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On the Iteration of Closed Geodesics and the Sturm Intersection Theory*

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Introduction

Although the original motivation of this paper is to be found in the Morse theory of closed geodesics, the results presented definitely belong into the realm of self-adjoint systems of ordinary differential equations. I have, therefore, written the main body of this paper purely in the context of this latter discipline, but would like to start here with a short account of the original problem as encountered in the Morse theory.

A closed geodesic g on a Riemannian manifold M is a map $g : R \to M$ of the reals into M, which satisfies the usual differential equations of a geodesic for all $x \in R$, and is periodic. The n-th iterate g^n of g is then defined in terms of g as the map

$$g^n(x) = g(nx), \qquad (n = \pm 1, \pm 2 \cdots), \ x \in R.$$

Thus g^1 represents the same geodesic as g; g^{-1} represents the oppositely oriented geodesic to g.

In [6] Morse assigns to each closed geodesic g two non-negative integers $\lambda(g)$ and $\nu(g)$; the index and nullity of the closed geodesic g respectively. ($\lambda(g)$ represents, roughly, the number of negative characteristic roots of the boundary value problem, associated to g, by Morse [6; p. 289]. $\nu(g)$, on the other hand, represents the multiplicity of the eigenvalue 0 of that same problem. In Section 1 these notions are precisely defined, for a general periodic operator L.) Our problem is to describe the nature of the sequences $\{\lambda(g^n);\ \nu(g^n)\}$ $\quad (n = 1, 2, \cdots)$.

The sequences are of interest because of the following result due to Morse:

On a compact Riemannian manifold M the number of closed geodesics with prescribed index k is greater than or equal to a topological invariant

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$R^k(M)$ of M (the k-th circular connectivity of M), provided the nullity of all closed geodesics on M is zero.

This theorem then furnishes a counting of the closed geodesics on M into which the nullity and index enter vitally. Since in this counting g^n, $n > 1$, counts as a distinct geodesic from g, the study of $\lambda(g^n)$, $\nu(g^n)$ is of some interest for purely computational reasons. On the other hand, if we consider g^n in some sense equivalent to g, which is usual when we speak of closed geodesics intuitively, the redundancy of the counting has to be adjusted, and this can only be done by setting bounds to the complexity of the sequence $\{\lambda(g^n)\}$.

The results of Section 1, when translated into this context can be summarized in the following theorems:

THEOREM A. *Every closed geodesic g determines non-negative integer-valued functions Λ_g and N_g on the unit circle, $|z| = 1$, such that*

$$\lambda(g^n) = \Sigma \Lambda_g(\omega),$$
$$\nu(g^n) = \Sigma N_g(\omega)$$

where ω ranges over the n-th roots of $+1$ or -1, depending on whether g is orientable or not.

Remark: g is orientable if the orientation of a coordinate system on M is not changed by parallel translation along a fundamental period of g. Otherwise g is non-orientable.

THEOREM B. *The function N_g [Λ_g] of Theorem A is in turn completely determined (determined up to an additive constant) by a certain $2n \times 2n$ matrix P_g, the Poincaré matrix of g. Here $n + 1 = dim.\ M$.*

The manner in which P_g determines Λ_g is unfortunately rather complicated and will not be discussed in detail here (see Theorems III and IV, Section 1). We will only list a few of the properties of these functions which follow from these theorems.

THEOREM C. *The functions Λ_g* and *N_g have, among others, the following properties:*

1. *$N_g(z)$ = dimension of null-space of $\{P_g - zI\}$ ($|z| = 1$).*
2. *$\Lambda_g(z)$ is constant at points at which $N_g(z) = 0$, and the jump of Λ_g at z is always bounded in absolute value by $N_g(z)$.*
3. *$\Lambda_g(z^*) = \Lambda_g(z)$; $N_g(z^*) = N_g(z)$*
   (z^* denotes complex conjugate of z.)

A real number ϱ, such that $N_g(e^{2\pi i\varrho}) \neq 0$ is called an imaginary (Poincaré) exponent of g. By part 1 of Theorem C, g has at most $2n$ distinct imaginary exponents mod 1. The following corollary to Theorems A and C

is now an immediate consequence of the definition of a Riemann integral.

COROLLARY I. $\lambda^*(g) = \lim_{n\to\infty} \frac{\lambda(g^n)}{n}$ *is given by the formula*

$$\lambda^*(g) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_g(e^{i\theta})\, d\theta = a_0 + \Sigma a_i \varrho_i$$

where the a_i *are integers, and* $\{\varrho_i\}$ *is a complete system of imaginary Poincaré exponents of* g *mod* 1.

$$\nu^*(g) = \lim_{n\to\infty} \frac{\nu(g^n)}{n} = \frac{1}{2\pi}\int_0^{2\pi} N_g(e^{i\theta})\, d\theta = 0.$$

If the imaginary exponents of g are all irrational, then the formula of Theorem A can be given more explicitly, as can be easily verified and is expressed in

COROLLARY II. *If the imaginary exponents* $\{\varrho_i\}$ *of* g *are all irrational, then*

$$\lambda(g^n) = a_{-1} + a_0 n + \Sigma\, a_i[n\varrho_i]$$

where ϱ_i *is again a complete system of imaginary exponents, and the* a_{-1}, a_i $(i = 0, \cdots)$ *are integers, while* $[x]$ *denotes the greatest integer* $\leqq x$.

In this case $\nu(g^n)$ is clearly 0 for all n.

These results augment, partly overlap, and partly generalize results due to Hedlund [4], and announced results of Morse and Pitcher [7]. Hedlund restricts his discussion to dim. $M = 2$, and the "nondegenerate case" i.e., $\nu(g^n) = 0$ for all n. Our formula does not add to his rather complete discussion of this case, except for the following remark.

It follows at once from Corollary I, that since dim. $M = 2$,

$$\lambda^*(g) = a_0 + a_1\varrho + a_2(-\varrho) = a_0 + (a_1 - a_2)\varrho,$$

or

$$\lambda^*(g) = a_0$$

depending on whether g has the imaginary exponents ϱ and $-\varrho$, or has none at all. Hence if $\lambda^*(g)$ is rational, but not an integer, then g must have an imaginary exponent ϱ, which is also rational and not an integer, and $\nu(g^n)$ must be different from zero for some values of n. Hence the conditions of theorem IV, case 1 in [4] are incompatible.

The Proceedings note of Morse and Pitcher, treating the n-dimensional nondegenerate case, announces among other results the existence of the limit $\lambda^*(g)$ and that $\lambda(g^n)$ is determined by P_g and $\lambda(g)$. The formula of Corollary I is foreshadowed there by the proposition that if all the imaginary exponents of g are rational then $\lambda^*(g)$ is also.

That these authors restricted themselves exclusively to the nondegenerate case, $\nu(g^n) = 0$, is explained by the fact that in the classical Morse theory $\lambda(g)$ has "local topological significance" only if $\nu(g) = 0$. However, if g is contained in a compact manifold of closed geodesics, say σ, and if dim. $\sigma = \nu(g)$ for all $g \in \sigma$, then $\lambda(g)$ is again topologically significant, as is shown in [2]. I was therefore interested in freeing the discussion from the hypothesis of nondegeneracy.

Theorem A is essentially proved (modulo the translation into differential geometry) in Theorem I of Section 1. The main idea of this simple proof is the transition from the given problem which can be, and usually is, treated over the real numbers, to a Hermitian problem. Theorem B on the other hand I have found quite difficult, and it is to the proof of this theorem that the bulk of this paper is devoted. To solve it we introduce, what I will call, the "Sturm intersection theory". Briefly stated, its main idea is the following: From the problem $Ly = \lambda y$, where L is a self-adjoint differential operator of the second order, we go over to the standardly associated first order matrix equation

$$\frac{dX_\lambda}{dt} = A_\lambda X_\lambda .$$

The fundamental solution of this equation, normalized by the condition $X_\lambda(0) = I$ (all real λ), then defines a map X^L of the t, λ-plane into a certain subgroup $\mathfrak{H}$ of the full linear group. On the other hand, a self-adjoint boundary condition B is seen to define a cycle γ_B in $\mathfrak{H}$. It is then shown that the spectrum of $Ly = \lambda y$ subject to B at $t = 0$ and $t = a > 0$, corresponds precisely to the intersection (in a topological sense) of the curve $\lambda \to X^L(a, \lambda)$ with γ_B. Once this is established, standard deformation arguments yield Theorem B.

This "Sturm intersection theory" is, I hope, of some independent interest, and I have therefore developed it here for general self-adjoint (or Hermitian) boundary-conditions, rather than for the special case needed in Theorem B. Having done this I also indicate, in Sections 8 and 9, how it can be used to prove (a) the general existence theorem of an infinite but discrete spectrum for a regular problem, (b) the continuous dependence of the spectrum on the boundary conditions, (c) a generalization of the oscillation theorem of Sturm and the focal point theorem of Morse. All these theorems seem to be consequences of the fact that the intersection of two cycles, of which one is homologous to zero, is again homologous to zero.

The proof that the topological intersections of the curve: $\lambda \to X_\lambda(a)$, with γ_B correspond to the spectrum of $Ly = \lambda y$ is complicated by the fact that some points of the spectrum may have a multiplicity greater than one.

This phenomenon reflects itself in the carrier B^0 of γ_B. In general, this set is not a submanifold of $\mathfrak{H}$. To construct γ_B, we are therefore led to introduce the "resolution of B^0" in Section 4. This is a submanifold $B^{(1)}$ in the cartesian product of $\mathfrak{H}$ with a complex projective space $G^{(1)}$. Under the canonical projection $f^{(1)}: \mathfrak{H} \times G^{(1)} \to \mathfrak{H}$, $B^{(1)}$ then maps onto B^0, and it is the image of the fundamental class of $B^{(1)}$ under $f^{(1)}$ which is defined as γ_B.

1. Periodic Hermitian Systems

Let E_1 be the unitary space of complex n-tuples $x = \{x_1, \cdots, x_n\}$, equipped with the Hermitian inner product

$$(x, y) = \sum_1^n x_i y_i^*.$$

Here the star of a complex number denotes its complex conjugate. In general, if $p: E_1 \to E_1$ is a linear transformation p^* shall denote its adjoint. Thus

$$(px, y) = (x, p^*y) \text{ all } x, y \in E_1.$$

In any unitary cordinate system on E_1, the matrix of p^* is therefore the complex conjugate transpose of the matrix of p. Throughout we will denote by $\nu(p)$ the dimension of the null space of a linear transformation p. S^1 shall stand for the circle $|z| = 1$, and R shall denote the real line.

Set $E = E_1 \oplus E_1$. The inner product in E is derived from the one in E_1 by the formula

$$(u, v) = (x_1, x_2) + (y_1, y_2)$$

if $u = \{x_1, y_1\}$, $v = \{x_2, y_2\}$, $x_i, y_i \in E_1$ $(i = 1, 2)$.

DEFINITION 1.1: *A Hermitian periodic operator L shall be defined as a second order differential operator on the vector functions* $t \to y(t) \in E_{(1)}$ $(t \in R)$ *of class* C'', *given by*

$$Ly \equiv -\{py' + qy\}' + q^*y' + ry. \tag{1.1}$$

Here, (as throughout the paper) the prime denotes differentiation with respect to t and $p(t)$, $q(t)$ *and* $r(t)$ *are continuously differentiable matrix functions defined on R, which satisfy the following conditions:*

(*a*) $p(t)$, $q(t)$ *and* $r(t)$ *are of period* 1,

(*b*) $p^*(t) = p(t)$; $r^*(t) = r(t)$ (*all t*),

(*c*) $(p(t)u, u) > 0$ *if* $u \neq 0$ (all t) (*i.e., p is positive definite — also denoted by* $p > 0$). *We reserve the symbol L for operators of the above type.*

Let

$$J\{x, y\} = \{-y, x\}, \qquad x, y \in E_1.$$

If y is a vector function from R to E_1 of class C'' and L is given, u_y shall always stand for the map of R into E defined in terms of the coefficients of L by

$$u_y(t) = \{y(t), p(t)y'(t) + q(t)y(t)\}. \tag{1.2}$$

The Hermitian character of L is then brought out by the following well-known identity, valid if $y(t)$, $x(t)$ are of class C'' :

$$(Ly, x) - (y, Lx) = (Ju_y, u_x)' . \tag{1.3}$$

We will study the eigenvalue problem

$$Ly = \lambda y \tag{1.4}$$

subject to a family of boundary conditions $[n, z]$ ($n = \pm 1, \pm 2$ etc; $z \in S^1$.) Here a function $y(t) \in E_1$ of class C'' satisfies $[n, z]$ if and only if

$$u_y(t + n) = zu_y(t) \qquad \text{all } t \in R.$$

With (1.4) subject to any boundary condition B we associate the spectral multiplicity function Θ^L_B. Θ^L_B is defined on the complex λ-plane by the condition

$$\Theta^L_B(\lambda) = \text{number of linearly independent solutions of } Ly = \lambda y \text{ subject to } B. \tag{1.5}$$

The points at which $\Theta^L_B(\lambda) \neq 0$ are called the spectrum of (1.4) subject to B.

Since each condition B of the family $\{[n, z]\}$ described above is self-adjoint, or Hermitian as we will call it, (1.4) subject to $B \in \{[n, z]\}$ has the following well-known properties.

PROPOSITION 1.1. *If $B \in \{[n, z]\}$, then $\Theta^L_B(\lambda) = 0$ if λ is not real. On the real axis Θ^L_B is a finite valued function which is different from zero only on an infinite discrete subset of R which is bounded from below.*

The following continuity theorem also holds. It can be found in the literature (see for example, [6]; p. 91) but will also follow from subsequent arguments in this paper (see Section 7).

If τ is an interval on R which is bounded from above, we set

$$[\Theta^L_B : \tau] = \sum_{\lambda \in \tau} \Theta^L_B(\lambda). \tag{1.6}$$

By proposition (1.1) this number is always finite. If the end-points of τ do not lie on the spectrum of (1.4) subject to B, τ shall be called admissible with respect to Θ^L_B.

The continuity theorem which we are after can now be stated as follows:

PROPOSITION 1.2. *If τ is an admissible interval with respect to $\Theta^L_{[n,z]}$, then there exists a neighborhood U of z on S^1 such that*

(1) *τ is admissible with respect to $\Theta^L_{[n,z]}$ for all $z \in U$,*

(2) $[\Theta^L_{[n,z]}: \tau] = [\Theta^L_{[n,w]}: \tau]$, $\quad z, w \in U$.

Having put these preliminaries down we can proceed to our first problem, which is to find the relations between the functions $\Theta^L_{[n,z]}$ when L is kept fixed and n, z are allowed to range over their respective domains. To simplify the notation we therefore drop the superscript L, and write Θ_z for $\Theta_{[1,z]}$. We then have the following Fourier Theorem, which is the basis of Theorem A of the introduction.

THEOREM I. *The spectral multiplicity functions $\Theta_{[n,z]}$ of (1.4) subject to $[n, z]$ are given by $\Theta_z = \Theta_{[1,z]}$, according to the formula*

$$\Theta[n, z] = \sum_{\omega} \Theta_{\omega} \tag{1.7}$$

where ω ranges over the n-th roots of z.

Proof: Every $y(t)$ subject to $[n, z]$ admits the unique Fourier expansion

$$y(t) = \sum \omega y_{\omega}(t)$$

where ω ranges over the n-th roots of z, and $y_{\omega}(t) = \dfrac{1}{n}\sum_{0}^{n-1} \omega^{-s} y(t+s-1)$.

Now

$$y_{\omega}(t+1) = \frac{1}{n}\sum_{0}^{n-1} \omega^{-s} y(t+s). \tag{1.8}$$

If we set $k = s + 1$, the right hand side of (1.8) becomes

$$\frac{\omega}{n}\left\{\sum_{1}^{n-1} \omega^{-k} y(t+k-1) + \omega^{-n} y(t+n-1)\right\},$$

$$\frac{\omega}{n}\left\{\sum_{1}^{n-1} \omega^{-k} y(t+k-1) + y(t-1)\right\}$$

since ω is an n-th root of z and $y(t+n) = zy(t)$. Hence $y_{\omega}(t+1) = \omega y_{\omega}(t)$; i.e., $y_{\omega}(t)$ satisfies $[1, \omega]$. Since L commutes with translations of length 1, Ly_{ω} again satisfies $[1, \omega]$. Hence y is a solution of (1.4) subject to $[n, z]$ if and only if each component $y_{\omega}(t)$ is subject to $[1, \omega]$. The theorem now follows.

Following Morse, the index and nullity (denoted by Λ^L_B and N^L_B, respectively) of the problem (1.4), subject to a Hermitian boundary condition B, is defined by

$$\Lambda^L_B = [\Theta^L_B: R^-], \qquad R^- = \{\lambda \mid \lambda < 0\}$$
$$N^L_B = \Theta^L_B(0).$$

Clearly Theorem I has the following

COROLLARY. *The index and nullity of $Ly = \lambda y$ subject to $[n, z]$ are given by*

$$\Lambda^L_{[n,z]} = \sum_\omega \Lambda^L_{[1,\omega]},$$

$$N^L_{[n,z]} = \sum_\omega N^L_{[1,\omega]}$$

where ω ranges over the n-th roots of z as before.

The sequence $\{\lambda(g^n)\}$ of the introduction is now seen easily to be identical with the sequence $\Lambda^L_{[n,1]}$ or $\Lambda^L_{[n,-1]}$ of the Hermitian operator L associated with g by Morse via the second variation. (The two cases correspond to the orientability of g.) We embark therefore on a detailed study of $\Lambda(z) = \Lambda^L_{[1,z]}$, $N(z) = N^L_{[1,z]}$ as functions on S^1.

PROPOSITION 1.3. *The functions $\Lambda(z)$, $N(z)$ defined above have the following properties:*

(1) *Both functions are non-negative integer valued functions on S^1.*

(2) *The inequality*

$$\lim_{z \to z_0^\pm} \Lambda(z) \geqq \Lambda(z_0) \tag{1.9}$$

holds.

(3) *$N(z) = 0$ except at at most $2n$ points $\{z_1, \cdots, z_r\} \in S^1$.*

(4) *The equality holds in (1.9) if z_0 is not one of these points $\{z_i\}$ (referred to as the Poincaré points of L). Furthermore, Λ is constant in the vicinity of points other than the Poincaré points and the jump of Λ at any point $z \in S^1$ is bounded in absolute value by $N(z)$.*

(5) *The sum $\sum N\{z_i\}$ extended over the Poincaré points of L does not exceed $2n$.*

(6) *If L is real, then $\Lambda(z) = \Lambda(z^*)$, $N(z) = N(z^*)$.*

Proof: (1) and (6) are clear from the definition of the functions. (2) is a consequence of the continuity theorem — Proposition 1.2. For, if 0 is not in the spectrum of (1.4) subject to $[1, z_0]$, then, by that proposition, Λ is constant in some vicinity of z_0. If 0 is in that spectrum, we can, because of the discreteness of the spectrum, define $\Lambda(z_0)$ by

$$\Lambda(z_0) = [\Theta^L_{[1,z_0]}\colon R^-_\varepsilon],\ R^-_\varepsilon = \{\lambda \mid \lambda \leqq -\varepsilon\},$$

for sufficiently small $\varepsilon > 0$. Let $\eta = \{\lambda \mid -\varepsilon \leqq \lambda < 0\}$. Then, since R^-_ε is admissible with respect to $\Theta^L_{[1,z_0]}$,

$$\Lambda(z) = \Lambda(z_0) + [\Theta^L_{[1,z]}\colon \eta]$$

in some vicinity of z_0. Since the last term on the right is only capable of positive values, property (2) follows. This argument clearly establishes (4) as well, since $N(z_0)$ can be defined by

$$N(z_0) = [\Theta^H_{[1,z_0]}\colon \delta] \text{ with } \delta = \{\lambda \mid -\varepsilon \leqq \lambda \leqq \varepsilon\}$$

for sufficiently small ε, and hence

$$N(z_0) \geqq [\Theta^L_{[1,z]}\colon \eta].$$

To establish (3), we make the usual transformation of the second order problem (1.4) over E_1 into a first order problem over E. As is easily checked, the equation

$$Ly = \lambda y$$

is quite equivalent to

$$u_v'(t) = A_\lambda(t)\, u_v(t) \tag{1.10}$$

where $A_\lambda(t)$ is the $2n \times 2n$ matrix constructed from the coefficients of L in the following manner:

$$A_\lambda(t) = \begin{cases} -p^{-1}(t)\,q(t), & p^{-1}(t) \\ r(t) - q^*(t)\,p(t)q(t) - \lambda\,, & q^*(t)p^{-1}(t). \end{cases} \tag{1.11}$$

From this point of view $\Lambda(z)$ and $N(z)$ are reinterpreted as:

$\Lambda(z)$ $[N(z)]$ = number of linearly independent solutions of $u'(t) = A_\lambda(t)u(t)$ with $\lambda < 0$ $[\lambda = 0]$, subject to $u(t+1) = zu(t)$.

Let $X_\lambda(t)$ be the fundamental matrix solution of (1.10), normalized by the condition

$$X_\lambda(0) = I \text{ (identity on } E) \text{ for all } \lambda; \tag{1.12}$$

then every solution of (1.10) is of the form

$$u(t) = X_\lambda(t)\,v$$

where v is a constant vector of E. Hence, recalling our convention that $\nu\{\ \}$ denotes the complex dimension of the null space of the matrix $\{\ \}$, we obtain the formula

$$\Theta^L_{[1,z]} = \nu\{X_\lambda(1) - zI\} \tag{1.13}$$

and in particular

$$N(z) = \nu\{X_0(1) - zI\}. \tag{1.14}$$

However, this interpretation of $N(z)$ clearly establishes properties (3) and (5) of Proposition 1.2.

The matrix $X_0(1)$ was originally introduced by Poincaré in his study of periodic systems. We denote it by P^L or just P in this paper and will refer to it as the Poincaré matrix of L. Thus the Poincaré points of L are precisely those characteristic roots of P which are of absolute value 1.

By (1.14) P completely determines $N(z)$. In view of Proposition 1.2, it therefore seriously restricts the behavior of Λ on S^1. Actually, however, the following stronger proposition holds:

THEOREM II. *The Poincaré matrix P^L of L determines $N(z)$ completely and specifies $\Lambda(z)$ up to an additive constant.*

It is this theorem which yields Theorem B of the introduction. The precise mechanism by which P determines Λ is described below.

For real numbers $\{t, \theta\}$, set

$$P(t, \theta; z) = R(t)P - ze^{i\theta}I$$

where
$$R(t) = \exp. (J^*t) = \begin{pmatrix} \cos t, & \sin t \\ -\sin t, & \cos t \end{pmatrix}$$

(each entry representing an $n \times n$ diagonal matrix).

A rectangle $\square$ of the t,θ-plane shall be called admissible with respect to P at z_0 if $\square$ is given by $-\delta \leqq t \leqq 0$, $-\varepsilon \leqq \theta \leqq \varepsilon$ (δ, $\varepsilon > 0$) and $P(t, \theta; z_0)$ is nonsingular:

(a) on the side $t = -\delta$,

(b) on the side $t = 0$ with the possible exception of the origin $(0, 0)$.

With an admissible $\square$ we associate the numbers

$$S_P^+(z_0; \square) = \sum_{-\delta \leqq t \leqq 0} \nu\{P(t, \varepsilon; z_0)\},$$

$$S_P^-(z_0; \square) = \sum_{-\delta \leqq t \leqq 0} \nu\{P(t, -\varepsilon; z_0)\}.$$

Theorem II is now made precise by the following result.

THEOREM III. *The numbers $S_P^\pm(z_0; \square)$ are independent of the admissible rectangle $\square$ chosen. They are, therefore, completely determined by P and can be denoted by $S_P^\pm(z_0)$. With this understanding, the function $\lambda(z)$ satisfies the condition*

$$\lim_{\theta \to 0\pm} \Lambda(ze^{i\theta}) = \Lambda(z) + S_P^\pm(z). \tag{1.15}$$

We call the numbers $S_P^\pm(z)$ the splitting numbers of P at z. Under certain conditions on P and z they can be computed by an infinitesimal method. Precisely, the following is true:

THEOREM IV. *If the Hermitian form $(\sqrt{-1}\,J u, u)$ restricted to the null-space of $\{P - zI\}$, is nondegenerate, then $S_P^+(z)$ and $S_P^-(z)$ are equal to the number of negative and positive characteristic roots of that restricted form, respectively. In particular, if the null-space of $\{P - zI\}$ is trivial, $S_P^\pm(z) = 0$.*

As remarked in the introduction, the proof of these theorems seems to necessitate the "Sturm intersection theory" which we develop in the next few sections. The theorems are established in Section 6.

We close this section with two examples which will be put to use in a subsequent paper on the closed geodesics of homogeneous spaces.

EXAMPLE I. $P = I$. In this case

$$S_P^{\pm}(z) = 0 \text{ if } z \neq 1,$$
$$S_P^{+}(1) = S_P^{-}(1) = n.$$

(Recall that the complex dimension of E is $2n$.) In this example Theorem IV is applicable since $\sqrt{-1}\, J$ is nondegenerate on E.

EXAMPLE II. $P = \begin{pmatrix} I_n, & 2\sigma I_n \\ 0\ , & I_n \end{pmatrix}$ (I_n = identity on E_1), $\sigma \neq 0$ and real.

Here

$$S_P^{\pm}(z) = 0 \qquad \text{if } z \neq 1,$$
$$S_P^{+}(1) = S_P^{-}(1) = \begin{cases} n \text{ if } \sigma > 0 \\ 0 \text{ if } \sigma < 0. \end{cases}$$

We have to use Theorem III at $z = 1$, since $(\sqrt{-1}\, Ju, u)$ is completely degenerate on the null space of $\{P - I\}$. The result stated follows fairly simply from the evaluation:

$$\text{det. } P(t, \theta, 1) = [2e^{i\theta}\{\cos\theta - \sqrt{1+\sigma^2}\cos(t+\varphi)\}]^n$$

where φ is the angle between $\pm\,\pi/2$ whose tangent is σ.

2. The Sturm Intersection Theory

For $t, \lambda \in R$ the matrix $X_\lambda(t)$ of (1.12) is easily seen to be nonsingular. The correspondence

$$(t, \lambda) \to X_\lambda(t)$$

therefore defines a map of the t, λ-plane, Δ, (λ real) into the full linear group $GL(n; C)$ [3]. We denote this map by X^L and refer to it as the map of L. In this context a well-known existence theorem yields

PROPOSITION 2.1. *The map $X^L : \Delta \to GL(n; C)$ is differentiable and, for fixed t, real analytic in λ.*

The following proposition is vital for our purposes.

PROPOSITION 2.2. *If L is Hermitian, the image of Δ under X^L is contained in the Lie sub-group $\mathfrak{H}$ of $GL(n; C)$ which is characterized by the condition*

$$X^* J X = J. \tag{2.1}$$

Proof: By definition X_λ satisfies, as a function of t, the differential equation

$$X_\lambda' = A_\lambda X_\lambda \tag{2.2}$$

with the initial condition

$$X_\lambda(0) = I. \tag{2.3}$$

Hence

$$\{X_\lambda^* J X_\lambda\}' = X_\lambda^* \{A_\lambda^* J + J A_\lambda\} X_\lambda. \tag{2.4}$$

However, it is easily checked that the Hermitian character of L implies that the relation

$$J A_\lambda + A_\lambda^* J = 0 \tag{2.5}$$

holds identically in λ and t (λ real from now on!). Hence $X_\lambda^* J X_\lambda$ is independent of t and reduces to J at $t = 0$. This proves the proposition.

In $\mathfrak{H}$ let $B^0(z)$ be the subset characterized by

$$\nu\{X - zI\} \neq 0. \tag{2.6}$$

It is then evident in view of (1.13) that the spectrum of $Ly = \lambda y$ subject to $[1, z]$ corresponds precisely to the intersections of the curve

$$\lambda \to X_\lambda^L(1)$$

with the set $B^0(z)$ in $\mathfrak{H}$. It is the whole purpose of the next two sections to establish the spectral theory of L as an intersection theory in the topological sense. In particular, we wish to show that if appropriately oriented, the set $B^0(z)$ carries a locally finite cycle $\gamma_{B(z)}$ such that for any compact admissible interval τ on the λ-axis,

$$[\Theta^L_{[1,z]} \colon \tau] = [\gamma_{B(z)} \colon \tau]_{\mathfrak{H}} \tag{2.7}$$

where the right-hand side of (2.7) is to be interpreted as the topological intersection number of the cycle $\gamma_{B(z)}$ and the positively oriented curve

$$\lambda \to X_\lambda^L(1), \qquad \lambda \in \tau,$$

in $\mathfrak{H}$.

Since the proposition turns out to be true for any Hermitian boundary condition imposed at $t = 0$, and since $t = a \neq 0$, we will give a precise formulation of our results in the general case. For this purpose we use a parametric representation of the most general Hermitian boundary conditions.

DEFINITION 2.1. *A Hermitian pair $\{M, N\}$ is an ordered pair of linear maps $\{M, N\}$ of E into itself having the following properties*:

$$Mv = Nv = 0 \Rightarrow v = 0, \tag{2.8}$$

$$M^* J M = N^* J N. \tag{2.9}$$

Two such pairs $\{M, N\}$ and $\{M_1, N_1\}$ shall be called equivalent if there exists a nonsingular T such that

$$\{MT, NT\} = \{M_1, N_1\}.$$

An equivalence class of Hermitian pairs shall be called a Hermitian boundary condition.

DEFINITION 2.2. *We say $u(t)$ is subject to the Hermitian boundary condition $B \supset \{M, N\}$* at $t = 0$, *and $t = a \neq 0$, if there exists a $v \in E$ such that*

$$u(0) = Mv, \; u(a) = Nv. \tag{2.10}$$

We notice that in this convention all the conditions $[n, z]$ are represented by the pair $\{I, zI\}$. They are, however, imposed at different places, depending on n.

DEFINITION 2.3. *If B is a Hermitian boundary condition represented by $\{M, N\}$, the carrier of B is the subset B^0 of $\mathfrak{H}$ defined by*

$$B^0 = \{X \in \mathfrak{H} \mid \nu\{XM - N\} \neq 0\}. \tag{2.11}$$

We are now in a position to state Theorem V which constitutes the central result of this paper. We remark that the assumed periodicity of L as defined in Section 1 is no essential restriction, since every regular operator on a finite interval of the t-axis can be extended to a periodic one.

THEOREM V. *If B is a self-adjoint boundary condition, the set B^0 carries a $(2n)^2 - 1$ dimensional, integral, locally finite cycle γ_B. γ_B has the property that if Θ_B^L is the spectral multiplicity function of the problem*

$$Ly = \lambda y$$

subject to B at $t = 0$ and $t = a$, and if τ is a compact admissible interval with respect to Θ_B^L, then

$$[\Theta_B^L \colon \tau] = [\gamma_B \colon \tau]_{\mathfrak{H}}$$

where the right hand side represents the topological intersection number of γ_B and the curve

$$\lambda \to X_\lambda^L(a), \qquad \lambda \in \tau.$$

The theorem is proved in Sections 4 and 5.

We digress here for a moment to make a remark about the set $\mathfrak{B}$, of all Hermitian boundary conditions. Every self-adjoint boundary condition (as defined above) defines a complex $2n$-dimensional subspace of $E \oplus E$. Namely, if B is represented by $\{M, N\}$ the plane associated with B

is the image of the map $E \to E \oplus E$ defined by

$$v \to \{Mv, Nv\}.$$

Condition 2.8 now implies that this map is an isomorphism. This correspondence is easily seen to be 1 to 1 and by it the set $\mathfrak{B}$ of all Hermitian boundary conditions is imbedded in the space G of complex $2n$-planes in complex $4n$-space. This Grassmann variety G carries the usual topology and we can therefore topologize $\mathfrak{B}$ by virtue of the imbedding $\mathfrak{B} \subset G$.

THEOREM VI. *The set $\mathfrak{B}$ of Hermitian boundary conditions (in the topology described above) is homeomorphic to $U(2n)$, the group of isometries of E.*

Proof: On $E \oplus E$ define the Hermitian form $\varphi(\{u_1, v_1\}, \{u_2, v_2\}) = (\sqrt{-1}\, Ju_1, u_2) - (\sqrt{-1}\, Jv_1, v_2)$. One checks easily that $\mathfrak{B} \subset G$ consists precisely of those $2n$-planes in $E \oplus E$ on which φ completely degenerates. Now it is seen that in a suitable decomposition of $E \oplus E$ into isomorphic factors $F^{(1)} \oplus F^{(2)}$, the form φ is given by

$$\varphi(u, v) = \begin{cases} (u, v) & \text{if } u, v \in F^1 \\ 0 & \text{if } u \in F^1, v \in F^2 \\ -(u, v) & \text{if } u, v \in F^2. \end{cases}$$

Let $p^{(i)} : E \oplus E \to F^{(i)}$ $(i = 1, 2)$ be the orthogonal projections on $F^{(i)}$. Then, if $h : E \to E \oplus E$ defines a $2n$-plane on which φ completely degenerates, the correspondence

$$p^{(1)}h(v) \xrightarrow{U_h} p^{(2)}h(v), \qquad v \in E,$$

defines an isometry of $p^{(1)}h(E)$ onto $p^{(2)}h(E)$. It follows that the kernel of $p^{(1)}h$ must coincide with that of $p^{(2)}h$, whence, since h is an isomorphism, they must both be trivial. U_h therefore defines an isometry of $F^{(1)}$ onto $F^{(2)}$ which completely characterizes the image of E under h. Conversely, every isometry U of $F^{(1)}$ onto $F^{(2)}$ defines a plane in $F^{(1)} \oplus F^{(2)}$ on which φ completely degenerates by the map

$$h_U : F^{(1)} \to \{F^{(1)}, UF^{(1)}\}.$$

Remark. The group $\mathfrak{H}$ is not compact. On the other hand, the pair $\{I, X\}$ $(X \in \mathfrak{H})$ is seen to be a Hermitian pair. Furthermore the correspondence $\mathfrak{H} \to \mathfrak{B}$ given by

$$X \to \{I, X\}$$

is seen to be a homeomorphism. By this device $\mathfrak{H}$ is, therefore, imbedded in the compact space $\mathfrak{B} = U(2n)$ and thus defines a compactification of $\mathfrak{H}$.

3. ⊕ Curves on $\mathfrak{H}$.

Theorem V is easily seen to imply that if X^L is the map of L and B is a Hermitian boundary condition, then the curve

$$\lambda \to X^L_\lambda(a), \qquad a \neq 0,$$

intersects B^0 at a discrete set of λ-values. In this section we bring the a-priori reason for this. It turns out that the "direction" of the curves $\lambda \to X^L_\lambda(a)$ is strongly restricted by the form of $A_\lambda(t)$ (see (1.11)). In particular, the direction of such a curve must always lie in a certain cone K, the "⊕ cone" of $\mathfrak{H}$, while tangent directions to B^0 never intersect this cone.

Let $\mathfrak{L}$ be the *real* vector space of linear maps $A\colon E \to E$ for which

$$JA + A^*J = 0. \tag{3.1}$$

The map $A \to JA$ thus sends $\mathfrak{L}$ isomorphically onto the set H of Hermitian maps on E. We will use the word dimension from now on to mean real dimension unless specially indicated. With this understanding, the dimension of H, and therefore $\mathfrak{L}$, is easily seen to be m^2, where $m = 2n$.

The defining relations (3.1) of $\mathfrak{L}$ are obtained from the defining relations of $\mathfrak{H}$, (2.1), by differentiation at the identity. Hence $\mathfrak{L}$ can be identified with $\mathfrak{H}_I$, the tangent spaces [3], to $\mathfrak{H}$ at I. We can compare directions globally on $\mathfrak{H}$ by first left-translating them to I. In this identification of the tangent space at I, the direction of a curve $X(t) \in \mathfrak{H}$ at $t = t_0$ is therefore given by the element $X^{-1}(t_0) \cdot X'(t_0) \in \mathfrak{L}$.

The ⊕ cone K of $\mathfrak{L}$, mentioned above, is defined as the subset of $A \in \mathfrak{L}$ for which JA is positive definite. K is convex and does not contain the origin.

DEFINITION 3.1. *A smooth curve $X(t)$ on $\mathfrak{H}$ is a ⊕ curve if its direction is contained in K for all t in its range of definition.*

PROPOSITION 3.1. *If L is Hermitian and a is a non-zero real number, then the curve*

$$\lambda \to X^L(\lambda, a), \qquad -\infty < \lambda < \infty,$$

is a ⊕ curve.

PROPOSITION 3.2. *If B is a Hermitian boundary condition, the direction of any smooth curve on B^0 is never contained in K.*

Proof of Proposition 3.1: Let $(\lambda, t) \to X_\lambda(t)$ be the map of L, and consider, for fixed λ and h, the expression $X^*_\lambda J X_{\lambda+h}$ as a function of t. Differentiation yields

$$\{X^*_\lambda J X_{\lambda+h}\}' = h\{X^*_\lambda J Q X_{\lambda+h}\}, \quad Q = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix};$$

whence

$$\{X_\lambda^* J X_{\lambda+h}\}(a) - \{X_\lambda^* J X_{\lambda+h}\}(0) = h \int_0^a X_\lambda^* T X_{\lambda+h}(\alpha)\, d\alpha$$

where

$$T = JQ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Due to (1.12) the second term of the left hand side reduces to J. Due to the definition of $\mathfrak{H}$ it can therefore be replaced by $X_\lambda^* J X_\lambda(a)$. Dividing by h and passing to the limit $h \to 0$ after this substitution, yields $\left\{X_\lambda^* J \dfrac{dX_\lambda}{d\lambda}\right\}(a)$ or equivalently $J\left\{X_\lambda^{-1} \dfrac{dX_\lambda}{d\lambda}\right\}(a)$ for the left hand side. Hence we obtain the evaluation

$$J\left\{X_\lambda^{-1} \frac{dX_\lambda}{d\lambda}\right\}(a) = \int_0^a X_\lambda^* T X_\lambda(\alpha)\, d\alpha. \tag{3.2}$$

We have to show that the left hand side is positive definite. Let therefore $v \neq 0$ be an element of E. Then there exists a unique $y(t) \in E_1$ such that $u_v(t) = X_\lambda(t)v$, i.e., y solves $Ly = \lambda y$ nontrivially. Evaluating the quadratic form in v on the right we obtain $\int_0^a (y(\alpha), y(\alpha))d\alpha$; this expression is > 0 since $a \neq 0$. This proves the assertion.

We proceed now to the proof of Theorem V. Proposition 3.2 will become clear on the way.

4. The Construction of γ_B.

Let $\{M, N\} \in B$ be a fixed representation of a fixed Hermitian boundary condition B. For $X \in \mathfrak{H}$ we set $0_B(X) = \nu\{XM - N\}$. Then $B^0 = \{X \mid 0_B(X) \geqq 1\}$. If $0_B(X)$ were equal to 1 for all $X \in B^0$ Theorems III to V would be quite straightforward. However, to deal with the singularities of B^0 rigorously, I seem to need the following construction.

For each integer $k \geqq 1$, set $G^{(k)}$ equal to the Grassmann variety of complex k-planes in E. $W^{(k)}$ shall denote the product space $\mathfrak{H} \times G^{(k)}$. Set $f^{(k)} : W^{(k)} \to \mathfrak{H}$ equal to the canonical projection of $\mathfrak{H} \times G^{(k)}$ onto its left factor. For $p \in G^{(k)}$, $E(p)$ shall stand for the orthogonal complement of the plane p in E. Finally $\mathfrak{N}_X$ shall denote the null-space of $\{XM - N\}$.

DEFINITION 4.1. *The k-resolution, $B^{(k)}$, of B^0 is defined as the subset of* points $\{X, p\} \in W^{(k)}$ *for which* $p \subset \mathfrak{N}_X$.

PROPOSITION 4.1. *If B is a Hermitian boundary condition, then for each $k \geqq 1$, its k-resolution $B^{(k)}$ is either vacuous, or a $(m^2 - k^2)$ dimensional* *real analytic submanifold of* $W^{(k)}$. *(Recall that* $m = 2n =$ *complex dimension of* E.)

Proof: In suitable coordinate systems over $W^{(k)}$ the defining relations of $B^{(k)}$ will be algebraic. It is sufficient, therefore, to show that if $B^{(k)}$ contains a point q, then the tangent space to $B^{(k)}$ at q is $(m^2 - k^2)$ dimensional.

Throughout the paper we will denote the tangent space to a manifold (say V) at a point q by the subscript q (i.e., V_q).

The differential $df^{(k)}$ of $f^{(k)}$ maps $B_q^{(k)}$ into $\mathfrak{H}_X$ where $X = f^{(k)}(q)$. Proposition 4.1 therefore is a consequence of the following two lemmas.

LEMMA 4.1. *The restriction of* $df^{(k)}$ *to* $B_q^{(k)}$ *maps* $B_q^{(k)}$ *onto a* $\{m^2 - 20_B(X)k + k^2\}$ *dimensional plane of* $\mathfrak{H}_X$. *Here* $q \in B^{(k)}$ *and* $f^{(k)}(q) = X$.

LEMMA 4.2. *The kernel of* $df^{(k)}$ *intersects* $B_q^{(k)}$ *in a subspace of dimension* $2\{0_B(X) - k\}k \quad (f^{(k)}(q) = X)$.

We introduce now, what we will call the *standard identification* of the tangent space $W_q^{(k)} \quad (q \in W^{(k)})$. $\mathfrak{H}_X$ $(X \in \mathfrak{H})$ was already identified with the fixed vector space $\mathfrak{L}$ in Section 3. $G_p^{(k)}$ will now be described in terms of "standard neighborhoods" of p. If $p \in G^{(k)}$ is a k-plane of E, a standard neighborhood U of p is given once a unitary basis $\{u_1, \cdots, u_k\}$ of p is chosen. U consists of the space

$$F(p) = E(p) \times E(p) \times \cdots \times E(p) \qquad (k \text{ factors})$$

together with the map

$$U\colon F(p) \to G^{(k)}$$

defined by $U\{v_1, \cdots, v_k\} =$ the plane spanned by $\{u_1 + v_1, \cdots, u_k + v_k\}$ $(v_i \in E(p))$. A standard neighborhood U of p therefore defines an identification of $G_p^{(k)}$ with $F(p)$. Finally $W_q^{(k)}$ is identified with $\mathfrak{H}_X \times G_p^{(k)}$, where $q = \{X, p\}$.

Under this identification, $B_q^{(k)}$ goes over (once U is chosen) into the subspace of $\mathfrak{L} \times F(p)$ characterized by the following conditions:

$\{A, v\}$ $(A \in \mathfrak{L},\ v = \{v_i\} \in F(p))$ is contained in $B_q^{(k)}$ under U, if and only if

$$-XAMu_i = \{XM - N\}v_i, \qquad i = 1, \cdots, k. \tag{4.1}$$

Here $q = \{X, p\}$, and $U = \{u_1, \cdots, u_k\}$ is a unitary base of p.

$df^{(k)}$ maps $\{A, v\}$ onto A. Hence we have to study the condition imposed on A by (4.1). Clearly this condition is equivalent to the demand that XAM map the plane p into the range of $\{XM - N\}$, or equivalently that $XAMp$ be orthogonal to $\mathfrak{N}_X^*$, the null-space of $\{XM - N\}^*$. Now

(2.8) and (2.9) are seen to imply that JXM maps $\mathfrak{N}_X$ isomorphically onto $\mathfrak{N}_X^*$. Hence $A \in \mathfrak{L}$ is in the image of $df^{(k)} B_q^{(k)}$ if and only if $(XAMp, JXM\mathfrak{N}_X) = 0$, or equivalently

$$(4.2) \qquad (JAMp,\ M\mathfrak{N}_X) = 0.$$

By the definition of $\mathfrak{L}$, JA is Hermitian, and conversely, for any Hermitian H, $J^*H \in \mathfrak{L}$. Hence the subspace of $\mathfrak{L}$ characterized by (4.2) coincides in dimension with the subspace of Hermitian matrices which map $Mp \subset M\mathfrak{N}_X$ into the orthogonal complement of $M\mathfrak{N}_X$. Let $\{u_1, \cdots, u_m\}$ be a unitary base of E such that $\{u_1, \cdots, u_k\}$ spans Mp and $\{u_1, \cdots, u_{k+s}\}$ $(k + s = 0_B(X))$ spans $M\mathfrak{N}_X$. In such a coordinate system a Hermitian matrix H having the above properties is precisely characterized by the condition that its entries h_{ji} and h_{ij} be zero for $i = 1, \cdots, k; j = 1, \cdots, k + s$. Hence (4.2) represents $k^2 + 2sk$ real independent conditions on A. Since the dimension of $\mathfrak{L}$ is m^2, Lemma 4.1 follows. Lemma 4.2 is more immediate. Clearly the intersection of $\mathfrak{N}_X$ and $E(p)$ has dimension $2\{0_B(X) - k\}$. Hence the kernel of $df^{(k)}$ intersects $B_q^{(k)}$ in a $k\{20_B(X) - 2k\}$ dimensional plane.

As a corollary to Lemma 4.1 we obtain a proof of Proposition 3.2. For, as is self-evident from (4.2), we have the following

COROLLARY I. *The image of* $B_q^{(k)}$ *under* $df^{(k)}$ *never intersects the* $\oplus$ *cone* $K \in \mathfrak{L}$.

Recall now that Proposition 3.2 states that the direction of a smooth curve $c(t)$ on B^0 never lies in K. Since the direction of such a curve would have to lie in the image of $B_q^{(1)}$ under $df^{(1)}$, the Corollary above implies:

COROLLARY II. *Proof of Proposition* 3.2.

PROPOSITION 4.2. $B^{(1)}$ *is an orientable submanifold of* $W^{(1)}$.

An orientation of a real vector space V is an equivalence class $\omega(V)$ of bases of V which are related by the transformation of a positive determinant. As usual, we set $-\omega(V)$ equal to the orientation opposite to $\omega(V)$. If $V_1, V_2 \subset V$ are subspaces of V, with $V_1 \cap V_2 = 0$, orientations $\omega(V_i)$ $(i = 1, 2)$ of V_i induce an orientation

$$\omega(V_1) \times \omega(V_2)$$

on $V_1 + V_2$, which is defined as the class of any basis $\{v_{(1)}^1, \cdots, v_{(1)}^{k_1}, v_{(2)}^1, \cdots, v_{(2)}^{k_2}\}$ of $V_1 + V_2$ for which $\{v_{(i)}^1, \cdots, v_{(i)}^{k_i}\} \in \omega(V_i)$, $i = 1, 2$. Clearly $\omega(V_1) \times \omega(V_2) = (-1)^{k_1 k_2} \omega(V_2) \times \omega(V_1)$.

$B^{(1)}$ becomes oriented if an orientation is assigned to each tangent space of $B^{(1)}$ in a continuous fashion.

Now $G^{(1)}$ carries an intrinsic orientation $\omega(G^{(1)})$ due to its complex structure. Namely, for any line $p \subset E$, $\omega(E(p))$ is defined as the class of any real base $\{u_1, iu_1; u_2, iu_2; \ldots; u_{m-1}, iu_{m-1}\}$ ($i=\sqrt{-1}$ here) where $\{u_1, \cdots, u_{m-1}\}$ is a complex base of $E(p)$.

With this understanding, every orientation of $\mathfrak{H}$ induces an orientation on $W^{(1)}$ by the formula $\omega(W_q^{(1)}) = \omega(\mathfrak{H}_X) \times \omega(G_p^{(1)})$, $q = \{X, p\}$.

Note. Here, as subsequently, $\mathfrak{H}_X[G_p^{(1)}]$ is identified with the subset $\mathfrak{H}_X \times 0[0 \times G_p^{(1)}]$ of $\mathfrak{H}_X \times G_p^{(1)}$.

For $A \in K$ let T^A denote the distribution on $W^{(1)}$ which assigns to each $q=\{X, p\} \in W^{(1)}$ the subspace T_q^A of $W_q^{(1)}$ spanned by $G_p^{(1)}$ and the vector A at X. We orient the 1-space generated by A by choosing A as a basis, and T_q^A by the formula $\omega(T_q^A) = \omega(A) \times \omega(G_p^{(1)})$. It is now easily checked that if $q = \{X, p\}$ is a point of $B^{(1)}$ with $0_B(X) = 1$, then $B_q^{(1)}$ and T_q^A span $W_q^{(1)}$. At such points then, an orientation of $\mathfrak{H}$ induces an orientation of $B_q^{(1)}$, by requiring the formula

$$\omega(B_q^{(1)}) \times \omega(T_q^A) = \omega(W_q^{(1)}) \tag{4.3}$$

to hold. Since K is convex this orientation is seen to be independent of the particular $A \in K$ chosen.

Let $B_*^{(1)}$ be the subset of $B^{(1)}$ which consists of points $q=\{X, p\}$ with $0_B(X) \geqq 2$. Formula (4.3) defines an orientation on $B^{(1)} - B_*^{(1)}$, which we refer to as the induced orientation on $B^{(1)} - B_*^{(1)}$.

Rather than Proposition 4.2 (which can be proved in a much simpler fashion) we really need the following refinement of that assertion:

PROPOSITION 4.3. *The induced orientation on $B^{(1)} - B_*^{(1)}$ can be extended to all of $B^{(1)}$.*

Proof. Let $B_j^{(1)} = \{q = \{X, p\} \mid 0_B(X) = j\}$. Then $B_*^{(1)} = \bigcup_{j\geqq 2} B_j^{(1)}$.

Now $B_j^{(1)}$ is seen to be a manifold of dimension $m^2 - 1 - (j-1)^2$, either by a tangent space argument, or more simply, by observing that $B_j^{(1)}$ can be fibered over a subset of $B^{(j)}$ into $2(j-1)$ dimensional fibers. Hence the induced orientation on $B^{(1)} - B_*^{(1)}$ extends to $\bigcup_{j\geqq 3} B_j^{(1)}$ because the dimension of these sets is too low. It remains to consider this extension to $B_2^{(1)}$. For this purpose we consider a regular curve $c(t)$, $-\varepsilon \leqq t \leqq \varepsilon$, on $B^{(1)} - \bigcup_{j\geqq 3} B_j^{(1)}$ which intersects $B_2^{(1)}$ only at $t = 0$, and there in such a fashion that its direction is transversal to $B_2^{(1)}$. Equation (4.3) then defines an orientation of $B_{c(t)}^{(1)}$ for $t \neq 0$ and hence limiting orientations of $B_{c(0)}^{(1)}$, as we approach $t=0$ from the right and the left. Our extension theorem is clearly equivalent to the proposition that these two orientations coincide.

Let c be given by $t \to q_t = \{X_t, p_t\}$, and let $u_t \in E$ be a regular covering of p_t; i.e., for each t in the domain of definition of $c(t)$, $u_t \in E$ is of unit length and spans p_t (over the complex numbers). u_t thus defines a standard neighborhood of p_t, and a corresponding identification of $W^{(1)}_{q_t}$ with $\mathfrak{L} \times E(p_t)$. Using this identification, and choosing $A = J^* \in K$, the distributions $T^{J^*}_{c(t)}$ and $B^{(1)}_{c(t)}$ have the following description:

$T^{J^*}_{c(t)} =$ subspace of $\mathfrak{L} \times E(p_t)$ of the form $\{\varrho J^*, v\}$ ϱ real, $v \in E(p_t)$.

$B^{(1)}_{c(t)} =$ subspace of $\mathfrak{L} \times E(p_t)$ defined by
$- X_t A M u_t = \{X_t M - N\} v, \qquad A \in \mathfrak{L}, \qquad v \in E(p_t).$

We have furthermore that u_t spans the null-space $\mathfrak{N}_{X_t}$ for $t \neq 0$, and that at $t = 0$, $\mathfrak{N}_{X_0}$ has a unitary basis $\{u_0, \check{u}_0\}$. Hence $T^{J^*}_q \cap B^{(1)}_q = 0$ along $c(t)$ for $t \neq 0$, and at $t = 0$, this intersection is spanned by $\{0, \check{u}_0\}$.

Finally, the fact that $c(t)$ lies in $B^{(1)}$ implies that the direction of X_t, i.e., $X_t^{-1} X'_t$, has the property

$$(JX_t^{-1} X'_t M u_t, M\mathfrak{N}_{X_t}) = 0 \tag{4.4}$$

while the transversality of c to $B^{(1)}_{\mathfrak{L}}$ at $t = 0$ implies that

$$(JX_0^{-1} X'_0 M \check{u}_0, M\mathfrak{N}_{X_0})$$

does not vanish identically, and hence in particular,

$$(JX_0^{-1} X'_0 M \check{u}_0, M\check{u}_0) \neq 0. \tag{4.5}$$

We now have the following lemma, which characterizes the intersection of the two distributions, $T^{J^*}_q$ and $B^{(1)}_q$, along $c(t)$, as, what I will call, a first order intersection.

LEMMA 4.3. *Let $\varphi_t = W^{(1)}_{q_t} \to T^{J^*}_{q_t}$ be any linear C^∞ projection, and let η be a nonvanishing C^∞ vector-field along c, with the following properties:*

(a) $\eta_t \in B^{(1)}_{q_t}$,

(b) $\eta_0 \in B^{(1)}_{q_0} \cap T^{J^*}_0$.

Then $\lim_{t \to 0} (\eta_t - \varphi_t \eta_t)/t \neq 0$.

Proof: Suppose $q_t \to \{A_t, v_t\}$ is the representation of η_t in the identification of $W^{(1)}_{q_t}$ defined by u_t above. Then

$$X_t A_t M u_t = \{X_t M - N\} v_t \tag{4.6}$$

and

$$\{A_0, v_0\} = \{0, \lambda u_0\}$$

($\lambda \neq 0$, a complex number).

If relation (4.6) is differentiated at $t = 0$ one therefore obtains

$$X_0 A'_0 M u_0 = X'_0 M \check{u}_0 + \{X_0 M - N\} v'_0.$$

Multiplication by $X_0^* J$ on the left yields

$$JA_0' M u_0 = (JX_0^{-1} X_0') M \check{u}_0 + X_0^* J\{X_0 M - N\} v_0' .$$

Now if we take the inner product with $M\mathfrak{N}_{X_0}$ on the right, the last term drops out (see (4.2)) and we end up with

$$(4.7) \qquad (JA_0' M u_0 , M\mathfrak{N}_{X_0}) = (JX_0^{-1} X_0' M \check{u}_0, M\mathfrak{N}_{X_0}).$$

It is easily checked that if the lemma is true for one C^∞ projection φ_t it is true also for all such projections. In our case we can therefore choose for φ_t the projection which, in our description of $W_{q_t}^{(1)}$, is given by

$$\varphi_t \colon \{A, v\} \to \left\{ \text{Trace} \left(\frac{JA}{m}\right) J^*, v \right\}.$$

The lemma will then be demonstrated if it is shown that

$$A_0' - \text{Trace} \left(\frac{JA_0'}{m}\right) J^* \neq 0.$$

Arguing by contradiction we assume that $A_0^1 = \alpha J^*$, with α some real number. Under this hypothesis (4.7) reduces to

$$(4.8) \qquad \alpha(Mu_0 , M\mathfrak{N}_{X_0}) = \lambda(JX_0^{-1} X_0' M\check{u}_0 , M\mathfrak{N}_{X_0}).$$

If in this expression $\mathfrak{N}_{X_0}$ is replaced by u_0 and $\check{u}_0$ (which together span $\mathfrak{N}_{X_0}$) one obtains in turn

$$(4.9) \qquad \alpha(Mu_0 , Mu_0) = \lambda(JX_0^{-1} X_0' M\check{u}_0 , Mu_0),$$

$$(4.10) \qquad \alpha(Mu_0 , M\check{u}_0) = \lambda(JX_0^{-1} X_0' M\check{u}_0 , M\check{u}_0).$$

The right hand side of (4.9) equals $\lambda(JX_0^{-1}X_0'Mu_0 , M\check{u})$ since $JX_0^{-1} X_0'$ is Hermitian.

However, this expression is zero by (4.4). Hence $\alpha = 0$. (Recall that $Mu_0 \neq 0$, since M is an isomorphism on $\mathfrak{N}_X$!) But then (4.10) contradicts (4.5) since $\lambda \neq 0$.

With the aid of Lemma 4.3 Proposition 4.3 is proved in the following manner:

Let $\eta = \{\eta_t^1 , \cdots , \eta_t^s\}$, $s = m^2 - 1$, be a family of C^∞ vector fields along c such that

(a) $\{\eta_t^1 , \cdots , \eta_t^s\}$ span $B_{q_t}^{(1)}$ (over the reals),

(b) $\{\eta_0^1 , \eta_0^2\}$ span $B_{q_0}^{(1)} \cap T_0^{J*}$.

Furthermore let $\gamma = \{\gamma_t^1 , \cdots , \gamma_t^k\}$ $(k = 2(m-1) + 1)$ be a family of C^∞ vector fields which span T_t^{J*}. These frame-fields taken in their natural order define orientations $\omega(B_{q_t}^{(1)})$ and $\omega(T_t^{J*})$. Since, moreover, for $t \neq 0$, $B^{(1)}_{q_t}$ and T^{J*}_t span $W^{(1)}_{q_t}$, the frame $\{\eta, \gamma\} = \{\eta^1_t, \cdots, \eta^s_t; \gamma^1_t, \cdots, \gamma^k_t\}$ defines the orientation $\omega(W^{(1)}_{q_t}) = \omega(B^{(1)}_{q_t}) \times \omega(T^{J*}_t)$ for $t \neq 0$. Our original assertion is clearly equivalent to the proposition that the limiting orientations on W_{q_0} defined by the frame $\{\eta, \gamma\}$ as $t \to \pm 0$ should be the same. To see that this is indeed the case, consider the frame

$$\left\{ \eta^1_t - \frac{\varphi_t \eta^1_t}{t}, \eta^2_t - \frac{\varphi_t \eta^2_t}{t}, \eta^3_t, \cdots, \eta^s_t; \gamma^1_t, \cdots, \gamma^k_t \right\},$$

where φ_t is the projection introduced earlier. This frame is in the same class as $\{\eta, \gamma\}$, and, by our construction, the limit of each individual vector exists as $t \to 0$. Also the limiting frames still span W_{q_0} by Lemma 4.3. These limiting frames as $t \to 0$ from the positive and negative side, therefore, define the limiting orientations on $W^{(1)}_{q_0}$ which we are after. Since these two frames are identical, except that the first two vectors are opposite in sign, they define the same orientation on $W^{(1)}_{q_0}$. This concludes the proof of Propositions 4.3 and 4.2.

The orientation of $B^{(1)}$ obtained by extending the induced orientation of $B^{(1)} - B^{(1)}_*$ will be referred to as the induced orientation of $B^{(1)}$. (It is, of course, induced by an orientation of $\mathfrak{H}$.) Γ_B shall denote the locally finite fundamental integral homology class defined by this orientation.

The map $f^{(1)}: W^{(1)} \to \mathfrak{H}$ is proper (that is $f^{-1}(C)$ is compact if C is compact). It therefore induces a homomorphism f_*, in locally finite homology, of $H(B^{(1)})$ into $H(B^0)$. The image of Γ_B under the homomorphism shall be denoted by γ_B. γ_B is carried by B^0 and coincides with the γ_B of Theorem V. We shall refer to $\gamma_B[\Gamma_B]$ as the class (resolved class) of the boundary condition B.

Remark: The construction of Γ_B and γ_B is admittedly rather tedious. To show that just γ_B is a cycle one could proceed in a much simpler fashion. For, since $f^{(1)}(B^{(1)}_*)$ is a subset of at least 3 dimensions less than $B^0 = f^{(1)}(B^{(1)})$ (see Lemma 4.1), the cycle on $B^{(1)} - B^{(1)}_*$ goes over into the cycle γ_B under $f^{(1)}_*$. However, our whole intersection argument in the next sections is based on the idea of working with Γ_B rather than with γ_B. It is for this reason that we had to show that the induced orientation of $B^{(1)} - B^{(1)}_*$ has an extension to $B^{(1)}$.

We close this section by describing certain positional properties of $B^{(1)} \subset W^{(1)}$ which will be needed subsequently. T^A shall denote the distribution on $W^{(1)}$ defined earlier. For $X \in \mathfrak{H}$, V^X shall stand for the set $f^{(1)-1}(X) \cap B^{(1)}$. It is then clear that V^X is a complex projective space of complex dimension $0_B(X) - 1 \quad (X \in B^{(0)})$.

PROPOSITION 4.4. *Let $X \in B^0$; $q \in V^X$. Then*

$$B^{(1)}_q \cap T^A_q = V^X_q. \tag{4.11}$$

This relation generalizes the relation $B_q^{(1)} \cap T_q^A = 0$ for X with $0_B(X) = 1$ used earlier. The proof follows easily from our descriptions of $B_q^{(1)}$ and T_q^A, the salient fact being that $A \in K \longleftrightarrow JA > 0$.

Let $T(V^X) = \bigcup_{q \in V^X} V_q^X$ be the tangent bundle of V^X [8]. We also define the "transversal bundle over V^X with respect to A", as the bundle of vector spaces over V^X whose fiber over $q \in V^X$ is the quotient space $W_q^{(1)}/(B_q^{(1)} + T_q^A)$. Let this bundle be denoted by $A(V^X)$.

PROPOSITION 4.5. *$T(V^X)$ and $A(V^X)$ are isomorphic bundles over V^X (i.e., there exists a map*

$$\psi_A \colon A(V^X) \to T(V^X),$$

whose restriction to the fiber over q in $A(V^X)$ is a linear isomorphism onto V_q^X).

Proof: To make the notation simpler we will drop the superscript X during the proof. By (4.11) the dimension of each fiber of $A(V)$ is equal to the dimension of V_q. Hence ψ_A can certainly be constructed at a given $q \in V$. The point of the proposition however is that ψ_A exists globally. Let $W(V)$ be the bundle $\bigcup_{q \in V} W_q^{(1)}$ over V. The map ψ_A is then equivalent to a map $\bar{\psi}_A \colon W(V) \to T(V)$ with the property that at any point $q \in V$, $\bar{\psi}_A$ maps W_q linearly onto V_q and its kernel is precisely $(B_q^{(1)} + T_q^A)$. Since the various $A(V)$ $(A \in K)$ are all easily seen to be isomorphic, it is sufficient to construct a $\bar{\psi} \colon W(V) \to T(V)$, whose kernel will have the desired properties for some $A \in K$. We construct $\bar{\psi}$ in the following fashion:

For $q = \{X, p\} \in V$, a unitary base u of p identifies $W_q^{(1)}$ with $\mathfrak{L} \times E(p)$ and V_q with $\mathfrak{N}_X \cap E(p)$. In this identification $\bar{\psi}$ is represented by

$$u\bar{\psi}\{A, v\} = M^{-1}\{\text{of the orthogonal projection of } JAMu \text{ on } M\{\mathfrak{N}_X \cap E(p).\}\} \tag{4.12}$$

(Since M is an isomorphism on $\mathfrak{N}_X$, M^{-1} is uniquely defined on $M \cdot \{\mathfrak{N}_X\}$.)

It is easily checked that (4.12) defines an intrinsic map $\bar{\psi} \colon W_q \to V_q^X$. Furthermore $\bar{\psi}$ is onto. For, as A ranges over $\mathfrak{L}$, $JAMu$ ranges over a subspace of E which includes $E(Mp)$ (i.e., the orthogonal complement of Mu). But $E(Mp)$ projects onto $M\{\mathfrak{N}_X \cap E(p)\}$.

Let K be the kernel of $\bar{\psi}$ at q. Under our identification an element $\{A, v\}$ is in $B_q^{(1)}$ for some $v \in E(p)$ if and only if $JAMp$ is orthogonal to $M\mathfrak{N}_X$ (see (4.2)). Such an $\{A, v\}$ therefore has a trivial projection on $M\mathfrak{N}_X$, let alone on $M\{\mathfrak{N}_X \subset E(p)\}$. Hence, $B_q^{(1)} \subset K$. It remains to show that $T_q^A \subset K$ for some $A \in K$, for if this happens for A_1, say, then a simple dimension-argument proves that K is spanned by $B_q^{(1)}$ and $T_q^{A_1}$.

We seek therefore an $A \in K$ for which $JAMu$ will stand perpendicular to $M\{\mathfrak{N}_X \cap E(p)\}$. Since $A \in K \Longleftrightarrow JA$ Hermitian positive, such an A exists as a consequence of the following lemma:

LEMMA 4.5. *Let $v \neq 0$ be an element of E, and let $\check{E}$ be a complex subspace of E which does not contain v. Then there exists a positive Hermitian transformation H which maps v into the orthogonal complement of $\check{E}$.*

Proof: Decompose v into $v_1 + v_2$, with $v_1 \in \check{E}$, $(v_1, v_2) = 0$. By hypothesis $v_2 \neq 0$. If $v_1 = 0$, set H equal to the identity. If $v_1 \neq 0$, complete $\{v_1, v_2\}$ to a unitary base of E, say $\{v_1, v_2; v_3, \cdots, v_m\}$. Now define H by

$$Hv_1 = 1/2v_1 - 1/2v_2,$$
$$Hv_2 = -1/2v_1 + 3/2v_2,$$
$$Hv_i = v_i, \qquad 3 \leqq i \leqq m.$$

This completes the proof of Proposition 4.5.

5. Clean Intersections of Manifolds; Proof of Theorem V.

We recall a few notions from the theory of intersection [5] adapted to our needs. All manifolds will be assumed to be of class C^∞ and paracompact. We again denote the tangent space to a manifold at a point q by the subscript q.

Let U, W be manifolds of dimension m and $m + n$, respectively. V shall be an n-dimensional compact manifold with boundary $\dot{V}$ $(n \geqq 1)$. Let $f: U \to W$, $g: V \to W$ be maps with $f(U) \cap g(\dot{V}) = 0$ and f proper. Under these circumstances orientations of U, $V - \dot{V}$ and W define an integer (properly speaking a zero-dimensional homology class) which we will denote by $[U_f : V_g]_W$, or with admitted ambiguity just by $[U : V]_W$. The integer $[U_f : V_g]_W$ is called the intersection number of U and V in W (under f and g).

Important properties of $[U : V]_W$ are the following:

(1) A reversal in orientation of any of the three spaces involved changes the sign of $[U : V]_W$.

(2) If f and g are deformed by f_t, g_t $(0 \leqq t \leqq 1)$ keeping $g_t(\dot{V}) \cap f_t(U)$ vacuous during the deformation, then $[U_{f_t} : V_{g_t}]_W$ is left invariant.

(3) If the image of the fundamental class of U (locally finite homology if V is not compact) under f is homologous to zero in $W - g(\dot{V})$, then $[U : V]_W = 0$.

If f and g satisfy certain regularity conditions, then $[U : V]_W$ can be computed by an infinitesimal method. A point $x = \{p, q\} \in U \times V$ will be called a regular intersection of f: $U \to W$, g: $V \to W$ $(g(\dot{V}) \cap f(U) = 0!)$ if

a) f [g] is regular in some neighborhood of p [q] in U [V],

b) $f(p) = g(q)$,

(5.1) c) $df(U_p) \cap dg(V_q) = 0$.

(A map f is called regular on an open set if it is of class C' and if df is an isomorphism into at all points of the set.) Note that (5.1) implies that W_y $(y = f(p) = g(q))$ is spanned by $df(U_p)$ and $dg(V_q)$.

With a regular intersection x we associate the intersection number $[U_f : V_g ; x]_W$. This number is ± 1, and is determined through orientations $\omega(U_p)$, $\omega(V_q)$, $\omega(W_y)$ by the relation

$$\omega(W_y)\ [U;\ V;\ x]_W = \omega\{df(U_p)\} \times \omega\{dg(V_q)\}. \tag{5.2}$$

If U and V have only regular intersections $\{x\}$ under f, g, then these intersection points $\{x\}$ are finite in number since V is compact. Hence the sum $\sum_x [U\colon V;\ x]$ makes sense.

(4) If f and g are maps with only regular intersections $\{x\}$, then

$$[U : V]_W = \sum_x [U\colon V;\ x]_W\,. \tag{5.3}$$

We will need the following generalization of (5.1).

DEFINITION 5.1. *A simply-connected compact submanifold $\overline{V} \subset U \times V$ is called a clean intersection of $f\colon U \to W$, $g\colon V \to W$ provided*

(1) *f [g] is regular in some neighborhood of the projection of $\overline{V}$ on U [V].*

(2) *$f(p) = g(q)$ for $\{p,\ q\} \in \overline{V}$.*

(5.4) (3) *$df(\overline{V}_x) = df(U_p) \cap dg(V_q)$ for $x = \{p,\ q\} \subset \overline{V}$.*

Note that a clean intersection $\overline{V}$ is a regular intersection if $\overline{V}$ is just a point. It is plausible therefore, in view of (5.3), that the contribution of a clean intersection $\overline{V}$ to $[U,\ V]_W$ should be computable in terms of local considerations in the vicinity of $\overline{V}$. This turns out to be the case, as we describe below; dim. $\overline{V} \geqq 1$.

On $\overline{V}$, f and g are equal and so define an immersion (i.e., an everywhere regular map) F of $\overline{V}$ into M. Let $T(W)$ denote the tangent bundle to W. Over $\overline{V}$ we now define the bundle $I(\overline{V})$ which is a quotient bundle of $F^{-1}T(W)$, the bundle induced from $T(W)$ by F over $\overline{V}$. Precisely, the fiber of $I(\overline{V})$ over $x = \{p,\ q\} \in \overline{V}$ is to be the quotient space

$$S_x = F^{-1}(W_y)/F^{-1}(dfU_p + dgV_q), \qquad y = F(x). \tag{5.5}$$

(To simplify the notation we will denote $F^{-1}\,dfU_p$ by U_p, and $F^{-1}dfV_q$ by V_q.) S_x is then, due to (5.4), of the same dimension as $\overline{V}_q$. Because $\overline{V}$ is simply connected $I(\overline{V})$ is orientable.

Orienting a quotient space E/E' is equivalent to orienting any complement of E' in E. With this in mind we say that orientations of U_p, V_q, W_y and $\overline{V}_x$ define orientations of $U_p/\overline{V}_x$ and S_x by the following formulas:

$$\omega(U_p/\overline{V}_x) \times \omega(\overline{V}_x) = \omega(U_p), \tag{5.6}$$

$$\omega(\overline{V}_x) \times \omega(V_p/\overline{V}_x) = \omega(V_p), \tag{5.7}$$

$$\omega(S_x) \times \omega(V_p/\overline{V}_x) = \omega(F^{-1}W_y). \tag{5.8}$$

Now, $I(\overline{V})$ is a manifold of twice the dimension of $\overline{V}$. Let $O\colon \overline{V} \to I(\overline{V})$ be the imbedding which maps x into the zero of S_x. $I(\overline{V})_x$ can then be identified with $\overline{V}_x \times S_x$. Hence setting $\omega(I(V)_x) = \omega(\overline{V}_x) \times \omega(S_x)$ defines an orientation of $I(\overline{V})$ which is seen to depend only on $\omega(W_y)$, $\omega(U_p)$ and $\omega(V_q)$. We call this the induced orientation of $I(\overline{V})$.

Consider now the self-intersection number of $O : \overline{V} \subset I(\overline{V})$; i.e., $[\overline{V} : \overline{V}]_{I(\overline{V})} = [\overline{V}_0 : \overline{V}_0]_{I(\overline{V})}$. This number depends only on the orientation of $I(\overline{V})$, since it is zero if dim. $\overline{V}$ is odd. We now have

THEOREM VII. *Let $f\colon U \to W$, $g\colon V \to W$ be given. Then if the intersection of (U, f) and (V, g) consists only of clean intersection manifolds $\{\overline{V}\}$, the intersection number of U and V is given by*

$$[U : V]_W = \sum_{\overline{V}} [\overline{V} : \overline{V}]_{I(\overline{V})} \tag{5.9}$$

where each $I(\overline{V})$ is assigned the induced orientation.

In particular, if each $I(\overline{V})$ is isomorphic to the tangent bundle of V, in an orientation preserving fashion, then

$$[U : V]_W = \sum_{\overline{V}} \chi(\overline{V}) \tag{5.10}$$

where $\chi(\overline{V})$ denotes the Euler characteristic of $\overline{V}$.

Although I know of no proof of this theorem in the literature, it will not surprise anyone acquainted with these matters. I will, therefore, only sketch a proof which, because of the condition: dim U + dim V = dim W, can be given in quite an elementary and geometric form. A generalization of (5.9) to higher dimensional intersections will be given elsewhere.

First one proves

LEMMA 5.1. *$I(\overline{V})$ admits a C^∞ cross section c which vanishes only at a finite number of points $\{x\}$, and is of the first order there.*

The lemma is proved by first construction a C^∞ cross section $c_1\colon \overline{V} \to I(\overline{V})$ which vanishes at isolated points, say at most at the center of every k-simplex (k = dim. $\overline{V}$) of a suitable subdivision on $\overline{V}$. This can be done since dim $\overline{V}$ = dim S_x ($x \in \overline{V}$), by standard obstruction theory. Then inside each simplex the zeros of c are made of the first order by a purely local argument.

For c to be of the first order at its zeros is however quite equivalent with the condition that $c: \overline{V} \to I(\overline{V})$ and $O: \overline{V} \to I(\overline{V})$ have only regular intersection points. Hence $[\overline{V}:\overline{V}]_{I(\overline{V})}$ can be computed by the formula

$$\sum_{x} [\overline{V}_c : \overline{V}_0 \, ; \, x]_{I(\overline{V})} \, .$$

Now let a fixed Riemannian structure be chosen for W. Once this is done the quotient space W_y/F_y (where F_y is some subspace of W_y) can be identified with the orthogonal complement of F_y in W_y. Hence $c:\overline{V}\to I(\overline{V})$ can be construed as a C^∞ function which assigns to every $x = \{p, q\} \in \overline{V}$ a vector in W_y $(y = F(x))$ perpendicular to both $df(U_p)$ and $dg(V_p)$. Now it is easy to see that c has an extension to some neighborhood of the projection of $\overline{V} \subset U$; i.e., there exists a C^∞ function $c_U(\overline{V})$ which assigns to every $p \in U$ a vector $c_U(p)$ in W_y $(y = f(p))$ satisfying the following conditions:

(1) $c_U(p)$ is perpendicular to $df(U_p)$,

(2) $c_U(p)$ vanishes outside a prescribed ϵ-neighborhood of the projection of $\overline{V} \subset U$.

(3) For points $x \in \overline{V}$, $c_U(x) = df\, c(x)$.

(A similar extension of c can of course be found over V.) Consider now the set $\{\overline{V}\}$ of clean intersection manifolds of the theorem. Each $\overline{V}$ of this set is surrounded by an $\epsilon > 0$ neighborhood, $\overline{V}(\epsilon)$, so small that the $\overline{V}(\varepsilon)$ are disjoint and that f and g are regular on the projections of $\overline{V}(\varepsilon)$ on U and V, respectively. For $\overline{V} \in \{\overline{V}\}$ we choose a cross section c as given by the lemma and the corresponding extension c_U of c over U; $c_U = 0$ outside $\overline{V}(\epsilon)$. Consider a deformation f_t of $f: U \to W$ in the direction of c_U. It is now checked that for a sufficiently small such deformation, (1) $f_t(U)$ will remain free from $g(\dot{V})$, (2) $f_{\varepsilon_0}(U)$ and $g(V)$ will have only regular intersection points, namely those points of $\overline{V} \in \{\overline{V}\}$ at which $c = 0$. Furthermore at any such point x, it is checked that

$$[\overline{V}_c : \overline{V}_0 \, ; \, x]_{I(\overline{V})} = [U_c : V; \, x]_W$$

if the induced orientation on $I(\overline{V})$ is used. Hence (5.3) yields the desired proof of the first part of Theorem VII. The second part now follows easily; for, it is well known that the self-intersection of $\overline{V}$ in its tangent bundle is precisely $\chi(\overline{V})$.

We apply Theorem VII to our original problem in the following fashion. Let $c : t \to X(t)$ $(t \in [a, b])$ be a regular curve in $\mathfrak{H}$. c defines an immersion $g_c : [a, b] \times G^{(1)} \to W, g_c = c \times 1$. $[a, b]$ shall always be oriented in the positive direction, and we set

$$\omega\{[a, b] \times G^{(1)}\} = \omega\{[a, b]\} \times \omega\{G^{(1)}\}.$$

In this orientation $[a, b] \times G^{(1)}$ shall be denoted by $V(a, b)$.

PROPOSITION 5.1. *Let B be a Hermitian boundary condition, and let c be a $\oplus$ curve in $\mathfrak{H}$ whose end points X_a, X_b do not lie on B^0. Then the intersection number of the immersions $B^{(1)} \subset W^{(1)}$, $g_c : V(a, b) \to W^{(1)}$, is given by the algebraic formula*

$$(5.11) \qquad [\Gamma_B, V(a, b)]_{W^{(1)}} = \sum_{a \leq t \leq b} \nu\{X_t M' - N\}.$$

Here Γ_B is $B^{(1)}$ with the induced orientation.

Proof: Since the end points of c miss B^0, the intersection number is well defined. Now one checks that the intersections of the two immersions considered are all clean. They are points $(0_B(X) = 1)$ or complex projective spaces $(0_B(x) > 1)$. This is an immediate consequence of (4.11), since the tangent space of $V[a, g]$ goes over into some T_q^A under dg_c. Moreover proposition 4.5, in our present context, then shows that each $I(\overline{V})$ is isomorphic to the tangent bundle of $\overline{V}$. (We leave the check that it is orientation preserving to the reader.) But the Euler characteristic of a complex projective space is precisely the complex dimension of the space $+ 1$. Hence $\chi(V^X) = 0_B(X)$. Therefore applying Theorem VII proposition 5.2 is demonstrated.

If we put the various parts of our theory together, the following "resolved" version of Theorem V emerges.

THEOREM V R. *Let L be a Hermitian operator, X^L its map. Let B be a Hermitian boundary condition and consider the problem*

$$Ly = \lambda y$$

subject to B at $t = 0$ and $t = a$. If Θ_B^L is the spectral multiplicity function of this problem, and $\tau = [\alpha, \beta]$ is an admissible compact interval of the λ-axis (with respect to Θ_B^L) then

$$[\Gamma_B : V(\tau)]_{W^{(1)}} = [\Theta_B^L : \tau].$$

Here Γ_B is the oriented resolved cycle $B^{(1)} \subset W^{(1)}$ and $V(\tau)$ is the manifold $[\alpha, \beta] \times G^{(1)}$ immersed in $W^{(1)}$ by the map $(\lambda, p) \to \{X_\lambda^L(a), p\}$.

To prove Theorem V, proper, the intersection of $f^{(1)} : B^{(1)} \to \mathfrak{H}$ with the curve $\tau : \lambda \to X_\lambda^L(a)$ has to be computed. However, this intersection number is seen to be the same as $[\Gamma_B : V(\alpha, \beta)]_{W^{(1)}}$ in the following fashion: $f^{(1)} : B^{(1)} \to \mathfrak{H}$ is not an immersion. However, the singularities of B^0 form a set whose dimension is by 3 smaller than dim. B^0 (Proposition 4.1).

Hence τ can be deformed away from these singularities, and the local intersections can be computed as before. However, at intersections of X with $0_B(X) = 1$ the intersection of τ with B^0 and $V(\alpha, \beta)$ with $B^{(1)}$ will both be regular, and the intersection numbers will be equal thanks to our orientations. Since $[\gamma_B : \tau]_{\mathfrak{H}}$ in Theorem V should, and always will subsequently, be interpreted as the intersection of $f^{(1)} : B^{(1)} \to \mathfrak{H}$, and $c : [\alpha, \beta] \to \mathfrak{H}$, Theorem V follows.

6. Proof of Theorems II to IV.

Recall the situation envisaged in Theorem II. The periodic operator L defines the $\oplus$ curve

$$\lambda \to X_\lambda^L(1) = P(\lambda), \qquad -\infty < \lambda < \infty,$$

in $\mathfrak{H}$ with $X_0^L(1) = P$. We have to study

$$\Lambda(z) = [\Theta_{B(z)} : R^-], \qquad z \in S^1. \tag{6.1}$$

$B(z)$ here stands for the boundary condition represented by $\{I, zI\}$ $(z \in S^1)$ and $R^- = \{\lambda \mid \lambda < 0\}$.

In view of the boundedness of the spectrum of L, the compactness of S^1 and Theorem V,

$$\Lambda(z) = [\gamma_{B(z)} : \tau]_{\mathfrak{H}} \text{ where } \tau : \lambda : \to P(\lambda), \quad \alpha \leqq \lambda \leqq \beta(z) \leqq 0,$$

with α sufficiently negative and independent of z, while $\beta(z)$ is sufficiently close to 0. (In particular if $P \notin B^0(z)$, then $\beta(z)$ can be chosen to be $= 0$.) We are interested in $\Lambda(z)$ as a function on S^1. Now if at a point z_0, $P \notin B^0(z)$, Λ is constant in the vicinity of z_0. It is sufficient therefore to study the situation where $P \in B^0(z_0)$. In that case $\beta(z_0) < 0$. Let σ be the curve $\sigma : \lambda \to P(\lambda)$ $(\beta(z_0) \leqq \lambda \leqq 0)$. For z sufficiently close to z_0, but not equal to z_0, $P \notin B^0(z)$, furthermore for z sufficiently close to z_0, $P(\beta(z_0)) \notin B^0(z)$. Hence for z satisfying both these conditions

$$\Lambda(z) = [\gamma_{B(z)} : \sigma]_{\mathfrak{H}} + \Lambda(z_0). \tag{6.2}$$

Since K is convex and does not contain the origin of $\mathfrak{L}$ all $\oplus$ curves are locally equivalent as far as intersections with γ_B are concerned. Hence the first term on the right hand side can be computed for the special $\oplus$ curve with direction $J^* \in K$ in some vicinity of $\lambda = 0$. But this is precisely what is done in Theorem III, with the aid of Theorem V.

It remains to explain Theorem IV. Here we need the resolved cycle $\Gamma_B(z_0)$. Recall that

$$B^{(1)}(z) = \{\{X, p\};\ X \in \mathfrak{H},\ p \in G^{(1)} \mid Xp = zp\}. \tag{6.3}$$

Hence if $e^{i\theta} : W^{(1)} \to W^{(1)}$ is the map sending $\{X, p\}$ into $\{e^{i\theta}X, p\}$, then $e^{i\theta}$ maps $B^{(1)}(z)$ onto $B^{(1)}(e^{i\theta}z)$. The derivative of this motion defines a vector field l on $W^{(1)}$. At any point $x = \{X, p\} \in W^{(1)}$, l is given by l_x, where l_x is represented (under any standard identification of $W^{(1)}_x$) by $\{\sqrt{-1}\, I, 0\}$. The manifold $V(\alpha, \varepsilon) = [\alpha, \varepsilon] \times G^{(1)}$ imbedded in $W^{(1)}$ by $\lambda \times G^{(1)} \to P(\lambda) \times G^{(1)}$ $(\alpha \leqq \lambda \leqq \varepsilon > 0)$ has, at $\lambda = 0$, a clean intersection with $B^{(1)}(z_0)$ (we are here assuming that $p \in B^0(z_0)$ as before) which is a point or a complex projective space V^P. Our problem is clearly to determine what happens to this clean intersection manifold as $B^{(1)}(z_0)$ is pushed in the direction of l. We will show that, under the hypothesis that P be nondegenerate at z_0, any deformation of $B^{(1)}(z_0)$ will, in general, split the possibly high dimensional clean intersection of $B^{(1)}(z_0)$ with $V(\alpha, \varepsilon)$ at $\lambda = 0$ into regular intersection points; the number of these points lying over negative λ-values is equal to the number of negative characteristic roots of the Hermitian form given in Theorem IV. Once this is established Theorem IV should, I hope, be evident.

As $V(-\varepsilon, \varepsilon)$ will, for $\varepsilon > 0$ sufficiently small, be imbedded in $W^{(1)}$ and $B^{(1)}(z)$ is already imbedded in $W^{(1)}$, we can identify the intersection manifold of $B^{(1)}(z)$ and $V(-\varepsilon, \varepsilon)$ with the actual intersection $\overline{V} = W^{(1)} \cap V(-\varepsilon, \varepsilon)$, and this will be done throughout this section. The tangent space to $V(-\varepsilon, \varepsilon)$ along $\overline{V}$ is then the same as some distribution T^A $(A \in K$; see Section 5). Now if at a point $q \in \overline{V}$, l_q does not lie in the space spanned by $B^{(1)}_q(z_0)$ and T^A_q, a small motion of $B^{(1)}(z_0)$ in the direction l will free q from $V(-\varepsilon, \varepsilon)$. If on the other hand $l_q \in (B^{(1)}_q(z_0) + T^A_q)$, then a motion of $B^{(1)}(z_0)$ in the direction l will displace the intersection q in the direction l^A_q on $V(-\varepsilon, \varepsilon)$, where l^A_q is the projection l_q on T^A_q along $B^{(1)}_q(z_0)$. We will call points $q \in \overline{V}$ for which $l_q \in (B^{(1)}_q(z_0) + T^A_q)$ the A-critical points of l on $\overline{V}$.

LEMMA 6.1. *$\bar{q} = \{X, q\} \in \overline{V}$ is an A-critical point of l if and only if there exists a real ϱ such that*

$$(\{\sqrt{-1}\, J - \varrho J A\}\, q,\ \mathfrak{N}_P) = 0. \tag{6.4}$$

Here $\mathfrak{N}_P =$ null-space of $\{P - z_0 I\}$, $q \in \mathfrak{N}_P$.

Proof: If $l_{\bar{q}} \in (B^{(1)}_{\bar{q}} + T^A_{\bar{q}})$, then we can subtract from $l_{\bar{q}}$ an element of $T^A_{\bar{q}}$ so as to end up in $B^{(1)}_{\bar{q}}$. Using a standard description of $W^{(1)}_{\bar{q}}$ this implies the existence of at least one $(\varrho A, v)$ $(A \in \mathfrak{L},\ v \in E(q))$ such that

$$\{\sqrt{-1}\, I, 0\} - \{\varrho A, v\} \in B^{(1)}_q .$$

By (4.2) this relation can be satisfied if and only if (6.4) holds.

Since $A \in K$, JA defines a nondegenerate positive form on $\mathfrak{N}_P$. Hence (6.4) has at most $s(=$ complex dim. $\mathfrak{N}_P)$ different solutions in ϱ which must be real. If now it is known that $\sqrt{-1}\,J$ is also nondegenerate on $\mathfrak{N}_P$ then none of these solutions is 0. Furthermore by varying $A \in K$ we can arrange it so that (6.4) has precisely s distinct non-zero solutions $\varrho_1, \cdots, \varrho_s$. Then the corresponding eigen directions $q_1 \cdot, \cdots, q_s$ are well defined, span $\mathfrak{N}_P$, and are precisely the A-critical points of l on $\overline{V}$. Moreover at such a critical point $\bar{q}_i$, $l_{\bar{q}_i}$ projects on $\{\varrho_i A, v\}$ via $B^{(1)}_{q_i}(z_0)$. Hence the intersection $\bar{q}_i$ will be moved in the direction $\{\varrho_i A, v\}$ along $V(-\varepsilon, \varepsilon)$ by a motion of $B^{(1)}(z_0)$ in the direction l. After such a small deformation the intersections will have to be regular as is checked readily.

Since we already know that the total contribution of $\overline{V}$ to $[B^{(1)}(z_0), V(-\varepsilon, \varepsilon)]_H$ is $\chi(\overline{V})$, the intersection numbers will have to be $+1$ (in order to add up to $\chi(\overline{V})$). But then the number of these intersections which occur over λ-values < 0 is clearly the number of the negative ϱ_i. This is precisely the statement of Theorem IV.

This completes the proofs of the theorems enunciated in Section 1. In the next section we discuss some general questions concerning the Sturm intersection theory, in particular, the comparison and oscillation theorems. Finally, the homology class of γ_B is computed.

7. The Comparison Theorem.

Let ${}_{(i)}L$ $(i = 0, 1)$ be two operators as in Section 1. Let ${}_{(i)}X'_\lambda(t) = {}_{(i)}A_\lambda(t)\,{}_{(i)}X_\lambda(t)$ be the differential equation of their maps. We define ${}_uX_\lambda(t)$ $(0 \leqq u \leqq 1)$ as the solution of

$$X'(t) = \{(1-u)\,{}_{(0)}A_\lambda(t) + u\,{}_{(1)}A_\lambda(t)\}\,X(t) \tag{7.1}$$

subject to

$$X = I, \text{ at } t = 0 \text{ for } -\infty < \lambda < \infty,\ 0 \leqq u \leqq 1. \tag{7.2}$$

The correspondence

$$(t, \lambda, u) \to {}_uX_\lambda(t)$$

is therefore a map of $\Delta \times [0,1] \to \mathfrak{H}$ which reduces to the map of ${}_{(i)}L$ on $\Delta \times [i]$ $(i = 0,1)$. Let B be a fixed Hermitian boundary condition imposed at $t = 0$ and $t = a \neq 0$, and set ${}_i\Theta_B$ $(i = 0,1)$ equal to the spectral multiplicity function of ${}_{(i)}Ly = \lambda y$ subject to B.

COMPARISON THEOREM (VIII). *If $\tau = [\lambda^- < \lambda < \lambda^+]$ is a bounded interval of the λ-axis which is admissible with respect to both ${}_0\Theta_B$ and ${}_1\Theta_B$, then*

$$[{}_1\Theta_B : \tau] - [{}_0\Theta_B : \tau] = [\gamma_B : C^+] - [\gamma_B : C^-] \tag{7.3}$$

where $C^\pm$ is the curve $u \to {}_uX_{\lambda^\pm}(a)$ $(0 \leqq u \leqq 1)$ and γ_B is the cycle of B in $\mathfrak{H}$.

Proof: Consider the map $(t, \lambda, u) \to {}_uX_\lambda(t)$ restricted to the rectangle $\square$ of the t, λ, u-space; the coordinates of the vertices in the order of orientation are:

$$(a, \lambda^-, 0),\ (a, \lambda^+, 0),\ (a, \lambda^+, 1),\ (a, \lambda^-, 1),\ (a, \lambda^-, 0).$$

Since our map is defined on all of $\Delta \times I$, the image of $\square$ is homologous to zero is $\mathfrak{H}$. Hence the total intersection of $\square$ with γ_B is zero. But this is precisely the content of the formula in the theorem where we have evaluated the intersections along the sides of $\square$ which are parallel to the λ-axis by means of Theorem V.

Notice that the vertices of $\square$ are not on γ_B because τ is admissible with respect to both ${}_{(1)}\Theta$ and ${}_{(2)}\Theta$. Since we know already by Theorem V that any $\oplus$ curve meets γ_B in a discrete set of λ-values this is no serious restriction on τ.

We write $L_1 < L_2$ if, for all λ, t, ${}_2A_\lambda(t) - {}_1A_\lambda(t) \in K$. Under the additional hypothesis that $L_1 < L_2$, it is easily checked (by the obvious analogue of the proof in Section 3) that the curves C^+ and C^- of the above Theorem are $\oplus$ curves. Hence in particular $[\gamma_B : C^\pm] \geqq 0$. An immediate consequence is the

COROLLARY. *If $L_1 < L_2$ and if the spectrum of L_1 is unbounded above (bounded below), then the same is true for L_2.*

Remark. This corollary can be used to prove part (a) of Proposition 1.1 in the following fashion. We construct an $L_1 < L$ with constant coefficients. For L_1, however, the positive definiteness of p easily yields the boundedness of the spectrum from below and the unboundedness from above.

Proposition 1.2 on the other hand is an immediate consequence of the invariance of intersections under restricted deformations.

8. The Oscillation Theorem

Here we envisage a single operator L and compare the problems

(1) $Ly = \lambda y$ subject to B at $t = 0$, $t = a$,

(2) $Ly = \lambda y$ subject to B at $t = 0$, $t = \varepsilon > 0$ $(\varepsilon < a)$.

Let Θ and Θ^ε denote the spectral multiplicity functions of these two problems. C_λ shall stand for the curve : $t \to X_\lambda(t)$ $(\varepsilon \leqq t \leqq a)$.

OSCILLATION THEOREM (IX). *Let $\tau = [\lambda^- < \lambda < \lambda^+]$ be an admissible interval with respect to Θ and Θ^ε. Then in the notation introduced above.*

$$[\Theta : \tau] = [\gamma_B : C_{\lambda^+}]_{\mathfrak{H}} - \{[\Theta^\varepsilon : \tau] - [\gamma_B : C_{\lambda^-}]_{\mathfrak{H}}\}. \tag{8.1}$$

Proof: Apply X^L to the rectangle □ of the t, λ-plane, Δ, whose vertices in order of orientation are:

$$(a, \lambda^-), \ (a, \lambda^+), \ (\epsilon, \lambda^+), \ (\epsilon, \lambda^-), \ (a, \lambda^-).$$

X^L (□) is again homologous to zero in $\mathfrak{H}$ as X^L is defined on all of Δ. Counting the contributions to the intersection of γ_B and □ on the sides of □ yields formula (8.1).

This theorem is called the oscillation theorem for the following reason. Since the spectrum of

$$Ly = \lambda y, \text{ subject to } B \text{ at } t = 0, \ t = x, \qquad 0 \leqq x \leqq a$$

is bounded from below for each x, λ^- can be chosen so negative that $[\gamma_B : C_{\lambda^-}]_{\mathfrak{H}} = 0$. Now if ε is small enough, the term $[\gamma_B : C_{\lambda^+}]_{\mathfrak{H}}$ represents the oscillation of $Ly = \lambda^+ y$ as t goes from 0 to a. Formula (8.1) then represents a relation between "the number of characteristic roots" of the original problem less than λ^+, and the number $[\gamma_B : C_{\lambda^+}]_{\mathfrak{H}}$ which generalizes the classical notion of the "number of zeros" of the general solution of $Ly = \lambda^+ y$ in the interval $\varepsilon \leqq t \leqq a$.

The interpretation of $[\gamma_B : C_{\lambda^+}]$ is not quite obvious, we will, therefore, discuss it in greater detail below, and show that for a certain class of Hermitian boundary conditions — the focal conditions — $[\gamma_B : C_{\lambda^+}]$ can be evaluated by the algebraic formula

$$(8.2) \qquad [\gamma_B : C_\lambda]_{\mathfrak{H}} = \sum_{\varepsilon \leqq t \leqq a} \nu\{C_\lambda(t)M - N\}.$$

In general, of course, no such formula holds; as a matter of fact $C_\lambda(t)$ need not even intersect B^0 in a discrete set of points.

9. Focal Boundary Conditions

Recall that a Hermitian boundary condition B is represented by a pair of maps $\{M, N\}$ of E into itself satisfying the two conditions

$$Mv = Nv = 0 \Longleftrightarrow v = 0,$$
$$M^*JM = N^*JN.$$

Furthermore $E = E_0^{(1)} + E_0^{(2)}$, as defined in Section 1. In this representation J maps $\{x, y\}$ into $\{-y, x\}$ $(x \in E_0^{(1)}, \ y \in E_0^{(2)} ; \ \{x, y\} \in E)$.

DEFINITION 9.1. *A Hermitian boundary condition* $B \supset \{M, N\}$ *shall be called focal if and only if*

$$(9.1) \qquad N(E) \subset E_0^{(2)} .$$

The important feature of the focal boundary conditions is given by the following proposition:

LEMMA 9.1. *Let L be as in Section 1, and let $(X_\lambda^L)'(t) = A_\lambda(t)X_\lambda^L(t)$ be the differential equation of its map. If $\{M, N\}$ represents a focal boundary condition, then*

$$(JA_\lambda(t)Nv, Nv) > 0 \text{ unless } Nv = 0 \quad (\lambda, t, \text{arbitrary}). \tag{9.2}$$

Proof: $JA_\lambda(t)$ has the form

$$\begin{pmatrix} * & * \\ * & p^{-1}(t) \end{pmatrix}.$$

Hence by (9.1) the lemma is clear, because $p^{-1}(t)$ is assumed positive definite.

PROPOSITION 9.1. *Let $B=\{M, N\}$ be focal, and let $C: t \to X_\lambda^L(t)$ $(\alpha \leqq t \leqq \beta)$ be a positively oriented curve whose end points do not intersect $B^{(0)}$. (X^L is the map of a Hermitian L as above.) Under these conditions*

$$[\gamma_B : C]_{\mathfrak{H}} = \sum_{\alpha \leqq t \leqq \beta} \nu\{X_\lambda^L(t)M - N\}. \tag{9.3.}$$

Proof: Let, for the sake of brevity, $X_\lambda^L(t) = X_\lambda(t)$. The direction of C at t is then given by $X_\lambda^{-1}(t) \cdot A_\lambda(t) \cdot X_\lambda(t)$. If C intersects γ_B at t, i.e., if $\{X_\lambda(t)M - N\}v = 0$ $(v \neq 0)$ then

$$(JX_\lambda^{-1}(t)A_\lambda(t)X_\lambda(t)Mv, Mv) = (JA_\lambda(t)Nv, Nv) > 0$$

by (9.2). Hence C is never tangent to γ_B. Suppose now C intersects γ_B at t_0.

Define:

$$C_u : t \to X_{\lambda+u}(t) \cdot \{X_{\lambda+u}(t_0)\}^{-1}X_\lambda(t_0) \qquad (u \text{ real}).$$

Then each C_u passes through $X_\lambda(t_0)$ and the direction of C_u at t_0 is given by

$$X_\lambda^{-1}(t_0) \cdot A_{\lambda+u}(t_0) \cdot X_\lambda(t_0).$$

But for u sufficiently large $X_\lambda^{-1}(t_0)A_{\lambda+u}(t_0)X_\lambda(t_0) \in K$ as is easily checked. If we consider u as a deformation parameter this shows that C can, in the vicinity of t_0, be deformed (through curves which are never "tangent" to B^0 at q) into a $\oplus$ curve through q. C therefore behaves like a $\oplus$ curve in respect to γ_B if B is focal. Theorem V now yields (9.3).

A completely similar argument shows that for ϵ small enough and B focal, the term $[\Theta^\varepsilon : \tau]_{\mathfrak{H}}$ in (8.1) is zero. In that special case (8.1), therefore, simplifies to

$$[\Theta : \tau]_{\mathfrak{H}} = \sum_{0 < t \leqq a} \nu\{X_{\lambda^+}^L(t)M - N\} \tag{9.4}$$

if λ^- is sufficiently negative. This formula, when appropriately reinterpreted, is precisely the focal point theorem of M.Morse [6, p. 58]. We make the translation into his terminology for a special focal boundary condition

B described below. It will serve at the same time to justify the claim that $[\gamma_B : C]_{\mathfrak{H}}$ represents oscillation.

The focal boundary condition B which we have in mind is described by

$$M = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

(This is equivalent to $y(0) = y(a) = 0$ and is the classical Sturm condition.) Now if $X^L_{\lambda^+}(t) = \begin{Bmatrix} \alpha(t)\,\beta(t) \\ \gamma(t)\,\delta(t) \end{Bmatrix}$, then $\nu\{X^L_{\lambda^+}(t)\,M - N\} = \nu\{\beta(t)\}$. In this case therefore the right hand side of (9.4) can be interpreted in the following fashion:

Let $y_1, \cdots, y_n$ be n linearly independent solutions of $Ly = \lambda^+ y$ which vanish at $t = 0$. Let $\beta(t)$ be the matrix whose columns are the y_i. Then if $\beta(a)$ is nonsingular, the sum $\sum_{0<t\leq a} \nu\{\beta(t)\}$ equals the weighted sum of eigenvalues (of $Ly = \lambda y$ subject to our particular B at $t = 0$ and $t = a$) less than λ^+.

This is precisely the focal point theorem of Morse for our B.

10. The Homology Class of γ_B

It is natural, for a topologist at least, to inquire what the homology class of the cycle γ_B in $\mathfrak{H}$ is. It is clear that γ_B is not homologous to zero; for, a closed $\oplus$ curve will have nontrivial intersection number with γ_B. However, to describe the class γ_B precisely, we have to find out how often γ_B intersects curves generating $H_1(\mathfrak{H}; I)$ (finite cycles).

We will only state our results. They are easily proved using Theorem V, and the proposition that the set, $\mathfrak{B}$, of Hermitian boundary conditions is homeomorphic to $U(2n)$, the unitary group — in particular that $\mathfrak{B}$ is connected.

THEOREM X. *The cycles γ_B of all Hermitian boundary conditions are homologous. Their common class γ is a locally finite integral class which, due to the Poincaré duality in $\mathfrak{H}$, is completely described by its intersection classes with $H_1(\mathfrak{H}; I)$ (finite integral homology).*

THEOREM XI. $H_1(\mathfrak{H}; I) = Z + Z$ *(is the direct sum of two copies of the integers). The classes a, $b \in H_1(\mathfrak{H}; I)$ represented by*

a: $S^1 \to \mathfrak{H};\ a(z) = zI,$

b: $S^1 \to \mathfrak{H};\ b(\exp i\,\theta) = \exp(\theta J^*)$

generate $H_1(\mathfrak{H}; I)$ and

$[\gamma : a]_{\mathfrak{H}} = 0,\ [\gamma : b]_{\mathfrak{H}} = 2n.$

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