The Calderón projector for the odd signature operator and spectral flow calculations in 3-dimensional topology

Hans U. Boden, Christopher M. Herald, and Paul Kirk

Abstract. We describe describe analytical aspects of the theory of spectral and boundary value problems for the odd signature operator twisted by a flat connection and outline how these are used to compute gauge-theoretic topological invariants of 3-manifolds.

1. Introduction

The Calderón projector and its range, the Cauchy data space, are well known and important tools in the study of boundary value problems and in the application of cut-and-paste techniques for Dirac operators. In the applications of the theory of Dirac operators to geometric topology, the odd signature operator as defined in the fundamental article of Atiyah, Patodi, and Singer [**2**] plays a prominent role. On a closed manifold, its kernel can be identified with cohomology and hence is independent of the choice of Riemannian metric. More subtly, spectral invariants such as spectral flow and η invariants also have topological properties for which no general analysis free definitions exist. One of the purposes of this article is to catalog some of the analytical properties of the Calderón projector which are special to the odd signature operator developed in the articles by Daniel and Kirk [**14**] and Kirk and Lesch [**24**].

The other purpose is to indicate the analytical aspects of the spectral flow calculations needed to compute $SU(3)$ Casson invariants defined by Boden and Herald in [**5**]. In a ground breaking article, Taubes [**30**] interpreted Casson's $SU(2)$ invariant of homology 3-spheres in gauge theory terms. The spectral flow of the odd signature operator entered into the formulation as a means to orient moduli spaces. Taubes' article has had a lasting impact on the study of analysis on 3-manifolds and forms the blueprint on which many constructions of topological invariants of 3-manifolds are based. This includes the work of the authors on $SU(3)$ generalizations of Casson's $SU(2)$ invariant, reported on in the articles [**5, 6, 7, 8**].

MSC-class: 57M27, 58J32. The first author was supported by the Natural Sciences and Engineering Research Council of Canada. The second author was supported by the National Science Foundation under grant no. DMS-0072564. The third author was supported by the National Science Foundation under grant no. DMS-0202148.

It is typical in geometric topology to carry out calculations of topological invariants using cut-and-paste constructions. The articles [**7**] and [**8**] contain calculations of the $SU(3)$ Casson invariant for Seifert-fibered homology 3-spheres. These calculations necessarily involve cut-and-paste problems for spectral flow, using techniques originating in articles by Yoshida [**31**], Nicolaescu [**27**], Cappell, Lee, and Miller [**12**], Kirk and Klassen [**22**], and Daniel and Kirk [**14**].

The calculations in [**7, 8**] also involve many other techniques, including the analysis of $SU(2)$ and $SU(3)$ flat moduli spaces on homology spheres and knot complements using algebraic-geometric techniques, analysis of perturbations of the flatness equations, calculations of Chern-Simons invariants, and enumeration of lattice points in convex polytopes. Consequently, the cut-and-paste analysis of the odd signature operator is treated only tersely there, and the present article gives an expanded and somewhat simplified exposition of the analytical methods used in those articles. The interested reader may wish to read [**8**] and [**7**] in conjunction with the present article.

We now outline the contents of this article. In Section 2 we recall the definition of the odd signature operator twisted by a connection and its tangential operator, the de Rham operator. We explain how the kernels of these operators and the scattering Lagrangian are determined topologically when the connection is flat. We discuss the spectral decomposition of the tangential operator and the corresponding Atiyah-Patodi-Singer (APS) boundary conditions. One of the main technical difficulties which is addressed in our work is the fact that the APS boundary conditions do not vary continuously as the twisting connection is varied.

In Section 3 we recall the definition of the Calderón projector and the Cauchy data space for the odd signature operator acting on a manifold with boundary. (We find it more intuitive to work with the Cauchy data spaces rather than the Calderón projector.) In Section 4 we state and outline a proof of a theorem which identifies the adiabatic limit of the Cauchy data space, using a theorem of Nicolaescu [**27**]. The adiabatic limit of the Cauchy data space for the odd signature operator twisted by a flat connection is described as the graded vector space associated to a filtration determined by the eigenspace decomposition of the tangential operator and the Cauchy data space. A precise description of the adiabatic limit is the key to the proofs of Theorem 5.1 and Theorem 5.2 concerning the behavior of Cauchy data spaces for the two parts in a decomposition of a closed manifold along a separating hypersurface. Theorems 5.1 and 5.2 illuminate the essential analytical property of the odd signature operator which distinguishes it from Dirac operators of general type. Roughly speaking it says that if one replaces a closed manifold with the two pieces in a decomposition along a separating hypersurface with infinite collars attached, then the L^2-solutions to $D_A\phi = 0$ on each piece do not interact. The precise statement is key to the proof of a splitting theorem for η and ρ_α invariants in [**24**] and simplifies the spectral flow calculations of [**8**].

Section 6 describes Taubes' approach to Casson's $SU(2)$ invariant and $SU(3)$ generalizations. We also discuss perturbations, which are needed to define generalized Casson invariants, but which lead to non-Dirac operators, indeed non-pseudodifferential operators. We discuss the results of an article by Kirk, Himpel, and Lesch [**21**] which show how to extend the construction of the Calderón projector to this generalized class of operators.

Sections 7, 8, and 9 contain a description of the cut-and-paste methods to compute the spectral flow terms which occur in the definitions of the $SU(3)$ Casson invariant, using the technology of Cauchy data spaces.

2. The odd signature operator

The odd signature operator takes as input a Hermitian $\mathbb{C}^n$ bundle $\mathscr{E} \to X$ over a compact, Riemannian manifold X with or without boundary of dimension $2k+1$, and a unitary connection A on $\mathscr{E}$. The connection A induces a *covariant derivative*

$$d_A : C^\infty(X;\mathscr{E}) \to C^\infty(X, \mathscr{E} \otimes T^*X),$$

which extends to a generalized exterior derivative

$$d_A : \Omega^p(X;\mathscr{E}) \to \Omega^{p+1}(X;\mathscr{E})$$

by the Leibniz rule

$$d_A(\omega \wedge \tau) = d_A(\omega) \wedge \tau + (-1)^{|\omega|}\omega \wedge d_A(\tau).$$

The Riemannian metric on X induces a (fiberwise) Hodge $*$ operator

$$* : \mathscr{E}_x \otimes \Lambda^p(T^*_x X) \to \mathscr{E}_x \otimes \Lambda^{2k+1-p}(T^*_x X).$$

The Hermitian metric on $\mathscr{E}$ and Riemannian metric on X induce inner products $\langle\ ,\ \rangle_x$ on the vector spaces $\mathscr{E}_x \otimes \Lambda^p(T^*_x X)$. These structures extend to sections, yielding the L^2-inner product on $\Omega^p(X;\mathscr{E})$:

$$(\omega, \tau) = \int_X \langle \omega_x, \tau_x \rangle_x,$$

and the Hodge $*$ isometry

$$* : \Omega^p(X;\mathscr{E}) \to \Omega^{2k+1-p}(X;\mathscr{E}).$$

We use the notation $\Omega^{ev}(X;\mathscr{E}) = \oplus_p \Omega^{2p}(X;\mathscr{E})$ and $\Omega^*(X;\mathscr{E}) = \oplus_p \Omega^p(X;\mathscr{E})$. If $Y \subset X$ is a submanifold then we denote forms on Y with values in the restriction $\mathscr{E}|_Y$ by $\Omega^p(Y;\mathscr{E})$. Similar notation will be used for cohomology groups.

With this notation in place, the *odd signature operator coupled to the connection* A

$$D_A : \Omega^{ev}(X;\mathscr{E}) \to \Omega^{ev}(X;\mathscr{E}),$$

is a formally self adjoint operator acting on sections of the bundle of even degree differential forms on X with values in $\mathscr{E}$, defined by

$$D_A(\beta) = i^{n+1}(-1)^{p-1}(*d_A - d_A*)(\beta) \quad \text{for} \quad \beta \in \Omega^{2p}(X;\mathscr{E}). \tag{2.1}$$

The odd signature operator D_A is a generalized Dirac operator, and as such enjoys several nice properties including the unique continuation property. (See the book [**10**] by Booss-Bavnbek and Wojciechowski for a comprehensive treatment of generalized Dirac operators and boundary value problems.) For the purpose of this article, the most important feature that distinguishes the odd signature operator from a general Dirac operator is the following consequence of the de Rham theorem.

Suppose that the connection A is flat (i.e. $d_A \circ d_A = 0$) and that X is closed. Then $D_A(\omega) = 0$ if and only if $d_A\omega = 0$ and $d^*_A(\omega) = 0$, where d^*_A denotes the L^2-adjoint of d_A and is given by the formula $d^*_A(\omega) = (-1)^{|\omega|} * d_A * \omega$. In other words, $D_A(\omega) = 0$ if and only if ω is a harmonic form in the elliptic complex

$$\cdots \xrightarrow{d_A} \Omega^{p-1}(X;\mathscr{E}) \xrightarrow{d_A} \Omega^p(X;\mathscr{E}) \xrightarrow{d_A} \Omega^{p+1}(X;\mathscr{E}) \xrightarrow{d_A} \cdots. \tag{2.2}$$

The Hodge and de Rham theorems identify the space of harmonic forms of the complex (2.2) with the singular cohomology $H^*(X;\mathbb{C}^n_\alpha)$ of X with local coefficients determined by the holonomy representation $\alpha = hol_A : \pi_1 X \to U(n)$ of the flat connection A. In particular, *the dimension of the kernel of D_A is independent of the choice of Riemannian metric on X*. This property gives us the flexibility to deform the metric to one suited to cut-and-paste operations without losing information. When A is not flat (2.2) is not a complex and the kernel of D_A is generally metric dependent.

The assumption that X be closed in the previous paragraph can be removed by imposing suitable boundary conditions. Classically this would be done using Dirichlet ($i^*(\omega) = 0$) or Neumann ($i^*(*\omega) = 0$) boundary conditions, where $i : \partial X \subset X$ denotes the inclusion of the boundary. This works well on the level of the elliptic complex (2.2); the space of harmonic forms of the resulting Dirichlet complex is isomorphic to $H^*(X,\partial X;\mathbb{C}^n_\alpha)$ and the space of harmonic forms of the resulting Neumann complex is isomorphic to $H^*(X;\mathbb{C}^n_\alpha)$ (see e.g. Duff and Spencer's article [**15**]). However, these local boundary conditions are not self adjoint. More precisely, the operator D_A with domain $\{\omega \mid i^*(\omega) = 0\}$ has as adjoint D_A with domain $\{\omega \mid i^*(*\omega) = 0\}$. In other words, the Neumann and Dirichlet boundary conditions are adjoint. Since our focus is on spectral properties of self adjoint realizations of D_A we are forced to consider non-local self adjoint elliptic boundary conditions, and this leads us to apply the corresponding Calderón-Seeley theory.

Suppose that ∂X is nonempty. Give X a Riemannian metric so that a collar of ∂X is isometric to $[0,\epsilon) \times \partial X$. Let A be a connection on X. We assume that A is in cylindrical form on the collar. This means that there is a $U(n)$ connection a on $\mathscr{E}|_{\partial X}$ so that the restriction of A to the collar $[0,\epsilon)\times\partial X$ is of the form

$$A_{[0,\epsilon)\times\partial X} = q^*(a),$$

where $q : [0,\epsilon)\times\partial X \to \partial X$ is the projection to the second factor. Let $d_a : \Omega^p(\partial X;\mathscr{E}) \to \Omega^{p+1}(\partial X;\mathscr{E})$ denote the associated coupled de Rham operator on ∂X, thus $d_A = d_a + dt \wedge \frac{\partial}{\partial t}$. Any connection which is flat in a neighborhood of the boundary can be gauge transformed into cylindrical form.

Define a restriction map

$$r : \Omega^{ev}(X;\mathscr{E}) \to \Omega^*(\partial X;\mathscr{E})$$

by the formula

$$r(\beta) = i^*(\beta) + i^*(*\beta) \tag{2.3}$$

where $i : \partial X \hookrightarrow X$ denotes the inclusion of the boundary.

To avoid confusion we denote the Hodge $*$ operator on the boundary by $\hat{*}$, thus

$$\hat{*} : \Omega^p(\partial X;\mathscr{E}) \to \Omega^{2k-p}(\partial X;\mathscr{E}).$$

We use $\hat{*}$ to define

$$J : \Omega^*(\partial X;\mathscr{E}) \to \Omega^*(\partial X;\mathscr{E})$$

by

$$J(\beta) = \begin{cases} i^{k-1}(-1)^p \hat{*}\,\beta & \text{if } \beta \in \Omega^{2p}(\partial X;\mathscr{E}), \\ i^{1-k}(-1)^q \hat{*}\,\beta & \text{if } \beta \in \Omega^{2q+1}(\partial X;\mathscr{E}). \end{cases} \tag{2.4}$$

We define the *de Rham operator* coupled to the connection a,

$$T_a : \Omega^*(\partial X;\mathscr{E}) \to \Omega^*(\partial X;\mathscr{E})$$

by

$$T_a(\beta) = (-1)^{p+1}(d_a\hat{*} + \hat{*}d_a)\beta \quad \text{for } \beta \in \Omega^p(\partial X; \mathscr{E}).$$

The operator T_a is a self adjoint generalized Dirac operator with respect to the L^2 inner product on the vector space of $\mathscr{E}|_{\partial X}$-valued differential forms on ∂X:

$$(\phi, \tau)_{\partial X} = \int_{\partial X} \langle \phi, \tau \rangle_x, \quad \phi, \tau \in L^2\big(\Omega^*(\partial X; \mathscr{E})\big). \tag{2.5}$$

Since ∂X is closed, T_a has discrete spectrum with each eigenvalue of finite multiplicity.

The following lemma is well known and routine to verify (see [**24**] for a proof).

LEMMA 2.1. *The restriction map r of Equation* (2.3) *induces an identification*

$$\Phi : \Omega^{ev}([0,\epsilon) \times \partial X; \mathscr{E}) \to C^\infty([0,\epsilon), \Omega^*(\partial X; \mathscr{E})),$$

which extends to a unitary isometry on the L^2 completions. Moreover,

$$\Phi D_A \Phi^{-1} = J(\tfrac{\partial}{\partial x} + T_a), \tag{2.6}$$

where x denotes the collar coordinate. T_a reverses the parity of forms, and the formulas

$$JT_a = -T_a J, \ J^2 = -I \tag{2.7}$$

hold.

Suppose furthermore that a is a flat connection on ∂X. Then $T_a d_a = -d_a T_a$ and $T_a d_a^* = -d_a^* T_a$, where $d_a^* = -\hat{*} d_a \hat{*}$ is the L^2 adjoint of d_a. Moreover, T_a^2 preserves the subspace $\Omega^p(\partial X; \mathscr{E})$ for each p, and the kernel of T_a is identified using the Hodge theorem with the cohomology of the complex

$$\cdots \xrightarrow{d_a} \Omega^{p-1}(\partial X; \mathscr{E}) \xrightarrow{d_a} \Omega^p(\partial X; \mathscr{E}) \xrightarrow{d_a} \Omega^{p+1}(\partial X; \mathscr{E}) \to \cdots. \tag{2.8}$$

The de Rham isomorphism identifies this cohomology with the singular cohomology $H^*(\partial X; \mathbb{C}^n_\alpha)$ with local coefficients determined by the holonomy $\alpha : \pi_1(\partial X) \to U(n)$ of the flat connection a. □

Equation (2.6) says that D_A is of Atiyah-Patodi-Singer type near the boundary of X, with *tangential operator* T_a. Equation (2.7) implies that the spectrum of T_a is symmetric, and J takes the subspace of λ eigenvectors isometrically to the subspace of $-\lambda$ eigenvectors. In particular J preserves $\ker T_a$.

REMARK 2.2. Since the vector bundle $\mathscr{E}$ is to be fixed throughout, to simplify notation we abbreviate $L^2\big(\Omega^{ev}(X; \mathscr{E})\big)$ by $L^2(X)$ and $L^2\big(\Omega^*(\partial X; \mathscr{E})\big)$ by $L^2(\partial X)$.

DEFINITION 2.3. A *Hermitian symplectic space* is a complex Hilbert space $(H, \langle\ ,\ \rangle)$ with an isometry $J : H \to H$ satisfying $J^2 = -I$ such that the i and $-i$ eigenspaces of J have the same dimension. Thus, if H is infinite dimensional, the i and $-i$ eigenspaces of J should be infinite dimensional.

A subspace $L \subset H$ is called *isotropic* if $JL \subset L^\perp$ and *Lagrangian* if $JL = L^\perp$. Note that a Lagrangian subspace is closed. A subspace $K \subset H$ is a *symplectic subspace* provided $JK = K$ and the i and $-i$ eigenspaces of $J|_K$ have the same dimension.

The symplectic space we will use is $L^2(\partial X)$ with L^2-inner product (2.5) and J defined by (2.4). We will often have to refer to subspaces of $L^2(\partial X)$ spanned by various eigenvectors of T_a. If λ is an eigenvalue of T_a, we let E_λ denote the corresponding eigenspace. If $\gamma \subset \mathbb{R}$ is a subset, we denote by E_γ the span of all eigenvectors of T_a with eigenvalue $\lambda \in \gamma$. Because the spectrum of T_a is discrete, E_γ is a finite dimensional subspace of $L^2(\partial X)$ whenever γ is bounded. When γ is not bounded, we take E_γ to be the span in the L^2 sense, meaning we take the L^2 closure. For example, the positive eigenspan of T_a is denoted by $E_{(0,\infty)}$, and $E_{(-\nu,\nu]}$ denotes the span of eigenvectors with eigenvalue λ satisfying $-\nu < \lambda \leq \nu$.

By Equation (2.7), the positive (resp. negative) eigenspan of T_a is isotropic in $L^2(\partial X)$. It is Lagrangian if and only if $\ker T_a = 0$.

If $\lambda > 0$ is an eigenvalue of T_a then $E_{-\lambda} \oplus E_\lambda$ is a finite dimensional symplectic subspace of $L^2(\partial X)$, and the subspaces $E_{-\lambda}$ and E_λ are Lagrangian in $E_{-\lambda} \oplus E_\lambda$. Other useful symplectic subspaces of $L^2(\partial X)$ are $E_{[-\mu,\mu]}$ and $E_{(-\infty,-\mu]} \oplus E_{[\mu,\infty)}$. That these subspaces are preserved by J follows from Equation (2.7).

Since $\ker T_a$ is preserved by J it is also a symplectic subspace. (The fact that the i and $-i$ eigenspaces are equidimensional relies on the assumption that $(\partial X, a)$ bounds (X, A).) The geometry of ∂X does not by itself specify a Lagrangian subspace of $\ker T_a$, but in our situation there is a natural Lagrangian subspace $S_A \subset \ker T_a$, called the *scattering Lagrangian*, defined in [**2**] as the limiting values of extended L^2-solutions to $D_A(\phi) = 0$. We will give another definition of the scattering Lagrangian in terms of the Cauchy data space below, but we mention the following folklore result. Associated to the flat connection A is its holonomy representation $\alpha : \pi_1(X) \to U(n) \subset GL(\mathbb{C}^n)$ and the corresponding twisted cohomology groups $H^*(X; \mathbb{C}^n_\alpha)$.

Theorem 2.4. *(Corollary 8.4 of* [**24**]*) With respect to the identification of* $\ker T_a$ *with the cohomology group* $H^*(\partial X; \mathbb{C}^n_\alpha)$ *given by the Hodge theorem, the scattering Lagrangian* S_A *equals the image of the restriction map*

$$H^*(X; \mathbb{C}^n_\alpha) \to H^*(\partial X; \mathbb{C}^n_\alpha).$$

Given any Lagrangian subspace $V \subset H^*(\partial X; \mathbb{C}^n_\alpha) \cong \ker T_a$ (e.g. $V = S_A$), the subspaces $V \oplus E_{(0,\infty)}$ and $E_{(-\infty,0)} \oplus V$ are Lagrangian subspaces of $L^2(\partial X)$. The Lagrangian $E_{(-\infty,0)} \oplus V$ provides a well posed boundary value problem for D_A acting on X in the sense of Seeley [**29**]. This is formalized in the following definition.

Definition 2.5. Denote by $L^2_s(X)$ the Sobolev completion of $\Omega^{ev}(X; \mathscr{E})$ to sections with s derivatives in L^2. Recall that restriction to the boundary defines a bounded linear map $L^2_t(X) \to L^2_s(\partial X)$ provided $t \geq s + 1/2$ and $s > 0$ (see e.g. [**10**]). This implies that the map r of Equation (2.3) extends to a bounded map $r : L^2_1(X) \to L^2(\partial X)$. The operator obtained by imposing *Atiyah-Patodi-Singer (APS) boundary conditions* $E_{(-\infty,0)} \oplus V$ *on* D_A means the operator D_A with domain restricted to the subspace

$$\{\phi \in L^2(X) \mid \phi \in L^2_1(X) \subset L^2(X) \text{ and } r(\phi) \in E_{(-\infty,0)} \oplus V\}$$

of $L^2(X)$, with the restriction map r defined in Equation (2.3).

We call $E_{(-\infty,0)} \oplus V$ and $V \oplus E_{(0,\infty)}$ *APS Lagrangians.*

The operator D_A with APS boundary conditions is an elliptic self adjoint operator with compact resolvent, and hence has discrete spectrum. When A is flat, the kernel of D_A with APS boundary conditions $E_{(-\infty,0)} \oplus V$ is related to cohomology by the following proposition. (See the last paragraph in the next section for its justification.)

PROPOSITION 2.6. *Suppose that A is a flat $U(n)$ connection on X with holonomy $\alpha : \pi_1(X) \to U(n)$ yielding local coefficients $\mathbb{C}^n_\alpha$. Write* $\ker(D_A, V)$ *for the kernel of D_A with APS boundary conditions $E_{(-\infty,0)} \oplus V$. Then there is an exact sequence*

$$0 \to \operatorname{image}\left(H^{ev}(X, \partial X; \mathbb{C}^n_\alpha) \to H^{ev}(X; \mathbb{C}^n_\alpha)\right) \to \ker(D_A, V) \to V \cap S_A \to 0. \quad \square$$

REMARK 2.7. If a collar of the boundary of X is parameterized as $(-\epsilon, 0] \times \partial X$, then the role of the positive and negative eigenspan of T_a are exchanged, and so in that case the Atiyah-Patodi-Singer boundary conditions which lead to a well posed problem are those of the form $V \oplus E_{(0,\infty)}$.

A useful homotopy theoretic invariant for Lagrangian subspaces of a (for the moment) finite dimensional symplectic space is the *Maslov index*. Its existence is based on the fact that the Grassmannian of Lagrangian subspaces has fundamental group isomorphic to $\mathbb{Z}$. We use the following manifestation of this fact. If L_t and M_t, $t \in [0, 1]$ are two continuous paths of Lagrangians, then $\mathrm{Mas}(L, M) \in \mathbb{Z}$ is the count, with sign, of how many times L_t and M_t pass through each other along the path. Some conventions must be set to deal with the situation when L and M are not transverse at $t = 0$ or $t = 1$ (see below). We refer the reader to Cappell, Lee, and Miller's article [**11**] for a comprehensive exposition of the Maslov index.

The Maslov index extends to infinite dimensional Lagrangians provided one is careful to choose (L_t, M_t) to be a path of *Fredholm pairs* of Lagrangian subspaces, i.e. so that $L_t \cap M_t$ is finite dimensional and $L_t + M_t$ has finite codimension. Intuitively one wants L_t and M_t to be sufficiently complementary (or equivalently L_t and JM_t to be sufficiently close) so that the finite dimensional constructions extend. This was explained in [**27**] and in detail in [**24**]. One can topologize the Grassmannian of (appropriate) Lagrangian subspaces of $L^2(\partial X)$ so that if L_t and JM_t are continuous paths in this Grassmannian, then the Maslov index gives a well defined integer $\mathrm{Mas}(L_t, M_t) \in \mathbb{Z}$. We omit the details here but mention that any time we refer to continuous paths of infinite dimensional Lagrangians, continuity is taken with respect to this topology. Furthermore, in any reference to the Maslov index in this infinite dimensional context, the Lagrangian subspaces will (for each parameter) be a Fredholm pair, so that the Maslov index makes sense. As usual with the Maslov index, some conventions must be set to deal with the situation when the Lagrangians are not transverse at $t = 0$ or $t = 1$.

Setting conventions is based on the observation that if L, M are Lagrangian subspaces of a Hermitian symplectic space, then for all small enough $\epsilon \neq 0$ the Lagrangians $e^{\epsilon J} L$ and M are transverse. The convention we choose is to replace the path $L(t)$ by the composite $\hat{L}(t)$ of the three paths $e^{(1-t)\epsilon J} L(0)$, $L(t)$, and $e^{t\epsilon J} L(1)$ for $\epsilon > 0$ very small. Then $\hat{L}(i)$ is transverse to $M(i)$ for $i = 0, 1$ and $\mathrm{Mas}(\hat{L}, M) = \mathrm{Mas}(L, M)$ whenever $L(i)$ and $M(i)$ are transverse at $i = 0, 1$. Hence we *define* $\mathrm{Mas}(L, M)$ to be $\mathrm{Mas}(\hat{L}, M)$ whenever L and M are not transverse at both endpoints. This convention has the virtue that it is additive with respect to composition of paths.

We take a moment to remind the reader of the notion of spectral flow. Given a (suitably continuous) path of (suitably nice) self adjoint operators D_t, the *spectral flow* $\mathrm{SF}(D_t) \in \mathbb{Z}$ is the net number of eigenvalues that change from negative to positive. Intuitively one defines the spectral flow to be the algebraic intersection number of the graph of the spectrum of D_t, $\Gamma = \{(t, \lambda) \mid \ker(D_t - \lambda) \neq 0\}$ with the segment $S = [0,1] \times \{0\}$ in $[0,1] \times \mathbb{R}$. To make this precise requires some work; a very careful construction is carried out by Booss-Bavnbek, Lesch, and Phillips in [**9**]. As in the case of the Maslov index, conventions must be set to deal with the situation when 0 is an eigenvalue of D_0 or D_1. A convenient choice is to take the intersection of Γ with $[0,1] \times \{-\epsilon\}$ for some small $\epsilon > 0$ (this is called the $(-\epsilon, -\epsilon)$ convention). This convention has the advantage that SF is additive with respect to composition of paths of operators.

We also briefly remind the reader of the η invariant of a Dirac operator D, as introduced in [**2**]. The sum

$$\eta(D, s) := \sum_{\lambda \in \operatorname{Spec} D, \lambda \neq 0} \operatorname{Sign} \lambda |\lambda|^{-s} \tag{2.9}$$

converges for $\operatorname{Re}(s) \gg 0$. For appropriate operators (including the twisted odd signature operator) the function $\eta(D, s)$ has a meromorphic continuation to $\mathbb{C}$ with no pole at 0. Then $\eta(D) \in \mathbb{R}$ is defined to be the value of $\eta(D, s)$ at $s = 0$.

The spectral invariants $\mathrm{SF}(D_{A_t})$ and $\eta(D_A) - \eta(D_\theta)$ (where θ denotes the trivial connection and A_t is a path from the trivial connection to a flat connection A_1) have no general analysis free definitions, despite the fact that they are independent of the choice of Riemannian metric and are basically topological invariants. The motivation for the authors' work with these invariants is to strip away as much analysis as possible from the formulas which compute these invariants. Our guiding principle is that computations of cohomology and representation varieties are quite reasonable, but direct calculation of the spectrum of the Dirac operator is difficult or impossible. To use topological methods to avoid direct calculations of the spectrum, however, requires cut-and-paste machinery. This leads directly to the study of boundary value problems for the Dirac operator.

3. The Cauchy data space

The *Calderón projector* for the operator D_A is a pseudodifferential projection in $L^2(\partial X)$ onto a distinguished Lagrangian subspace called the *Cauchy data space* for D_A. The Calderón projector is fundamental in the study of boundary value problems for Dirac operators, as evidenced by its prominent role in the lectures of this conference. In the context of the odd signature operator, it is a (very) close relative of the image of the cohomology restriction map $H^*(X) \to H^*(\partial X)$, a space of fundamental importance in geometric topology.

First, the definition of the Cauchy data space. Informally, the Cauchy data space for D_A is the subspace of $L^2(\partial X)$ consisting of restrictions to the boundary of solutions ϕ to the equation $D_A \phi = 0$. We give a more formal definition; the statements and proofs of facts invoked in this definition can be found in [**10**, Part II].

Definition 3.1. Define the *null space* $N(D_A, \frac{1}{2})$ to be

$$N(D_A, \tfrac{1}{2}) = \{\phi \in L^2_{1/2}(X) \mid D_A \phi = 0 \text{ in } X - \partial X\}.$$

The restriction map on smooth sections $r : \Omega^{ev}(X; \mathscr{E}) \to \Omega^*(\partial X; \mathscr{E})$ of Equation (2.3) does not in general extend to $L^2_{1/2}(X)$. However, it is defined on $N(D_A, \frac{1}{2})$ yielding a bounded operator

$$r : N(D_A, \tfrac{1}{2}) \to L^2(\partial X) \tag{3.1}$$

(see [**10**, Theorem 12.4]). One can construct a bounded left inverse $K : L^2(\partial X) \to N(D_A, \frac{1}{2})$ called the *Poisson operator* using an invertible extension of D_A to a closed manifold. The composite $P = r \circ K : L^2(\partial X) \to L^2(\partial X)$ is a projection ($P^2 = P$) called the *Calderón projector*. Its image, denoted by $\Lambda(D_A)$, is a closed subspace of $L^2(\partial X)$ called the *Cauchy data space*.

The Calderón projector is a pseudodifferential operator of order 0 which has the same principal symbol as the projection to the subspace $E_{(0,\infty)}$ (in fact Scott [**28**] and Grubb [**17**] proved that the difference of these projections is a smoothing operator). This implies $(\Lambda(D_A), E_{(-\infty,0]})$ is a Fredholm pair.

The scattering Lagrangian $S_A \subset \ker T_a$ satisfies

$$S_A = \mathrm{Proj}_{\ker T_a}(\Lambda(D_A) \cap E_{(-\infty,0]}). \tag{3.2}$$

Indeed, this can be taken as the definition of S_A.

The unique continuation property for generalized Dirac operators implies that the map of Equation (3.1) is injective. This implies that $\ker(D_A, V)$ is isomorphic to $\Lambda(D_A) \cap (E_{(-\infty,0)} \oplus V)$. Combining this observation, Equation (3.2), and [**2**, Proposition 4.9] (which corresponds to the case $V = 0$) yields the proof of Proposition 2.6.

4. The adiabatic limit of the Cauchy data space

Having set up our definitions and notation, we can now turn to a study of the analytical properties of the odd signature operator on a manifold with boundary. We begin with a structure theorem for the adiabatic limit of the Cauchy data space for D_A as the collar neighborhood of the boundary is stretched.

Let

$$X_R = X \cup_{\partial X} \big([-R, 0] \times \partial X\big)$$

so that $\partial X_R = \{-R\} \times \partial X$. The equation (2.6) shows how to extend D_A to X_R. Let $\Lambda_R(D_A)$ denote the Cauchy data space for this extension of D_A to X_R. For simplicity we write Λ_R instead of $\Lambda_R(D_A)$, and Λ for $\Lambda(D_A) = \Lambda_0(D_A)$.

In [**27**], Nicolaescu shows that the limit

$$\Lambda_\infty = \lim_{R\to\infty} \Lambda_R$$

exists and gives a description of it. Moreover, the appendix to [**14**] shows that the path of Lagrangians

$$t \mapsto \begin{cases} \Lambda_{1/(1-t)} & 0 \le t < 1, \\ \Lambda_\infty & t = 1 \end{cases}$$

is a continuous map (into the appropriate Grassmannian).

Fix $\nu \ge 0$. One has the following increasingly fine decompositions of $L^2(\partial X)$ into orthogonal direct sums of symplectic subspaces:

$$\begin{aligned} L^2(\partial X) &= \big(E_{(-\infty,0)} \oplus E_{(0,\infty)}\big) \oplus \ker T_a \\ &= \big(E_{(-\infty,-\nu)} \oplus E_{(\nu,\infty)}\big) \oplus \big(E_{[-\nu,0)} \oplus E_{(0,\nu]}\big) \oplus \ker T_a \\ &= \big(E_{(-\infty,-\nu)} \oplus E_{(\nu,\infty)}\big) \\ &\quad \oplus \big(d_a(E_{(0,\nu]}) \oplus d_a^*(E_{[-\nu,0)})\big) \oplus \big(d_a^*(E_{(0,\nu]}) \oplus d_a(E_{[-\nu,0)})\big) \\ &\quad \oplus \ker T_a. \end{aligned} \tag{4.1}$$

The last equality follows from the orthogonal decomposition $E_\lambda = d_a(E_{-\lambda}) \oplus d_a^*(E_{-\lambda})$ for $\lambda \neq 0$. Notice that in these decompositions, only the first symplectic subspace is infinite dimensional.

The following theorem was proven in [**24**] as a refinement of Nicolaescu's theorem in the special case of the odd signature operator.

THEOREM 4.1. *Let $\nu \geq 0$ be any number large enough so that $\Lambda \cap E_{(-\infty,-\nu)} = 0$. Then there exists a subspace $W \subset d_a(E_{(0,\nu]})$ isomorphic to the image of the map*

$$H^{ev}(X, \partial X; \mathbb{C}^n_\alpha) \to H^{ev}(X; \mathbb{C}^n_\alpha)$$

so that, letting $W^\perp$ denote the orthogonal complement of W in $d_a(E_{(0,\nu]})$, with respect to the orthogonal symplectic decomposition (4.1) the adiabatic limit Λ_∞ decomposes as a direct sum of Lagrangian subspaces:

$$\Lambda_\infty = E_{(\nu,\infty)} \oplus \big(W \oplus JW^\perp\big) \oplus d_a(E_{[-\nu,0)}) \oplus S_A \tag{4.2}$$

where S_A is the scattering Lagrangian of D_A (see Theorem 2.4).

If $W = 0$ then ν can be taken to be zero and $\Lambda_\infty = E_{(0,\infty)} \oplus S_A$.

Sketch of proof. The starting point for the proof is Nicolaescu's theorem ([**27**, Theorem 4.9]) which asserts that

$$\Lambda_\infty = E_{(0,\infty)} \oplus \lim_{R\to\infty} e^{RT_a} L_\nu,$$

where L_ν is the orthogonal projection to $E_{[-\nu,\nu]}$ of $\Lambda \cap E_{(-\infty,\nu]}$:

$$L_\nu = \mathrm{Proj}_{E_{[-\nu,\nu]}} \big(\Lambda \cap E_{(-\infty,\nu]}\big).$$

Since $\Lambda \cap E_{(-\infty,-\nu)} = 0$, the (restriction of the) projection

$$\Lambda \cap E_{(-\infty,\nu]} \to E_{[-\nu,\nu]}$$

is injective, and hence the projection

$$\Lambda \cap E_{(-\infty,\mu]} \to E_{[-\nu,\nu]}$$

is also injective for $\mu \leq \nu$.

Denote by $L_\infty \subset E_{[-\nu,\nu]}$ the Lagrangian subspace $\lim_{R\to\infty} e^{RT_a} L_\nu$. Thus

$$\Lambda_\infty = E_{(\nu,\infty)} \oplus L_\infty.$$

Hence the problem is reduced to the finite dimensional problem of identifying the dynamics of the family e^{RT_a} acting on the Lagrangian subspace L_ν of $E_{[-\nu,\nu]}$. The key point is that the largest eigenvalue dominates.

To make this precise, let $\mu_1 < \mu_2 < \cdots < \mu_q$ denote the complete list of eigenvalues of T_a in the range $[-\nu,\nu]$ (thus $\mu_i = -\mu_{q-i}$). Given $\ell \in L_\nu$, write $\ell = \ell_1 + \ell_2 + \cdots + \ell_q$ where ℓ_i lies in the μ_i eigenspace E_{μ_i}. Define $m(\ell)$ to be the largest i so that ℓ_i is nonzero. Thus

$$\ell = \ell_1 + \cdots + \ell_{m(\ell)}.$$

Then

$$\lim_{R\to\infty} e^{RT_\alpha}\big(\tfrac{1}{e^{R\mu_{m(\ell)}}}\ell\big) = \ell_{m(\ell)} \tag{4.3}$$

and so $\ell_{m(\ell)} \in L_\infty$.

This observation can be reinterpreted as follows. Intersecting L_ν with the filtration

$$0 \subset E_{[-\nu,\mu_1]} \subset E_{[-\nu,\mu_2]} \subset \cdots \subset E_{[-\nu,\mu_q]}$$

gives a filtration

$$0 \subset L_\nu(\mu_1) \subset L_\nu(\mu_2) \subset \cdots \subset L_\nu(\mu_q) = L_\nu$$

of L_ν. One can view the quotient $L_\infty(k) := L_\nu(\mu_k)/L_\nu(\mu_{k-1})$ as a subspace of E_{μ_k}. Equation (4.3) implies

$$L_\infty = L_\infty(1) \oplus \cdots \oplus L_\infty(q).$$

The space $\Lambda \cap E_{(-\infty,0)}$ consists exactly of solutions to $D\phi = 0$ which extend to exponentially decaying (and hence L^2) solutions on X_∞. These are L^2 harmonic even forms on X_∞. Proposition 4.9 of [**2**] shows that the space of L^2 harmonic even degree forms is isomorphic to the image of $H^{ev}(X, \partial X; \mathbb{C}^n_\alpha) \to H^{ev}(X; \mathbb{C}^n_\alpha)$.

Notice that if k is the largest index so that $\mu_k < 0$, then $\Lambda \cap E_{(-\infty,0)}$ is isomorphic to $L_\nu(\mu_k)$, since $\Lambda \cap E_{(-\infty,-\nu)} = 0$. Thus $L_\nu(\mu_k)$ is isomorphic to the image of $H^{ev}(X, \partial X; \mathbb{C}^n_\alpha) \to H^{ev}(X; \mathbb{C}^n_\alpha)$. It is also clearly isomorphic to $L_\infty(1) \oplus \cdots \oplus L_\infty(k)$. We write

$$W = L_\infty(1) \oplus \cdots \oplus L_\infty(k).$$

Similarly, if the kernel of T_a is non-trivial, then $\mu_{k+1} = 0$ and it follows from the definition and the fact that $\Lambda \cap E_{(-\infty,-\nu)} = 0$ that $L_\infty(k+1)$ is the scattering Lagrangian S_A. Denoting the sum of rest of the $L_\infty(i)$ by V we conclude

$$L_\infty = W \oplus S_A \oplus V. \tag{4.4}$$

The proof is completed by showing that since W corresponds to harmonic forms which exponentially decay on X_∞, W lies in $d_a(E_{(0,\nu]})$. The space V is a subspace of $E_{(0,\nu]} = d_a^*(E_{[-\nu,0)}) \oplus d_a(E_{[-\nu,0)})$. Since L_∞ is a Lagrangian subspace of $E_{[-\nu,\nu]}$ this forces

$$V = JW^\perp \oplus d_a(E_{[-\nu,0)}).$$

□

5. Adiabatic limits and manifold decompositions

Theorem 4.1 is very useful when computing spectral flow and η invariants of the odd signature operator over split manifolds. Consider the decomposition

$$M = X \cup_\Sigma Y$$

of a closed manifold M^{2k+1} along a separating hypersurface Σ with the collar neighborhood parameterized as $(-\epsilon, \epsilon) \times \Sigma$ and with $\{-\epsilon\} \times \Sigma \subset X$ and $\{\epsilon\} \times \Sigma \subset Y$. Let A be a flat connection on M in cylindrical form on the collar. Replacing the collar $(-\epsilon, \epsilon) \times \Sigma$ by $(-\epsilon - R, \epsilon + R) \times \Sigma$ yields manifolds M_R, X_R, and Y_R. Denote by $\Lambda_R(X)$ and $\Lambda_R(Y)$ the corresponding Cauchy data spaces of the twisted odd signature operators acting on X_R and Y_R, and denote by $\Lambda_\infty(X)$ and $\Lambda_\infty(Y)$ their adiabatic limits, identified in Theorem 4.1.

The paths of Cauchy data spaces obtained by stretching $\Lambda_R(X)$ and $\Lambda_R(Y)$ to infinity are continuous paths in the Lagrangian Grassmannian [**14**]. The dimension of the kernel of D_A acting on M_R is, on the one hand, independent of $R \in [0, \infty)$ since it can be identified with the metric independent cohomology group $H^{ev}(M; \mathbb{C}^n_\alpha)$, where $\alpha : \pi_1 M \to U(n)$ is the holonomy representation of the flat connection A. On the other hand, the kernel of D_A on M_R is identified with $\Lambda_R(X) \cap \Lambda_R(Y)$. Thus $\dim\big(\Lambda_R(X) \cap \Lambda_R(Y)\big)$ is independent of R. In fact, the next theorem states that this dimension does not jump up even at $R = \infty$.

THEOREM 5.1. *Let W_X, W_Y be the spaces of Theorem 4.1 for X and Y, and let $S_A(X), S_A(Y)$ be the scattering Lagrangians, as in Theorem 2.4. Then, for any $R \geq 0$,*

$$\Lambda_R(X) \cap \Lambda_R(Y) \cong \Lambda_\infty(X) \cap \Lambda_\infty(Y) \cong \Lambda_\infty(X) \cap \Lambda_R(Y) \cong \Lambda_R(X) \cap \Lambda_\infty(Y)$$

and these intersections are isomorphic to

$$H^{ev}(M; \mathbb{C}^n_\alpha)$$

and also to

$$W_X \oplus W_Y \oplus \big(S_A(X) \cap S_A(Y)\big).$$

The proof follows from Theorem 4.1 and the consequences of Remark 2.7. For a complete argument, see [**24**, Lemmas 8.9 and 8.10].

In the following theorem, let $R(t)$ be a continuous, monotonic function $[0, \frac{1}{2}] \to [0, \infty]$, e.g. $R(t) = t/(1-2t)$.

THEOREM 5.2. *Consider the path of Lagrangian subspaces obtained by stretching $\Lambda_R(D_A, X)$ to its adiabatic limit and then rotating W_X to JW_X in $d_a(E_{(0,\nu]}) \oplus d_a^*(E_{[-\nu,0)})$:*

(5.1)
$$L(X)(t) = \begin{cases} \Lambda_{R(t)}(D_A, X) & t \in [0, \frac{1}{2}] \\ E_{(\nu,\infty)} \oplus \big(e^{(2t-1)J} W_X \oplus JW_X^\perp\big) \oplus d_a(E_{[-\nu,0)}) \oplus S_A(X) & t \in [\frac{1}{2}, 1] \end{cases}$$

Thus $L(X)(0)$ is the Cauchy data space $\Lambda(D_A, X)$ for D_A and $L(X)(1)$ is the APS Lagrangian $E_{(0,\infty)} \oplus S_A(X)$.

Similarly define $L(Y)(t)$ by following the stretching of $\Lambda_R(D_A, Y)$ to its adiabatic limit and then rotating W_Y to JW_Y in $d_a(E_{[-\nu,0)}) \oplus d_a^*(E_{(0,\nu]})$. Hence $L(Y)(0) = \Lambda(D_A, Y)$ and $L(Y)(1) = E_{(-\infty,0)} \oplus S_A(Y)$.

Then, (with appropriate conventions for the Maslov index)

$$\operatorname{Mas}(L(X), L(Y)) = 0.$$

PROOF. First, Theorem 5.1 shows that stretching does not change the dimension of the kernel, and hence the Maslov index for the first half of the path is zero. As for the second path, there is a technical matter of conventions for the Maslov index when the Lagrangians are not transverse at the endpoints, but the crucial point is that in the decomposition (4.1), the rotations on the X and Y sides occur in *different* symplectic summands of $L^2(\partial X)$, and hence do not run into each other.

More precisely, the Maslov index is additive under direct sum of symplectic spaces and Lagrangian subspaces. Thus the Maslov index $\operatorname{Mas}(L(X), L(Y))$ equals

$$\operatorname{Mas}(e^{tJ} W_X \oplus JW_X^\perp, d_a(E_{(0,\nu]})) + \operatorname{Mas}(d_a(E_{[-\nu,0)}), e^{tJ} W_Y \oplus JW_Y^\perp), \tag{5.2}$$

where the first term in Equation (5.2) is the Maslov index in the finite dimensional symplectic space $d_a(E_{(0,\nu]}) \oplus d_a^*(E_{[-\nu,0)})$ and the second term is the Maslov index in $d_a^*(E_{(0,\nu]}) \oplus d_a(E_{[-\nu,0)})$ (see the decomposition (4.1)) and $t \in [0, 1]$.

The first term of Equation (5.2) can be simplified further; in fact using additivity of the Maslov index and the symplectic decomposition

$$d_a(E_{(0,\nu]}) \oplus d_a^*(E_{[-\nu,0)}) = (W_X \oplus JW_X) \oplus (W_X^\perp \oplus JW_X^\perp),$$

one sees that

$$\operatorname{Mas}(e^{tJ} W_X \oplus JW_X^\perp, d_a(E_{(0,\nu]})) = \operatorname{Mas}(e^{tJ} W_X, W_X),$$

where the Maslov index on the right is taken in $W_X \oplus JW_X$. Our conventions are chosen so that this Maslov index is zero. Similarly the other term in (5.2) vanishes. □

REMARK 5.3. Any other convention for defining the Maslov index will differ from ours by the dimensions of the intersection of the Lagrangians at the endpoints. Thus, no matter what convention is chosen, the conclusion of Theorem 5.2 reads

$$\mathrm{Mas}(L(X), L(Y)) = \epsilon_1 \dim W_X + \epsilon_2 \dim W_Y$$

where $\epsilon_i \in \{-1, 0, 1\}$. This causes no computational difficulties since the W_X are identified with cohomology groups in Theorem 4.1.

This is a very satisfying result. It enables one to stretch with impunity and then to rotate to the APS Lagrangians $E_{(0,\infty)} \oplus S_A(X)$ and $E_{(-\infty,0)} \oplus S_A(Y)$, without worrying about introducing any Maslov index.

In practice this allows one to pretend that $\Lambda(D_A, X)$ and $\Lambda(D_A, Y)$ *are* the APS Lagrangians $E_{(0,\infty)} \oplus S_A(X)$ and $E_{(-\infty,0)} \oplus S_A(Y)$ determined by the scattering Lagrangians. Thus a simple cohomological invariant, the scattering Lagrangian, has been substituted for a complicated analytic invariant, the Cauchy data space. This greatly simplifies spectral flow and η invariant computations. For example the splitting formula for η invariants

$$\widetilde{\eta}(D, M) = \widetilde{\eta}(X, \Lambda(D, X)) + \widetilde{\eta}(Y; J\Lambda(D, X))$$

(valid for any generalized Dirac operator) established in [**24**] reduces to

$$\widetilde{\eta}(D, M) = \widetilde{\eta}(X, E_{(-\infty,0)} + S_A(X)) + \widetilde{\eta}(Y; E_{(0,\infty)} + JS_A(X))$$

in the special case of the odd signature operator D_A twisted by a flat connection.

One must take care when working with the adiabatic limits of Cauchy data spaces in a parameterized context. The convergence $\Lambda_R \to \Lambda_\infty$ is not uniform with respect to a parameter. In fact there exist continuous (even analytic) paths of flat connections for which $\Lambda_\infty(D_{A_t})$ is not continuous in t (see e.g. [**8**]).

We refer the interested reader to the articles of the authors and their collaborators for applications of this machinery [**24, 25, 8, 7, 21**]. Theorem 5.2 is crucial to carrying out spectral flow computations for paths of connections on split manifolds when the dimension of the kernel of T_a jumps up along a path as in [**8**] and B. Himpel's thesis [**20**].

6. Casson's $SU(2)$ invariant and its generalizations.

Casson's $SU(2)$ invariant and its generalizations have two contrasting interpretations, one topological and the other analytical. That these interpretations are equivalent is based on the correspondence between the space $R(\pi_1 X, SU(2))$ of conjugacy classes of $SU(2)$ representations of $\pi_1 X$ and the moduli space $\mathscr{M}(X, P)$ of flat connections on a principal $SU(2)$ bundle P over X.

Roughly speaking, a Casson-type invariant is one that can be defined in the following way. First fix a compact Lie group G. The space $R(\pi_1 X, G)$ of representations modulo conjugation is compact and generically zero dimensional. By defining an orientation on $R(\pi_1 X, G)$, the Casson-type invariant is obtained by counting these points with sign.

In general, there are serious challenges to constructing an invariant according to this recipe. Before discussing these difficulties, we first describe the simplest nontrivial example. Namely, we recall the definition of Casson's $SU(2)$ invariant for 3-manifolds X with $H_*(X;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$ (we say "X is a ZHS," meaning X is a $\mathbb{Z}$-homology sphere).

As mentioned above, there are both topological and analytic descriptions of the invariant, by Casson and Taubes, respectively. The relative benefits of the two approaches are the following. The topological approach fits immediately with cut-and-paste constructions, such as Heegaard decompositions and surgery descriptions of the 3-manifolds. But since it relies on a Heegaard decomposition (to define the orientations of the points in $R(\pi_1 X, SU(2))$), the topological method requires more work to show it is a topological invariant. (See Akbulut and McCarthy's book [**1**] for an exposition of Casson's construction.) In the analytical approach, topological invariance is fairly straightforward, but the definitions of the orientations involves spectral data of linear operators associated to flat connections. In as much as efforts to generalize Casson's invariant to higher rank groups ($SU(n)$, $n > 2$) have only succeeded using the analytic approach, the only known means of calculating these generalized Casson invariants is via spectral flow calculations. This paper provides a survey of the techniques that have been used to produce these calculations.

Taubes' analytic approach to the $SU(2)$ Casson invariant involves the space $\mathscr{A}$ of all connections on the trivial principal $SU(2)$ bundle $X \times SU(2)$ over a ZHS X. Notice that since $SU(n)$ is simply connected, every principal $SU(n)$ bundle P over a 3-manifold X is trivial and so if $\mathscr{E}$ is an associated $\mathbb{C}^n$ vector bundle, $\Omega^p(X;\mathscr{E}) \cong \Omega^p(X) \otimes \mathbb{C}^n$. Moreover, $\mathscr{M}(X,P)$ is homeomorphic to $R(\pi_1 X, SU(n))$. The gauge group $\mathscr{G}$ of bundle automorphisms acts on connections by pulling back horizontal subspaces. The trivial connection (denoted by θ) gives $\mathscr{A}$ a distinguished base point.

There is a metric independent function, the *Chern-Simons* function

$$cs : \mathscr{A} \to \mathbb{R},$$

which is defined by transgressing the second Chern class, i.e. given a connection A_1, choose a path A_t from the trivial connection $A_0 = \theta$ to A_1, and view the path as a connection $\mathbf{A}$ on $X \times [0,1]$. Then

$$cs(A_1) = \int_{X \times I} c_2(\mathbf{A}) = \frac{1}{4\pi^2} \int_{X \times I} \operatorname{Tr}(F(\mathbf{A}) \wedge F(\mathbf{A})) \in \mathbb{R}$$

where $F(\mathbf{A})$ denotes the curvature of $\mathbf{A}$. Then cs descends to a $\mathbb{R}/\mathbb{Z}$-valued function on $\mathscr{B} := \mathscr{A}/\mathscr{G}$. Taubes' approach is to adapt finite dimensional manifold topology to $\mathscr{B}$. He uses the L^2-gradient vector field of the Chern-Simons function to define an Euler characteristic, i.e. to define Casson's invariant as the signed sum over the zeros of the gradient vector field. The gradient vector field (suitably interpreted) is given by $\mathscr{B} \ni [A] \mapsto *F(A) \in T_{[A]}\mathscr{B} \subset \Omega^1(X; su(2))$. Thus the set of zeros is exactly the moduli space of flat connections. A first technical problem arises in that $\mathscr{B}$ is not a (Banach or Hilbert) manifold (even after suitably completing in a Sobolev norm), due to the fact that the gauge group $\mathscr{G}$ does not act freely on $\mathscr{A}$. To examine this point more closely, consider more generally a principal G-bundle P for some compact Lie group G. Let $\mathscr{A}_P$ denote the space of connections on P, $\mathscr{G}_P$ the gauge group of bundle automorphisms, and $\mathscr{B}_P = \mathscr{A}_P/\mathscr{G}_P$ its orbit space. The stabilizer $\operatorname{Stab}_A \subset \mathscr{G}_P$ of a connection $A \in \mathscr{A}_P$ is isomorphic to a subgroup of G, namely the centralizer in G of the holonomy subgroup of A. Compactness of G can be used to show that a slice theorem holds in this case, so that a neighborhood of $[A]$

in $\mathscr{B}_P$ is homeomorphic to a neighborhood of 0 in a linear quotient $\mathrm{Slice}/\mathrm{Stab}_A$. Gauge equivalent connections have conjugate stabilizers. A connection is called *irreducible* if its stabilizer is as small as possible, namely the center of G, and the subspace $\mathscr{B}_P^*$ of gauge equivalence classes of irreducible connections is a Banach manifold and an open dense subset of $\mathscr{B}_P$.

In the case of $G = SU(2)$, there are three possible conjugacy classes of stabilizers of subgroups of $SU(2)$, the center $\pm I$, the maximal torus S^1, and the entire group $SU(2)$. For *flat connections* on a ZHS X, there is a single, isolated gauge equivalence class of reducible connections, namely the class containing the trivial connection. By ignoring it, Taubes considers cs as a function on the *manifold* $\mathscr{B}^*$, and its critical points form a compact space, since it is homeomorphic to $R(\pi_1 X, SU(2)) - \{\theta\}$.

A second technical problem arises, namely that the critical points of cs might not be a finite set of isolated points, more precisely, $cs : \mathscr{B}^* \to \mathbb{R}/\mathbb{Z}$ may not be Morse. Taubes addresses this problem by introducing a class of *holonomy perturbations* $h : \mathscr{B} \to \mathbb{R}$. These perturbations have three important properties:

(1) For generic h, $cs+h$ is Morse in the sense that its critical points are finite, isolated and the Hessian of $cs+h$ at each critical point is nondegenerate.
(2) The L^2-adjoint of the Hessian of $cs+h$ is a relatively compact perturbation of the L^2-adjoint of the Hessian of cs at each connection.
(3) The holonomy perturbations achieve the kind of perturbations required in Casson's construction, so that the two invariants (Taubes' and Casson's) can be compared.

The holonomy perturbations can be constructed for any principal G bundle. They are roughly defined as follows: Given a solid torus $D^2 \times S^1$ embedded in X, one gets a function $hol : D^2 \times \mathscr{A} \to G$ which sends (x, A) to the holonomy of A around the loop $\{x\} \times S^1$. Then, given a cut-off function $c : D^2 \to \mathbb{R}$ and a conjugation invariant function $f : G \to \mathbb{R}$, define $h : \mathscr{A} \to \mathbb{R}$ by

$$h(A) = \int_{D^2} f(hol(x, A))c(x)dx.$$

Since f is ad-invariant, h descends to $h : \mathscr{B} \to \mathbb{R}$. More generally, one chooses embeddings of n solid tori in X and an invariant function $f : G^n \to \mathbb{R}$ to define a perturbation h.

The last hurdle is to define the sign at each critical point. On a finite dimensional manifold, the Poincaré-Hopf theorem identifies the Euler characterstic as the signed sum of zeros of a gradient vector field $\mathrm{grad}(f)$, i.e. the critical points of f. The sign of a critical point p is taken to be $(-1)^{i(p)}$, with $i(p)$ the dimension of the negative eigenspace of the Hessian of f at p. In the present setting, however, even if A is a nondegenerate critical point of cs, the Hessian of cs, H_A, has infinitely many negative and positive eigenvalues. However, Taubes showed that H_A is a closed self adjoint operator with discrete spectrum. The spectral flow of H_{A_t} along a path A_t joining two critical connections defines a *sign difference*, or *relative orientation* between two critical connections $[A_0]$ and $[A_1]$:

$$\mathrm{Sign}([A_0], [A_1]) = (-1)^{\mathrm{SF}(H_{A_t})}.$$

Here, one uses the affine structure of $\mathscr{A}$ to define the Hessian at non-critical points; this is a continuous path of self adjoint operators provided A_t is irreducible for all t.

To circumvent the lack of continuity of H_A as A passes through reducible connections (where $\mathscr{B}$ is not even a manifold), Taubes observed that stabilizing H_A by adding an operator with symmetric spectrum yields the odd signature operator D_A. (To fit the operator in the context of the present article, it is also necessary to complexify since the Hessian acts on the real vector space of $su(2)$-valued forms. The induced action on $\mathbb{C}^3 \cong su(2) \otimes \mathbb{C}$ is unitary. Complexifying does not change the spectral flow provided one counts complex eigenvalues.) Thus

$$\mathrm{Sign}([A_0],[A_1]) = (-1)^{\mathrm{SF}(D_{A_t})}.$$

In the case when the Chern-Simons function is not Morse, a suitable perturbation h is chosen, and the Hessian of cs is replaced by the Hessian of $cs + h$. This has the effect of changing D_A to an operator $D_{A,h} := D_A + V_{A,h}$, where the perturbation $V_{A,h}$ is a bounded (in L^2) self adjoint operator, which has a pseudolocality property: letting $S \subset X$ denote the union of the solid tori along which the various holonomy perturbations are defined,

$$\varphi V_{A,h} = 0 \text{ for all } \varphi \in C_0^\infty(X \setminus S). \tag{6.1}$$

This property says that $V_{A,h} f$ depends only on the restriction of f to S and vanishes outside S.

If A_0 and A_1 are critical points of $cs + h$, then define

$$\mathrm{Sign}([A_0],[A_1]) = (-1)^{\mathrm{SF}(D_{A_t,h})}. \tag{6.2}$$

To obtain an *absolute* sign, use the trivial connection as a base point, i.e. set

$$\mathrm{Sign}(A) = (-1)^{\mathrm{SF}(D_{A_t,h_t})}, \quad A_0 = \theta, A_1 = A, \ h_0 = 0, h_1 = h.$$

An application of the Atiyah-Patodi-Singer theorem shows that if g is a gauge transformation, then $\mathrm{Sign}(A, g \cdot A) = 8 \deg(g)$. Thus $\mathrm{Sign}(A)$ is well defined on $\mathscr{B}$. With these hurdles cleared, Taubes defines an invariant by

$$\lambda(X) = \sum_{[A] \in \mathscr{M}_h^*} \mathrm{Sign}(A), \tag{6.3}$$

where $\mathscr{M}_h^*$ denotes the compact 0-dimensional manifold of *perturbed flat* connections, i.e. the space of critical points of $cs + h$ modulo gauge transformations. The assumption that X is a ZHS implies that $\mathscr{M}^*$ is compact (as is $\mathscr{M}_h^*$ for fixed h), and under variation of h along a generic path $\mathscr{M}_h^*$ changes by a cobordism, so the sum (6.3) is independent of h.

We next indicate how Casson's $SU(2)$ invariant is generalized to $SU(3)$. The important difference between the set up considered by Taubes in the $SU(2)$ case and the situation for other Lie groups is that the moduli space $\mathscr{M}$ of flat G connections has a more complicated stratification because the collection of orbit types is larger. This leads to equivariant transversality problems and complicates the search for a topological invariant generalizing Casson's. For $SU(3)$ the local analysis was worked out by Boden and Herald in [**5**] (see also [**19**]).

For $G = SU(3)$ and X an integer homology 3-sphere there are three relevant strata: the stratum of irreducible flat connections $\mathscr{M}^*(X)$, the stratum of reducible, nontrivial flat connections $\mathscr{M}^{\mathrm{red}}(X)$, and the stratum containing the trivial connection. As in the $SU(2)$ case, the trivial connection is isolated (since X is a homology sphere) but, in contrast to the $SU(2)$ case, the closure of $\mathscr{M}^*$ in $\mathscr{M}$ can intersect $\mathscr{M}^{\mathrm{red}}$. This complicates the construction of a well defined invariant. At this time there are three competing versions of the $SU(3)$ Casson invariant [**5, 6, 13**], each one different from the others but all defined using the same basic approach, which we now explain.

First, a small holonomy perturbation h (adapted to the $SU(3)$ setting) is chosen so that the moduli spaces $\mathscr{M}_h^*$ and $\mathscr{M}_h^{\text{red}}$ are discrete and regular. (The existence of an appropriate h is a delicate matter, see [**5, 19**].) Ideally, one would like an $SU(3)$ Casson invariant to be a signed count of points in the irreducible perturbed flat moduli space, so that its nonvanishing would imply the existence of irreducible $SU(3)$ representations of $\pi_1(X)$. Thus one sets

$$\lambda'(X,h) = \sum_{[A]\in\mathscr{M}_h^*(X)} (-1)^{\text{SF}(\theta,A)}.$$

In this formula $\text{SF}(\theta,A)$ denotes the spectral flow of the path of perturbed odd signature operators

$$D_{A_t,h_t} : \Omega^{ev}(X)\otimes su(3) \to \Omega^{ev}(X)\otimes su(3)$$

where A_t is a path from the trivial connection θ to the connection A, and h_t is a path of perturbations from 0 to h.

The fact that the closure of the irreducible flat connections can contain reducible connections has the unhappy consequence that the integer $\lambda'(X,h)$ is not in general independent of the choice of h. One must correct for the ambiguity arising from this choice. The three invariants differ only by the choice of correction term.

In [**5**] the correction term

$$\lambda_1''(X,h) = \tfrac{1}{2} \sum_{[A]\in\mathscr{M}_h^{\text{red}}} (-1)^{\text{SF}(\theta,A)}(\text{SF}_{\mathbb{C}^2}(\theta,A) - 4cs(\widehat{A}) + 2) \tag{6.4}$$

is shown to have the desired property: namely that $\lambda_1(X) := \lambda'(X,h) + \lambda_1''(X,h)$ is a well defined topological invariant. In Equation (6.4), since A is a reducible connection, there is a path of reducible connections from the trivial connection θ to A. This path can be gauge transformed to preserve the decomposition $su(3) \cong su(2)\oplus\mathbb{C}^2\oplus\mathbb{R}$. Thus $\text{SF}_{\mathbb{C}^2}(\theta,A)$ refers to the spectral flow of the path D_{A_t,h_t} restricted to $\Omega^{ev}(X)\otimes\mathbb{C}^2$. Also, $\widehat{A}$ refers to an (honest) flat connection near A, and $cs(\widehat{A})$ is its Chern-Simons invariant, a real number. Thus $\lambda_1(X)$ is a real number. The term $4cs(\widehat{A})$ was added to restore gauge invariance to λ_1'': the term $\text{SF}_{\mathbb{C}^2}(\theta,A)$ eliminates the dependence on the perturbation but unfortunately is not gauge invariant. The formulas $\text{SF}_{\mathbb{C}^2}(\theta, g\cdot A) = \text{SF}_{\mathbb{C}^2}(\theta,A) + 4\deg(g)$ and $cs(g\cdot\widehat{A}) = cs(\widehat{A}) + \deg(g)$ explain why this combination works.

Another construction for a correction term was presented in [**6**], yielding an integer valued invariant with many nice properties. One sets

$$\lambda_2''(X,h) = \tfrac{1}{4} \sum_{[A]\in\mathscr{M}_h^{\text{red}}} (-1)^{\text{SF}(\theta,A)} \left(\text{SF}_{\mathbb{C}^2}(\widehat{A}^+,A) + \text{SF}_{\mathbb{C}^2}(\widehat{A}^-,A) + h^1_{\widehat{A}^-}\right). \tag{6.5}$$

In this formula $\widehat{A}^\pm$ are flat connections in the unique path component of the space of flat connections closest to A. They are specified by the condition that $\widehat{A}^+$ (resp. $\widehat{A}^-$) maximizes (resp. minimizes) the $\mathbb{C}^2$ spectral flow from A.

Informally speaking, a perturbed flat connection emerges from a path component of flat connections as the perturbation is turned on. Thus, for small enough perturbations, to each perturbed flat connection one can associate the component of flat connections from which it originated. This implies that if $\widehat{A}$ is a flat connection very close to A, then for any gauge transformation g, $g \cdot \widehat{A}$ is a flat connection very close to $g \cdot A$. This ensures that the correction term λ_2'' is gauge invariant. It follows from an examination of how the moduli space changes under perturbations that the sum $\lambda_2(X) := \lambda'(X,h) + \lambda_2''(X,h) \in \mathbb{Z}$ is a well defined smooth invariant of the homology sphere X, independent of the choice of perturbation h.

A third version of a correction term is defined by Cappell, Lee, and Miller in [**13**]. We refer the reader to their article for a definition of their correction term, which also involves spectral flow.

Having defined $SU(3)$ generalizations of Casson's invariant, a basic question which remains is: What topological information do these invariants measure? To answer this, one needs to carry out computations and develop cut-and-paste techniques. These invariants are very difficult to compute because spectral flow is difficult to compute, and because analyzing carefully how the flat moduli space changes under perturbations is a delicate problem.

In [**8**], λ_1 is computed for Brieskorn spheres of the form $\Sigma(2,q,r)$, and in [**7**], λ_2 is computed for all Brieskorn spheres $\Sigma(p,q,r)$. In the following two sections, we outline the crucial role played by the Calderón projector and Maslov index in carrying out the spectral flow portion of these computations.

7. Splitting the spectral flow of the odd signature operator

In [**31**] Yoshida introduced a useful strategy for computing the spectral flow of the odd signature operator by decomposing the manifold along a separating hypersurface. For this to work, one needs to establish a splitting formula for the spectral flow, which involves developing the theory for manifolds with boundary and requires sufficient knowledge of the representation varieties of both parts of the decomposition. A general method to obtain such splitting formulas was derived in [**14**], and we outline how it works for the odd signature operator.

Let $X = Y \cup_T Z$ be a decomposition of a closed manifold along a separating hypersurface T. Let A_t be a path of connections on X in cylindrical form on a neighborhood of T, with A_0, A_1 flat, and the restriction of A_t to Z flat. Let D_t be the corresponding path of odd signature operators. Nicolaescu's theorem states that $\mathrm{SF}(D_t) = \mathrm{Mas}(\Lambda(D_t,Y),\Lambda(D_t,Z))$. To simplify notation let $M_1(t) = \Lambda(D_t,Z)$ and $N_1(t) = \Lambda(D_t,Y)$. Thus (M_1,N_1) is a path of Fredholm pairs of Lagrangian subspaces of $L^2(T)$.

(1) Let M_2 be the reverse of the path given in Theorem 5.2 for Z. Thus M_2 is a path from $E_{(0,\infty)}(0) \oplus S_{A_0}(Z)$ to $M_1(0) = \Lambda(D_0,Z)$. Similarly let N_2 be the corresponding path for Y, starting at $E_{(-\infty,0)}(0) \oplus S_{A_0}(Y)$ and ending at $\Lambda(D_0,Y)$.

Similarly, at $t=1$, one obtains paths M_3 from $\Lambda(D_1,Z)$ to $E_{(0,\infty)}(1) \oplus S_{A_1}(Z)$ and N_3 from $\Lambda(D_1,Y)$ to $E_{(-\infty,0)}(1) \oplus S_{A_1}(Y)$. Theorem 5.2 implies that

$$\mathrm{SF}(D_t) = \mathrm{Mas}(M_2 * M_1 * M_3, N_2 * N_1 * N_3).$$

(2) Note that $M_2(0)$, $M_3(1)$, $N_2(0)$, and $N_3(1)$ are APS Lagrangians. The next step is to find continuous paths of APS Lagrangians M_4 from $M_2(0)$ to $M_3(1)$ and N_4 from $N_2(0)$ to $N_3(1)$. This step is easy to carry out if the kernel of the tangential operator is constant along the path. In the

general case, the existence of such a path is established in [**25**, Lemma 7.3]. To compute spectral flow one needs precise control over this path.

(3) Now replace the path $M_2 * M_1 * M_3$ by the homotopic path $M_4 * \overline{M}_4 * (M_2 * M_1 * M_3)$ and replace $N_2 * N_1 * N_3$ by the homotopic path $(N_2 * N_1 * N_3) * N_4 * \overline{N}_4$. (Here $\overline{M}_4$ means the reverse of M_4.) Homotopy invariance and path additivity of the Maslov index imply that

$$\mathrm{SF}(D_t) = \mathrm{Mas}(M_4, N_2 * N_1 * N_3) + \mathrm{Mas}(\overline{M}_4, \overline{N}_4) + \mathrm{Mas}(M_2 * M_1 * M_3, N_4).$$

The first term is essentially the spectral flow of the restriction of D_t to Y with APS boundary conditions. The second term is a finite dimensional Maslov index since the APS Lagrangians are orthogonal away from the kernel of the tangential operator. The last term is essentially the spectral flow of the restriction of D_t to Z. In summary we have a *splitting formula*

$$\mathrm{SF}(D_t, X) = \mathrm{SF}(D_t, Y; M_4) + \mathrm{SF}(D_t, Z; N_4) + \tau \tag{7.1}$$

where τ is a (hopefully explictly computable) finite dimensional Maslov index taking place in the kernel of the tangential operator, a symplectic space identified with the cohomology of the separating hypersurface T, and the spectral flow terms are with respect to the well behaved APS boundary conditions.

We will illustrate this in the next section for the special case which is needed to calculate the term $\mathrm{SF}_{\mathbb{C}^2}(\theta, A)$ in Equation (6.4). In that case the kernel of the tangential operator is trivial except at $t = 0$ when A_0 is the trivial connection. Thus finding the paths M_4, N_4 as above is only tricky at $t = 0$. For $t > 0$ one can take $M_4(t) = E_{(0,\infty)}(t)$ and $N_4(t) = E_{(-\infty,0)}(t)$. The problem that arises is that one must compute $\lim_{t\to 0^+} M_4(t)$ and adjust for the fact that this limit might not equal $M_2(0)$. See the comment preceding the proof of Lemma 8.1 below.

Recent work of Himpel [**20**] uses this approach in a more complicated setting to prove a conjecture in topological quantum field theory. The basic problem is to understand the limit of the APS Lagrangians as a path approaches a point where the kernel of the tangential operator jumps up in dimension.

8. The spectral flow from the trivial connection: calculations when the tangential operator jump up in dimension

In [**8**] the invariant $\lambda_1(\Sigma) = \lambda'(\Sigma, h) + \lambda_1''(\Sigma, h)$ is computed for Σ a Brieskorn sphere of the form $\Sigma(2, q, r)$. (Recall that for p, q, r relatively prime, $\Sigma(p, q, r)$ is the homology 3-sphere constructed by intersecting the hypersurface $x^p + y^q + z^r = 0$ in $\mathbb{C}^3$ with S^5.) For $\Sigma(2, q, r)$ it turns out that the unperturbed flat moduli space is regular, i.e. one can take $h = 0$, and this motivated looking at these examples first.

To compute $\lambda_1(\Sigma(2, q, r))$, several types of spectral flow calculations are necessary. The most challenging of these is the quantity $\mathrm{SF}_{\mathbb{C}^2}(\theta, A)$ in Equation (6.4), which must be calculated for each reducible flat connection A. The greatest difficulty here is at the beginning of the path, where it leaves the trivial connection.

Let K be the r-fiber in the Seifert-fibered homology sphere $\Sigma(2, q, r)$. Let Y denote the closed tubular neighborhood of K; fix a homeomorphism of Y with the solid torus $D^2 \times S^1$. Let Z denote the closure of $\Sigma - Y$, so Σ is the union of Y and Z along a torus T. Pulling back the volume form of S^1 via the two coordinate projections $T \to S^1$ defines forms dx and dy on T. These are chosen so that the 1-form dx extends to a closed form on Z which generates $H^1(Z)$, which we continue to denote by dx. We fix a Riemannian metric on Σ so that $\{dx, dy\}$ is an orthonormal basis for the restriction of this metric to T, and so that a neighborhood of T in Σ is isometric to a product $[-1,1] \times T$.

Given a flat reducible $SU(2) \times \{1\} \subset SU(3)$ connection, the twisted odd signature operator on $su(3)$-valued forms decomposes as described in Section 6. The problem of computing the $\mathbb{C}^2$ component of the spectral flow reduces to one concerning $SU(2)$ connections on a trivial $\mathbb{C}^2$ bundle and the spectral flow of the associated twisted odd signature operator.

For each irreducible flat $SU(2)$ connection A on $\Sigma = \Sigma(2,q,r)$, one can find a path A_t of $SU(2)$ connections such that:

(1) $A_0 = \theta$ and $A = A_1$.
(2) The restriction of A_t to Z is flat.
(3) The restriction, a_t, of A_t to the separating torus has connection 1-form $p(t)M dx + q(t)M dy$, where M is the matrix
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
and p, q are real valued functions on $[0,1]$ so that $p(t) = t$ and $q(t) = 0$ for $t \in [0,\epsilon]$ for some small $\epsilon > 0$; and in general $(p(t), q(t)) \notin \mathbb{Z}^2$ for $t > 0$. Moreover, A_t is in cylindrical form in a neighborhood $[-1,1] \times T$ of T.
(4) The restriction of A_t to Z is of the form $tM dx$ for $t \in [0,\epsilon]$.

The term $\mathrm{SF}_{\mathbb{C}^2}(\theta, A)$ in Equation (6.4) refers to the spectral flow of the family D_t of odd signature operators obtained by coupling the odd signature operator to the path A_t of connections. We denote by T_t the path of tangential operators on the torus:
$$T_t : \Omega^*(T) \otimes \mathbb{C}^2 \to \Omega^*(T) \otimes \mathbb{C}^2$$
$$T_t(\alpha, \beta, \gamma) = (\hat{*} d_{a_t} \beta, -d_{a_t} \hat{*} \gamma - \hat{*} d_{a_t} \alpha, d_{a_t} \hat{*} \beta).$$

Since the restriction a_t of A_t to the separating torus T is flat, Lemma 2.1 implies that the kernel of the tangential operator T_t is isomorphic to the cohomology group $H^*(T; \mathbb{C}^2_{\alpha_t})$ where $\alpha_t : \pi_1(T) \to SU(2)$ denotes the holonomy representation of the flat connection a_t.

A simple calculation shows that
$$\ker T_t = \begin{cases} 0 & \text{if } t > 0, \\ H^*(T) \otimes \mathbb{C}^2 \cong \mathbb{C}^8 & \text{if } t = 0. \end{cases}$$

We will show in the next lemma that as t approaches zero, four positive eigenvalues become zero and four negative eigenvalues become zero. Since we are looking for a continuous family $M_4(t)$ of APS Lagrangians, we choose $M_4(t)$ to be the positive eigenspan $E_{(0,\infty)}(t)$ of T_t for $t = 0$, and at $t = 0$ we must choose $M_4(0)$ to be $K \oplus E_{(0,\infty)}(0)$, where $K \subset \ker T_0$ is the span of those four eigenvectors that correspond to the positive eigenvalues which become zero at $t = 0$. This will ensure that the path $M_4(t)$ is continuous at $t = 0$. We will prove the following lemma below:

LEMMA 8.1. *The kernel of T_0 is the space of harmonic $\mathbb{C}^2$ valued forms on the torus,*

$$\ker T_0 = \{a + bdx + cdy + edxdy \mid a, b, c, e \in \mathbb{C}^2\}$$

Let $K \subset \ker T_0$ be the span of the 4 positive eigenvectors which become zero at $t = 0$, i.e.

$$K = \operatorname{span}\{\psi_i(0) \mid T_t(\psi_i(t)) = \mu_i(t)\psi_i(t),\ \mu_i(0) = 0,\ \mu_i(t) > 0 \text{ for small } t > 0\}_{i=1}^4.$$

Then

$$K = \operatorname{span}\left\{\begin{pmatrix}1 - i\ dy\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1 + i\ dy\end{pmatrix}, \begin{pmatrix}-i\ dx + dxdy\\ 0\end{pmatrix}, \begin{pmatrix}0\\ idx + dxdy\end{pmatrix}\right\}$$

Before we prove this lemma, we expand on a technical point which was glossed over in the previous section. Note that M_4 should be a path of APS Lagrangians starting at $\lim_{R\to\infty} \Lambda_R(D_0, Z) = S_{A_0}(Z) \oplus E_{(0,\infty)}$. In general there is no reason why the subspaces K (of Lemma 8.1) and

$$S_{A_0}(Z) = \operatorname{im} H^*(Z; \mathbb{C}^2) \to H^*(T; \mathbb{C}^2) = \{a + bdx \mid a, b \in \mathbb{C}^2\} \tag{8.1}$$

of $H^*(T, \mathbb{C}^2)$ coincide. In fact Lemma 8.1 shows that they do not, and this leads to the need for a correction term: the term $\operatorname{Mas}(V(t), S_{A_0}(Y))$, where $V(t)$ is a path of Lagrangians in $H^*(T; \mathbb{C}^2)$ from $S_{A_0}(Z)$ to K. We justify our sloppiness in the previous section by observing that this is a finite dimensional Maslov index which can be computed by hand (see [**8**] where this is done) and which we absorb in the term τ in the splitting formula (7.1).

In any case, our method is now fully exposed:

- Stretch to replace the Cauchy data space $\Lambda(D_0, Z)$ by its adiabatic limit $S_{A_0}(Z) \oplus E_{(0,\infty)}$.
- Rotate the finite dimensional scattering Lagrangian $S_{A_0}(Z)$ in the kernel of T_0 to line it up with K.
- Proceed along the path using the positive eigenspan of T_t for $t > 0$.

In light of Lemma 8.1 this gives a continuous path of Lagrangians which starts at the Cauchy data space of D_{A_0} acting on Z and proceeds using the APS Lagrangians.

PROOF OF LEMMA 8.1. The operator T_t^2 is the twisted Laplacian acting on the torus

$$T_t^2(\alpha, \beta, \gamma) = (d_{a_t}^* d_{a_t}\alpha, (d_{a_t}^* d_{a_t} + d_{a_t} d_{a_t}^*)\beta, d_{a_t} d_{a_t}^*\gamma) = (\Delta_{a_t}\alpha, \Delta_{a_t}\beta, \Delta_{a_t}\gamma).$$

For small t, $a_t = itMdx$, and a simple calculation reveals that

$$T_t^2(\alpha, \beta, \gamma) = (\Delta\alpha, \Delta\beta, \Delta\gamma) + t^2(\alpha, \beta, \gamma),$$

where Δ denotes the ordinary (untwisted) Laplacian acting on $\mathbb{C}^2$-valued forms.

The eigenvalues of Δ acting on p-forms are $m^2 + n^2$ for $m, n \in \mathbb{Z}$. Thus if $\Delta w = (m^2 + n^2)w$ for $w \in \Omega^p(T)$,

$$\Delta_{a_t}\begin{pmatrix}w\\ 0\end{pmatrix} = (m^2 + n^2 + t^2)\begin{pmatrix}w\\ 0\end{pmatrix} \text{ and } \Delta_{a_t}\begin{pmatrix}0\\ w\end{pmatrix} = (m^2 + n^2 + t^2)\begin{pmatrix}0\\ w\end{pmatrix}.$$

Since the $\mathbb{C}^2$ valued p-forms are spanned (over $\mathbb{C}$) by $\begin{pmatrix}w\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\ w\end{pmatrix}$ if w spans the ordinary $\mathbb{C}$-valued p-forms, it follows that the set $\{m^2 + n^2 + t^2 \mid m, n \in \mathbb{Z}\}$ is the entire spectrum of Δ_{a_t}. In particular, $\ker T_t = \ker T_t^2$ is trivial for small $t \neq 0$.

Let t be small and positive. Let w be a $\mathbb{C}$ valued function on the torus with $\Delta w = (m^2+n^2)w$ (i.e. $w = e^{i(mx+ny)}$). Let $\mu = m^2+n^2+t^2$. Then, for

$$\alpha = \begin{pmatrix} w \\ 0 \end{pmatrix} \text{ or } \alpha = \begin{pmatrix} 0 \\ w \end{pmatrix} \tag{8.2}$$

we have

$$T_t\left(\alpha, \pm\tfrac{1}{\sqrt{\mu}}\hat{*}d_{a_t}\alpha, 0\right) = \left(\mp\tfrac{1}{\sqrt{\mu}}\Delta_{a_t}\alpha, -\hat{*}d_{a_t}\alpha, 0\right) = \mp\sqrt{\mu}\left(\alpha, \pm\tfrac{1}{\sqrt{\mu}}\hat{*}d_{a_t}\alpha, 0\right) \tag{8.3}$$

and similarly

$$T_t\left(0, \pm\tfrac{1}{\sqrt{\mu}}d_{a_t}\alpha, \hat{*}\alpha\right) = \mp\sqrt{\mu}\left(0, \pm\tfrac{1}{\sqrt{\mu}}d_{a_t}\alpha, \hat{*}\alpha\right). \tag{8.4}$$

This shows that the eigenvalues of T_t are $\pm\sqrt{m^2+n^2+t^2}$ and computes the corresponding 4-dimensional μ eigenspace when $\mu \neq 0$, e.g. for $t > 0$ small enough. For $t = 0$ the formulas apply except when $\mu = 0$, i.e. $m = n = 0$. In that case, if $c_1 : T \to \mathbb{C}^2$ (resp. $c_2 : T \to \mathbb{C}^2$) denotes the constant function on the torus with value $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$), then the kernel of T_0 is spanned (over $\mathbb{C}$) by the 8 vectors of the form

$$(c_i, 0, 0), (0, 0, c_i\ dx\ dy), (0, c_i\ dx, 0), \text{ and } (0, c_i\ dy, 0),\ i = 1, 2.$$

Notice that

$$d_{a_t}c_j = it\ dx\ Mc_j = (-1)^{j+1}it\ dx\ c_j.$$

Taking $m = n = 0$ and $t > 0$ small in Equation (8.3) (so $\sqrt{\mu} = t$) one sees that the vector $v = (c_j, (-1)^j i\ dy\ c_j, 0)$ satisfies $T_t v = tv$ for all $t > 0$. But it also satisfies this equation for $t = 0$, and so v is a positive eigenvector of T_t for $t > 0$ and a zero eigenvector for $t = 0$. Thus v lies in the space K. A similar computation using Equation (8.4) shows that K is spanned by the four vectors

$$(c_1, -i\ dy\ c_1, 0), (c_2, i\ dy\ c_2, 0), (0, -i\ dx\ c_1, c_1\ dxdy), (0, i\ dx\ c_2, c_2\ dxdy).$$

This implies Lemma 8.1.

□

Thus the paths of Lagrangian subspaces that enter in the splitting formula (7.1) can be constructed, and all three terms can be computed. The term $\mathrm{SF}(D_t, Y; M_4)$ can be calculated because Y is a solid torus and so using homotopy invariance and path additivity of spectral flow one can reduce the calculation to three simple calculations on lens spaces.

The term $\mathrm{SF}(D_t, Z; N_4)$ can be computed because the connections A_t restrict to *flat* connections on Z. Since N_4 is a path of APS Lagrangians the kernel of D_t on Z with N_4 boundary conditions can be computed cohomologically using Proposition 2.6. Thus one knows exactly when the kernel jumps up in dimension and topological arguments can be invoked to decide which way the eigenvalues cross zero at each such parameter.

Finally, the term τ corresponds to a sum of finite dimensional Maslov indices constructed from the Lagrangian subspaces $S_{A_0}(Y), S_{A_0}(Z)$, and K in $H^*(T;\mathbb{C}^2) = \ker T_0$. These can be computed explictly from the definition using Lemma 8.1 and Equation (8.1).

9. The spectral flow from a flat connection to a perturbed flat connection

We now turn to a description of the most interesting spectral flow calculations in [**7**]. In that article we consider Brieskorn spheres $\Sigma = \Sigma(p, q, r)$ with $p, q, r > 2$. For these 3-manifolds, the flat $SU(3)$ moduli space is degenerate, and so perturbations are necessary. According to Formula (6.5), the quantity $\mathrm{SF}_{\mathbb{C}^2}(\widehat{A}^{\pm}, A)$ must be calculated for each perturbed flat reducible connection A. Here, the reducible flat connections $\widehat{A}^{\pm}$ are basepoints in the component of the flat moduli space closest to $[A]$ which maximize/minimize the spectral flow.

In the present context, the reducible flat moduli space consists of isolated points, so that $\widehat{A}^{+} = \widehat{A}^{-}$, and we henceforth denote this flat connection by $\widehat{A}$. In the interesting cases, the kernel of $D_{\widehat{A}}$ acting on $\mathbb{C}^2$ valued forms has (real) dimension 4. One can show that, for small perturbations, the perturbed flat reducible connections vary smoothly with the perturbation, so in particular there is one perturbed flat reducible connection near each reducible flat connection $\widehat{A}$.

One must first solve the transversality problem, i.e. find a pertubation for which the perturbed moduli space is nondegenerate. For this perturbation, one must then calculate the spectral flow $\mathrm{SF}_{\mathbb{C}^2}(\widehat{A}, A)$ for each flat reducible connection $\widehat{A}$ and nearby perturbed flat reducible A.

The basic property of holonomy perturbations that makes computations possible is the fact that perturbed flat connections are flat on the complement of a tubular neighborhood of the curves along which the perturbations are defined and the perturbed flat connection on one of the solid tori can be described concretely.

Viewing $\Sigma(p, q, r)$ as a Seifert-fibered space with three singular fibers, it turns out that a perturbation in a neighborhood of one of the singular fibers suffices to make the reducible perturbed flat moduli space nondegenerate. The data one ends up with after a careful analysis of the perturbation and the flat moduli space of the knot complement are:

(1) A path $h_t,\ t \in [0, \epsilon]$ of perturbations.
(2) A path $A_t,\ t \in [0, \epsilon]$ of $U(2)$ connections on Σ with A_0 flat and A_t h_t-perturbed flat.
(3) The restriction of A_t to the complement Z of a tubular neighborhood Y of a singular fiber is flat for all $t \in [0, \epsilon]$.
(4) The holonomy $\alpha_t : \pi_1(T) \to U(2)$ of the restriction a_t of A_t to the separating torus satisfies:

$$\alpha_t(\mu) = \tfrac{t}{3}(\cos(c) - 1)I,\ \ \alpha_t(\lambda) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

for some constant c. Here $\mu, \lambda \in \pi_1(T)$ are the meridian and longitude (hence $\mu \in \ker \pi_1(T) \to \pi_1(Z)$).

The action of $U(2)$ on $\mathbb{C}^2$ is not the canonical one, but rather the tensor product of the canonical representation and the square of the determinant. The reason for this comes from the way one passes from reducible $SU(3)$ to $U(2)$ connections, and we refer to [**7**] for details. The following proposition is proven by computing various cohomology groups using topological methods. We omit the calculations, but emphasize that it is precisely because these cohomology calculations are routine (for topologists, in any case) that the spectral flow can be calculated with the techniques described in this paper. Indeed, this is the main point we wish to impart on the reader.

Proposition 9.1. *Let* $\rho_t : \pi_1(Z) \to U(2)$ *denote the holonomy representation of the flat connection* A_t *for* $t \in [0, \epsilon]$. *Note that* ρ_0 *extends to* $\pi_1(\Sigma)$.

1. *The cohomology group* $H^*(T; \mathbb{C}^2_{\alpha_t}) = 0$ *for all* $t \in [0, \epsilon]$. *Thus the kernel of the tangential operator* T_t *is zero for all* $t \in [0, \epsilon]$.
2. *The cohomology group* $H^1(Z; \mathbb{C}^2_{\rho_t})$ *is isomorphic to* $H^1(Z, \partial Z; \mathbb{C}^2_{\rho_t})$ *and has (real) dimension* 4 *for* $t = 0$ *and is zero for* $t > 0$. *Thus the operator* $D_{A_t} : \Omega^{0+1}(Z) \otimes \mathbb{C}^2 \to \Omega^{0+1}(Z) \otimes \mathbb{C}^2$ *with APS boundary condition* $E_{(-\infty,0)}(t)$ *is self adjoint and by Proposition 2.6 has trivial kernel for* $t > 0$, *and its kernel has dimension* 4 *for* $t = 0$.

Since the connections A_t are not flat for $t > 0$ but rather are perturbed flat, the dimension of the kernel of the Hessian of the Chern-Simons function at these points is not determined by cohomology. It is still, however, the cohomology of a perturbed version of the de Rham complex (2.2). This perturbed cohomology still satisfies a Mayer-Vietoris sequence for the decomposition $X = Y \cup_T Z$. Therefore, the information in Proposition 9.1 can be combined with the following perturbed cohomological information for the Y side of the decomposition.

Proposition 9.2. *The perturbed flat cohomology* $H^1_{A_t,h_t}(Y; \mathbb{C}^2)$ *is trivial for all small* t.

Propositions 9.1 and 9.2 allow us to apply the splitting formula (7.1). Things are much easier than in the previous section since the kernel of the tangential operator is trivial along the entire path; one does not need Lemma 8.1, and the finite dimensional term τ vanishes. The term $\mathrm{SF}(D_t, Y, N_4)$ vanishes because of Proposition 9.2. Thus

$$\mathrm{SF}(D_{A_t}, \Sigma) = \mathrm{SF}(D_{A_t}, Z; E_{(-\infty,0)}). \tag{9.1}$$

This is certainly a promising formula since the path of connections A_t restricts to a path of *flat* connections on Z. In particular, $\mathrm{SF}(D_{A_t}, Z; E_{(-\infty,0)})$ is a homotopy invariant (Theorem 7.4 of [**25**]) and so should be calculable by the methods of algebraic topology. There are various approaches to carrying out such calculations, e.g. via cup products as in [**22**].

We took a more elementary route to computing $\mathrm{SF}(D_t, Z; E_{(-\infty,0)})$ in [**7**]. The structure of the space of $U(2)$ representations of the space Z is completely understood. This is because Z is Seifert-fibered with 2 singular fibers, $U(2)$ representations are obtained by twisting $SU(2)$ representations, and the structure of the space of $SU(2)$ representations of $\pi_1(Z)$ is known (see Klassen [**26**]). Using the correspondence between flat connections and representations one easily finds a 2-parameter family of flat connections $A_{t,s}$ on Z (with holonomies $\rho_{t,s}$) so that:

1. $A_{t,0} = A_t$.
2. $H^{ev}(Z; \mathbb{C}^2_{\rho_{t,s}})$ is 4-dimensional for $t = 0$, 0-dimensional for $t > 0$.
3. $\rho_{t,1}$ is diagonal (and hence abelian).

Thus $\mathrm{SF}(D_{A_t}, Z; E_{(-\infty,0)})$ is equal to $\mathrm{SF}(D_{A_{t,1}}, Z; E_{(-\infty,0)})$. But computing spectral flow along a path of *abelian* $U(2)$ flat connections is simple. One can use the index theorem as in [**23**]. By this approach one shows, finally, that two of the eigenmodes become negative and two become positive. Hence, with our conventions we conclude that

Theorem 9.3.

$$\mathrm{SF}(D_{A_t}, \Sigma) = \mathrm{SF}(D_{A_t}, Z; E_{(-\infty,0)}) = -2.$$

Armed with this theorem the calculation of the $SU(3)$ Casson invariant can proceed. An index theory calculation computes the signs in the term $\lambda'(\Sigma, h)$ (this was done by Boden in [**4**]). The irreducible $SU(3)$ moduli space is identified using algebraic-geometric methods, and finally the path components are counted using enumeration methods for lattice points in polyhedra. We refer the interested reader to the article [**7**].

References

[1] Akbulut, S., and McCarthy, J.D., Casson's invariant for oriented homology 3-spheres. An exposition. Mathematical Notes, 36. Princeton University Press, Princeton, NJ, 1990. xviii+182 pp.

[2] Atiyah, M.F., Patodi, V.K., and Singer, I.M., *Spectral asymmetry and Riemannian geometry. I.* Math. Proc. Cambridge Philos. Soc. 77 (1975), 43–69.

[3] Bleeker, D., and Booss–Bavnbek, B., *Spectral invariants of operators of Dirac type on partitioned manifolds.* Preprint, math.AP/0304214.

[4] Boden, H.U., *Unitary representations of Brieskorn spheres.* Duke Math. J. 75 (1994), no. 1, 193–220.

[5] Boden, H.U., and Herald, C.M., *The* SU(3) *Casson invariant for integral homology* 3-*spheres.* J. Differential Geom. 50 (1998), no. 1, 147–206.

[6] Boden, H.U., Herald, C.M., and Kirk, P., *An integer valued* SU(3) *Casson invariant.* Math. Res. Lett. 8 (2001), no. 5-6, 589–603.

[7] Boden, H.U., Herald, C.M., and Kirk, P., *The integer valued* SU(3) *Casson invariant for Brieskorn spheres.* Preprint, math.GT/0311173.

[8] Boden, H.U., Herald, C.M., Kirk, P., and Klassen, E.P., *Gauge theoretic invariants of Dehn surgeries on knots.* Geom. Topol. 5 (2001), 143–226.

[9] Booss–Bavnbek, B., Lesch, M., and Phillips, J., *Unbounded Fredholm operators and spectral flow.* To appear in Canadian J. Math, math.FA/0108014.

[10] Booss-Bavnbek, B., and Wojciechowski, K.P., Elliptic boundary problems for Dirac operators. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1993. xviii+307 pp.

[11] Cappell, S., Lee, R., and Miller, E.Y., *On the Maslov index.* Comm. Pure Appl. Math. 47 (1994), 121–186.

[12] Cappell, S., Lee, R., and Miller, E.Y., *Self-adjoint elliptic operators and manifold decompositions. II. Spectral flow and Maslov index.* Comm. Pure Appl. Math. 49 (1996), no. 9, 869–909.

[13] Cappell, S., Lee, R., and Miller, E.Y., *A perturbative* SU(3) *Casson invariant.* Comment. Math. Helv. 77 (2002), no. 3, 491–523.

[14] Daniel, A.M., and Kirk, P., *A general splitting formula for the spectral flow.* With an appendix by K. P. Wojciechowski. Michigan Math. J. 46 (1999), no. 3, 589–617.

[15] Duff, G.F.D., and Spencer, D.C., *Harmonic tensors on Riemannian manifolds with boundary.* Ann. of Math. (2) 56, (1952), 128–156.

[16] Floer, A., *An instanton-invariant for* 3-*manifolds.* Comm. Math. Phys. 118 (1988), no. 2, 215–240.

[17] Grubb, G., *Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems.* Ark. Mat. 37 (1999), no. 1, 45–86.

[18] Herald, C.M., *Legendrian cobordism and Chern-Simons theory on* 3-*manifolds with boundary.* Comm. Anal. Geom. 2 (1994), no. 3, 337–413.

[19] Herald, C.M., *Transversality for equivariant gradient systems and gauge theory on 3-manifolds.* Preprint (2003).

[20] Himpel, B., *A splitting formula for spectral flow on closed 3-manifolds.* Thesis, Indiana University (2004).

[21] Himpel, B., Kirk, P., and Lesch, M., *Calderon projector for the Hessian of the perturbed Chern-Simons function on a 3-manifold with boundary.* to appear in J. London Math Society, math.GT/0302234.

[22] Kirk, P., and Klassen, E.P., *Computing spectral flow via cup products.* J. Differential Geom. 40 (1994), no. 3, 505–562.

[23] Kirk, P., Klassen, E.P., and Ruberman, D., *Splitting the spectral flow and the Alexander matrix.* Comment. Math. Helv. 69 (1994), no. 3, 375–416.

[24] Kirk, P., and Lesch, M., *The eta-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary.* To appear in *Forum Math.*, math.DG/0012123.

[25] Kirk, P., and Lesch, M., *On the rho invariant for manifolds with boundary.* Algebr. Geom. Topol. 3 (2003), 623–675.

[26] Klassen, E.P., *Representations of knot groups in* SU(2). Trans. Amer. Math. Soc. 326 (1991), no. 2, 795–828.

[27] Nicolaescu, L.I., *The Maslov index, the spectral flow, and decompositions of manifolds.* Duke Math. J. 80 (1995), no. 2, 485–533.

[28] Scott, S.G., *Determinants of Dirac boundary value problems over odd-dimensional manifolds.* Comm. Math. Phys. 173 (1995), no. 1, 43–76.

[29] Seeley, R.T., *Topics in pseudo-differential operators.* C.I.M.E. Conference on pseudo-differential operators 1968, Edizioni Cremonese, Roma, 1969, pp. 169–305.

[30] Taubes, C.H., *Casson's invariant and gauge theory.* J. Differential Geom. 31 (1990), no. 2, 547–599.

[31] Yoshida, T., *Floer homology and splittings of manifolds.* Ann. of Math. (2) 134 (1991), no. 2, 277–323.

Department of Mathematics & Statistics, McMaster University, Hamilton, Ontario L8S-4K1, Canada
E-mail address: boden@mcmaster.ca
URL: http://www.math.mcmaster.ca/∼boden/

Mathematics Department, University of Nevada, Reno NV 89557, U.S.A.
E-mail address: herald@unr.edu
URL: http://unr.edu/homepage/herald/

Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.
E-mail address: pkirk@indiana.edu
URL: http://mypage.iu.edu/∼pkirk