ON HOMOMORPHISMS FROM A FIXED REPRESENTATION
TO A GENERAL REPRESENTATION OF A QUIVER

WILLIAM CRAWLEY-BOEVEY

ABSTRACT. We study the dimension of the space of homomorphisms from a given representation X of a quiver to a general representation of dimension vector β. We prove a theorem about this number, and derive two corollaries concerning its asymptotic behaviour as β increases. These results are related to work of A. Schofield on homological epimorphisms from the path algebra to a simple artinian ring.

In this paper we prove a number of results about $\hom(X, \beta)$, the dimension of the space of homomorphisms from a fixed representation X of a quiver to a general representation of dimension vector β. Our basic result relates this number to the dimension of a subset of a Grassmannian of submodules of X. This result is in the spirit of A. Schofield's paper on general representations of quivers [S2], and generalizes one of his results. We derive two corollaries concerning the asymptotic behaviour of $\hom(X, r\beta)$, of interest in themselves, and also because of their connection with another theorem of Schofield, that the homological epimorphisms from a path algebra to a simple artinian ring are in 1–1 correspondence with indivisible Schur roots. (A homological epimorphism is a ring homomorphism $R \to S$ with $S \otimes_R S \cong S$ and $\mathrm{Tor}_i^R(S, S) = 0$ for $i > 0$. This correspondence was announced by Schofield in a lecture in March 1995 in Krippen, Germany, but actually dates from 1991.) In fact our second corollary is already known, proved by Schofield and used in the proof of the correspondence, but his proof of the corollary involves difficult results about semi-invariants of quivers, whereas the proof here is quite elementary.

Let K be an algebraically closed field, Q a finite quiver with vertex set I, and let $\langle -, - \rangle$ be the Ringel form on $\mathbb{Z}^I$. If $\beta \in \mathbb{N}^I$ we write

$$\mathrm{Rep}_{KQ}(\beta) = \prod_{a: i \to j} \mathrm{Hom}(K^{\beta(i)}, K^{\beta(j)})$$

for the configuration space of representations of Q of dimension vector β, and if $y \in \mathrm{Rep}_{KQ}(\beta)$ we write K_y for the corresponding left KQ-module.

If X is a finitely generated left KQ-module and $y \in \mathrm{Rep}_{KQ}(\beta)$ then $\mathrm{Hom}(X, K_y)$ is a finite dimensional vector space. Moreover the function $y \mapsto \dim \mathrm{Hom}(X, K_y)$ is an upper semicontinuous function on $\mathrm{Rep}_{KQ}(\beta)$, and it follows that the minimum value of this function, denoted $\hom(X, \beta)$, is also its general value. The *rank* of a homomorphism $X \to K_y$ is the dimension vector of its image. For any $\alpha \in \mathbb{N}^I$ the set of homomorphisms of rank at most α is a closed subset of $\mathrm{Hom}(X, K_y)$. It follows that there is a unique maximal rank $\gamma_{X,y}$ of homomorphisms from X to K_y,

and that the set of homomorphisms of rank $\gamma_{X,y}$ is an open subset of $\mathrm{Hom}(X, K_y)$. The function $y \mapsto \gamma_{X,y}$ is constant on a non-empty open subset of $\mathrm{Rep}_{KQ}(\beta)$, and its general value is denoted $\gamma_{X,\beta}$, the *general rank* of a homomorphism from X to a representation of dimension β. (The general rank was introduced by Schofield [S2, Lemma 5.1] in case X is a general representation of some dimension vector. His argument adapts to prove that the function $y \mapsto \gamma_{X,y}$ is constant on a non-empty open subset, as claimed above, using the setup of our §4 in case X is an infinite dimensional finitely generated module.)

Our basic result is as follows. We hasten to say that its interest lies in its corollaries.

Theorem. *If X is a finite dimensional KQ-module and $\beta \in \mathbb{N}^I$ then*

$$\mathrm{hom}(X, \beta) = \langle \gamma_{X,\beta}, \beta \rangle + \dim U$$

for some non-empty open subset U of $\mathrm{Gr}_{KQ}(X, \gamma_{X,\beta})$.

Here, if α is a dimension vector, then $\mathrm{Gr}_{KQ}(X, \alpha)$ is the variety of submodules of X of codimension α (i.e. submodules $Y \subseteq X$ with X/Y of dimension vector α). In fact the theorem holds more generally for X a finitely generated module. The main problem is to make sense of $\mathrm{Gr}_{KQ}(X, \alpha)$. We discuss this in §4.

The theorem generalizes Theorem 5.2 of Schofield [S2] since for a sufficiently general representation X of dimension α the variety $\mathrm{Gr}_{KQ}(X, \gamma_{X,\beta})$ has all irreducible components of dimension $\langle \alpha - \gamma_{X,\beta}, \gamma_{X,\beta} \rangle$ (see the last line of [S2, p52]), so any non-empty open subset of this variety also has this dimension.

Let $K_0(KQ)$ be the Grothendieck group of finitely presented left KQ-modules modulo short exact sequences (or equivalently of finitely generated projective left KQ-modules modulo direct sums). If $\beta \in \mathbb{Z}^I$ there is a homomorphism $K_0(KQ) \to \mathbb{Z}$, also denoted β, defined by $\beta(KQe_i) = \beta(i)$ where e_i is the idempotent in the path algebra KQ corresponding to the vertex i. If X is finite dimensional then in fact $\beta(X) = \langle \underline{\dim} X, \beta \rangle$. A finitely presented KQ-module X is said to be β-*semistable* if $\beta(X) = 0$, and $\beta(Y) \geq 0$ for all finitely presented submodules Y of X. It is β-*stable* if in addition $\beta(Y) > 0$ whenever $Y \neq 0, X$.

In §2 we prove the following corollaries in the special case when X is finite dimensional. We deduce the general case in §3 using a universal localization argument suggested by A. Schofield.

Corollary 1. *If X is a finitely presented KQ-module and $\beta \in \mathbb{N}^I$ then*

$$\lim_{r \to \infty} \frac{1}{r} \mathrm{hom}(X, r\beta) = \max\{\beta(X/Y) \mid Y \subseteq X \text{ a finitely presented submodule}\}.$$

If β is an indivisible Schur root then there is a corresponding homological epimorphism from KQ to a simple artinian ring given by Schofield's theorem, and this has an associated Sylvester module rank function [S1, Theorem 7.12]. It can be constructed as follows: if χ_β is the function which sends a finitely presented module X to the number given by either side of the equation in Corollary 1, then the Sylvester module rank function is obtained by normalising χ_β so as to have value 1 at KQ.

Corollary 2. *If X is a finitely presented KQ-module which is β-semistable with $\beta \in \mathbb{N}^I$ then $\mathrm{hom}(X, r\beta) = 0$ for some integer $r > 0$.*

As an example, taking KQ to be a free algebra, Corollary 2 states that if A is an $n \times n$ matrix with entries in $K\langle x_1, \ldots, x_m \rangle$ and which is full, meaning that it

cannot be written as a product of an $n \times (n-1)$ and an $(n-1) \times n$ matrix, then the x_i can be specialized to $r \times r$ matrices over K in such a way that A becomes an invertible $(rn) \times (rn)$ matrix.

Some of these results were presented in lectures I gave at the Krippen meeting mentioned above, and I would like to thank the organizers, D. Happel, C. M. Ringel, and L. Unger. I would also particularly like to thank A. Schofield.

1. Proof of the theorem

Let X be a finite dimensional left KQ-module and let $\alpha, \beta \in \mathbb{N}^I$. If $e_i \in KQ$ is the idempotent corresponding to a vertex $i \in I$, then X has a decomposition $X = \bigoplus_i X_i$ where $X_i = e_i X$. If V is a finite dimensional vector space we write $\mathrm{Gr}(V, d)$ for the Grassmannian of subspaces of codimension d in V. The variety $\mathrm{Gr}_{KQ}(X, \alpha)$ of submodules of X of codimension α is the closed subset of

$$\prod_i \mathrm{Gr}(X_i, \alpha(i))$$

consisting of the tuples (Y_i) which are subrepresentations of X. We define a vector space $\mathrm{Hom}(X, \beta)$ via

$$\mathrm{Hom}(X, \beta) = \bigoplus_i \mathrm{Hom}_K(X_i, K^{\beta(i)}).$$

We write $\mathrm{Hom}(X, \beta)_\alpha$ for the locally closed subset consisting of those $\theta \in \mathrm{Hom}(X, \beta)$ with each θ_i of rank $\alpha(i)$. It is standard that the map

$$\rho : \mathrm{Hom}(X, \beta)_\alpha \longrightarrow \prod_i \mathrm{Gr}(X_i, \alpha(i))$$

sending θ to $(\mathrm{Ker}\,\theta_i)$ is a locally trivial bundle with fibre

$$\prod_i \mathrm{Inj}(K^{\alpha(i)}, K^{\beta(i)}),$$

where $\mathrm{Inj}(U, V)$ is the set of injective linear maps $U \to V$. Let $\mathrm{Hom}_{KQ}(X, \beta)_\alpha$ be the subset of $\mathrm{Hom}(X, \beta)_\alpha$ consisting of those θ with $(\mathrm{Ker}\,\theta_i)$ a subrepresentation of X. It is a closed subset of $\mathrm{Hom}(X, \beta)_\alpha$ since

$$\mathrm{Hom}_{KQ}(X, \beta)_\alpha = \rho^{-1}(\mathrm{Gr}_{KQ}(X, \beta)).$$

The following lemma is immediate.

Lemma 1.1. *The map* $\kappa : \mathrm{Hom}_{KQ}(X, \beta)_\alpha \to \mathrm{Gr}_{KQ}(X, \alpha)$ *sending* θ *to* $(\mathrm{Ker}\,\theta_i)$ *is a locally trivial bundle with fibre* $\prod_i \mathrm{Inj}(K^{\alpha(i)}, K^{\beta(i)})$.

We write $\mathrm{RepHom}_{KQ}(X, \beta)$ for the closed subset of $\mathrm{Rep}_{KQ}(\beta) \times \mathrm{Hom}(X, \beta)$ consisting of those pairs (y, θ) with θ a KQ-homomorphism from X to K_y. Similarly we define $\mathrm{RepHom}_{KQ}(X, \beta)_\alpha$.

Lemma 1.2. *If* $\alpha \leq \beta$ *then the map* $\phi : \mathrm{RepHom}_{KQ}(X, \beta)_\alpha \to \mathrm{Hom}_{KQ}(X, \beta)_\alpha$ *sending* (y, θ) *to* θ *is a locally trivial bundle with fibre*

$$\prod_{a:i \to j} \mathrm{Hom}(K^{\beta(i)-\alpha(i)}, K^{\beta(j)}).$$

Proof. $\operatorname{Hom}_{KQ}(X,\beta)_\alpha$ has a covering by open subsets of the form

$$U_{\epsilon\pi} = \{\theta \in \operatorname{Hom}_{KQ}(X,\beta)_\alpha \mid \prod_i \det(\pi_i\theta_i\epsilon_i) \neq 0\}$$

where $\epsilon_i : K^{\alpha(i)} \to X$ are injective linear maps and $\pi_i : K^{\beta(i)} \to K^{\alpha(i)}$ are surjective linear maps. We show that

$$\phi^{-1}(U_{\epsilon\pi}) \longrightarrow U_{\epsilon\pi}$$

is a trivial bundle. Observe that the map sending θ to $(\pi_i\theta_i\epsilon_i)^{-1}$ is a regular map of varieties from $U_{\epsilon\pi}$ to the set of $\beta(i) \times \beta(i)$ matrices. One can choose a complement to $\operatorname{Ker}(\pi_i)$ in $K^{\beta(i)}$, and using this one can find maps

$$\mu_i : K^{\beta(i)-\alpha(i)} \to K^{\beta(i)} \quad \text{and} \quad \lambda_i : K^{\beta(i)} \to K^{\beta(i)-\alpha(i)}$$

with $\lambda_i\mu_i$ the identity on $K^{\beta(i)-\alpha(i)}$ and $\operatorname{Im}(\mu_i) = \operatorname{Ker}(\pi_i)$. If $a : i \to j$ is an arrow, write x_a for the linear map $X_i \to X_j$ induced by multiplication by a. We show that the constructions
  (1) $z_a = y_a\mu_i$, and
  (2) $y_a = z_a\lambda_i + (\theta_j x_a - z_a\lambda_i\theta_i)\epsilon_i(\pi_i\theta_i\epsilon_i)^{-1}\pi_i$
define a 1–1 correspondence between

$$(y,\theta) \in \phi^{-1}(U_{\epsilon\pi}) \quad \text{and} \quad (\theta,z) \in U_{\epsilon\pi} \times \prod_{a:i\to j} \operatorname{Hom}(K^{\beta(i)-\alpha(i)}, K^{\beta(j)}).$$

This gives an isomorphism of varieties over $U_{\epsilon\pi}$, so proves that $\phi^{-1}(U_{\epsilon\pi}) \to U_{\epsilon\pi}$ is a trivial bundle, and the lemma follows.

First, given (θ,z), the element (y,θ) constructed according to (2) belongs to $\operatorname{RepHom}_{KQ}(X,\beta)$, and hence to $\phi^{-1}(U_{\epsilon\pi})$. To see this, we need to prove that $y_a\theta_i$ and $\theta_j x_a$ are equal. We have a decomposition $X_i = \operatorname{Im}\epsilon_i \oplus \operatorname{Ker}\theta_i$ since θ_i has rank $\alpha(i)$. Now $y_a\theta_i$ and $\theta_j x_a$ both vanish on $\operatorname{Ker}\theta_i$, for the fact that $\theta \in \operatorname{Hom}_{KQ}(X,\beta)$ means that the spaces $\operatorname{Ker}\theta_i$ form a subrepresentation of X, so $x_a(\operatorname{Ker}\theta_i) \subseteq \operatorname{Ker}\theta_j$. Also $y_a\theta_i$ and $\theta_j x_a$ agree on $\operatorname{Im}\epsilon_i$ since they have the same composition with ϵ_i.

Next if (θ,z) is given, one constructs (y,θ) using (2) and (θ,z') using (1), then

$$z_a' = z_a\lambda_i\mu_i + (\theta_j x_a - z_a\lambda_i\theta_i)\epsilon_i(\pi_i\theta_i\epsilon_i)^{-1}\pi_i\mu_i = z_a.$$

Finally if (y,θ) is given, one constructs (θ,z) using (1) and (y',θ) using (2), then the fact that $\theta_j x_a = y_a\theta_i$ implies that

$$y_a' = (y_a\mu_i)\lambda_i + (y_a\theta_i - (y_a\mu_i)\lambda_i\theta_i)\epsilon_i(\pi_i\theta_i\epsilon_i)^{-1}\pi_i$$
$$= y_a(\mu_i\lambda_i + \xi_i - \mu_i\lambda_i\xi_i)$$

where ξ_i is the idempotent endomorphism $\theta_i\epsilon_i(\pi_i\theta_i\epsilon_i)^{-1}\pi_i$ of $K^{\beta(i)}$. Now $\xi_i\mu_i = 0$ so $K^{\beta(i)} = \operatorname{Im}\xi_i \oplus \operatorname{Im}\mu_i$ by dimensions, and by checking compositions with ξ_i and μ_i we obtain

$$\mu_i\lambda_i + \xi_i - \mu_i\lambda_i\xi_i = 1_{K^{\beta(i)}}.$$

Thus $y_a' = y_a$. $\qquad\square$

Lemma 1.3. *Suppose that $\theta : Y \to Z$ is a locally trivial map of varieties with fibre F, an irreducible variety. If U is an open subset of Y, then $\theta(U)$ is open in Z and $\dim\theta(U) = \dim U - \dim F$.*

Proof. We may assume that θ is trivial, so $Y = Z \times F$ and θ is the projection. Then

$$\theta(U) = \bigcup_{f \in F} i_f^{-1}(U)$$

where i_f is the map $Z \to Z \times F$, $z \mapsto (z, f)$, so $\theta(U)$ is open in Z. Now every fibre of the map $U \to \theta(U)$ is a non-empty open subset of F, so has dimension $\dim F$. The claim follows. $\qquad\square$

Proof of the theorem. Let $\alpha = \gamma_{X,\beta}$. There is a non-empty open subset R of $\mathrm{Rep}_{KQ}(\beta)$ with $\dim \mathrm{Hom}(X, K_y) = \hom(X, \beta)$ and $\gamma_{X,y} = \alpha$ for all $y \in R$. Let

$$V = \{(y, \theta) \in \mathrm{RepHom}_{KQ}(X, \beta)_\alpha \mid y \in R\},$$

an open subset of $\mathrm{RepHom}_{KQ}(X, \beta)_\alpha$. Now all fibres of the projection $V \to R$ have dimension $\hom(X, \beta)$, so

$$\dim V = \dim R + \hom(X, \beta) = \dim \mathrm{Rep}_{KQ}(\beta) + \hom(X, \beta).$$

Applying Lemma 1.3 to the locally trivial map of Lemma 1.2 and the open subset V we have

$$\dim \phi(V) = \dim V - \dim \prod_{a:i \to j} \mathrm{Hom}(K^{\beta(i)-\alpha(i)}, K^{\beta(j)}).$$

Now applying Lemma 1.3 to the locally trivial map of Lemma 1.1 and the open subset $\phi(V)$ we have

$$\dim \kappa(\phi(V)) = \dim \phi(V) - \dim \prod_i \mathrm{Inj}(K^{\alpha(i)}, K^{\beta(i)}).$$

Combining these formulae we see that $\kappa(\phi(V))$ is an open subset of $\mathrm{Gr}_{KQ}(X, \alpha)$ of dimension

$$\hom(X, \beta) + \sum_{a:i \to j} \beta(i)\beta(j) - \sum_{a:i \to j} (\beta(i) - \alpha(i))\beta(j) - \sum_i \alpha(i)\beta(i)$$

This simplifies to $\hom(X, \beta) - \langle \alpha, \beta \rangle$, which proves the theorem. $\qquad\square$

2. Proofs of the corollaries in the finite dimensional case

Proof of Corollary 1 (for X finite dimensional). For any vectors $\rho, \sigma \in \mathbb{N}^I$ we have $\hom(X, \rho + \sigma) \leq \hom(X, \rho) + \hom(X, \sigma)$. Now if $\frac{1}{d}\hom(X, d\beta) = \lambda$ and we write $r = ad + b$ with $a, b \in \mathbb{N}$ and $b < d$, then

$$\frac{1}{r}\hom(X, r\beta) \leq \frac{1}{ad}\hom(X, (ad + b)\beta)$$
$$\leq \frac{1}{ad}\big(a\hom(X, d\beta) + b\hom(X, \beta)\big)$$
$$\leq \lambda + \frac{1}{a}\hom(X, \beta)$$

and this tends to λ as $r \to \infty$. It follows that the sequence $\frac{1}{r}\hom(X, r\beta)$ converges to its infimum μ.

If $Y \subseteq X$ then $\hom(X, r\beta) = \dim \mathrm{Hom}(X, M)$ for some M of dimension $r\beta$, so

$$
\begin{aligned}
\frac{1}{r} \hom(X, r\beta) &= \frac{1}{r} \dim \mathrm{Hom}(X, M) \\
&\geq \frac{1}{r} \dim \mathrm{Hom}(X/Y, M) \\
&\geq \frac{1}{r} \dim \mathrm{Hom}(X/Y, M) - \frac{1}{r} \dim \mathrm{Ext}(X/Y, M) \\
&= \frac{1}{r} \langle \underline{\dim}\, X/Y, r\beta \rangle = \langle \underline{\dim}\, X/Y, \beta \rangle = \beta(X/Y).
\end{aligned}
$$

Thus $\mu \geq \max\{\beta(X/Y) \mid Y \text{ is a submodule of } X\}$. As r increases the vector $\gamma_{X, r\beta}$ is bounded by the dimension vector of X, so some dimension vector α arises infinitely often; say for $r = r_1, r_2, \ldots$. By assumption there are maps from X to representations of dimension $r_i \beta$ of rank α, so there must be a subrepresentation Y of X of codimension α. By the theorem we have

$$
\frac{1}{r_i} \hom(X, r_i \beta) \leq \langle \alpha, \beta \rangle + \frac{1}{r_i} \dim \mathrm{Gr}_{KQ}(X, \alpha).
$$

Thus $\mu \leq \langle \alpha, \beta \rangle = \langle \underline{\dim}\, X/Y, \beta \rangle = \beta(X/Y)$. This proves the corollary. $\square$

Proof of Corollary 2 (for X finite dimensional). Any β-semistable representation has a filtration with β-stable quotients, so we may assume that X is β-stable. By Corollary 1 we have $\lim_{r \to \infty} \frac{1}{r} \hom(X, r\beta) = 0$.

As in the previous corollary let $\alpha = \gamma_{X, r\beta}$ for $r = r_1, r_2, \ldots$ and let Y be a submodule of X of codimension α. By the theorem

$$
\frac{1}{r_i} \hom(X, r_i \beta) = \langle \alpha, \beta \rangle + \frac{1}{r_i} \dim U_i
$$

with U_i open in $\mathrm{Gr}_{KQ}(X, \alpha)$. Letting $r_i \to \infty$ we must have $\langle \alpha, \beta \rangle = 0$. Thus $\beta(Y) = 0$, which contradicts the β-stability of X unless $\alpha = 0$ or $\alpha = \underline{\dim}\, X$. In either case $\mathrm{Gr}_{KQ}(X, \alpha)$ is reduced to a point, so that $\dim U_i = 0$, and hence $\hom(X, r_i \beta) = 0$, as required. $\square$

3. Passage to finitely presented modules

In this section we extend the corollaries from the case when X is finite dimensional to X being finitely presented (in case the quiver has oriented cycles, so that there are infinite dimensional finitely presented modules). The idea of using universal localization here was suggested to the author by A. Schofield.

We define a new quiver Q' as follows. If $I = \{1, \ldots, n\}$ then Q' has vertex set $I' = \{1', \ldots, n', 1'', \ldots, n''\}$. For each vertex $i \in I$ there is an arrow $v_i : i' \to i''$ in Q', and for each arrow $a : i \to j$ in Q there is an arrow $a' : i' \to j''$ in Q'. Observe that the vertices $1', \ldots, n'$ are sources while $1'', \ldots, n''$ are sinks. Thus KQ' is finite dimensional.

Let $A = KQ'$ and $B = M_2(KQ)$. There is an equivalence $KQ\text{-Mod} \to B\text{-Mod}$ sending a KQ-module M to the vector space $X = M^2$ with B acting as a matrix on a column vector. In what follows we usually make no distinction between a KQ-module and the corresponding B-module. (This is perfectly acceptable for categorical notions like Hom and Ext, but care is needed for traditional notions like the elements of a module.)

Let $\phi : A \to B$ be the homomorphism defined by

$$\phi(e_{i'}) = \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi(e_{i''}) = \begin{pmatrix} 0 & 0 \\ 0 & e_i \end{pmatrix}, \quad \phi(v_i) = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \quad \phi(a') = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

We write $B \otimes -$ for $B \otimes_A -$, and we write $_A X$ for the restriction of a B-module via ϕ. Restriction by ϕ sends a representation M of Q to the representation Z of Q' with $Z_{i'} = Z_{i''} = M_i$, with v_i the identity map $Z_{i'} \to Z_{i''}$, and with the linear map corresponding to an arrow a' the same as the linear map for M corresponding to a. Thus restriction by ϕ induces an equivalence from B-Mod to the full subcategory of KQ'-Mod consisting of those representations in which the v_i are isomorphisms. This observation implies that $\phi : A \to B$ is the universal localization of KQ' with respect to the maps $KQ'e_{i''} \to KQ'e_{i'}$ corresponding to the v_i. The next lemma collects some properties of universal localization, see [S1, Theorems 4.5, 4.7, 4.8].

Lemma 3.1. (1) *Any finitely presented B-module is induced from a finitely presented* (*hence finite dimensional*) *A-module.*

(2) $\operatorname{Hom}_B(X, Y) \cong \operatorname{Hom}_A(X, Y)$ *and* $\operatorname{Ext}^1_B(X, Y) \cong \operatorname{Ext}^1_A(X, Y)$ *for any B-modules X and Y.*

(3) $B \otimes_A X \cong X$ *and* $\operatorname{Tor}^A_1(B, X) = 0$ *for any B-module X.*

If $\beta \in \mathbb{Z}^I$ we define $\beta' \in \mathbb{Z}^{I'}$ by $\beta'(i') = \beta'(i'') = \beta(i)$. Clearly if X is a B-module of dimension vector β then $_A X$ has dimension vector β'.

Lemma 3.2. *If Z is a finite dimensional A-module and $\beta \in \mathbb{N}^I$, then*

(1) $\hom(B \otimes Z, \beta) = \hom(Z, \beta')$, *and*

(2) $\beta(B \otimes_A Z) \geq \beta'(Z)$, *with equality if* $\operatorname{Tor}^A_1(B, Z) = 0$.

Proof. (1) Since $\operatorname{Hom}_A(Z, {}_A Y) \cong \operatorname{Hom}_B(B \otimes Z, Y)$ we certainly have the relation $\hom(Z, \beta') \leq \hom(B \otimes Z, \beta)$. On the other hand, in a general representation of Q' of dimension β' the arrows v_i are isomorphisms, so the representation is the restriction of a B-module. Thus there is a B-module Y of dimension β with

$$\hom(Z, \beta') = \dim \operatorname{Hom}_A(Z, {}_A Y)$$
$$= \dim \operatorname{Hom}_B(B \otimes Z, Y) \geq \hom(B \otimes Z, \beta).$$

(2) If Z is projective we have equality $\beta(B \otimes_A Z) = \beta'(Z)$ since

$$B \otimes A e_{i'} \cong B \otimes A e_{i''}$$

corresponds to the KQ'-module $KQ'e_i$. In general if $0 \to P' \to P \to Z \to 0$ is a projective resolution then there is an exact sequence

$$0 \to \operatorname{Tor}^A_1(B, Z) \to B \otimes P' \to B \otimes P \to B \otimes Z \to 0,$$

and the kernel $\operatorname{Tor}^A_1(B, Z)$ must be a projective B-module since it embeds in $B \otimes P'$. Thus $\beta(\operatorname{Tor}^A_1(B, Z)) \geq 0$, and the result follows by additivity. $\square$

Lemma 3.3. *If Z is an A-module and $\overline{Z}$ is the image of Z under the natural map $Z \to B \otimes Z$, then $B \otimes Z \cong B \otimes \overline{Z}$ and $\operatorname{Tor}^A_1(B, \overline{Z}) = 0$.*

Proof. Tensoring the maps $Z \twoheadrightarrow \overline{Z} \hookrightarrow B \otimes Z$ with B we have

$$B \otimes Z \twoheadrightarrow B \otimes \overline{Z} \to B \otimes (B \otimes Z) \cong B \otimes Z$$

so $B \otimes Z \cong B \otimes \overline{Z}$. Also $\operatorname{Tor}^A_1(B, \overline{Z}) \hookrightarrow \operatorname{Tor}^A_1(B, B \otimes Z)$ since A is hereditary, and this last space is zero by Lemma 3.1. $\square$

Proof of Corollary 1 (*for X finitely presented*). As in the finite dimensional case $\frac{1}{r}\hom(X, r\beta)$ converges to its infimum μ and

$$\mu \geq \max\{\beta(X/Y) \mid Y \text{ a finitely presented submodule of } X\}.$$

It only remains to find a finitely presented submodule Y of X with $\beta(X/Y) \geq \mu$. Now X is induced from a finite dimensional A-module W. By the finite dimensional case of Corollary 1 we have

$$\inf\{\frac{1}{r}\hom(W, r\beta') \mid r \geq 1\} = \beta'(Z)$$

for some quotient Z of W. By Lemma 3.2 this becomes

$$\inf\{\frac{1}{r}\hom(X, r\beta) \mid r \geq 1\} \leq \beta(B \otimes Z)$$

and since $B \otimes Z$ is isomorphic to the quotient of X by a finitely presented submodule (using that path algebras are left coherent), we have the result.   □

Proof of Corollary 2 (*for X finitely presented*). By Lemma 3.1, the module X is induced from a finite dimensional A-module Z, and by Lemma 3.3 we may suppose that $\operatorname{Tor}_1^A(B, Z) = 0$. By assumption X is β-semistable. Now $\beta'(Z) = \beta(X) = 0$ by Lemma 3.2, and if Z has a quotient W with $\beta'(W) > 0$ then X has a quotient $B \otimes W$ with $\beta(B \otimes W) > 0$, contradicting the fact that X is β-semistable. Thus Z is β'-semistable. By the finite dimensional case of Corollary 2 there is some $r > 0$ with $\hom(Z, r\beta') = 0$, and then $\hom(X, r\beta) = 0$, as required.   □

4. EXTENSION OF THE THEOREM TO FINITELY GENERATED MODULES

In this section we extend the theorem to the case when X is a finitely generated module (assuming that Q has oriented cycles, so that not all finitely generated modules are finite dimensional). In order even to state the theorem, we need to define a variety $\operatorname{Gr}_{KQ}(X, \alpha)$ of submodules of X of codimension α. It is convenient to use the functorial language of Demazure and Gabriel [DG] in which a scheme (over K) is a certain type of functor from commutative K-algebras to sets. Recall that a module for a commutative ring has *rank r* if it is finitely generated and at each prime it is free of rank r. It is then automatically projective by [B, II, §5, no. 2]. If X is a finitely generated KQ-module and α is a dimension vector, we write $\mathbf{Gr}_{KQ}(X, \alpha)$ for the functor sending a commutative K-algebra R to the set of $R \otimes_K KQ$-submodules Y of $R \otimes_K X$ with the property that for each vertex i the R-module $e_i((R \otimes_K X)/Y)$ has rank $\alpha(i)$. We prove that $\mathbf{Gr}_{KQ}(X, \alpha)$ is a separated algebraic scheme. Its reduced induced structure is then the variety $\operatorname{Gr}_{KQ}(X, \alpha)$.

Theorem. *If X is a finitely generated KQ-module and $\beta \in \mathbb{N}^I$ then*

$$\hom(X, \beta) = \langle \gamma_{X,\beta}, \beta \rangle + \dim U$$

for some non-empty open subset U of $\operatorname{Gr}_{KQ}(X, \gamma_{X,\beta})$.

In order to prove the theorem (and indeed to prove that $\mathbf{Gr}_{KQ}(X, \alpha)$ is an algebraic scheme), we construct schemes $\mathbf{RepHom}_{KQ}(X, \beta)_\alpha$ and $\mathbf{Hom}_{KQ}(X, \beta)_\alpha$ and others corresponding to constructions in §1. The theorem then follows by the same argument as §1. We skip the details. (Note that an algebraic scheme has the same dimension as its reduced induced structure.)

We work in slightly greater generality. Let K be any field, let V be a vector space over K, possibly infinite dimensional, let A be a finitely generated K-algebra, and

let X be a finitely generated A-module. Fix a complete set of (not necessarily primitive) orthogonal idempotents e_i ($i \in I$, a finite set), so $e_i e_j = \delta_{ij} e_i$ and $\sum_{i \in I} e_i = 1$. Any A-module M has a decomposition $M = \bigoplus_i M_i$ where $M_i = e_i M$. We define functors as follows:

- $\mathbf{Gr}(V, r)$ is the functor which sends a commutative K-algebra R to the set of R-submodules W of $R \otimes_K V$ with $(R \otimes_K V)/W$ of rank r.
- $\mathbf{Hom}(V, m)$ is the functor sending R to $\mathrm{Hom}_R(R \otimes_K V, R^m)$.
- $\mathbf{Hom}(V, m)_r$ is the subfunctor of $\mathbf{Hom}(V, m)$ sending R to the set of θ with $\mathrm{Im}(\theta)$ a direct summand of R^m of rank r.

- $\mathbf{Gr}_A(X, \alpha)$ is the subfunctor of $\prod_i \mathbf{Gr}(X_i, \alpha(i))$ consisting of those tuples $Y_i \subseteq R \otimes_K X_i$ with $\bigoplus_i Y_i$ an $R \otimes_K A$-submodule Y of $R \otimes_K X$.
- $\mathbf{Hom}_A(X, \beta)_\alpha$ is the subfunctor of $\prod_i \mathbf{Hom}(X_i, \beta(i))_{\alpha(i)}$ sending R to the set of tuples (θ_i) with $\bigoplus_i (\mathrm{Ker}\, \theta_i)$ an $R \otimes_K A$-submodule of $R \otimes_K X$.
- $\mathbf{Rep}_A(\beta)$ sends R to the set of $R \otimes_K A$-module structures M on the R-module $\bigoplus_i R^{\beta(i)}$, with e_i acting as the projection onto $R^{\beta(i)}$.
- $\mathbf{RepHom}_A(X, \beta)$ is the subfunctor of $\mathbf{Rep}_A(\beta) \times \prod_i \mathbf{Hom}(X_i, \beta(i))$ sending R to the pairs $(M, (\theta_i))$ with the θ_i defining an $R \otimes_K A$-module homomorphism $R \otimes_K X \to M$.
- $\mathbf{RepHom}_A(X, \beta)_\alpha$ is the subfunctor of $\mathbf{RepHom}_A(X, \beta)$ defined by also demanding that $\theta_i \in \mathbf{Hom}(X_i, \beta(i))_{\alpha(i)}(R)$.

The first lemma is presumably already known, but I couldn't find a suitable reference. Of course it is all standard in case V is finite dimensional.

Lemma 4.1. (1) $\mathbf{Hom}(V, m)$ *is an affine scheme.*

(2) $\mathbf{Gr}(V, r)$ *is a separated scheme.*

(3) $\mathbf{Hom}(V, m)_r$ *is a locally closed subscheme of* $\mathbf{Hom}(V, m)$.

(4) *The natural transformation* $\kappa : \mathbf{Hom}(V, m)_r \to \mathbf{Gr}(V, r)$, *which sends a homomorphism* $R \otimes_K V \to R^m$ *to its kernel, is a locally trivial bundle with fibre* $\mathbf{Hom}(K^r, m)_r$.

Proof. (1) If V has basis v_λ ($\lambda \in \Lambda$) then
$$\mathrm{Hom}_R(R \otimes_K V, R^m) \cong \mathrm{Hom}_{K\text{-alg}}(K[x_{\lambda, i} \mid \lambda \in \Lambda, 1 \le i \le m], R).$$

(2) We show that $\mathbf{G} = \mathbf{Gr}(V, r)$ is a local functor by modifying the argument of [DG, I, §1, 3.13]. Suppose that R is a commutative K-algebra, (f_i, x_i) is a partition of unity, and $Y_i \in \mathbf{G}(R_{f_i})$ are compatible in the sense that Y_i and Y_j have the same image in $\mathbf{G}(R_{f_i f_j})$ for all i and j. We need to show that there is some $Y \in \mathbf{G}(R)$ inducing Y_i in each $\mathbf{G}(R_{f_i})$. Let $B = \prod_i R_{f_i}$. By the argument of [DG], the B-module $\prod_i Y_i$ is induced from an R-submodule Y of $R \otimes_K V$. It remains to show that the R-module $M = (R \otimes_K V)/Y$ has rank r. Now the induced module $B \otimes_R M$ is isomorphic to $\prod_i (R_{f_i} \otimes_K V)/Y_i$, so it has rank r, and it follows that M is finitely generated projective by [B, I, §3, no. 6, Proposition 12]. By faithful flatness any prime $\mathfrak{q}$ of R comes from a prime $\mathfrak{p}$ of B, and $B_{\mathfrak{p}} \otimes_{R_{\mathfrak{q}}} M_{\mathfrak{q}} \cong (B \otimes_R M)_{\mathfrak{p}}$ is a free $B_{\mathfrak{p}}$-module of rank r, so $M_{\mathfrak{q}}$ is free of rank r. Thus M does have rank r.

Next we find a covering of $\mathbf{G}$ by affine open subfunctors. Let Q be an r-dimensional subspace of V, and define
$$\mathbf{U}_Q(R) = \{Y \in \mathbf{G}(R) \mid Y \oplus (R \otimes_K Q) = R \otimes_K V\}.$$

The argument of [DG, I, §1, Example 3.9] shows that $\mathbf{U}_Q$ is an open subfunctor of $\mathbf{G}$. Let P be a complement to Q in V, and let p_λ ($\lambda \in \Lambda$) be a basis of P. The

submodules Y of $R \otimes_K V$ complementary to $R \otimes_K Q$ are in 1–1 correspondence with homomorphisms from $R \otimes_K P$ to $R \otimes_K Q$, so

$$
\begin{aligned}
\mathbf{U}_Q(R) &\cong \operatorname{Hom}_R(R \otimes_K P, R \otimes_K Q) \\
&\cong \operatorname{Hom}_K(P, R \otimes_K Q) \\
&\cong \operatorname{Hom}_{K\text{-alg}}(K[x_{\lambda,i} \mid \lambda \in \Lambda, 1 \le i \le r], R).
\end{aligned}
$$

Thus $\mathbf{U}_Q$ is an affine scheme.

For any extension field F of K and any $Y \in \mathbf{G}(F)$ there is some Q such that $Y \in \mathbf{U}_Q(F)$. Thus the $\mathbf{U}_Q$ cover $\mathbf{G}$. It follows that $\mathbf{G}$ is a scheme.

Suppose R is a commutative K-algebra and $Y, Y' \in \mathbf{G}(R)$. If $\phi : R \to S$ is a homomorphism of commutative K-algebras then $S \otimes_R Y \subseteq S \otimes_R Y'$ if and only if $\phi(\psi(Y' + y)) = 0$ for all $y \in Y$ and $\psi : (R \otimes_K V)/Y' \to R$. Thus $S \otimes_R Y = S \otimes_R Y'$ if and only if $\phi(I) = 0$ where I is the ideal of R generated by the elements $\psi(Y' + y)$ and $\psi'(Y + y')$ with $y \in Y$, $y' \in Y'$, $\psi : (R \otimes_K V)/Y' \to R$ and $\psi' : (R \otimes_K V)/Y \to R$. This implies that $\mathbf{G}$ is separated.

(3) Let $\theta : R \otimes_K V \to R^m$ be a homomorphism and write I_d for the ideal in R generated by all $d \times d$ minors of θ (after choosing a basis of V). If $\phi : R \to S$ is a homomorphism of commutative K-algebras then $S \otimes_R \operatorname{Im}(\theta)$ is a summand of S^m of rank r if and only if $S \otimes_R \operatorname{Coker}(\theta)$ has rank $m - r$. By the theory of Fitting ideals (see for example [E, §20.2], but this only deals with noetherian rings) this holds if and only if $S\phi(I_r) = S$ and $\phi(I_{r-1}) = 0$. The claim follows.

(4) $\kappa^{-1}(\mathbf{U}_Q)(R) = \{\theta \in \mathbf{Hom}(V, m)_r(R) \mid \operatorname{Ker}(\theta) \oplus (R \otimes_K Q) = R \otimes_K V\}$ and the map to $\mathbf{Hom}(Q, m)_r(R) \times \mathbf{Gr}(V, r)(R)$ sending θ to $(\theta|_{R \otimes Q}, \operatorname{Ker}\theta)$ is a bijection. Thus $\kappa^{-1}(\mathbf{U}_Q) \to \mathbf{U}_Q$ is a trivial bundle with fibre $\mathbf{Hom}(Q, m)_r$. $\qquad\square$

Lemma 4.2. $\mathbf{Rep}_A(\beta)$ and $\mathbf{RepHom}_A(X, \beta)$ are affine algebraic schemes.

Proof. For $\mathbf{Rep}_A(\beta)$ this is fairly standard, see for example [P, Chapter 4]. To study $\mathbf{RepHom}_A(X, \beta)$, write X as a quotient of A^k, for then $\mathbf{RepHom}_A(X, \beta)$ is a closed subfunctor of $\mathbf{RepHom}_A(A^k, \beta)$. Now a homomorphism from $R \otimes_K A^k$ to an $R \otimes_K A$-module M is given by a k-tuple of elements of M, so $\mathbf{RepHom}_A(A^k, \beta)$ is isomorphic to the product of $\mathbf{Rep}_A(\beta)$ with $k \times \sum_i \beta(i)$-dimensional affine space. Thus it is an affine algebraic scheme. $\qquad\square$

Lemma 4.3. $\mathbf{Gr}_A(X, \alpha)$, $\mathbf{Hom}_A(X, \beta)_\alpha$ and $\mathbf{RepHom}_A(X, \beta)_\alpha$ are separated algebraic schemes and the natural transformation $\mathbf{Hom}_A(X, \beta)_\alpha \to \mathbf{Gr}_A(X, \alpha)$ is a locally trivial bundle with fibre $\prod_i \mathbf{Hom}(K^{\alpha(i)}, \beta(i))_{\alpha(i)}$.

Proof. $\mathbf{RepHom}_A(X, \beta)_\alpha$ is a locally closed subscheme of $\mathbf{RepHom}_A(X, \beta)$, so it is separated and algebraic.

Let R be a commutative K-algebra and let $Y_i \in \mathbf{Gr}(X_i, \alpha(i))(R)$. Write $Y = \bigoplus_i Y_i$. If $\phi : R \to S$ is a homomorphism of commutative K-algebras, then $S \otimes_R Y$ is an $S \otimes_K A$-submodule of $S \otimes_K X$ if and only if $\phi(\psi(Y + ay)) = 0$ for all $y \in Y$, all $a \in A$, and all $\psi : (R \otimes_K X)/Y \to R$. This implies that $\mathbf{Gr}_A(X, \alpha)$ is a closed subscheme of $\prod_i \mathbf{Gr}(X_i, \alpha(i))$.

Now $\mathbf{Hom}_A(X, \beta)_\alpha$ is the pullback of $\mathbf{Gr}_A(X, \alpha)$ along

$$
\prod_i \mathbf{Hom}(X_i, \beta(i))_{\alpha(i)} \longrightarrow \prod_i \mathbf{Gr}(X_i, \alpha(i)),
$$

so it is a closed subscheme of the product $\prod_i \mathbf{Hom}(X_i, \beta(i))_{\alpha(i)}$ and also the natural transformation $\mathbf{Hom}_A(X, \beta)_\alpha \to \mathbf{Gr}_A(X, \alpha)$ is a locally trivial bundle with fibre $\prod_i \mathbf{Hom}(K^{\alpha(i)}, \beta(i))_{\alpha(i)}$.

Now we have $\mathbf{RepHom}_A(X, \alpha)_\alpha \cong \mathbf{Hom}_A(X, \alpha)_\alpha$ since if $\theta_i : R \otimes_K X_i \to R^{\alpha(i)}$ are surjective maps and $\bigoplus_i \operatorname{Ker} \theta_i$ is an $R \otimes_K A$-submodule of $R \otimes_K X$, then there is a unique $R \otimes_K A$-module structure on $\bigoplus_i R^{\alpha(i)}$ such that (θ_i) is an $R \otimes_K A$-module homomorphism $R \otimes_K X \to \bigoplus_i R^{\alpha(i)}$. Thus $\mathbf{Hom}_A(X, \alpha)_\alpha$ is algebraic, so the bundle $\mathbf{Hom}_A(X, \alpha)_\alpha \to \mathbf{Gr}_A(X, \alpha)$ implies that $\mathbf{Gr}_A(X, \alpha)$ is algebraic. Now for general β the bundle $\mathbf{Hom}_A(X, \beta)_\alpha \to \mathbf{Gr}_A(X, \alpha)$ implies that $\mathbf{Hom}_A(X, \beta)_\alpha$ is algebraic. $\qquad\square$

References

[B]   N. Bourbaki, *Algèbre Commutative*, Hermann, Paris, 1961–83.

[DG] M. Demazure and P. Gabriel, *Groupes Algébriques*, Tome 1, Masson, Paris, 1970. MR **46:**1800

[E]   D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Math. 150, Springer-Verlag, New York, 1995. CMP 95:10

[P]   C. Procesi, *Rings with polynomial identities*, Marcel Dekker, New York, 1973. MR **51:**3214

[S1] A. Schofield, *Representations of rings over skew fields*, London Math. Soc. Lec. Note Ser. 92, Cambridge Univ. Press, 1985. MR **87c:**16001

[S2] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. **65** (1992), 46–64. MR **93d:**16014

Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England

E-mail address: w.crawley-boevey@leeds.ac.uk