ON THE SIGNATURES OF TORUS KNOTS

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Abstract. We study properties of the signature function of the torus knot $T_{p,q}$. First we provide a very elementary proof of the formula for the integral of the signatures over the circle. We obtain also a closed formula for the Tristram–Levine signature of a torus knot in terms of Dedekind sums.

1. Preliminaries

Let K be a knot in S^3 with a Seifert matrix S. Let also $z \in S^1$, $z \neq 1$ be a complex number. The *Tristram–Levine* signature $\sigma(z)$ is the signature of the hermitian form

$$(1-z)S + (1-\bar{z})S^T.$$

This is obviously an integer-valued piecewise constant function. It does not depend on a particular choice of Seifert matrix. If we substitute $z = -1$ we get an invariant σ_{ord}, which is called the *(ordinary) signature*. We define also the integral I_K

$$I_K = \int_0^1 \sigma(e^{2\pi i x})\, dx.$$

Signatures are very strong knot cobordism invariants, which can be used to bound the four-genus and the unknotting number of K. The integral I_K of the signature function is one of the so called ρ invariants of knots (see [COT1, COT2]) and is of independent interest.

For a torus knot $T_{p,q}$, where $\gcd(p,q) = 1$, the signature function can be expressed in the following nice way (see [Li] or [Kau, Chapter XII])

Proposition 1.1. *Let*

$$\Sigma = \left\{ \frac{k}{p} + \frac{l}{q} \colon 1 \leq k \leq p-1,\ 1 \leq l \leq q-1 \right\}. \tag{1.1}$$

Then for any $x \in (0,1) \setminus \Sigma$ we have

$$\sigma(e^{2\pi i x}) = |\Sigma \setminus (x, x+1)| - |\Sigma \cap (x, x+1)|, \tag{1.2}$$

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where $|\cdot|$ *denotes the cardinality of a set. In particular* $\sigma_{ord} = |\Sigma\setminus(1/2,3/2)| - |\Sigma \cap (1/2,3/2)|$.

The explicit formulae for σ_{ord} and I_K of torus knots have been known in the literature for quite a long time. In fact, σ_{ord} by a result of Viro (see (2.4)) is equal to τ_2, which was computed in [HZ] for p and q odd, and (denoted as $\sigma(f+z^2)$) in [Nem] in general case. On the other hand, Kirby and Melvin [KM, Remark 3.9] and [Nem, Example 4.3] provided a formula for I_K. Nevertheless all the above-mentioned results are related more to singularity theory and low-dimensional topology, than to knot theory itself.

After the discovery of ρ invariants, the interest of computing I_K for various families of knots grew significantly. Two independent new proofs of the formula for I_K of torus knots [Bo, Co] appeared in 2009. In particular [Bo] provided a bridge between the I_K and cuspidal singularities of plane curves.

In this paper we present an elementary proof of the formula for I_K (Proposition 2.1). We also cite a formula of Némethi and draw some consequences from it. In Section 4 we use a theorem of Rosen to obtain the explicit value of the signature $\sigma(z)$ of a torus knot not only for $z=-1$, but also for any $z \in S^1 \setminus \{1\}$ (Proposition 4.2). This result seems to be new. In Section 5 we show that the formula for $\sigma_{ord}(T_{p,q})$ cannot be written as a rational function of p and q.

2. Formula for the integral

Proposition 2.1. *For a torus knot* $T_{p,q}$ *we have*

$$I = -\frac{1}{3}\left(p-\frac{1}{p}\right)\left(q-\frac{1}{q}\right). \tag{2.1}$$

This proposition was first proved in [KM, Remark 3.9]. Refer to [Nem, Bo, Co] for other proofs.

Proof. Let $f(x) = -\sigma(e^{2\pi i x})$ and $J = \int_0^1 f(x)\,dx = -I$. Then

$$f(x) = \sum_{y\in\Sigma} \mathbf{1}_{(y,y+1)}(x) - \sum_{y\in\Sigma} \mathbf{1}_{\mathbb{R}\setminus(y,y+1)}(x).$$

(Here, for a set $A \subset \mathbb{R}$, $\mathbf{1}_A$ denotes the function which is equal to 1 on A and 0 away from A.) Hence

$$J = \sum_{y\in\Sigma}\int_0^1 \left(\mathbf{1}_{(y-1,y)}(x) - \mathbf{1}_{\mathbb{R}\setminus(y-1,y)}(x)\right)\,dx = \sum_{y\in\Sigma}(1-2|y-1|).$$

It follows that

$$J = \sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\left(1-2\left|\frac{k}{p}+\frac{l}{q}-1\right|\right).$$

As for any $u, v \in \mathbb{R}$ we have $1-2|u+v-1| = 2\min(1-u, v)+2\min(u, 1-v)-1$,

$$J = 2\sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\min\left(\frac{p-k}{p}, \frac{l}{q}\right) + 2\sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\min\left(\frac{k}{p}, \frac{q-l}{q}\right) - (p-1)(q-1) =$$

$$= 4\sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\min\left(\frac{k}{p}, \frac{l}{q}\right) - (p-1)(q-1) = \frac{4}{pq}\sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\min(qk, pl) - (p-1)(q-1).$$

Now, obviously,

$$\sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\min(qk, pl) =$$

$$= \sum_{s=0}^{\infty} |\{\{1, \dots, p-1\} \times \{1, \dots, q-1\} \colon qk > s \text{ and } pl > s\}| =$$

$$= \sum_{s=0}^{pq-1}(p-1-\lfloor s/q \rfloor)(q-1-\lfloor s/p \rfloor).$$

We can multiply the expression in parentheses. Then, as $\sum_{s=0}^{pq-1}\lfloor s/p \rfloor = p\sum_{l=0}^{q-1} l = \dfrac{1}{2}pq(q-1)$ we get

$$\sum_{s=0}^{pq-1}(p-1-\lfloor s/q \rfloor)(q-1-\lfloor s/p \rfloor) = pq(p-1)(q-1) - \frac{1}{2}pq(p-1)(q-1) -$$

$$\frac{1}{2}pq(p-1)(q-1) + \sum_{s=0}^{pq-1}\lfloor s/p \rfloor \lfloor s/q \rfloor = \sum_{s=0}^{pq-1}\lfloor s/p \rfloor \lfloor s/q \rfloor.$$

It remains to compute $\sum_{s=0}^{pq-1}\lfloor s/p \rfloor \lfloor s/q \rfloor$. To this end let us denote by $R_p(s)$ the remainder of s modulo p. We then have

$$\sum_{s=0}^{pq-1}\lfloor s/p \rfloor \lfloor s/q \rfloor = \sum_{s=0}^{pq-1}\left(\frac{s-R_p(s)}{p} \cdot \frac{s-R_q}{q}\right) =$$

$$\frac{1}{pq}\left(\sum_{s=0}^{pq-1} s^2 - \sum_{s=0}^{pq-1} sR_p(s) - \sum_{s=0}^{pq-1} sR_q(s) + \sum_{s=0}^{pq-1} R_p(s)R_q(s)\right) =$$

$$\frac{1}{3}p^2q^2 + \frac{1}{4}pq - \frac{1}{4}p^2q - \frac{1}{4}pq^2 - \frac{1}{12}p^2 - \frac{1}{12}q^2 + \frac{1}{12},$$

where we used the fact that $\sum_{s=0}^{pq-1} R_p(s)R_q(s) = \sum_{k=0}^{p-1}\sum_{l=0}^{q-1} kl$ by the Chinese remainder theorem.

Putting all the pieces together we obtain the desired formula. □

Let us now present another proof, due to Némethi [Nem], see also [Br, HZ]. Before we do this, let us recall some facts from topology.

Assume that the knot K is drawn on $S^3 = \partial B^4$ and consider a Seifert surface F of K. Let us push it slightly into B^4 and for an integer m let N_m be the m fold cyclic cover of B^4 branched along F. Then the quantity $\tau_m = \sigma(N_m)$ (here σ is a signature of a four-manifold) is independent of the choices made. We have the formula essentially due to Viro (see [GLM, Section 2] or [Vi]).

$$\tau_m = \sum_{k=1}^{m-1} \sigma_K(\xi^k), \tag{2.2}$$

where ξ is a primitive root of unity of order m. In particular, since σ is a Riemann integrable function, we have

$$I = \int_0^1 \sigma(e^{2\pi i x}), dx = \lim_{m\to\infty} \frac{1}{m}\tau_m. \tag{2.3}$$

On the other hand

$$\tau_2(K) = \sigma_{ord}(K). \tag{2.4}$$

If K is a torus knot $T_{p,q}$ and m, p, q are pairwise coprime, then the m-fold cover of S^3 branched along K is diffeomorphic to the Brieskorn homology sphere $B(p, q, m)$ (see [Br], [GLM, Section 5]). Then τ_m turns out [HZ, Section 10.2 and 11] to be the signature of the manifold $X_{p,q,m}$ defined as the intersection of $z_1^p + z_2^q + z_3^m = \varepsilon$ with $B(0, 1) \subset \mathbb{C}^3$. In this context τ_m was computed by [HZ, Formula 11 on page 122] and by [Nem, Example 4.3]. Especially the last formula is worth citing (Némethi uses $m(S(f))$ to denote the limit (2.3)).

$$I = -4(s(p, q) + s(q, p) + s(1, pq)). \tag{2.5}$$

Here $s(a, b)$ is the Dedekind sum (see Section 3). As by elementary computations $s(1, pq) = \dfrac{(pq - 1)(pq - 2)}{12pq}$, we get that

$$s(p, q) + s(q, p) = -\frac{I}{4} - \frac{(pq - 1)(pq - 2)}{12pq}.$$

Now we can look at the above equation as defining I in terms of $s(p, q) + s(q, p)$, but if we know I, we know $s(p, q) + s(q, p)$. In other words we get the following observation.

Corollary 2.2. *Any proof of Proposition 2.1 provides a proof of the Dedekind reciprocity law.*

3. Lattice points in the triangle

Let us recall basic definitions. For a real number x, $\lfloor x \rfloor$ denotes the integer part and $\{x\} = x - \lfloor x \rfloor$ the fractional part. The *sawtooth* function is defined as

$$\langle x \rangle = \begin{cases} \{x\} - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z}. \end{cases}$$

Sometimes $\langle x \rangle$ is denoted $((x))$. We prefer this notation because it does not lead to confusion with ordinary parenthesis. We can now define the functions (below p, q and m are integers and x, y are real numbers):

$$s(p,q) = \sum_{j=0}^{p-1} \left\langle \frac{j}{q} \right\rangle \left\langle \frac{pj}{q} \right\rangle$$

$$s(p,q;x,y) = \sum_{j=0}^{p-1} \left\langle \frac{j+y}{q} \right\rangle \left\langle p\frac{j+y}{q} + x \right\rangle.$$

These functions satisfy the following reciprocity laws (see [RG, HZ]). If m, p and q are pairwise coprime, then

$$s(p,q) + s(q,p) = \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right) - \frac{1}{4} \tag{3.1}$$

$$\begin{aligned} s(p,q,x,y) + s(q,p,y,x) = &-\frac{1}{4}d(x)d(y) + \langle x \rangle \langle y \rangle + \\ &+\frac{1}{2}\left(\frac{q}{p}\Psi_2(y) + \frac{1}{pq}\Psi_2(py+qx) + \frac{p}{q}\Psi_2(x)\right) \end{aligned} \tag{3.2}$$

Here

$$d(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Psi_2(x) = B_2(\{x\}) = \{x\}^2 - \{x\} + \frac{1}{6}$$

is the second Bernoulli polynomial. Now for a fixed $C \in [0,1)$ and p, q coprime, let

$$A(p,q;C) = \{(k,l) \in \mathbb{Z}^2_{\geq 0} \colon 0 \leq \frac{k}{p} + \frac{l}{q} < 1 - C\}$$

and

$$N(p,q;C) = |A(p,q;C)|.$$

We have the following result due to Rosen [Ro, Theorem 3.4].

Proposition 3.1. *In this case*

$$\begin{aligned} N(p,q;C) = {} & \frac{(1-C)^2}{2}pq + \frac{(1-C)}{2}(p+q) + \frac{q}{12p} + \frac{p}{12q} + K - \\ & - s(p,q;Cp,0) - s(q,p;Cq,0) + \langle Cp \rangle + \langle Cq \rangle + \\ & + (1-C)\langle Cpq \rangle - (\frac{7}{8}\delta_0 + \frac{3}{8}\delta_1 - \frac{1}{8}\delta_2) + \frac{1}{4}, \end{aligned} \tag{3.3}$$

where

$$K = \begin{cases} \frac{1}{12pq} - \frac{1}{8} & \text{if } Cpq \in \mathbb{Z} \\ \frac{1}{2pq}\Psi_2(Cpq) & \text{otherwise} \end{cases}$$

And for $r = 0,1,2$, δ_r *is the number of non-negative integers* k,l *such that* $\frac{k}{p} + \frac{l}{q} + C = r$.

This proposition admits an important corollary [Ro, Corollary 3.5].

Corollary 3.2. *If* p *and* q *are odd and coprime, then*

$$N(p,q;\frac{1}{2}) = \frac{pq}{8} + \frac{p+q}{4} + \frac{q}{6p} + \frac{p}{6q} + \frac{1}{24pq} - s(2p,q) - s(2q,p).$$

If p *and* q *are coprime and* q *is even, then*

$$N(p,q;\frac{1}{2}) = \frac{pq}{8} + \frac{p+q}{4} - s(2p,q) + 2s(p,q). \tag{3.4}$$

We shall use these results to compute the signature of the torus knots. We need a following trivial lemma

Lemma 3.3. *The number of points* $(k,l) \in A(p,q;C)$ *such that* $kl = 0$ *is equal to*

$$Z(p,q;C) = \lfloor (1-C)p \rfloor + \lfloor (1-C)q \rfloor + 1 - d((1-C)p) - d((1-C)q),$$

where $d(x)$ *again is* 1*, if* $x \in \mathbb{Z}$*, and* 0 *otherwise.*

If Cp and Cq are not integers,

$$Z(p,q;C) = (1-C)(p+q) - \langle (1-C)p \rangle - \langle (1-C)q \rangle .$$

4. Explicit formulae for the signatures

We begin with computing the value of the ordinary signature. As it was already mentioned, $\sigma_{ord} = \tau_2$ (see (2.4)) so the first result below is in general known [HZ, Nem], but not necessarily in the context of knot theory.

Proposition 4.1. *If p and q are both odd and coprime, then the ordinary signature of the torus knot $T_{p,q}$ satisfies*

$$\sigma_{ord}(T_{p,q}) = -\frac{pq}{2} + \frac{2p}{3q} + \frac{2q}{3p} + \frac{1}{6pq} - 4(s(2p,q) + s(2q,p)) - 1,$$

where $s(x,y)$ is the Dedekind sum (see Section 3 or [RG]*) (compare with* [HZ, Formula 11 on page 122]*). If p is odd and $q > 2$ is even, then*

$$\sigma_{ord}(T_{p,q}) = -\frac{pq}{2} + 1 + 4s(2p,q) - 8s(p,q).$$

Proof. Let us consider the torus knot $T_{p,q}$ and let Σ be as in (1.1). We can write σ_{ord} as

$$\sigma_{ord} = 4|\Sigma \cap (0, \frac{1}{2})| - |\Sigma|. \tag{4.1}$$

Since $|\Sigma| = (p-1)(q-1)$, we need to find a closed formula for

(4.2)
$$S(p,q) = |\Sigma| \cap (0, \frac{1}{2}) = \left| \left\{ \frac{k}{p} + \frac{l}{q} < \frac{1}{2},\ 1 \le k \le p-1,\ 1 \le l \le q-1 \right\} \right|.$$

From the definition we get immediately that

$$S(p,q) = N(p,q;\frac{1}{2}) - Z(p,q;\frac{1}{2}).$$

Now $Z(p,q;\frac{1}{2}) = \frac{1}{2}(p+q)$ if p and q are both odd and $\frac{1}{2}(p+q-1)$ if q is even and $q > 2$. Hence, for p and q odd we have

$$S(p,q) = \frac{pq}{8} - \frac{p+q}{4} - s(2p,q) + 2s(p,q),$$

while for q even we have by (3.4)

$$S(p,q) = \frac{pq}{8} - \frac{p+q}{4} + \frac{1}{2} - s(2p,q) + 2s(p,q).$$

and using (4.1) we complete the proof. □

To express explicitly the values of Tristram–Levine signatures at other points let us assume that Cpq is not an integer. Define

$$\begin{aligned} M(p,q;C) = N(p,q;C) - Z(p,q;C) &= \frac{(1-C)^2}{2}pq - \frac{(1-C)}{2}(p+q) \\ &+ \frac{q}{12p} + \frac{p}{12q} - s(p,q;Cp,0) - s(q,p;Cq,0) + \frac{1}{4} - \\ &- \frac{1}{2}(\langle Cp \rangle + \langle Cq \rangle) + (1-C)\langle Cpq \rangle + \frac{1}{2pq}\Psi_2(Cpq). \end{aligned}$$

Now it is a trivial consequence of Proposition 1.1 that if $C \in [0,1)$ and $e^{2\pi iC} = z$, then

$$\sigma(z) = -(p-1)(q-1) + 2M(p,q;C) + 2M(p,q;1-C).$$

Now, since for any integer k and real x we have $\langle (1-x)k\rangle + \langle xk\rangle = 0$, the formula for $M(p,q;C) + M(p,q;1-C)$ can be simplified to

$$\frac{1-2C+2C^2}{2}pq - \frac{1}{2}(p+q) + \frac{q}{6p} + \frac{p}{6q} + (1-2C)\langle Cpq\rangle + \frac{1}{pq}(\langle Cpq\rangle^2 - \frac{1}{12}) + \frac{1}{2} -$$
$$- s(p,q;Cp,0) - s(q,p;Cq,0) - s(p,q;(1-C)p,0) - s(q,p;(1-C)q,0).$$

Hence we prove the following result.

Proposition 4.2. *If $z = e^{2\pi iC}$ where $C \in [0,1)$ is such that Cpq is not an integer, then the signature of the torus knot $T_{p,q}$ can be expressed in the following formula.*

$$\sigma(z) = -2(C-C^2)pq + \frac{q}{3p} + \frac{p}{3q} + (2-4C)\langle Cpq\rangle + \frac{2}{pq}(\langle Cpq\rangle^2 - \frac{1}{12}) -$$
$$-2\left(s(p,q;Cp,0) + s(q,p;Cq,0) + s(p,q;(1-C)p,0) + s(q,p;(1-C)q,0)\right).$$

In particular we see rigorously that for large p and q the shape of the function $\sigma(e^{2\pi ix})$ resembles that of the function $2pq(x^2-x)$.

5. Expressing $\sigma_{ord}(T_{p,q})$ as a rational function

Proposition 5.1. *There* does not *exist a rational function $R(p,q)$ such that for all odd and coprime positive integers*

$$R(p,q) = \sigma_{ord}(T_{p,q}).$$

Proof. Assume that $R(p,q) = \sigma(T_{p,q})$. Then $S(p,q) = \frac{1}{4}(R(p,q)+(p-1)(q-1))$ is also a rational function and

$$S(p,q) = \left|\Sigma \cap (0,\frac{1}{2})\right| = \left|\left\{\frac{k}{p} + \frac{l}{q} < \frac{1}{2},\ 1 \le k \le p-1,\ 1 \le l \le q-1.\right\}\right|$$

(cf. formulae (4.1) and (4.2)). If $p|(q-1)$ the value of $S(p,q)$ can be easily computed:

$$S(p,q) = \sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{q}{2} - \frac{qk}{p} \right\rfloor = \sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{q-1}{2} - \frac{(q-1)k}{p} + \frac{p-k}{2p} \right\rfloor =$$
$$= \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{q-1}{2} - k\frac{q-1}{p}\right) = \frac{(q-1)(p-1)^2}{8p}.$$

Since for infinitely many values (p,q) with $q = np+1$ with p odd and n even, we have $p|(q-1)$, it follows that $S(p,q) = \frac{(q-1)(p-1)^2}{8p}$ on each line $q = np+1$. Since these rational functions agree on infinitely many lines, they must be equal.

But now assume that $p = nq+1$ for some even n. Similar arguments as above show that $S(p,q)$ must also be identical to the function $\frac{(p-1)(q-1)^2}{8q}$. This leads to a contradiction, since these two rational functions are different. $\square$

Remark 5.2. We can also compute values of $S(p,q)$ in many other cases, like $q = np-1$, $q = p+2$. With more care we can prove that e.g. $S(p,q) - \lfloor \frac{q}{p} \rfloor$ is not a rational function.

The proof carries over to show that no such rational function exists for the case p even and q odd. We leave the obvious details to the reader.

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