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Spectrum of plane curves via knot theory

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Abstract

In this paper, we use topological methods to study various semicontinuity properties of the local spectrum of singular points of algebraic plane curves and spectrum at infinity of polynomial maps in two variables. Using the Seifert form, the Tristram–Levine signatures of links, and the associated Murasugi-type inequalities, we reprove (in a slightly weaker form) a result obtained by Steenbrink and Varchenko on semicontinuity of the spectrum of singular points under deformations and result of Némethi and Sabbah on semicontinuity of the spectrum at infinity regarding families of polynomial maps. We also relate the spectrum at infinity of a polynomial map with the collection of the spectra of singular points of a chosen fiber.

1. *Introduction*

The Hodge spectrum of a local isolated hypersurface singularity $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ can be derived from the mixed Hodge structure of the vanishing cohomology of the singular germ [**1**, **36**, **37**, **40**, **41**]. Usually, it is not topological, it is one of the finest analytic invariants of the germ. Although it does not characterize the singularity completely, it gives extremely strong information about it. As was conjectured by Arnold [**1**], and proved by Varchenko [**40**, **41**] and Steenbrink [**37**], the spectrum behaves semicontinuously under deformations, which makes it, for example, a very strong tool in attempts to solve the adjacency problem (that is, to determine which singularities can specialize to a given one).

A more precise picture is the following: the algebraic monodromy acts on the vanishing cohomology, which supports the Seifert form (which can be identified with the variation map) and the mixed Hodge structure polarized by the intersection form. The equivariant Hodge numbers were codified by Steenbrink (see [**36**]) in the spectral pairs; if one deletes the information about the weight filtration one gets the spectrum/spectral numbers $\mathrm{Sp}(f)$. They are (in some normalization) rational numbers in the interval $(0,n+1)$. In the presence of a deformation f_t, where t is the deformation parameter $t\in(\mathbb{C},0)$, the semicontinuity guarantees that $|\mathrm{Sp}(f_0)\cap I|\geqslant|\mathrm{Sp}(f_{t\neq 0})\cap I|$ for certain semicontinuity domains I. Arnold in [**1**] conjectured that $I=(-\infty,x]$ is a semicontinuity domain for any $x\in\mathbb{R}$, Steenbrink and Varchenko proved the statement for $I=(x,x+1]$, which implies Arnold's conjecture. Additionally, for some cases, Varchenko verified the stronger version, namely semicontinuity for $I=(x,x+1)$ (see [**40**]).

The semicontinuity property (with any domain) cannot be extended to the spectral pairs. Therefore, in studies targeting these kinds of applications one usually works with the spectrum only. This is what we will do in the present article as well.

On the other hand, one of the strongest topological invariants of f is its Seifert form, for terminology see, for example, [**2**]. The relation between the Hodge invariants and the Seifert form was established by the second author in [**24**], proving that the collection of mod 2 spectral

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pairs are equivalent with the real Seifert form. In this way, the real Seifert form is in strong relationship with the mod 2 spectrum, that is, with the collection of numbers $x \bmod 2$ in $(0,2]$, where x runs over $\mathrm{Sp}(f)$. Clearly, for plane curve singularities, that is, when $n=1$, by taking mod 2 reduction, we lose no information.

Our primary goal is to *extend the above correspondence for an arbitrary link* (S^3_R, L), where S^3_R is the boundary of some ball with radius R in $\mathbb{C}^2$, and L is the intersection of S^3_R with some affine algebraic curve C in $\mathbb{C}^2$. The primary interest is the *link at infinity* of such affine curve (hence $R \gg 0$), but we also wish to develop a method to study *any* general (S^3_R, L), for which the available methods in the literature are rather sparse.

Let us consider a complex polynomial map $F : \mathbb{C}^2 \to \mathbb{C}$. For its topology at infinity, see Neumann's article [**31**]. Our first main result *recovers the spectrum at infinity associated with the limit mixed Hodge structure at infinity (supported by the cohomology of the generic fiber) from the real Seifert form of the regular link at infinity associated with* F. In particular, we reobtain the spectrum at infinity topologically in a pure link-theoretical language.

The key bridge which connects the link-theoretical language and invariants with the Hodge theoretical spectrum is the Tristram–Levine signatures [**18**, **39**]. For example, for the weighted homogeneous singularity given by $\{x^p - y^q = 0\}$ with p and q relative prime integers, the spectrum is $\mathrm{Sp}_{p,q} = \{i/p + j/q,\ 1 \leqslant i \leqslant p-1,\ 1 \leqslant j \leqslant q-1\}$, while the Tristram–Levine signature function of the (p,q)-torus knot, evaluated at $e^{2\pi i x}$ with $x \in (0,1)$, $pqx \notin \mathbb{Z}$, is equal to $2 \cdot |\mathrm{Sp}_{p,q} \cap (x, x+1)| - (p-1)(q-1)$, see, for example, [**19**]. In [**6**], we made this relation rigorous, showing a direct translation between the spectrum of singularities and Tristram–Levine signatures of their links.

In this correspondence, what is really surprising, and this is the *second main message* of the article, is the fact that the *semicontinuity of the* mod 2 *spectrum is topological*: it can be recovered independently of analytic (Hodge theoretical) tools, it follows from pure link theory. More precisely, we prove that length 1 'intervals' intersected by the mod 2 spectrum, namely sets of type $\mathrm{Sp} \cap (x, x+1)$ and $(\mathrm{Sp} \cap (0,x)) \cup (\mathrm{Sp} \cap (x+1, 2])$, for $x \in [0,1]$, satisfy semicontinuity properties, whenever this question is well-posed.

In this article, we exemplify this by three cases: we recover the semicontinuity (in the above form, with certain weak assumptions) for deformations of local plane curve singularities, corresponding to the above-mentioned results of Varchenko and Steenbrink, and we also prove the semicontinuity of the spectrum at infinity associated with a family of polynomials in two variables, in the spirit of [**28**]. The third case targets a new phenomenon: in the context of an affine curve $C \subset \mathbb{C}^2$, we show a semicontinuity property which connects the local spectrum of the singularities of C with the spectrum at infinity of C.

In all these cases, the key link-theoretical ingredient is a Murasugi-type inequality, which controls the modification of the Tristram–Levine signatures under those types of surgeries which appear when we pass from $C \cap S^3_r$ to $C \cap S^3_R$ via Morse theory ($r < R$). This was studied by the first author in [**5**].

The article is organized as follows. In Section 2, we review the theory of hermitian variation structures (HVS) from [**24**], their relation with the spectrum, and the methods how one associates such a structure to a link [**6**]; furthermore, we connect the spectrum with the Tristram–Levine signatures [**6**]. We also recall some of the main results of [**5**] about surgery inequalities of links of type $S^3_R \cap C$. Section 3 contains the study of the spectrum at infinity of a polynomial map in terms of the Seifert form at infinity. In Section 4, we prove semicontinuity results regarding the spectrum.

For a finite set A, we denote by $|A|$ the cardinality of A.

2. *Hermitian variations structures of links*

In Section 2.1, we recall the definition of an abstract *hermitian variation structures* and its *spectrum*, while in Section 2.2 the definition of the HVS and the spectrum associated with links in a 3-sphere. Section 2.3 reviews the definition of mixed Hodge structures and their 'Hodge' spectrum. Finally, in Section 2.4, we draw a relationship observed in [6] between the spectrum and Tristram–Levine signatures of links. In Section 2.5, we recall some results from [5] which are crucial ingredients in the proof of the semicontinuity results of the last section.

2.1. *Hermitian variation structures*

These structures were introduced in [24], they generalize the ε-symmetric isometric structures. Here, we review the minimal basics, for more details, see [24, 25].

Recall that a structure $(U = \mathbb{C}^n; b, h)$, where b is an ε-symmetric hermitian form on U preserved by the automorphism h of U, is called an *isometric structure* (for $\varepsilon = \pm 1$). The classification of isometric structures when b is non-degenerate was established by Milnor [21] (see also [29, 30]). Any ε–hermitian variation structure (in short ε–HVS) can be regarded as an isometric structure together with an operator $V\colon U^* \to U$ such that

$$\overline{V^*} = -\varepsilon V \overline{h^*} \quad \text{and} \quad V \circ \tilde{b} = h - \mathrm{Id}, \tag{2.1}$$

where $\tilde{b}$ is the form b regarded as a map from U to U^*. We denote it by $\mathcal{V} = (U; b, h, V)$. Here $\cdot^*$ denotes passing to the dual space, while $\bar{\cdot}$ the complex conjugation.

Definition 2.1. We say that the isometric structure $(U; b, h)$ can be completed to an HVS if there exists $V : U^* \to U$ such that (2.1) is satisfied.

If b is non-degenerate, then the isometric structure can be uniquely completed to an HVS: $V = (h - \mathrm{Id}) \circ \tilde{b}^{-1}$. In general, not every isometric structure can be completed (see, for example, (3.7)(c)). Moreover, if a completion exists, in general, it is not unique (even if we restrict ourselves to non-degenerate matrices V, see, for example, [24, (2.7.7)]).

An HVS is called *simple* if V is an isomorphism. The classification of simple HVSs is established in [24]. Each simple variation structure is a direct sum of indecomposable simple variation structures. Indecomposable structures can be listed: for each positive integer k, and for each $\lambda \in \mathbb{C}$ such that $0 < |\lambda| \leqslant 1$, there exist:

(1) a unique simple indecomposable variation structure $\mathcal{V}_\lambda^{2k}$ if $|\lambda| < 1$;
(2) two simple indecomposable structures, denoted by $\mathcal{W}_\lambda^k(+1)$ and $\mathcal{W}_\lambda^k(-1)$, if $|\lambda| = 1$.

This classification is a refinement of the Jordan block decomposition of the matrix h (or of Milnor's classification of non-degenerate isometric structures). More precisely, the matrix h corresponding to $\mathcal{W}_\lambda^k(\pm 1)$ is a single Jordan block of size k and eigenvalue λ, while the one corresponding to $\mathcal{V}_\lambda^{2k}$ has two Jordan blocks of size k: one with eigenvalue λ, the other with eigenvalue $1/\overline{\lambda}$. For their precise form, see [24].

Let us write a simple variation structure $\mathcal{V}$ as the unique sum of the indecomposable ones:

$$\mathcal{V} = \bigoplus_{\substack{0<|\lambda|<1 \\ k\geqslant 1}} q_\lambda^k \cdot \mathcal{V}_\lambda^{2k} \oplus \bigoplus_{\substack{|\lambda|=1 \\ k\geqslant 1,\ u=\pm 1}} p_\lambda^k(u) \cdot \mathcal{W}_\lambda^k(u) \tag{2.2}$$

for certain non-negative integers q_λ^k and $p_\lambda^k(u)$. Here, we write $m \cdot \mathcal{V}$ for $\mathcal{V} \oplus \ldots \oplus \mathcal{V}$ (m-times).

The numbers $\{q_\lambda^k\}_{|\lambda|<1}$ and $\{p_\lambda^k(\pm 1)\}_{\lambda \in S^1}$ are called the *H-numbers* of the HVS $\mathcal{V}$.

Using H-numbers we can define the *spectrum of* $\mathcal{V}$. Sometime, in order to emphasize the source of the definition, we call it *HVS-spectrum*.

DEFINITION 2.2 ([23] or [6, (2.3.1)–(2.3.3)]). Consider the H-numbers $\{q^k_\lambda\}_{|\lambda|<1}$ and $\{p^k_\lambda(\pm 1)\}_{\lambda\in S^1}$ of $\mathcal{V}$. The *extended spectrum* ESp is the union $\mathrm{ESp} = \mathrm{Sp}\cup\mathrm{ISp}$, where

(a) Sp, the *spectrum*, is a finite set of real numbers from the interval $(0,2]$ such that any real number α occurs in Sp precisely $s(\alpha)$ times, where

$$s(\alpha)=\sum_{n=1}^{\infty}\sum_{u=\pm 1}\left(\frac{2n-1-u(-1)^{\lfloor\alpha\rfloor}}{2}p^{2n-1}_\lambda(u)+np^{2n}_\lambda(u)\right),\quad (e^{2\pi i\alpha}=\lambda);$$

(b) ISp is a subset of complex numbers from $(0,2]\times i\mathbb{R}$, $\mathrm{ISp}\cap\mathbb{R}=\emptyset$, such that $z=\alpha+i\beta$ occurs in ISp precisely $s(z)$ times, where

$$s(z)=\begin{cases}\sum k\cdot q^k_\lambda & \text{if } \alpha\leqslant 1,\ \beta>0 \ \text{ and } e^{2\pi i z}=\lambda,\\ \sum k\cdot q^k_\lambda & \text{if } \alpha>1,\ \beta<0 \ \text{ and } e^{2\pi i z}=1/\bar\lambda,\\ 0 & \text{if } \alpha\leqslant 1 \text{ and } \beta<0, \text{ or } \alpha>1 \ \text{ and } \beta>0.\end{cases}$$

Since the dimension of the vector space supporting $\mathcal{V}^{2k}_\lambda$ is $2k$ and of $\mathcal{W}^k_\lambda(\pm 1)$ is k, one obtains

$$|\mathrm{ESp}|=\dim U=\deg\det(h-t\,\mathrm{Id}). \tag{2.3}$$

2.2. *The HVS and the spectrum of a link*

The *variation structure* and *H-numbers* of a link in S^3 were defined in [6]. Let us review shortly how the construction is performed.

Let S be a Seifert matrix of a link L. (For the conventions used in its definition, see Section 3.2.) By Keef's result [16] S is S-equivalent either to an empty matrix, or to a matrix S', which can be decomposed into a direct sum

$$S'=S_0\oplus S_{\mathrm{ndeg}}, \tag{2.4}$$

where S_0 is a zero matrix and S_{ndeg} is non-degenerate, that is $\det S_{\mathrm{ndeg}}\neq 0$. Moreover, any two such non-degenerate models S_{ndeg} of the same link are congruent over $\mathbb{Q}$. The size of S_0 is also determined by L (it is equal to $\dim(\ker S\cap\ker S^T)$), we call it the *irregularity of* L, and we denote it by

$$\mathrm{Irr}=\mathrm{Irr}(L):=\mathrm{size}(S_0). \tag{2.5}$$

Let n be the size of S_{ndeg}. The quadruple $\mathcal{V}=(U,b,h,V)$, where $U=\mathbb{C}^n$, $V=(S^T_{\mathrm{ndeg}})^{-1}$, $h=VS$, $b=S-S^T$, constitutes an HVS with the sign choice $\varepsilon=-1$. (Here $\cdot^T$ denotes the transposition.) As changing a Seifert matrix results in congruency of S_{ndeg}, which leads to an isomorphism of variations structures, the structure $\mathcal{V}$ does not depend on the choice of a Seifert matrix, so it is a well-defined link invariant, called $\mathcal{V}_L$. Additionally, $\mathcal{V}_L$ is *simple*. Note that $\mathcal{V}_L$ is defined over the rational numbers $\mathbb{Q}$. The characteristic polynomial $\Delta^h=\det(h-t\,\mathrm{Id})$ of h will be called the *characteristic polynomial of the link*. Its connection with Alexander polynomials is as follows (see, for example, [6, § 4]):

LEMMA 2.3. *Let $\mathcal{V}_L$ be as above. If the Alexander polynomial Δ of L is non-zero, then $\Delta=\Delta^h$ up to multiplication by an invertible element of $\mathbb{Q}[t,t^{-1}]$. If the Alexander polynomial is zero, then Δ^h is proportional to the first higher Alexander polynomial Δ_k, which is not identically zero: $\Delta_k=0$ for $0\leqslant k<\mathrm{Irr}$ and $\Delta_{\mathrm{Irr}}=\Delta^h$ (up to an invertible element).*

DEFINITION 2.4. Consider the integers $\{q^k_\lambda\}_{|\lambda|<1}$ and $\{p^k_\lambda(\pm 1)\}_{\lambda\in S^1}$ provided by the direct sum decomposition (2.2) of $\mathcal{V}_L$. They are called the *H-numbers* of the link L. The associated (extended) spectrum is called the (*extended*) *spectrum of the link*.

From (2.3) one has $|\mathrm{ESp}| = \deg \Delta^h$. Moreover, $\mathrm{Sp} \setminus \mathbb{Z}$ is symmetric with respect to 1.

2.3. *Mixed Hodge structures and their spectrum*

The name of the spectrum in Definition 2.2 is motivated by the fact that if L is an *algebraic link*, that is, the link of (local) isolated plane curve singularity, then ISp is empty and Sp is the 'classical' spectrum associated with the mixed Hodge structure of the vanishing cohomology (for this see, for example, [**17**, **22**, **36**, **37**, **40**]).

More generally, let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated singularity at 0, and let Y be the Milnor fiber and $U = \tilde{H}_n(Y, \mathbb{R})$. (For details regarding the Milnor fibration, see, for example, [**2**, **20**, **24**].) One takes the monodromy operator $h\colon U \to U$, the intersection form $b\colon U \times U \to \mathbb{R}$ and the variation operator $V\colon U^* \to U$. One checks (see, for example, [**2**] or [**24**, § 5]) that the complexification of $(U; b, h, V)$ constitutes a $(-1)^n$-HVS. If S is the Seifert matrix of the Milnor fibration, then at the level of matrices $V = (S^T)^{-1}$. Since S is unimodular, V is an isomorphism, hence the variation structure is simple. For plane curves, one has $\varepsilon = -1$, hence $h = (S^T)^{-1}S$ and $b = S - S^T$. The structure $(U; b, h, V) \otimes \mathbb{C}$ is called the 'homological HVS' of the germ.

There is a dual HVS as well, the 'cohomological HVS' associated with the germ, which is supported on $H^* := \tilde{H}^n(Y, \mathbb{C})$. Additionally, $\tilde{H}^n(Y, \mathbb{C})$ carries a limit mixed Hodge structure with Hodge filtration F and weight filtration W such that the semisimple part h^*_{ss} of the cohomological monodromy operator acts on (H^*, F, W). They define spectral pairs. In order to eliminate any confusion about the existing different normalizations, we provide some details.

One considers the generalized λ-eigenspaces U^*_λ for all the eigenvalues λ of the Gauss–Manin monodromy operator $h_{\mathrm{GM}} = (h^*_{ss})^{-1}$ and the equivariant (Gauss–Manin) Hodge numbers $h^{p,q}_\lambda := \dim Gr^p_F Gr^W_{p+q} U^*_\lambda$.

Then these numbers can be codified in a different way in the collection of *Hodge spectral pairs* of $(U^*, F, W; h^*_{ss})$. This is a collection of pairs (α, w) from $\mathbb{R} \times \mathbb{N}$ defined by

$$\mathrm{Spp}_{\mathrm{GM}}(f) = \sum_{(\alpha,w)} h^{n+[-\alpha], w+s-n-[-\alpha]}_{\exp(-2\pi i\alpha)}(\alpha, w) \in \mathbb{N}[\mathbb{R} \times \mathbb{N}], \tag{2.6}$$

where $s = 1$ if $\lambda = \exp(-2\pi i\alpha) = 1$ and $s = 0$ otherwise.

This can be transformed in several ways. If, by some geometric reason, one wishes to emphasize more the cohomological monodromy operator h^*_{ss} (instead of h_{GM}), one considers

$$\mathrm{Spp}_*(f) = \sum_{(\alpha,w)} h^{n+[-\alpha], w+s-n-[-\alpha]}_{\exp(2\pi i\alpha)}(\alpha, w) \in \mathbb{N}[\mathbb{R} \times \mathbb{N}]. \tag{2.7}$$

If one forgets the weight filtration, then from the equivariant Hodge filtration one can read the *Hodge spectrum*, namely

$$\mathrm{Sp}_*(f) = \sum \alpha \in \mathbb{N}[\mathbb{R}] \quad \text{(the sum over the spectral pairs } (\alpha, w) \text{ of } \mathrm{Spp}_*(f)). \tag{2.8}$$

Any such spectral number α is in the interval $(-1, n)$. Another normalization of the spectrum defines the spectral numbers in the interval $(0, n+1)$: $\mathrm{Sp}_{\mathrm{MHS}}(f)$ is the collection of numbers $(\alpha + 1)$, where α runs over the entries of $\mathrm{Sp}_*(f)$.

The identification of the Hodge invariants with the associated HVS goes through the crucial polarization property of the mixed Hodge structure. In this way, the cohomological HVS of f can be obtained from (U^*, F, W) by collapsing the Hodge filtration mod 2, having the collapsed spectral numbers in $(-1, 1]$. The corresponding H-numbers are, in fact, the equivariant primitive Hodge numbers of (U^*, F, W) under this collapsing procedure. Usually, the homological and cohomological HVSs do not agree; in the case $\varepsilon = (-1)^n = -1$, they differ by a sign: $\mathcal{V}_{\text{coh}} = -\mathcal{V}_{\text{hom}}$. This explains the two slightly different definitions of the spectral numbers (Definition (2.2)(a)) and (2.7). Nevertheless, one has the following identification:

PROPOSITION 2.5 [24, (6.5)]. *The HVS-spectrum* Sp_{HVS} *is a* $\text{mod}\, 2$ *reduction of the Hodge spectrum* Sp_{MHS} *considered in* $(0, 2]$. *In other words,*

$$\text{Sp}_{\text{HVS}} = \{x \text{ mod } 2 \colon x \in \text{Sp}_{\text{MHS}}\}.$$

Therefore, for a germ of an isolated plane curve singularity one gets $\text{Sp}_{\text{HVS}} = \text{Sp}_{\text{MHS}}$. This means that the Hodge spectrum can be described completely in terms of the (real) Seifert form of the link. This is the model of our further investigation.

2.4. *Spectrum of a link and the Tristram–Levine signatures*

The Tristram–Levine signatures (defined first in [18, 39]) turn out to be a knot-theoretic counterpart of the spectrum of singular points. We recall how they can be explicitly expressed from the spectrum of the link.

DEFINITION 2.6. Let L be a link and S its Seifert matrix. The *Tristram–Levine signature* function is the mapping from $S^1 \setminus \{1\} = \{\zeta \in \mathbb{C} \colon |\zeta| = 1, \zeta \neq 1\}$ to $\mathbb{Z}$ given by

$$\sigma_L(\zeta) = \text{signature}[(1-\zeta)S + (1-\overline{\zeta})S^T].$$

The *nullity* $n_L(\zeta)$ is the nullity of the same form $(1-\zeta)S + (1-\overline{\zeta})S^T$, while the *normalized nullity*, $\widetilde{n}_L(\zeta)$, is defined as $n_L(\zeta) - \text{Irr}$. For completeness, we extend the definitions for $\zeta = 1$, too. First, we set $\sigma_L(1) = 0$. Then note that, for any $\zeta \neq 1$, $\widetilde{n}_L(\zeta)$ equals the multiplicity of the root of Δ^h at ζ. We define $\widetilde{n}_L(1)$ by this characterization for $\zeta = 1$.

We have the following relation between H-numbers, signatures and nullities of the link.

PROPOSITION 2.7 [6, (4.4.6) and (4.4.9)]. *Let* $\text{Sp} = \text{Sp}_{\text{HVS}}$ *be the real part of the spectrum as in Definition* 2.2. *Let* $x \in (0, 1)$ *and* $\zeta = e^{2\pi i x}$. *Then*

$$\sigma_L(\zeta) = -|\text{Sp} \cap (x, x+1)| + |\text{Sp} \setminus [x, x+1]| + \sum_{n=1}^{\infty} \sum_{u=\pm 1} u p_\zeta^{2n}(u),$$
$$\widetilde{n}_L(\zeta) = \sum_{k,u} p_\zeta^k(u).$$

In particular,

$$-\sigma(\zeta) + \widetilde{n}(\zeta) \geqslant |\text{Sp} \cap (x, x+1)| - |\text{Sp} \setminus [x, x+1]|. \tag{2.9}$$

REMARK 2.8. In the cases $x \in \{0, 1\}$, the inequality (2.9) still holds. Indeed, $\sigma_L(1) = 0$, and the right-hand side is zero, as well, because $\text{Sp} \setminus \mathbb{Z}$ is symmetric. Moreover, if 1 is not a root of Δ^h, then (2.9) is an equality for $x = 1$.

Let us denote

$$D = |\text{Sp} \cap \{x, x+1\}| \geqslant 0.$$

Assume that Δ^h, the characteristic polynomial of the link, has no roots outside the unit circle. Then $\deg \Delta^h = |\mathrm{Sp}| = |\mathrm{Sp} \cap (x, x+1)| + |\mathrm{Sp} \setminus [x, x+1]| + D$, hence one also has

$$\deg \Delta^h - \sigma(\zeta) + \widetilde{n}(\zeta) = 2|\mathrm{Sp} \cap (x, x+1)| + \sum_{\substack{k \text{ odd} \\ u=\pm 1}} p^k_\zeta(u) + \sum_{k \text{ even}} 2p^k_\zeta(-1) + D. \tag{2.10}$$

For any $x \in [0, 1]$, parallel to the set $\mathrm{Sp} \cap (x, x+1)$, we will also consider the set $\mathrm{Sp} \setminus [x, x+1] = \mathrm{Sp} \cap (0, x) + \mathrm{Sp} \cap (1+x, 2]$. These two types cover all the 'length 1 open intervals' of the mod 2 spectrum.

The following corollary will be used extensively in the sequel.

Corollary 2.9. *Let L be a link and Δ^h its characteristic polynomial. Assume that Δ^h has no roots outside the unit circle. If $\zeta = e^{2\pi i x}$, for $x \in [0, 1)$, is not a root of Δ^h, then*

$$|\mathrm{Sp} \cap (x, x+1)| = \tfrac{1}{2}(\deg \Delta^h - \sigma(\zeta)) \quad \textit{and} \quad |\mathrm{Sp} \setminus [x, x+1]| = \tfrac{1}{2}(\deg \Delta^h + \sigma(\zeta)).$$

Moreover, for arbitrary $x \in [0, 1]$:

$$\begin{aligned} \tfrac{1}{2}\left(\deg \Delta^h - \sigma(\zeta) + \widetilde{n}(\zeta)\right) &\geqslant |\mathrm{Sp} \cap (x, x+1)|, \\ \tfrac{1}{2}\left(\deg \Delta^h + \sigma(\zeta) + \widetilde{n}(\zeta)\right) &\geqslant |\mathrm{Sp} \setminus [x, x+1]|. \end{aligned} \tag{2.11}$$

2.5. *Morse theory of plane curves*

For any $\xi \in \mathbb{C}^2$ and $r > 0$, let $B(\xi, r)$ be the ball centred at ξ and with radius r, also $S^3(\xi, r) := \partial B(\xi, r)$. For an algebraic curve C sitting in $\mathbb{C}^2$, we write $(C \cap B(\xi, r))^\wedge$ for the normalization of $C \cap B(\xi, r)$, and the genus of $C \cap B(\xi, r)$ is the genus of its normalization.

For any link L, we denote by c_L its number of components, and we set

$$\begin{aligned} w_L(\zeta) &:= -\sigma_L(\zeta) + 1 - c_L + n_L(\zeta), \\ -u_L(\zeta) &:= \sigma_L(\zeta) + 1 - c_L + n_L(\zeta). \end{aligned}$$

Remark 2.10. The convention used in [5] is that n_L is the dimension of the kernel of $(1-\zeta)S + (1-\overline{\zeta})S^T$ *increased by* 1, this explains the formal differences compared with [5].

We also fix $\zeta \in S^1 \setminus \{1\}$. Let us begin by citing a result from [5].

Proposition 2.11 [5, Proposition 6.8]. *Let ξ be a generic point of $\mathbb{C}^2$ and $r_0 < r_1$ two values such that the intersections $L_i := C \cap S^3(\xi, r_i)$ are transverse ($i = 0, 1$). With the notation $c_i = c_{L_i}$, $g_i =$ the genus of $C_i := C \cap B(\xi, r_i)$ and $k_i =$ the number of connected components of $C_i{}^\wedge$, one has*

$$\begin{aligned} w_{L_1}(\zeta) - \sum w_{L_k^{\mathrm{sing}}}(\zeta) - w_{L_0}(\zeta) &\geqslant -2(g_1 - g_0 + c_1 - c_0 - k_1 + k_0), \\ -(u_{L_1}(\zeta) - \sum u_{L_k^{\mathrm{sing}}}(\zeta) - u_{L_0}(\zeta)) &\geqslant -2(g_1 - g_0 + c_1 - c_0 - k_1 + k_0), \end{aligned} \tag{2.12}$$

where L_k^{sing} are the links of singularities of C, which lie in $B(\xi, r_1) \setminus B(\xi, r_0)$.

We use Proposition 2.11 in two special cases.

Corollary 2.12. *Let C_0 and C_1 be as in Proposition 2.11. If $C_{01} = C_1 \setminus C_0$ is smooth, then*

$$-\sigma_{L_1}(\zeta) + n_{L_1}(\zeta) - (-\sigma_{L_0}(\zeta) + n_{L_0}(\zeta)) \geqslant \chi(C_{01}),$$
$$\sigma_{L_1}(\zeta) + n_{L_1}(\zeta) - (\sigma_{L_0}(\zeta) + n_{L_0}(\zeta)) \geqslant \chi(C_{01}).$$

Proof. Use the definition of w, (2.12) and $C_{01}{}^\wedge = C_{01}$ for the first inequality. For the second one, we use $-u$ instead of w. □

The other important application is in the case when r_0 is small, hence L_0 is an unknot.

Proposition 2.13. *Fix r such that the intersection $C \cap S(\xi, r)$ is transverse, and set $L := C \cap S(\xi, r)$. Let C_{smooth} be the smoothing of $C \cap B(\xi, r)$ (for example, if C is given by $F^{-1}(0)$ for some reduced polynomial, then C_{smooth} can be taken as $F^{-1}(\varepsilon) \cap B(\xi, r)$ for ε sufficiently small). Let $z_1, \dots, z_k$ be the singular points of $C \cap B(\xi, r)$ with links $L_1^{\text{sing}}, \dots, L_k^{\text{sing}}$, Milnor numbers $\mu_1, \dots, \mu_k$, number of branches $c_1, \dots, c_k$, and signatures $\sigma_1(\zeta), \dots, \sigma_k(\zeta)$. Then*

$$\begin{aligned} -\sigma_L(\zeta) + n_L(\zeta) + (1 - \chi(C_{\text{smooth}})) &\geqslant \sum_{j=1}^{k} (-\sigma_{L_j^{\text{sing}}}(\zeta) + n_j(\zeta) + \mu_j), \\ \sigma_L(\zeta) + n_L(\zeta) + (1 - \chi(C_{\text{smooth}})) &\geqslant \sum_{j=1}^{k} (\sigma_{L_j^{\text{sing}}}(\zeta) + n_j(\zeta) + \mu_j). \end{aligned} \tag{2.13}$$

Proof. We prove only the first part, in the second one, we use $-u_L$ instead of w_L.

Let $r_{\min}$ be minimal with $C \cap S(\xi, r)$ non-empty, and set $r_0 := r_{\min} + \varepsilon$ for ε sufficiently small. Then L_0 is an unknot with $w_{L_0}(\zeta) \equiv 0$, $c_0 = k_0 = 1$, thus (2.12) gives

$$\begin{aligned} -\sigma_L(\zeta) + n_L(\zeta) + 1 - c_L \geqslant &\sum_{j=1}^{k} (-\sigma_j(\zeta) + n_j(\zeta) + \mu_j) \\ &- \sum_{j=1}^{k} (\mu_j + c_j - 1) - 2g(C) - 2c_L + 2k_1. \end{aligned}$$

The proof is completed by applying the genus formula $2(g(C_{\text{smooth}}) - g(C)) = \sum_{j=1}^{k} (\mu_j + c_j - 1)$, the fact that $b_1(C_{\text{smooth}}) = 2g(C_{\text{smooth}}) + c_L - 1$ and observing that $b_0(C_{\text{smooth}}) \leqslant 2k_1$ (it is even bounded by k_1 alone). □

Remark 2.14. The cited result (that is, Proposition 2.11) does not really require Morse theoretical arguments, although they are very convenient. We could deduce it, with approximately the same amount of work, from the Murasugi inequality [**15**, Theorem 12.3.1], too. The argument is that $C_{01} = C \cap (B(\xi, r_1) \setminus B(\xi, r_0))$ induces a cobordism between the links $L_0' := L_0 \amalg L_1^{\text{sing}} \amalg \dots \amalg L_j^{\text{sing}}$ and L_1. In this way, we do not use anywhere that C is a complex curve, only that its genus is the difference of the genera of the minimal Seifert surfaces of L_1 and L_0'.

3. *The Seifert form and the MHS of a polynomial at infinity*

In this section, we compare the Hodge spectrum associated with the limit mixed Hodge structure of a polynomial map at infinity with the HVS-spectrum provided by its regular link at infinity. In this way, *we recover the Hodge-spectrum from the 'Seifert form at infinity'*. For results concerning the limit mixed Hodge structure (MHS) and the Hodge spectrum at infinity the reader might consult [7, 9, 11, 13, 35].

3.1. *Basic definitions*

Let $F\colon \mathbb{C}^2 \to \mathbb{C}$ be a reduced polynomial with critical values $x_1, \ldots, x_N$. Since $\mathbb{C}^2$ is not compact, the topology of a fiber $F^{-1}(y)$ can be different for different regular values y of F.

DEFINITION 3.1 [31]. The fiber $F^{-1}(c)$ is called *regular at infinity* if there exists a (small) disk $D \ni c$ in $\mathbb{C}$ and a (large) ball $B \subset \mathbb{C}^2$ such that F restricted to $F^{-1}(D) \setminus B$ is a C^∞ trivial fibration. The fiber is called *irregular at infinity* if it is not regular at infinity.

Consider all the values $y_1, \ldots, y_M$ such that $F^{-1}(y_k)$ is not regular at infinity. Set $\rho \in \mathbb{R}$ with $\rho > \max_{k,l}\{|x_k|, |y_l|\}$ and set $\gamma = \{z \in \mathbb{C}\colon |z| = \rho\}$. Then F restricted to $F^{-1}(\gamma)$ is a C^∞ locally trivial fibration, called the *fibration of F at infinity*. It will be denoted Fib_∞. The fiber of Fib_∞ is the (generic) fiber $Y_\infty := F^{-1}(\rho)$ of F. The induced algebraic monodromy over γ, called the *monodromy of F at infinity*, will be denoted by

$$h_\infty\colon H_1(Y_\infty, \mathbb{Z}) \longrightarrow H_1(Y_\infty, \mathbb{Z}). \tag{3.1}$$

Furthermore, we also consider on $H_1(Y_\infty, \mathbb{Z})$ the intersection form b_∞. Already this *isometric structure* $(H_1(Y_\infty); b_\infty, h_\infty)$ contains important information about the behaviour of F at infinity, nevertheless, we will enhance it in two different ways. The first is topological: we investigate the possibility of extending the pair (b_∞, h_∞) to a variation structure (this, strictly speaking, in general, is only 'partially' possible). The candidate for the variation operator is the inverse transpose of the Seifert matrix of the link at infinity. The second is algebraic: one lifts the pair (b_∞, h_∞) to the level of a polarized mixed Hodge structure by considering the limit mixed Hodge structure of F at infinity.

First, we start with the topological part.

Fix a fiber $F^{-1}(c)$ which is regular at infinity. For sufficiently large R the intersection $F^{-1}(c)$ with $\partial B(R)$ is transverse. This link $F^{-1}(c) \cap \partial B(R) \subset \partial B(R)$, denoted by L^∞_{reg}, is independent (up to an isotopy) of R and c. It is called the *regular link at infinity* of F.

According to [31, Theorem 5], we can associate with L^∞_{reg} the so-called fundamental *multilink* at infinity L_{fund}, which is fibred. This means the following: there exists a link L_{fund} with components $\{L_{\mathrm{fund}}, i\}_{i=1}^\nu$ and positive multiplicities $\mathbf{n} = \{n_i\}_{i=1}^\nu$ such that there is a fibration $\phi : S^3 \setminus L_{\mathrm{fund}} \to S^1$ with the following property: for any closed loop $\tau \in S^3 \setminus L_{\mathrm{fund}}$, $\phi_*([\tau]) \in H_1(S^1) = \mathbb{Z}$ equals the linking number of $[\tau]$ with $\sum_i n_i L_{\mathrm{fund},i}$. Furthermore, the closure $\overline{Y_t}$ of the fiber $Y_t = \phi^{-1}(e^{2\pi i t})$ $(t \in [0, 1])$ is not a manifold with boundary, but homologically $\overline{Y_t} \setminus Y_t$ is the multilink $\sum_i n_i L_{\mathrm{fund},i}$.

Finally, the connection between the multilink L_{fund} and the link at infinity L^∞_{reg} is the following. Let $T = T(L_{\mathrm{fund}})$ be a closed small tubular neighbourhood of L_{fund}. Then L^∞_{reg} is the intersection of a fiber Y_0 with $\partial T(L_{\mathrm{fund}})$.

LEMMA 3.2. *For any $i \in \{1, \ldots, \nu\}$, let l_i be the linking number*

$$l_i = \mathrm{lk}\left(L_{\mathrm{fund},i}, \sum_{j \neq i} n_j L_{\mathrm{fund},j}\right)$$

and n'_i the (positive) greatest common divisor of n_i and l_i. Then the number of components of $Y_0 \cap \partial T(L_{\mathrm{fund},i})$ is exactly n'_i. Hence, L^∞_{reg} has $\sum_{i=1}^\nu n'_i$ components. Moreover, the components of $Y_0 \cap \partial T(L_{\mathrm{fund},i})$ are cyclically permuted by the geometric monodromy of ϕ.

Proof. See [12, § 3 and 4]. □

Another important point about L_{fund} is that its fiber Y_0 can be identified with the generic fiber Y_∞ of the polynomial F [31, Theorem 4]. In fact, by Bartolo and Cassou-Noguès [3, Theorem 1.1], one has:

LEMMA 3.3. *The multilink fibration $S^3 \setminus L_{\text{fund}} \to S^1$ associated with $(L_{\text{fund}}, \mathbf{n})$ and the fibration Fib_∞ of F are isomorphic.*

By [12, p. 37], Y_0 has $d = \gcd_i\{n_i\}$ connected components. In the sequel, we will assume that $d = 1$, that is, *the generic fiber of F is connected.*

3.2. *The multilink Seifert form of L_{fund}*

The surface Y_0, the fiber of the multilink $(L_{\text{fund}}, \mathbf{n})$, is a generalized Seifert surface of the multilink, cf. [12, pp. 28–29]. In the sequel, we refer to it as the *multilink Seifert surface*. Using this surface, one can define the *multilink Seifert form* associated with Y_0, cf. [12, § 15]. It is a bilinear form on $H_1(Y_0, \mathbb{Z})$ defined similarly as the classical Seifert form, namely $S_{\text{fund}}(\alpha, \beta)$ for $\alpha, \beta \in H_1(Y_0, \mathbb{Z})$ is the linking number $\text{lk}(\alpha, \beta^+)$, where β^+ is the push-forward of β in the positive direction.

If all the multiplicities $\{n_i\}_i$ equal 1, then L_{fund} is a fibred link, and S_{fund} is its classical Seifert form, hence it has determinant ± 1. In the case of general multiplicity system $\mathbf{n}$ this is not the case anymore. In fact, S_{fund} can be even degenerate. Nevertheless, some parts of the classical theory survive.

LEMMA 3.4. *Let H^* denote the dual of H, T^o the interior of T, and $\bar{Y}_{[a,b]} := \bigcup_{a \leqslant t \leqslant b} \overline{Y_t}$.*

(a) *The groups $H_1(\overline{Y_0}, \mathbb{Z})$ and $H_1(Y_0, \mathbb{Z})^*$ are isomorphic. In fact, one has the following sequence of isomorphisms, denoted by s:*

$$\begin{aligned} H_1(\overline{Y_0}) &\xrightarrow{\partial^{-1}} H_2(S^3, \overline{Y_0}) \xrightarrow{(1)} H_2(S^3, \overline{Y}_{[0,1/2]}) \xrightarrow{(2)} H_2(\overline{Y}_{[1/2,1]}, \overline{Y_{1/2}} \cup \overline{Y_1}) \\ &\xrightarrow{(3)} H_2(\bar{Y}_{[1/2,1]}, \overline{Y_{1/2}} \cup \overline{Y_1} \cup (T \cap \bar{Y}_{[1/2,1]})) \xrightarrow{(4)} H_2(\bar{Y}_{[1/2,1]} \setminus T^o, \partial(\bar{Y}_{[1/2,1]} \setminus T^o)) \\ &\xrightarrow{(5)} H_1(\bar{Y}_{[1/2,1]} \setminus T^o)^* \xrightarrow{(6)} H_1(\overline{Y_1} \setminus T^o)^* \xrightarrow{(7)} H_1(Y_1)^* = H_1(Y_0)^*. \end{aligned}$$

(b) *Let $j : H_1(Y_0, \mathbb{Z}) \to H_1(\overline{Y_0}, \mathbb{Z})$ be induced by the inclusion. Then the composition*

$$H_1(Y_0, \mathbb{Z}) \xrightarrow{j} H_1(\overline{Y_0}, \mathbb{Z}) \xrightarrow{s} H_1(Y_0, \mathbb{Z})^*$$

can be identified with the multilink Seifert form S_{fund}.

(c) *Identify the isometric structure (b_∞, h_∞) with the intersection form and monodromy of $H_1(Y_0)$ (by 3.3). Then, in matrix notation,*

$$b_\infty = S_{\text{fund}} - S_{\text{fund}}^T \quad \text{and} \quad S_{\text{fund}}^T h_\infty = S_{\text{fund}}.$$

In particular, h_∞ is an automorphism of S_{fund}, that is, $h_\infty^T S_{\text{fund}} h_\infty = S_{\text{fund}}$.

Proof. In the sequence of isomorphisms ∂^{-1} comes from the exact sequence of the pair; (1), (3), (6) and (7) are induced by deformation retracts; (2) and (4) are excisions, while (5) is provided by duality of the manifold with boundary $\bar{Y}_{[1/2,1]} \setminus T^o$. Parts (b) and (c) follow by similar argument as in the classical case, see, for example, the survey [27, (3.15)]. □

In the above composition, although s is an isomorphism, j in general is not. Since, by our assumption $\tilde{H}_0(Y_0,\mathbb{Z})=0$, the morphism j can be inserted in the following long exact sequence:

$$0 \longrightarrow H_2(\overline{Y_0}) \to H_2(\overline{Y_0},Y_0) \longrightarrow H_1(Y_0) \xrightarrow{j} H_1(\overline{Y_0}) \longrightarrow H_1(\overline{Y_0},Y_0) \longrightarrow 0. \tag{3.2}$$

LEMMA 3.5.

(a) $H_2(\overline{Y_0},Y_0,\mathbb{Z}) = \bigoplus_{i=1}^{\nu}\mathbb{Z}^{n_i'-1}$,

(b) $H_1(\overline{Y_0},Y_0,\mathbb{Z}) = \bigoplus_{i=1}^{\nu}(\mathbb{Z}^{n_i'-1}\oplus\mathbb{Z}_{n_i/n_i'})$.

In particular, $H_2(\overline{Y_0},\mathbb{Z})=0$ *and*

$$\dim\ker j = \sum_{i=1}^{\nu}(n_i'-1). \tag{3.3}$$

Proof. By excision and deformation retract argument $H_q(\overline{Y_0},Y_0)=\oplus_i H_q(A_i,B_i)$, where

$$(A_i,B_i) := (\overline{Y_0}\cap T(L_{\text{fund},i}),\quad Y_0\cap\partial T(L_{\text{fund},i})).$$

Note that the homotopy type of A_i is $L_{\text{fund},i}$, while of B_i is n_i' copies of S^1. Each of these copies maps (via the inclusion $B_i\hookrightarrow A_i$) onto $L_{\text{fund},i}$ as the n_i/n_i'-covering. Therefore, the inclusion $B_i\hookrightarrow A_i$ at H_1-level is $\mathbb{Z}^{n_i'}\to\mathbb{Z}$, $\{a_1,\dots,a_{n_i'}\}\mapsto(n_i/n_i')\cdot\sum_k a_k$. This gives (a) and (b). The rest follow by rank computation argument from (3.2). □

Next, we will consider another compactification $\widetilde{Y}_0$ of Y_0. Denote $Y_0^o:=Y_0\setminus T(L_{\text{fund}})^o$, the complement of the interior of the tube. The boundary ∂Y_0^o consists of $\sum_i n_i'$ copies of S^1. Let $\widetilde{Y}_0$ be obtained from Y_0^o by gluing to each boundary circle a 2-disc, in this way, obtaining a compact smooth surface. In fact, the fibration at infinity F over γ can be compactified (even algebraically) to a fibration $\tilde{F}$ over γ with smooth compact fibers $\widetilde{Y}_0$, where in this language the compact fiber consists of Y_0 with additionally $\sum_i n_i'$ 'points at infinity'. This point of view is used in Hodge theoretical computations, see, for example, [8] or [9, § 3].

One has the following exact sequence:

$$0\longrightarrow\mathbb{Z}\longrightarrow\oplus_i\mathbb{Z}^{n_i'}\longrightarrow H_1(Y_0,\mathbb{Z})\longrightarrow H_1(\widetilde{Y}_0,\mathbb{Z})\longrightarrow 0. \tag{3.4}$$

Above, $\bigoplus_i\mathbb{Z}^{n_i'}$ is generated by the discs, their images in $H_1(Y_0)$ are the classes of the circles ∂Y_0^o. The group $\mathbb{Z}$ from the left is $H_2(\widetilde{Y}_0)$; its image is generated by ∂Y_0^o. The monodromy extends to $H_1(\widetilde{Y}_0)$ (or to $\tilde{F}$) (and will be denoted by $\tilde{h}_\infty$), and also to the discs/points at infinity: it acts trivially on $\mathbb{Z}$, on $\mathbb{Z}^{n_i'}$ acts by permutation of the base elements (denoted by h_{per}).

Let $\tilde{b}_\infty$ be the intersection form on $H_1(\widetilde{Y}_0)$.

LEMMA 3.6. (a) *The monodromy operator* $\tilde{h}_\infty$ *has no eigenvalue* 1, *and all its Jordan blocks have size not larger than* 2.

(b) *The exact sequence* (3.4), *together with the algebraic monodromy action on it, splits. That is,* h_∞ *has no Jordan block of size* 3, *and the blocks of size* 2 *of* h_∞ *and* $\tilde{h}_\infty$ *agree. In other words, over* $\mathbb{Q}$, *one has a direct sum decomposition*:

$$(H_1(Y_0);b_\infty,h_\infty) = (H_1(\widetilde{Y}_0);\tilde{b}_\infty,\tilde{h}_\infty)\oplus(\oplus_i\mathbb{Q}^{n_i'}/\mathbb{Q};0,h_{\text{per}}). \tag{3.5}$$

Moreover, $(\tilde{b}_\infty,\tilde{h}_\infty)$ *is a non-degenerate isometric structure.*

(c) *All roots of* h_∞ *are roots of unity.*

Proof. The statements follow from the mixed Hodge theory of the degeneration at infinity of F and $\tilde{F}$. Part (a) is proved, for example, in [**8**, **9**]. Part (b) follows from the spectral pair computation of the mixed Hodge structure carried on $H^1(Y_0,\mathbb{C})$. More precisely, there is a cohomological analogue of the sequence (3.4) which carries mixed Hodge structure compatible with the action of the monodromy, see again [**9**, § 3]. The number of Jordan blocks of size 2 correspond to those spectral pairs (α, w) for which $w = 0$. These are computed for both $H^1(Y_0,\mathbb{C})$ and $H^1(\widetilde{Y}_0,\mathbb{C})$ in [**7**], and their numbers agree. For (c) use the Monodromy Theorem for $\tilde{F}$ at infinity. □

Finally, we summarize the properties of L_{fund} in the following proposition. As usual, if h is an automorphism of the vector space V, then $V_{\lambda=1}$ denotes the generalized eigenspace corresponding to eigenvalue 1, while $V_{\lambda\neq 1}$ is the direct sum of the other generalized eigenspaces.

PROPOSITION 3.7. *Set $U := H_1(Y_0,\mathbb{Q})$ and let b_∞ and h_∞ be the intersection form and the algebraic monodromy induced by the multilink fibration $\phi : S^3 \setminus L_{\text{fund}} \to S^1$.*
Then the following facts hold.

(a) *The surface Y_0 is the minimal multilink Seifert surface of the multilink $(L_{\text{fund}}, \mathbf{n})$, and all minimal multilink Seifert surfaces of $(L_{\text{fund}}, \mathbf{n})$ are isotopic to Y_0.*
(b) *One has a direct sum decomposition (Keef decomposition, cf. (2.4)):*

$$(U, S_{\text{fund}}) = (U_0 \oplus U_{\text{ndeg}}, S_{\text{fund},0} \oplus S_{\text{fund,ndeg}}) \tag{3.6}$$

such that $S_{\text{fund},0} = 0$ of size $\text{Irr} = \sum_{i=1}^{\nu}(n'_i - 1)$, and $S_{\text{fund,ndeg}}$ is non-degenerate.
A possible free generator set for U_0 is the collection of the cycles $L^\infty_{\text{reg},i,k} - L^\infty_{\text{reg},i,k+1}$ $(1 \leqslant i \leqslant \nu; 1 \leqslant k < n'_i)$, where $\{L^\infty_{\text{reg},i,k}\}_{k=1}^{n'_i}$ are the components of $Y_0 \cap \partial T(L_{\text{fund},i})$.
(c) *The compatibility of the decompositions (3.6) and (3.5) is the following:*

- (c.1) *$(U_{\text{ndeg}})_{\lambda\neq 1} = H_1(\widetilde{Y}_0)$. On this space, $(S_{\text{fund,ndeg}})_{\lambda\neq 1}$ completes the non-degenerate isometric structure $(\tilde{b}_\infty, \tilde{h}_\infty)$ to a simple (-1)-variation structure.*
- (c.2) *$(\bigoplus_i \mathbb{Q}^{n'_i}/\mathbb{Q}; 0, [h_{\text{per}}]) = ((U_{\text{ndeg}})_{\lambda=1}; 0, \text{Id}) \oplus (U_0; 0, h_\infty|_{U_0})$ (and this is an eigenspace decomposition).*
- (c.3) *$(U_{\text{ndeg}})_{\lambda=1}$ has dimension $\nu - 1$, on it the restriction of b_∞ is trivial, the restriction of h_∞ is the identity, and this degenerate isometric structure is completed by $(S_{\text{fund,ndeg}})_{\lambda=1}$ to a simple variation structure.*
- (c.4) *On U_0, the restrictions of b_∞ and S_{fund} are trivial (hence all the equivariant signature-type invariants including the Tristram–Levine signatures of restrictions of S_{fund} and (b_∞, h_∞) are the same). Nevertheless, the restriction of h_∞ is non-trivial (in fact, it has no eigenvalue 1), hence the isometric structure cannot be completed to a variation structure. The characteristic polynomial of the restriction of h_∞ is*

$$\det(h_\infty|_{U_0} - t\,\text{Id}) = \prod_i \frac{t^{n'_i} - 1}{t - 1}.$$

Proof. (a) follows from [**12**, (4.1)]. For (b) note that the generators listed are in the kernel of j. For this use, for example, the proof of (3.5), where $\{L^\infty_{\text{reg},i,k}\}_{k=1}^{n'_i}$ are exactly the components of B_i. Another possibility is a direct verification of the fact that $S_{\text{fund}}(L^\infty_{\text{reg},i,k} - L^\infty_{\text{reg},i,k+1}, \beta) = S_{\text{fund}}(\beta, L^\infty_{\text{reg},i,k} - L^\infty_{\text{reg},i,k+1}) = 0$, for any β. Indeed, if T is a sufficiently small tubular neighbourhood, then it does not intersect β, on the other hand, inside of T the circles $L^\infty_{\text{reg},i,k}$ and $L^\infty_{\text{reg},i,l}$ are homologous. For part (c) use Lemmas 3.4(c) and 3.6; for the characteristic polynomial use the fact that the components $\{L^\infty_{\text{reg},i,k}\}_k$ are cyclically permuted, cf. Lemma 3.2. □

The multilink structure $(L_{\text{fund}}, S_{\text{fund}})$ now will be used in two different aspects. First, it can be related with the link L^∞_{reg}; in fact, one can recover it from L^∞_{reg}, see (3.3). On the other hand, the multilink fibration of L_{fund} can be identified with the fibration at infinity Fib_∞ of F, cf. Lemma 3.3. In this way, L_{fund} creates the bridge between L^∞_{reg} and Fib_∞.

3.3. *The Seifert form of L^∞_{reg}*

Set $Y_0^o := Y_0 \setminus T(L_{\text{fund}})^o$ as above. Obviously, $Y_0^o \hookrightarrow Y_0$ admits a deformation retract, hence $H_1(Y_0^o, \mathbb{Z}) = H_1(Y_0, \mathbb{Z})$ canonically.

LEMMA 3.8. *One has the following facts.*

(a) *The surface Y_0^o is the minimal Seifert surface of L^∞_{reg}, and all minimal Seifert surfaces of L^∞_{reg} are isotopic to Y_0^o.*
(b) *The Seifert form S_{reg} of L^∞_{reg} associated with Y_0^o is identical with S_{fund} (under the identification $H_1(Y_0^o, \mathbb{Z}) = H_1(Y_0, \mathbb{Z})$). In particular, all the result listed in Proposition 3.7 about S_{fund} are valid for S_{reg} as well.*
(c) *Let h_{reg} be the monodromy of the variation structure associated with $S_{\text{reg,ndeg}} = S_{\text{fund,ndeg}}$ (as in Section 2.2). Then the higher Alexander polynomials Δ_k of L^∞_{reg} satisfy the following identities: $\Delta_k \equiv 0$ for $0 \leqslant k < \text{Irr}$, $\Delta_{\text{Irr}}(t) = \det(h_{\text{reg}} - t\ \text{Id})$.*
(d) *All the roots of the (higher) Alexander polynomial Δ_{Irr} of L^∞_{reg} are roots of unity.*

Proof. (a)–(c) follow from Proposition 3.7 and from the construction of Y_0^o. For (d), use either Lemma 3.6(c) or note that the multilink fibration of L_{fund} can be represented by a splice diagram [**31**, **32**], hence the characteristic polynomial of h_{fund} is a product of cyclotomic polynomials by Eisenbud and Neumann [**12**, Theorem 13.6]. □

3.4. *The HVS-spectrum of the regular link at infinity, L^∞_{reg}*

For local isolated plane curve singularities, we have the following classical result (consequences of the Monodromy Theorem and polarization properties), which in the language of H-numbers $p^k_\lambda(\pm 1)$ and q^k_λ of their local links can be formulated as follows (see, for example, [**24**, Proposition 6.14] or [**6**, Proposition 3.1.5, Lemma 3.1.6], compare also with [**42**]).

PROPOSITION 3.9. *Let L be an algebraic link and $p^k_\lambda(\pm 1)$, q^k_λ its H-numbers. Then*

(a) *$q^k_\lambda = 0$ for all $k > 0$ and $|\lambda| < 1$; moreover $p^k_\lambda(\pm 1) = 0$ unless λ is a root of unity;*
(b) *$p^k_\lambda(\pm 1) = 0$ for all $k > 2$. Moreover $p^2_1(\pm 1) = 0$;*
(c) *$p^2_\lambda(-1) = 0$ and $p^1_1(-1) = 0$.*

The fundamental mulitlink at infinity L_{fund}, or the regular link at infinity L^∞_{reg}, in general, cannot be realized by a local algebraic link. However, their H-numbers share similar properties as the H-numbers of local links.

PROPOSITION 3.10. *For the H-numbers of L^∞_{reg} the following facts hold:*

(a) *$q^k_\lambda = 0$ for all $k > 0$ and $|\lambda| < 1$, moreover $p^k_\lambda(\pm 1) = 0$ unless λ is a root of unity.*
(b) *$p^k_\lambda(\pm 1) = 0$ for all $k > 2$, and $p^2_1(\pm 1) = 0$.*
(c) *$p^1_1(-1) = 0$.*
(d) *$p^2_\lambda(1) = 0$ for $\lambda \neq 1$.*

Proof. (a) and (b) follow from Lemma 3.6. Next, we prove (c). First, we recall that $\mathcal{W}_1^1(\pm 1) = (\mathbb{C}; 0, \mathrm{Id}, \mp 1)$, hence we have to show that the restriction of S_{fund} on $U_{\lambda=1}$ is negative definite. This follows from the more general Proposition 3.17 of Section 3.7.

(d) By Neumann and Rudolph [**31**, **33**], L_{fund} can be represented by a splice diagram where all edges determinants are negative. Thus, L_{fund} has uniform twists (all positive) (see [**12**, Chapter 14]). Therefore, by the discussion in [**29**, Section 2], we have, for each $x \in U_\lambda$,

$$S_{\text{fund}}(\lambda x, (h_\infty - \lambda)x) \geqslant 0.$$

This shows that $p_\lambda^2(+1)$ cannot occur. □

Let $\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}})$ be the HVS-spectrum associated with the link L^∞_{reg}.

Corollary 3.11. (a) *The HVS-spectrum of* $U_{\lambda=1}$ *consists of* $(\nu - 1)$ *copies of* (1).

(b) *All elements of* $\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}})$ *are situated in* $(0, 2)$, *and* $\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}})$ *is symmetric with respect to 1.*

Remark 3.12. The proofs of Propositions 3.9 and 3.10 rely on some key properties of the splice diagrams of the corresponding links. The common properties (which imply the common $p_1^1(-1) = 0$) are that in both cases the 'multiplicities of the nodes' and the '(near) weights' are positive. The crucial difference between the diagrams is that in the local case the edge determinants are positive, while for the diagram at infinity they are negative. This implies the sign difference in the $p_\lambda^2(\pm 1)$-vanishing. For more detail, see Section 3.7.

The next identity will often be used in the sequel.

Corollary 3.13. *If* Y_∞ *is a regular fiber of* F, *then* $1 - \chi(Y_\infty) = \deg \Delta_{\mathrm{Irr}}(L^\infty_{\text{reg}}) + \mathrm{Irr}$.

Proof. By Lemma 3.8(a), the size of S_{reg} is equal to $1 - \chi(Y_\infty)$. On the other hand, $\deg \Delta_{\mathrm{Irr}}$ is equal to the size of S_{ndeg} by Lemma 3.8(c). The difference of the sizes of the two matrices is equal to Irr by Proposition 3.7(b) (cf. also Lemma 3.8(b)). □

3.5. *The Hodge spectrum of the fibration of F at infinity.*

Let $\mathrm{Sp}_{\mathrm{MHS},\infty}$ be the spectrum associated with the limit mixed Hodge structure of F at infinity defined in a similar way as in Section 2.3. The main result of this subsection shows that $\mathrm{Sp}_{\mathrm{MHS},\infty}$ can be recovered from the rational Seifert form of L^∞_{reg} and from the integers $\{n'_i\}_i$. Conversely, $\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}})$ is the maximal subset of $\mathrm{Sp}_{\mathrm{MHS},\infty}$, which is symmetric with respect to 1.

More precisely, in $\mathbb{Z}[\mathbb{Q}]$ one has:

Theorem 3.14.

$$\mathrm{Sp}_{\mathrm{MHS},\infty} = \mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}}) + \sum_{i=1}^{\nu} \left(\frac{1}{n'_i}\right) + \ldots + \left(\frac{n'_i - 1}{n'_i}\right).$$

Proof. Let us consider the decomposition given in Proposition 3.7:

$$(U_0; 0, h_\infty|_{U_0}) \oplus ((U_{\text{ndeg}})_{\lambda=1}; 0, \mathrm{Id}) \oplus ((U_{\text{ndeg}})_{\lambda\neq 1}; \tilde{b}_\infty, \tilde{h}_\infty). \tag{3.7}$$

The last component carries a limit mixed Hodge structure which is polarized by $\tilde{b}_\infty$, and it extends to a simple HVS with $(S_{\mathrm{fund,ndeg}})_{\lambda\neq 1}$. In such a situation, the HVS-spectrum agrees with the Hodge spectrum. The proof is absolutely the same as in the local case, see Proposition 2.5, or the original source [24, (6.5)] (or the affine polynomial case in [14]).

For the middle component both the HVS and Hodge spectra consist of $(\nu-1)$ copies of (1): in the HVS case, see Corollary 3.11 as a consequence of Proposition 3.10(c), while for the Hodge case, see [7] or [9].

These two components provide the contribution from $\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\mathrm{reg}})$. The remaining part, provided by the first summand, is computed in [7], and it is the sum on the right-hand side of the identity of Theorem 3.14. □

EXAMPLE 3.15. Recall that F is 'good at infinity' if and only if L^∞_{reg} is a fibred link, that is, $n_i = 1$ for all i, cf. [33, Theorem 6.1]. By our result, in such a case one has $\mathrm{Sp}_{\mathrm{MHS},\infty} = \mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\mathrm{reg}})$.

COROLLARY 3.16. *For any $x \in [0,1]$, one has*

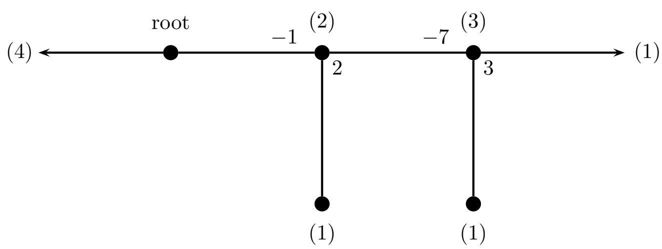
$$|\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\mathrm{reg}}) \cap (x, x+1)| \leqslant |\mathrm{Sp}_{\mathrm{MHS},\infty} \cap (x, x+1)| \leqslant |\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\mathrm{reg}}) \cap (x, x+1)| + \mathrm{Irr}$$

and the analogous inequality holds for $\mathrm{Sp} \setminus [x, x+1]$.

3.6. *An example*

The above discussion might have been technically quite involved. We want to illustrate the occurring phenomena by investigating one example, the Briançon polynomial, which appeared in [4, 7, 9, 10] (we remark that in [3, Example 4.14] there is a different polynomial called the Briançon polynomial, which has a different link at infinity and different irregular fibers).

The splice diagram of the fundamental link at infinity is as follows:



Here, the numbers in parentheses are the multiplicities of the vertices and arrowheads (link components). The numbers not in parentheses denote the weights of corresponding edges (those omitted are equal to 1). We have $n_1 = 4$, $n'_1 = \gcd(4,6) = 2$, and $n_2 = n'_2 = 1$.

Computing the Euler characteristics of a minimal Seifert surface of L_{fund} (as in [12]), we get that this surface is a three times punctured torus (3 is the number of components of L_{fund}). The rank of $H_1(Y_\infty)$ is 4, while the ranks of U_0 and $(U_{\mathrm{ndeg}})_{\lambda=1}$ are 1. The monodromy at infinity permutes $L_{1,1}$, $L_{1,2}$ and fixes L_2. The characteristic polynomial of the monodromy on boundary components is therefore t^2-1. The Alexander polynomial of L_{fund}, hence the characteristic polynomial of the monodromy at infinity, is $(t^2-1)(t^2+t+1)$.

The equivariant signatures (which correspond to jumps of the Tristram–Levine signatures) of L_{fund} can be computed using [30, Theorem 5.3 and Section 6]. Using [30, Theorem 5.3] for the left-most splice component, we compute that $\sigma^-_{e^{2\pi i/3}} = -1$ and $\sigma^-_{e^{-2\pi i/3}} = 1$ so the jumps of the Tristram–Levine signature function are, respectively, -2 and 2, in other words,

$p^1_{e^{2\pi i/3}}(+1) = 0$, $p^1_{e^{2\pi i/3}}(-1) = 1$, $p^1_{e^{-2\pi i/3}}(+1) = 1$, $p^1_{e^{-2\pi i/3}}(-1) = 0$ (compare [7, Sections 3.5, 3.6]). On the other hand, a straightforward computation shows that the right splice component does not contribute to the equivariant signature at all. Hence, the non-trivial H-numbers are $p^1_{e^{2\pi i/3}}(-1) = p^1_{e^{-2\pi i/3}}(+1) = p^1_1(1) = 1$.

Concluding, the spectrum at infinity is equal to $\{\frac{2}{3}, \frac{4}{3}, \frac{1}{2}, 1\}$ (cf. [9, Example 3.6(ii)]), where $\{\frac{2}{3}, 1, \frac{4}{3}\}$ is the contribution from L^∞_{reg}.

3.7. *The definiteness of 'linking matrix': the proof of Proposition* 3.10(c)

We wish to prove that the restriction of S_{fund} on $U_{\lambda=1}$ is negative definite. This follows from a more general combinatorial result, which we now state.

Let Γ be a rooted Eisenbud–Neumann diagram, cf. [31]. For an edge, we call the weight which is closer to the root vertex the *near weight* and the other one the *far weight*. For any two nodes v and w, if the geodesics connecting w and the root vertex contain v then we say that w is *beyond* v. We allow more than one near weight at each node to have weight different than 1. The linking numbers and multiplicities are determined from the diagram as in [12, § 10,11]. The arrowhead vertices will be denoted by $L_1, \dots, L_\nu$, their multiplicities are $n_1, \dots, n_\nu$.

Let $\mathbb{Q}^\nu$ be the $\mathbb{Q}$-vector space generated by $\{L_i\}_i$. The *linking matrix* $\{\text{lk}(L_i, L_j)\}_{ij}$ is defined as follows: for $i \neq j$, it is the standard linking pairing, while the self-linking $\text{lk}(L_i, L_i)$ is defined via the identity $\text{lk}(L_i, \sum_j n_j L_j) = 0$. Equivalently,

$$\text{lk}(n_i L_i, n_i L_i) = -\sum_{j \neq i} \text{lk}(n_i L_i, n_j L_j). \tag{3.8}$$

In particular, the null-space of the linking matrix is at least one-dimensional.

PROPOSITION 3.17. *Let Γ be a rooted connected graph with the following properties.*

(a) *All near weights are positive and no far weight is allowed to be zero.*

(b) *If the far weight at a node v is negative, then all far weights of nodes beyond v are also negative (this property is weaker than negativity of edge determinants).*

(c) *The multiplicities of all arrowhead and non-arrowhead vertices are positive.*

Then, the linking matrix $\text{lk}(L_i, L_j)$ is negative semi-definite with one-dimensional null-space.

Proof. We begin with a following special case.

LEMMA 3.18. *The statement of Proposition 3.17 holds if $\text{lk}(L_i, L_j) > 0$, for all $i \neq j$.*

Proof. The reasoning is exactly as in [29, § 3]: for $L = \sum \ell_j n_j L_j$, one has

$$\text{lk}(L, L) = \sum_{i<j} 2\ell_i \ell_j \,\text{lk}(n_i L_i, n_j L_j) + \sum_i \ell_i^2 \,\text{lk}(n_i L_i, n_i L_i).$$

Substituting (3.8), we get

$$\text{lk}(L, L) = -\sum_{i<j} (\ell_i - \ell_j)^2 \,\text{lk}(n_i L_i, n_j L_j). \tag{3.9}$$

Hence, $\text{lk}(L, L)$ is zero if $\ell_1 = \ldots = \ell_n$, and negative otherwise. □

In general, if some far weights are negative, some of the linking numbers $\text{lk}(L_i, L_j)$ might be negative as well; in these cases the proof is more involved.

LEMMA 3.19. *If $\nu \geqslant 2$, then the self-linking number* $\operatorname{lk}(L_i, L_i)$ *is negative for any* i.

Proof. For each i, let v_i be a node supporting L_i, α_i denotes the far weight at v_i and $\beta_{i1}, \dots, \beta_{ik_i}$ the near weights at v_i, with β_{i1} the near weight on the edge supporting L_i.

If $\operatorname{lk}(L_i, L_j) > 0$, for all $j \neq i$, then the statement follows from (3.8). Hence, assume that $\operatorname{lk}(L_i, L_j) < 0$, for some j. Assume that L_i and L_j are supported by nodes v_i and v_j, respectively (the case $v_i = v_j$ is also possible). Let γ be a path in Γ joining L_i to L_j. Since $\operatorname{lk}(L_i, L_j) < 0$, one of the vertices lying on γ, call it v_γ, must have a negative weight. This, by assumption (a), must be a far weight, hence there is a unique v_γ along the path with this property. Now, if v_i is beyond v_γ, then by (b), we have $\alpha_i < 0$. Otherwise, $v_i = v_\gamma$ and $\alpha_i < 0$ by the definition of v_γ. Next, let M_i be the multiplicity of v_i, namely

$$M_i = \sum_j \operatorname{lk}(v_i, n_j L_j) = \alpha_i \beta_{i2} \dots \beta_{ik_i} n_i + \sum_{j \neq i} \operatorname{lk}(v_i, n_j L_j).$$

But, for $j \neq i$, one has $\operatorname{lk}(v_i, n_j L_j) = \beta_{i1} \operatorname{lk}(L_i, n_j L_j)$, hence

$$-\operatorname{lk}(L_i, L_i) = \frac{1}{\beta_{i1} n_i} \sum_{j \neq i} \operatorname{lk}(v_i, n_j L_j) = \frac{M_i}{n_i \beta_{i1}} - \frac{\alpha_i \beta_{i2} \dots \beta_{ik_i}}{\beta_{i1}} > 0 \tag{3.10}$$

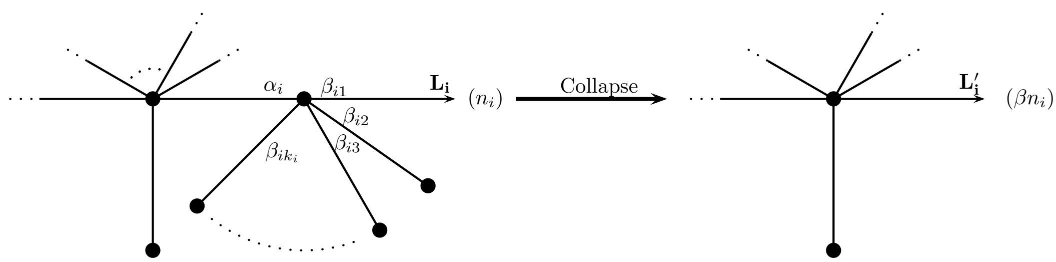
as $M_i > 0$. □

COROLLARY 3.20. *If the diagram has one or two arrowheads, then the statement of Proposition* 3.17 *holds.*

Proof. Use Lemma 3.19 and the fact that the null-space is not trivial. □

The proof of Proposition 3.17 is based on induction via reduction of the diagram (via two operations).

DEFINITION 3.21. Let Γ be a rooted graph. Assume that the supporting node v_i of the arrowhead vertex L_i has the following properties: it is not the root vertex, there is no node beyond it, L_i is the unique arrowhead supported by v_i. Hence, all its adjacent vertices except L_i and another one (in the direction of the root) are leaves. As above, denote the valency of v_i by $k_i + 1$ (see the picture given below).



A *collapse* of v_i is a graph Γ' with v_i replaced by an arrowhead vertex L'_i with multiplicity $n_i\beta$, where $\beta = \beta_{i2} \dots \beta_{ik_i}$ and all other weights and multiplicities are unchanged.

LEMMA 3.22. *The linking matrices of* Γ *and* Γ' *are congruent. Moreover, if* Γ *satisfies the assumptions* (a)–(c) *of the proposition, then so does* Γ'.

Proof. We shall use the notation lk_Γ and $\mathrm{lk}_{\Gamma'}$ for the linking forms on Γ and Γ'.

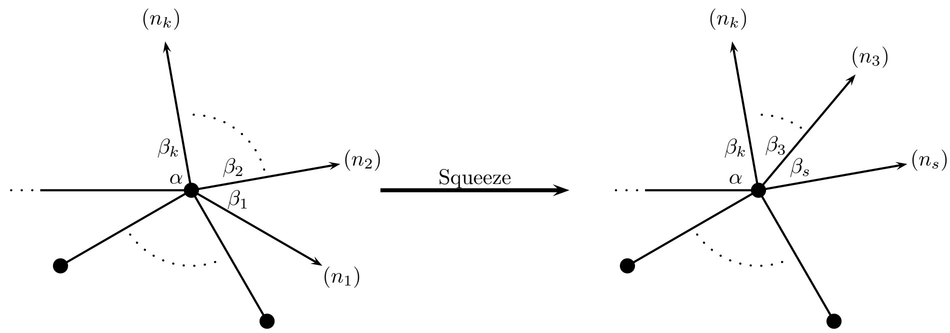
For any vertex v (node or arrowhead), different from the deleted ones, we have $\mathrm{lk}_\Gamma(v, L_i) = \mathrm{lk}_{\Gamma'}(v, \beta L_i')$. We claim that $\mathrm{lk}_{\Gamma'}(\beta L_i', \beta L_i') = \mathrm{lk}_\Gamma(L_i, L_i)$. This follows from (3.8) applied for Γ and Γ', and from the fact that in Γ' the relation $\beta n_i L_i' + \sum_{j\neq i} n_j L_j' = 0$ holds. Thus, the linking matrix of Γ written in the basis $L_1, \dots, L_i, \dots, L_\nu$ is the same as the linking matrix of Γ' in the basis $L_1, \dots, \beta L_i', \dots, L_\nu$. This proves the first part. As for the other part, the multiplicities of all vertices (besides the deleted ones) are preserved. This shows that if Γ satisfies (c), then so does Γ', while (a) and (b) are obvious. □

DEFINITION 3.23. Let v_0 be a node with no other node beyond it. Let $L_1, \dots, L_k$ be the arrowheads adjacent to v_0 ($k \geqslant 2$), denote their multiplicities by $n_1, \dots, n_k$. The node v_0 might have several adjacent leaves as well. We denote by β the product of their near weights. Assume that the overall number of vertices of Γ is at least 3.

A *squeeze* of Γ is a graph arising from Γ by replacing two arrowheads supported by v_0 (say, L_1 and L_2) by a single one, denoted by L_s, with multiplicity defined as

$$n_s := n_2\beta_1 + n_1\beta_2$$

and the near weight $\beta_s := \beta_1\beta_2$.



LEMMA 3.24. *Let* Γ' *be a squeeze of arrowheads* L_1 *and* L_2 *from* Γ. *If* Γ *satisfies the assumptions* (a)–(c) *of the proposition, then so does* Γ'. *Moreover, the rational linking matrix of* Γ *is a direct sum of the linking matrix of* Γ' *and a negative definite one-dimensional matrix.*

Proof. As for the first part, we observe that β_s and n_s were chosen in such a way that all multiplicities of vertices are preserved. Moreover, by construction, we have

$$\begin{aligned} &\mathrm{lk}_{\Gamma'}(L_i, L_j) = \mathrm{lk}_\Gamma(L_i, L_j) \quad \text{if } \{i,j\}\cap\{1,2\} = \emptyset \quad \text{and} \quad i \neq j, \\ &\mathrm{lk}_{\Gamma'}(n_s L_s, L_j) = \mathrm{lk}_\Gamma(n_1L_1 + n_2L_2, L_j) \quad \text{if } j \geqslant 3. \end{aligned} \tag{3.11}$$

We claim that $\mathrm{lk}_{\Gamma'}(L_j, L_j) = \mathrm{lk}_\Gamma(L_j, L_j)$ for $j \geqslant 3$. Indeed, this follows from (3.11) and (3.8) applied for both graphs. Now, let us define,

$$\Lambda_1 = \beta_1 L_1 - \beta_2 L_2 \quad \text{and} \quad \Lambda_2 = xL_1 + yL_2,$$

where the rational numbers x and y will be determined later. By definition,

$$\operatorname{lk}_\Gamma(\Lambda_1, L_j) = 0 \quad \text{for any } j \geqslant 3.$$

The self-linking of Λ_1 is equal to

$$\operatorname{lk}_\Gamma(\Lambda_1, \Lambda_1) = \beta_1^2 \operatorname{lk}_\Gamma(L_1, L_1) + \beta_2^2 \operatorname{lk}_\Gamma(L_2, L_2) - 2\alpha\beta\beta_1\beta_2 \dots \beta_k.$$

If $\alpha > 0$, then the above expression is negative, because $\operatorname{lk}_\Gamma(L_1, L_1)$ and $\operatorname{lk}_\Gamma(L_2, L_2)$ are negative by Lemma 3.19. If $\alpha < 0$, we use (3.10) to show that $\operatorname{lk}_\Gamma(\Lambda_1, \Lambda_1) = -M_{v_0}(\beta_1/n_1 + \beta_2/n_2) < 0$, because the multiplicity of M_{v_0} is positive. Hence, in all cases, $\operatorname{lk}_\Gamma(\Lambda_1, \Lambda_1) < 0$.

Since $\operatorname{lk}_\Gamma(\Lambda_1, \Lambda_1) < 0$ and $\operatorname{lk}_\Gamma(L_1, L_2) \neq 0$, there exist x and y such that $\operatorname{lk}_\Gamma(\Lambda_2, \Lambda_1) = 0$ and Λ_1, Λ_2 are linearly independent. Such x and y are determined up to a multiplicative constant. To choose it, observe that

$$\operatorname{lk}_\Gamma(\Lambda_2, L_j) = x \operatorname{lk}_\Gamma(L_1, L_j) + y \operatorname{lk}_\Gamma(L_2, L_j) = \left(x + y\frac{\beta_1}{\beta_2}\right) \operatorname{lk}_\Gamma(L_1, L_j)$$

and $\operatorname{lk}_{\Gamma'}(L_s, L_j) = (1/\beta_2) \operatorname{lk}_\Gamma(L_1, L_j)$. We normalize the rational numbers x and y so that $x + y(\beta_1/\beta_2) = (1/\beta_2)$. Then, we have for all $j \geqslant 3$

$$\operatorname{lk}_\Gamma(\Lambda_2, L_j) = \operatorname{lk}_{\Gamma'}(L_s, L_j). \tag{3.12}$$

Finally, we show that $\operatorname{lk}_\Gamma(\Lambda_2, \Lambda_2) = \operatorname{lk}_{\Gamma'}(L_s, L_s)$. This is done as follows. First, on Γ we have the relation $n_1 L_1 + n_2 L_2 + \sum n_j L_j = 0$, which can be rewritten as

$$\lambda_1 \Lambda_1 + \lambda_2 \Lambda_2 + \sum_{j \geqslant 3} n_j L_j = 0$$

for some λ_1 and λ_2. On the other hand, on Γ' we have $n_s L_s + \sum_{j \geqslant 3} n_j L_j = 0$. Now, taking the linking numbers with L_r, for some $r \geqslant 3$, we obtain

$$0 = \sum_{j \geqslant 3} n_r \operatorname{lk}_\Gamma(L_r, L_j) + \lambda_2 \operatorname{lk}_\Gamma(L_r, \Lambda_2) = \sum_{j \geqslant 3} n_r \operatorname{lk}_{\Gamma'}(L_r, L_j) + n_s \operatorname{lk}_{\Gamma'}(L_r, L_s).$$

Now, by (3.11), since $r \geqslant 3$ the above equation simplifies to

$$\lambda_2 \operatorname{lk}_\Gamma(L_r, \Lambda_2) = n_s \operatorname{lk}_{\Gamma'}(L_r, L_s).$$

From (3.12) and $\operatorname{lk}_{\Gamma'}(L_r, L_s) \neq 0$, it follows that $n_s = \lambda_2$. But then we have

$$\operatorname{lk}_\Gamma(\Lambda_2, \lambda_2 \Lambda_2) = -\sum_{j \geqslant 3} \operatorname{lk}_\Gamma(\Lambda_2, n_j L_j) = -\sum_{j \geqslant 3} \operatorname{lk}_{\Gamma'}(L_s, n_j L_j) = \operatorname{lk}_{\Gamma'}(L_s, n_s L_s).$$

As $n_s = \lambda_2$, we conclude that $\operatorname{lk}(\Lambda_2, \Lambda_2) = \operatorname{lk}(L_s, L_s)$. Hence the linking form on Γ restricted to $\Lambda_2, L_3, \dots, L_n$ is the same as the linking form on Γ' written in basis $L_s, L_3, \dots, L_n$, while the element Λ_1 splits out completely as an orthogonal summand. □

Finishing the proof of Proposition 3.17. By applying collapses and squeezes to Γ, we end up with a diagram, for which no further collapse or squeeze is possible. This diagram has one or two arrowheads and we conclude the proof by Corollary 3.20. □

REMARK 3.25. If we assume that the multiplicities of nodes of Γ are only non-negative (not just positive), we can still prove semidefiniteness of the linking matrix, possibly with higher dimensional null-space. We omit the details.

4. *Semicontinuity results*

Now, we are ready to prove various semicontinuity results. In Section 4.1, we recover (in a slightly weaker form) the classical semicontinuity results valid in the local case of algebraic plane curve singularities (proved by Varchenko [**40**], see also [**37**]). Next, in Section 4.2, we analyse the behaviour of the spectrum under a degeneration of affine plane curves in the spirit of [**28**]. Finally, we consider an affine plane curve, and we relate its spectrum at infinity with the spectrum of its local singularities, see Section 4.3. This type of comparison is unknown in Hodge theory.

4.1. *Semicontinuity of the local singularity spectrum*

Recall that in the local case $\mathrm{Sp}_{\mathrm{MHS}} = \mathrm{Sp}_{\mathrm{HVS}}$ (cf. 2.5), which will be denoted just by Sp.

Let us consider now the following situation. Let $f_t(x, y)$ be a smooth family of holomorphic functions in two local coordinates depending on a local parameter t. Assume that $f_0(x, y) = 0$ has an isolated singularity at the origin. Let us introduce the following notation.

(1) We fix a small ball B centered at the origin such that $f_0^{-1}(0)$ is transverse to ∂B and $f_0(z)/|f_0(z)|\colon \partial B \setminus f_0^{-1}(0) \to S^1$ is a Milnor fibration.

(2) We denote by $L_0 = f_0^{-1}(0) \cap \partial B$ the link of f_0 at 0, and Sp_0 the spectrum of the link.

(3) The parameter $t \neq 0$ and $|t|$ is sufficiently small so that $f_t^{-1}(0) \cap \partial B$ is a transversal intersection, and this link is isotopic in ∂B to L_0.

(4) Set $C = f_t^{-1}(0) \cap B$.

(5) We denote by $z_1, \dots, z_k$ the singular points of C, $L_1^{\mathrm{sing}}, \dots, L_k^{\mathrm{sing}}$ the corresponding local links of these singularities, and $\mathrm{Sp}_1, \dots, \mathrm{Sp}_k$ denote the spectra of $L_1^{\mathrm{sing}}, \dots, L_k^{\mathrm{sing}}$ respectively.

Proposition 4.1. *Fix $x \in [0, 1]$ such that $e^{2\pi i x}$ is not a root of the Alexander polynomial of L_0. Then*

$$\begin{aligned} |\mathrm{Sp}_0 \cap (x, x+1)| &\geqslant \sum_j |\mathrm{Sp}_j \cap (x, x+1)|, \\ |\mathrm{Sp}_0 \setminus [x, x+1]| &\geqslant \sum_j |\mathrm{Sp}_j \setminus [x, x+1]|. \end{aligned} \tag{4.1}$$

Proof. Assume $x \neq 0, 1$. We shall prove only the first inequality, the second one is completely analogous (in Section 2.5 all inequalities are given in pairs, we use the first one to prove results about $\mathrm{Sp} \cap (x, x+1)$, the other one to prove results about $\mathrm{Sp} \setminus [x, x+1]$). As L_j^{sing} is an algebraic link, μ_j is the degree of the Alexander polynomial of L_j^{sing}. Hence, $\mu_j - \sigma_{L_j^{\mathrm{sing}}}(\zeta) + n_{L_j^{\mathrm{sing}}}(\zeta) \geqslant 2|\mathrm{Sp}_j \cap (x, x+1)|$ by Corollary 2.9.

By assumption $n_{L_0}(\zeta) = 0$. Since L_0 is also an algebraic link, $1 - \chi(C_{\mathrm{smooth}})$ is the degree of the Alexander polynomial of L_0. Thus, again by Corollary 2.9, one gets $-\sigma_{L_0}(\zeta) + (1 - \chi(C_{\mathrm{smooth}})) = 2|\mathrm{Sp}_0 \cap (x, x+1)|$. Then we conclude by the inequality (2.13).

Next, assume that $x \in \{0, 1\}$. The assumption that $e^{2\pi i x}$ is not a root of the Alexander polynomial means that L_0 is a knot and $|\mathrm{Sp}_0 \cap (0, 1)| = |\mathrm{Sp}_0 \cap (1, 2)|$ is the delta invariant δ_0. For any singularity link, hence for L_j^{sing}, $\delta_j = |\mathrm{Sp} \cap (0, 1]| \geqslant |\mathrm{Sp}_j \cap (0, 1)|$. Hence the statement follows from $\delta_0 \geqslant \sum \delta_j$. □

4.2. *Semicontinuity of spectrum at infinity of families of affine curves*

The methods described in this paper allow us also to prove the results on semicontinuity of the spectrum at infinity in the sense of Némethi and Sabbah [**28**].

Let $F_t\colon \mathbb{C}^2 \to \mathbb{C}$ be a smooth family of polynomials with a local deformation parameter t. Let Sp_t be the corresponding Hodge spectrum at infinity and Irr_t be the irregularity of the link at infinity $L^{\infty}_{t,\mathrm{reg}}$ associated with F_t.

Note that, over a small punctured disc $D^* \ni t$, the spectrum Sp_t is constant.

THEOREM 4.2. *Fix $x \in [0,1]$ such that $\{x, x+1\} \cap \mathrm{Sp}_t = \emptyset$ for $t \in D^*$. Then*

$$|\mathrm{Sp}_t \cap (x, x+1)| + \mathrm{Irr}_t \geqslant |\mathrm{Sp}_0 \cap (x, x+1)|$$

and the same statement holds for $\mathrm{Sp} \setminus [x, x+1]$ *instead of* $\mathrm{Sp} \cap (x, x+1)$.

Proof. Let us assume first that x is not an integer. We write $\zeta := e^{2\pi i x} \in S^1 \setminus \{1\}$.

Let us choose c such that $C_0 = F_0^{-1}(c)$ is smooth and regular at infinity. Furthermore, choose ξ and r_0 such that $S^3(\xi, r_0) \cap C_0$ is the regular link of F_0 at infinity, denoted by L_0. By openness of the transversality condition, there exist D, an open neighbourhood of 0, and W_c, an open neighbourhood of c, such that, for any $w \in W_c$ and $t \in D$, the intersection $F_t^{-1}(w) \cap S^3(\xi, r_0)$ is transverse and isotopic to L_0. Let us take any $t \in D^*$ and choose $w \in W_c$ such that $C_t = F_t^{-1}(w)$ is smooth and regular at infinity. Finally, choose r_t such that $L_t := S^3(\xi, r_t) \cap C_t$ is the regular link at infinity of C_t.

Since $C_t \cap B(\xi, r_0)$ is isotopic to $C_0 \cap B(\xi, r_0)$, by Corollary 2.12, we get

$$-\sigma_{L_t}(\zeta) + n_{L_t}(\zeta) + 1 - \chi(C_t \cap B(\xi, r_t)) \geqslant -\sigma_{L_0}(\zeta) + n_{L_0}(\zeta) + 1 - \chi(C_0 \cap B(\xi, r_0)).$$

By assumption, ζ is not a root of $\Delta_{\mathrm{Irr}_t}(L_t)$. Hence, applying Proposition 3.13, for F_t and F_0, we obtain

$$-\sigma_{L_t} + \deg \Delta_{\mathrm{Irr}_t}(L_t) + 2\,\mathrm{Irr}_t \geqslant -\sigma_{L_0}(\zeta) + \tilde{n}_{L_0}(\zeta) + \deg \Delta_{\mathrm{Irr}_0} + 2\,\mathrm{Irr}_0\,.$$

Then Corollary 2.9 implies

$$|\mathrm{Sp}_{\mathrm{HVS}}(L_t) \cap (x, x+1)| + \mathrm{Irr}_t \geqslant |\mathrm{Sp}_{\mathrm{HVS}}(L_0) \cap (x, x+1)| + \mathrm{Irr}_0\,.$$

Finally, Corollary 3.16 provides the result. To show the statement for $\mathrm{Sp} \setminus [x, x+1]$, we use the same argument.

If $x = 0$ (or $x = 1$) and x satisfies the assumption of the theorem, then for any $t \neq 0$, and for any $\theta > 0$ sufficiently small, we have

$$|\mathrm{Sp}_t \cap (\theta, 1+\theta)| = \mathrm{Sp}_t \cap (0,1)|.$$

On the other hand, for $\theta > 0$ sufficiently small, we have $|\mathrm{Sp}_0 \cap (\theta, 1+\theta)| \geqslant |\mathrm{Sp}_0 \cap (0,1)|$ (we have an equality if and only if $\mathrm{Sp}_0 \cap \{1\} = \emptyset$). Hence the statement of the theorem for $x = 0, 1$ follows from the statement for $x \in (0,1)$. □

4.3. *Spectrum at infinity of a singular curve*

Let $C \subset \mathbb{C}^2$ be an irreducible plane algebraic curve given by zero set of a reduced polynomial F. Let $z_1, \ldots, z_k$ be its singular points and $\mathrm{Sp}_1, \ldots, \mathrm{Sp}_k$ their (Hodge or HVS) spectra. Let Sp_∞ be the Hodge spectrum of F at infinity. Similarly, let $L^{\infty}_{\mathrm{reg}}$ be the regular link of F at infinity, $\mathrm{Sp}_{\mathrm{HVS}}(L^{\infty}_{\mathrm{reg}})$ its HVS-spectrum and Irr be as defined in (3.3).

THEOREM 4.3. *With the above notation, for all $x \in [0,1]$ such that $e^{2\pi ix}$ is not a root of the Alexander polynomial of L^∞_{reg}, we have*

$$\begin{aligned} |\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}}) \cap (x,x+1)| + \mathrm{Irr} &\geqslant \sum_j |\mathrm{Sp}_j \cap (x,x+1)|,\\ |\mathrm{Sp}_\infty \cap (x,x+1)| + \mathrm{Irr} &\geqslant \sum_j |\mathrm{Sp}_j \cap (x,x+1)|. \end{aligned} \tag{4.2}$$

Moreover, the analogous statement holds if we replace $\mathrm{Sp} \cap (x,x+1)$ *by* $\mathrm{Sp} \setminus [x,x+1]$.

In the good case, if the regular link at infinity is fibred (for example, if it is a knot), then $\mathrm{Irr} = 0$ and the second inequality of (4.2) takes the form $|\mathrm{Sp}_\infty \cap (x,x+1)| \geqslant \sum_{k=1}^n |\mathrm{Sp}_k(x,x+1)|$.

Proof. First, we assume that $x \in (0,1)$. We focus on the case $\mathrm{Sp} \cap (x,x+1)$; the case $\mathrm{Sp} \setminus [x,x+1]$ is analogous.

If C is regular at infinity, the inequality (2.13) reads as

$$-\sigma_{L^\infty_{\text{reg}}}(\zeta) + n_{L^\infty_{\text{reg}}}(\zeta) + (1 - \chi(C_{\text{smooth}})) \geqslant \sum_j (-\sigma_j(\zeta) + n_j(\zeta) + \mu_j), \tag{4.3}$$

where C_{smooth} is the smoothing of C. Since each link L^{sing}_j is algebraic, $-\sigma_j(\zeta) + n_j(\zeta) + \mu_j \geqslant 2|\mathrm{Sp}_j \cap (x,x+1)|$. On the other hand, by Proposition 3.13, we get

$$1 - \chi(C_{\text{smooth}}) = \mathrm{Irr} + \deg \Delta_{\mathrm{Irr}}. \tag{4.4}$$

By Lemma 3.8(d), $\Delta^h_{L^\infty_{\text{reg}}}$ has no roots outside the unit circle, hence Corollary 2.9 applies. Since ζ is not a root of $\Delta^h_{L^\infty_{\text{reg}}}$, $\widetilde{n}(\zeta) = 0$, hence

$$-\sigma_{L^\infty_{\text{reg}}}(\zeta) + \deg \Delta^h_{L^\infty_{\text{reg}}} = 2|\mathrm{Sp}_{L^\infty_{\text{reg}}} \cap (x,x+1)|, \tag{4.5}$$

and $n(\zeta) = \mathrm{Irr}$ (see Section 2.4). Then (4.3)–(4.5) prove the statement in this case.

If C is not regular at infinity, we argue as follows. We take an r_0 such that $C \cap S^3(\xi, r_0)$ is the link of C at infinity, denoted by L_C. Then (2.13) yields

$$-\sigma_{L_C}(\zeta) + n_{L_C}(\zeta) + (1 - \chi(C^{r_0}_{\text{smooth}})) \geqslant \sum_j (-\sigma_j(\zeta) + n_j(\zeta) + \mu_j), \tag{4.6}$$

where $C^{r_0}_{\text{smooth}} := C_\varepsilon \cap B(\xi, r_0)$ is the smoothing of C in $B(\xi, r_0)$. Here, $C_\varepsilon := F^{-1}(\varepsilon)$ is smooth and regular at infinity (for ε non-zero and sufficiently small). Moreover, we can assume that the links $C \cap S^3(\xi, r_0)$ and $C_\varepsilon \cap S^3(\xi, r_0)$ are isotopic. Let r_1 be such that $C_\varepsilon \cap S^3(\xi, r_1)$ is the regular link of F at infinity. Corollary 2.12 applied to C_ε yields

$$-\sigma_{L^\infty_{\text{reg}}}(\zeta) + n_{L^\infty_{\text{reg}}}(\zeta) - (-\sigma_{L_C}(\zeta) + n_{L_C}(\zeta)) \geqslant \chi(C_{01}), \tag{4.7}$$

where $C_{01} = C_\varepsilon \cap (B(\xi, r_1) \setminus B(\xi, r_0))$. A combination of (4.6) and (4.7) yields

$$-\sigma_{L^\infty_{\text{reg}}}(\zeta) + n_{L^\infty_{\text{reg}}}(\zeta) + (1 - \chi(C_\varepsilon)) \geqslant \sum_j (-\sigma_j(\zeta) + n_j(\zeta) + \mu_j).$$

This inequality is identical to (4.3) and we proceed further as in the previous case.

Assume that $x = 0$. Then, by the assumption, 1 is not a root of the Alexander polynomial of L^∞_{reg}, hence L^∞_{reg} is a knot (because $U_{\lambda=1}$ is trivial, but its dimension is $\nu - 1$ by Proposition 3.7). Therefore, the link at infinity is good, $\mathrm{Irr} = 0$ and $\mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}}) = \mathrm{Sp}_\infty$.

For $\theta > 0$ sufficiently small $|\mathrm{Sp}_\infty \cap (0,1)| = |\mathrm{Sp}_\infty \cap (\theta, 1+\theta)|$ (because $1 \notin \mathrm{Sp}_{\mathrm{HVS}}(L^\infty_{\text{reg}}) = \mathrm{Sp}_\infty$). On the other hand, in the local case, $|\mathrm{Sp}_j \cap (0,1)| \leqslant |\mathrm{Sp}_j \cap (\theta, 1+\theta)|$, hence the statement follows from the case $x \in (0,1)$.

The case $x = 1$ follows by the same argument with $\theta < 0$. □

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