is again obtained from the product framing of $V_1 \times V_2 \times \cdots \times V_n$.

For $n = 2$, this is the construction of [10]. When $M = \mathbf{R}^{m-1}$, $\lambda_2$ gives a homomorphism, which we may write as

$$\lambda_2 : \pi_m(\Sigma^k) \to \pi_{m+1}(\Sigma^{2k}).$$

There it was shown that this is the usual Hopf invariant of [12], followed by suspension (apart from sign). This fact also follows from 5.12 and previously known results.

Let us summarize.

THEOREM 6.8. *The Hopf invariant*

$$\lambda_n : \pi^k(M \times \mathbf{R}, \infty) \to \pi^{nk}(M \times \mathbf{R}^n, \infty) \quad (n \geq 1; k \geq 1)$$

*can be interpreted geometrically when $M$ is a smooth manifold, by the preceding discussion and conditions* 6.6 *and* 6.7, *in terms of framed submanifolds.*

In particular we may make use of all the properties of the Hopf invariants $\lambda_n$ developed in § 2, § 3, and § 4. It is quite possible to prove all these results directly, for the case of framed submanifolds, by using transversality and framed-cobordism methods. These properties produce some interesting interaction between homotopy theory and differential topology.

*The transfer homomorphism.* In cohomotopy theory we can define Gysin-type transfer homomorphisms having the usual properties.

Let $g: M \subset N$ be an embedding of smooth manifolds, and suppose this embedding is framed. (We do not require $M$ to be compact.) Then any framed submanifold $V$ of $M$ gives rise to a framed submanifold $V$ of $N$, where we frame $V$ in $N$ by taking the restriction to $V$ of the given framing of $M$ in $N$, *followed* by the framing of $V$ in $M$.

LEMMA 6.9. *Let $g: M \subset N$ be a framed embedding of manifolds. Then $g$ induces*

$$g_! : \pi^p(M \times \mathbf{R}^n, \infty) \to \pi^{p+k}(N \times \mathbf{R}^n, \infty),$$

*where $k$ is the codimension of $M$ in $N$. It satisfies*

(a) *$g_!$ is a homomorphism (if $n \geq 1$);*

(b) *If $M$ is compact, and represents $\alpha \in \pi^k(N, \infty)$, then $g_! 1 = \alpha$, where $1 \in \pi^0(M, \infty)$ is the obvious identity class;*

(c) *Suppose $\alpha \in \pi^p(M \times \mathbf{R}^n, \infty)$ and $\beta \in \pi^q(N \times \mathbf{R}^m, \infty)$, then $g_!(\alpha \cdot g^* \beta) = g_! \alpha \cdot \beta$, the usual formula for products;*

(d) *If also $l: N \subset P$ is a framed embedding, then $(lg)_! = l_! g_!$;*

(e) *$g_!$ commutes with suspension.*

*Proof.* The proof is entirely trivial. Properties such as (c), and many others, become obvious once it is noted that $g_!$ may be induced by a suitably defined Thom map $g' : N_c \to \Sigma^k \wedge M_c$ of the framed embedding $g$, even when $g$ is not proper.

REMARK. Take the unit spheres $S^p \subset \mathbf{R}^{p+1}$ and $S^q \subset \mathbf{R}^{q+1}$; then in $\mathbf{R}^{p+q+2} = \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$ we find the framed embedding $i: S^p \times S^q \subset S^{p+q+1}$, where $S^{p+q+1}$ is the sphere radius $\sqrt{2}$ in $\mathbf{R}^{p+q+2}$. It induces the transfer homomorphism

$$i_! : \pi^k(S^p \times S^q) \to \pi^{k+1}(S^{p+q+1}),$$

if we take a basepoint in $S^p \times S^q$, or, in more familiar notation,

$$i_!:[S^p \times S^q, S^k] \to [S^{p+q+1}, S^{k+1}] \cong \pi_{p+q+1}(S^{k+1}).$$

As such, it is simply the Hopf construction (apart from sign), as given in 1.5 of [18].

*The self-linking class*. Consider again the construction of $\lambda_n$. We start with a framed submanifold $V$ of $M \times \mathbf{R}$ representing $\alpha \in \pi^k(M \times \mathbf{R}, \infty)$. We construct copies $V_i$ of $V$ in $M \times \mathbf{R}$, and from $V_i$ we construct a submanifold $W_i \cong V \times \mathbf{R}^{n-1}$ of $M \times \mathbf{R}^n$ by the condition: $(x, t_1, t_2, \ldots, t_n) \in W_i$ if and only if $(x, t_i) \in V_i$. The submanifold $W_i$ inherits a framing from $V_i$. Under a suitable transversality condition, the intersection $W$ of the submanifolds $W_i$ is again a framed submanifold, framed by taking the framings of the $W_i$ in order of increasing $i$. Denote by $W_{12\ldots n}$ that part of $W$ on which $t_1 < t_2 < \cdots < t_n$; then $W_{12\ldots n}$ represents $\lambda_n \alpha$.

If $n = 2$, we have a framed submanifold $W_{12}$ of $M \times \mathbf{R}^2$, which we may also regard as a framed submanifold of $W_1 \cong V \times \mathbf{R}$.

Definition 6.10. We define the *self-linking class* $\gamma \in \pi^k(V \times \mathbf{R}, \infty)$ as represented by the framed submanifold $W_{12} \subset W_1 \cong V \times \mathbf{R}$.

It measures, to some extent, the linking of $V$ in $M \times \mathbf{R}$ with another copy of $V$ pushed along a framing section. The submanifold $W_{12}$ also defines a cohomology linking class, as a function on the cycles of $V$, which has been used by Haefliger [9]. However, $\gamma$ is defined directly, and contains more information, as we shall see.

Let us return to the general case. Denote by $W_{1i}$ the submanifold of $W_1 \cap W_i$ on which $t_1 < t_i$. If we work in $W_1 \cong V \times \mathbf{R}^{n-1}$, we find that if the pushed-off submanifolds $V_i$ for $i > 2$ are chosen sufficiently close to $V_2$, then the framed submanifolds $W_{1i}$ $(2 < i \leq n)$ are all diffeomorphic to $W_{12}$, and moreover, are precisely those needed for constructing $\lambda_{n-1}\gamma$. Hence $\lambda_{n-1}\gamma$ is the class in $\pi^{(n-1)k}(W_1, \infty)$ of that part of the intersection $W_{12} \cap W_{13} \cap \cdots \cap W_{1n}$ on which $t_2 < t_3 < \cdots < t_n$. Inclusion of this framed submanifold in $M \times \mathbf{R}^n$, by the embedding $W_1 \subset M \times \mathbf{R}^n$, gives us back $W_{12\cdots n}$, with the same framing as before. Finally, we may write the embedding $W_1 \subset M \times \mathbf{R}^n$ in the form $g \times 1: V \times \mathbf{R}^{n-1} \subset M \times \mathbf{R} \times \mathbf{R}^{n-1} \simeq M \times \mathbf{R}^n$, where $g: V \subset M \times \mathbf{R}$.

Let us summarize this result.

Theorem 6.11. *Suppose the framed submanifold $g: V \subset M \times \mathbf{R}$ represents $\alpha \in \pi^k(M \times \mathbf{R}, \infty)$. Then its self-linking class $\gamma \in \pi^k(V \times \mathbf{R}, \infty)$ can be defined canonically, and it satisfies*

$$g_! \lambda_{n-1} \gamma = \lambda_n \alpha \quad \textit{for} \quad n \geq 1$$

*or, formally,*

$$g_! e^\gamma = e^\alpha - 1 .$$

This result is reminiscent fo the Riemann-Roch theorems [2] due to Atiyah and Hirzebruch.

We can describe the self-linking class $\gamma$ in a simpler way. We observe that it is the class of the composite Thom map

$$V \times \mathbf{R} \subset M \times \mathbf{R}^2 \to M \times \mathbf{R} \to \Sigma^k,$$

where the last map is the Thom map of $V_2$. The composite $V \times \mathbf{R} \to M \times \mathbf{R}$ sends $((x, t_1), t_2)$ to $(x, t_2)$, if $t_1 < t_2$. Therefore put $t = t_2 - t_1$, which is to be positive. Then we have the map $u: V \times \mathbf{R}_+ \to M \times \mathbf{R}$, where $\mathbf{R}_+ = [0, \infty]$, given by

$$u((x, t_1), t) = (x, t_1 + t). \tag{6.12}$$

We have therefore proved

LEMMA 6.13. *The self-linking class $\gamma$ is the class of the composite*

$$V \times \mathbf{R}_+ \xrightarrow[u]{} M \times \mathbf{R} \to \Sigma^k,$$

*where u is given by* 6.12 *and the second map is the Thom map of $V_2 \subset M \times \mathbf{R}$.*

*Orientation.* In order to discuss spheres, we need a convention for identifying the two kinds $S^n$ and $\Sigma^n = D^n/S^{n-1}$ of $n$-sphere (see [37]).

DEFINITION 6.14. A *boundary convention* consists of the choice of one of the two classes of homotopy equivalences $S^n \simeq \Sigma^n$, one for each $n$.

In view of 6.1, such a homotopy equivalence is the class of some framed point in $S^n$. It is useful to be more general.

DEFINITION 6.15. Let $M$ be a smooth connected $n$-manifold. An *orientation* $\mathcal{O}(M)$ of $M$ is the class in $\pi^n(M, \partial M, \infty)$ of some framed point in $M$.

(This terminology is convenient here but unfortunate; an orientable manifold has two possible orientations, whereas a 'non-orientable' manifold has only one.)

Now $S^n = \partial D^{n+1}$, and $D^{n+1}$ has a canonical orientation. Therefore what we need for a boundary convention is some systematic method of relating the orientations of a manifold $M$ and its boundary $\partial M$. For our present purposes the most convenient convention is the 'homotopy' convention, as follows:

$$\mathcal{O}(M) = \mathcal{O}(\partial M) \oplus \mathbf{n}, \tag{6.16}$$

where $\mathbf{n}$ is an outward normal. This means that to frame a point of $\partial M$ in $M$, we take a framing in $\mathcal{O}(\partial M)$, *followed* by $\mathbf{n}$. (For other conventions, we refer to the Appendix.)

*The J-homomorphism.* We need the preceding conventions in order to define the $J$-homomorphism precisely.

Take the standard embedding $S^n \subset \mathbf{R}^{n+1} \subset \mathbf{R}^{n+k}$ (with $k > 1$), and give it the 'standard' framing $\mathscr{F}_0$, consisting of the outward normal to $S^n$ in $\mathbf{R}^{n+1}$, followed by the canonical framing of $\mathbf{R}^{n+1}$ in $\mathbf{R}^{n+k}$. Given $\alpha \in \pi_n(SO(k))$, choose $f: S^n \to SO(k)$ representing $\alpha$, so that we may let $f$ act on the framing $\mathscr{F}_0$, to give a new framing $f \cdot \mathscr{F}_0$ of $S^n$. Then we define

$$J^*: \pi_n(SO(k)) \to \pi_{n+k}(S^k) \tag{6.17}$$

by taking $J^*\alpha$ as the class in $\pi^k(\mathbf{R}^{n+k}, \infty)$ of $S^n$ with the twisted framing $f\cdot\mathscr{F}_0$. This agrees with the homomorphism $J'$ defined by KERVAIRE in 1.8 of [18], except that we are orienting spheres differently, and twisting the framing differently. The question of signs is discussed in the Appendix.

It is easy to determine from 6.13 the self-linking class of $S^n$ with the framing $f\cdot\mathscr{F}_0$. As in 6.12 we map $S^n \times \mathbf{R}_+$ into $\mathbf{R}^{n+k}$ using the direction of the last coordinate vector. The Thom map of $S^n$ is obtained from the map $N \cong S^n \times D^k \to D^k$ taking $(x, y)$ to $f(x)^{-1}y$, where $N$ is a tubular neighbourhood of $S^n$ constructed using $\mathscr{F}_0$. Let $z \in \partial D^k = S^{k-1}$ be the point such that $S^n \times \mathbf{R}_+$ cuts $S^n \times S^{k-1}$ in $S^n \times z$. We consider the composite

$$l: S^n \cong S^n \times z \subset S^n \times S^{k-1} \to S^{k-1},$$

where the last map sends $(x, y)$ to $f(x)^{-1}y$. To obtain $V_2$, we push $V_1 = S^n$ out along a framing section until it lies in $\partial N \cong S^n \times S^{k-1}$. We then see that the self-linking class $\gamma$ is the class of the suspension

$$S^n \times \mathbf{R}_+ \to ES^n \xrightarrow[El]{} ES^{k-1} \cong \Sigma^k.$$

Now $l$ may be expressed simply as the composite

$$S^n \xrightarrow[f_1]{} SO(k) \xrightarrow[\varphi]{} S^{k-1},$$

where $\varphi$ is the evaluation map at $z$, and the map $f_1$ is the inverse in $SO(k)$ of $f$, which therefore represents $-\alpha$.

LEMMA 6.18. *Given $f \in \alpha \in \pi_n(SO(k))$, let $\gamma \in \pi^k(S^n \times \mathbf{R}, \infty)$ be the self-linking class of the sphere $S^n$ with the twisted framing $f\cdot\mathscr{F}_0$. Then $\gamma = -E\varphi_*\alpha$, where $\varphi: SO(k) \to S^{k-1}$ is the evaluation map at a point.*

THEOREM 6.19. *Given $\alpha \in \pi_n(SO(k))$, we have*

$$\lambda_2 J^*\alpha = -E^{k+1}\varphi_*\alpha,$$
$$\lambda_n J^*\alpha = 0 \quad \text{for} \quad n > 2.$$

*Proof.* We apply 6.11 to the embedding $g: S^n \subset \mathbf{R}^{n+k}$ with the twisted framing $f\cdot\mathscr{F}_0$. Then $\lambda_2 J^*\alpha = g_! \gamma$, where $\gamma$ is given by 6.18; and if $n > 2$, $\lambda_n J^*\alpha = g_! \lambda_{n-1}\gamma = 0$, since $\gamma$ is a suspension.

To find $g_!$, let $j: \mathbf{R}^n \subset S^n$ be an orientation-preserving embedding. In the required dimension, we have the diagram

$$\begin{array}{ccccc}
\pi^{k-1}(\mathbf{R}^n, \infty) & \underset{j_!}{\cong} & \pi^{k-1}(S^n, \infty) & \xrightarrow[g_!]{} & \pi^{2k-1}(\mathbf{R}^{n+k}, \infty) \\
\downarrow E & & \downarrow E & & \downarrow E \\
\pi^k(\mathbf{R}^n \times \mathbf{R}, \infty) & \underset{j_!}{\cong} & \pi^k(S^n \times \mathbf{R}, \infty) & \xrightarrow[g_!]{} & \pi^{2k}(\mathbf{R}^{n+k} \times \mathbf{R}, \infty)
\end{array}$$

which commutes by 6.9. This shows that the composite

$$\pi^{k-1}(\mathbf{R}^n, \infty) \to \pi^{2k}(\mathbf{R}^{n+k} \times \mathbf{R}, \infty) = \pi^{2k}(\mathbf{R}^{n+k+1}, \infty)$$

agrees with $(-1)^{k(k-1)} E^{k+1} = E^{k+1}$, in view of our conventions. It follows that $g_! \gamma = -E^{k+1} \varphi_* \alpha$.

This result is a slight desuspension of Lemma 6.5 of [18].

*An illustration.* We show how the geometric Hopf invariant $\lambda_2$ occurs in differential topology.

Consider a smooth $(n-1)$-connected $2n$-manifold $V$ with boundary $\partial V$ a homotopy $(2n-1)$-sphere, and suppose we have a framed embedding

$$g:(V, \partial V) \subset (\mathbf{R}_+^{2n+k}, \mathbf{R}^{2n+k-1}),$$

where $\mathbf{R}_+^{2n+k}$ denotes the positive half-space with the last coordinate positive. This situation has been extensively studied, e.g. by Wall [33]. Certain facts emerge from this investigation: Smale theory [26] shows that if $n>2$ $V$ contains a wedge $K$ of $n$-spheres as deformation retract, and that because $V$ is framed in $\mathbf{R}_+^{2n+k}$, the first suspension of the map $\partial V \subset V \to K$ is nullhomotopic. This implies that the Puppe exact sequence in cohomotopy of $(V, \partial V)$ breaks up to yield the short exact sequence

$$0 \to \pi^{n+1}(\partial V \times \mathbf{R}^2, \infty) \to \pi^{n+1}(V \times \mathbf{R}, \partial V \times \mathbf{R}, \infty) \to \pi^{n+1}(V \times \mathbf{R}, \infty) \to 0.$$

Naturality of the transfer homomorphisms, naturality of $\lambda_2$, Hopf isomorphisms (from the Hopf classification theorem [14]), and various elementary observations, yield the commutative diagram, which contains the short exact sequence,

$$\begin{array}{ccccc}
\pi^{n+k+1}(\mathbf{R}^{2n+k+1}, \infty) & \cong & \pi^{n+k+1}(\mathbf{R}_+^{2n+k+1}, \mathbf{R}^{2n+k}, \infty) & \to & 0 \\
\uparrow g_! & & \uparrow g_! & & \uparrow \\
\pi^{n+1}(\partial V \times \mathbf{R}^2, \infty) & \to & \pi^{n+1}(V \times \mathbf{R}, \partial V \times \mathbf{R}, \infty) & \to & \pi^{n+1}(V \times \mathbf{R}, \infty) \\
\downarrow \lambda_2 & & \downarrow \lambda_2 & & \wr\Vert \\
 & & & & H^n(V, \partial V; \mathbf{Z}) \\
\pi^{2n+2}(\partial V \times \mathbf{R}^3, \infty) & \to & \pi^{2n+2}(V \times \mathbf{R}^2, \partial V \times \mathbf{R}^2, \infty) & & \\
\wr\Vert & & \wr\Vert & & \\
H^{2n-1}(\partial V, \mathbf{Z}) & \cong & H^{2n}(V, \partial V; \mathbf{Z}) \cong \mathbf{Z}. & &
\end{array}$$

The left side may be identified with

$$\pi_{2n+k+1}(S^{n+k+1}) \xleftarrow[E^k]{} \pi_{2n+1}(S^{n+1}) \xrightarrow[\lambda_2]{} \pi_{2n+2}(S^{2n+2}) \cong \mathbf{Z}.$$

We know all about $\lambda_2$ here, since it is the suspended Hopf invariant, in fact the original Hopf invariant [13]. From the work of Adams [1] its image is

$$\begin{cases} \text{zero if } n \text{ is even}, \\ \mathbf{Z} \text{ if } n = 1, 3, \text{ or } 7, \\ \text{the even integers, for other odd } n. \end{cases}$$

Let us write $G$ for the group $H^n(V, \partial V; \mathbf{Z})$, which is free abelian. We may use the diagram to define a function

$$\varphi: G \to \operatorname{Coker}(EH),$$

by lifting an element of $G$ to $\pi^{n+1}(V \times \mathbf{R}, \partial V, \infty)$ and taking its image under $\lambda_2$ in $H^{2n}(V, \partial V; \mathbf{Z})$; the indeterminacy is in $\operatorname{Im}(EH)$.

We can do slightly better, if we insist on choosing only those elements of $\pi^{n+1}(V \times \mathbf{R}, \partial V \times \mathbf{R}, \infty)$ that give zero in $\pi^{n+k+1}(\mathbf{R}^{2n+k+1}, \mathbf{R}^{2n+k}, \infty)$. The kernel of $E^k$ is generated by the Whitehead product $[\iota, \iota]$, where $\iota$ denotes the identity class of $S^{n+1}$; it has Hopf invariant $\pm 2$ if $n$ is odd, 0 if $n$ is even. We therefore find a function

$$\varphi: G \to \begin{cases} \mathbf{Z} \text{ if } n \text{ is even}, \\ \mathbf{Z}_2 \text{ if } n \text{ is odd}. \end{cases}$$

From the Cartan formula 2.1 (c), $\varphi$ is not linear, but instead satisfies

$$\varphi(\alpha + \beta) = \varphi\alpha + \varphi\beta + \alpha \cdot \beta,$$

and also, from 3.17,

$$\alpha \cdot \alpha = 2\varphi\alpha \quad \text{if } n \text{ is even}.$$

Hence $\varphi$ is a quadratic form on $G$.

This function appears in many different disguises. If we attempt to do framed surgery on $V$, by killing the cohomology class $\alpha$, $\varphi\alpha$ is the obstruction (see [33]). When $n$ is odd, it gives the Arf invariant of $V$. (Indeed, we have here essentially the original approach, through cohomotopy groups, used by KERVAIRE in [19].) An easy geometric argument shows that if an embedded sphere $S^n \subset V$ represents the cohomology class $\alpha$, its normal bundle in $V$ is determined by $\varphi\alpha$ (for the sphere can certainly be framed in $V \times \mathbf{R}$).

REMARK. It is evident that everything we have said about cohomotopy sets can be generalized. We may consider submanifolds $V$ of a manifold $M$ whose normal bundle need not be framed, but has a more general structure group, and replace the sphere $\Sigma^k$ by the universal Thom complex of this group. This is still a special case of the general geometric theory of § 5.

## 7. Appendix on signs

The purpose of this Appendix is to compare the signs of the various definitions of Hopf invariant, on homotopy groups of spheres, as promised. We also consider the $J$-homomorphism. The situation is further confused by the use in the literature of two different boundary conventions.

There is usually no difficulty with signs when working in a sufficiently general context as in § 2 to § 5, when shuffles may safely be omitted from the notation. When one specializes to spheres, however, it is not always clear what signs have been introduced; the shuffle $E^m A \wedge E^n B \cong E^{m+n}(A \wedge B)$ is apt to be overlooked when $A$ is a sphere $S^p$, because both sides are already identified with $E^{m+n+p}B$, in a way depending on the convention used. For the general philosophy on management of signs, we cannot do better than refer to J. H. C. WHITEHEAD [37].

Spheres appearing in homotopy theory tend to be identified canonically (up to homotopy) with either the unit sphere $S^n$ in $\mathbf{R}^{n+1}$, or the sphere $\Sigma^n = D^n/S^{n-1}$. As explained in § 6, a *boundary convention* (compare [37]) consists of a choice for each $n$ of one of the two classes of homotopy equivalences $S^n \simeq \Sigma^n$, or equivalently, an *orientation* of $S^n$, in the sense of 6.15. The homotopy groups $\pi_n(X)$ are always defined as $[\Sigma^n, X]$, so that one needs a boundary convention even to *define* the composition

$$\pi_q(S^r) \times \pi_p(S^q) \to \pi_p(S^r).$$

Let $M$ be a manifold with boundary $\partial M$ (in particular $M = D^n$, which is canonically oriented, and $\partial M = S^{n-1}$), with outward normal $\mathbf{n}$ at a point of $\partial M$. The *homotopy convention* is determined by taking, as in 6.16,

$$\mathcal{O}(M) = \mathcal{O}(\partial M) \oplus \mathbf{n}, \tag{7.1}$$

whereas the *homology convention* is determined by taking

$$\mathcal{O}(M) = \mathbf{n} \oplus \mathcal{O}(\partial M) \tag{7.2}$$

(see 2.6 of [10]). The resulting maps $S^n \to \Sigma^n$ differ by the sign $(-1)^n$.

When comparing formulae proved according to different conventions, it is clearly necessary to be precise as to which kinds of sphere are involved. If the formulae need to have functors applied to them before being compared, one must also state the convention used in defining the functors. In this Appendix we work in the homotopy convention.

*Smash products and suspension.* The smash products of maps of spheres come from the canonical homotopy equivalences $\Sigma^m \wedge \Sigma^n \simeq \Sigma^{m+n}$, or equivalently, $\Sigma^n \simeq \Sigma^1 \wedge \Sigma^1 \wedge \cdots \wedge \Sigma^1$. In this paper we defined the suspension functor (in effect) by $EA = A \wedge \Sigma^1$, which is found to work well with the homotopy convention. The alternative definition, $EA = \Sigma^1 \wedge A$, works well with the homology convention.

It should be noted that the Freudenthal suspension homomorphism $E: \pi_r(S^n) \to \pi_{r+1}(S^{n+1})$ depends on the boundary convention, because it uses

$$E S^n = S^n \wedge \Sigma^1 \simeq \Sigma^n \wedge \Sigma^1 \cong \Sigma^{n+1} \simeq S^{n+1}.$$

The Barratt-Hilton formula, which merely expresses the naturality of the smash product, is unaffected by the choice of boundary convention, and is disturbed only by changing the definition of the suspension $E$; it reads, as in Theorem 3.2 of [7],

$$\alpha \wedge \beta = (-1)^{p(q+j)} E^j \alpha \circ E^p \beta = (-1)^{i(q+j)} E^i \beta \circ E^q \alpha, \tag{7.3}$$

where $\alpha \in \pi_p(S^i)$ and $\beta \in \pi_q(S^j)$.

*Whitehead products.* There are at least three different conventions in the literature concerning the Whitehead product. Suppose $\alpha \in \pi_p(X)$, and $\beta \in \pi_q(X)$.

The original definition [36] by J. H. C. WHITEHEAD was by means of a canonical map $S^{p+q+1} \to \Sigma^p \vee \Sigma^q$, and used the homology convention; we denote the Whitehead product of $\alpha$ and $\beta$ formed according to this convention by $[\alpha, \beta]''$.

Instead, one can use the homotopy convention; we denote this product by $[\alpha, \beta]'$. This convention was used explicitly by BARCUS and BARRATT [3], and earlier by G. W. WHITEHEAD [35].

A third convention (which could be called the transgression convention) was used by BARRATT [5]. We denote the product so defined by $[\alpha, \beta]$. This is the product we used in § 4 of this paper. It is the definition amenable to generalization. The idea of defining the product this way and the feasibility of doing so are due to FOX [8] and SAMELSON [25].

The three products are related as follows:

$$[\alpha, \beta] = (-1)^q [\alpha, \beta]' = (-1)^{p+1} [\alpha, \beta]'' . \tag{7.4}$$

Take also $\gamma \in \pi_r(X)$. The commutation and Jacobi identities 4.3 and 4.4 yield

$$[\beta, \alpha] = (-1)^{pq+p+q} [\alpha, \beta]$$

and

$$(-1)^{rp+q} [[\alpha, \beta], \gamma] + (-1)^{pq+r} [[\beta, \gamma], \alpha] + (-1)^{qr+p} [[\gamma, \alpha], \beta] = 0 .$$

If we substitute from 7.4, these take the more familiar forms

$$[\beta, \alpha]'' = (-1)^{pq} [\alpha, \beta]'' \tag{7.5}$$

and

$$(-1)^{rp} [[\alpha, \beta]'', \gamma]'' + (-1)^{pq} [[\beta, \gamma]'', \alpha]'' + (-1)^{qr} [[\gamma, \alpha]'', \beta]'' = 0 , \tag{7.6}$$

as in Theorem B of [12]. The Jacobi identity in the products $[\alpha, \beta]'$ is more complicated.

*Hopf invariants.* For Hopf invariants the situation is less simple. We have traced seven fundamentally different homotopy definitions of the generalized Hopf invariant. Initially they were homomorphisms $\Psi : \pi_r(S^n) \to \pi_r(S^{2n-1})$ (or $\pi_{r+1}(S^{2n})$). Later they appeared as homomorphisms

$$\psi : \pi_r(S^i \vee S^j) \to \pi_r(S^{i+j-1}) \text{ (or } \pi_{r+1}(S^{i+j})) ,$$

from which homomorphisms $\Psi$ were recovered by putting $i=j=n$ and composing with the pinch map $\rho_2 : S^n \to S^n \vee S^n$. We compare the various definitions by evaluating them on fixed elements $\alpha \in \pi_r(S^n)$ and $\beta \in \pi_r(S^i \vee S^j)$.

In this paper we introduced the homomorphisms

$$\lambda_2 : \pi_r(S^n) \to \pi_{r+1}(S^{2n}), \quad \text{and} \quad \mu_2 : \pi_r(S^i \vee S^j) \to \pi_{r+1}(S^{i+j}) .$$

(Note that we have already used the homotopy boundary convention to write them in this form.) The latter may be obtained from

$$\lambda_2 : \pi_r(S^i \vee S^j) \to \pi_{r+1}(\Lambda^2(S^i \vee S^j))$$

by projection (see 5.5). As the definition 2.1 of $\lambda_2$ is axiomatic, it is particularly convenient for comparing with the other Hopf invariants. We take the opportunity of simplifying some of the signs by observing that by 3.18

$$2\lambda_2 \alpha = 0 \quad \text{if } n \text{ is odd}. \tag{7.7}$$

(1). G. W. WHITEHEAD [35] defined a homomorphism

$$H_W : \pi_r(S^n) \to \pi_r(S^{2n-1}),$$

only for $r < 3n-3$. It uses the homotopy convention for the boundary homomorphism and the Whitehead product.

(2). HILTON [11] defined a homomorphism

$$h^* : \pi_r(S^i \vee S^j) \to \pi_{r+1}(S^{i+j}),$$

and hence

$$H^* : \pi_r(S^n) \to \pi_{r+1}(S^{2n}),$$

for all $r$. We defined $H^*$ in 5.11, and can obtain $h^*$ also from 5.11 by taking $B = S^{i-1} \vee S^{j-1}$ and projecting. The definitions use the homotopy convention. HILTON observes [11] that

$$H^* \alpha = E H_W \alpha$$

whenever $H_W$ is defined. From 5.12 we find that

$$\lambda_2 \alpha = -H^* \alpha, \qquad \mu_2 \beta = -h^* \beta.$$

(3). HILTON defined [12] a homomorphism

$$h_H : \pi_r(S^i \vee S^j) \to \pi_r(S^{i+j-1}),$$

and hence

$$H_H : \pi_r(S^n) \to \pi_r(S^{2n-1}).$$

The definition (compare 4.8) uses the decomposition theorem for the homotopy groups of a wedge of spheres. It depends on the convention for Whitehead products, and on a choice between $[\iota_1, \iota_2]$ and $[\iota_2, \iota_1]$. We suppose it defined by the product $[\iota_1, \iota_2]''$. From 4.17 we find (remembering that it conceals a shuffle)

$$\lambda_2 \alpha = E H_H \alpha, \qquad \mu_2 \beta = (-1)^{i+j} E h_H \beta.$$

(BARCUS and BARRATT used $[\iota_2, \iota_1]'$ in [3]. Any two definitions, corresponding to different conventions, differ by an appropriate power of U – the class of a map of degree $-1$ on $S^{i+j-1}$ – easily determined by the relation 7.4 between the Whitehead products.)

(4). JAMES defined in [16] a very general homomorphism, applying to maps of suspensions of connected CW-complexes, which we gave in 3.10. The definition uses the reduced product spaces introduced by him in [15], and depends on the choice of one of the 8 possible conventions regarding the order of the terms in 3.9. JAMES uses lexicographic ordering from the right. We, and TODA [32], use lexicographic ordering from the left. According to 3.12, all choices coincide after one suspension. Let us call the resulting homomorphisms for spheres

$$H_J: \pi_r(S^n) \to \pi_r(S^{2n-1}), \quad \text{and} \quad h_J: \pi_r(S^i \vee S^j) \to \pi_r(S^{i+j-1}).$$

Then from 3.15 we have (noting the shuffle involved in 3.13),

$$\lambda_2 \alpha = (-1)^{n+1} E H_J \alpha, \quad \mu_2 \beta = (-1)^{j+1} E h_J \beta.$$

(5). KERVAIRE defined [18] a homomorphism

$$H_K: \pi_r(S^n) \to \pi_{2r}(S^{2n+r-1})$$

by a geometric construction, and used the homology convention. If we compare $H_K$ with the geometric form of $\lambda_2$ given in § 6, we find that in either case we start from the same two framed submanifolds $V_1$ and $V_2$ of in $\mathbf{R}^r$, and construct the *same* manifold $W'$, embedded in $\mathbf{R}^{2r}$ or $\mathbf{R}^{r+1}$. It remains to compute the framings, having regard to the differing conventions. We find

$$H_K \alpha = (-1)^{r+1} E^{r-1} \lambda_2 \alpha.$$

(6). HAEFLIGER and STEER defined [10] a geometric Hopf invariant

$$h_{HS}: \pi_r(S^i \vee S^j) \to \pi_{r+1}(S^{i+j}),$$

and hence

$$H_{HS}: \pi_r(S^n) \to \pi_{r+1}(S^{2n}).$$

It differs from the geometric form of $\lambda_2$ given in § 6 only to the extent that $I \times S^n$ was used in § 2 of [10] rather than $S^n \times I$, which results in a different orientation. Thus

$$\lambda_2 \alpha = (-1)^r H_{HS} \alpha, \quad \mu_2 \beta = (-1)^r h_{HS} \beta.$$

Let us combine these formulae. On $\alpha \in \pi_r(S^n)$ we have

$$\left.\begin{aligned} \lambda_2 \alpha = - H^* \alpha = E H_H \alpha = - E H_J \alpha = (-1)^r H_{HS} \alpha = - E H_W \alpha, \\ H_K \alpha = (-1)^{r+1} E^{r-1} \lambda_2 \alpha, \end{aligned}\right\} \tag{7.8}$$

and on $\beta \in \pi_r(S^i \vee S^j)$ we have

$$\mu_2 \beta = - h^* \beta = (-1)^{i+j} E h_H \beta = (-1)^{j+1} E h_J \beta = (-1)^r h_{HS} \beta. \tag{7.9}$$

Of course many of these relations were already known. Further, it follows from 3.2 of [28] and 3.12 of [29] that $h_J\beta = U^{i+1} h_H\beta$, and hence $H_J\alpha = U^{n+1} H_H\alpha$. Formulae with different signs appearing in the literature (such as Theorem 6.2 of [12] and Theorem 7.1 of [18]) are often proved, we believe, by combining formulae valid only under different conventions: not all of these are excused by 7.7. We claim that [18] also contains three other sign errors (see below).

Let us note that on spheres the composition formula 3.16 takes the simple form

$$\left.\begin{aligned}\lambda_2(\alpha\circ\gamma) &= \lambda_2\alpha\circ E\gamma + (\alpha\wedge\alpha)\circ\lambda_2\gamma\,,\\ \mu_2(\beta\circ\gamma) &= \mu_2\beta\circ E\gamma + (\beta_1\wedge\beta_2)\circ\lambda_2\gamma\,,\end{aligned}\right\}\tag{7.10}$$

where $\gamma\in\pi_p(S^r)$, $\alpha\in\pi_r(S^n)$, $\beta\in\pi_r(S^i\vee S^j)$, and $\beta_1=\pi_1\circ\beta\in\pi_r(S^i)$, $\beta_2=\pi_2\circ\beta\in\pi_r(S^j)$. Also that if $\alpha\in\pi_p(EX)$ and $\beta\in\pi_q(EX)$, 4.6 gives

$$\lambda_2[\alpha,\beta]'' = (-1)^{p+q}(\alpha\wedge\beta + (-1)^{pq}\beta\wedge\alpha)\in\pi_{p+q}(EX\wedge EX)\,.\tag{7.11}$$

*The J-homomorphism.* G. W. WHITEHEAD defined the homomorphism $J:\pi_n(SO(k))\to\pi_{n+k}(S^k)$ in [34] by using the Hopf construction (slightly generalized from [13])

$$G:[S^p\times S^q, S^m]\to\pi_{p+q+1}(S^{m+1})\,.$$

We use the explicit form given in [35], which adopts the homotopy boundary convention.

KERVAIRE defined [18] a geometric Hopf construction $G'$ and a geometric homomorphism $J_k$, using the homology convention. In the definition 6.17 of our homomorphism $J^*$, we started with the same framed embedding $S^n\subset\mathbf{R}^{n+k}$, but we used the homotopy convention, and also twisted the framing differently. In comparing $G'$ and $J'$ with $G$ and $J$, KERVAIRE overlooks the fact that different conventions are used (see below). Let $G_g$ and $J_g$ be the homomorphisms defined as by G. W. WHITEHEAD in [35], but using the homology convention. We find that on $\gamma\in[S^p\times S^q, S^m]$,

$$G\gamma = (-1)^{q+1}G'\gamma = (-1)^{p+q+1}G_g\gamma\,,\tag{7.12}$$

and on $\alpha\in\pi_n(SO(k))$

$$J\alpha = (-1)^{n+k}U^{k+1}J'\alpha = (-1)^k J_g\alpha = (-1)^{k+1}UJ^*\alpha\,.\tag{7.13}$$

From this and 6.19 we have

$$\lambda_2 J\alpha = E^{k+1}\varphi_*\alpha = EH_H J\alpha\,,\tag{7.14}$$

where $\varphi:SO(k)\to S^{k-1}$ is the map in 6.19 or in Lemma 6.5 of [18]. We have used here 7.7 and the fact that $\lambda_2 U\beta = \lambda_2\beta$ (from 3.16), to simplify the sign.

*Discussion of the signs in* [18]. The trouble taken over signs in [18] justifies a detailed examination. First of all, KERVAIRE uses the homology convention, but fails to note that G. W. WHITEHEAD in [35] uses the homotopy convention. Let $E_g:\pi_r(S^n)\to$

$\pi_{r+1}(S^{n+1})$ and $*_g$ be the suspension and join operations defined using the homology convention; then $E_g\alpha = -E\alpha$ and $\alpha *_g \beta = -\alpha * \beta$. (The smash product is unaltered.) Given $\alpha \in \pi_p(S^m)$ and $\beta \in \pi_q(S^n)$, we claim that $E_g(\alpha \wedge \beta) = (-1)^{q+n}\alpha *_g \beta$, not with sign $(-1)^{q+m}$ as asserted in 1.11 of [18]. (The sign given in [18] is not explained.)

Next take a map $t: S^p \times S^q \to S^m$ as in Lemma 6.7 of [18], of type $(\alpha, \beta)$ according to the homotopy convention. Then $t$ has type $(\alpha_g, \beta_g)$, say, according to the homology convention, where $\alpha_g = (-1)^p \alpha \in \pi_p(S^m)$ and $\beta_g = (-1)^q \beta \in \pi_q(S^m)$. Let $\tau$ be the class of $t$. Then KERVAIRE proves correctly (p. 363) that $G'\tau = (-1)^{pm+q+p}\beta' \wedge \alpha'$, where $\beta' = (-1)^{p+q+pq}E^{p+1}\beta$ and $\alpha' = (-1)^{p+q}E_g^{q+1}\alpha$, but goes on to use the incorrect version of 1.11, which introduces a sign wrong by $(-1)^{p+q}$. Thus Lemma 6.7 should read

$$H_K G'\tau = (-1)^{p+1} E_g^{p+q+1}(\alpha_g *_g \beta_g),$$

or, by 7.12,

$$H_K G\tau = (-1)^{p+q} E^{p+q+1}(\alpha * \beta).$$

In the proof of Theorem 7.1, KERVAIRE compares Lemma 6.7 with the formula $HG\tau = -\alpha * \beta$ of [35] (with sign corrected by J. H. C. WHITEHEAD [37]), but omits to note that 6.7 is proved for $G'$, not $G$. This fully accounts for the discrepancy from 7.8.

In Lemma 6.5 of [18]. KERVAIRE actually proves $H_K J'\alpha = (-1)^{n+1}E_g^{n+2k}\varphi_*\alpha$, where $\alpha \in \pi_n(SO(k))$. Now substitution of $J'\alpha = (-1)^n U^{k+1} J_g \alpha$ from 7.13 yields $H_K J_g \alpha = -E_g^{n+2k}\varphi_*\alpha$, not with sign $(-1)^k$ as asserted, because 3.16 gives $H_K U\beta = H_K\beta$, not $-H_K\beta$. Thus $H_K J\alpha = (-1)^{n+1}E^{n+2k}\varphi_*\alpha$, which suspends 7.14 (thanks to 7.7).

*University of Warwick,*
*Hertford College, Oxford.*

REFERENCES

[1] J. F. ADAMS: *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) *72* (1960), 20–104.

[2] M. F. ATIYAH and F. HIRZEBRUCH: *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. *65* (1959), 276–281.

[3] W. D. BARCUS and M. G. BARRATT: *On the homotopy classification of the extensions of a fixed map*, Trans. Amer. Math. Soc. *88* (1958), 57–74.

[4] M. G. BARRATT: *Track groups* (I), Proc. London Math. Soc. (3) *5* (1955), 71–106.

[5] M. G. BARRATT: *Spaces of finite characteristic*, Quart. J. Math. Oxford (2) *11* (1960), 124–136.

[6] M. G. BARRATT: Higher Hopf invariants (mimeographed notes), University of Chicago (Summer 1957).

[7] M. G. BARRATT and P. J. HILTON: *On join operations in homotopy groups*, Proc. London Math. Soc. (3) *3* (1953), 430–445.

[8] R. H. FOX: *Homotopy groups and torus homotopy groups*, Ann. of Math. (2) *49* (1948), 471–510.

[9] A. HAEFLIGER: *Knotted* (4k-1)-*spheres in* 6*k-space*, Ann. of Math. (2) *75* (1962), 452–466.

[10] A. HAEFLIGER and B. STEER: *Symmetry of linking coefficients*, Comment. Math. Helv. *39* (1965), 259–270.

[11] P. J. HILTON: *Suspension theorems and the generalized Hopf invariant*, Proc. London Math. Soc. (3) *1* (1951, 462–493.
[12] P. J. HILTON: *On the homotopy groups of the union of spheres*, J. London Math. Soc. *30* (1955), 154–172.
[13] H. HOPF: *Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension*, Fund. Math. *25* (1938), 427–440.
[14] H. HOPF: *Die Klassen der Abbildungen der n-dimensionalen Polyeder auf die n-dimensionale Sphäre*, Comment. Math. Helv. *5* (1933), 39–54.
[15] I. M. JAMES: *Reduced product spaces*, Ann. of Math. (2) *62* (1955), 170–197.
[16] I. M. JAMES: *On the suspension triad*, Ann. of Math. (2) *63* (1956), 191–247.
[17] I. M. JAMES: *The transgression and Hopf invariant of a fibration*, Proc. London Math. Soc. (3) *11* (1961), 588–600.
[18] M. A. KERVAIRE: *An interpretation of G. Whitehead's generalization of H. Hopf's invariant*, Ann. of Math. (2) *69* (1959), 345–365.
[19] M. A. KERVAIRE: *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. *34* (1960), 257–270.
[20] J. W. MILNOR: The construction *FK* (mimeographed notes), Princeton University.
[21] J. W. MILNOR: *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. *90* (1959), 272–280.
[22] J. C. MOORE: *The double suspension and p-primary components of the homotopy groups of spheres*, Bol. Soc. Mat. Mexicana *1* (1956), 28–37.
[23] L. S. PONTRJAGIN: *Smooth manifolds and their applications in homotopy theoty* (in Russian), Trudy Mat. Inst. Steklov. *45* (1955), English translation in Amer. Math. Soc. Translations (2) *11*, 1–114.
[24] D. PUPPE: *Homotopiemengen und ihre induzierten Abbildungen*. I, Math. Z. *69* (1958), 299–344.
[25] H. SAMELSON: *A connection between the Whitehead and Pontrjagin product*, Amer. J. Math. *75* (1953), 744–752.
[26] S. SMALE: *On the structure of manifolds*, Amer. J. Math. *84* (1962), 387–399.
[27] N. E. STEENROD: *Cohomology invariants of mappings*, Ann. of Math. (2) *50* (1949), 954–988.
[28] B. STEER: *Extensions of mappings into H-spaces*, Proc. London Math. Soc. (3) *13* (1963), 219–272.
[29] B. STEER: *transgression in sphere-bundles*, Topology *2* (1963), 1–10.
[30] R. THOM: *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. *28* (1954), 17–68.
[31] R. THOM: *Un lemme sur les applications différentiables*, Bol. Soc. Mat. Mexicana *1* (1956), 59–71.
[32] H. TODA: *On the double suspension* $E^2$, J. Inst. Poly. Osaka City Univ. *7* (1956), 103–145.
[33] C. T. C. WALL: *Classification of* $(n\text{-}1)$*-connected* $2n$*-manifolds*, Ann. of Math. (2) *75* (1962), 163–189.
[34] G. W. WHITEHEAD: *On the homotopy groups of spheres and rotation groups*, Ann. of Math. (2) *43* (1942), 634–640.
[35] G. W. WHITEHEAD: *A generalization of the Hopf invariant*, Ann. of Math. (2) *51* (1950), 192–237.
[36] J. H. C. WHITEHEAD: *On adding relations to homotopy groups*, Ann. of. Math. (2) *42* (1941), 409–428.
[37] J. H. C. WHITEHEAD: *On certain theorems of G. W. Whitehead*, Ann. of Math. (2) *58* (1953), 418–428.

(Received, September 8 1966)