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Morse theory for plane algebraic curves

Maciej Borodzik

Abstract

We use Morse theoretical arguments to study algebraic curves in $\mathbb{C}^2$. We take an algebraic curve $C \subset \mathbb{C}^2$ and intersect it with spheres with fixed origin and growing radii. We explain in detail how the embedded type of the intersection changes if we cross a singular point of C. Then we apply link invariants such as Murasugi's signature and Tristram–Levine signature to obtain information about possible singularities of the curve C in terms of its topology.

1. *Introduction*

By a plane algebraic curve we understand a set

$$C = \{(w_1, w_2) \in \mathbb{C}^2 \colon F(w_1, w_2) = 0\},$$

where F is an irreducible polynomial. Let $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, and $r \in \mathbb{R}$ be positive. If the intersection of C with a 3-sphere $S(\xi, r)$ is transverse, it is a link in $S(\xi, r) \simeq S^3$. We denote it by L_r.

If ξ happens to be a singular point of C and r is sufficiently small, L_r is a link of a plane curve singularity of C at ξ. On the other hand, for any $\xi \in \mathbb{C}^2$ and for any sufficiently large r, L_r is the link of C at infinity.

Links of plane curve singularities have been perfectly understood for almost 30 years (see [**10**] for topological or [**39**] for algebro-geometrical approach). Possible links at infinity are also well described (see [**28**, **29**]). The most difficult case to study, as it was pointed out in a beautiful survey [**35**], is the intermediate step, that is, possible links L_r for r neither very small nor very large.

Our idea is to study the differences between the links of singularities of a curve and its link at infinity via Morse theory: we begin with r small and let it grow to infinity. The isotopy type of the link changes, when we pass through critical points. If C is smooth, the theory is classical (see, for example [**14**, Chapter V] or [**22**]), yet if C has singular points, the analysis requires more care and is a new element in the theory.

To obtain numerical relations we apply some knot invariants. Namely, we study changes of Murasugi's signature in detail and then pass to Levine–Tristram signatures, which give a new set of information. Our choice is dictated by the fact that these invariants are well behaved under the 1-handle addition (this is Murasugi's Lemma, see Lemma 4.2). From a knot theoretical point of view, Morse theory provides inequalities between signatures, which are very closely related to those in [**16**, **17**] (cf. Corollary 5.22 and a discussion below it). What is important, are the applications in algebraic geometry. In this paper, we show only a few of them. First of all, we present an elementary proof of Corollary 5.19. The only known proof up to now [**5**, **6**] relies heavily on algebraic geometry techniques. This result is of interest not only for algebraic geometers, but also in the theory of bifurcations of ODEs (see [**6**, **9**] and references therein). We also reprove Varchenko's estimate on the number of cusps of a degree d curve in

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$\mathbb{C}P^2$ (see Corollary 6.10). Corollary 5.21 and Lemma 6.9 show also a different, completely new application of our method. We refer to [**1**] for a brand new application in studying deformations of singularities of plane curves.

We also want to point out that the methods developed in this article have been used in [**3**] to show various semicontinuity results for singularities of plane curves, including establishing a relationship between a spectrum of a polynomial in two variables at infinity and spectra of singular points of one of its fibres, in a purely topological way. The application of (generalized) Tristram–Levine signatures in higher-dimensional singularity theory is also possible, even though the details somehow differ from those developed in the present paper. This latter work is in progress.

Although Tristram–Levine signatures turn out to be an important tool for extracting data about plane curves, they are surely not the only one. One of the main messages of the article is that *any knot cobordism invariant can be used to obtain global information about possible singularities which may occur on a plane curve*. Altough the s invariant of Rasmussen [**34**] and the τ invariant of Ozsváth–Szabo [**30**] apparently do not give any new obstructions (they are equal to the four genus for positive knots) and Peters' invariant [**31**] seems to be very much related to the Tristram–Levine signature at least for torus knots, the author is convinced that the application of full Khovanov homology in this context will lead to brand new discoveries in the theory of plane curves.

Convention 1.1. Throughout the paper, we use the standard Euclidean, metric on $\mathbb{C}^2$. The standard ball with centre ξ and radius r will be denoted $B(\xi, r)$. We may assume, to be precise, that it is a closed ball, but we never appeal to this fact. The boundary of the ball $B(\xi, r)$ is the sphere denoted $S(\xi, r)$.

2. *Handles related to singular points*

Let C be a plane algebraic curve given by equation $F = 0$, where F is a reduced polynomial. Let $\xi \in \mathbb{C}^2$. Let $z_1, \ldots, z_n$ be all the points of C such that either C is not transverse to $S(\xi, \|z_k - \xi\|)$ at z_k, or z_k is a singular point of C. We shall call them critical points. Let

$$\rho_k = \|z_k - \xi\|.$$

We order $z_1, \ldots, z_n$ in such a way that $\rho_1 \leqslant \rho_2 \leqslant \ldots \leqslant \rho_n$. We shall call $\rho_1, \ldots, \rho_n$ *critical values*. We shall pick a *generic* ξ which means that

(G1) $\rho_1 < \rho_2 < \ldots < \rho_n$, that is, at each level set of the distance function

$$g = g_\xi(w_1, w_2) = |w_1 - \xi_1|^2 + |w_2 - \xi_2|^2 \tag{2.1}$$

restricted to C there is at most one critical point (this is not a very serious restriction and it is put here rather for convenience).

(G2) If z_k is a smooth point of C, then $g|_C$ is of Morse type near z_k.

(G3) If z_k is a singular point of C, we assume the condition (2.4) holds.

Generic points always exist. Obviously G3 and G1 are open-dense conditions. For G2 see, for example, [**22**, Theorem 6.6].

We want to point out that we assume here tacitly, that the overall number of critical points is finite. This follows from the algebraicity of the curve C (see Remark 3.3). If C is not algebraic, this does not hold automatically, because even the number of singular points of C can be infinite and the link at infinity may not even be defined; consider, for example, a curve $\{(z_1, z_2) \in \mathbb{C}^2 \colon z_1 \sin z_2 = 0\}$. Using methods of Forstneric, Globevnik and Rosay [**11**, Proposition 2], one can produce other amusing, albeit not explicit, examples.

REMARK 2.1. From the condition G3, we see in particular that if ξ does not lie on C, then z_1 is a smooth point of C. Indeed, $g|_C$ attains local minimum of z_1, so the tangent space $T_{z_1}C$ is not transverse to $T_{z_1}S(\xi,\rho_1)$. If z_1 is not smooth, this violates G3.

It is well known that, if r_1 and r_2 are in the same interval (ρ_k,ρ_{k+1}), then links L_{r_1} and L_{r_2} are isotopic, where

$$L_r = C \cap S(\xi,r) \subset S(\xi,r).$$

The next definition provides very handy language.

DEFINITION 2.2. Let ρ_k be a critical value. The links L_{ρ_k+} and L_{ρ_k-} (or, if there is no risk of confusion, just L_+ and L_-) are the links $L_{\rho_k+\varepsilon}$ and $L_{\rho_k-\varepsilon}$ with $\varepsilon > 0$ such that $\rho_k+\varepsilon < \rho_{k+1}$ and $\rho_k-\varepsilon > \rho_{k-1}$. We shall say, informally, that the change from L_- to L_+ is a crossing or a passing through a singular point z_k.

The following result is classical. It can be found e.g. in [**14**, Chapter V].

LEMMA 2.3. *Assume that z_k is a smooth point of C. Then L_{ρ_k+} arises from L_{ρ_k-} by addition of a* 0*-handle, an* 1*-handle or a* 2*-handle according to the Morse index at z_k of the distance function g restricted to C.*

A 0-handle corresponds to adding an unlinked unknot to the link. A 2-handle corresponds to deleting an unlinked unknot. The addition of a 1-handle is a hyperbolic operation, which we now define.

DEFINITION 2.4 (see [**15**, Definition 12.3.3]). Let L be a link with components $K_1,\dots,K_{n-1},K_n$. Let us join the knots K_{n-1} and K_n by a band, so as to obtain a knot K'. Let $L' = K_1 \cup \ldots \cup K_{n-2} \cup K'$. We shall then say, that L' is obtained from L by a *hyperbolic transformation*.

The hyperbolic transformation depends heavily on the position of the band, for example, by adding a band to a Hopf link we can obtain a trivial knot, but also a trefoil and, in fact, infinitely many different knots.

REMARK 2.5. Assume again that $\xi \notin C$. We know that z_1 is a smooth point. As for $r < \rho_1$ the link L_r is empty and for $r > r_1$ it is not, the first handle must be a birth. In particular, for $r \in (\rho_1,\rho_2)$ the link L_r is an unknot.

LEMMA 2.6. *If C is a complex curve, there are no* 2*-handles.*

Proof. A 2-handle corresponds to a local maximum of a distance function (2.1) restricted to C. The functions $w_1-\xi_1$ and $w_2-\xi_2$ are holomorphic on C, hence $|w_1-\xi_1|^2+|w_2-\xi_2|^2$ is subharmonic on C, and as such, it does not have any local maxima on C. □

1-handle might occur in three forms.

DEFINITION 2.7. Let $C_- = C \cap B(\xi,\rho_k-\varepsilon)$. A 1-handle attached to two different connected components of the normalization of C_- is called a *join*. A 1-handle attached to a single component of the normalization of C_- but to two different components of L_- is called a *marriage*. And finally, if it is attached to a single component of L_-, it is called a *divorce*.

If the point z_k is not smooth, the situation is more complicated.

DEFINITION 2.8. The *multiplicity* of a singular point z of C is the local intersection index of C at z with a generic line passing through z.

PROPOSITION 2.9. *Let z_k be a singular point of C with multiplicity p. Let L^{sing} be the link of the singularity at z_k. Then $L_+ (= L_{\rho_k+})$ can be obtained from the disconnected sum of L_- $(= L_{\rho_k-})$ with L^{sing} by adding p 1-handles.*

Proof. This is the most technical and difficult proof in the article. First, we shall introduce the notation, then we shall outline the proof, which in turn consists of four steps.

Introducing the notation. Up to an isometric coordinate change we can assume that $\xi = (0, 0)$ and $z_k = (\rho_k, 0)$.

Let $G_1, \ldots, G_b$ be the branches of C at z_k. By Puiseux theorem (see, for example, [**39**, Section 2]), each branch G_j can be locally parametrized in a Puiseux expansion

$$w_1 = \rho_k - \beta_j \tau^{p_j}, \quad w_2 = \alpha_j \tau^{p_j} + \ldots, \ \ \tau \in \mathbb{C}, \ |\tau| \ll 1, \tag{2.2}$$

that is, it is a topological disk. Let $\psi_j \colon \{|\tau| \ll 1\} \to \mathbb{C}^2$ be the parametrization given by (2.2).

The (generalized) tangent line to G_j at z_k is the line Z_j defined by

$$Z_j = \{(w_1, w_2) \in \mathbb{C}^2 \colon \alpha_j(w_1 - \rho_k) + \beta_j w_2 = 0\}. \tag{2.3}$$

The tangent space to C at z_k is then the union of lines $Z_1, \ldots, Z_b$. By genericity of ξ, we may assume that

$$\alpha_j \beta_j \neq 0 \quad \text{for any } j. \tag{2.4}$$

This means that neither the line $\{(w_1, w_2) \colon w_1 - \rho_k = 0\}$ nor $\{w_2 = 0\}$ is tangent to C at z_k. In other words, we can choose ε, λ and μ in such a way that the following conditions are satisfied.

(S1) The intersection of each tangent line Z_j with $S(0, \rho_k - \varepsilon)$ is non-empty (we use $\beta_j \neq 0$).

(S2) The intersection $B(0, \rho_k - \varepsilon) \cap B(z_k, \mu\varepsilon)$ is non-empty and *omits* each tangent line Z_j (that is, $\mu > 1$, μ is very close to 1 and we use $\alpha_j \neq 0$).

(S3) The two-sphere $S(0, \rho_k - \varepsilon) \cap S(z_k, \lambda\varepsilon)$ is not disjoint with Z_j (this is a refinement of (S1)).

(S4) $\lambda\varepsilon$ is sufficiently small (in the sense which will be made precise later).

(S5) In particular, if we choose

$$\tilde{r} = \sqrt{\rho_k^2 + \lambda^2 \varepsilon^2},$$

then z_k is the only point at which the intersection of C with $S(0, r)$ is not transverse, for $r \in [\rho_k - \varepsilon, \tilde{r}]$.

It is important to note that the two conditions $\alpha_j \neq 0$ and $\beta_j \neq 0$ are of a different nature. Namely, if for some j, $\beta_j = 0$, the proposition fails. On the other hand, the condition $\alpha_j \neq 0$ is used only to make the exposition clearer and easier to understand. The proof given below works if for some j, $\alpha_j = 0$, but we would have to use less transparent arguments in two places.

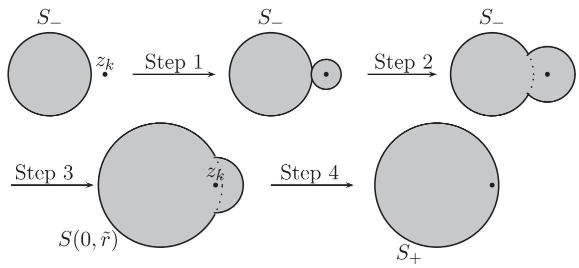


FIGURE 1. *Schematic presentation of the proof of Proposition* 2.9. *The curve* C (*not drawn on the figure*) *is intersected with boundaries of shaded sets providing links* L_-, L^1, L^2, L^3, *and, finally,* L_+.

Let us define the following sets:

$$B_- = B(0, \rho_k - \varepsilon) \quad B_+ = B(0, \tilde{r}) \quad L^2_s = C \cap \partial(B_- \cup B(z_k, s\varepsilon)),$$
$$S_\pm = \partial B_\pm \quad L^1 = L^2_\mu \quad L^3 = C \cap \partial(B(0,\tilde{r}) \cup B(z_k, \lambda\varepsilon)).$$

Here $s \in [\mu, \lambda]$ is a parameter.

Outline of the proof. The proof of the proposition will consist of the following steps.

Step 1. The link L^1 is a disconnected sum of L_- and the link of singularity L^{sing};

Step 2. The link L^2_λ arises from L^2_μ by adding p 1-handles;

Step 3. The link L^3 is isotopic to L^2_λ;

Step 4. The link L_+ is isotopic to L^3.

The most important part is Step 2, all others are technical. The notation L^1, L^2 and L^3 suggests in which step the given link appears (Figure 1).

In proving Steps 2–4 we will use the following lemma, which is a slight generalization of a standard result about isotopies. For the convenience of the reader, we also present a sketch of proof.

LEMMA 2.10 (Transverse isotopy). *Let* $S^3 = W_N \cup W_S$ *be a decomposition of* S^3 *into upper 'northern' and lower 'southern' closed hemispheres and let* $S^2_{\text{eq}} = W_N \cup W_S$ *be the 'equator'. We denote by* W_N^o *and* W_S^o *the interiors of* W_N *and* W_S, *respectively. Assume that* $\phi_s\colon S^3 \to \mathbb{C}^2$ *is a family of embeddings with the following assumptions:*

(Is1) $\phi\colon S^3 \times [0,1] \to \mathbb{C}^2 \times [0,1]$ *given by* $\phi(x,s) = (\phi_s(x), s)$ *is continuous, that is,* ϕ_s *is a continuous family*;

(Is2) ϕ_s *is a smooth family when restricted to* W_N *and to* W_S, *in particular it is smooth when restricted to* S^2_{eq};

(Is3) *the image* $\phi_s(W_N^o)$ *and* $\phi_s(W_S^o)$ *is transverse to* C;

(Is4) (*the crucial in our applications*) *the image* $\phi_s(S^2_{\text{eq}})$ *is transverse to* C.

Then the links $\phi_0^{-1}(C)$ *and* $\phi_1^{-1}(C)$ *are isotopic.*

Proof of Lemma 2.10. If ϕ_s is C^1 smooth, the statement is standard. The proof in this case is slightly more technical, but follows the same pattern. Namely, we shall prove that for any $s \in [0,1]$ and for any s' sufficiently close to s, the links $\phi_s^{-1}(C)$ and $\phi_{s'}^{-1}(C)$ are isotopic and the statement shall follow from compactness and connectedness of the interval $[0,1]$.

Let us then consider a particular $s \in [0,1]$. Recall that C was given by an equation $\{F = 0\}$. Let S^3_{reg}, respectively, $S^2_{\text{eq,reg}}$, be the set of points $x \in S^3$ (respectively $x \in S^2_{\text{eq}}$) such that $\phi_s(S^3)$ (respectively $\phi_s(S^2_{\text{eq}})$) is transverse to $F^{-1}(F(\phi_s(x)))$ at $\phi_s(x)$.

Now for each $x \in W_N \cap S^3_{\mathrm{reg}}$, we can choose a vector $v^N_s(x)$ such that

$$DF \cdot \left(\frac{\partial \phi_s}{\partial s} + v^N_s(x)\right) = 0 \tag{2.5}$$

(here DF means the derivative regarded as a 4×2 real matrix). This property means that $F \circ \phi_s$ is constant along the integral curves of the (non-autonomous) vector field v^N_s. Now two different vectors $v^N_s(x)$ and $\tilde{v}^N_s(x)$ satisfying (2.5) differ by a vector which is tangent to $(F \circ \phi_s)^{-1}(F(\phi_s(x)))$. In particular, we can pick $v^N_s(x)$ to be a smooth vector field, and, whenever $x \in S^2_{\mathrm{eq,reg}}$, we can make $v^N_s(x)$ tangent to S^2_{eq}. As each fibre $F^{-1}(F(\phi_s(x)))$ which is transverse to S^2_{eq} intersects S^2_{eq} in finitely many points, we see that the vector fields v^N_s are then uniquely defined on $S^2_{\mathrm{eq,reg}}$.

Similarly, we construct a vector field $v^S_s(x)$. The two vector fields v^S_s and v^N_s agree on $S^2_{\mathrm{eq,reg}}$ and therefore they can be glued to produce a vector field v_s defined on $U = (S^3_{\mathrm{reg}} \setminus S^2_{\mathrm{eq}}) \cup S^2_{\mathrm{eq,reg}}$. As v^S_s and v^N_s are smooth, v_s is locally Lipschitz. By Cauchy's theorem, v_s can be integrated to a local diffeomorphism. This diffeomorphism maps fibres of $F \circ \phi_s$ to fibres of $F \circ \phi_{s'}$, for s' sufficiently close to s.

Now the assumptions (Is3) and (Is4) guarantee that $\phi_s^{-1}(C)$ lies in the interior of U. Therefore, $\phi_s^{-1}(C)$ is isotopic to $\phi_{s'}^{-1}(C)$ for s' close to s and we conclude the proof. □

Before we pass to the core of the proof of Proposition 2.9, let us make an obvious, but important, remark. The order of tangency of each branch of G_j of C to Z_j (see (2.3)) is, by (2.2), $p_j \geqslant 2$. Therefore, a point $z \in C$ sufficiently close to z_k, the tangent space T_zC is very close to Z_j for some j. In particular, if we can show transversality of some space $X \subset \mathbb{C}^2$ to all of Z_j, we can often claim the transversality of X to C.

Step 1. By condition (S2) above, the intersection of B_- and $B(z_k, \mu\varepsilon)$ is disjoint from C. Therefore, $C \cap (S_- \setminus B(z_k, \mu\varepsilon)) = C \cap S_- = L_-$ and $C \cap (S(z_k, \mu\varepsilon) \setminus B_-) = C \cap S(z_k, \mu\varepsilon) = L^{\mathrm{sing}}_k$. Thus the intersection of C with $\partial(B_- \cup B(z_k, \mu\varepsilon))$ is indeed a disjoint sum of L_- and L^{sing}_k.

Step 2. For any $s \in [\mu, \lambda]$, C is transverse to $B(z_k, s\varepsilon)$ (because each of $Z_1, \dots, Z_b$ is transverse and ε is sufficiently small). We are in a situation covered by Lemma 2.10: $\partial(B_- \cap B(z_k, s\varepsilon))$ can be regarded as an image of a piecewise smooth map from S^3 to $\mathbb{C}^2$, which maps S^3_S to S_-, S^3_N to $S(z_k, \varepsilon)$ and S^2_{eq} to $S_- \cap S(z_k, \varepsilon)$. Nevertheless, as the links L^2_μ and L^2_λ are non-isotopic, some of the assumptions of Lemma 2.10 must fail. Indeed, we shall show below that (Is4) is not satisfied (see Remark 2.11) and we accomplish Step 2 by studying the intersection of C with $S_- \cap S(z_k, s\varepsilon)$.

Consider a branch G_j of C (see (2.2)). The idea is that up to terms of order τ^{p_j+1} or higher, the image of the branch G_j is a p_j -times covered disk, which lies in Z_j, so the situation described in Figure 2 happens precisely p_j times, which gives p_j 1-handles. Since the multiplicity of a

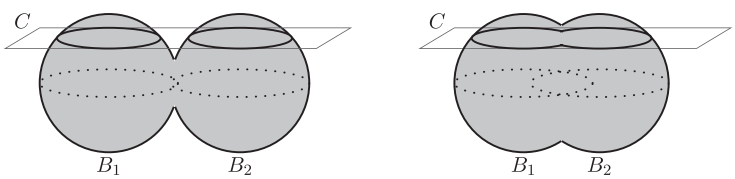


FIGURE 2. *Toy model in three dimensions, which should help us to understand Step 2. Two balls B_1 and B_2. A plane C intersects the boundary of $\partial(B_1 \cup B_2)$ in two disjoint circles (left picture). If we push the ball B_2 inside B_1, this intersection becomes one circle. This is precisely a one-handle attachment that occurs in Step 2.*

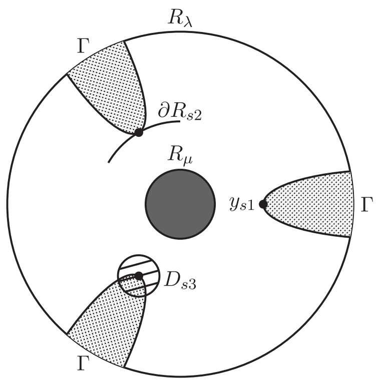


FIGURE 3. *Schematic presentation of notation used in Step 2. The branch in question as multiplicity $p_j = 3$. For clearness of the picture, we draw only one disk D_{ja} and do not label all objects. We also draw only a part of ∂R_{s2}, the whole ∂R_{s2} is the full circle.*

singular point is equal to the sum of multiplicities of branches, this will conclude the proof (Figure 3).

To be more rigorous, consider a disk

$$G_j \cap B(z_k, \lambda\varepsilon),$$

which can be presented as $\psi_j(R_\lambda)$, where ψ_j is the parametrization of G_j (see (2.2)) and

$$R_\lambda = \{\tau \in \mathbb{C}\colon (|\beta_j|^2 + |\alpha_j|^2)|\tau|^{2p_j} + \ldots \leqslant \lambda^2\varepsilon^2\},$$

where $\ldots$ denotes higher order terms in τ. Let

$$\Gamma = {\psi_j}^{-1}(B_-) \cap R_\lambda$$

and for $s \in [\mu, \lambda]$, let

$$R_s = {\psi_j}^{-1}(B(z_k, \varepsilon s)) \cap R_\lambda.$$

Observe that

$$ {\psi_j}^{-1}(L_s^2) = \partial(\Gamma \cup R_s). \tag{2.6}$$

It is also useful to have in mind the following fact.

REMARK 2.11. The intersection of the branch G_j with $S_- \cap S(z_k, s\varepsilon)$ is not transverse (and so the condition (Is4) of Lemma 2.10 is not satisfied, and so one may expect a change of topology of link L_s^2) if and only if $\partial\Gamma$ is not transverse to ∂R_s.

Using the local parametrization, we can see that R_s, up to higher order terms, is given by

$$|\tau|^2 \leqslant \left(\frac{s^2\varepsilon^2}{|\alpha_j|^2 + |\beta_j|^2}\right)^{1/p_j} + \ldots,$$

that is, this is, up to higher order terms, a disk. In particular, it is a convex set (see Remark 2.12). On the other hand, we can compute explicitly the parametrization of $\partial\Gamma$. By plugging (2.2) into the condition $|w_1|^2 + |w_2|^2 = (\rho_k - \varepsilon)^2$, and neglecting the terms of order $p_j + 1$ or higher in τ (and with ε^2), we get

$$\partial\Gamma = \{\tau\colon \text{ Re } \beta_j\tau^{p_j} = \tfrac{1}{2}\varepsilon\rho_k\}.$$

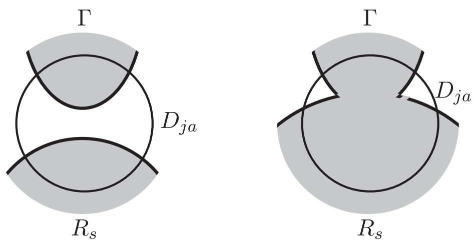


Figure 4. *Passing through s_{ja}. The picture presents $\psi_j^{-1}(B_- \cup B(z_k, s\varepsilon)) \cap R_\lambda = \Gamma \cup R_s$, lying inside the disk D_{ja}. On the left $s < s_{ja}$ and Γ is disjoint from R_s, on the right $s > s_{ja}$ and $\Gamma \cap R_s \neq \emptyset$. The boundary of $\Gamma \cup R_s$ is mapped onto link L_s^2: we see that the topology changes by the 1-handle addition as s crosses s_{ja}.*

Chosing η_j such that $\eta_j^{p_j} = \beta_j$, and writing in polar coordinates (r, ϕ) on R_λ

$$\eta_j^{-1}\tau = r(\cos\phi + i\sin\phi),$$

we finally obtain

$$\partial\Gamma = \{(r, \phi) \in R_\lambda \colon r^{p_j}\cos p_j\phi = \tfrac{1}{2}\varepsilon\rho_k\}, \tag{2.7}$$

modulo higher order terms. We can see that $\partial\Gamma$ consists of p_j connected components, indeed, for $\cos p_j\phi < 0$ equation (2.7) cannot hold. It follows that Γ has also p_j connected components, let us call them $\Gamma_{j1} \dots, \Gamma_{jp_j}$. Each set Γ_{ja} is convex. This follows from (2.7) and a simple analytic observation, which we now state explicitly.

Remark 2.12. In general, the convexity of the connected subset of a disk given by $\{f \geqslant 0\}$ for some f depends only on second derivatives of f. So if a function g is C^2-close enough to f, and the set $\{f \geqslant 0\}$ is convex, then $\{g \geqslant 0\}$ is convex, as well. Since the terms we neglect in the discussion above are of order τ^{p_j+1} and $p_j \geqslant 2$, the convexity of R_s follows from the convexity of a disk of radius $\varepsilon(|\alpha_j|^2 + |\beta_j|^2)s^{1/p_j}$ and the convexity of Γ_{ja} follows from the convexity of the set with boundaries parametrized by (2.7) without higher order terms. Here we use implicitly condition (S4).

Now consider a single $a \in \{1, \dots, p_j\}$. By conditions (S2) and (S3) above, $\Gamma_{ja} \cap R_\mu = \emptyset$ and $\Gamma_{ja} \cap R_\lambda \neq \emptyset$. Thus, by convexity, there exists a single $s = s_{ja}$ such that $\partial\Gamma_{ja}$ is tangent to $R_{s_{ja}}$. In particular, there are p_j points on R_λ such that $\partial\Gamma$ is tangent to R_s for some s. Let us call them $y_{j1}, \dots, y_{jp_j}$. Let us pick a very small disk D_{ja} near y_{ja}. Then for $s < s_{ja}$ close to s_{ja}, $\psi_j^{-1}(L_s^2) \cap D_{ja}$ (cf. (2.6)) consists of two arcs: one on $\partial\Gamma$ and the other on ∂R_s, see Figure 4. On the other hand, for $s > s_{ja}$ close to s_{ja}, $\psi_j^{-1}(L_s^2) \cap D_{ja}$ consists of two arcs, each lies partially on $\partial\Gamma$ and partially on ∂R_s. It follows that a 1-handle addition occurs in D_{ja} when s passes through s_{ja}.

Step 3. We isotope the ball $S_- = S(0, \rho_k - \varepsilon)$ to $S_+ = S(0, \tilde{r})$ and use Lemma 2.10. More precisely, consider a family of sets

$$B_s^3 := B(z_k, \lambda\varepsilon) \cup B(0, s),$$

where $s \in [\rho_k - \varepsilon, \tilde{r}]$. We can easily find a piecewise smooth family of maps $\phi_s^3 \colon S^3 \to \partial B_s^3$, such that $\phi_s^3(W_N) \to S(0, s)$, $\phi_s^3(W_S) \to S(z_k, \lambda\varepsilon)$ and $\psi_s^3(S^2_{\mathrm{eq}}) = S(z_k\lambda\varepsilon) \cap S(0, s)$ (notation from Lemma 2.10). Now $\phi_s^3(W_N^o)$ is transverse to C. Indeed, this follows by (S5) and the fact that z_k is not in the image $\phi_s^3(W_N^o)$. Obviously $\phi_s^3(W_S^o)$ is transverse to C, because C is transverse

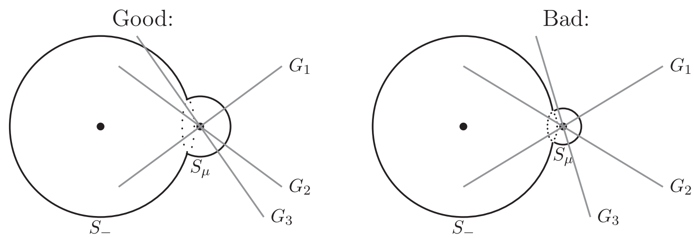


FIGURE 5. *Step* 3. *We explain why the condition* (S3) *is important.* S_μ *is shorthand for* $S(z_k, \mu\varepsilon)$. *The dotted ellipse represents* $S_- \cap S_\lambda$. *On the right-hand side, there is one branch of* C, *namely* G_3, *which does not intersect* $S_- \cap S_\lambda$, *if we start enlarging* S_-, *the intersection of* $S_- \cap S_\lambda$ *will eventually become non-empty, so we shall meet a non-transversality point. If we choose* λ *large enough, then all non-transversality points are dealt in with Step* 2.

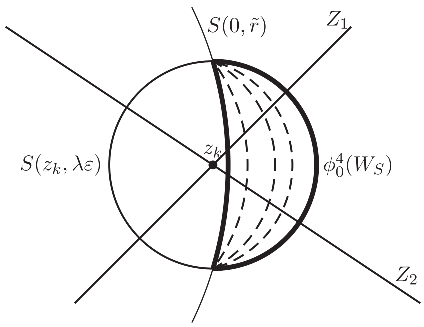


FIGURE 6. *Step* 4. *A schematic presentation of an isotopy of* ϕ^4_s. *The consecutive images* $\phi^4_s(W_S)$ *are drawn with dashed lines, only* $\phi^4_0(W_S)$ *and* $\phi^4_1(W_S)$ *(not labelled on the picture) are bold solid lines. The lines* Z_1 *and* Z_2 *are examples of possible tangent lines to* C, *they are all transverse to images* $\phi^4_s(W_S)$ *for* $s \in [0,1]$.

to $S(z_k, \lambda\varepsilon)$. Therefore, condition (Is3) of Lemma 2.10 is satisfied. We need to show (Is4). But observe that

$$S(0,\tilde r) \cap S(z_k,\lambda\varepsilon) = S(z_k,\varepsilon) \cap \{w_2 = 0\}. \tag{2.8}$$

Each tangent line Z_j (see (2.3)) is in fact transverse to $S(z_k,\lambda\varepsilon) \cup S(0,s)$ for all $s \in [\rho_k - \varepsilon, \tilde r]$. (This follows from elementary geometric argument which we leave as an exercise. Figure 5 explains the key point of the argument, namely that λ has been chosen large enough.) Then by choosing ε small enough we can ensure that C is transverse to $S(z_k,\lambda\varepsilon) \cup S(0,s)$ so (Is4) is satisfied and the step is accomplished.

Step 4. Let $B^4_0 = B(0,\tilde r) \cup B(z_k,\varepsilon)$. With the notation of Lemma 2.10, let us consider a family of maps $\phi^4_s\colon S^3 \to \mathbb{C}^2$ such that $\phi^4_s(W_N) = S_+ \setminus B(z_k,\varepsilon)$ (in fact, we may assume that $\phi^4_s|_{W_N}$ does not depend on s), $\phi^4_0(W_S) = S(z_k,\varepsilon) \setminus B_+$ and $\phi^4_1(S^3) = S_+$. Then the transversality of $\phi^4_s(W^o_N)$ and of $\phi^4_s(S^2_{\mathrm{eq}})$ to C (part of condition (Is3) and the condition (Is4) is obvious). It is not difficult to choose ϕ_s so that $\phi^4_s(W^o_S)$ is transverse to C. For example, one can observe that, for any $s = [0,1]$, the sphere

$$S_s = S(s \cdot z_k, \sqrt{(1-s)^2\rho_k^2 + \lambda^2\varepsilon^2})$$

passes through the intersection of $S(0,\tilde r) \cap S(z_k,\lambda\varepsilon)$, for $s=0$, we have $S_0 = S(z_k,\lambda\varepsilon)$ and for $s=1$, $S_1 = S(0,\tilde r)$. Then we can easily construct ϕ^4_s such that $\phi^4_s(W_S)$ lies on S_s. It is a matter of direct computations to check that $\phi^4_s(W_S)$ is transverse to each tangent line Z_j (see (2.3)) so, if ε is small enough, also to C. See Figure 6. □

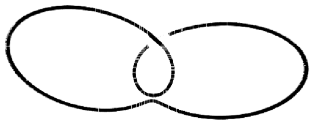


FIGURE 7. *Curve* $\{x^3 - x^2 - y^2 = 0\}$ *intersected with a sphere* $S((-1,0),0.95)$ *on the left and* $S((-1,0),1.04)$ *on the right. For radius* $r = 1$*, we cross an ordinary double point. The trivial knot (on the left) becomes a trefoil after a change of one undercrossing to an overcrossing. (Figures 7 and 8 have been drawn using a C++ computer program written by the author. The author can provide the source code.)*

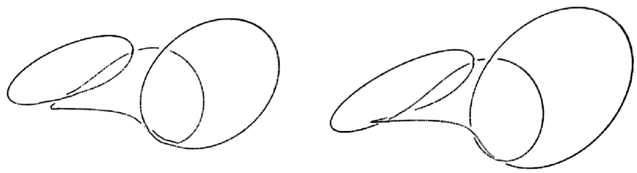


FIGURE 8. *Swallowtail curve (given in parametric form by* $x(t) = t^3 - 3t$*,* $y(t) = t^4 - 2t^2$*) intersected with a sphere* $S((0,0),2.15)$ *on the left and* $S((0,0),2.5)$ *on the right. We cross two* A_2 *singularities at* $r = \sqrt{5}$*. The two external circles on the left twist around the middle one, after crossing a singular point.*

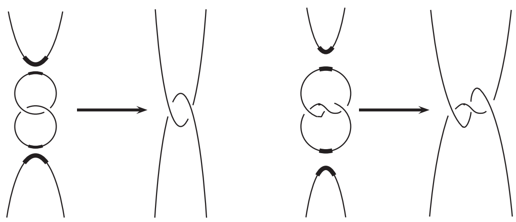


FIGURE 9. *The transformation of links shown on Figures 7 and 8 explained as taking a sum with a Hopf link (respectively torus knot* $T_{2,3}$*) and gluing two* 1*-handles to the result. The bold parts of links represent places where the handles are attached. Remark that on Figure 8 the procedure is applied twice, because we cross two singular points at one time (that is,* $(0,0)$ *violates the genericity condition* G1 *in this case).*

Let us fix an arbitrary ordering of 1-handles at a given singular point once and for all. We shall then denote them $\tilde{H}_1, \dots, \tilde{H}_p$. We can think of the procedure described in Proposition 2.9 as follows: first we take the disconnected sum of L_- with L^{sing}. After that we glue the handle $\tilde{H}_1$, then $\tilde{H}_2$ and so on. In this setting $\tilde{H}_1$ is a join handle and others are divorces or joins or marriages. Such handles will be called *fake joins*, *fake divorces* and *fake marriages*, respectively. The total number of such handles at a point z_k will be denoted f_{j}^k, f_{d}^k and f_{m}^k. These numbers can be computed by studying changes of the number of components and the Euler characteristics between C_- and C_+ and between L_- and L_+ (see the proof of Proposition 5.8) and, as such, they are independent of the ordering of handles.

EXAMPLE 2.13. If z_k is an ordinary double point (locally defined by $\{xy = 0\}$), then L_+ arises from L_- by changing a negative crossing on some link diagram to a positive crossing (see Figure 7 and its explanation in Figure 9).

3. *Number of non-transversality points*

This section is auxillary in the sense that it provides some control over the number of non-transversality points, which might be useful in the future. We use only one result from this section, namely the finiteness of critical points of an algebraic curve.

Let us consider a curve $C = \{F = 0\}$ in $\mathbb{C}^2$, such that F is a reduced polynomial of degree d. Let $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$ be a fixed point (a ball centre). Let $S_r = S(\xi, r)$ be a three-sphere of radius r centred at ξ. Let $w = (w_1, w_2)$ be an arbitrary point in $C \cap S_r$. Assume that C is smooth at w.

LEMMA 3.1. *The intersection $C \cap S_r$ is transverse at w if and only if the determinant*

$$J_\xi(w) = \det \begin{pmatrix} \overline{\dfrac{\partial F}{\partial w_1}}(w) & \overline{\dfrac{\partial F}{\partial w_2}}(w) \\ w_1 - \xi_1 & w_2 - \xi_2 \end{pmatrix}$$

does not vanish.

Proof. Assume that C is not transverse to S_r at w. This means that

$$T_w C + T_w S_r \neq \mathbb{C}^2.$$

Since $T_w S_r$ is a real three dimensional, $T_w C + T_w S_r = T_w S_r$, thus

$$T_w C \subset T_w S_r.$$

Taking the orthogonal complements of these spaces we see that

$$N_w S_r \subset N_w C.$$

But $N_w C$ is a complex space. Thus $\mathbf{i} \cdot N_w S_r \subset N_w C$ and by dimension arguments we get that

$$N_w S_r \otimes \mathbb{C} = N_w C.$$

Now $N_w S_r \otimes \mathbb{C}$ is spanned over $\mathbb{C}$ by a vector $(w_1 - \xi_1, w_2 - \xi_2)$. The lemma follows (the above reasoning can be reversed to show the 'if' part). □

If w is a singular point of C, $J_\xi(w) = 0$ by the definition.

COROLLARY 3.2. *For a curve C of degree d and a generic point $\xi \in \mathbb{C}^2$ there are $d(d-2)$ such points (counted with multiplicities) $w \in C$ where the intersection*

$$C \cap S(\xi, \|w - \xi\|),$$

is not transverse at w.

Proof. For a fixed ξ, $J_\xi(w)$ is a polynomial of degree $d-1$ in w and 1 in $\bar{w}$. Intersecting $\{J_\xi = 0\}$ with C of degree d yields $d^2 - 2d$ points (counted with multiplicities) by generalized Bézout theorem (see, for example [8, Theorem 1]). □

REMARK 3.3. The number of intersection points can be effectively larger than $d^2 - 2d$: as the curve $\{J_\xi = 0\}$ is not complex, there might occur intersection points of multiplicity -1. Anyway, this number is always finite, because both C and J_ξ are real algebraic.

The local intersection index of C with $\{J_\xi(w) = 0\}$ at a singular point z can be effectively calculated. We have the following lemma.

LEMMA 3.4. *Assume that $0 \in \mathbb{C}^2$ is a singular point of C. The local intersection index of C with $\{J_\xi = 0\}$ at 0 is equal to the Milnor number μ of C at 0 minus 1.*

Proof. This follows from Teissier lemma (see [**33**] or [**12**]), which states that

$$(f, J(f,g))_0 = \mu(f) + (f,g)_0 - 1,$$

where $(a,b)_0$ denotes the local intersection index of curves $\{a = 0\}$ and $\{b = 0\}$ at 0 and $J(f,g)$ is the Jacobian

$$\frac{\partial f}{\partial w_1}\frac{\partial g}{\partial w_2} - \frac{\partial f}{\partial w_2}\frac{\partial g}{\partial w_1}.$$

We shall apply this lemma to the case when $f = F$ is the polynomial defining the curve C, whereas g is the distance function:

$$g(w_1, w_2) = |w_1 - \xi_1|^2 + |w_2 - \xi_2|^2.$$

Then $(f,g)_0 = 0$. In fact, intersection of $\{f = 0\}$ and $\{g = 0\}$ is real one dimensional. But if we perturb g to $g - \mathbf{i}\varepsilon$, the intersection set becomes empty.

The issue is that the Teissier lemma holds when f and g are holomorphic. To see that nothing bad happens, if g is as above, we have to skim through a part of the proof of Teissier lemma (see, for example [**33**]). Assume for a while that the curve $\{f = 0\}$ can be parameterized near 0 by

$$w_1 = t^n, \quad w_2 = w_2(t),$$

where $w_2(t)$ is holomorphic and n is the local multiplicity of $\{f = 0\}$ at 0. (The case of many branches does not present new difficulties.) Then

$$\begin{aligned} &\frac{\partial f}{\partial w_1}(t^n, w_2(t)) \cdot nt^{n-1} + \frac{\partial f}{\partial w_2}(t^n, w_2(t)) = 0, \\ &\frac{\partial g}{\partial w_1}(t^n, w_2(t)) \cdot nt^{n-1} + \frac{\partial g}{\partial w_2}(t^n, w_2(t)) = \frac{d}{dt} g(t^n, w_2(t)). \end{aligned} \tag{3.1}$$

The first equation follows from differentiating the identity $f(t^n, w_2(t)) \equiv 0$. The second is simply the chain rule applied to its right-hand side. On its left-hand side, we could have terms with $(\partial g / \partial \bar{w}_2)(\partial \bar{w}_2 / \partial t)$. But they vanish, as w_2 is holomorphic.

From (3.1), we get

$$nt^{n-1} J(f,g)(t^n, w_2(t)) = -\frac{dg(t^n, w_2(t))}{dt} \cdot \frac{\partial f}{\partial w_2}(t^n, w_2(t)). \tag{3.2}$$

Now we can compare orders with respect to t. On the left-hand side of (3.2), we have

$$(n-1) + (f, J(f,g))_0,$$

whereas on the right-hand side, we get

$$(f,g)_0 - 1 + \left(f, \frac{\partial f}{\partial w_2}\right)_0.$$

And we use another lemma, also due to Teissier, that $(f, (\partial f / \partial w_2))_0 = \mu(f) + n - 1$. This can be done directly as f is holomorphic. □

4. *Signature of a link and its properties*

Let $L \subset S^3$ be a link and V a Seifert matrix of L (see, for example [14] for necessary definitions).

Definition 4.1. Let us consider the symmetric form

$$V + V^T. \tag{4.1}$$

The *signature* $\sigma(L)$ of L is the signature of the above form. The *nullity* (denoted $n(L)$) is 1 plus the dimension of a maximal null-space of the form (4.1).

The signature is an important knot cobordism invariant. Unlike many other invariants, signature behaves well under a 1-handle addition. More precisely, we have the following.

Lemma 4.2 (see [23]).

(a) *Let L and L' be two links such that L' can be obtained from L by a hyperbolic transformation (see Definition 2.4). Then*

$$|n(L) - n(L')| = 1 \quad \text{and} \quad \sigma(L) = \sigma(L'); \;\; \text{or}$$
$$|\sigma(L) - \sigma(L')| = 1 \quad \text{and} \quad n(L) = n(L').$$

(b) *Signature is additive under the connected sum. The nullity of a connected sum of links L_1 and L_2 is equal to $n(L_1) + n(L_2) - 1$.*

(c) *Let L be a link and L' be a link resulting in the change from an undercrossing to an overcrossing on some planar diagram of L. Then either*

$$\sigma(L') - \sigma(L) \in \{0, -2\} \quad \text{and} \quad n(L) = n(L'); \;\; \text{or}$$
$$\sigma(L') = \sigma(L) - 1 \quad \text{and} \quad |n(L) - n(L')| = 1.$$

(d) *The nullity n does not exceed the number of components of the link.*

(e) *The signature and nullity are additive under the disconnected sum.*

The signature of a torus knot was computed for example in [14, 21].

Lemma 4.3. *Let $p, q > 1$ be coprime numbers and $T_{p,q}$ be the (p, q)-torus knot. Let us consider a set*

$$\Sigma = \left\{ \frac{i}{p} + \frac{j}{q}, 1 \leqslant i < p, 1 \leqslant j < q \right\}$$

(note in passing that this is the spectrum of the singularity $x^p - y^q = 0$, see [2] for a detailed discussion of this phenomenon). Then

$$\sigma(T_{p,q}) = \#\Sigma - 2\#\Sigma \cap (\tfrac{1}{2}, \tfrac{3}{2}). \tag{4.2}$$

This means that σ counts the elements in Σ with a sign -1 or $+1$ according to whether the element lies in $(\frac{1}{2}, \frac{3}{2})$ or not.

Example 4.4. We have

$$\begin{aligned}\sigma(T_{2,2n+1}) &= -2n,\\ \sigma(T_{3,n}) &= 4\left\lfloor\frac{n}{6}\right\rfloor - 2(n-1),\\ \sigma(T_{4,n}) &= 4\left\lfloor\frac{n}{4}\right\rfloor - 3(n-1).\end{aligned} \tag{4.3}$$

Moreover, for p and q large, $\sigma(T_{p,q}) = -pq/2 + \dots$, where $\dots$ denotes lower order terms in p and q.

Lemma 4.3 holds even if p and q are not coprime (see [**14**]): then we have a torus link instead of a knot.

Next result is a direct consequence of the discussion in [**26**]. It holds, in fact, for any graph link with non-vanishing Alexander polynomial.

Lemma 4.5. *Let L be an algebraic link. Then $n(L) = c(L)$.*

The following result of A. Némethi (private communication) will also be useful.

Proposition 4.6. *Let f be a reduced polynomial in two variables such that the curve $\{f = 0\}$ has an isolated singularity at $(0,0)$. Let $f = f_1 \cdot f_2$ be the decomposition of f locally near $(0,0)$, such that $f_1(0,0) = f_2(0,0) = 0$. Let L, L_1 and L_2 be the links of singularities of $\{f = 0\}$, $\{f_1 = 0\}$ and $\{f_2 = 0\}$ at $(0,0)$ and σ, σ_1 and σ_2 be its signatures. Then we have*

$$\sigma \leqslant \sigma_1 + \sigma_2.$$

We could use the proof from [**24**]. Nevertheless, we shall show a topological proof at the end of the next section.

Lemma 4.7. *Let L be a link of a plane curve singularity with r branches. Then $\sigma(L) \leqslant 1 - r$. Moreover, the equality holds only for the Hopf link and a trivial knot.*

Proof. Let G be a germ of a singular curve bounding L. Let μ be the Milnor number of the singularity of G and $\delta = \frac{1}{2}(\mu + r - 1)$ be the δ-invariant of the singular point. There is a classical result (see, for example [**25**]) that $-\sigma(L) \geqslant \delta$. This settles the case if $r = 1$. If $r > 2$, we use the inequality $\delta \geqslant \frac{1}{2}r(r-1) > r$ (which holds because $2\delta \geqslant \sum_{i\neq j}(C_i \cdot C_{\mathrm{j}})$, where $(C_i \cdot C_{\mathrm{j}})$ is the intersection index of two branches at a given singular point) and we are done. If $r = 2$, we know that $\delta \geqslant 1$, with equality only for an ordinary double point. □

Corollary 4.8. *Let $L = K_1 \cup \ldots \cup K_{n+1}$ be a link of a plane curve singularity with $n+1$ branches. Then*

$$\sigma(L) \leqslant \sigma(K_{n+1}) + 1 - n.$$

Proof. Let $L' = K_1 \cup \ldots \cup K_n$. By Proposition 4.6, $\sigma(L) \leqslant \sigma(L') + \sigma(K_{n+1})$. By Lemma 4.7, $\sigma(L') \leqslant 1 - n$. □

5. *Changes of signature upon an addition of a handle*

In order to study the behaviour of some invariants of knots let us introduce the following notation. Here, $r \in \mathbb{R}$, $r > 0$ and $r \notin \{\rho_1, \dots, \rho_n\}$:

(1) L_r is the link $C \cap S(\xi, r)$;
(2) C_r is the surface $C \cap B(\xi, r)$ and $\hat{C}_r$ is its normalization;
(3) $k(C_r)$ is the number of connected components of $\hat{C}_r$;
(4) $c(C_r)$ or $c(L_r)$ is the number of boundary components of C_r;
(5) $\chi(C_r)$ is the Euler characteristic of C_r;
(6) $p_g(C_r)$ is the genus of C_r, which for smooth C_r satisfies $2k - 2p_g = \chi + c$;
(7) $\sigma(L_r)$ is the signature of L_r
(8) $n(L_r)$ is the nullity of L_r.

If C_r is singular, we are interested in the geometric genus of C_r, that is, the genus of normalization of C_r. This explains the notation p_g for a genus.

Table 1 describes the change of the above quantities upon attaching a handle.

TABLE 1. *Changes of $c(C_r)$, $k(C_r)$, $\chi(C_r)$, $p_g(C_r)$, $\sigma(L_r)$ and $n(L_r)$ upon crossing a smooth non–transversality point.*

Name	Index	Δc	Δk	$\Delta\chi$	Δp_g	$\Delta\sigma$	Δn
Birth	0	1	1	1	0	0	1
Death	2	-1	0	1	0	0	-1
Join	1	-1	-1	-1	0	s	s'
Divorce	1	1	0	-1	0	s	s'
Marriage	1	-1	0	-1	1	s	s'

Here $s, s' \in \{-1, 0, 1\}$ and $|s| + |s'| = 1$ by Lemma 4.2 (a).

Let

$$\begin{aligned} w(L) &= -\sigma(L) + n(L) - c(L), \\ u(L) &= -\sigma(L) - n(L) + c(L). \end{aligned} \tag{5.1}$$

LEMMA 5.1. *If L is a non-trivial link of singularity, then $u(L) > 0$ and $w(L) \geqslant 0$. Moreover, $w(L) = 0$ if and only if L is a Hopf link.*

Proof. We use Lemma 4.7 to prove this for $w(L)$. For $u(L)$, we use the fact that the signature is negative and Lemma 4.2(d). □

For a knot, by Lemma 4.2(d), we have $w(L) = u(L) = -\sigma(L)$. In the general case of links we have

$$-\sigma(L) + (c(L) - 1) \geqslant u(L) \geqslant -\sigma(L) \geqslant w(L) \geqslant -\sigma(L) - (c(L) - 1). \tag{5.2}$$

LEMMA 5.2. *The invariants $w(L)$ and $u(L)$ are additive under the disconnected sum.*

LEMMA 5.3. *Attaching a birth, death, marriage or join handle does not decrease $w(L)$.*

Proof. Only the case of 1-handles requires some attention. The number of components decreases by 1 and either the nullity or the signature can change, and only by 1. □

REMARK 5.4. The divorce handle might decrease the quantity $w(L)$ at most by 2.

LEMMA 5.5. *Attaching a birth, death, marriage or join handle does not increase $u(L)$. The divorce might increase $u(L)$ at most by 2.*

LEMMA 5.6. *Let z_k be a singular point of C, L_k^{sing} the link of its singularity and f_{d}^k the number of fake divorces (see comment after the proof of Proposition 2.9) at z_k. Let, for $\varepsilon > 0$ small enough $L_\pm = L_{\rho_k \pm \varepsilon}$, where $\rho_k = \|z_k - \xi\|$. Then*

$$w(L_+) \geqslant w(L_-) + w(L_k^{\mathrm{sing}}) - 2f_{\mathrm{d}}^k,$$
$$u(L_+) \leqslant u(L_-) + u(L_k^{\mathrm{sing}}) + 2f_{\mathrm{d}}^k.$$

Proof. We use the notation from the proof of Proposition 2.9. We have

$$w(L^1) = w(L_-) + w(L_k^{\mathrm{sing}}) \quad \text{step 1}$$
$$w(L^2) \geqslant w(L^1) - 2f_{\mathrm{d}}^k \quad \text{step 2}$$
$$w(L_+) = w(L^2) \quad \text{steps 3 and 4.}$$

In the middle equations, we have used the fact that a fake divorce can lower the invariant at most by 2. The proof for u is identical. □

LEMMA 5.7. *Assume that C is smooth. Let p_g be the genus of the curve C and d the number of its components at infinity. Let also $a_{\mathrm{b}}, a_{\mathrm{m}}, a_{\mathrm{d}}$, and a_{j} denote the number of birth, marriage, divorce and join handles. The following formulae hold*:

$$\begin{aligned} a_{\mathrm{m}} &= p_g, \\ a_{\mathrm{b}} + a_{\mathrm{d}} - a_{\mathrm{j}} - a_{\mathrm{m}} &= d, \\ a_{\mathrm{b}} - a_{\mathrm{j}} &= 1. \end{aligned} \tag{5.3}$$

In particular,

$$a_{\mathrm{d}} = d + p_g - 1. \tag{5.4}$$

Proof. For $r < \rho_1$, L_r is empty. Thus the first handle must be a birth and for $r \in (\rho_1, \rho_2)$, L_r is an unknot. It has $p_g = 0$, $c = 1$ and $k = 1$. When we next cross critical points, these quantities change according to Table 1. For $r > \rho_n$, we have the link at infinity and C_r is isotopic to C. □

PROPOSITION 5.8. *Let C be an algebraic curve in $\mathbb{C}^2$, not necessarily smooth. For a generic point ξ, let $S_0 = S(\xi, r_0)$ and $S_1 = S(\xi, r_1)$ (with $r_0 < r_1$) be two spheres intersecting transversally with C. For $i = 0, 1$, we define $p_{g_i} = p_g(C_{r_i})$, $c_i = c(C_{r_i})$ and $k_i = k(C_{r_i})$.*

Let a_{d}^{01} and f_{d}^{01} be the numbers of divorces and fake divorces respectively, on C, which lie between S_0 and S_1. Then

$$a_{\mathrm{d}}^{01} + f_{\mathrm{d}}^{01} \leqslant p_{g_1} - p_{g_0} + c_1 - c_0 - (k_1 - k_0).$$

Proof. Let $\pi : \hat{C} \to C$ be the normalization map. The composition of π with the distance function g (see (2.1)) restricted to C yields a function $\hat{g} : \hat{C} \to \mathbb{R}$. This function does not have to be a Morse function on $\hat{C}$, but we can take a small subharmonic perturbation of $\hat{g}$ on $\hat{C_{r_1}}$, such that the resulting function is Morse in the preimage $\pi^{-1}B(\xi, r_1)$. This perturbation we shall still denote by $\hat{g}$. Let $\hat{a}_{\mathrm{b}}$, $\hat{a}_{\mathrm{d}}$, $\hat{a}_{\mathrm{j}}$ and $\hat{a}_{\mathrm{m}}$ be the number of births, divorces, joins and marriages of $\hat{g}$ in $U = \pi^{-1}(B(\xi, r_1) \setminus B(\xi, r_0))$. We need the following result:

LEMMA 5.9. *There is a bound*

$$\hat{a}_{\mathrm{d}} \geqslant a_{\mathrm{d}}^{01} + f_{\mathrm{d}}^{01}. \tag{5.5}$$

Proof. If $z_k \in C$ is a smooth point of C and critical point of g, then $\pi^{-1}(z_k)$ is a critical point of $\hat{g}$ of the same index. Moreover, if z_k is a divorce, join or marriage, then $\pi^{-1}(z_k)$ will also be, respectively, a divorce, join or a marriage.

Next we show that any fake divorce on C corresponds to a divorce on $\hat{C}$. This is done by comparing the changes of topology when crossing a singular point with the changes of topology of normalization. So let z_k be a singular point of C. Let us define

$$C_{\pm} = C \cap B(\xi, \rho_k \pm \varepsilon) \quad \text{and} \quad L_{\pm} = \partial C_{\pm}$$

Let $\hat{C}_{\pm}$ be the normalization. Define also

$$\Delta_g = p_g(C_+) - p_g(C_-), \quad \Delta_k = k(C_+) - k(C_-), \quad \Delta_c = c(L_+) - c(L_-).$$

Observe that from a topological (as opposed to smooth) point of view, passing through a singular point of multiplicity p and r branches amounts to picking r disks and attaching them to $\hat{C}_-$ with p 1-handles. Analogously to (5.3), we then get $f_{\mathrm{m}}^k = \Delta_g$, $f_{\mathrm{d}}^k - f_{\mathrm{j}}^k - f_{\mathrm{m}}^k = \Delta_c$ and $f_{\mathrm{j}}^k = \Delta_k$. Hence

$$f_{\mathrm{d}}^k = \Delta_c + \Delta_g - \Delta_k.$$

The number of divorces on $\hat{C}$ that are close to $\pi^{-1}(z_k)$ (denote this number by $\hat{a}_{\mathrm{d}}^k$) can be computed in the same way. Since the number of boundary components of $\hat{C}_{\pm}$ is the same as $c(C_{\pm})$, and Δ_g measures also the change of genus between $\hat{C}_+$ and $\hat{C}_-$, we have

$$\hat{a}_{\mathrm{d}}^k = \Delta_c + \Delta_g - \Delta_k = f_{\mathrm{d}}^k.$$

□

Finishing the proof of Proposition 5.8. Let us consider the changes of the topology of $\hat{C} \cap \hat{g}^{-1}((-\infty, r^2))$ as r changes from r_0 to r_1. The number of components of the boundary changes by $c_1 - c_0$, while the genus by $g_1 - g_0$ and the number of connected components of normalization by $k_1 - k_0$. Using Table 1 (compare the argument in the proof of Lemma 5.7), we get $\hat{a}_{\mathrm{d}} = g_1 - g_0 + c_1 - c_0 - (k_1 - k_0)$. □

REMARK 5.10. In most applications we will have $k_0 = k_1 = 1$, for example, in the case when L_1 is a link at infinity of a reduced curve and L_0 is a trivial knot.

EXAMPLE 5.11. Let C be a curve given by $x^3 - x^2 - y^2 = 0$ (see Figure 7, but now the centre is in a different place), $\xi = (0, 0)$, r_0 is small and let us take r_1 large enough. Then L_0 is the Hopf link, L_1 is the trefoil, $p_{g_1} = p_{g_0} = 0$ (C is rational), $c_0 = 2$, $c_1 = 1$, $k_1 = 1$ but $k_0 = 2$ ($\hat{C}_0$ consists of two disks). Then the number of divorces is bounded by 0 and, indeed, there is only one critical value between r_0 and r_1 and the corresponding handle is a join.

COROLLARY 5.12. *If $C \subset \mathbb{C}^2$ is a reduced plane algebraic curve and its link at infinity has d components, then for any generic ξ the total number of divorces on C (including the fake divorces) satisfies*

$$a_{\mathrm{d}} + f_{\mathrm{d}} \leqslant p_g(C) + d - 1.$$

Proof. Let us pick a generic ξ and choose $r_0 \in (\rho_1, \rho_2)$ while r_1 is sufficiently large. Then S_0 is an unknot, because the first handle that occurs when coming from $r = 0$, is always a birth. Moreover, $S_1 \cap C$ is the link of C at infinity and so it has d components. The statement follows from Proposition 5.8 $\square$

THEOREM 5.13. *Let C be a curve with link at infinity L_∞ and with singular points $z_1, \dots, z_n$, such that the link at the singular point z_k is L_k^{sing}. Then*

$$w(L_\infty) \geqslant \sum_{k=1}^{n} w(L_k^{\mathrm{sing}}) - 2(p_g(C) + d - 1),$$
$$u(L_\infty) \leqslant \sum_{k=1}^{n} u(L_k^{\mathrm{sing}}) + 2(p_g(C) + d - 1),$$

where d is the number of components of L_∞.

Proof. The proof now is straightforward. Let us take a generic ξ. Then, for $r \in (\rho_1, \rho_2)$, L_r is an unknot (see Remark 2.5), so $w(L_r) = u(L_r) = 0$. Then, as we cross subsequent singular points, $w(L_r)$ and $u(L_r)$ change (see Lemmas 5.3–5.6). We obtain

$$w(L_\infty) \geqslant \sum_{k=1}^{n} (w(L_k^{\mathrm{sing}}) - 2f_{\mathrm{d}}^k) - 2a_{\mathrm{d}}$$

and similar expression for u. The theorem now follows from Corollary 5.12. $\square$

REMARK 5.14. Observe that the first inequality in Theorem 5.13 (as applications below show, the more important one) 'does not see' ordinary double points, because if z_k is an ordinary double point, then $w(L_k^{\mathrm{sing}}) = 0$ (however $u(L_k^{\mathrm{sing}}) = 2$).

As the whole discussion leading to Theorem 5.13 was quite involved, we present some examples.

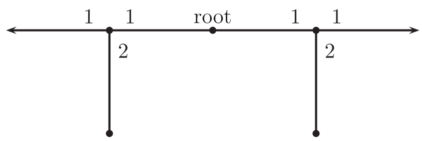
EXAMPLE 5.15. Consider a curve $\{x^3 - x^2 - y^2 = 0\}$, see Example 5.11. An ordinary double point at $(0, 0)$ is the only singular point (it has $w_L = 0$ and $u_L = 2$). The link at infinity is a trefoil with $w = u = 2$. The geometric genus of a curve is equal to 0.

EXAMPLE 5.16. Let C be a swallowtail curve as in Figure 8. It has two ordinary cusps (the corresponding links of singularities are trefoils) and one ordinary double point, its geometric genus is 0 and the link at infinity is the torus knot $T_{3,4}$, with $w = u = 6$. The inequalities in Theorem 5.13 read $6 \geqslant 4$ (the first one) and $6 \leqslant 6$ (the second one) (Figure 9).

EXAMPLE 5.17. Consider a curve parameterized by $x(t) = t^4$, $y(t) = t^6 + t^9$. It has a singular point at $(0, 0)$. According to Eisenbud and Neumann [**10**], the link of this singularity (let us call it L_1) is a $(15, 2)$ cable on the trefoil. The curve also has three other ordinary double points (corresponding to $t = \sqrt[3]{1 + i}$, which can be found by solving the equations $x(t) = x(s)$, $y(t) = y(s)$, $t \neq s$). The link at infinity L_{inf} (see [**28**]) is a $(4, 9)$ torus knot. According to Lemma 6.6 below, $\sigma(L_1) = \sigma(T_{15,2}) = -14$. By Example 4.4, we have $\sigma(L_{inf}) = -16$. Hence $w(L_{inf}) = u(L_{inf}) = 16$ and $w(L_1) = u(L_1) = 14$. Theorem 5.13 holds because $16 \geqslant 14 + 3 \cdot 0$ (inequality for w) and $16 \leqslant 14 + 3 \cdot 2$ (inequality for u).

A good number of possible examples can also be found in [**4**, **7**], where a detailed list of plane algebraic curves with the first Betti number 1 is presented, and singularities are given explicitly for each curve on the list. We provide one example (point (w) in the list of [**7**]), where a divorce handle occurs.

EXAMPLE 5.18. Consider a curve parameterized by $x(t) = t^2 - 2t^{-1}$, $y(t) = 2t - t^{-2}$. It has three ordinary cusps and no other singularities. It follows that $\sum w(L_k^{\mathrm{sing}}) = \sum u(L_k^{\mathrm{sing}}) = 6$. The curve has two branches at infinity, corresponding to $t \to \infty$ and $t \to 0$. Each branch is smooth at infinity and tangent to the line at infinity with the tangency order 2. An application of the algorithm of [**28**] shows that the link at infinity can be represented by the following splice diagram.



Then, the algorithm of Neumann [**27**] shows that the signature of the link at infinity is equal to -5, so $w(L_\infty) = 4$ and $u(L_\infty) = 6$. There is one divorce handle, and indeed $w(L_\infty) = \sum w(L_k^{\mathrm{sing}}) - 2$.

From Theorem 5.13, we can deduce many interesting corollaries. First of all, we use it in showing that some curves with given singularities might not exist. The point (a) of the corollary below is almost a restatement of the result of Petrov [**32**], which can be interpreted as in [**5**] as a bound for k with $p = 3$. The point (c) gives the same estimate as in [**6**], but we use here only elementary facts, not the BMY inequality.

COROLLARY 5.19. *Let $x(t)$ and $y(t)$ be polynomials of degree p and q with p and q coprime. Let C be the curve given in the parametric form by*

$$\{w_1 = x(t), w_2 = y(t), t \in \mathbb{C}\}. \tag{5.6}$$

Assume that the singularity of C at the origin has a branch with singularity A_{2k} (that is, A_{2k} is a singularity of a parametrisation). Then $2k$ is less than or equal to the signature of the torus knot $T_{p,q}$. In particular,

(a) *$k \leqslant q - 1 - 2\lfloor q/6 \rfloor$ if $p = 3$;*
(b) *$k \leqslant \frac{3}{2}(q-1) - 2\lfloor q/4 \rfloor$ if $p = 4$;*
(c) *$k \lesssim pq/4$ in general.*

Proof. Let L_0 be the link of singularity of C at 0. Let $c(L_0)$ be the number of its components. By assumption, one of its components is a link $T_{2,2k+1}$ with signature $-2k$. By Corollary 4.8

$$-\sigma(L_0) \geqslant 2k + c(L_0) - 1.$$

Hence

$$w(L_0) \geqslant 2k.$$

The link at infinity L_∞ is a knot $T_{p,q}$. Hence $w(L_\infty) = \sigma(L_\infty) = \sigma(T_{p,q})$. This, in turn, is computed in Lemma 4.3. The result is then a direct consequence of Theorem 5.13, since $p_g(C) = 0$ by assumption (see (5.6)). □

Remark 5.20. Corollary 5.19(c) holds even if p and q are not coprime. We can compute the signature of the knot at infinity by Lemma 6.6.

The next result is somewhat unexpected, especially if we compare it with [**36**, Proposition 87] stating that no invariant coming from a Seifert matrix of the knot, including the signature, can tell whether a link is a $\mathbb{C}$-link.

Corollary 5.21. *If a $\mathbb{C}$-link L with m components bounds an algebraic curve of geometric genus p_g, then*

$$-\sigma(L) \geqslant 2 - 2m - 2p_g.$$

In particular, if a knot bounds a rational curve, its signature is non-positive.

Now we can rephrase Theorem 5.13 in a Kawauchi-like inequality.

Corollary 5.22. *Let C be as in Theorem 5.13. Let b be the first Betti number of C (that is, the rank of $H_1(C;\mathbb{Q})$. We stress here that we consider the homology of $C \subset \mathbb{C}^2$, not of its compactification in $\mathbb{C}P^2$). Then*

$$\left|\sigma(L_\infty) - \sum_{k=1}^{n} \sigma(L_k^{\text{sing}})\right| \leqslant b + n(L_\infty) - 1.$$

Proof. Let r_k be the number of branches of the link L_k^{sing} and d be the number of branches at infinity. By Theorem 5.13 and the fact that $w(L_k^{\text{sing}}) \geqslant -\sigma(L_k^{\text{sing}}) - (r_k - 1)$, we get.

$$-\sigma(L_\infty) - d + n(L_\infty) \geqslant -\sum \sigma(L_k^{\text{sing}}) - \sum (r_k - 1) - 2(p_g(C) + d - 1).$$

Denoting $R = \sum (r_k - 1)$, we get

$$\sigma(L_\infty) - \sum \sigma(L_k^{\text{sing}}) \leqslant 2p_g + R + d + n(L_\infty) - 2 = b + n(L_\infty) - 1,$$

as $b = 2p_g + R + d - 1$. The inequality in the other direction is proved in an identical way, using the invariant u instead of w. □

With not much work, Corollary 5.22 can be deduced from [**16**, **17**] (see [**15**, Theorem 12.3.1]), *without ever using the holomorphicity of C*. Roughly speaking, we pick a ball $B \subset \mathbb{C}^2$ disjoint from C and pull (by an isotopy) all the singular points of C inside B, so as to get a real surface C' with the property that $C' \cap \partial B$ is a disjoint union of links $L_1^{\text{sing}}, \dots, L_n^{\text{sing}}$. Then $C' \setminus B$ realizes a cobordism between this sum and the link of C at infinity. Then [**15**, Theorem 12.3.1] provides Corollary 5.22.

The main drawback of that approach is that C' is no longer holomorphic. In short, it works for the signature (and Tristram–Levine signatures as well), but if we want at some moment to go beyond and use some more subtle invariant, holomorphicity of C might be crucial. At present we do not know any such invariant, but we are convinced that without exploiting thoroughly the holomorphicity of C we cannot get a full understanding of the relation between the link at infinity and the links of singularities of C.

We finish this section by showing a topological proof of Proposition 4.6. For the convenience of the reader we recall the statement.

Proposition 5.23. *Let f be a reduced polynomial in two variables such that the curve $\{f=0\}$ has an isolated singularity at $(0,0)$. Let $f=f_1\cdot f_2$ be the decomposition of f locally near $(0,0)$, such that $f_1(0,0)=f_2(0,0)=0$. Let L, L_1 and L_2 be the links of singularities of $\{f=0\}$, $\{f_1=0\}$ and $\{f_2=0\}$ at $(0,0)$ and σ,σ_1,σ_2 its signatures. Then we have*

$$\sigma \leqslant \sigma_1+\sigma_2.$$

Proof. Let $r>0$ be small enough, so that $L=\{f=0\}\cap S(0,r)$ is the link of the singularity of f. For a generic vector $v\in\mathbb{C}^2$ sufficiently close to 0, the intersection of $S(0,r)$ with $C=C^v=\{F_v=0\}$ is isotopic to L, where $F_v(w)=f_1(w)f_2(w-v)$. By definition, $C=C_1\cup C_2$ where

$$C_1=\{f_1(w)=0\}\cap B(0,r) \text{ and } C_2=\{f_2(w-v)=0\}\cap B(0,r).$$

Let $\varepsilon\ll r$. The link $C\cap S(0,\varepsilon)$ is clearly the link L_1 of the singularity given by $\{f_1=0\}$. Consider a change of the isotopy type of $C\cap S(0,s)$ as s increases from ε to r.

Claim. There are neither divorce nor fake divorce handles on C for $s\in[\varepsilon,r]$.

The claim follows from Proposition 5.8: we put $r_0=\varepsilon$ and $r_1=r$. Then $p_{g_1}=p_{g_0}=0$, indeed, the normalization of C is a union of disks. Moreover, in the notation from Proposition 5.8, $c_1=k_1$ and $c_0=k_0$. In fact, to show $c_0=k_0$ we observe that $C\cap S(0,\varepsilon)$ is the link of singularity, and both c_0 and k_0 are the numbers of branches of the singular point. The same argument shows that $c_1=k_1$ is equal to the number of branches of singularity of f at $(0,0)$. This shows the claim.

Now the Morse theoretical arguments show that

$$w(L)\geqslant w(L_1)+\sum_k w(L_k^{\mathrm{sing}}),$$

where we sum over all singular points of C, which lie in $B(0,r)\setminus B(0,\varepsilon)$. These singular points are easy to describe. Indeed, there are no singular points which lie only on C_1, there is one singular point, at v, that lies only on C_2 and the corresponding link is the link L_2. Moreover, there are double points arising as intersections of C_1 with C_2. The number of these double points can be effectively computed as the local intersection index of $\{f_1=0\}$ with $\{f_2=0\}$, alternatively as the linking number of L_1 with L_2, but we content ourselves by pointing out that for each double point $w(L_k^{\mathrm{sing}})=0$ (see Remark 5.14). Therefore, we get

$$w(L)\geqslant w(L_1)+w(L_2).$$

And the statement of proposition follows from Lemma 4.5, because then $w(L)=-\sigma(L)$, $w(L_1)=-\sigma(L_1)$ and $w(L_2)=-\sigma(L_2)$. □

6. *Application of Tristram–Levine signatures*

The notion of signature was generalized by Tristram and Levine [**19**, **38**]. The Tristram–Levine signature turns out to be a very strong tool in the theory of plane algebraic curves. In what follows ζ will denote a complex number of module 1 and different than 1.

Definition 6.1. Let L be a link and S be a Seifert matrix. Consider the Hermitian form

$$(1-\zeta)V+(1-\bar{\zeta})V^T. \tag{6.1}$$

The *Tristram–Levine* signature $\sigma_\zeta(L)$ is the signature of the above form. The *nullity* $n_\zeta(L)$ is the nullity of the above form increased by 1.

The addition of 1 is a matter of convention. This makes the nullity additive under disconnected and not connected sum.

REMARK 6.2. For a link L, let us define $n_0(L)$ as a minimal number such that the $n_0(L)$th Alexander polynomial is non-zero. Let $\Delta_{\min}(L) = \Delta_{n_0(L)}(L)$. Then, it is a matter of elementary linear algebra to prove that $n_\zeta(L) \geqslant n_0(L) + 1$ and $n_\zeta(L) > n_0(L) + 1$ if and only if $\Delta_{\min}(\zeta) = 0$ (we owe this remark to A. Stoimenow, see [2] for a thorough discussion).

EXAMPLE 6.3. For $\zeta = -1$, we obtain the classical signature and nullity.

We have, in general, scarce control on the values of n_ζ if ζ is a root of the Alexander polynomial. However, many interesting results can be obtained already by studying invariants σ_ζ and n_ζ when ζ is not a root of the Alexander polynomial. To simplify the formulation of these results let us define the functions σ^*_ζ and n^*_ζ as

$$\sigma^*_\zeta = \begin{cases} \sigma_\zeta & \text{if } \zeta \text{ is not a root of } \Delta_{\min}, \\ \lim\limits_{\rho \to \zeta^+} \sigma_\rho & \text{otherwise.} \end{cases} \tag{6.2}$$

Here $\rho \to \zeta^+$ if we can write $\rho = \exp(2\pi i y)$, $\zeta = \exp(2\pi i x)$ and $y \to x^+$. Similarly we can define n^*_ζ. By Remark 6.2, $n^*_\zeta \equiv n_0(L) + 1$, but we keep this function in order to make the notation consistent with previous sections.

Tristram–Levine signatures share similar properties to the classical signature.

LEMMA 6.4 (see [19, 38], compare also [37]). *Lemma 4.2 holds if we exchange $\sigma(L)$ and $n(L)$ with $\sigma^*_\zeta(L)$ and $n^*_\zeta(L)$.*

Litherland [21] computes also the signature of torus knot $T_{p,q}$:

LEMMA 6.5. *Let p and q be coprime and Σ as in Lemma 4.3. Let $\zeta = \exp(2\pi i x)$ with $x \in (0, 1)$. Then*

$$\sigma^*_\zeta(T_{p,q}) = \#\Sigma - 2\#\Sigma \cap (x, 1 + x]. \tag{6.3}$$

The choice of the closure of the interval $(x, 1 + x]$ in formula (6.5) agrees with taking the right limit in formula (6.2). Indeed, if $x_k \to x^+$, then the number of points in $\Sigma \cap (x_k, x_k + 1]$ converges to the number of points in $\Sigma \cap (x, x + 1]$.

The signature of an iterated torus knot can be computed inductively from the result of [21].

LEMMA 6.6. *Let K be a knot and $K_{p,q}$ be the (p, q)-cable on K. Then, for any ζ, we have*

$$\sigma_\zeta(K_{p,q}) = \sigma_{\zeta^q}(K) + \sigma_\zeta(T_{p,q}).$$

This allows recursive computation of signatures of all possible links of unibranched singularities. In the case of an arbitrary singularity one uses results of Neumann [26, 27].

Because of Lemma 6.4 we can repeat the reasoning from Section 5 to obtain a reformulation of Theorem 5.13, Corollary 5.21 and Corollary 5.22.

THEOREM 6.7. *Let C be an algebraic curve with singular points $z_1, \dots, z_n$, with links of singularities $L_1^{\mathrm{sing}}, \dots, L_n^{\mathrm{sing}}$. Let L_∞ be the link of C at infinity. Let also b be the first Betti number of C. Then*

$$\left|\sigma_\zeta^*(L_\infty) - \sum \sigma_\zeta^*(L_k^{\mathrm{sing}})\right| \leqslant b + n_0(L_\infty). \tag{6.4}$$

The proof goes along the same line as the proof of Corollary 5.22. We introduce the quantities $w_\zeta = -\sigma_\zeta^*(L) + n_\zeta^*(L) - c(L)$ and $u_\zeta = -\sigma_\zeta^*(L) - n_\zeta^*(L) + c(L)$ and study their changes on crossing different singular handles. We remark only that $n_\zeta^*(L_\infty) = n_0(L_\infty) + 1$.

Using the same argument as in Proposition 5.8 we obtain a result which relates the signatures at two intermediate steps.

PROPOSITION 6.8. *For any generic parameter ξ, let r_0 and r_1 be two non-critical parameters. For $i = 0, 1$ let L_i and c_i be, respectively, the link $C \cap S(\xi, r_i)$ and its number of components. Let Δp_g be the difference of genera of $C \cap B(\xi, r_1)$ and $C \cap B(\xi, r_0)$ and Δk the difference between numbers of connected components of corresponding normalizations. We have then*

$$\begin{aligned} w_\zeta(L_1) - \sum w_\zeta(L_k^{\mathrm{sing}}) - w_\zeta(L_0) &\geqslant -2(\Delta p_g + c_1 - c_0 - \Delta k),\\ -(u_\zeta(L_1) - \sum u_\zeta(L_k^{\mathrm{sing}}) - u_\zeta(L_0)) &\geqslant -2(\Delta p_g + c_1 - c_0 - \Delta k), \end{aligned}$$

where we sum only over those critical points that lie in $B(\xi, r_1) \setminus B(\xi, r_0)$.

Corollary 5.21 generalizes immediately to the following, apparently new result.

LEMMA 6.9. *If K is a $\mathbb{C}$-knot bounding a rational curve, then $\sigma_\zeta^*(K) \leqslant 0$ for any ζ.*

Another application of Theorem 6.7 is in the classical problem of bounding the number of cusps of a plane curve of degree d, see [**13**] for the discussion of this problem. Our result is a topological proof of Varchenko's bound.

COROLLARY 6.10. *Let $s(d)$ be a maximal number of A_2 singularities on an algebraic curve in $\mathbb{C}P^2$ of degree d. Then*

$$\limsup \frac{s(d)}{d^2} \leqslant \frac{23}{72}.$$

Proof (*Sketch*). Let C be a curve of degree d in $\mathbb{C}P^2$. Let us pick up a line H intersecting C in d distinct points. We chose an affine coordinate system on $\mathbb{C}P^2$ such that H is the line at infinity. Let C_0 be the affine part of C. Then C_0 can be defined as a zero set of a polynomial F of degree d. Let $z_1, \dots, z_s$ be the singular points of C_0 of type A_2.

Case 1. C_0 has no other singular points.

Then, by the genus formula, $b_1(C_0) = d^2 - 2s + O(d)$. Let us take $\zeta = e^{\pi i/6}$. Then $\sigma_\zeta^*(L_i^{\mathrm{sing}}) = 2$. On the other hand, the link of C_0 at infinity is the torus link $T_{d,d}$ and its signature

$$\sigma_\zeta^*(T_{d,d}) = 2d^2 \cdot \tfrac{1}{6}(1 - \tfrac{1}{6}) + O(d) = \tfrac{5}{18}d^2 + O(d).$$

(For $\zeta = e^{2\pi ix}$ we have asymptotics $\sigma^*_\zeta(T_{d,d}) = 2d^2x(1-x) + O(d)$ by results [26, 27].) Then (6.4) provides

$$2s - \tfrac{5}{18}d^2 \leqslant d^2 - 2s + O(d).$$

Case 2. C_0 has other singular points. Let $\xi \in \mathbb{C}^2$ be a generic point of $\mathbb{C}^2$ and let r_∞ be sufficiently large, so that the intersection of C_0 with a sphere $S(\xi, r_\infty)$ is transverse. Let G be a generic polynomial of very high degree vanishing at each of z_k with up to order at least 4 (that is, generic among polynomials sharing this property). For $\varepsilon > 0$ small enough this guarantees that the curve

$$C_\varepsilon = \{F + \varepsilon G = 0\}$$

has singularities of type A_2 at each z_k, is smooth in $B(\xi, r_\infty)$ away from $z_1, \dots, z_s$ and its intersection with the sphere $S(\xi, r_\infty)$ is the same as the intersection of C_0. Now we can repeat the proof in Case 1. $\square$

The above estimate is very close to the best estimate known to the author, that the limit is bounded from above by $(125 + \sqrt{73})/432$ (see [18]).

Theorem 6.7 can be used together with results (especially Lemma 3 and Theorem 3) in [21]. We can get another proof of classical Zajdenberg–Lin theorem (see [20]), if we put $b = 0$ (we defer the details to a subsequent paper). It is, presumably, possible to go beyond this theorem and classify all plane curves with small first Betti number (compare [4, 7]). We can also hope to prove some results concerning the maximal possible number of singular points of the algebraic curve with given first Betti number, the problem that is known as the Lin conjecture.

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References

1. M. Borodzik, 'Deformations of singularities of plane curves. Topological approach', Preprint, 2009, arxiv: 0907.4129.

2. M. Borodzik and A. Némethi, 'Hodge-type structures as link invariants', Preprint, 2010, arxiv:1005.2084, to appear in Ann. Institut Fourier.

3. M. Borodzik and A. Némethi, 'Spectrum of plane curves via knot theory', Preprint, 2011, arxiv:1101.5471, to appear in J. of London Math. Soc.

4. M. Borodzik and H. Żołądek, 'Complex algebraic plane curves via the Poincaré–Hopf formula. I. Parametric lines', *Pacific J. Math.* 229 (2007) 307–338.

5. M. Borodzik and H. Żołądek, 'Complex algebraic plane curves via Poincaré–Hopf formula. III. Codimension bounds', *J. Math. Kyoto Univ.* 48 (2008) 529–570.

6. M. Borodzik and H. Żołądek, 'Small amplitude limit cycles in the polynomial Liénard system', *J. Differential Equation* 243 (2008) 2522–2533.

7. M. Borodzik and H. Żołądek, 'Complex algebraic plane curves via the Poincaré–Hopf formula. II. Parametric annuli', *Israeli J. Math.* 175 (2010) 301–347.

8. K.-T. Chen, 'On the Bezout theorem', *Amer. J. Math.* 106 (1984) 725–744.

9. C. Christopher and S. Lynch, 'Small-amplitude limit cycle bifurcations for Liénard systems with quadratic damping or restoring forces', *Nonlinearity* 12 (1999) 1099–1112.

10. D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematical Studies 110 (Princeton University Press, Princeton, NJ, 1985).

11. F. Forstneric, J. Globevnik and J. P. Rosay, 'Nonstraightenable complex lines in $\mathbb{C}^2$', *Ark. Mat.* 34 (1996) 97–101.

12. E. Garcia Barroso and A. Płoski, 'Pinceaux de courbes planes et invariants polaires', *Ann. Polon. Math.* 83 (2004) 113–128.

13. F. Hirzebruch, 'Singularities of algebraic surfaces and characteristic numbers', *The Lefschetz Centennial Conference, Part I (Mexico City, 1984)*, Contemporary Mathematics 58 (American Mathematical Society, Providence, RI, 1986) 141–155.
14. L. Kauffman, *On knots*, Annals of Mathematical Studies 115 (Princeton University Press, Princeton, NJ, 1987).
15. A. Kawauchi, *A survey on knot theory* (Birkhäuser, Basel, 1996).
16. A. Kawauchi, T. Shibuya and S. Suzuki, 'Description on surfaces in four-space. I. Normal forms', *Math. Sem. Notes Kobe Univ.* 10 (1982) 75–125.
17. A. Kawauchi, T. Shibuya and S. Suzuki, 'Description on surfaces in four-space. II. Singularities and cross-sectional links', *Math. Sem. Notes Kobe Univ.* 11 (1983) 31–69.
18. A. Langer, 'Logarithmic orbifold Euler numbers of surfaces with applications', *Proc. London Math. Soc.* (3) 86, 358–396.
19. J. Levine, 'Knot cobordism groups in codimension two', *Comment. Math. Helv.* 44 (1969) 229–244.
20. V. Ya. Lin and M. G. Zaidenberg, 'An irreducible, simply connected algebraic curve in $\mathbb{C}^2$ is equivalent to a quasihomogeneous curve', *Dokl. Akad. Nauk SSSR* 271 (1983) 1048–1052 (in Russian).
21. R. A. Litherland, 'Signatures of iterated torus knots' *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, Lecture Notes in Mathematics 722 (Springer, Berlin, 1979) 71–84.
22. J. Milnor, *Morse theory*, Annals of Mathematical Studies 51 (Princeton University Press, Princeton, NJ, 1963).
23. K. Murasugi, 'On a certain numerical invariant of link types', *Trans. Amer. Math. Soc.* 117 (1965) 387–422.
24. A. Némethi, 'On the spectrum of curve singularities', *Singularities (Oberwolfach, 1996)*, Progress in Mathematics 162 (Birkhäuser, Basel, 1998) 93–102.
25. A. Némethi, 'The signature of $f(x,y)+z^n$', *Proceedings of real and complex singularities (C.T.C Wall's 60th birthday meeting), Liverpool (England), 1996*, London Mathematical Society Lecture Notes Series 263 (London Mathematical Society, London, 1999) 131–149.
26. W. D. Neumann, 'Invariants of plane curve singularities', *Knots, braids and singularities, (Plans-sur-Becs, 1982)*, Monographies de L'Enseignement Mathématique 31 (Enseignement Mathematique, Geneva, 1983) 223–232.
27. W. D. Neumann, 'Splicing algebraic links', *Adv. Stud. Pure Math.* 8 (1986) 349–361.
28. W. D. Neumann, 'Complex algebraic plane curves via their links at infinity', *Invent. Math.* 98 (1989) 445–489.
29. W. D. Neumann and L. Rudolph, 'Unfolding in knot theory', *Math. Ann.* 278 (1987) 409–439; Corrigendum to 'Unfoldings in knot theory', *Math. Ann.* 282 (1988) 349–351.
30. P. Ozsváth and Z. Szabó, 'Knot Floer homology and the four-ball genus', *Geometry and Topology* 7 (2003) 625–639.
31. Th. Peters, 'A concordance invariant from the Floer homology of ± 1 surgeries', Preprint, 2010, arxiv: 1003.3038.
32. G. S. Petrov, 'Number of zeroes of complete elliptic integrals', *Funct. Anal. Appl.* 18 (1984) 148–149 [Russian: 18 (1984) 73–74].
33. A. Płoski, 'The Milnor Number of a plane algebroid curve', "Materiały XVI Konferencji Szkoleniowej z Analizy i Geometrii Zespolonej", Łódź, 1995, 73–82.
34. J. Rasmussen, 'Khovanov homology and the slice genus', *Invent. Math.* 182 (2010) 416–447.
35. L. Rudolph, 'Some knot theory of complex affine curves', *L'Enseign. Math.* 29 (1983) 185–208; (new version: arXiv:math.GT/0106058).
36. L. Rudolph, 'Knot theory of complex plane curves', *Handbook of knot theory* (Elsevier, Amsterdam, 2005) 349–427.
37. A. Stoimenow, 'Some applications of Tristram–Levine signatures and relations to Vassiliev invariants', *Adv. in Math.* 194 (2005) 463–484.
38. A. G. Tristram, 'Some cobordism invariants for links', *Proc. Cambridge Philos. Soc.* 66 (1969) 251–264.
39. C. T. C. Wall, *Singular points of plane curves* (Cambridge University Press, Cambridge, 2004).

Maciej Borodzik
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warsaw
Poland

mcboro@mimuw.edu.pl