ANNALS OF MATHEMATICS
Vol. 60, No. 2, September, 1954
*Printed in U.S.A.*

# NONDEGENERATE CRITICAL MANIFOLDS

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(Received February 4, 1953)

## Introduction

Let $J$ be a smooth[1] function defined on an open set $U$ of Euclidean $n$ space $E_n$. A point $x \in U$ is called a critical point of $J$ if the gradient of $J$ vanishes at $x$.[2] The nullity and index of a critical point $x$ of $J$ are defined as the dimension of the null space and the number of negative characteristic roots of the Hessian[3] of $f$ evaluated at $x$, respectively.

It is a basic result of the Morse theory [1] that if $x$ is a critical point of nullity 0, then for all sufficiently small neighborhoods $V$ of $x$ (in $U$) the following isomorphisms hold:

$$H_k\,(J_x \cap V,\ J_x^- \cap V) \approx H_{k-\lambda_x}\,(x) \tag{I}$$

In this formula $J_x[J_x^-]$ denotes the subset of $U$ on which $J \leqq J(x)$ $[J < J(x)]$, while $\lambda_x$ is the index of $x$. $H_k\,(J_x \cap V,\ J_x^- \cap V)$ and $H_k(x)$ $(k = 0, 1, \cdots)$ refer to the $k^{\text{th}}$ singular homology group [9] of the pair $(J_x \cap V,\ J_x^- \cap V)$ at the left and of the point $x$ at the right. The coefficients are immaterial; however, we shall assume throughout the paper that all homology groups considered are computed mod 2.

The purpose of this paper is the proof and application of the following extension of (I).

DEFINITION. Let $M \subset U$ be a compact manifold smoothly imbedded in $U$ such that (a) all points of $M$ are critical points of a smooth function $J$, defined on $U$, and (b) the nullity of all $x \in M$ equals the dimension of $M$; then $M$ will be called a nondegenerate critical manifold of $J$ on $U$.

THEOREM I. *If $M$ is a nondegenerate critical manifold of $J$ on $U$, then for all sufficiently small neighborhoods $V$ of $M$*

$$H_k\,(J_M \cap V,\ J_M^- \cap V) \approx H_{k-\lambda_M}(M) \tag{I*}$$

*where $J_M[J_M^-]$ refers to the subset of $U$ on which $J \leqq J(x)$ $[J < J(x)]$ $x \in M$, and $\lambda_M$ denotes the index of some $x \in M$.*[4]

As an application we compute the "sensed circular connectivities" of the $n$-sphere. Our result is the following one: (see §2 for exact definitions) if we de-

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[1] The term "smooth" will mean "at least three times continously differentiable" throughout the paper.

[2] i.e. $\partial J/\partial x_i(x) = 0$ $(i = 1 \cdots n)$.

[3] The Hessian of $J$ is the $n \times n$ matrix of second partial derivatives of $J$ with respect to $x_i$ $(i = 1 \cdots n)$.

[4] $\lambda_M$ and $J_M$, $[J_M^-]$ are independent of $x \in M$, as our conditions guarantee that $J$ and the nullity are constants on $M$.

note by $\mathfrak{M}(S^n; z) = \sum_0^\infty p_k z^k$ the formal power series with $p_k$ equal to the $k^{\text{th}}$ sensed circular connectivity of $S^n$ we show that

$$\text{(II)} \qquad \mathfrak{M}(S^n; z) = P(G^2_{n-1}; z)\, z^{n-1}\,(1 - z^{2(n-1)})^{-1}$$

where $P(G^2_{n-1}; z)$ is the Poincaré polynomial of the mod 2 Betti numbers of the Grassmann manifold of oriented 2-planes in Euclidean $n + 1$ space. For example, $\mathfrak{M}(S^2; z) = (1 + z^2)(z + z^3 + z^5 \cdots)$.

Our method also yields a formula for the circular connectivites $\bar{p}_k$ of the $n$-sphere, namely:

$$\text{(III)} \qquad \overline{\mathfrak{M}}(S^n; z) = P(\bar{G}^2_{n-1}; z)\, z^{n-1}(1 - z^{2(n-1)})^{-1}$$

where now $\overline{\mathfrak{M}}(S^n; z) = \sum_0^\infty \bar{p}_k z^k$, while $\bar{G}^2_{n-1}$ stands for the Grassmann manifold of unoriented 2-planes in $E_{n+1}$.

In [1] Morse characterised $\bar{p}_k$ as the number of solutions of certain diophantine equations ([1]; p. 349). It appears therefore that in his classic derivation of $\overline{\mathfrak{M}}(S^n; z)$ Morse obtained the Betti numbers of $\bar{G}^2_{n-1}$ as a byproduct.

The factorization of $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ as given by II and III is not without a certain intuitive appeal. Recall that Morse introduced the numbers $\bar{p}_k$ for the purpose of counting closed geodesics on manifolds. On the other hand the $n$-sphere in its usual representation in $E_{n+1}$ admits a continuum of closed geodesics, which in a rather obvious topology is homeomorphic to $G^2_{n-1}$ if the geodesics are taken as oriented, and to $\bar{G}^2_{n-1}$ if they are taken as unoriented. Furthermore, a given geodesic can be traversed an integral number of times; this would then account for the factor $z^{n-1}(1 - z^{2(n-1)})^{-1}$.

The sensed circular connectivities $p_k$ introduced here can be shown to have the same relation to the "nonreversible problem" that the $\bar{p}_k$ have to the "reversible problem" considered by Morse. This relationship and some of its applications will be reported at another time.

## 1. Proof of (I*)

We recall from [1] that the proof of (I) is constructed as follows: Consider the linear manifold $E_x$ of vectors $x + y$ passing through the critical point $x$, spanned by those vectors $y$ which are linear combinations of the characteristic vectors associated with the negative characteristic roots of the Hessian of $J$ at $x$. The dimensionality of this manifold is $\lambda_x$. The formula (I) is established by the construction of a homotopy retraction of the pair $(J_x, J_x^-) \cap V_\varepsilon$ onto the pair $(E_x, E_x - x) \cap V$ valid for all sufficiently small spherical neighborhoods $V_\varepsilon$ of $x$ (see [2] p. 34).

To prove (I*) we first surround $M$ by an $\varepsilon$ neighborhood $V_\varepsilon$ so small that the mapping $p: V_\varepsilon \to M$ defined by the orthogonal projection of points of $V_\varepsilon$ onto $M$ is well defined and constitutes a fiber decomposition of $V_\varepsilon$, with base space $M$, and fiber the orthogonal cells $F'_x$ of dimension $n - \dim M$. Thus $\bar{V}_\varepsilon$ can be considered as the mapping cylinder of the normal sphere bundle of $M$. For each $x \in M$ the manifold $E_x$ is well defined and will, as a consequence of condition $b$,

be completely orthogonal to $M$ at $x$. Let $A_\varepsilon \subset V_\varepsilon$ be the subbundle of $V_\varepsilon$ consisting of those $x \in V_\varepsilon$ for which $x \in E_{p(x)}$. $A_\varepsilon$ is thus a bundle whose fibers $F_x$ are $\lambda_M$-cells. $M$ is imbedded in $A_\varepsilon$. We set $A^-$ equal to $A_\varepsilon - M$. Now compare the pair $(A_\varepsilon, A_\varepsilon^-)$ with $(J_M, J_M^-) \cap V_\varepsilon$. Clearly $(A_\varepsilon, A_\varepsilon^-) \subset (J_M, J_M^-) \cap V_\varepsilon$ for sufficiently small $\varepsilon$. However, the following much stronger proposition holds:

PROPOSITION. *For $0 < \varepsilon$ small enough $(A_\varepsilon, A_\varepsilon^-)$ is a fiber preserving deformation retract of $(J_M, J_M^-) \cap V_\varepsilon$.*

*In particular, therefore*

$$H_m\{(J_M, J_M^-) \cap V_\varepsilon\} \approx H_m(A_\varepsilon, A_\varepsilon^-)$$

*for small enough $\varepsilon$.*

PROOF. The proof proceeds in entire analogy to that of (I) given in [2. p. 34]. We set $V$ equal to the "abstract" normal bundle [8] of $M \subset U$. $E_x$, $x \in M$ is a well defined subspace of the fiber of $V$ at $x$. Let $D_x$ be the complementary subspace to $E_x$ and let $E$, $D$ denote the subbundle consisting of $\bigcup_{x \in M} E_x$, $\bigcup_{x \in M} D_x$ respectively. We set $H^+$ and $H^-$ equal to the projections defined on $V$ which map the normal space at $x$ into $D_x$ and $E_x$ respectively. $\mathrm{H}^+$ and $\mathrm{H}^-$ are continuous fiber mappings of $V$ onto $D$ and $E$ respectively. Consider now the following deformation which establishes $E$ as a fiber and norm preserving deformation retract of $V - D$. This deformation will be parametrized by $\pi/4 \leqq \theta \leqq \pi/2$, and assigns to each $y \in V$ the element $y_\theta = \dfrac{\cos \theta H^+ y + \sin \theta H^- y}{|\cos \theta H^+ y + \sin \theta H^- y|} \mid y \mid$ for $y \neq 0$, and $y_\theta = y$ for $y = 0$. (We use a Euclidean metric in the fiber and $\mid y \mid$ denotes the length of a vector $y$.)

On each fiber this deformation reduces to the one used by Seifert and Threlfall in [2].

Now for small enough $\varepsilon$ we can identify $V$ and $V_\varepsilon$ so as to take radial lines into radial lines. Then the restriction of $J$ to a given fiber $F$ will have a critical point of index 0 at the intersection of $F$ and $M$. Hence the argument of [2] applies and for sufficiently small $\varepsilon$ the proposition to be proved will be true on each individual fiber. By continuity and compactness it follows that the proposition as stated is correct.

Hence (I*) will be demonstrated if we prove that $H_k(A_\varepsilon, A_\varepsilon^-) \approx H_{k-\lambda_M}(M)$. As we are using mod 2 coefficients throughout, it is sufficient to prove the equality of the ranks of the groups in question.

Consider now the exact sequence of the pair $(A_\varepsilon, A_\varepsilon^-)$:

$$(1.1) \qquad \to H_n(A_\varepsilon^-) \to H_n(A_\varepsilon) \to H_n(A_\varepsilon, A_\varepsilon^-) \to H_{n-1}(A_\varepsilon^-) \to$$

$A_\varepsilon^-$ is a bundle with fiber deformable into a $\lambda_M - 1$ sphere. Accordingly we have the exact Gysin, Thom, Chern – Spanier [3] [4] [5] sequence of $A_\varepsilon^-$ as follows:

$$(1.2) \qquad \to H_n(A_\varepsilon^-) \to H_n(M) \to H_{n-\lambda_M}(M) \to H_{n-1}(A_\varepsilon^-) \to.$$

Comparing the two sequences it is easily seen that $H_{n-\lambda_M}(M)$ and $H_n(A_\varepsilon, A_\varepsilon^-)$ are equivalent group extensions and therefore have the same rank. (We merely have to establish the commutativity of the square

$$\begin{array}{ccc} H_n(A_\varepsilon^-) & \to & H_n(A_\varepsilon) \\ \uparrow & & \uparrow \\ H_n(A_\varepsilon^-) & \to & H_n(M) \end{array}$$

which of course is well known [4]. We might add that in Thom's abstract [4] of the proof of (1.1) the isomorphism $H_{m-\lambda_M}(M) \overset{\phi^*}{\leftarrow} H_m(A_\varepsilon, A_\varepsilon^-)$ is stated as the basic lemma and (1.2) deduced from it. The Thom result asserts the isomorphism in question to exist for a suitable local system of coefficients on $M$.)

REMARK. The local nature of the whole problem makes it clear that the theorem is equally true if we replace the imbedding space $U$ by an open subset of an arbitrary Riemann manifold.

## 2. The circular connectivities

The $m^{\text{th}}$ circular connectivity $\bar{p}_m$ of a Riemann manifold $R$ is defined by Morse [1] as the rank of the limit group $\overline{\mathfrak{M}}_m(R)$ of the $m^{\text{th}}$ mod 2 homology groups of a directed system of pairs [9] $\{(\bar{\pi}_k, \bar{\Delta}_k); h\}$. $\bar{\pi}_k(R)$ (written $\bar{\pi}_k$ when there is no danger of confusion!) is defined as follows: In $R^k = R \times \cdots \times R$ ($k$ factors) consider the subset of points $(x) = \{x_1 \cdots x_k\}$ such that the distance between consecutive points in the array $x_1, x_2, \cdots x_n, x_1$ on $R$ is less than the elementary length $\rho$ of geodesics on $R$. (See [1] p. 196.) We denote this subset in $R^k$ by $\Pi_k$. A point $(x)$ of $\Pi_k$ uniquely determines a closed geodesic polygon on $R$, namely the polygon obtained by joining consecutive points of the above array by elementary geodesics. If in $\Pi_k$ we identify those points which differ either by a cyclic permutation of the factors of $R^k$, or by a reversal of the order of the factors of $R^k$, we obtain the space $\bar{\pi}_k$ (i.e. $(x_1, \cdots, x_k) \sim (x_k, x_1 \cdots x_{k-1}) \sim (x_k, x_{k-1}, \cdots x_1)$ in $\bar{\pi}_k$). The points of $\bar{\pi}_k$ therefore correspond to closed unsensed $k$ edged geodesic polygons on $R$, none of whose vertices are distinguished.

By introducing $s$ equally spaced vertices on each of the sides of a polygon $(x) \in \bar{\pi}_k$ a new polygon with $sk$ vertices is obtained. This subdivision operation gives rise to maps $h_s$ of $\bar{\pi}_k$ into $\bar{\pi}_{sk}$ which make the family $\bar{\pi}_k$ into the directed system $\{\bar{\pi}_k ; h\}$. If we now set $\bar{\Delta}_k$ equal to the diagonal in $\bar{\pi}_k$, (that is, the set of points of the form $(x) = (x, x, \cdots x)$) we obtain the desired family of pairs $\{(\bar{\pi}_k, \bar{\Delta}_k); h\}$.

The length of a polygon is a continuous function $J$ defined on all $\bar{\pi}_k$. We clearly have $J(x) = J(h_s(x))$, $x \in \bar{\pi}_k$ (all $s$); hence if $\bar{\pi}_k^c$ $[\bar{\pi}_k^{c-}]$ denotes the subspace of $\bar{\pi}_k(R)$ on which $J \leqq c$ $[J < c]$; $c \geqq 0$, $h$ restricted to $\bar{\pi}_k^c$ $[\bar{\pi}_k^{c-}]$ makes the family $\{\bar{\pi}_k^c, \bar{\Delta}_k ; h\}$ $[\{\bar{\pi}_k^{c-}, \bar{\Delta}_k ; h\}]$ a directed system. The group $\overline{\mathfrak{M}}(J_c ; R)$ $[\overline{\mathfrak{M}}(J_{c-} ; R)]$ is defined as the limit homology group of $\{(\bar{\pi}_k^c, \bar{\Delta}_k); h\}$ $[\{(\bar{\pi}_k^{c-}, \bar{\Delta}_k); h\}]$.

By the symbol $\overline{\mathfrak{M}}(J_c, J_{c'} ; R)$ $c > c'$ we will mean the limit relative homology group of the family of pairs $\{\bar{\pi}_k^c, \bar{\pi}_k^{c'} ; h\}$.

REMARK. The definition of $\overline{\mathfrak{M}}(R)$ seems to be dependent on Riemann structure of $R$. However, Morse showed that $\overline{\mathfrak{M}}(R)$ is a topological invariant of $R$. [See [1] p. 297.] His proof can easily be modified to yield also the topological invariance of the group $\mathfrak{M}(R)$ defined in the next section:

### 3. The sensed circular connectivities

In the previous construction $\bar{\pi}_k(R)$ was obtained from $\Pi_k(R)$ be identifying on $\Pi_k(R)$ points which were images under the group of cyclic permutations, or reversal of order. Let $\pi_k(R)$ be the space obtained from $\Pi_k(R)$ by only identifying those points which are cyclically equivalent. Then if we replace $\bar{\pi}_k$ by $\pi_k$ in the construction of §2, we obtain the "sensed" counterparts of $\overline{\mathfrak{M}}(R)$ etc. which we denote by $\mathfrak{M}(R)$, etc. We will adhere to the notation that the barred letters stand for the equivalent in the "unoriented" theory of the unbarred letters in the "oriented" theory.

### 4. On the computation of $\mathfrak{M}(S^n; z)$

Let $S^n$ be represented by the subset $\sum_1^{n+1} x_i^2 = 1/(2\pi)^2$ in Euclidean $n+1$ space. The metric induced on $S^n$ by the Euclidean metric makes $S^n$ into an analytic Riemann manifold. It is well known that then the closed geodesics on $S^n$ "are" the intersection of $S^n$ with two-planes $E^2 \subset E^{n+1}$. The length of these geodesics is $t$, $t$ being the positive integral value of the number of times the point set $E^2 \subset S^{n+1}$ is traversed. For $\rho$ we take the number $\frac{1}{3}$.

The computation of $\mathfrak{M}(S^n; z)$ is broken up into two steps. First we compute the group $E_1 = \sum_1^\infty \mathfrak{M}(J_t, J_{t-1})$ (direct sum). $E_1$ will turn out to be finitely generated in each dimension.

Next let $E_\infty = \sum_1^\infty D_t/D_{t-1}$ (direct sum) where $D_t$ is the image of $\mathfrak{M}(J_t)$ in $\mathfrak{M}(R)$ under the inclusion map induced by the inclusions $(\pi_k^t, \Delta) \to (\pi_k, \Delta)$. As $\pi_k^0 = \Delta$ and $\pi_k^t = \pi_k$ for sufficiently large $t$, we have

$$0 = D_0 \subset D_1 \subset \cdots \subset \mathfrak{M}(R) = \bigcup_0^\infty D_t .$$

The second step of the computation consists of showing that $E_1$ and $E_\infty$ are isomorphic in a dimension preserving fashion. It follows then that $\mathfrak{M}_n(R)$ is also finitely generated in each dimension and as we are working over a field of coefficients $E_\infty$ is (unnaturally) isomorphic to $\mathfrak{M}(R)$ in a dimension preserving fashion. The procedure of going from $E_1$ to $E_\infty$ is of course the basic method of the classical work of Morse as well as of the modern spectral theory of Leray. $E_1$ is obtained by local considerations while the step from $E_1 \to E_\infty$ usually involves arguments "in the large". In the case under consideration however, it turns out very fortuitously that $E_1 \approx E_\infty$ for purely "dimensional" reasons.

### 5. Computation of $E_1$

Let $\tilde{\mu}_k$ $[\mu_k]$ denote the subset of $\Pi_k$ $[\pi_k]$, each point of which represents a polygon whose $k$ edges are equal in length. A point $z \in \tilde{\mu}_k$ $[\mu_k]$ is called critical if it represents a closed geodesic. Clearly in our case $J$ takes on only integral values on critical points of $\tilde{\mu}_k$ $[\mu_k]$. We set $\tilde{\sigma}_{t,k}$ $[\sigma_{t,k}]$ equal to the totality of critical points

of length $t$ in $\tilde{\mu}_k$ [$\mu_k$]. As $\rho = \frac{1}{3}$, $\tilde{\sigma}_{t,k}$ [$\sigma_{t,k}$] will be nonvacuous for $t < 3k$, and we assume in the following that this condition is always satisfied.

It is shown by Morse ([1], Theorem 5.1, p. 261) that for sufficiently small neighborhoods $\tilde{V}_{t,k}$ of $\tilde{\sigma}_{t,k}$ in $\tilde{\mu}_k$, $\tilde{V}_{t,k}$ is an analytic manifold (in the subset topology induced by the imbedding of $\tilde{\mu}_k \subset \Pi_k$). As points of $\tilde{V}_{t,k}$ are well away from the diagonal in $\Pi_k$, the projection $\tilde{\pi}_k \to \pi_k$ induces a manifold structure on the image $V_{t,k}$ of $\tilde{V}_{t,k}$.

The following basic proposition due to Morse establishes the local character of $E_1$. (We have phrased it to fit into our context.)

THEOREM A. (Morse) *Let $V_{t,k}$ be any family of sufficiently small neighborhoods of $\sigma_{t,k} \subset \mu_k$ ($t < 3k$), such that $h_s: V_{t,k} \subset V_{t,sk}$ and let $V^*_{t,k}[V^-_{t,k}]$ be the subset of $V_{t,k}$ on which $J \leqq t$, [$J < t$]. Then*

$$\mathfrak{M}_m(J_t, J_{t-1}) \approx \lim_{(k)} \{H_m(V^*_{t,k}, V^-_{t,k}); h\}.$$

REMARK. Theorem A can be deduced roughly in the following fashion: One first shows that $\mathfrak{M}_m(J_t, J_{t-1}) \approx \mathfrak{M}_m(J_t, J_{t-})$ using the fact that there are no critical points of $J$ between $t$ and $t - 1$. Next one shows that for any system of neighborhoods $\dot{V}_{t,k}$ of $\sigma_{t,k}$ on $\pi_k$ with $h_s \dot{V}_{t,k} \subset \dot{V}_{t,ks}$,

$$\mathfrak{M}_m(J_t, J_{t-}) \approx \lim_k \{H_m(\dot{V}^*_{t,k}, \dot{V}^-_{t,k}); h\}$$

(here the * and $^-$ superscript have the same meaning as in Theorem A). Finally for a sufficiently small family $\dot{V}_{t,k}$, the pair $(\dot{V}^*_{t,k}, \dot{V}^-_{t,k})$ can be $J$ deformed into $(\dot{V}^*_{t,k}, \dot{V}^-_{t,k}) \cap u_k = (V^*_{t,k}, V^-_{t,k})$.

All the deformations needed to carry this program out are given by Morse in [1]. (p. 250–270.) I am planning to give a more detailed expository account of this phase of the Morse theory in the near future.

With the aid of Theorem A the computation of $E_1$ is reduced to the study of the groups $H_m(V^*_{t,k}, V^-_{t,k})$ and the homomorphisms

$$h_{s*}: H_m(V^*_{t,k}, V^-_{t,k}) \to H_m(V^*_{t,ks}, V^-_{t,ks})$$

induced by the restriction of $h_s$ to $V_{t,k}$.

It is clear that if we choose $V_{t,k}$ small enough $\sigma_{t,k}$ will be the only critical set of the function $J$ on $V_{t,k}$. We propose to show that $\sigma_{t,k}$ is a nondegenerate critical manifold of $J$ on $V_{t,k}$ in the sense of §1.

If we assign to each oriented closed geodesic $x \in \sigma_{t,k}$ the oriented two plane in which it lies, we obtain a representation of $\sigma_{t,k}$ as a 1-sphere bundle over $G^2_{n-1}$ the Grassmann manifold of oriented 2 planes in $E^{n+1}$. The dimension of $\sigma_{t,k}$ is therefore $2n - 1$.

Now let $x$ be a point of $\sigma_{t,k}$. As a critical point of $J$ on $V_{t,k}$ $x$ has a well defined index and nullity.

LEMMA. *The nullity and index of any point $x \in \sigma_{t,k}$ ($t < 3k$) are respectively given by*

$$N(x) = 2n - 1$$

$$\lambda(x) = (n - 1)(2t - 1).$$

*It follows that* $\sigma_{t,k}$ *is a nondegenerate critical manifold of* $V_{t,k}$ *and hence by* (I*)

$$H_m(V^*_{t,k}, V^-_{t,k}) \approx H_{m-(2t-1)(n-1)}(\sigma_{t,k})$$

*for sufficiently small neighborhood* $V_{t,k}$.

PROOF. In the neighborhood of $\sigma_{t,k}$, $\pi_k$ is an analytic manifold on which $J$ is well defined. $\mu_k$ is smoothly imbedded in $\pi_k$. Every point $x$ of $\sigma_{t,k}$ is a critical point of $J$ on $\pi_k$ as well as of $J$ on $\mu_k$. It follows that $N(x)$ and $\lambda(x)$ can be interpreted as the nullity and number of negative characteristic roots of the Hessian of $J$ as a function on $\pi_k$ restricted to the tangent space of $\mu_k$ in $\pi_k$ at $x$.

At each vertex $p_i$ $(i = 1 \cdots k)$ of $x$ we introduce a local analytic coordinate system $v^i_1 \cdots, v^i_n$ covering a neighborhood $U_i$ of $p_i$ in such a manner that the point $p$ on $x$, whose signed distance from $p_i$ is equal to $d$ (recall that $x$ is oriented) has coordinates $v^i_1 = d; v^i_2 = v^i_3 \cdots v^i_n = 0$, for sufficiently small $|d|$. To emphasize this choice of coordinates notationally we shall write $d^i$ for $v^i_1$. Clearly a neighborhood $U$ of $x$ in $\pi_k$ is given by those polygons whose $i^{\text{th}}$ vertex lies on $U_i$ $(i = 1 \cdots k)$. (As we are working locally and away from the diagonal in $\pi_k$ it is clearly "all right" to number our vertices, starting at any vertex we please and then adhering to the choice made.) Consider now the expansion of $J$ in terms of the local coordinates introduced. As $J$ is independent of $d^i$ when $v^i_2 = \cdots = v^i_n = 0$, all partials of $J$ with respect to the $d^i$'s are 0. Also as $x$ is a closed geodesic for any choice of the $d^i$ when $v^i_2 = \cdots = v^i_n = 0$, $\partial J/\partial v^i_j = \partial^2 J/\partial d^i \partial v^{i'}_j = 0$ for $(i, i' = 1 \cdots k, j = 2 \cdots n)$. Hence $J$ has an expansion which starts off with $J = J(x) + Q_x$, where $Q_x$ is a quadratic form in the variables $(v^i_2, \cdots v^i_n)$ $(i = 1 \cdots k)$ only.

This fact can also be expressed in the following fashion. Let $W \subset U$ be the submanifold of $U$ given by the equations $d^i = 0$ $(i = 1 \cdots k)$. $W$ is called by Morse a proper section of $x$. Let $f: U \to W$ be the projection of $U$ onto $W$ defined by: $f(v^1_1, v^1_2 \cdots; v^2_2, v^2_2 \cdots; \cdots; v^k_1, v^k_2 \cdots v^k_n) = (0, v^1_2, \cdots; 0, v^2_2 \cdots; \cdots; 0, v^k_2 \cdots)$. The above result is then equivalent to the assertion that $J(f(x))$ and $J(x)$ coincide on $U$ up to terms of the second order.

The index and nullity respectively of a geodesic $x$ with respect to a proper section $W$ is defined by Morse (see [1], pg. 288) as

$$N_W(x) = \text{nullity of } Q_x$$

(as a function of $v^i_2 \cdots v^i_n$ only!) and

$$\lambda_W(x) = \text{index of } Q_x.$$

Now the numbers $N_W(x)$; $\lambda_W(x)$; $N(x)$, $\lambda(x)$ are related in the following fashion

$$N(x) = N_W(x) + 1 \tag{5.1}$$

$$\lambda(x) = \lambda_W(x). \tag{5.2}$$

This is seen as follows. To compute $N(x)$ we have to consider the nullity of $Q_x$ in the tangent space of $\mu_k \subset \pi_k$, at $x$. However in the local coordinate systems

which we have chosen, the tangent space of $u_k$ at $x$ is clearly characterized by the equations $d^1 = d^2 = \cdots = d^k$. As the $d^i$'s do not occur at all in $Q_x$ the formulae (5.1) and (5.2) become self evident.

A fundamental result of the Morse theory is an invariance theorem (see [1], p. 289) which in our context reads as follows.

THEOREM 6. (Morse) *Let $x \in \sigma_{t,k}$, and let $x' \in \sigma_{t,ks}$ be the image of $x$ under the subdivision map $h_s$ (for $s = 1$, $h_1$ is the identity map). Let $W$ be a proper section of $x$ and $W'$ a proper section of $x'$. Then*

$$N_W(x) = N_W(x') \text{ and } \lambda_W(x) = \lambda_{W'}(x'). \tag{5.3}$$

*It follows that $N(x)$ and $\lambda(x)$ are independent of the particular proper section used and invariant under subdivision; i.e. independent of $k$.*

We can therefore choose $k = 4t$ for the explicit computations. This choice of $k$ makes the length between consecutive points of $x$ equal to $\frac{1}{4}$. We now compute $N_W(x)$ and $\lambda_W(x)$ for a special proper section $W$, which in effect has already been constructed by Seifert and Threlfall in another connection (see [2], p. 61).

We recall their construction briefly for the case of $S^2$. Let $x \in \sigma_{t,4t}$ be represented by the polygon which starting at $p_0$ (the point where the $x_3$ axis intersects $S^2$), circles the sphere $t$ times in the (2, 3) plane. The second vertex of $x$ will then lie on the first intersection of $x$ with the (12) plane, $p_3$ will be the antipode of $p_1$, $p_4$ the antipode of $p_2$ etc. To construct the proper section $W$ we take open intervals about the vertices of $x$ on geodesics cutting these vertices at right angles to $x$, and use as local coordinates $\{v_2^i\}$ on these, the geodesic length. The well known cosine law of spherical trigonometry then easily yields for $Q_x$ the quadratic form

$$Q_x = -(v_2^1 v_2^2 + v_2^2 v_2^3 + \cdots + v_2^{4t-1} v_2^{4t} + v_2^{4t} v_2^1)$$

whose nullity is seen to be 2 and whose index is $(2t - 1)$. Using the analogous construction for $S^n$, $Q_x$ is seen to take the form

$$\begin{aligned} Q_x = &-(v_2^1 v_2^2 + v_2^2 v_2^3 + \cdots + v_2^{4t-1} v_2^{4t} + v_2^{4t} v_2^1) \\ &-(v_3^1 v_3^2 + \quad \cdots \quad + v_3^{4t-1} v_3^{4t} + v_3^{4t} v_3^1) \\ &-(v_n^1 v_n^2 + \quad \cdots \quad + v_n^{4t-1} v_n^{4t} + v_n^{4t} v_n^1) \end{aligned}$$

whence its index is $(2t - 1)(n - 1)$ and its nullity $2(n - 1)$. Hence

$$N(x) = 2(n - 1) + 1 = 2n - 1$$

and

$$\lambda(x) = (2t - 1)(n - 1).$$

This completes the proof of the lemma.

The next step in the computation of $E_1$ is the investigation of the subdivision homomorphism

$$h_{s*}: H(V_{t,k}^*, V_{t,k}^-) \to H(V_{t,ks}^*, V_{t,ks}^-).$$

The Thom construction of the isomorphism $\phi_*$ of (I*) makes it evident that $\phi_*$ can be so chosen that commutativity holds in the following diagram:

$$\begin{array}{ccc} H(V^*_{t,k}, V^-_{t,k}) & \xrightarrow{h_{s*}} & H(V^*_{t,ks}, V^-_{t,ks}) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ H(\sigma_{t,k}) & \xrightarrow{(h_s \mid \sigma_{t,k})_*} & H(\sigma_{t,ks}). \end{array} \tag{5.4}$$

Hence it is sufficient to study the limit group of the directed system $\{H_m(\sigma_{t,k}), (h \mid \sigma_{t,k})_*\}$ ($t$ fixed!).

It is convenient at this point to transfer the computation to cohomology in order to exploit the multiplicative properties of the cohomology ring of $\sigma_{t,k}$. As the underlying coefficient system is the compact field of integers mod 2 the direct limit homology groups and inverse limit cohomology groups will be isomorphic, assuming the limit groups to be finite.

We recall that $\sigma_{t,k}$ is a 1-sphere bundle over the simply connected manifold $G^2_{n-1}$. Let $\hat{G}^2_{n-1}$ be the Stiefel manifold of two-frames in $E^{n+1}$ (i.e. the tangent bundle of $S^n$). $\hat{G}^2_{n-1}$ is also a 1-sphere bundle over $G^2_{n-1}$. We further have a bundle map of

$$\hat{G}^2_{n-1} \longrightarrow \sigma_{t,k}$$

which is constructed by assigning to each tangent vector $v$ on $S^n$, the $x \in \sigma_{t,k}$ which has one of its vertices at the base point of $v$ and which lies in the two-plane determined by $v$ and the origin of $E^{n+1}$. This map is fiber preserving and induces the identity map of the base spaces. (It is in fact a covering projection.)

If $p^1, \cdots, p^k$ are the vertices of $x \in \sigma_{t,k}$ then only $a_{t,k}$ of the points $p^i$ will be distinct, where $a_{t,k}$ is defined as the numerator of $k/t$ written in its lowest terms. Hence in the mapping $\hat{G}^2_{n-1} \longrightarrow \sigma_{t,k}$ the fiber $S^1$ of $\hat{G}^2_{n-1}$ is mapped onto the fiber $S^1$ of $\sigma_{t,k}$ with degree $a_{t,k}$.

Consider now the Leray spectral sequence of the bundle $\sigma_{t,k} \longrightarrow G^2_{n-1}$. (See Serre [6] in particular p. 470.) Using cohomology with mod 2 coefficients and Serre's notation and terminology throughout, we have the canonical ring isomorphism

$$E_2^{p,q}(\sigma_{t,k}) \approx H^p(G^2_{n-1}) \otimes H^q(S^1).$$

We define $c_{t,k} \in H^2(G^2_{n-1})$ as the class $d_2 a$ where $a$ generates $H^1(S^1)$. We let $c \in H^2(G^2_{n-1})$ be $d_2 a$ in $E_2(\hat{G}^2_{n-1})$. The mapping

$$\sigma_{t,k} \longrightarrow \hat{G}^2_{n-1}$$

induces a homomorphism of $E_2^{p,q}(G^2_{n-1}) \longrightarrow E_2^{p,q}(\sigma_{t,k})$ which is the identity on $H(G^2_{n-1})$, has degree $a_{t,k}$ on $H(S^1)$, and commutes with $d_2$. (See [10], p. 130, also [11].) It follows that $c_{t,k} = a_{t,k}c$. In particular if $a_{t,k}$ is even $c_{t,k} = 0$. Hence in that case $d_2$ is trivial in all of $E_2(\sigma_{t,k})$ and, as the higher differentials are trivial

for dimensional reasons, $E_\infty(\sigma_{t,k}) \approx E_2(\sigma_{t,k})$. We have therefore established the following proposition:

PROPOSITION. *If $a_{t,k}$ is even*

$$E_\infty^{p,q}(\sigma_{t,k}) \approx H^p(G_{n-1}^2) \otimes H^q(S^1) \qquad \textit{(ring isomorphism)}.$$

REMARK. Recall that the additive structures of $E_\infty(\sigma_{t,k})$ and of $H(\sigma_{t,k})$ are (unnaturally) isomorphic as we are dealing with a field of coefficients.

The evaluation of the inverse limit $\lim_{k\in I} H(\sigma_{t,k})$ now proceeds in two steps. First we replace the directed system $I$ (the integers ordered by divisibility) over which $k$ ranges, by $I_t$, the subset of $I$ for which $a_{t,k}$ is even. For any fixed $t$ $I_t$ is cofinal with $I$.

Next we compute the inverse limit $\lim_{I_t} E_\infty(\sigma_{t,k})$, and show that the additive structure of this group and $\lim_{I_t} H(\sigma_{t,k})$ are, again unnaturally, isomorphic. (The groups $E_\infty(\sigma_{t,k})$; $k \subset I_t$ become an inverse system by means of the homomorphisms $E_\infty^{p,q}(\sigma_{t,ks}) \to E_\infty^{p,q}(\sigma_{t,k})$ induced by the fiber mappings $h_s : \sigma_{t,k} \to \sigma_{t,ks}$.) Thanks to the proposition above this inverse system can be described in detail: Each group of the family is isomorphic to $H(G_2^{n-1}) \otimes H(S^1)$ and $h_s^*$ is the identity on $H(G_2^{n-1}) \otimes H^0(S^1)$, while $h_s^*(e \otimes a) = s(e \otimes a)$. (Here $e$ generates $H^0(G_2^{n-1})$ and $a$ generates $H^1(S^1)$ as above.) As we are working mod 2 it follows that no element of the inverse limit can have a component in any group which is of the form $e \otimes a$. As the homomorphisms involved are ring homomorphisms, neither can a component of the type $u \otimes a$ $(u \neq 0 \in H(G_2^{n-1}))$ occur. It follows that only components contained in $H(G_2^{n-1}) \otimes H^0(S')$ occur, whence $\lim_I E_\infty^{p,0}(\sigma_{t,k}) \approx H^p(G_2^{n-1})$ and $\lim_I E_\infty^{p,q}(\sigma_{t,k}) = 0$ for $q > 0$. It remains to prove the additive isomorphism of the two groups $\lim_I E_\infty(\sigma_{t,k})$ and $\lim_I H(\sigma_{t,k})$.

Recall that the fibering of $\sigma_{t,k}$ over $G_2^{n-1}$ induces a filtering of $H(\sigma_{t,k})$ whose associated graded group $\mathfrak{G}H(\sigma_{t,k})$ is canonically isomorphic to $E_\infty(\sigma_{t,k})$. The homomorphisms $h_s$ preserve the filtration and thereby induce the maps on $E_\infty$ which we have been discussing above. We can introduce a filtration in $\lim_I H(\sigma_{t,k})$ by taking the filtration of a limit element to be $\leqq p$ if the filtration of every one of its components is $\leqq p$. We denote the associated graded group $\mathfrak{G} \lim_I H(\sigma_{t,k})$. As all groups in question are finite dimensional vector spaces over a field, Theorem 5.7, p. 226 of [9][5] yields a canonical isomorphism of $\lim_k E_\infty(\sigma_{t,k}) = \lim_k \mathfrak{G}H(\sigma_{t,k})$ onto $\mathfrak{G} \lim_I H(\sigma_{t,k})$. It follows that the additive structure of $H(\sigma_{t,k})$ is (unnaturally) isomorphic to $\lim_I E_\infty(\sigma_{t,k})$. Assembling these results with Lemma I and reverting to homology we obtain the evaluation:

$$\lim_I H_m(V_{t,k}^*, V_{t,k}^-) \approx H_{m-(2t-1)(n-1)}(G_{n-1}^2).^6$$

---

[5] This theorem states that inverse limit operation preserves exactness in the category of finite dimensional vector spaces over a field.

[6] I am indebted to the referee for pointing out an error in the original derivation of this formula, as well as suggesting a remedy—which has been essentially followed.

Hence by the Morse result (Theorem A)

$$\mathfrak{M}_m(J_t, J_{t-1}) \approx H_{m-(2t-1)(n-1)}(G^2_{n-1}).$$

## 6. The step from $E_1$ to $E_\infty$

The group $E_1$ is a direct sum of the groups $\mathfrak{M}_m(J_t, J_{t-1})$; $t \geqq 1$. We set

$$E_1^{p,q} = \mathfrak{M}_{p+q}(J_p, J_{p-1}) \qquad p = 1, 2, \cdots$$

$$E_1^{p,q} = 0 \text{ for } p \leqq 0.$$

Thus $E_1$ becomes the direct sum of the groups $E_1^{p,q}$ over all $p$, $q$. From the evaluation of $\mathfrak{M}_m(J_t, J_{t-1})$ and the fact that $G^2_{n-1}$ has dimension $(2n-2)$ it is seen that $E_1^{p,q}$ is trivial except possibly when

$$0 \leqq p + q - (2p-1)(n-1) \leqq 2(n-1) \tag{6.1}$$

or equivalently for fixed $p > 0$ when $q$ lies in the range:

$$(2n-3)p - (n-1) \leqq q \leqq (2n-3)p + (n-1). \tag{6.2}$$

Let $Q_p$ be the interval of integers characterized by the above inequality. We observe that $Q_p \cap Q_{p-1}$ consists of just two elements, namely $q = (2n-3)p - (n-1)$ and $q = (2n-3)p - (n-1) + 1$. In general the difference of the lowest point of $Q_p$ and the highest point of $Q_{p-r}$ is given by $r(2n-3) - 2(n-1)$. Hence if we raise the interval $Q_p$ by $r-1$ units it fails to intersect $Q_{p-r}$, for $r > 1$.

An obvious transcription of this result is the following proposition with $r > 1$.

PROPOSITION. *If $d_r$ is an endomorphism of $E_1$ which maps $E_1^{p,q}$ into $E_1^{p-r,q+r-1}$, then $d_r$ annihilates all elements of $E_1^{p,q}$.*

Actually this proposition is equally valid for $r = 1$ because of the simple connectedness of $G^2_{n-1}$. The vanishing of the first and therefore by the duality in manifolds the vanishing of the $(2n-3)^{\text{rd}}$ homology groups of $G^2_{n-1}$ implies that $E_1^{p,(2n-3)p-(n+1)+1}$ and $E_1^{p-1,(2n-3)p-(n-1)}$ are trivial. Hence in the only range of $q$ where $d_1$ could be nontrivial (i.e. for the two values of $q$ where $Q_p$ and $Q_{p-1}$ overlap) once the image and once the range of $d_1$ is trivial. Therefore $d_1$ again annihilates every element of $E_1^{p,q}$.

Now let $A$ denote the limit singular chain groups of the directed system $\{\pi_k, h\}$. Let $D \subset A$ be the limit chain group of the set $\{\Delta, h\}$. If we now define $A_p \subset A$ as the limit chain groups of the system $\{(\pi_k^p, h\}$, we get a grading of $A/D$ in the sense of Leray. (We are again using the notations and conventions as given in Serre [6].) In the resulting spectral sequence Serre's $E_1^{p,q}$ $[E_\infty^{p,q}]$ is easily identified with our $E_1^{p,q}$ $[E_\infty^{p,q}]$. Above we have just shown that $d_1$ must be trivial in $E_1^{p,q}$. Hence $E_1 \approx E_2$. But $d_2$ must again be trivial by our dimension argument. Hence $E_3 \approx E_1$. Proceeding in this fashion we see that $E_\infty \approx E_1$.

Hence, as pointed out earlier, the rank of $H_m(A/D)$ equals the rank of the $m$-dimensional elements of $E_1^{p,q}$ (that is, the sum of all the ranks of $E_1^{p,q}$ with $p + q = m$). It follows that

$$\mathfrak{M}(S^n; z) = P(G_{n-1}^2; z)\, z^{(n-1)}(1 - z^{2(n-1)})^{-1}.$$

REMARK 1. For readers not acquainted with the Leray spectral sequence it should be pointed out that the use of this powerful tool in the last section can be circumvented in a sense just because all the differentials $d_r$ are trivial. Essentially, the "Lückenverfahren" of Seifert and Threlfall (see [2], p. 73) applies. Thus the statement that all the $d_r$'s are trivial is equivalent to the statement in the Morse terminology that the critical manifolds are of "increasing type", or again, equivalently, the "linking cycles" can be extended "below" the critical level.

REMARK 2. The circular connectivities (as distinguished from the *sensed* circular connectivities!) of the $n$-sphere, are obtained in a precisely analogous fashion. If we bar all the letters in §§4 and 5 these sections can as a matter of fact be read in that context.

The only essential difference occurs in §6. Here the $d_r$'s are trivial for $r > 1$ as before, but as $\bar{G}_{n-1}^2$ is not simply connected it is not a priori obvious that $d_1$ is trivial. Nevertheless $d_1$ turns out to be trivial, and we therefore get the following formula for $\overline{\mathfrak{M}}(S^n; z)$:

$$\overline{\mathfrak{M}}(S^n; z) = P(\bar{G}_{n-1}^2; z)\, z^{n-1}(1 - z^{2(n-1)})^{-1}.$$

REMARK 3. It might be of interest to compare the procedure given here with that of Morse in [1].

Morse uses for his model of $S^n$ ellipsoids on which certain closed geodesics are completely isolated. We on the other hand exploit precisely the degeneracy of the $n$ sphere in its usual representation. This avoids first of all the study of geodesics on ellipsoids. In particular, in the Morse treatment no *fixed* model will do for the computation in all dimensions. (On any fixed ellipsoid, for instance, the indices of the iterates of certain geodesics behave regularly only for a finite number of iterations.)

Secondly we avoid the explicit construction of the linking cycles $\lambda_{ij}^r$ (see [1], p. 323. As mentioned earlier their existence is equivalent to the vanishing of all the $d_r$'s.

Thirdly, the computation of $\mathfrak{M}(J_c, J_{c-})$ differs from that of Morse. The notion of a nondegenerate critical manifold enters essentially here, as even if $\sigma \in \mu_k$ represents only a single isolated geodesic it will still be a 1-sphere of critical points.

These then are roughly the main deviations. They are of a more or less technical nature. The extent to which the present derivation of $\overline{\mathfrak{M}}(S^n; z)$ is based on the Morse theory is amply given by the basic theorems which we quote from Morse's treatise.

## 7. The general theory

It is evident that in certain parts of our computation no reference was made to the special case of the $n$-sphere which we had in mind. The purpose of this section is to abstract from the preceding pages those features which are valid in general. We state the result as a theorem and corollary.

THEOREM II. *Let $R$ be a Riemann manifold. Let $\Pi_k(R)$ $\cdots$ etc. and $J$ be defined as in §2 and let $\mu_k$ $\cdots$ etc. be defined as in §5. Let $\{\tilde{\sigma}_{t,k}\}$ be critical sets of $J$ on $\Pi_k(R)$ (for $k$ sufficiently large) such that $h_s : \tilde{\sigma}_{t,k} \to \tilde{\sigma}_{t,ks}$, on which $J = t$. Suppose further that* (a) *$\sigma_{t,k}$ is a nondegenerate critical manifold of $Ju_k(R)$, of index $\lambda_0$.* (b) *$\sigma_{t,k}$ can be fibered by one spheres $S^1$ in the obvious fashion. ($S^1$ acts on $\sigma_{t,k}$ by rotating the closed geodesics.) Let the base-space of this fibering be $\sigma_t^* = \sigma_{t,k}^*$.*

*Then if $\{V_{t,k}\}$ is a sufficiently small family of neighborhoods of $\sigma_{t,k}(h_s V_{t,k} \subset V_{t,ks})$ in $\mu_k(R)$ and $V_{t,k}^* = \pi_k^t(R) \cap V_{t,k}$ ; $V_{k,k}^- = \pi_k^{t-}(R) \cap V_{t,k}$*

$$\lim_k H_m(V_{t,k}^*, V_{t,k}^-) \approx H_{m-\lambda_\sigma}(\sigma_t^*)$$

COROLLARY. *Let $R$ be a Riemann manifold such that:*

a. *The critical values of $J$ (on any $\tilde{\pi}_k$) form a countable sequence of numbers $n_1 < n_2 < n_3$* etc.

b. *The critical set corresponding to $n_r$ on $\mu_k$ is a nondegenerate critical manifold $\sigma_{n_r,k}$ of index $\lambda_{n_r}$, satisfying condition* b *above.*

*Then $\mathfrak{M}(R)$ has an associated graded group $\mathfrak{GM}(R)$ (see* [6]) *which is the terminal group $E_\infty$ of a spectral sequence $E_1$, $E_2$ $\cdots$ $E_\infty$ of which the first term $E_1$ is determined by the formula:*

$$E_1^{p,q} = H_{p+q-\lambda_{n_p}}(\sigma_{n_p}^*), \qquad p > 0$$

$$E_1^{0,q} = \phi,$$

*provided $E_1^{p,q}$ is finitely generated in each dimension (i.e., for $p + q$ = constant).*

REMARK. A given Riemann manifold may of course give rise to quite different groups $E_1$ when in different metrics. The passage from $E_1$ to $E_\infty$ differs correspondingly. Thus for instance if we take a nonspherical ellipsoid for our model of $S^n$ the vanishing of all the $d_r$'s in the resulting $E_1$ is by no means a priori obvious, and is, as a matter of fact, *never* true. The proof of this proposition is connected with the study of the "Iteration of closed geodesics" (see [7], and will be treated at another time.

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