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LECTURES ON K(X)

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PREFACE

These are the terse notes for a graduate seminar which I conducted at Harvard during the Fall of 1963.

By and large my audience was acquainted with the standard material in bundle theory and algebraic topology and I therefore set out directly to develop the theory of characteristic classes in both the standard cohomology theory and K-theory.

Since 1963 great strides have been made in the study of K(X), notably by Adams in a series of papers in Topology. Several more modern accounts of the subject are available. In particular the notes of Atiyah, "Notes on K-theory" not only start more elementarily, but also carry the reader further in many respects. On the other hand, those notes deal only with K-theory and not with the characteristic classes in the standard cohomology.

The main novelty of these lectures is really the systematic use of induced representation theory and the resulting formulae for the KO-theory of sphere bundles. Also my point of view toward the J-invariant, $\theta(E)$ is slightly different from that of Adams. I frankly like my groups $H^1(\mathbb{Z}^+; KO(X))$ and there is some indication that the recent work of Sullivan will bring them into their own.

Reprints of several papers have been appended to the notes. The first of these is a proof of the periodicity for KU, due to Atiyah and myself, which is, in some ways, more elementary than our final version of this work in "On the periodicity theorem for complex vector bundles" (1964), Acta Mathematica, vol. 112, pp. 229-247.

The second paper, on Clifford modules, deals with the Spinor groups from scratch and relates them to K-theory.

Finally, we have appended my original proof of the periodicity theorem based on Morse theory.

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LECTURES ON K(X)

§1. Introduction. Two vector bundles E and F over a finite CW-complex X are called J-equivalent if their sphere bundles $\mathbb{S}(E)$ and $\mathbb{S}(F)$ are of the same fiber-homotopy type. If they become J-equivalent after a suitable number of trivial bundles is added to both of them, they are called stably J-invariant, and the stable J-equivalence classes of bundles over X is denoted by J(X) .

The primary aim of these notes is to discuss a J-invariant of vector bundles $\theta(E)$, which is computable once the group of stable bundles over X, - that is - K(X) is known. The invariant $\theta(E)$ is clearly suggested by the recent work of Atiyah-Hirzebruch [4], [5] and especially F. Adams [1]. In fact $\theta(E)$ bears the same relation to the Adams operations as the Whitney class, a known J-invariant bears to the Steenrod operation. Further Adams' beautiful solutions of the vector-field problem may be interpreted as the explicit computation of the order of $\theta(E)$ where E is the line-bundle over real projective space.

The guiding principle of these notes is then to construct the analogue of the theory of characteristic classes in the K-theory and as this analogue is much simpler in the KU-theory, (complex stable bundles) this case is taken up first, in Sections 1 to 8. For the KO-theory I had to be considerably less elementary, in the sense that I used some explicit results from representation-theory, especially of the Spinor groups.

The contents of the notes may be summarized as follows: Sections 2 to 4 are devoted to the standard material on Chern classes etc. of complex vector-bundles. I have here essentially specialized Grothendieck's account in the Seminar Bourbaki, to the topological case.

In Section 5, K(X) is defined and its first properties are derived, again following Grothendieck's point of view, especially in the definition of the exterior powers. These, in turn lead to an easy definition of the Adams operations. I also very briefly recount the cohomological properties of K(X) in this section. Here as well as in Section 6 the appropriate reference is Atiyah-Hirzebruch [5].

Section 6 introduces the periodicity theorem for the KU-theory and deduces the first consequences from it. In Section 7 the KU-analogue of the Thom isomorphism between the cohomology of the base-space and the compact reduced cohomology of the total-space of a vector-bundle is defined. Section 8 then employs this Thom isomorphism to construct and in some sense compute the obstruction, $\theta(E)$, to a fiber homotopy trivialization of a sphere-bundle derived from a complex vector-bundle E. In Section 8, this θ is used to obtain the results of Kervaire-Milnor on the classical J-homomorphism.

Section 9 discusses the complex representative ring of a Lie group, RU(G) and relates it to the representative ring of one of its maximal tori. I here state some of the classical results of representation theory, and go into considerable detail for the groups U(n), SU(n), SO(n) and Spin(n). In Section 10 the real representative ring is compared to the complex one, especially for the Spinor-groups. Section 11 gives some basic isomorphism in the theory of fiber-bundles, and induced representations which lead to a different interpretation of some of the results on the KU-theory. In Section 12 the periodicity for KO is stated and used to identify the generators of $KO(S_{8n})$ as bundles induced by certain Spin-representations.

Section 13 finally brings the KO analogue of the invariant θ and derives some of its properties. Section 14 reinterprets the results of 13 in terms of the Thom-isomorphism in the KO-theory, while Section 15 goes on to give the Gysin-sequence for the KO-theory.

When $\widetilde{KO}(X)$ has no torsion, the invariant $\theta(E)$ is equivalent to a J-invariant $\Omega(E) \in KO(X) \otimes \mathbb{Q}/KO(X)$. The definition of Ω and the proof of this equivalence is carried out in Section 16, while in Section 17 we show that the character of $\Omega(E)$ is essentially the $\hat{\mathfrak{A}}$ genus of E as defined by Hirzebruch.

Section 18 deals with the projective space bundle associated to a vector bundle. In Section 19 we sketch two methods for computing $KO(P_n)$ where P_n is the real projective space, and then compute $J(P_n)$. We also sketch the way in which the isomorphism $KO(P_n) \simeq J(P_n)$ implies the solution of the vector-field problem on spheres. Section 20 is a technical appendix on the difference element.

§2. Notation and some preliminaries. We write $\mathfrak{A}$ for the category of finite CW-complexes and $\hat{\mathfrak{A}}$ for the category of finite CW-complexes with base points, and will in general follow the notation of [5]. If E is a vector bundle over $X \in \mathfrak{A}$ (the dimension of the fibers may vary, on the components of X) we write $\mathbb{D}(E)$ for the unit disc bundle of E (relative to some Riemann structure) and denote its boundary by $\mathbb{S}(E)$. The pair $(\mathbb{D}(E), \mathbb{S}(E))$ as well as the quotient space $\mathbb{D}(E)/\mathbb{S}(E)$ will be denoted by X^E. In the latter interpretation, X^E will be thought of as an element of $\hat{\mathfrak{A}}$, $\mathbb{S}(E)$ playing the role of the base point. When $\dim E = 0$, it is convenient to set $X^E = X \cup p$ where p is a disjoint point playing the role of base point. We also have occasion to use the object $\mathbb{P}(E)$ whose points are the 1-dimensional subspaces of the fibers E_x, $x \in X$. Thus $\mathbb{P}(E) \xrightarrow{\pi} X$ is a fibering over each component of X, the fibers being $(n - 1)$ dim projective spaces, $n = \dim E_x$.

The constructions we have just described make sense both, for real and for complex vector bundles and have certain pretty clear functorial properties, e.g., if $f : Y \to X$ is a map one has induced maps of $\mathbb{P}(f^{-1}E)$ into $\mathbb{P}(E)$. In addition the following "tautologous" bundles are canonically defined over $\mathbb{P}(E)$:

S_E - the sub line-bundle, whose fiber over $\ell_x \in \mathbb{P}(E)$ consists of the points of the line $\ell_x \subset E_x$

Q_E -the quotient bundle, whose fiber over $\ell_x \in \mathbb{P}(E)$ consists of the vector space E_x/ℓ_x .

If $\pi : \mathbb{P}(E) \to X$ denotes the projection, then we clearly have the exact sequence:

$$0 \longrightarrow S_E \longrightarrow \pi^{-1}E \longrightarrow Q_E \longrightarrow 0 \quad . \tag{2.1}$$

It is for many purposes useful to study the space X^E as a quotient of $\mathbb{P}(E+1)$. (1 denotes the trivial bundle relative to the field over which $\mathbb{P}(E)$ is constructed, endowed with the canonical section $x \to (x, 1)$.) This identification proceeds via the following map

$$\eta : \mathbb{D}(E) \longrightarrow \mathbb{P}(E+1)$$

defined by: $\eta(e_x)$ = line generated by $\{e_x - \{1 - |e_x|^2\} 1_x\}$ in $(E+1)_x$. (Here $|e_x|$ denotes the Riemann length of e_x and 1_x is the value of the canonical section of 1 at x .)

Clearly η is a homeomorphism of $\mathbb{D}(E) - \mathbb{S}(E)$ onto $\mathbb{P}(E+1) - \mathbb{P}(E)$, and maps $\mathbb{S}(E)$ onto $\mathbb{P}(E)$ by the Hopf fibering. Thus $\mathbb{P}(E+1)/\mathbb{P}(E) = X^E$ under η .

Note also that for $e_x \in \mathbb{D}(E) - \mathbb{S}(E)$, the projection

$$E_x \longrightarrow (E+1)_x/\eta(e_x)$$

is an isomorphism, and further that under this projection e_x maps into a positive multiple of the coset of 1_x .

The first observation implies that the map η induces an isomorphism:

$$\pi_1^{-1} E \approx \eta^{-1} Q_{(E+1)} \qquad \text{over } \mathbb{D}(E) - \mathbb{S}(E) \tag{2.2}$$

where π_1 denotes the projection $\mathbb{D}(E) \to X$. Now the injection $\mathbb{D}(E) \to E$ may be interpreted as a section of $\pi_1^{-1} E$ which is non-vanishing on $\mathbb{D}(E) - X$. We call this the tautologous section of $\pi_1^{-1}E$. On the other hand the section "1" of $\pi^{-1}(E+1)$ projects onto a section of Q_E ; the second remark may now be interpreted as asserting that the isomorphism (2.2) takes this section into a positive multiple of the tautologous section in $\pi_1^{-1}(E)$.

§3. The Chern classes and allied functions on bundles. Throughout this section we will only consider complex vector bundles. We recall that the complex line bundles over $X \in \mathfrak{A}$ are classified by their first obstructions which are contained in $H^2(X;\mathbb{Z})$. If L is a line-bundle, this obstruction for L is denoted by $c_1(L)$. One has $c_1(L \otimes L') = c_1(L) + c_1(L')$, $c_1(L^*) = - c_1(L)$. (*denotes the dual operation.) Recall also that if E is a vector bundle over a point (i.e., a complex vector space) then $x = c_1(S_E^*)$ generates $H^2(\mathbb{P}(E))$, and hence the powers $1, x, \cdots, x^{n-1}$, $n = \dim E$, give a free additive basis for $H^*\{\mathbb{P}(E)\}$. Finally $x^n = 0$. More generally the following holds:

PROPOSITION 3.1. *Let* $E \to X$, *be a vector bundle. Then as an* $H^*(X;\mathbb{Z})$-*module,* $H^*\{\mathbb{P}(E)\}$ *is freely generated by* $1, x_E, \cdots, x_E^{n-1}$, $n = \dim E$, *where* $x_E \in H^2(\mathbb{P}(E))$ *is equal to* $c_1(S_E^*)$.

Proof: As the restrictions of x_E^i, $i = 0, \cdots, (n-1)$ to a given fiber $\mathbb{P}_x(E)$ of $\mathbb{P}(E)$ over X form a base for $H^*(\mathbb{P}_x E)$, the fiber is totally non-homologous to zero and the proposition is a standard consequence of the Leray spectral sequence. Q.E.D.

COROLLARY 1. *There exist unique classes* $c_i(E) \in H^{2i}(X;\mathbb{Z})$, $i = 0, \cdots, \dim E = n$, $c_0(E) = 1$, *such that the equation*

$$\sum_{k=0}^{n} x_E^{n-k} c_k(E) = 0 \tag{3.1}$$

holds in $H^*(\mathbb{P}(E))$. *We call this relation the defining equation of* $\mathbb{P}(E)$.

This is clear. The $c_i(E)$ are called the Chern classes of E, and one defines $c(E)$ by:

$$c(E) = \Sigma\, c_i(E) .$$

Thus $c(E)$ is an element of $1 + \tilde{H}(X)$ the multiplicative group of elements in $H^*(X)$ which start with $1 \in H^0(X)$.

The functorial properties of $E \to \mathbb{P}(E)$ now easily yield the following:

COROLLARY 2. *If* $Y \xrightarrow{f} X$ *is a map, then* $f^*c(E) = c(f^{-1}E)$ *for any bundle* E *over* X.

PROPOSITION 3.2. *If* E *is the direct sum of line bundles:* $E = L_1 + \cdots + L_n$. *Then* $c(E) = \Pi\, c(L_i)$. *Thus, the defining equation of* $\mathbb{P}(E)$ *is given by*

$$\Pi\,(x_E + c_1(L_i)) = 0 .$$

Proof: Consider $0 \to S_E \to \pi^{-1}E \to Q_E \to 0$. Tensoring by S_E^* we obtain $0 \to 1 \to (\pi^{-1}E) \otimes S_E^* \to Q_E \otimes S_E^* \to 0$. Thus $(\pi^{-1}E) \otimes S_E^* = \Sigma_1^n L_i \otimes S_E^*$ has a nonvanishing section s. Let s_i be the projection of s on $L_i \otimes S_E^*$, and let $u_i \subset X$ be the closed set on which $s_i = 0$. Then

$$\bigcap_1^n \mathcal{U}_i = \emptyset$$

as s is nonvanishing. Now it follows from obstruction theory that $c_1(L_i \otimes S_E^*)$ can be pulled back to $H^2(X; X - \mathcal{U}_i)$. Hence

$$\prod_1^n c_1(L_i \otimes S_E^*)$$

can be pulled back to $H^{2n}(X, \cup\{X - \mathcal{U}_i\})$. However this group is 0, as $\cup\{X - \mathcal{U}_i\} = X$. Now

$$\prod_1^n c(L_i \otimes S_E^*) = \prod_1^n \{c(L_i) + x_E\} .$$

Hence the defining equation of $\mathbb{P}(E)$ is as given in the proposition. But this equation defines $c(E)$ uniquely and so implies the special Whitney formula

$$\prod_1^n c(L_i) = c(E) .$$

<u>The splitting principle</u>: We have already seen that when lifted to $\mathbb{P}(E)$ the bundle E splits off a line bundle S_E. Further $H^*(X)$ is imbedded by π^* into $H^*\{\mathbb{P}(E)\}$. Set $E_1 = Q_E$ over $\mathbb{P}(E)$ and consider $\mathbb{P}(E_1)$ over $\mathbb{P}(E)$. When E is lifted to $\mathbb{P}(E_1)$ it splits off 2 line bundles and it is still true that $H^*(X)$ is imbedded in $H^*(E_1))$ by the

projection. If we continue this process: Set $E_{n+1} = QE_n$, over $\mathbb{P}(E_n)$, $n = 1, \cdots, \dim E = m$, we finally obtain a space $\mathbb{P}(E_m)$ over X, with the property that when lifted to $\mathbb{P}(E_m)$, E splits into a direct sum of line bundles, and $H^*(X)$ is imbedded in $H^*\{\mathbb{P}(E_m)\}$ by the projection. We denote $\mathbb{P}(E_m)$ by $\mathbb{F}(E)$. By the naturality of the Chern class, and Proposition 3.2, $c(E)$ will therefore split into linear factors:

$$c(E) = \prod c(L_i) \qquad \text{in } H^*\{\mathbb{F}(E)\} .$$

An easy consequence of this fact and (3.2) is now the general Whitney formula

$$c(E + F) = c(E) \cdot c(F) .$$

More generally, let $F(x)$ be a formal power series in x with coefficients in Λ. Then F can be extended to an additive function from bundles on X to $H^*(X; \Lambda)$ by setting:

1. $F(L) = F\{c_1(L)\}$ L a line bundle.
2. $F(E) = \Sigma F\{c_1(L_i)\}$, where L_i are the components of E lifted to $\mathbb{F}(E)$.

(Note, the $F(E)$ can be expressed in terms of the $c_i(E)$, by expressing $F(x_1) + \cdots + F(x_m)$, $m = \dim E$ in terms of the elementary symmetric functions in the x_i, and then replacing these by the $c_i(E)$.)

The Whitney formula now shows that $F(E + E') = F(E) + F(E')$, i.e., that F is additive. Similarly we may extend F to a multiplicative function from bundles to $H^*(X; \Lambda)$.

One defines:

$$F(E) = \Pi F\{c_1(L_i)\}, \qquad \text{where } E = \Sigma L_i \text{ on } \mathbb{F}(E) \quad .$$

Examples of this construction are:

1. If $F(x) = 1 + x$, then the multiplicative extension of F is $c(E)$.
2. If $F(x) = \dfrac{x}{1-e^{-x}}$, then the multiplicative extension of F is called the "Todd class of E", and is denoted by $T(E)$.
3. If $F(x) = e^x$, then the additive extension of F is called the character of E, and is denoted by $ch(E)$.

In these examples $\Lambda = \mathbb{Z}$ in the first case, and $\Lambda = \mathbb{Q}$ in the other two.

PROPOSITION: If E and E' are bundles over X, then

$$ch(E \otimes E') = ch(E) \cdot ch(E') \quad .$$

Proof: By the splitting principle we may assume that $E = \Sigma L_i$, $E' = \Sigma L_i'$ whence $E \otimes E' = \Sigma L_i \otimes L_j$. Therefore

$$\begin{aligned} ch(E \otimes E') &= \Sigma\ e^{c_1(L_i \otimes L_j')} \\ &= \Sigma\ e^{c_1(L_i) + c_1(L_j')} \\ &= \Sigma\left(e^{c_1(L_i)}\right)\Sigma\left(e^{c_1(L_j')}\right) \\ &= ch(E) \cdot ch(E') \qquad \text{Q.E.D.} \end{aligned}$$

§4. The Thom isomorphism in $H^*(X; \mathbb{Z})$. Consider the sequence $\mathbb{P}(E) \xrightarrow{\alpha} \mathbb{P}(E+1) \xrightarrow{\beta} X^E$ where β is induced by the identification $\eta : X^E \to \mathbb{P}(E+1)/\mathbb{P}(E)$ of Section 2. We assume X connected in the following, however the extension to the general case is obvious.

PROPOSITION 4.1. In cohomology with integer coefficients we have the exact sequence

$$0 \longleftarrow H^*\{\mathbb{P}(E)\} \xleftarrow{\alpha^*} H^*\{\mathbb{P}(E+1)\} \xleftarrow{\beta^*} \tilde{H}^*(X^E) \longleftarrow 0 \quad .$$

Further $\operatorname{im} \beta^*$ = ideal generated by U in $H^*(\mathbb{P}(E+1))$ where

$$U = \sum_{k=1}^{n} x_{(E+1)}^{n-k} \cdot c_k(E) \qquad n = \dim E \ ,$$

and $x_{(E+1)} \cdot U = 0$.

Proof: Clearly $\alpha^* x_{(E+1)} = x_E$. Hence by Proposition 3.1 α^* is onto. This proves the exactness of the sequence in question. Now let $g = \Sigma_0^n a_i x^i_{(E+1)}$ be an element of the kernel of α^*. Then in $H^*\{\mathbb{P}(E)\}$ we have $\Sigma_0^n a_i x_E^i = 0$. But the defining equation of $\mathbb{P}(E)$ is

$$x_E^n = - \sum_1^n c_i(E) x_E^{n-i} .$$

Thus we have $0 = a_i - a_n c_{n-i}(E)$, $i = 0, \cdots, n-1$, and so

$$g = \sum_1^n a_n c_{n-i}(E) x^i_{(E+1)} = a_n \cdot U .$$

Thus the kernel of α^* is a free module of rank one over $H^*(X)$ with generator U. Thus U generates the image of β^* over $H^*(X)$. It remains to show that $x_{(E+1)} U = 0$. The defining equation for $\mathbb{P}(E+1)$ is

$$\sum x^{n+1-k} c_k(E+1) = 0 .$$

But by "Whitney" $c_k(E+1) = c_k(E)$ whence $c_{n+1}(E+1) = 0$. Therefore the defining equation of $\mathbb{P}(E+1)$ is precisely

$$x_{(E+1)} \cdot U = 0 . \qquad \text{Q.E.D.}$$

We now define the Thom isomorphism

$$i_* : H^*(X) \longrightarrow \tilde{H}(X^E)$$

by the formula $\beta^* \circ i_* a = a \cdot U$, in $H^*\{\mathbb{P}(E)\}$. By Proposition (4.1) i_* is a bijection.

§5. The functor K(X). We consider the additive functions from bundles over X into abelian groups, i.e., functions $E \to F(E)$ with values in g, so that $F(E + E') = F(E) + F(E')$. There is then a minimal universal object $K(X)$ - which solves the universal problem posed here, i.e., $K(X)$ is an abelian group with a natural additive function, γ, from bundles to $K(X)$ such that if F is any additive function as above, then F induces a unique homomorphism

$$F_* : K(X) \longrightarrow g$$

with the property: $F(E) = F_*\{\gamma(E)\}$.

Indeed one may take for $K(X)$ the free group generated by the bundles over X modulo the subgroup generated by the following relations; whenever $0 \to E \to E' \to E'' \to 0$ is an exact sequence of bundles over X, and $[E]$, $[E']$, $[E'']$ are respective generators in the free group, then

$$[E'] - ([E] + [E''])$$

precisely $\gamma(E)$. We will, for the most part, omit the symbol γ, and write E for both a bundle and its class in $K(X)$ unless the confusion caused by this convention becomes unmanageable. The elements of $K(X)$ are sometimes called virtual bundles.

Elementary properties of K(X)

5.1. $K(X)$ is a contravariant functor from $\mathfrak{A}$ to the catagory of Abelian groups. (If $f : Y \to X$, is a map, and E a bundle over X, then $f^{-1}E$ is a bundle over Y. As this operation is additive it induces a homomorphism $K(Y) \to K(X)$ which is denoted by $f^!$.)

5.2. There exists an (infinite) CW complex, $\underline{\underline{K}}$ which represents the functor K, i.e., there is a natural isomorphism between $K(X)$ and $\pi[X ; \underline{\underline{K}}]$ denotes homotopy classes of maps of X into $\underline{\underline{K}}$. Further $\underline{\underline{K}}$ may be endowed with an H-structure which induces the additive structure on $K(X)$. (This proposition follows readily from the following facts:

a. The functor $\underline{\underline{E}}_n : X \to n$ plane bundles over X is representable.

b. $\underline{\underline{E}}_n(X) \cong \underline{\underline{E}}_{n+1}(X)$ for $n >> \dim X$.

c. If E is a bundle over X, then there exists a bundle $E^{\perp}$ over X so that $E + E^{\perp}$ is isomorphic to a trivial bundle.)

5.3. Let $X \in \tilde{\mathfrak{A}}$, with base point p_X. One defines $\tilde{K}(X)$ as the kernel of the natural projection: $\mathbb{Z} \approx K(p_X) \longleftarrow K(X)$, which we denote by dim. Thus $\tilde{K}(X)$ corresponds to the virtual bundles of dim 0. $\tilde{K}(X)$ is thus an ideal in $K(X)$. It is also a direct summand as the homomorphism induced by projection $X \to p_X$ splits the exact sequence:

$$0 \longleftarrow K(p_X) \longleftarrow K(X) \longleftarrow \tilde{K}(X) \longleftarrow 0 \ .$$

The trivial zero-dimensional bundle corresponds to a point in a suitable component of $\underline{\underline{K}}$. If we consider this point the base point of $\underline{\underline{K}}$, then for objects in $\tilde{\mathfrak{A}}$, $\tilde{K}(X)$ is represented by $\pi[X, \underline{\underline{K}}]$ where now $\pi[X, \underline{\underline{K}}]$ denotes homotopy classes of basepoint preserving maps.

In a sense $\tilde{K} : \tilde{\mathfrak{A}} \rightsquigarrow g$, is the more basic functor. Indeed, if $A \xrightarrow{i} X$ is a pair in $\mathfrak{A}$ (or $\tilde{\mathfrak{A}}$) one defines the relative groups

$$K(X, A) \cong \tilde{K}(X, A) \quad \text{as} \quad \tilde{K}(X/A)$$

where X/A is considered as an element of $\tilde{\mathfrak{A}}$ with base-point A. If A is vacuous X/A is defined as the space $X^+ = X$ union a disjoint point p_X which plays the role of basepoint. Thus

$$K(X) \cong \tilde{K}(X^+)$$

and K on $\mathfrak{A}$ is seen to be the composition of the functor $X \to X^+$ and $\tilde{K}$.

5.4. As $\tilde{K}$ is representable one now has an exact sequence:

(5.4.1) $\tilde{K}(A) \xleftarrow{i^!} \tilde{K}(X) \xleftarrow{j^!} \tilde{K}(X, A)$ for (X, A) a pair in $\tilde{\mathfrak{A}}$

and more generally if we define

$$\tilde{K}^i(X, A) \text{ by } \tilde{K}(X/A \# \Sigma^{(-i)}) , \qquad i \leq 0 ,$$

(Σ^i denotes the i-sphere with basepoint, $\#$ denotes the product in $\tilde{\mathfrak{A}}$), then the Puppe exact sequence which extends (5.4.1) holds:

$$\tilde{K}^i(A) \longleftarrow \tilde{K}^i(X) \longleftarrow \tilde{K}^i(X, A) \xleftarrow{\delta} \tilde{K}^{i-1}(A) \longleftarrow \cdots .$$

We write $\tilde{K}^*$ for the graded functor $\tilde{K}^i$, $i \leq 0$. This functor shares many properties with the functor H^* — more or less by definition: they are exactness, and excision. $\tilde{K}^*$ differs at this point from H^* in that it is not defined for all integers, and that K^* of the 0-sphere S^0 in $\tilde{\mathfrak{A}}$ is not trivially computable.

5.5. *The graded ring structure on* $\tilde{K}^*(X)$. The functor $\tilde{K}^*$ has various elementary properties which are the consequence of the definition of $K(X)$ as a solution of

a universal problem, rather than of the representability. The first of these is the ring structure induced on $\tilde{K}^*(X)$ by the tensor product of bundles.

If E and E' are bundles over X and $Y \in \mathfrak{A}$ respectively then $E \otimes E'$ is a bundle over $X \times Y$. This operation is seen to define a natural transformation

$$K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$

which we still refer to as the (exterior) tensor product and denote by $\otimes$.

When $X = Y$, the diagonal map $\Delta: X \to X \times X$, defines a ring structure on $K(X)$ by :

$$u \cdot v = \Delta^*(u \otimes v) \qquad u \in K(X), \quad v \in K(X) .$$

This is the interior tensor product and is usually written with a dot. Clearly this operation converts $K(X)$ into a commutative ring. To extend this operation to $\tilde{K}$ on $\tilde{\mathfrak{A}}$, one needs the following fact:

PROPOSITION 5.1. *Let* $X, Y \in \tilde{\mathfrak{A}}$, *and let* $X \times Y$ *be their Cartesian product, and consider the sequence:*

$$0 \longrightarrow X \vee Y \xrightarrow{i} X \times Y \xrightarrow{j} X \# Y \longrightarrow 0$$

where $X \vee Y = p_X \times Y \cup X \times p_Y$. *Then the sequence*

$$0 \longleftarrow K^i(X \vee Y) \xleftarrow{i^!} K^i(X \times Y) \xleftarrow{j^!} K^i(X \# Y) \longleftarrow 0, \quad i \leq 0$$

is exact.

Proof: Let $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ and $\pi : X \times Y \to p_X \times p_Y$ be the natural projections. We have

$$\tilde{K}(X \vee Y) \approx \tilde{K}(X) \oplus \tilde{K}(Y)$$

and

$$K(X \vee Y) \approx \tilde{K}(X) \oplus \tilde{K}(Y) \oplus K(p_X \times p_Y) .$$

Now define $\sigma : K(X \vee Y) \to K(X \times Y)$ by:

$$\sigma(\alpha + \beta + \gamma) = \pi_1^! \alpha + \pi_2^! \beta + \pi^! \gamma , \qquad \begin{array}{l} \alpha \in \tilde{K}(X), \\ \beta \in \tilde{K}(Y), \\ \gamma \in K(p_X \times p_Y) . \end{array}$$

It is then clear that $i^! \cdot \sigma$ = identity. Now the Puppe exact sequence yields the result.

It is easy to see that if $u \in \tilde{K}(X)$ and $v \in \tilde{K}(Y)$ then $b = u \otimes v \in \tilde{K}(X \times Y)$ is in the kernel of $i^!$. Hence there is a unique element (again written) $u \otimes v \in \tilde{K}(X \# Y)$ which maps into b under $j^!$. This is the extension of the tensor product to $\tilde{K}$ on $\tilde{\mathfrak{A}}$.

We have $\tilde{K}^i(X) = \tilde{K}(X \# \Sigma^{-i})$, $\tilde{K}^j(Y) \approx \tilde{K}(Y \# \Sigma^{-j})$. Hence $\tilde{K}^i(X) \otimes \tilde{K}^j(X)$ is paired to $\tilde{K}(X \# \Sigma^{-i} \# Y \# \Sigma^{-j})$ by

our product. Now $(X \# \Sigma^{-i} \# Y \# \Sigma^{-j}) \simeq (X \# Y \# \Sigma^{-(i+j)})$ by the homotopy commutativity of the product in $\tilde{\mathfrak{A}}$. Hence our product extends to a pairing,

$$\tilde{K}^i(X) \otimes \tilde{K}^j(Y) \longrightarrow \tilde{K}^{i+j}(X \# Y) .$$

This is the extended (exterior) tensor product. By the diagonal construction one now deduces a graded ring structure on $\tilde{K}^*(X)$ and this product turns out to be commutative, i.e.:

$$u \cdot v = (-1)^{pq} v \cdot u \qquad u \in \tilde{K}^p(X), \quad v \in \tilde{K}^q(Y).$$

Remarks: 1). If $X \in \mathfrak{A}$, one defines $K^*(X)$ by $\tilde{K}^*(X^+)$ and if (X, A) is a pair in $\mathfrak{A}$ (or $\tilde{\mathfrak{A}}$) $K^*(X, A)$ is defined as $\tilde{K}^*(X/A)$. 2) Observe that $K^*(X, A)$ is a graded $K^*(X)$ module, as the diagonal map $X \to X/A \# X^+$ factors through X/A in the obvious manner. 3) The 0-sphere S^0 acts as a unit in $\mathfrak{A}: X \# S^0 = X$. Hence $\tilde{K}^*(X)$ is in a natural way a graded $\tilde{K}^*(S^0)$ module. In fact $K^*(p)$ — as we may call $\tilde{K}^*(S^0)$ acts on all the functors $K^*(X)$, $\tilde{K}^*(X)$, $K^*(X, A)$ etc. in a natural way and commutes with the natural transformations linking them. For a more detailed exposition of the material covered in this section consult [5].

The operations λ^i on K(X).

If V is a module (over $\mathbb{C}$, or $\mathbb{R}$) and $V^n = V\otimes\cdots\otimes V$ (n factors) then the permutation group $\mathfrak{S}_n$ acts on V^n in the obvious manner. Let $Q \subset V^n$ be the subspace generated by the elements $\sigma \cdot w - (-1)^\sigma w$, $w \in V^n$, $\sigma \in \mathfrak{S}_n$, $(-1)^\sigma = +1$, -1, according to the parity σ. The quotient space V^n/Q is denoted by $\lambda^n(V)$ and is called the nth exterior power of V. We set $\lambda^0(V)$ = base field. The λ^i are clearly co-varient functors from the category of modules to the category of modules.

They further satisfy the identity:

$$(5.4) \qquad \lambda^n(V+W) = \sum_{i+j=n} \lambda^i(V)\otimes\lambda^j(W) .$$

We can now extend the λ^i as operations on vector bundles in the obvious way. If E is a bundle over X, $\lambda^i E$ will be the bundle over X whose fiber at $x \in X$ is $\lambda^i E_x$. Further the identity (5.4) will still be valid in the broader context, and one may use it to define natural transformations $\lambda^i : K(X) \to K(X)$ in the following manner.

Consider $K(X)[[t]]$, the formal power series in t with coefficients in $K(X)$, and let $1 + \tilde{K}(X)[[t]]$ be the multiplicative group of elements in $K(X)[[t]]$ which start with 1. If E is a bundle, define

$$\lambda_t(E) \in 1 + \tilde{K}(X)[[t]]$$

by

$$\lambda_t(E) = \sum_0^\infty t^i \gamma(\lambda^i E) .$$

Now (5.4) implies that

$$\lambda_t(E)\cdot\lambda_t(E') = \lambda_t(E+E') .$$

Hence, $E \to \lambda_t(E)$ is an additive function from bundles to $1 + \tilde{K}(X)[[t]]$. Hence by the universal property of K(X), there is a unique operation

$$\lambda_t : K(X) \to 1 + \tilde{K}(X)[[t]]$$

which "agrees" with λ_t as defined on bundles:

$$\lambda_t(\gamma E) = \lambda_t(E) .$$

The component of $\lambda_t(E)$ whose coefficient is t^i is now defined to be $\lambda^i(E)$.

Examples. $\lambda_t(L) = 1 + tL$ if L is a line bundle.

$$\lambda_{-t}(-L) = \frac{1}{1-tL} = 1 + tL + t^2L^2 + \cdots$$

Note that in general $\lambda_a(x)$, $x \in K(X)$, $a \in \mathbb{Z}$, is not a well defined element of $K(X)$. However if $x = \gamma(E)$ then $\lambda_t(E)$ is a polynomial in t, and $\lambda_a(x)$ is well defined, by substituting a for t . In fact in that case a may be taken to be an element of $K(X)$ and of course $\lambda_a(x+y) = \lambda_a(x) \cdot \lambda_a(y)$. $x = \gamma(E)$, $y = \gamma(E')$, $a \in K(X)$.

The Adams Operations

We have just seen that the λ^i define operations in $K(X)$ subject to the relation

$$\lambda_t(x + y) = \lambda_t(x) \cdot \lambda_t(y) \qquad x, y \in K(X) \ .$$

We now define operations $\psi_i : K(X) \to K(X)$, $i = 1, \cdots$ in terms of the λ_i which will be additive:

$$\psi_i(x + y) = \psi_i(x) + \psi_i(y) \ .$$

To do this, set $\psi_t(x) = t\psi_1(x) + t^2\psi_2(x) + \cdots$, $x \in K(X)$ and define ψ_t by the formula:

$$(5.5) \qquad \psi_{-t}(x) = -t \cdot d/dt\ \lambda_t(x)/\lambda_t(x) = -t\lambda_t'(x)/\lambda_t(x) \ .$$

Because $\lambda_t(x) = 1 + t\,\lambda^1(x) + \cdots$ the R.H.S. is a well defined element of $K(X)[[t]]$ and so determines ψ_t .

Let us now compute $\psi_{-t}(x + y)$. This equals:

$$-t\lambda_t'(x+y)/\lambda_t(x+y) = -t\{\lambda_t'(x)\lambda_t(y) + \lambda_t(x)\lambda_t'(y)/\lambda_t(x) \cdot \lambda_t(y)\}$$
$$= \psi_{-t}(x) + \psi_{-t}(y) \ .$$

Thus the ψ_i are additive as asserted, and these are the operations Adams introduced recently. They are in many ways more tractable than the λ^i , principally because they will be seen to be ring homomorphisms of $K(X)$. If one solves for the ψ_i in (5.5) explicitly one obtains the following formulae, which may serve if one wishes as a definition of the ψ_i :

$$\psi_1 - \lambda^1 = 0$$
$$\psi_2 - \psi_1 \cdot \lambda^1 + 2\lambda^2 = 0$$
$$\psi_3 - \psi_2 \cdot \lambda^1 + \psi_1 \cdot \lambda^2 - 3\lambda^3 = 0$$
$$\psi_i - \psi_{i-1}\ \lambda^1 \ \pm \cdots \pm i\lambda^i = 0 \ .$$

Note: 1. The expression $t\lambda_t'/\lambda_t$ can be written $t d/dt \log \lambda_t$. Now as λ_t behaves multiplicatively, $\log \lambda_t$ will behave additively and hence its derivative also. This point of view makes the definition of ψ_t quite plausible. The operation ψ_t is to be preferred to just $\log \lambda_t$ because the latter has meaning only over rationals, due to the rational numbers which occur in the expansion of $\log(1 + x)$.

2. The formulae are precisely the ones linking the elementary symmetric functions with the power sums, (Newton's formula), and the precise analogues of the ψ_i in the framework of characteristic classes was used quite frequently.

3. The following formula is one of the main reasons why the ψ_i are so useful:

PROPOSITION: *Let* L *be a line-bundle. Then*

$$\psi_k(L) = L^k .$$

Proof: $\lambda_t(L) = 1 + tL$, therefore $\psi_{-t} = \frac{-tL}{1 + tL}$

whence $\psi_t L = \Sigma t^k L^k$, Q. E. D.

§ 6. *The ring* $K^*(p)$. The properties of K^* and $\tilde{K}^*$ which we have reviewed in the last section are direct consequences either of the representability of these functors, or of the fact that the functorial operations of linear algebra extend in a natural way to vector-bundles. These properties are shared by the "real" and the "complex" K.

In this section we discuss the implications of the periodicity theorem on the *complex* K-theory.

We write simply ξ for the virtual bundles $(S_E^* - 1)$ over $\mathbb{P}(E)$, dim E = 2. Thus ξ is an element $\tilde{K}(S_2) = K^{-2}(p)$.

PERIODICITY THEOREM I. $K^*(p) \simeq \mathbb{Z}[\xi]$. This theorem will be assumed. For a proof see [6].

COROLLARY 1. *Let* $\xi_* : K^i(X) \to K^{i-2}(X)$ *denote the operation of* $\xi \in K^*(p)$ *on* $K^*(X)$. *Then* ξ_* *is a bijection.*

Proof: ξ_* may be thought of a natural transformation of one cohomology theory into another which induces an isomorphism on points. Hence ξ_* is bijective in $\mathfrak{A}$ by general nonsense.

COROLLARY 2. ξ_* *also induces bijections* $\tilde{K}^i(X) = \tilde{K}^{i-2}(X)$, $X \in \tilde{\mathfrak{A}}$ *and* $K^i(X, A) \to K^{i-2}(X, A)$, *for* (X, A) *a pair in* $\mathfrak{A}$ *or* $\tilde{\mathfrak{A}}$.

Same proof.

One may now define $\mathbb{K}(X) = K^0(X) + K^{-1}(X)$. Using ξ_*, $\mathbb{K}(X)$ is made into a graded ring (over $\mathbb{Z}_2$) in the obvious manner. $\xi_*^{-1}(u \cdot v)$, is in $K^0(X)$ when $u, v \in K^{-1}(X)$. Similarly we convert our other constructions to operations on $\mathbb{K}$, $\tilde{\mathbb{K}}$ etc. In terms of this functor the periodicity theorem then states that:

$$\tilde{\mathbb{K}}(X) \otimes \tilde{\mathbb{K}}(S^i) \approx \tilde{\mathbb{K}}(X \# S^i), \quad X \in \tilde{\mathfrak{A}}, \quad S^i \text{ the i-sphere in } \tilde{\mathfrak{A}},$$

where on the left we mean the graded tensor product.

Similarly one obtains

$$\mathbb{K}(X) \otimes \mathbb{K}(S^i) \approx \mathbb{K}(X \times S^i) \ , \ X \in \mathfrak{A}, \quad S^i \text{ the i-sphere in } \mathfrak{A} .$$

Now, as $\tilde{\mathbb{K}}(S^i) = \mathbb{Z}$ for $i \geq 0$, we see that $\mathbb{K}$ and $\tilde{\mathbb{K}}$ satisfy all the axioms of Eilenberg, Steenrod, for a cohomology and reduced cohomology theory, *provided* we assume these axioms are asserted for a graded theory indexed by the group of order 2 .

First consequences.

THEOREM 6.1. *Let* ξ_n *generate* $\tilde{K}(S_{2n})$, *and let* u_n *generate* $H_{2n}(S_{2n})$ *then* $\langle ch(\xi_n), u_n \rangle = \pm 1$.

Proof: For ξ (i.e., the case $n = 1$) this proposition is clear. Now $\pi : S_2 \times \cdots \times S_2 \to S_2 \# \cdots \# S_2 = S_{2n}$ maps ξ_n onto $\xi \otimes \cdots \otimes \xi$, and if $ch(\xi) = x$ where x generates $H^2(S_2)$, then $ch(\xi \otimes \cdots \otimes \xi) = x \otimes \cdots \otimes x$ which is π^* of a generator of $H^{2n}(S_{2n})$. Q.E.D.

COROLLARY 1. *A class* $u \in H_{2n}(X, \mathbb{Z})$ *is spherical only if for all* $\xi \in K(X)$, $\langle ch(\xi), u \rangle$ *is an integer*.

Clear.

We may extend ch to a homomorphism $ch : \mathbb{K}(X) \to H^*(X)$ by setting ch on $\mathbb{K}^{-1}(X)$ equal to the composition

$$\tilde{K}(\Sigma' X) \xrightarrow{ch} H^*(\Sigma' X) \xrightarrow{(\Sigma')^{-1}} \tilde{H}^*(X) \ .$$

COROLLARY 2. $ch : \mathbb{K}(X) \to H^*(X)$ *is a ring homomorphism.*

Proof: This is clear on $K(X)$. For $u \in K^{-1}(X)$, $v \in K(X)$ it is also easy. If $v \in K^{-1}(X)$, then $u \cdot v$ in $\mathbb{K}(X)$ is the class $\xi_*^{-1} u \cdot v$. Hence it has only to be shown that $ch\ \xi_* = \Sigma_*^2\ ch$ where Σ_*^2 is the suspension in cohomology. But this is clear because ch is multiplicative and $ch\ \xi$ generates $H^2(S^2)$.

§7. *The Thom homomorphism for* $\mathbb{K}(X)$. Let $E \to X$ be a complex vector bundle, and consider the sequences:

(7.1)
$$\begin{tikzcd} \mathbb{K}(\mathbb{P}(E)) \arrow[rr, bend right, "\delta"'] & \mathbb{K}\{\mathbb{P}(E+1)\} \arrow[l, "\alpha!"'] & \mathbb{K}(X^E) \arrow[l, "\beta!"'] \end{tikzcd} .$$

The following is an analogue of Proposition 3.1.

THEOREM 7.1. a) $\mathbb{K}\{\mathbb{P}(E)\}$ *is a free module over* $\mathbb{K}(X)$ *with generator*, $1, \xi_E, \cdots, \xi_E^{n-1}$, $n = \dim E$, *where* $\xi_E = S_E^* - 1 \in \mathbb{K}^*\{\mathbb{P}(E)\}$. *Further* $\lambda_{-S_E} \cdot \pi^! E^* = 0$ *whence we have a defining relation of the form:*

$$\xi_E^n + \xi_E^{n-1} \cdot C^1(E) + \cdots + C^n(E) = 0$$

where the $C^i(E)$ *are elements of* $\mathbb{K}^0(X)$ *expressible in terms of the* $\lambda^i E^*$. *In particular* $C^n(E) = \lambda_{-1}(E^*)$.

b) *The sequence* (7.1) *has* $\delta = 0$ *and* β^* *imbeds* $\mathbb{K}(X^E)$ *onto the ideal generated by* $U = \lambda_{-S_{(E+1)}} \pi^! E^*$ *in* $\mathbb{K}\{\mathbb{P}(E+1)\}$.

The proof is broken up into several stages:

LEMMA 1 . *The element* $\lambda_{-S_E} \cdot \pi^! E^*$ *in* $\mathbb{K}^0\{\mathbb{P}(E)\}$ *is* 0 .

Proof: We have the sequence of bundles over E .

$$0 \to S_E \to \pi^! E \to Q_E \to 0 \quad .$$

If we dualize we obtain:

$$0 \longleftarrow S_E^* \longleftarrow \pi^! E^* \longleftarrow Q_E^* \longleftarrow 0 \quad .$$

Apply λ_t to obtain:

$$(1 + tS_E^*) \cdot \lambda_t Q_E^* = \lambda_t \pi^! E^*$$

set $t = -S_E$. Then the first factor vanishes. Q.E.D.

LEMMA 2. *The theorem is true where* X *a point* p .

Proof: Assume the theorem for $\dim E \leq n$, and consider the sequence (7.1) with $\dim E = n$. In this situation $X^E = S_{2n}$. Hence (7.1) goes over into

$$0 \longleftarrow \mathbb{K}^0\{\mathbb{P}(E)\} \xleftarrow{\alpha!} \mathbb{K}^0\{\mathbb{P}(E+1)\} \xleftarrow{\beta!} \mathbb{Z} \longleftarrow 0, \mathbb{K}^{-1}(\mathbb{P}(E+1)) = 0.$$

Now, $U = \lambda_{-S_{(E+1)}} \cdot \pi! E^*$ maps onto 0 under $\alpha!$ by Lemma 1 . Hence $U = \beta^! \lambda \cdot \xi_n$ where $\lambda \in \mathbb{Z}$ and ξ_n is our generator of $\tilde{K}(S_{2n})$. We next show that λ is +1 by applying the character to both sides. To see this we will prove the more general formula:

PROPOSITION 7.1. *Let* U *be as defined in Theorem* (7.1) . *Then*

$$\operatorname{ch} U = i_* \cdot T^{-1}(E)$$

where i_* *denotes the Thom isomorphism of Section* 2 *and* T *the Todd class also defined in that section.*

Proof: By the splitting principle we may assume that $E = \Sigma E_i$, whence $E^* = \Sigma E_i^*$. Let $\ell_i = c_1(E_i)$. Then:

$$U = \Pi(1 - S_{E+1} \cdot E_i^*)$$

whence

$$\mathrm{ch}\, U = \Pi(1 - e^{-(x+\ell_i)})$$

$$\frac{\Pi(1 - e^{-(x+\ell_i)})}{\Pi(x + \ell_i)} \cdot \Pi(x + \ell_i) .$$

(Here $x = x_{(E+1)}$) .

On the other hand $i_*(1) = \Pi(x + \ell_i)$ and $(i_* 1) \cdot x = 0$. Hence

$$\mathrm{ch}\, U = \Pi \frac{(1 - e^{-\ell_i})}{\ell_i} i_* 1 = i_* T^{-1}(E). \quad \text{Q.E.D.}$$

Now then, in our case E is the trivial bundle. Hence $T(E) = 1$. It follows that ch U generates $H^2(\mathbb{P}(E + 1)) : \mathrm{ch}\, U = (x_{E+1})^n$. However $\mathrm{ch}(\beta^! \xi_n)$ also equals $(x_{E+1})^n$. This proves Lemma 2 .

The theorem in general now follows from the *functorial* nature of the constructions we are performing in 2 stages.

Stage 1. Take $X \in \mathfrak{A}$, E trivial over X . To establish the theorem in this case one has to extend the Kunneth theorem from (7.1) to $\mathbb{K}\{X \times \mathbb{P}(E)\} = \mathbb{K}(X) \otimes \mathbb{K}\{\mathbb{P}(E)\}$, which is easily done by induction on the dim of E .

Stage 2. Take a finite covering $\{\mathcal{U}_i\}_{i=1}^{n}$ on X so that $E|\mathcal{U}_i$ is trivial. Assume the theorem for E over $X_k = \mathrm{Union}_{1 \leq i \leq k}\, \mathcal{U}_i$, and prove it for X_{k+1} by the Meyer Vietoris sequence.

Remarks. In my lectures I outlined a different proof for this theorem. Essentially I started with different statement of the periodicity theorem, namely with the assertion that when p is a point, then a generator of $\tilde{K}(S_{2n})$ goes (under the $\beta!$ of 7.1) over into $U = \lambda_{-S_{(E+1)}} \cdot E^*$. That is, I described an explicit trivialization of U on $\mathbb{P}(E)$ and thus a bundle on X^E , which I asserted to be the generator of $\tilde{K}(S_{2n})$. One may of course work backwards from this assumption to the periodicity theorem as stated here. The present analysis works because, as we now see, a posteriori, it does not matter *how one trivializes* U on $\mathbb{P}(E)$; the result will always generate $\tilde{K}(S_{2n})$. (The difference of two elements in $K(X/A)$ obtained by trivializing a bundle E on $A \subset X$, is in the image of $\delta: K^{-1}(A) \to K(X/A)$ and in this case $K^{-1}(A) = 0$.)

DEFINITION 7.1. *Let* $E \to X$ *be a complex vector bundle over* X . *Define*

$$i_! : \mathbb{K}(X) \to \tilde{\mathbb{K}}(X^E)$$

by the relation

$$\beta^! i_! u = \pi^! u \cdot U \qquad u \in K(X)$$

where

$$\beta : \mathbb{P}(E + 1) \longrightarrow X^E , \quad \text{and} \quad U = \lambda_{-S(E+1)}\pi^! E^* .$$

This additive homomorphism will be referred to as the "Thom homomorphism".

THEOREM 7.2. *The Thom homomorphism*

$$i_! : \mathbb{K}(X) \longrightarrow \tilde{\mathbb{K}}(X^E)$$

is a bijection. Further if $i^!\ \mathbb{K}(X^E) \to \mathbb{K}(X)$ *is induced by the inclusion* $X \longrightarrow X^E$, *then*:

(7.2) $$i^! i_! u = (\lambda_{-1}E^*) \cdot u .$$

We also have:

(7.3) $$\mathrm{ch}\ i_! u = i_* T^{-1}(E) \cdot \mathrm{ch}\ u ,$$

where T denotes the Todd class of Section 3 .

Except for the last two formulas, this theorem is a clear consequence of Theorem 7.1. The last formula follows from Proposition 7.1. To see (7.2) we observe that by the remarks in Section 1, $i = \beta \circ \sigma$ where σ is the map $X \to \mathbb{P}(E + 1)$ induced by the trivial section of 1 . Now it is clear that $\sigma^!(S_{E+1}) = 1$. Hence

$$i^! i_! u = \sigma^! \beta^! i_! u = \sigma^!(\lambda_{-S_{E+1}} \pi^! E^*)u = \lambda_{-1}E^* \cdot u . \quad \text{Q.E.D.}$$

Note: If we compare this with $i^* i_* u = c_n(E)u$ in the H^* case, we see that $\lambda_{-1}(E^*)$ plays the role of the n-th Chern class of the n-dimensional bundle E . By the way, $i_!$ could equally well have been defined so that $i^! i_! 1 = \lambda_{-1}(E)$, however the present definition coincides with the usual sign conventions which come from algebraic geometry.

COROLLARY 1. (*The splitting principle*). *Let* $\mathbb{F}(E)$ *be defined as in Section* 2, $\pi : \mathbb{F}(E) \to X$. *Then* $\pi^!$ *imbeds* $\mathbb{K}(X)$ *in* $\mathbb{K}\{\mathbb{F}(E)\}$; *further* $\pi^! E$ *splits into a sum of line bundles* $\pi^! E = \Sigma L_i$. *Hence* $\pi^! \lambda^i E = \Sigma L_1 \otimes \cdots \otimes L_i$ *the ith elementary function in the* L_i . *Thus the remarks concerning the extension of functors from line bundles to* $H^*(X)$ *apply equally well to the extension of functors from line bundles to* $\mathbb{K}(X)$.

COROLLARY 2. *The Adams operations* ψ_k *are ring homomorphisms*: $K(X) \to K(X)$.

We have already seen that if L is a line bundle, then:

$$\psi_k(L) = L^k .$$

Hence if $E = \Sigma L_i$, $E' = \Sigma L'_j$ are direct sums of line bundles, then

$$\psi_k(E \otimes E') = \psi_k(\Sigma L_i \otimes L'_j) = \Sigma (L_i)^k \otimes (L'_j)^k$$
$$= (\Sigma(L_i)^k)\otimes(\Sigma(L'_j)^k) = \psi_k(E)\otimes\psi_k(E') .$$

By the splitting principle this special case now implies the general one. Q.E.D.

The natural question arises of how $i_!$ commutes with the operations λ^i and ψ_k . We will answer this question for the ψ_k-which being additive and ring-homomorphisms - are much easier to handle. With this end in view we introduce the multiplicative functions θ_k, from bundles to K(X) , defined by :

(7.4) $\theta_k(L) = 1 + L^* + \cdots + L^{*k-1}$ if L is a line bundle

(7.5) $$\theta_k(E + F) = \theta_k(E) \cdot \theta_k(F) .$$

By the splitting principle, $\theta_k(E)$ is uniquely determined by these two conditions.

PROPOSITION 7.2. *The function* $E_k \to \theta_k(E)$ *has in addition to* 7.4, *and* 7.5, *the following properties:*

(7.6) $$\dim \theta_k(E) = k^{\dim E}$$

(7.7) $$\theta_{ts}(E) = \psi_t\theta_s(E) \cdot \theta_t(E)$$ (*cocycle condition*).

Proof: $\theta_k(L) = L + (\xi + 1) + \cdots + (\xi + 1)^{k-1}$, when $\xi = L - 1$. Hence $\dim \theta_k(L) = k$. As θ_k is multiplicative we obtain (7.6) . Finally, (7.7) is again trivial for line bundles:

$$\frac{L^{ts} - 1}{L^t - 1} \cdot \frac{L^t - 1}{L - 1} = \frac{L^{ts} - 1}{L - 1} ,$$

is preserved under multiplication, and hence holds in general.

Note that $\theta_2(E) = \Lambda_1(E)$.

THEOREM 7.3. *Let* $i_! : K(X) \to \tilde{K}(X^E)$ *be the Thom isomorphism. Then*

(7.8) $$i_! u \cdot i_! v = i_! \lambda_{-1}(E^*) \cdot u \cdot v$$

(7.9) $$\psi_k i_! u = i_! \theta_k(E)\psi_k(u) \qquad u, v \in K(X) .$$

Proof: (7.8) is a consequence of the fact that $U = \lambda_{-1}E^* + \cdots + \xi^n_{E+1}$. Hence $U^2 = \lambda_{-1}E^* \cdot U$. Now $\beta^!(i_! u \cdot i_! v) = U^2 u \cdot v = U \lambda_{-1}E^* uv$ whence $\beta^! i_! \lambda_{-1}E^* uv = i_! u \cdot i_! v$. Q.E.D.

For (7.9) we argue as follows: as ψ_k is a ring homomorphism it is sufficient to show that $\psi_k i_! 1 = i_! \theta_k(E)$.

We may, as usual, assume that $E = \Sigma L_i$. Then

$$\beta^! i_! 1 = U = \prod_i (1 - SL_i^*), \qquad S = S_{E+1} .$$

Hence

$$\psi_k U = \prod_i (1 - S^k L_i^{*k})$$

$$= U \cdot \prod_i (1 + SL_i^* + \cdots + S^{k-1} L_1^{*k-1}) .$$

On the other hand over $\mathbb{P}(E + 1)$ we have

$$\lambda_{-S}(E^* + 1) = 0 \Rightarrow (1 - S)\lambda_{-S}E^* = 0$$

which implies that $SU = U$. Hence

$$\psi_k U = U \cdot \prod_i (1 + L_i^* + \cdots + L_i^{*(k-1)})$$

$$= U \cdot \pi^! \theta_k(E) . \qquad \text{Q.E.D.}$$

Note: This proof is the precise analogue of the proof for the formula of Proposition 7.1: $ch(i_! 1) = i_* T^{-1}(E)$.

COROLLARY 3. If $\xi \in \tilde{K}(S_{2n})$ then:

$$\xi^2 = 0 , \ \psi_k \xi = k^n \xi , \ k \lambda^k \xi = (-1)^{k-1} \psi_k \xi .$$

Proof: Interpret S_{2n} as X^E with X a point, $\dim E = n$. Then $\lambda_{-1} E^* = 0$, and $\theta_k(E) = k^n$. This yields

the first two formulae. Now the last follows from the relations between λ_k and ψ_k, which whenever the multiplication is trivial reduce to:

$$k \lambda^k = (-1)^{k-1} \psi_k .$$

8. Applications: The obstruction to coreducibility.

If $E \to X$ is a (complex) bundle over a connected $X \in \mathfrak{A}$ then E is called coreducible if the sequence

$$P_X^E \xrightarrow[j]{} X^E \longrightarrow X^E / P_X^E$$

splits: i.e., if there exists a map $f : X^E \to P_X^E$ so that $f \cdot j$ = identity.

E is called S-coreducible if $(E + m \cdot 1)$ is coreducible for some u. The first positive integer n for which nE is S-coreducible is called the J-order of E. (This integer is the order of the J-class of E under the generalized J-homomorphism $J : K(X) \to J(X)$. (See [13].)

THEOREM 8.1. Let E be a complex vector bundle over $X \in \tilde{\mathfrak{A}}$ where we now assume that X is connected. Then E is coreducible only if there exists an invertible element $u^* \in K^+(X)$ so that for all $k \in \mathbb{Z}^+$,

$$(8.1) \qquad \theta_k(E) = k^{\dim E} \cdot \psi_k u^*/u^* \quad .$$

Proof: Assume that X^E is coreducible. Then we have a map: $f : X^E \to p_X^E$ such that $f \circ j$ = identity. Consider the commutative diagram:

$$\begin{tikzcd}
\tilde{K}(p_X^E) & \tilde{K}(X^E) \arrow[l, "j^!"'] \\
K(p_X) \arrow[u, "i'_!"] & K(X) \arrow[l, "\dim"'] \arrow[u, "i_!"']
\end{tikzcd}$$

and define $u \in K(X)$ by

$$i_! u = f^! i'_! 1 \quad .$$

Then $j^! i_! u = i'_! 1$ whence $\dim u = 1$. Further as $\psi_k i'_! 1 = k^{\dim E} \cdot i'_! 1$ by (7.7), it follows from (7.9) that

$$i_! \theta_k(E) \cdot \psi_k(u) = \psi_k(i_! u) = \psi_k(f^! i'_! 1) = i_! k^{\dim E} u \quad .$$

Thus $\theta_k(E) \cdot \psi_k(u) = k^{\dim E} \cdot u$. Now it is easy to see that the elements of $K(X)$, $X \in \mathfrak{A}$ which are invertible are precisely the elements with dim 1. Clearly ψ_k maps these elements into themselves. Hence our condition may be written in the form:

$$\theta_k(E) = k^{\dim E} \cdot u/\psi_k(u) , \qquad \dim u = 1 .$$

Finally if $u^* = 1/u$, we obtain:

$$\theta_k(E) = k^{\dim E} \cdot \psi_k u^*/u^* . \qquad \text{Q.E.D.}$$

For the stable theory the "obstruction" to S-coreducibility may be put in this form:

DEFINITION 8.1. Let $\mathbb{Z}^+$ denote the multiplicative monoid of the positive integers. A function $f : \mathbb{Z}^+ \to K(X)$ will be called a cocycle if:

$$(8.1) \qquad f(ts) = \psi_t f(s) \cdot f(t) \qquad s, t \in \mathbb{Z}^+$$

$$(8.2) \qquad \dim f(s) = s^{n(f)} \quad \text{where} \quad n(f) \in \mathbb{Z}^+ .$$

Clearly the cocycles form a monoid under pointwise multiplication. We call two cocycles f, g equivalent if there exist $n, m \in \mathbb{Z}^+$ such that

$$s^n f(s) = s^m g(s) \qquad s \in \mathbb{Z}^+ .$$

These equivalence classes form a monoid under multiplication, and we call these the stable cocycles.

PROPOSITION 8.1. *The stable cocycles form an Abelian group.*

Proof: Let $\hat{K}(X) = \mathfrak{m}$ be the ideal of elements of dim 0. From the fact that X has finite category, it follows that $\hat{K}(X)$ is nilpotent:

$$\hat{K}(X) = \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^n = 0 .$$

Now let f be a cocycle. Thus

$$f(s) = s^n + a(s), \qquad a(s) \in \hat{K} .$$

Let $f_1(s) = s^n - a(s)$. This will again be a cocycle. Hence

$$f(s) \cdot f_1(s) = s^{2n} + a(s)^2 .$$

We now replace f by the cocycle $f \cdot f_1$ and perform the same operation. After a finite number of steps one obtains a cocycle $g(s)$ so that

$$f(s) \cdot g(s) = s^n .$$

Hence the stable cocycle represented by g determines an inverse to the one represented by f. Q.E.D.

DEFINITION 8.2. *A stable cocycle which is represented by a function of the form:* $t \to \psi_t u^*/u^*$, *where* u^* *is an invertible element of* $K(X)$ *is called a stable coboundary. The group of stable cocycles modulo stable coboundaries is denoted by*

$$H^1(\mathbb{Z}^+ ; K(X)) .$$

There is now a natural homomorphism

$$\Theta: K(X) \to H^1(\mathbb{Z}^+; K(X))$$

defined as follows: If E is a bundle over X then $t \to \theta_t(E)$ defines a cocycle, and we define $\Theta(E)$ to be its class in $H^1(\mathbb{Z}^+; K(X))$. (As $\theta_t(E + n1) = t^n \cdot \theta_t(E)$, we see that $\Theta(E)$ depends only on the stable class of E.)

One has $\Theta(E + F) = \Theta(E) + \Theta(F)$ by (7.5). Hence Θ is additive, and therefore extends to a unique homomorphism

$$\Theta: K(X) \to H^1(\mathbb{Z}^+; K(X)) .$$

The image of $K(X)$ under Θ will be denoted by $\Theta(X)$.

THEOREM 8.2. *The kernel of* $J : K(X) \to J(X)$ *is contained in the kernel of* $\Theta: K(X) \to \Theta(X)$. *In other words* Θ *factors through* J, *and so induces a surjection*

$$\Theta_* : J(X) \to \Theta(X) .$$

Thus $\Theta(X)$ *furnishes a lower bound for* $J(X)$.

Proof: S-coreducibility of a bundle E means that for some n, $E + n \cdot 1$ be coreducible. Our necessary condition for this is then that there exist an integer n and an invertible u^* in $K(X)$ so that

$$\theta_k(E + n \cdot 1) = k^{\dim E} \psi_k u^*/u^*$$

i.e.,

$$k^n \theta_k(E) = k^{\dim E} \psi_k u^*/u^* .$$

That is, the stable cocycle represented by $k \to \theta_k(E)$ should be 0 in $\Theta(X)$. Q.E.D.

Example: *The classical* J-*homomorphism*

$J : K(S_{2n}) \to J(S_{2n}) \subset \pi_{m+2n}(S_m)$, $m >> n$.

We recall that $\tilde{K}(S_{2n}) \cong \mathbb{Z}$, and $\psi_k u = k^n u$ for $u \in \tilde{K}(S_{2n})$. Let ξ be a generator of this group, and as a first step to determining the group $H^1(\mathbb{Z}^+; K(S_{2n}))$, consider the form which a stable cocycle must take. As there is no torsion, we may extend to the rationals and write every cocycle in the form:

$$f(t) = t^\sigma(1 + a(t) \cdot \xi), \quad a(t) \in \mathbb{Q}, \; t^\sigma a(t) \in \mathbb{Z} .$$

The cocycle condition then yields:

$$\begin{aligned} f(ts) &= (ts)^\sigma(1 + a(ts)\xi) = \psi_t f(s) \cdot f(t) \\ &= s^\sigma(1 + a(s)t^\sigma\xi)(1 + a(t)\xi)t^\sigma , \end{aligned}$$

so that, $a(ts) = a(s)t^n + a(t)$. On the other hand $a(ts) = a(st)$ whence:

$$a(s)t^n + a(t) = a(t)s^n + a(s)$$

or

$$a(s)(t^n - 1) = a(t)(s^n - 1) .$$

It follows that f is completely determined by σ, and $a(2)$, (or indeed any $a(k)$ would do with $k > 1$.)

$$a(s) = \frac{a(2)}{(2^n - 1)} \cdot (s^n - 1) .$$

We set $A(f) = a(2)/(2^n - 1)$. Thus f is determined by the pair $\{\sigma, A(f)\}$, and clearly equivalent cocycles differ only in their σ-component. Thus the stable class of f is determined by the rational number $A(f)$. This number is not arbitrary. We have to have : $s^\sigma \cdot a(s) \in \mathbb{Z}$, (large σ) or:

$$A(f) \cdot s^\sigma(s^n - 1) \in \mathbb{Z} \quad \text{for all } s \in \mathbb{Z}^+, \quad \sigma \text{ large} .$$

Now the greatest common factor of $s^\sigma(s^n - 1)$ (σ large) is a well defined integer $\rho(n)$. Hence the stable cocycles may be identified with the integral multiples of $1/\rho(n)$ in $\mathbb{Q}$. Now, $A(f)$ will represent 0 in $\Theta(S_{2n})$ if and only if there exists integers ν, λ so that

$$t^\sigma(1 + a(t)\xi) = t^\nu(1 - \lambda\xi)(1 - \lambda t^n\xi)^{-1}$$

i.e., if and only if :

$$a(t) = \lambda(t^n - 1)$$

or

$$A(f) \cdot (t^n - 1) = \lambda(t^n - 1) \quad \Leftrightarrow A(f) \text{ is an integer.}$$

Thus: $H^1(\mathbb{Z}^+; K(S_{2n})) \cong \mathbb{Z}_{\rho(n)}$.

Determination of $\Theta(S_{2n})$.

From the preceding it is clear that we only need to choose a representative cocycle for $\Theta(\xi)$ a generator of $\tilde{K}(S_{2n})$ say f, and then determine the value $A(f)$, which we denote by $A(\xi)$. This amounts to choosing a bundle E with $E - \dim E \cdot 1 = \xi$ and determining $\theta_2(E) = \lambda_{+1}(E)$. Now

$$\lambda_t(E) = \lambda_t(\xi) \cdot (1+t)^{\dim E} .$$

Write $\lambda_t(\xi) = 1 - \varphi_n(t)$ where $\varphi_n(t)$ is a power series in $\mathbb{Q}[[t]]$. Because

$$\lim_{t\to+1} \lambda_t(E) \text{ exists, } \lim_{t\to+1} \varphi_n(t) \text{ will have to exist,}$$

whence

$$\theta_2(E) = 2^{\sigma}\{1 - \lim_{t\to+1} \varphi_n(t) \cdot \xi\} .$$

Now comparing this to $A(f)$ we see that

$$A(\xi) = \lim_{t\to+1} \varphi_n(t)/2^n - 1 .$$

Thus the problem reduces to computing $\lambda_t\xi$. Recall now (Corollary 3 of Theorem 7.3) that $\psi_k\xi = k^n\xi$, whence $\lambda^k\xi = (-1)^{k-1}(k^{n-1})\xi$, $k \geq 1$. Thus $\lambda_t\xi = 1 - (\Sigma_{t\geq 1} (-t)^k k^{n-1}) \cdot \xi$. Or $\varphi_n(t) = \Sigma(-t)^k k^{n-1}$. This implies

$$t\varphi_n'(t) = \varphi_{n+1}(t) .$$

Set $q_n(u) = \varphi_n(e^u)$. Then the above goes over into

$$q_n'(u) = q_{n+1}(u) \quad \text{and} \quad \lim_{t\to 1} \varphi_n(t) = q_n(0) .$$

Now $q_1 = \dfrac{-e^u}{1+e^u}$, whence

$$q_n(0) = (n-1)! \times \text{coefficient of } u^{n-1} \text{ in } q_1 .$$

We next observe that:

$$q_1 + 1/2 = 1/2 \tanh (u/2)$$

and that $1/2 \tanh u/2 = \Sigma\, 2^{(2k-1)}(2^{2k} - 1)\{B_{2k}/(2k)!\}(u/2)^{2k-1}$ where B_{2k} are the Bernoulli #'s. Hence $q_{2n-1}(0) = 0$, $q_{2n}(0) = (2^{2n} - 1) \cdot B_{2n}/2n$, whence finally

$$A(\xi) = B_{2n}/2n .$$

Thus we obtain:

$$\Theta(S_{4n}) \approx \mathbb{Z}_{d(n)}$$

where $d(n)$ is the denominator of $B_{2n}/2n$.

Remarks 1. This lower bound was first obtained by Milnor and Kervaire by rather geometric methods. One obtains the same bound if one applies the character criterion (Theorem 6.1). The argument would be as follows follows: Suppose that X^{mE} is coreducible, $m \in \mathbb{Z}$, E-dim E generating $\tilde{K}(S_{2n})$. Now as a CW complex $X^{mE} = S_{2m} \cup e_{2(m+n)}$. Hence coreducibility ⇒

$$X^{mE} = S_{2m} \vee S_{2(m+n)} \quad .$$

(Splitting off the top cell is called coreducibility, and, as we see, over the spheres the two conditions are equivalent.)

Consider now the bundle $i_! 1 \in K(X^{mE})$. We have the implication: the coreducibility of X^{mE}

⇒ top cocycles of X^{mE} spherical .

⇒ ch $i_!$ 1 is integral on this cycle (Theorem 6.1)

⇒ $i_*(T^{-1}E)^m$ is integral on this cycle by (7.3)

⇒ $\{T^{-1}(E)\}^m$ is integral on the top cycle of S_{2n} .

Now we know by (Theorem 6.1) that $ch(E) = \dim E + u_n$ where u_n generates $H^{2n}(S_{2n})$.

However it is clear from the earlier discussion that $ch(E)$ determines $T^{-1}(E)$ in a purely algebraic way. If one carries out this determination in the present case one obtains the same lower bound on m .

2. The lower bound which we described can be improved by a factor of 2 with the aid of the real K-theory, i.e., the K-theory obtained by starting with real vector-bundles. This theory will be denoted by KO , and it is the purpose of the next sections to prove the KO-analogues of the theorems we have developed for K . In particular we seek an $i_! : KO(X) \xrightarrow{\approx} \widetilde{KO}(X^E)$ when E is any real vector bundle. Unfortunately such an $i_!$ does not exist in general, and I know of no way to extend the elementary arguments of the preceeding section to define $i_!$ even when it does exist. We will therefore have to switch our point of view a little and discuss the Lie-group phenomena which underly the construction of $i_!$.

§9. The representative ring of a group.

In the following G will denote a compact Lie group. By a G-module we mean a vector space W (over the field $\mathbb{R}$ or $\mathbb{C}$) together with an action of G as a group of continuous automorphisms of W . Two such modules are called isomorphic if there is a isomorphism between them which commutes with the G action.

One denotes by $RU(G)$ the free group generated by the irreducible isomorphy classes of complex G-modules and by $RO(G)$ the corresponding group over the real numbers. We write simply $R(G)$ when either of these will do and use the symbols $KU(X)$, $KO(X)$, $K(X)$ correspondingly. There are several additional structures on $R(G)$. The tensor product of modules induces a commutative ring structure on $R(G)$ and the exterior powers $\lambda^i W$ of a G-module extend to operations $\lambda^i : R(G) \to R(G)$ by the same principle used in the K-theory. This becomes clear if one uses the alternate definition of $R(G)$ as the ring obtained from the category of G-modules via the K-construcion, i.e., as the solution of a universal problem. These two definitions coincide because every G-module is a direct sum of irreducible G-modules in view of the compactness of G.

The rings $R(G)$ are useful because the "mixing process" defines a functor

$$\alpha : H^1(X ; \underline{G}) \times R(G) \longrightarrow K(X)$$

from principal G-bundles over X - $H^1(X ; \underline{G})$-cross $R(G)$, to $K(X)$. To see this recall that a (principal) G-bundle E over X is a space on which G acts on the right so that

locally this action corresponds to the right translations of G on $U \times G$. Suppose now that E is such a G-bundle over X, and that F is a space on which G acts on the left. Then we have the mixing diagram:

(9.1)
$$\begin{tikzcd}
E \arrow[d, "\pi"'] & E \times F \arrow[l] \arrow[r] \arrow[d, "\tau"] & F \arrow[d] \\
X & E \times_G F \arrow[l, "\sigma"] \arrow[r] & p
\end{tikzcd}$$

where τ is obtained by identifying $eg \times g^{-1}f$ with $e \times f$ in $E \times F$. Thus $E \times_G F \to X$ is a locally trivial fibering with F as fiber.

Now in the case when F is a G-module $E \times_G F$ is a vector bundle over X, which we denote by $\alpha(E, F)$ or $\alpha_E(F)$ or $F(E)$. The linear extension of this function defines the functor α.

The following are quite obvious properties of α:

(9.2) For fixed E, the homomorphism $\alpha_E : R(G) \to K(X)$ a λ^i-homomorphism of the two rings.

(9.3) The following diagram is commutative:

$$\begin{tikzcd} H^1(X;\underline{G}) \times \underline{R}(G) \arrow[r, "\alpha"] & K(X) \\ H^1(Y;\underline{H}) \times \underline{R}(H) \arrow[r, "\alpha"] \arrow[u, "i_* \circ f^{-1} \times i^!"] & K(Y) \arrow[u, "f^!"'] \end{tikzcd} .$$

Here $i : H \to G$ is a homomorphism of groups, $i_* : H^1(X; \underline{H}) \to H^1(X;\underline{G})$ the induced homomorphism, $i^! \underline{R}(G) \to \underline{R}(H)$ the restriction homomorphism, $f : X \to Y$, a map, and f^{-1} and $f^!$ the induced homomorphisms of f in $H^1(X; \underline{H})$ and $K(Y)$ respectively.

In the next section certain elements of $R(G)$ will have to be singled out when G is one of the classical groups. For this purpose we review some of the basic facts concerning $R(G)$. All of these are essentially due to E. Cartan.

PROPOSITION 9.1. *Every irreducible complex* $U(1)$ *module is one dimensional. Hence* $RU\{U(1)\} \cong$ *group-ring of* $\mathrm{Hom}\,\{U(1), C^*\}$.

Here, of course, $U(1)$ denotes the circle group of complex numbers of norm 1.

COROLLARY. *Let* x *denote the* $\mathbb{C}$ *module of* $U(1)$ *given by the inclusion* $U(1) \to C^*$. *Then*

$$RU\{U(1)\} = \mathbb{Z}[x, x^{-1}] \quad .$$

Thus in this case RU is the ring of finite Laurent series in x.

More generally let $T = U(1) x \cdots x\, U(1)$ be a torus, and let $f_i : T \to U(1)$, $i = 1, \cdots, k$, be the various projections. Then $x_i = f_i^! x \in RU(T)$ and

$$RU(T) \approx \mathbb{Z}[x_i, x_i^{-1}] \qquad i = 1, \cdots, k .$$

These facts are quite elementary. The following two theorems are not.

THEOREM I : *Let* $T = U(1)x \cdots x\, U(1)$, k *factors, be a maximal torus of* G. *Let* $W = W(G, T)$ *be the group of automorphisms of* T *induced by inner automorphisms of* G. *Then* W *acts on* $RU(T)$ *and we let* $RU(T)^W$ *denote the ring of invariants under this action. We also denote the restriction homomorphism from* $RU(G)$ *to* $RU(T)$ *by* ch,

In this notation ch *induces a bijection of* $RU(G)$ *onto* $RU(T)^W$:

$$\underline{ch} : RU(G) \cong RU(T)^W \quad .$$

THEOREM II . *If* G *is compact connected and simply connected, then* $RU(G)$ *is a polynomial ring*.

In view of Theorem I one may describe the elements of RU(G) in RU(T) once W(G, T) is known. In the following section we make certain standard choices for T in G and describe the action of W(G) on a standard basis for RU(T) .

THE UNITARY GROUP U_n, and SU_n .

We interpret U_n as the $n \times n$ matrices with complex coefficients which satisfy the identity:

$$A\,\overline{A}^t = 1 \quad ,$$

SU_n is the subgroup with determinant 1 .

The diagonal matrices in U_n form a maximal torus $T(U_n)$. Let x_i be the character on $T : x_i : T \to \mathbb{C}^*$, which assigns to $t \in T(U_n)$ its ith diagonal entry. We also let x_i stand for the element in $RU\{T(U_n)\}$ determined by the $T(U$ structure defined on $\mathbb{C}$ via: $t \cdot z = x_i(t) \cdot z$, $z \in \mathbb{C}$, $t \in T(U$ Thus

$$RU\{T(U_n)\} = \mathbb{Z}[x_i, x_i^{-1}] \qquad i = 1, \cdots, n .$$

We have further:

(9.4) $W(U_n)$ acts as the permutation group of the x_i in $RU\{T(U_n)\}$.

(9.5) $RU(U_n) \cong$ (under ch) the finite invariant Laurent series in $x_1, \cdots, x$.

(9.6) Let ρ_n be the standard representation of U_n on $\mathbb{C}^n$. Then $\underline{ch}\,\rho_n = x_1 + \cdots + x_n$, and hence

$$RU(U_n) \cong \mathbb{Z}[\rho_n, \lambda^2\rho_n, \cdots, \lambda^n\rho_n; \{\lambda^n\rho_n\}^{-1}] \quad .$$

Remarks:

1. The implications (9.4) ⇒ (9.5) ⇒ (9.6) are quite straightforward.
2. The $\lambda^i\rho_n$ are irreducible because $\underline{ch}\ \lambda^i\rho_n$ consists of "one orbit" of the action of W .

(9.7) $$RU\{SU_n\} = \mathbb{Z}[\rho_n, \lambda^2\rho_n, \cdots, \lambda^{n-1}\rho_n]$$

with $\lambda^n\rho_n = 1$. Here ρ_n denotes the restriction of the standard representation to SU_n .

THE GROUPS SO_n

This group is a subgroup of U_n on which

$$\overline{A} = A\ , \quad \det A = 1, \quad A \in U_n \ .$$

Thus SO_n consists of the real $n \times n$ matrices subject to

$$A \cdot A^t = 1\ , \quad \det A = 1\ .$$

We now have to treat these groups separately depending on the parity of n .

Case 1 . The odd orthogonal groups, SO(2k+1). We may imbed SO(2) × ··· × SO(2) (k factors) in SO(2k + 1) as the k diagonal boxes:

$$\begin{pmatrix} SO(2) & & & \\ & SO(2) & & \\ & & SO(2) & \\ & & & 1 \end{pmatrix}$$

followed by a 1. This will be our standard maximal torus: $T(SO_{2k+1})$. We now choose isomorphisms $\underline{y}_i : SO(2) \to \mathbb{C}^*$ and let $y_i \in RU(T\{SO(2k+1)\})$ be the corresponding classes. Thus

(9.8) $\quad RU(T\{SO(2k+1)\}) \cong \mathbb{Z}[y_i; y_i^{-1}] \qquad i = 1, \cdots, k$.

Further

(9.9) $W\{SO(2k+1)\}$ acts as the group generated by permutations of the y_i and transformations $y_i \to y_i^{\epsilon_i}$, $\epsilon_i = \pm 1$.

<u>Case</u> 2. The even orthogonal groups. We include $SO(2k)$ in $SO(2k+1)$ as the matrices with last diagonal entry 1. Then $T\{SO(2k)\} = T\{SO(2k+1)\}$.

(9.10) $W\{SO(2k)\}$ acts as the group generated by permutations of the y_i and transformations $y_i \to y_i^{\epsilon_i}$, $\epsilon_i = \pm 1$, $\prod_1^k \epsilon_i = 1$.

THE SPIN-GROUPS

The double covering of $SO(n)$ is denoted by $Spin(n)$. Let $\pi : Spin(n) \to SO(n)$ be the projection and choose

$\tilde{T} = T\{Spin(n)\}$ as $\pi^{-1}T\{SO(n)\}$. We now have, setting $T = T\{SO(n)\}$.

(9.11) The homomorphism $\pi^!$: $RU(T) \to RU(\tilde{T})$ extends to a bijection of $RU(T)[u]/(u^2 = y_1 \cdots y_k)$ onto $RU(\tilde{T})$, (i.e., $RU(\tilde{T})$ is a quadratic extension over $RU(T)$.) Further this isomorphism is compatible with the action of the W of the two groups on the respective rings.

It is customary to write $y_1^{1/2}, \cdots, y_k^{1/2}$ for the element u. With this understood, we define $\Delta_{2n}^{\pm} \in RU(Spin(2n))$ and $\Delta_{2n+1} \in RU\{Spin(2n+1)\}$ by:

$$\underline{ch}\,\Delta_{2n}^{+} = \Sigma\, y_1^{\epsilon_1} \cdots y_n^{\epsilon_n}, \quad \epsilon_i = \pm 1/2, \quad \prod_1^n \epsilon_i = 1/2^n$$

$$\underline{ch}\,\Delta_{2n}^{-} = \Sigma\, y_1^{\epsilon_1} \cdots y_n^{\epsilon_n}, \quad \epsilon_i = \pm 1/2, \quad \prod_1^n \epsilon_i = -1/2^n$$

$$\underline{ch}\,\Delta_{2n+1} = \Sigma\, y_1^{\epsilon_1} \cdots y_n^{\epsilon_n}, \quad \epsilon_i = \pm 1/2, \quad \prod_1^n \epsilon_i = \pm 1/2^n.$$

These are the so-called spin-representations of the Spin-groups. Under restriction it is clear that Δ_{2n+1} goes over into $\Delta_{2n}^{+} + \Delta_{2n}^{-}$ while Δ_{2n}^{+} and Δ_{2n}^{-} restrict to Δ_{2n-1}.

From (9.9), (9.10) and (9.11) one concludes that:

(9.12) $\quad RU\{Spin(2n+1)\} \cong \mathbb{Z}[\rho, \cdots, \lambda^n\rho, \Delta_{2n+1}]$

(9.13) $$RU\{\mathrm{Spin}\,(2n)\} = \mathbb{Z}[\rho, \cdots, \lambda^{n-1}\rho; \Delta^{+}_{2n}, \Delta^{-}_{2n}]$$

where now ρ denotes $\pi^{!}$ of the ρ_{2n+1} and ρ_{2n} restricted to $SO(2n+1)$ and $SO(2n)$ respectively.

Exercise: Let $\mathbb{Z}_2 \subset \mathrm{Spin}(n) \times U(1)$ be the subgroup generated by $\epsilon \times (-1)$ where ϵ generates the Kernel of $\pi : \mathrm{Spin}(n) \to SO(n)$. This group is in the center of $\mathrm{Spin}(n) \times U(1)$ and the quotient $\mathrm{Spin}(n) \times U(1)/\mathbb{Z}_2$ is denoted by $\mathrm{Spin}^c(n)$. Give a description of $RU\{\mathrm{Spin}^c(n)\}$. Also show that there exists a homomorphism $\varphi : U(n) \to \mathrm{Spin}^c(2n)$ which makes the following diagram commutative:

$$\begin{tikzcd}
& \mathrm{Spin}^c(2n) \arrow[d, "\pi"] \\
U(n) \arrow[ur, "\varphi"] \arrow[r, "i"'] & SO(2n)
\end{tikzcd}$$

where i is the usual imbedding.

§10. The RO of a compact Lie-group. If V is a real G-module $V \otimes_{\mathbb{R}} \mathbb{C}$ is in an obvious way a complex G-module. This operation defines a λ^i-ring homomorphism

$$\epsilon^* : RO(G) \to RU(G) .$$

Conversely we may pass from a complex G-module to the underlying real G-module, thus obtaining an additive homomorphism

$$\epsilon_* : RU(G) \to RO(G) .$$

These two operations are linked by the standard identity

(10.1) $$\epsilon_* \circ \epsilon^* W = 2W ; \quad \epsilon^* \circ \epsilon_* V = V + V^* .$$

From the fact that R(G) is a free module it now follows that:

(10.2) Both $\epsilon^* : RO(G) \to RU(G)$ and $\epsilon_* : RU(G) \to RO(G)$ are injective.

We already know a considerable amount about RU(G). It is therefore natural to consider RO(G) as imbedded in RU(G) via ϵ^* and this will be our point of view. We next describe a criterion for an element x of RU(G) to be contained in $RO(G) \subset RU(G)$.

CRITERION: The class of a complex G-module W is contained in RO(G) if and only if W admits a non-degenerate G-invariant quadratic form ϕ.

Proof: Let V be a real G-module. Because G is compact we may integrate a positive definite form over G and so obtain a nondegenerat

$\emptyset : V \to \mathbb{R}$. The complexification of $\emptyset$ then is a form with the same properties on $\epsilon^* V$.

Conversely assume that W is a complex G-module with nondegenerate quadratic form $\emptyset$. Choose an invariant positive definite hermitian form on W and denote the inner product it defines by $\langle u, v \rangle$.

Consider the $\mathbb{R}$-linear map $T : W \to W$, defined by:

$$\langle Tx, y \rangle = \overline{\emptyset(x, y)} \ .$$

Clearly we have:

(10.3) $\qquad T\lambda x = \bar{\lambda} Tx , \qquad \lambda \in \mathbb{C}, \quad x \in W .$

(10.4) $\qquad$ T is nonsingular, and commutes with the action of G .

Properly speaking, T is thus defined on $\epsilon_* W$. Now the formula $\{x, y\} = \langle x, y \rangle + \overline{\langle x, y \rangle}$ defines a positive definite inner product on $\epsilon_* W$ and it is easily seen that T is self-adjoint with respect to it.

Let $W^+ \subset \epsilon_* W$ be the subspace spanned by the eigenvectors of T corresponding to the positive eigenvalues. Similarly, define W^- . Then these spaces are real G-modules and span $\epsilon_* W$ by (10.4) . On the other hand by (10.3) we see that $W^+ \cdot \sqrt{-1} = W^-$. Hence the natural map

$$W^+ \otimes \mathbb{C} \to W$$

given, by the $\mathbb{C}$-structure of W is a bijection of G-modules and so exhibits W as $\epsilon^* W^+$. Q.E.D.

COROLLARY 10.1. If $W = \epsilon^* V$, then $W^* \cong W$.

COROLLARY 10.2. *Let* W *be an irreducible complex* G-*module with* $W^* \cong W$. *Then* $W = \epsilon^* V$, *where* V *is a (necessarily irreducible)* G-*module over* $\mathbb{R}$, *if and only if* $\lambda^2 W$ *does not contain the trivial representation.*

Proof: By Schur's lemma $W^* \otimes W$ contains the trivial G-module precisely once. Now, as $W^* \cong W$, we have:

$$W^* \otimes W \cong W \otimes W \cong S^2 W^* \oplus \lambda^2 W$$

where $S^2(W)$ denotes the second symmetric product of W^*. We see then that the trivial G-module occurs either in $S^2 W^*$ or in $\lambda^2 W$. In the former case W will have a (necessarily nondegenerate) quadratic form. In the latter case it will not. Q.E.D.

Thus if one knows the expansion of $\lambda^2 W$ in terms of the irreducible G-modules one may decide the question of whether W is in $\epsilon^* RO(G)$.

COROLLARY 10.3. *Let* A *denote the set of isomorphism classes of irreducible* G-*modules* $\{W\}$ *for which* $W^* \neq W$, *and let* B *denote the complementary set. Let* $A_{1/2}$ *denote a "fundamental domain" for the action of* $*$ *in* A, i.e., *of every pair* w, w^*, *let* $A_{1/2}$ *contain precisely one member. Let* B^+ *denote those modules in* B, *for which* $\lambda^2 W$ *does not contain the trivial representation, and set* $B^- = B - B^+$. *Then an additive base for* $\epsilon^* RO(G)$ *is given by:*

$$\{W + W^* \mid W \in A_{1/2}\} \cup \{W \mid W \in B^+\} \cup \{2W \mid W \in B^-\} .$$

The proof should be clear.

An example: $RO\{\mathrm{Spin}(n)\} \subset RU\{\mathrm{Spin}(n)\}$.

To study this inclusion we will use the notation of Section 9 and also abbreviate $RU\{\mathrm{Spin}(n)\}$ to $RU(n)$. Similarly $RO(n)$ denotes $RO\{\mathrm{Spin}(n)\}$. Recall then that:

$$RU(2n) = \mathbb{Z}[\lambda^1\rho_{2n}, \cdots, \lambda^{n-1}\rho_{2n}; \Delta^+_{2n}, \Delta^-_{2n}] .$$

Now ρ_{2n} and hence $\lambda^i\rho_{2n}$ are clearly in $RO(2n)$. Hence the only question which remains is when the spin representations $\Delta^{\pm}_{2n}$ are in $RO(2n)$.

To apply our criterion we need the following facts:

$$(10.5) \qquad (\Delta^{\pm})^* = \begin{cases} \Delta^{\pm}_{2n} & \text{if } n \text{ is even} \\ \Delta^{\mp}_{2n} & \text{if } n \text{ is odd} . \end{cases}$$

$$\lambda^2 \cdot \Delta^{\pm}_{2n} = \sum_{i=0}^{i=n-1} \lambda^i\rho_{2n} \qquad i = (n+2) \bmod 4$$

$$(10.6) \qquad \Delta^+_{2n} \cdot \Delta^-_{2n} = \sum_{i=0}^{i=n-1} \lambda^i\rho_{2n} \qquad i = (n+1) \bmod 4$$

$$S^2 \circ \Delta^{\pm}_{2n} = \sum_{i=0}^{i=n-1} \lambda^i\rho_{2n} + \lambda^n_{\pm}\rho_{2n}, \quad i \equiv (n) \bmod 4 .$$

In the last formula, S^2 denote the symmetric square, and $\lambda^n_{\pm}\rho_{2n}$ are the two pieces into which $\lambda^n\rho_{2n}$ splits: Thus if we set

$$\prod_1^n (1 + ty_i)(1 + uy_i^{-1}) = \Sigma A_{ij} t^i u^j ,$$

then

$$\underline{\mathrm{ch}}\, \lambda^n_+ \rho_{2n} = \sum_{i+j=n} A_{ij} , \qquad i \text{ even} .$$

These formulae are relatively straightforward combinatorial identities in $\mathbb{Z}[y_i, y_i^{-1}]$.

PROPOSITION 10.1. *The elements* $\Delta^{\pm}_{2n}$, $\lambda^i\rho_{2n}$, $i \leq n-1$ *are represented by irreducible* Spin(2n)-*modules*.

This result is nontrivial - for instance one has to construct the spin-representations. We will assume this statement. [See [10]] .

Applying these formulae to our criterion we conclude:

(10.7) $$\Delta^{\pm}_{8n} \in RO(8n)\,, \quad \Delta^{\pm}_{8n+4} \notin RO(8n+4) \quad .$$

We turn next to the odd case. Recall then that

$$\underline{ch}(\Delta_{2n+1}) = \underline{ch}(\Delta^{+}_{2n} + \Delta^{-}_{2n})$$

$$\underline{ch}(\rho_{2n+1}) = \underline{ch}(\rho_{2n} + 1) \quad .$$

Hence one may again use the formulae 10.5, 10.6, to obtain:

$$\lambda^2 \circ \Delta_{2n+1} = \sum_{i=1}^{n-1} \lambda^i \rho_{2n+1} \qquad i = n+3 \text{ or } n+2 \bmod 4$$

$$S^2 \circ \Delta_{2n+1} = \sum_{i}^{n} \lambda^i (\rho_{2n+1} - 1) \quad i = n \text{ or } n+1 \bmod 4$$

and thereby conclude that:

(10.8) $\Delta_{2n+1} \subset RO(2n+1)$ only if $n = 0, 3 \pmod 4$.

In particular then, combining (10.7) with (10.8), we have:

(10.9) $RO(n) \cong RU(n)$ for $n \equiv -1, 0, 1 \bmod 8$.

PROPOSITION 10.2. *Let* $\iota^{!} : RO(8n+1) \to RO(8n)$ *be induced by the inclusion* $\text{Spin}(8n) \to \text{Spin}(8n+1)$. *Then*

(10.10) $\iota^{!}$ *is an injection* .

(10.11) $RO(8n)$ *is freely generated by* 1 *and* Δ^{+}_{8n} *over* $RO(8n+1)$.

From this last observation we conclude immediately that:

PROPOSITION 10.3. *There are unique elements* $A, B, \theta_k, \Gamma_k \in RO(8n+1)$ *which satisfy the equations:*

(10.12)
$$(\Delta^{+})^2 = (\iota^{!} A)\Delta^{+} + \iota^{!} B, \qquad \Delta^{+} = \Delta^{+}_{8n}$$
$$(\psi_k \Delta^{+}) = (\iota^{!}\theta_k)\Delta^{+} + \iota^{!}\Gamma_k \quad .$$

Further one has:

$$\theta_2 = A = \Delta_{8n+1}\,; \quad B = -\sum_{i=1}^{2n} \lambda^{2i-1}(\rho_{8n+1} - 1)$$

$$\underline{ch}\ \theta_k = \prod_{1}^{4n} \left\{ y_i^{(k-1)/2} + \cdots + y_i^{-(k-1)/2} \right\}$$

We conclude by tabulating our results concerning the real spin representations in terms of the complex ones:

RO(n)	Real Spin Representations	a_h-their dimension	$\widetilde{KO}(S_n)$
1	Δ_1	1	Z_2
2	$\Delta_2^+ + \Delta_2^-$	2	Z_2
3	$2\Delta_3$	4	O
4	$2\Delta_4^+$, $2\Delta_4^-$	4	Z
5	$2\Delta_5$	8	O
6	$\Delta_6^+ + \Delta_6^-$	8	O
7	Δ_7	8	O
8	Δ_8^+, Δ_8^-	8	Z

This table is periodic in the sense that $a_{n+8} = 16a_n$ and that the pattern is preserved in the first and last column. Note that comparison with the last column gives us the empirical fact that

$$a_n/a_{n+1} = \begin{cases} 1 & \text{if } \widetilde{KO}(S_n) = 0 \\ 1/2 & \text{if } \widetilde{KO}(S_n) \neq 0 \end{cases} .$$

This strange relation between the integers $\{a_i\}$ - the so-called Radon-Hurwitz numbers and $\widetilde{KO}(S_n)$ was noticed by Shapiro and myself last year. It essentially expresses the fact that the generators of $KO(S^n)$ are given by induced representations [8] .

§11. *Induced representations*. Let $i : H \to G$ be the inclusion of a closed subgroup of G . Thus G acts on G/H on the left, and we may, by the mixing construction, interpret G/H as a functor from G-bundles over X to spaces over X on which a certain H-bundle is singled out. For example, if $G = U(n)$, $H = U(n-1) \times U(1)$ this construction will specialize to our earlier $\mathbb{P}$ - functor $E \to \mathbb{P}(E)$. For this reason we will, in general, denote this construction by $\mathbb{P}$. Precisely: If E is a G-bundle over X, $\mathbb{P}(E)$ is defined by

$$\mathbb{P}(E) = E \underset{G}{\times} G/H .$$

In other words $\mathbb{P}(E)$ is the associated bundle to E with fiber G/H .

The following three theorems are standard in the theory of fiber bundles. As they express different ways of looking at the same thing I propose to call them tautologies.

TAUT. 1. *Consider the quotient space* E/H . *There is a natural isomorphism* $E/H \cong \mathbb{P}(E)$ *as spaces over* X .

Proof: Clearly $E = E \underset{G}{\times} G$. Dividing both sides by H we obtain $E/H = (E \underset{G}{\times} G)_H = E \underset{G}{\times} G/H$. Q.E.D.

Thus we have the following diagram:

$$\begin{tikzcd} E \arrow[r, "\rho"] \arrow[d, "\pi"'] & E/H = \mathbb{P}(E) \arrow[dl, "\sigma"] \\ X & \end{tikzcd}$$

where each map is a fibering, and ρ exhibits E as an H-bundle over $\mathbb{P}(E)$. *This bundle is denoted by* $\hat{E}$.

TAUT. 2. *In the situation envisaged above there is a canonical isomorphism*:

$$\sigma^{-1}E = \hat{E} \times_H G .$$

In words we have: The G-extension of $\hat{E}$ is isomorphic to the inverse image of E under σ. Or again, $\sigma^{-1}E$ admits a canonical reduction to the H-bundle $\hat{E}$.

Proof: By the definition of $\sigma^{-1}E$ one has the "exact sequence":

$$0 \longrightarrow \sigma^{-1}E \longrightarrow E \times E/H \overset{\pi'}{\underset{\sigma}{\rightrightarrows}} X$$

where $\pi' : E \times E/H \to E \xrightarrow{\pi} X$ and σ' projects the other way. Now define $\tilde{f} : E \times G \to E \times E$ by $\tilde{f}(e, g) = (eg, e)$. Then $\tilde{f}$ induces a map $f : E \times_H G \to E \times E/H$ which may be lifted

to $\sigma^{-1}E$. Using local triviality one easily constructs an inverse. Q.E.D.

Note: In the context of our "old" $\mathbb{P}(E)$ this proposition corresponds to the fact that when lifted to $\mathbb{P}(E)$, E became the direct sum of S_E and Q_E.

TAUT. 3. *The G-bundle* E *can be reduced to an H-bundle if and only if* $\mathbb{P}(E) \xrightarrow{\sigma} X$ *admits a section.*

Proof: Let $s : X \to \mathbb{P}(E)$ be a section. Then, by Taut. 1, $S^{-1} \circ \sigma^{-1}E = S^{-1}(\hat{E} \times_H G)$. Thus, as $\sigma \circ s = 1$, we obtain $E = (S^{-1}\hat{E}) \times_H G$ and $S^{-1}\hat{E}$ is an H-reduction of E. Conversely, assume that $E = F \times_H G$ where F is an H-bundle over X. Then we have $\mathbb{P}(E) = F \times_H G \times_G G/H$ $= F \times_H G/H$, and the identity coset of G/H in each fiber yields a section of $\mathbb{P}(E)$ over X. Q.E.D.

We next relate this situation with the functors discussed in Section 9. Fixing E, G and H, we have the following three homomorphisms canonically defined:

$$\alpha_{\hat{E}} : R(H) \longrightarrow K\{\mathbb{P}(E)\}$$

$$\alpha_E : R(G) \longrightarrow K(X)$$

$$i^! : R(H) \longrightarrow R(G) .$$

Apart from the obvious functorial relations between these there are two identities connecting them: The first we will call the permanence law:

PERMANENCE. *Let* $x \in R(H)$, $y \in R(G)$ *and denote the projection* $\mathbb{P}(E) \to X$ *by* σ . *Then*

$$\alpha_{\hat{E}}(x \cdot i^! y) = \alpha_{\hat{E}}(x) \cdot \sigma^! \alpha_E(y) \quad .$$

There is a more palatable form for this identity. We may consider $R(H)$ as an $R(G)$ module via $i^!$, and also consider $K\{\mathbb{P}(E)\}$ as an $R(G)$ module via $\sigma^! \circ \alpha_E$. With this agreed the premanence states simply that

$$\alpha_{\hat{E}} : R(H) \to K\{\mathbb{P}(E)\}$$

is an $R(G)$-homomorphism.

Proof: Using a somewhat sloppy notation the steps are as follows: Assume that V is an H-module and that W is a G module. Our problem is to identify the following two bundles over $\mathbb{P}(E)$:

$$A = \sigma^{-1}(E \times_G W) \otimes (E \times_H V), \quad B = E \times_H (V \otimes W) \quad .$$

Now $A = \{(\sigma^{-1}E) \times W\} \otimes (E \times_H V)$ by naturality. Hence by Taut. 2, $A = \{E \times_H G \times_G W\} \otimes (E \times_H V)$. But $E \times_H G \times_G W = E \times_H W$ whence $A = (E \times_H W) \otimes (E \times_H (V \otimes W) = B$. Q.E.D.

Remarks: When $X = p$ is a point, $\mathbb{P}(E)$ is just G/H over p . In this case the permanence is equivalent to the statement that if W is a G-module, then $G \times_H W \to G/H$ is the trivial bundle over G/H . In this case

$$\alpha_{\hat{E}} : R(H) \to K(G/H)$$

may be considered as a localized form of the induced representation $i_* : R(H) \to R(G)$ defined for finite groups. Indeed, in our terminology, $i_* U$, where U is an H-module can be defined as the G-module of *sections* of $G \times_H U \to G/H$. (When G is finite this space is finite-dimensional.) In this context $i_*(x \cdot i^! y) = i_*(x) \cdot y$ is still valid, however i_* is only an additive homomorphism.

The second identity involving $\alpha_{\hat{E}}$ describes the behavior of this homomorphism under the action of the normalizer of H in G . Thus let $N(H) = \{g \in G \mid gHg^{-1} \subset H\}$ and define $\underline{N}(H)$ as $N(H)/H$.

Each $n \in N(H)$ acts on H by sending $h \to n h n^{-1}$ and so induces an action of $N(H)$ on $R(H)$, which factors through $\underline{N}(H)$, because two modules which differ by an inner automorphism are isomorphic. In short $R(H)$ is canonically a $\underline{N}(H)$-module.

Next let E be a G-bundle. Then if $n \in N(H)$ the right translation of E by n, $e \to e \cdot n$ preserves the H cosets of E and hence induces a map of $\mathbb{P}(E) \to \mathbb{P}(E)$, which again only depends on the H coset of n in $N(H)$. Thus $\underline{N}(H)$ acts on $\mathbb{P}(E)$ and hence on $K\{\mathbb{P}(E)\}$. With this agreed we have the plausible:

EQUIVARIANCE. The induced representation

$$\alpha_E : R(H) \to K\{\mathbb{P}(E)\}$$

commutes with the action of $N(H)$ on these two rings.

Proof: Let V be an H-module, and let $n \in N(H)$. Now define V^n as the H-module with the same underlying vector-space but the new action $h * v = nhn^{-1} \cdot v$. This module then represents the action of n on $V \in R(H)$. Also let $f : E \to E$ be the right translation $e \to e \cdot n$. Then our problem is to construct an isomorphism of the bundles $E \times_H V^n$ and $f^{-1} \cdot (E \times_H V)$. In other words we have to find an isomorphism ψ, which makes the following sequence exact

$$E \times V^n \xrightarrow{\psi} \mathbb{P}(E) \times (E \times_H V) \rightrightarrows \mathbb{P}(E) .$$

Define $\tilde{\psi} : E \times V^n \to E \times (E \times V)$ by $\tilde{\psi}(e, v) = (e, e \cdot n \times v)$. Then $\tilde{\psi}$ is easily seen to induce the desired ψ. Q.E.D.

§ 12. The periodicity theorem for KO. We let KO^* denote the cohomological extension of the functor KO. Thus

$$KO^* = \sum_{i \leq 0} KO^i ,$$

with $KO^0 = KO$ and this functor shares all the general properties of KU.

The starting point of its more special properties in the following periodicity theorem:

PERIODICITY THEOREM II. The tensor product of bundles induces a bijection:

$$KO^*(X) \otimes KO(S^8) \xrightarrow{\cong} KO^*(X \times S^8) . \tag{12.1}$$

This is the Kunneth formulation. The corresponding relative theorem may be stated as follows:

Let $\eta_8 \in KO^{-8}(p)$ be a generator. Then multiplication with η induces an isomorphism of $KO^i(X)$ with $KO^{i-8}(X)$.

The ring $KO^*(p)$ is also known: It is generated by 1 and elements $\eta_i \in KO^{-i}(p)$, $i = 1, 4, 8$ which are subject to

$$2\eta_1 = 0 \ , \quad \eta_1^3 = 0 \ , \quad \eta_4^2 = 4\eta_8 \quad .$$

The pertinent references here are ([6], [7]).

One may compare KO and KU by means of the complexification of bundles : $\epsilon^* : KO(X) \to KU(X)$, and then disregarding of the complex structure: $\epsilon_* : KU(X) \to KO(X)$, and just as in Section 10 these two operations are related by: by:

$$\epsilon_* \circ \epsilon^* u = 2u$$
$$\epsilon^* \circ \epsilon_* u = u + u^*$$

just as in RO and RU .

Hence we see that $KO^*(X) \cong \{KU^*(X)\}^{\mathbb{Z}_2}$ mod 2 primary material, if the superscript $\mathbb{Z}_2$ denotes the fixed elements under the conjugation automorphism of $KU^*(X)$.

A slightly more detailed look at the periodicity theorem yields a more detailed relation between these two functors. Indeed if B_U and B_0 denote the classifying spaces of $\widetilde{KU}$ and $\widetilde{KO}$, the map ϵ^* is realized by a fibering

$$U/O \to B_O \to B_U$$

with $U/O = \text{limit } U_n/O_n$ as fiber. On the other hand the

periodicity theorem as stated in [6] asserts that $U/O \cong \Omega^{-1}B_O$. Hence the fibering above gives rise to an exact sequence:

$$(12.2) \quad \cdots \ \widetilde{KO}^{i-1}(X) \to KO^{i-1}(X) \xrightarrow{\epsilon^*} \widetilde{KU}^{i-1}(X) \to \widetilde{KO}^{i+1}(X) \to \cdots$$

from which one immediately concludes that

$$(12.3) \qquad KO(S_{8n}) \overset{\epsilon^*}{\cong} KU(S_{8n}) \ .$$

For our purposes we will require the following description of the generators of $KU(S_{8n})$ and $KO(S_{8n})$.

THEOREM III. *Let* $H_n = \text{Spin}\,(2n)$, $G_n = \text{Spin}(2n+1)$ *so that* $G_n/H_n = S_{2n}$. *Let* $\Delta_n^+ \in RU(H_n)$ *be one of the* Spin *representations and let* $y_n = \alpha_{\hat{E}}(\Delta_n^+)$ *be the induced element in* $KU(S_{2n})$. *Then* 1 *and* y_n *form a base for* $KU(S_{2n})$.

Proof: Let n and m be fixed and set $G = \text{Spin}\,(2\{m + n + 1\})$. Also let $W_{m+n} = G/H_{(m+n)}$. We may arrange the various inclusions involved here so that the following diagram is commutative:

$$\begin{tikzcd}
H_m \times H_n \arrow[r, "①"] \arrow[d, "④"'] & G_m \times G_n \arrow[d, "②"] \\
H_{(m+n)} \arrow[r, "③"'] & G
\end{tikzcd}$$

Thus there is an induced map

$$f : S_{2m} \times S_{2n} \to W_{m+n}$$

Now W_{m+n} is fibered by $S_{2(m+n)}$-spheres over $S_{2(m+n)+1}$, and $G_{m+n}/H_{m+n} \xrightarrow{i} W_{m+n}$ represents the fiber. It follows that there exists a map $g : S_{2m} \times S_{2n} \to G_{m+n}/H_{m+n}$ which makes the following diagram homotopy commutative:

$$\begin{tikzcd}
S_{2m} \times S_{2n} \arrow[r, "g"] \arrow[dr, "f"'] & G_{m+n}/H_{m+n} = S_{2(m+n)} \arrow[d, "i"] \\
 & W_{m+n}
\end{tikzcd}$$

Furthermore it is not difficult to see that g has degree 2.

Next, let $Y_{m+n} \in KU(W_{m+n})$ be the bundle induced by $\Delta^+_{m+n} \in RU(H_{m+n})$. Then clearly $i^! Y_{m+n} = y_{m+n}$ as defined in the theorem.

We first propose to compute $f^! Y_{m+n}$. By the naturality of the inducing procedure this amounts to understanding

$$f^! : RU(H_{m+n}) \to RU(H_m \times H_n) \cong RU(H_m) \otimes RU(H_n) .$$

Now, from our discussion in Section 10 it is apparent that

$$f^!(\Delta^+_{m+n} - \Delta^-_{m+n}) = (\Delta^+_m - \Delta^-_n) \otimes (\Delta^+_n - \Delta^-_n) .$$

Hence if ξ_m is the bundle induced by $(\Delta^+_m - \Delta^-_m)$ over S S_{2m}, and we set $\hat{\xi}_{m+n}$ equal to the bundle induced over W_{m+n} by $\Delta^+_{m+n} - \Delta^-_{m+n}$, we obtain

$$f^! \hat{\xi}_{m+n} = \xi_m \otimes \xi_n ,$$

whence

$$g^! \xi_{m+n} = \xi_m \otimes \xi_n ,$$

because $i^! \hat{\xi}_{m+n} = \xi_{m+n}$. On the other hand using the permanence law and the fact that $\Delta^+_m + \Delta^-_m$ is in the image of $KU(G_m)$ we have:

$$\xi_m = 2(y_m - \dim y_m) .$$

Hence if we assume our theorem for m and n, ξ_m and ξ_n are twice the generators of $\widetilde{KU}(S_{2n})$ and $\widetilde{KU}(S_{2m})$ respectively.

Now the formula $g^!\xi_{m+n} = \xi_m \otimes \xi_n$ proves the same assertion for ξ_{m+n} because of the periodicity theorem for $\widetilde{KU}$ and the fact that g has degree 2 . Q.E.D.

Remark. If one is familiar with theory of characteristic classes it is not difficult to compute the character of y_n directly and so prove Theorem 3. See [11] .

COROLLARY 1. $KO(S_{8n})$ *is generated by* 1, *and the bundle induced by the real spin representation* $\Delta^+ \in RO\{Spin(8n)\}$.

Proof: Clear in view of 12.3, Theorem III and 10.6.

COROLLARY 2. *If* y *denotes the bundle induced by* Δ^+ *in* $KO(S_{8n})$, *then the* 8n'th *component of* ch y *generates* $H^{8n}(S_{8n})$.

Proof: By Corollary 1 of Theorem 6.1, the character of a generator of $\widetilde{KU}(S_{2n})$ always generate $H^{2n}(S_{2n})$. Hence Corollary 1 and (12.3) prove the assertion.

§13. *Sphere-bundles.* Consider the following situation:

$$G = Spin(8n+1)$$

$$H = Spin(8n)$$

$$E = \text{a principal } G\text{-bundle over } X .$$

In this case $\mathbb{P}(E)$ is therefore a sphere bundle over X . Precisely: Let $\rho \in RO\{Spin(8n+1)\}$ be the standard representation. Then $\alpha_E(\rho)$ is a vector bundle, V, over X, and its unit sphere-bundle may be identified with $\mathbb{P}(E)$:

$$\mathbb{P}(E) \cong \mathbb{S}(V) .$$

By our general remarks, there is an H-bundle $\hat{E}$ defined over $\mathbb{P}(E)$. We let $y \in KO\{\mathbb{P}(E)\}$ be the induced bundle:

$$y = \alpha_{\hat{E}}(\Delta^+)$$

where Δ^+ is one of the real Spin representations in $\mathbb{R}O(H)$. We now have the following extension of the periodicity theorem:

THEOREM A. *In the situation envisaged above,* $KO^*\{\mathbb{S}(V)\}$ *is a free module over* $KO^*(X)$ *with generators* 1 *and* y .

Proof: When X = point, this theorem reduces to Corollary 1 of Theorem III . Hence by the Kunneth formula (12.1), the theorem is true when E is a trivial G-bundle. But the Meyer Vietoris argument, together with the cohomological property of KO^* proves the general case.

COROLLARY 1. *There exist unique elements in* KO(X) *which make the following formulae valid in* KO{S(V)}:

$$y^2 = A(E) \cdot y + B(E)$$

$$(13.1) \qquad \psi_k y = \theta_k(E) \cdot y + \Gamma_k(E) \; .$$

This is clear. One thus has four invariants of E in KO(X).

COROLLARY 2. *Suppose that* E *and* E' *are two* Spin(8n + 1) *bundles over* X. *Then* $\mathbb{P}(E)$ *and* $\mathbb{P}(E')$ *are of the same fiber-homotopy type only if:*

$$(13.2) \qquad \theta_k(E) = \theta_k(E') \cdot \psi_k u/u \qquad u \in KO(X), \ \dim u = 1,$$

Proof: Let $f : \mathbb{P}(E) \to \mathbb{P}(E')$ be a fiber homotopy equivalence. Then $f^{!} : KO^*(\mathbb{P}(E')) \to KO^*\{\mathbb{P}(E)\}$ is a $KO^*(X$ isomorphism. Hence $f^{!}y' = ay + b$, with $\dim a = 1$. Thus $\psi_k f^{!} y' = \psi_k a \cdot \psi_k y + \psi_k b = (\psi_k a)\,\theta_k(E)y + \psi_k b + \Gamma_k(E)$. On the other hand $f^{!}\psi_k y' = f^{!}\{\theta_k(E')y' + \Gamma_k(E')\} = a\;\theta_k(E')y + \theta_k(E')a + \Gamma_k(E)$. Q.E.D.

COROLLARY 3. *The invariants* $\theta_k(E)$ *have the property:*

$$(13.3) \qquad \{\psi_k\theta_s(E)\}\theta_k(E) = \theta_{sk}(E) \quad .$$

The proof is clear. We note that we have here the 2nd part of the cocycle condition of Section 8. The first part still has no analogue, as we do not know how to "compute" the invariants $\theta_s(E)$. The following theorem solves this problem:

THEOREM B. *Consider the elements* A, B, θ_k, Γ_k *in* RO{Spin(8n + 1)} *defined in Proposition* 10.5. *Then the invariants of* (13.1) *are given by:*

$$A(E) = \alpha_E(A) \, , \qquad B(E) = \alpha_E(B)$$
$$\theta_k(E) = \alpha_E(\theta_k) \, , \qquad \Gamma_k(E) = \alpha_E(B) \; .$$

Proof: This is a clear consequence of the permanence law. For instance:

$$y^2 = \alpha_{\hat{E}}(\Delta^+)^2 = \alpha_{\hat{E}}(\Delta^+ \cdot i^{!}A + i^{!}B)$$
$$= \alpha_E(A) \cdot y + \alpha_E(B) \; . \qquad \text{Q.E.D.}$$

COROLLARY 4. $\dim \theta_k(E) = k^{4n}$.

Proof: $\underline{ch}\,\theta_k = \Pi_1^{4n}(y^{(k-1)/2} + \dots + y_1^{-(k-1)/2}) \quad k \geq 2$ whence $\dim \underline{ch}\,\theta_k = k^{4n}$. Q.E.D.

COROLLARY 5. S(V) *has the same fiber homotopy type as the trivial sphere-bundle only if*

$$\theta_k(E) = k^{4n}\psi_k u/u \qquad \dim u = 1,\ u \in KO(X)\ .$$

Here we now have a complete analogue of the formula (8.1), developed for the KU-theory. There we obtained this criterion for the coreducibility of a Thom-complex, here it arises from the J-triviality of a sphere-bundle. However, these are closely related:

If E is a real vector bundle, then X^E is coreducibl $\Leftrightarrow$ $S(E+1)$ has trivial fiber homotopy type.

We may now precisely mimic the construction of (8.2), and so define the group, $H^1(\mathbb{Z}^+; KO(X))$.

Further the function $k \to \theta_k(E)$ defines a cocycle and hence a class $\theta(E) \in H^1(\mathbb{Z}^+; KO(X))$. Hence Corollary 1 implies that:

PROPOSITION 13.1. The element $\theta(E) \in H^1(\mathbb{Z}^+; KO(X$ is an invariant of the stable fiber homotopy type of $\mathbb{P}(E) = S(V)$.

Note: Our θ in the complex case was defined directly on the vector bundle. The construction of the present θ depends on the principal G-bundle E and not only on its associated vector-bundle V . Thus if we start with a real $(8n+1)$ dimensional bundle V, over X, the

θ invariant can only be defined for it if V is of the form $\rho(E)$ for some principal $\mathrm{Spin}(8n+1)$ bundle. On the other hand if $\rho(E_1) \cong \rho(E_2)$ as vector bundles, then $\mathbb{P}(E_1) \cong \mathbb{P}(E_2)$ whence $\theta(E_1) = \theta(E_2)$. Thus θ does depend only on V , provided V is of the form $\rho(E)$. Vector bundles of this type are said to have a Spin reduction, and V has a spin-reduction if and only if $w_1(V), w_2(V) = 0$ as is well-known.

In short, $\theta(V)$ may be thought of as the second obstruction to trivialization of the fiber-homotopy type of $S(V)$, $w_1(V) + w_2(V)$ denote the first two Whitney classes of V .

If we let $K\,\mathrm{Spin}(X)$ = subgroup of $KO(X)$ on which w_1 and $w_2 = 0$, then it is easily seen that θ extends to a homomorphism

$$\theta : K\,\mathrm{Spin}(X) \to H^1(X^+, KO(X))\ .$$

We return now to the computation of the $\theta_k(E)$.

PROPOSITION 13.2. Let $A(E), \cdots, \Gamma_k(E)$ be the 4 invariants of E described by (13.1). Also let $V = \rho(E)$. Then in $KO(X)$ these invariants are given by universal polynomials in the $\lambda^i V$, and an auxiliary element, $\Delta(V)$, where $\Delta(V)$ satisfies the equation:

(13.4) $$2\Delta(V)^2 = \lambda_1(V) \quad .$$

Proof: We set $\Delta(V) = \alpha_E(\Delta)$ where Δ is the spin-representation in $RO\{Spin(8n+1)\}$. Then, as we know that $RO\{Spin(8n+1)\} = \mathbb{Z}[\lambda^i\rho;\Delta]$, $i \leq 4n$ it follows that the elements A, B, θ_k, Γ_k of this ring can be expressed as polynomials in the $\lambda^i\rho$ and Δ. Applying α_E we obtain the first part of the proposition.

To obtain the identity (13.4) recall that

$$\underline{ch}\,\Delta = \prod_1^{4n} (y_i^{1/2} + y_i^{-1/2})$$

whence

$$(\underline{ch}\,\Delta)^2 = \prod_1^{4n} (y_i + 2 + y_i^{-1})$$

$$= \prod_1^{4n} (1 + y_i)(1 + y_i^{-1})$$

$$= \underline{ch}\,\lambda_1(\rho - 1) = \left(\frac{1}{2}\right)\lambda_1\rho \quad \text{Q.E.D.}$$

We give now some explicit examples:

PROPOSITION 13.3.

$$A(E) = \theta_2(E) = \Delta(V)$$

$$B(E) = \sum_{i=1}^{2n} \lambda^{2i-1}(V - 1)$$

while in general $\theta_k(E)$ may be computed by the following algorithm:

Let $L = \mathbb{Z}[z_i; z_i^{-1}]$, $i = 1, \cdots, 4n$ be the ring of finite Laurent series. Define elements γ^i, ω, η_k in L by:

$$\sum_0^{\infty} \gamma^i t^i = (1+t) \prod_1^{4n} (1 + tz_i^2)(1 + tz_i^{-2})$$

$$\omega = \prod_1^{4n} (z_i + z_i^{-1})$$

$$\eta_k = \Pi\{z_i^{(k-1)} + \cdots + z_i^{-(k-1)}\} \quad .$$

Write $\eta_k = P_k(\gamma^i, \omega)$ where P_k is a polynomial. Then $\theta_k(E) = P_k(\lambda^i V; \Delta(V))$.

Proof: This should be clear in view of our results on $KO\{Spin(8n+1)\}$. We have really just disguised the isomorphism $\underline{ch}$, and replaced y_i by z_i^2 to make the computations directly in L.

This algorithm is clearly quite difficult to carry out in general. However if additional information about V is at hand the computations are much easier. For us the following example is of special importance.

PROPOSITION 13.4. Let $V = 8nL + 1$ where L is a line-bundle. Then $w_1(V) = w_2(V) = 0$ and we have:

$$(13.5)\qquad \theta_k(V) = \begin{cases} k^{4n} + \frac{k^{4n}}{2}(L - 1) & k \text{ even} \\ k^{4n} + \left\{\frac{k^{4n} - 1}{2}\right\}(L - 1) & k \text{ odd} . \end{cases}$$

Proof: Let ξ be the principal $\mathbb{Z}_2$-bundle of L, and let η be the one-dimensional representation in $RO(\mathbb{Z}_2)$, so that $L = \alpha_\xi(\eta)$. So then $V = \alpha_\xi\{(8n + 1)\eta\}$. Put differently, let $\mathbb{Z}_2 \to SO(8n)$ be defined by sending the generator of $\mathbb{Z}_2$ into minus the identity, and let $f : \mathbb{Z}_2 \to SO(8n + 1)$ be this homomorphism followed by the inclusion. Let $f_*\xi$ be the extension of ξ to $SO(8n + 1)$. Then $V = \alpha_{f_*\xi}(\rho)$ where ρ is the standard representation of $SO(8n + 1)$. Now, because we are in $\dim(8n + 1)$, f can be lifted to $\mathrm{Spin}(8n + 1)$:

$$\begin{tikzcd} & \mathrm{Spin}(8n + 1) \arrow[d] \\ \mathbb{Z}_2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & SO(8n + 1) \end{tikzcd}$$

and our problem is to compute $\tilde{f}^! : RO\,\mathrm{Spin}(8n + 1) \to RO(\mathbb{Z}_2)$. Indeed we have: $\theta_k(E) = \alpha_{\tilde{f}_*\xi}(\theta_k) = \alpha_\xi(\tilde{f}^!\theta_k)$. On the other hand one sees quite easily that, in terms of the notation introduced in Section 9, $\tilde{f}^! y = \tilde{f}^! y^{-1} = \epsilon^*\eta$ while

$\tilde{f}^!\{(y_1 \cdots y_{4n})^{1/2}\} = \epsilon^* 1$. Hence, under $\tilde{f}^!$, the element $\theta_k = \Pi_1^{4n}(y_i^{(k-1)/2} + \ldots + y_i^{-\{(k-1)/2\}}$ goes over into $(1 + \eta + \cdots \eta^{k-1})^{4n}$. Thus

$$\tilde{f}^!\theta_k = \begin{cases} (s + s\eta)^{4n} & k = 2s \\ (\overline{s + 1} + s\eta)^{4n} & k = 2s + 1 . \end{cases}$$

Let $\sigma = \eta - 1$. Then $\sigma^2 = -\sigma$. Hence the identity

$$(A + B\sigma)^m = A^m + \left\{\frac{A^m - (A - 2B)^m}{2}\right\}\sigma$$

holds. It follows that

$$\tilde{f}^!\theta_k = \begin{cases} (2s)^{4n} + \frac{(2s)^{4n}}{2}\cdot\sigma & k = 2s \\ (2s + 1)^{4n} + \left\{\frac{(2s + 1)^{4n} - 1}{2}\right\}\sigma & k = 2s + 1 . \end{cases}$$

Now applying α_ξ we obtain 13.5.

Exercises. 1. Let $\theta_t^c(V)$, where V is a complex bundle, denote the θ_t of Section 7. Thus θ_t^c is characterized by: $\theta_t^c(L) = 1 + L + \cdots + L^{t-1}$ for line bundles and $\theta_t^c(V + V') = \theta_t^c(V) \cdot \theta_t^c(V')$.

Now suppose that $\dim V = 4n$, and $\lambda^{4n}V = 1$. Then the real bundle $\epsilon_* V$ will have a $\mathrm{Spin}(8n)$ reduction, so that $\theta_t(\epsilon_* V)$ is well-defined. Prove the formula:

$$\epsilon^*\theta_t(\epsilon_* V) = \theta_t^c(V) .$$

2. Using the invariant θ, of the KO-theory and in particular formula (13.4) refine our earlier estimates on $J : \widetilde{KO}(S_{4n}) \to J(S_{4n})$ by a factor of 2 .

3. Prove the analogue of Theorem A, B etc. when E is a $\mathrm{Spin}^c(2n+1)$ bundle, $H = \mathrm{Spin}^c(2n)$, and KO is replaced by KU .

§14. *The Thom isomorphism*. We adhere to the notation of the last section but assume that in addition $E = i_* E'$ where E' is a principal Spin(8n)-bundle --- that is to say E' is an H-reduction of E . The corresponding section of $\mathbb{P}(E)$ is denoted by S . We thus have the split exact sequence of spaces:

$$0 \longrightarrow X \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{P}(E) \xrightarrow{j} \mathbb{P}(E)/s(X) \longrightarrow 0 \quad . \tag{14.1}$$

In terms of the associated vector-bundles over X one has: $W = \rho_{8n}(E') = \alpha_E(\rho_{8n})$, $V = \rho_{8n+1}(E)$ so that $V = W + 1$, and hence (14.1) goes over into

$$0 \longrightarrow X \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{B}(W+1) \xrightarrow{j} X^W \longrightarrow 0 \, . \tag{14.2}$$

i.e., $\mathbb{P}(E)/sX$ may be identified with X^W .

Because (14.2) splits $\widetilde{KO}^*(X^W)$ may be identified with its image under $j^!$ and hence with the kernel of $s^!$ in the KO(X)-module $KO^*(\mathbb{P}(E))$. With this understood, let $z \in \widetilde{KO}(X^W)$ be the element $y - s^! y$ where y is the bundle of the previous section. Then we have:

THEOREM C' . *$\widetilde{KO}(X^W)$ is freely generated by* z *over* $KO^*(X)$. *Further,*

$$z^2 = \{\Delta^-(E') - \Delta^+(E')\} \cdot z$$

and

$$\psi_k z = \theta_k(E) \cdot z$$

where $\theta_k \in RO\{\mathrm{Spin}(8n+1)\}$ *is given by Theorem B* .

The proof is trivial, one just computes in $KO^*(\mathbb{P}(E))$ whose ring and ψ_k-structure are given by Theorems A and B.

Let $i : X \to X^W$ be the imbedding given by $\overline{s}$, the antipodal section s, followed by j . We associate the additive homomorphism $x \to -z \cdot x$, $x \in KO(X)$ with i and denote it by $i_!$. With this terminology Theorem C' may be stated as follows:

THEOREM C'' . *Let* W *be a* 8n-*dimensional vector-bundle which admits a reduction to* Spin(8n) . Then *the homomorphism*

$$i_! : KO^*(X) \to \widetilde{KO}^*(X^W)$$

is a bijection, and satisfies the formulae:

$$\begin{aligned} (i_! u)(i_! v) &= i_! \Delta_{-1}(W) \cdot u \cdot v \\ \psi_k i_! u &= i_! \theta_k(W) \cdot \psi_k u \\ i^! \, i_! u &= \Delta_{-1}(W) \cdot u \quad . \end{aligned}$$

(Here we have abbreviated $\Delta^+(F) - \Delta^{-1}(F)$ to $\Delta_{-1}(W)$, and $\theta_k(E)$ to $\theta_k(W)$, where F is the principal Spin(8n) bundle associated to W and E is its Spin(8n+1)-extension. Only the last statement needs verification. For this purpose consider the action of $N(H)/H$ (see Section 11) in our case. This group is $\mathbb{Z}_2$ and acts on $RO(H)$ by exchanging Δ^+ and Δ^- and it acts on $\mathbb{S}(V)$ as the antipodal map. Let us write $a: S(V) \to S(V)$ for this map. Clearly $a^!$ is a $KO^*(X)$ automorphism of $KO^*\{\mathbb{S}(V)\}$. Hence by the equivariance property (see Section 11) we have:

$$a^! y = a^! \alpha_{\hat{E}}(\Delta^+) = \alpha_{\hat{E}}(\Delta^-) \quad .$$

On the other hand by the permanence law,

$$\alpha_{\hat{E}}(\Delta^-) = - \alpha_{\hat{E}}(\Delta^+) + \Delta(E) \quad .$$

Thus $a^! y = -y + \Delta(E)$. Hence $\overline{s}^!(y - s^! y) = s^! a^! (y - s^! y) = s^!(-y + \Delta(E) - s^! y) = \Delta^-(F) - \Delta^+(F)$. This formula now yields the relation in question directly.

Exercise. Follow-up Exercise 3 of Section 13 in the present context.

§15. The Gysin sequence.

We now assume that W is an n-dimensional vector-bundle over X, and let $\mathbb{S}(W)$ denote the associated sphere-bundle.

THEOREM 15.1. *If* W *admits a reduction to* Spin(m), *then the following Gysin sequence is valid:*

$$\begin{aligned} &\longleftarrow KO^{p-m+1}(X) \longleftarrow KO^p\{\mathbb{S}(W)\} \xleftarrow{\pi^*} \\ &KO^p(X) \xleftarrow{\Phi} KO^{p-m}(X) \longleftarrow \ , \qquad p \in \mathbb{Z} \end{aligned}$$

where now KO^p *is defined for all integers by the periodicity*: $KO^{p-8} \approx KO^p$.

Proof: Let $\mathbb{D}(W)$ denote the unit disc-bundle of W as in Section 1. Then as we saw there, one has the exact sequence of spaces:

$$\mathbb{S}(W) \longrightarrow \mathbb{D}(W) \xrightarrow{f} X^W$$

which gives rise to the exact sequence:

$$\widetilde{KO}^{p+1}(X^W) \leftarrow KO^p\{S(W)\} \leftarrow KO^p(X) \xleftarrow{f^!} \widetilde{KO}^p(X^W) \leftarrow \quad .$$

We will therefore be done once $\widetilde{KO}^p(X^W)$ is identified with $KO^{p-m}(X)$.

Choose an integer $k \geq 0$, so that $m + k = 8n$. Then $W + k \cdot 1$ is an 8n-dimensional bundle which admits a reduction to Spin(8n). Hence the Thom isomorphism:

$$KO^p(X) \xrightarrow{\cong} \widetilde{KO}^p(X^{(W+k\cdot 1)})$$

is well defined. On the other hand

$$X^{(W+k\cdot 1)} = \Sigma^k X^W$$

whence

$$\widetilde{KO}^p(X^{(W+k\cdot 1)}) \approx \widetilde{KO}^{p-k}(X^W) \quad .$$

Composing these two isomorphisms one obtains the isomorphism:

$$\widetilde{KO}^p(X^W) \to KO^{p+k}(X) \quad ,$$

which goes over into

$$\widetilde{KO}^p(X^W) \approx KO^{p-m}(X)$$

by applying the periodicity law n-times.

Note that when $\dim W = 8n$, we have already determined the homomorphism Φ: $\Phi : KO^p(X) \to KO^{p+m}(X)$

is multiplication by $\Delta_{-1}(W) = \Delta_+(W) - \Delta_-(W)$, as follows from Theorem C". It seems a reasonable conjecture that Φ is always given by multiplication with $\Phi(1) \in KO^m(X)$.

§16. The rational J-invariant derived from $\theta(V)$.

In Section 13 we defined the cocycle $k \to \theta_k(V)$ for an (8n + 1) dimensional bundle with a Spin-reduction, and showed that the J-type of V was trivial only if there exists a $u \in KO(X)$, $\dim u = 1$ such that:

$$(16.1) \qquad \theta_k(V) = k^{4n}\psi_k u/u \qquad \text{for all } k \in \mathbb{Z}^+ \quad .$$

PROPOSITION 16.1. The equation (16.1) can always be solved for u in $KO(X) \otimes \mathbb{Q}$.

In $KO(X)$ (16.1) can of course have no solution as examples show. This proposition depends vitally upon the nilpotence of $\widetilde{KO}(X)$ i.e., upon the finiteness of X . To see the implications of this assumption consider the general situation of Section 11. Thus $E \to X$ is a G-bundle and $\alpha_E : R(G) \to K(X)$ the corresponding homomorphism. Also let $I \subset R(G)$ be the ideal of elements of dimension 0 . Then $\alpha_E(I) \subset KO(X)$. Hence under our finiteness assumption α_E annihilates a high enough power of I . It follows that α_E extends uniquely to the I-adic completion $\hat{R}(G)$ of $R(G)$. In other words, if Σa_i is an infinite series of elements in R(G) with

$$a_i \in I^{n_i}, \quad \lim_{i\to\infty} n_i = \infty$$

then $\alpha_E(\Sigma a_i)$ is a well defined element in K(X).

Consider now the cocycle $k \to \theta_k$ where $\theta_k \in RO\{Spin(8n+1)\}$ are the elements defined by 10.13, i.e., by:

$$\underline{ch}\,\theta_k = \prod_1^{4n} \left\{ y_i^{(k-1)/2} + \cdots + y_i^{(1-k)/2} \right\}.$$

We will construct an element $\Omega \in R\hat{O}\{Spin(8n+1)\} \otimes \mathbb{Q}$ with the property that:

(16.2) $$\dim \Omega = 1, \quad \theta_k = k^{4n}\psi_k\Omega/\Omega, \qquad \text{all } k \in \mathbb{Z}^+ .$$

If such an element can be found, Proposition 16.1 will clearly have been proved, one simply sets $u = \alpha_E(\Omega)$, where E is the principal Spin(8n + 1) bundle of V.

To describe elements in $R\hat{O}$ of G = Spin(8n + 1), we start with the imbedding

$$RO\{SO(8n+1)\} \xrightarrow{\underline{ch}} \mathbb{Z}[y_i, y_i^{-1}] \quad i = 1, \cdots, 4n$$

described in Section 10. For convenience we abbreviate the LHS to RO and the R.H.S. to L. In L the ideal which corresponds to I(T) → RO(T) is generated by the element $(x_i - 1)$ and $(x_i^{-1} - 1)$. We set

$$\eta_i = 1 - x_i .$$

Now $\underline{ch}$ extends to a homomorphism

$$R\hat{O} \to \hat{L}$$

which identifies $R\hat{O}$ with the formal power series in the η_i which are invariant under permutations and the operations $1 - \eta_i \to \frac{1}{1-\eta_i}$ (corresponding to $x_i \to x_i^{-1}$). Hence the element Ω determined by

(16.3) $$\underline{ch}\,\Omega = \prod_1^{4n} \left\{ \frac{\eta_i}{\sqrt{1-\eta_i}} \log(1-\eta_i) \right\}$$

is a well determined element of $R\hat{O}$.

We have $\psi_k \cdot y_i = y_i^k$ whence $\psi_k \eta_i = 1 - (1-\eta_i)^k$. Therefore

$$\underline{ch}\,\psi_k\Omega = \prod_1^{4n} \left\{ \frac{1-(1-\eta_i)^k}{(1-\eta_i)^{k/2}} \cdot k \cdot \log(1-\eta_i) \right\},$$

and

$$\underline{ch}\,\psi_k\Omega/\Omega = k^{4n} \prod_1^{4n} \frac{(1-\eta_i)^{-k/2} - (1-\eta_i)^{k/2}}{(1-\eta_i)^{-1/2} - (1-\eta_i)^{1/2}} = ch^{4n}\theta_k .$$

Q.E.D.

Before completing our discussion of the element Ω we bring another application of the fact that α_E extends to $R\hat{O}(G)$.

THEOREM 16.1. *Let* $M_p \subset KO(X) \otimes \mathbb{Q}$ *be the subspace on which* ψ_k *acts by multiplication with* k^p. *Then,*

$$KO(X) \otimes \mathbb{Q} = \sum_{p=0}^{\infty} M_p$$

is a direct sum decomposition.

Proof: It will be sufficient to decompose every bundle W into its components in M_{4p}. Let then W be given, and let E be the principal $SO(2n)$ bundle associated to $2W$. (Note that $2W$ always has a reduction to SO.) Thus $2W = \rho(E) = \alpha_E(\rho)$ where $\rho \in RO\{SO(2n)\}$ is the standard representation.

Now in $\hat{RO}\{SO(2n)\} \otimes \mathbb{Q}$ we have, in our earlier notation, the following obvious identity:

$$\underline{ch}\,\rho = \sum_{1}^{n} \left\{ e^{\log(1-\eta_i)} + e^{-\log(1-\eta_i)} \right\} .$$

Hence if we define $\rho_p \in \hat{RO}\{SO(2n)\} \otimes \mathbb{Q}$ by

$$\underline{ch}\,\rho_p = \frac{1}{p!} \sum_{1}^{n} \left[\left\{ \log(1-\eta_i) \right\}^p + \left\{ \log \frac{1}{1-\eta_i} \right\}^p \right] .$$

Then

$$\rho = \sum_{p=0}^{\infty} \rho_p \quad \text{and} \quad \psi_k \rho_p = k^p \cdot \rho_p .$$

Hence in $KO(X) \otimes \mathbb{Q}$ we have

$$W = \frac{1}{2} \sum_{p=0}^{\infty} \alpha_E(\rho_p)$$

giving the desired decomposition of W. Of course we see also that $M_p = 0$ if p is odd.

To continue with our class Ω. Note first that an element Ω may be defined in each of the rings $\hat{RO}\{SO(2n)\}$ by the formula:

$$\underline{ch}\,\Omega = \prod_{1}^{4n} \frac{\eta_i}{\sqrt{1-\eta_i}} \log(1-\eta_i) \qquad i = 1, \cdots, n .$$

Hence for any $SO(2n)$-bundle E we obtain a well determined element $\Omega(E) \in 1 + \tilde{KO}(X) \otimes \mathbb{Q}$. Further it is clear that

$$\Omega(E + E') = \Omega(E) \cdot \Omega(E') .$$

Hence Ω extends to a homomorphism

$$\Omega : KO(X) \to 1 + \tilde{KO}(X) \otimes \mathbb{Q} .$$

(Note. If W is an $SO(n)$ bundle, define $\Omega(W)$ as $\sqrt{\Omega(2W)}$.)

THEOREM 16.2. *Let* W *and* W' *be two vector-bundles over* X. *Then* W *and* W' *are stably* J-*equivalent only if*

$$\Omega(W) = \Omega(W')\cdot U, \quad U \in KO(X), \quad \dim U = 1 .$$

Thus $\Omega(W) \in 1 + \widetilde{KO}(X) \otimes \mathbb{Q}/1 + \widetilde{KO}(X)$ *is a stable J-invariant of* W .

Proof: Assume first that $\dim W = \dim W' = (8n+1)$ and that they admit spin-reductions. Then W and W' are of the same stable J-type only if there exists a $U \in 1 + \widetilde{KO}(X)$ so that

$$\frac{1}{k^{4n}}\,\theta_k(W) = \frac{1}{k^{4n}}\,\theta_k(W')\cdot\{\psi_k U/U\} \text{ in } KO(X)\otimes\mathbb{Q} .$$

This implies $\psi_k\,\Omega(W)/\Omega(W) = \psi_k\{\Omega(W')\cdot U\}/\Omega(W')\cdot U$ and hence by Theorem 16.1, that $\Omega(W) = \Omega(W')\cdot U$.

This settles this special case. In general, suppose W and W' are J-equivalent without necessarily having a Spin-reduction. Choose $W^{\perp}$ so that $W + W^{\perp}$ is a trivial bundle of dimension $(8n+1)$. Then $W' + W^{\perp}$ will be J-equivalent to the trivial bundle and hence have a Spin-reduction. So then $\Omega(W')\cdot\Omega(W^{\perp}) \in 1 + \widetilde{KO}(X)$ which implies $\Omega(W) \equiv \Omega(W') \bmod 1 + \widetilde{KO}(X)$. Q.E.D.

§17. *The* $\hat{\mathfrak{A}}$ *class.* In the last sections we have found the analogues in the KO-theory of the θ which we had constructed in the complex case by elementary

considerations. It is now natural to try and find an analogue for the Todd class which was encountered there. The purpose of this section is to discuss this question.

We continue to use the notation of Section 13. We also recall that *ch* : $KO(X) \to H^*(X ; \mathbb{Q})$ is defined as the composition $KO(X) \xrightarrow{\epsilon^*} KU(X) \xrightarrow{ch} H^*(X; \mathbb{Q})$, and

$$ch\ O(X) \subset H^*(X; \mathbb{Q})$$

as the image of this homomorphism.

THEOREM A'. *Consider the sphere bundle* $\mathbb{S}(V) \to X$ *of Section* 13, *and let* $Y = ch_{8n}(y)$ *be the* 8n-th *component of the character of* y . *Then* $H^*(\mathbb{S}(V); \mathbb{Q})$ *is a free module over* $H^*(X; \mathbb{Q})$ *with* 1 *and* Y *as generators.*

Proof: When X is a point, Corollary 2 of Theorem 3, Section 10 proves this assertion. Hence it is true always by the usual Meyer-Vietoris argument.

COROLLARY 1. *There exist elements unique in* $H^*(X; \mathbb{Q})$ *which make the following equations valid in* $H^*(\mathbb{S}(V); \mathbb{Q})$:

$$Y^2 = \alpha(E)Y + \beta(E)$$

$$ch\ y = \mathfrak{A}(E)Y + \mathfrak{B}(E) .$$

COROLLARY 2. *Let* E *and* E' *be two* Spin (8n + 1)-*bundles over* X . *Then their associated sphere-bundles* $\mathbb{P}(E)$ *and* $\mathbb{P}(E')$ *are of the same fiber homotopy type only if*

$$\mathfrak{A}(E) \cdot \{\mathfrak{A}(E')\}^{-1} \in \mathrm{ch}\, O(X) .$$

Proof: Assume $f : \mathbb{P}(E) \to \mathbb{P}(E')$ is a fiber-homotopy equivalence. Then $f^! y' = ay + b$ where a, $b \in KO(X)$, dim a = 1, by Theorem A .

Hence $\mathrm{ch}\, f^! y' = \mathrm{ch}(a)\, \mathfrak{A}(E) Y + K_1$, $K_1 \in H^*(X; \mathbb{Q})$. On the other hand $f^* \mathrm{ch}\, y' = \mathfrak{A}(E') \cdot f^* Y' + K_2$, $K_2 \in H^*(X;\mathbb{Q})$. Now when E is a point it follows from Corollary 2 of Theorem III , Section 12, that $\mathfrak{A}(E) = 1$. Hence the constant term of $\mathfrak{A}(E) = 1$. In other words:

$$\mathfrak{A}(E) = 1 + \tilde{\mathfrak{A}}(E) \quad \tilde{\mathfrak{A}}(E) \in \tilde{H}^*(X; \mathbb{Q}) .$$

Also, because dim a = 1, we have:

$$\mathrm{ch}\, a = 1 + \tilde{\mathrm{ch}}\, a \quad \tilde{\mathrm{ch}}\, a \in \tilde{H}^*(X;\mathbb{Q}) .$$

Hence

$$f^* Y' = \mathrm{ch}_{8n} f^! y' = Y + K_3 , \quad K_3 \in H^*(X; \mathbb{Q}) .$$

Now if we compare coefficients of Y , we obtain

$$\mathrm{ch}\,(a) \cdot \mathfrak{A}(E) = \mathfrak{A}(E') . \qquad \text{Q.E.D.}$$

Thus the invariant corresponding to θ in $H^*(X;\mathbb{Q})$ is the element $\mathfrak{A}(E) \in H^*(X; \mathbb{Q})/\mathrm{ch}\, O(X)$. In view of the results of the preceding section it is not surprising that $\mathfrak{A}(E)$ should be related to the invariant Ω of the preceding section:

THEOREM. *Let* $V = \rho_n(E)$ *be the vector bundle associated to* E *by the regular representation. Then*

$$\mathrm{ch}\, \Omega(V) = \mathfrak{A}(E) .$$

Proof: We will first show that the coboundary of $\mathfrak{A}(E)$ is the cocycle:

$$k \to \mathrm{ch}\, \frac{\theta_k(E)}{k^{4n}} .$$

Precisely let ψ_k operate on $H^{2n}(X;\mathbb{Q})$ by multiplication by k^n . With this understood we have:

PROPOSITION 17.2. *Let* $\theta_k(E)$ *be the cocycle of* E . *Then*

$$\mathrm{ch}\, \theta_k(E) = k^{4n} \cdot \psi_k\{\mathfrak{A}(E)\}/\mathfrak{A}(E) . \tag{17.1}$$

Proof: We have $\psi_k y = \theta_k(E) y + \Gamma_k(E)$. Hence $\mathrm{ch}\, \psi_k y = \mathrm{ch}\, \theta_k(E) \cdot \mathrm{ch}\, y + \mathrm{ch}\, \Gamma_k(E) = \mathrm{ch}\, \theta_k(E)\, \mathfrak{A}(E) \cdot Y + K_1$ where $K_1 \in H^*(X : \mathbb{Q})$. On the other hand $\psi_k \mathrm{ch} = \mathrm{ch}\, \psi_k$ as follows directly from the splitting principle for KU. Hence

$$\text{ch}\,\psi_k y = \psi_k \text{ch}\, y = \psi_k\{\mathfrak{A}(E)Y\} + \psi_k \mathfrak{B}(E)$$

$$= \{\psi_k \mathfrak{A}(E)\}k^{4n} \cdot Y + \psi_k \mathfrak{B}(E) \quad .$$

Comparing coefficients of Y we obtain:

$$\text{ch}\,\theta_k(E) = k^{4n}\psi_k\, \mathfrak{A}(E)/\mathfrak{A}(E) \qquad \text{Q.E.D.}$$

To return to the proof of the theorem: Combining (16.2) and (17.1) we see that $\mathfrak{A}(E)/\text{ch}\,\Omega(E)$ is invariant under ψ_k. As both these expressions start with one, we may conclude that $\mathfrak{A}(E) = \text{ch}\,\Omega(E)$.

One may express $\mathfrak{A}(E)$ in terms of $\text{ch}(V)$, $(V = \rho(E))$ or, as is usually done in terms of the Pontryagin classes p_i of V. (Recall that $p_i(V) = (-1)^i c_{2i}(\epsilon^* V)$ where c_i is the ith Chern-class of V.) Indeed, we know that if the Chern-class $c(\epsilon^* V)$ is represented formally by $\Pi(1+y_i)(1-y_i)$, then $\text{ch}(V)$ is represented by

$$1 + \sum_{1}^{4n} \{e^{y_i} + e^{-y_i}\} \ ,$$

and hence $\text{ch}\{\Omega(V)\}$ by

$$\prod_{1}^{4n} \frac{1 - e^{y_i}}{y_i} \cdot e^{-y_i/2} = \prod_{1}^{4n} \frac{\sin h(y_i/2)}{(y_i/2)} \quad .$$

In other words if the last formal power series is expressed in terms of the elementary symmetric functions of the y_i^2, $p_1, \cdots, p_{4n}$, and these are then replaced by the Pontryagin classes of V we obtain $\mathfrak{A}(E)$.

This recipe is thus the analogue of Proposition 13.3. In their work [4, 5], Atiyah and Hirzebruch use the class $\mathfrak{A}^{-1}(E) = \text{ch}\,\Omega^{-1}(V)$ and denote it by $\hat{\mathfrak{A}}(V)$. Their derivation of the algorithm relating the Pontryagin class of V to $\hat{\mathfrak{A}}(V)$ is quite different from ours. They were led to the study of $\hat{\mathfrak{A}}(V)$ through their investigation of the cohomology of G/U where U is a subgroup of maximal rank in G [11]. In a sense, their computation is the proper analogue in the $H^*(X; \mathbb{Q})$ theory of our derivation of a recipe for $\theta_k(E)$.

<u>Exercise</u>. Let $X \xrightarrow{f} Y$ be a smooth inclusion of compact oriented differentiable manifolds. Let N be the normal bundle of X in Y, and let $j : Y \to X^N$ be the natural projection. Assume now that N has a Spin-reduction, so that we have the Thom isomorphism:

$$\varphi : KO(X) \to KO^n(X^N) \qquad n = \dim N \ .$$

One defines the "Umkehrungs" homomorphism $f_!$ in the KO-theory by:

$$f_! u = j^! \varphi(u) \quad .$$

Thus $f_! : KO(X) \to KO(Y)$.

Prove the formula $f_!(u f^! v) = (f_! u) \cdot v$, and the Riemann-Roch formula:

$$\mathrm{ch}\{f_! u\} = f_*\{\mathfrak{A}(N) \cdot \mathrm{ch}\, u\} \qquad u \in KO(X) .$$

This formula may also be written in the form

$$\{\mathrm{ch}\, f_!(u)\}\, \mathfrak{A}^{-1}(t_y) = f_*\{\mathrm{ch}(u) \cdot \mathfrak{A}^{-1}(t_x)\} , \qquad u \in KO(X) ,$$

t_x, t_y the respective tangent bundles of X and Y . Using this expression, an imbedding of $X \subset S^{8n}$ (high n) and the periodicity theorem define $f_!$ for any map $X \to Y$ for which $f^! t_y - t_x$ admits a Spin-reduction and show that the above formula persists. This is the differentiable Riemann-Roch theorem of [4] .

Carry out the analogue for the KU theory also using the $\mathrm{Spin}^c(n)$ bundles.

§18. *Real projective bundles.* Consider the exact sequence

$$(18.1) \qquad \mathrm{Spin}(n) \to \widehat{\mathrm{Spin}(n)} \to \mathbb{Z}_2$$

where $\widehat{\mathrm{Spin}(n)}$ is the normalizer of $\mathrm{Spin}(n)$ in $\mathrm{Spin}(n+1)$. The nontrivial $\mathbb{Z}_2$-module then pulls back to an element

$\eta \in RU\{\widehat{\mathrm{Spin}(n)}\}$.

PROPOSITION 18.1. *Let* $\alpha : \widehat{\mathrm{Spin}(n)} \to \mathrm{Spin}(n+1)$; $n \geq 3$ *be the inclusion, and let* Δ^+, Δ^- *be the Spin representations of* $\mathrm{Spin}(n+1)$. *(We set* $\Delta^+ = \Delta^-$ *if* $n+1$ *is odd.) Then*

$$(18.2) \qquad (\alpha^! \Delta^{\pm}) \otimes \eta = \alpha^! \Delta^{\mp} .$$

Proof: The sequence (18.1) is obtained by covering the corresponding sequence

$$(18.3) \qquad SO(n) \to \widehat{SO(n)} \xrightarrow{\pi} \mathbb{Z}_2$$

which exhibits $\widehat{SO(n)}$ as $O(n)$, by the way. To obtain a splitting of (18.1) we proceed as follows. Given $n+1$ integers $\{\epsilon_i\} = \epsilon$ let $d(\epsilon)$ be the diagonal matrix in $O(n+1)$ with *ith* entry $(-1)^{\epsilon_i}$. Then $\widehat{SO(n)} \hookrightarrow SO(n+1)$ is the subgroup which commutes with the element $d(1, \cdots, 1; -1)$. Let

$$\underline{a} = d(1, \cdots, 1; -1, -1, -1, -1) \in SO(n+1) .$$

This element is clearly in $\widehat{SO(n)}$. Further $\pi\, \underline{a}$ generates $\mathbb{Z}_2$. Hence $\underline{a}$ splits (18.3). Let a be a lifting of $\underline{a}$ to $\widehat{\mathrm{Spin}(n)}$. Then we assert that a^2 = identity in $\widehat{\mathrm{Spin}(n)}$. Indeed the shortest closed 1-parameter group in $SO(n+1)$ containing $\underline{a}$ as its midpoint represents the trivial element

of $\pi_1\{SO(n+1)\}$ and hence lifts to a closed curve in $\widehat{\mathrm{Spin}(n)}$.
Q. E. D.

Thus a $\widehat{\mathrm{Spin}(n)}$ module, V, is specified by the action of Spin(n) on V and the action of the element a on V.

Suppose now that $(n+1)$ is even. Then Δ^+ and Δ^- are distinct elements of RU which both restrict to the irreducible module Δ of RU{Spin(n)}. Further, the restriction of Δ^+ to the group generated by a can be computed:

We choose the "obvious" maximal torus $T \subset \mathrm{Spin}(n+1)$ containing a and write y_i for the characters on T as before. Then for a proper choice of the numbering and orientations of the y_i we have:

$$y_i(a) = \begin{cases} -1 & \text{if} \quad i = 1, 2 \\ +1 & \text{if} \quad i \neq 1, 2 \end{cases}$$

while $\sqrt{y_1 \cdots y_m(a)} = +1$.

It follows that $\underline{ch}\,\Delta^+(a) = \dim \Delta^+ \cdot +1$, $\underline{ch}\,\Delta^-(a) =$ $= \dim \Delta^- \cdot (-1)$ or more precisely the restrictions of Δ^+ and Δ^- to the subgroup generated by a are respectively $\dim \Delta^+ \times$ trivial representation and $\dim \Delta^- \times$ the representation η. Thus if V is a representation space for $\alpha^! \Delta^+$ then a acts by $+1 \times$ identity and $\alpha^! \Delta^-$ is described on V by changing the action of $\widehat{\mathrm{Spin}(n)}$ only at a, namely by letting a act as -1. But this action is precisely the one given by $\alpha^! \Delta^+ \otimes \eta$.

Q. E. D.

Suppose next that n + 1 is odd. Then $\alpha^! \Delta, \Delta = \Delta^+ = \Delta^-$ can be described in this manner. Let V be a representation space for $\Delta_+ \in RU\,\widehat{\mathrm{Spin}(n)}$, and define an action of Spin(n) on V + V by setting

$$\begin{aligned} g(e_1, e_2) &= (ge_1, aga^{-1}e_2) \qquad g \in \mathrm{Spin}(n) \\ a(e_1, e_2) &= (e_2, e_1) \ . \end{aligned}$$

This is true because the automorphism induced by a on Spin(n) exchanges Δ_+ with Δ_-. Now then $\alpha^! \Delta \otimes \eta$ will be given by the same representation on Spin(n) however a will now send (e_1, e_2) into $-(e_2, e_1)$. The problem is therefore to show that these two actions are equivalent, and this will be demonstrated, once we construct an element c in the center of Spin(n) with the property that

$$c a c^{-1} = a \cdot \epsilon$$

where ϵ generates the kernel of Spin(n) → SO(n). Indeed, in each spin representation ϵ acts by -1, so that the inner automorphism by c would take the first action into the second one.

Let $\underline{c} = d(-1, \cdots, -1, 1)$. This element is in the center of $SO(n)$. We set c equal to a lifting of $\underline{c}$. Then if $2m = n$ we have:

$$c^2 = \epsilon^m$$

$$(c \cdot a)^2 = \epsilon^{m-1}$$

as follows from the fact that the shortest closed 1-parameter subgroup of $SO(n+1)$ containing $\underline{c}$, respectively $\underline{ca}$ represents m times respectively $(m-1)$ times the generator of $\pi_1\{SO(n+1)\}$. Hence

$$ca \cdot ca = c^2 \epsilon \quad ,$$

or equivalently

$$c^{-1}ac = a\epsilon \qquad \text{as} \quad a^{-1} = a \, . \quad \text{Q.E.D.}$$

COROLLARY 1. *The formula (18.2) holds in* $RO\{\widehat{\mathrm{Spin}(n)}\}$ *when the* Δ^+, Δ^- *are interpreted as the real spin representations of* $RO\{\mathrm{Spin}(n+1)\}$.

This is clear from the results of Section 10 because η is the complexification of a real bundle.

If we apply the permanence law to these relations we obtain the following theorem.

THEOREM 18.1. *Let* E *be a principal* $\mathrm{Spin}(n+1)$ *bundle over* X. *Let* $\widehat{\mathrm{Spin}(n)} \hookrightarrow \mathrm{Spin}(n+1)$ *be the inclusion and consider the projective space bundle* $\mathbb{P}(E)$ *over* X *associated to this subgroup. Then if* $\Delta^{\pm} \in RO\{\mathrm{Spin}(n+1)\}$ *are the Spin representations and* $\eta \in KO(X)$ *is the sub-bundle over* $\mathbb{P}(E)$ *(see Section 1), the following relation holds in* $KO\{\mathbb{P}(E)\}$:

$$\Delta^+(E) \otimes \eta = \Delta^-(E) \quad . \tag{18.4}$$

Proof: All that is needed is to identify $\eta(\hat{E})$ with the sub-bundle η over $\mathbb{P}(E)$ and then to apply the permanence law.

COROLLARY: *Consider* P_n = *real projective space of* $(n-1)$ *dimensions, and let* $\eta \in KO(P_n)$ *be the sub-bundle. Then if* $a_n = \dim \Delta_n^+$ *where* Δ_n^+ *is the real spin representation of* $\mathrm{Spin}(n)$, *we have*

$$a_n \eta = a_n \cdot 1$$

or

$$a_n(1 - \eta) = 0 \quad .$$

Proof: Just let X be a point in the previous theorem.

REMARKS. 1. The same result of course holds in $KU(P_n)$: one has $(1 - \eta \otimes \mathbb{C}) \cdot \dim_C \Delta_n^+ = 0$ where we now let Δ^+ be the complex spin representation.

2. We have carried out the proof of Proposition 18.1 only for $n \geq 3$. When Spin(n) is properly defined for $n = 2$ as the double covering of SO(2) everything is still valid in that case also.

§19. *Some examples*. In view of the last proposition of Section 18 the following is not quite surprising.

THEOREM 19.1. *Let* P_n *denote the real projective space of dimension* $n - 1$. *Then*

(19.1) $$\widetilde{KU}(P_n) = \mathbb{Z}_{b_n}$$

(19.2) $$\widetilde{KO}(P_n) = \mathbb{Z}_{a_n}$$

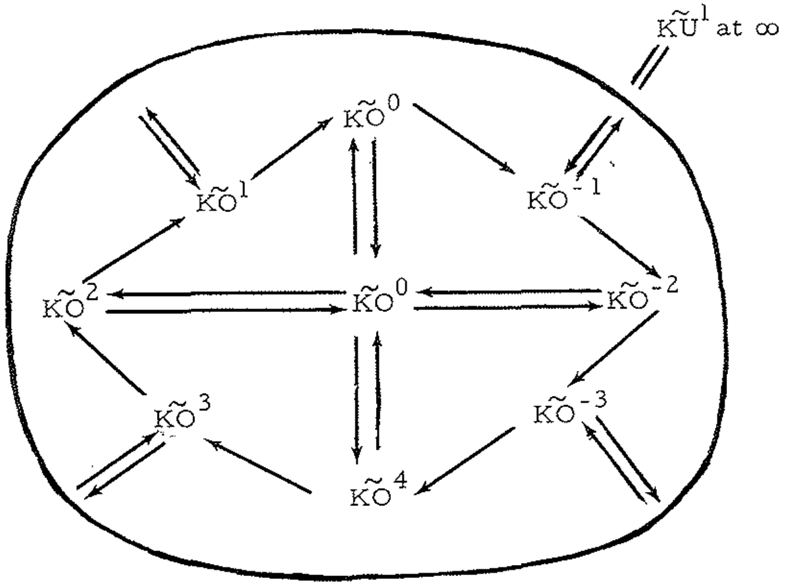
where a_n and b_n are the dimensions of the Spin representations in RO{Spin(n)} and RU{Spin(n)} respectively. Further $\widetilde{KO}(P_n)$ is generated by $\xi = 1 - \eta$ and $\widetilde{KO}(P_n)$ by $(1 - \eta) \otimes \mathbb{C}$ where η is the sub-bundle over P_n . Thus, as $\eta^2 = 1$, we have $\xi^2 = -2\xi$.

This theorem has several proofs, none of which are really quite satisfactory. In a way the most straightforward one is the following procedure of Milnor. By the spectral sequence for KU(X), see [5], it is clear that $\widetilde{KU}(P_n)$ has order b_n and that $KU^{-1}(P_n) = \mathbb{Z}$ if n is even and is 0 otherwise. To prove that $\widetilde{KU}(P_n)$ is in fact cyclic one uses the universal coefficient theorem which gives rise to an exact sequence:

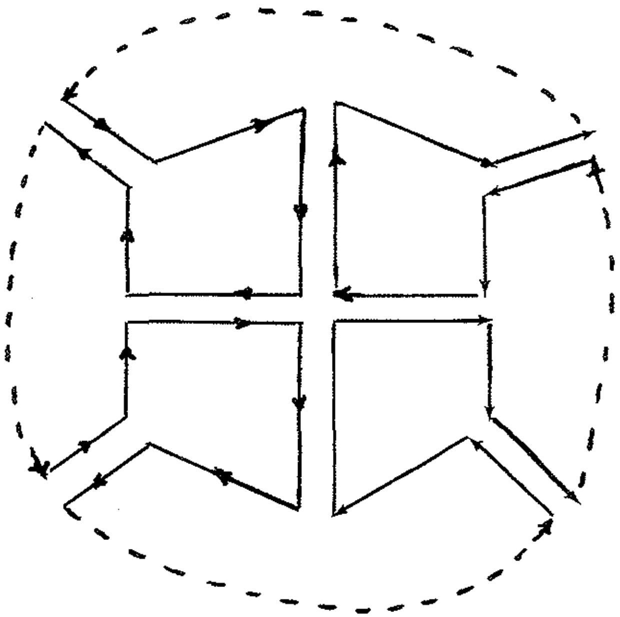
$$0 \longleftarrow \operatorname{Tor}(KU^{i+1}(X); \mathbb{Z}_2) \leftarrow KU^i(X; \mathbb{Z}_2) \leftarrow KU^i(X) \otimes \mathbb{Z}_2 \leftarrow 0$$

where $KU^*(X; \mathbb{Z}_2)$ is defined as $KU^*(X \# P_3)$, P_3 being the Moore-space for the group $\mathbb{Z}_2$. Now there is a spectral sequence covering to $KU^*(X; \mathbb{Z}_2)$ with E_2 term $H^*(X; KU^*(p; \mathbb{Z}_2))$, and $KU^*(p; \mathbb{Z}_2)$ is seen to be $\mathbb{Z}_2$ in every dimension. Finally it turns out that already the first differential operator, $d_3 = Sq^1 Sq^2 + Sq^2 Sq^1$, kills the spectral sequence yielding $\widetilde{KU}(X; \mathbb{Z}_2) = \mathbb{Z}_2$. Thus $\widetilde{KU}(P_n)$ is cyclic. That ξ is a generator then follows by induction. To get at $\widetilde{KO}(P_n)$ Milnor now uses the sequence (12.2) relating KU and KO .

One may arrange this sequence in the following manner,



so that sequence shaped as follows is exact:



Another approach is to systematically use the Spin representations to build bundles on the spaces P_n/P_k and then to use a double-induction. This was the point of view used by A. Shaprio and myself in [8]. The gist of the argument is as follows: Let $M_k \subset RO\{Spin(k)\}$ be the additive subgroup generated by the Spin-representations in $RO\{Spin(k)\}$. Thus $M_k \cong \mathbb{Z}$ for $k \neq 4n$, and $M_k \cong \mathbb{Z} + \mathbb{Z}$ for $k = 4n$. We further have natural restriction homomorphisms: $M_k \to M_{k-r}$.

Now, let η be the sub-bundle over P_n, and consid consider $P_k \subset P_r$. Then on $P_k a_k \cdot \eta$ is isomorphic to a trivial bundle by the corollary to Theorem 18.1. In fact every spin representation on $RO\{Spin(k)\}$ is seen to define a definite trivialization of $a_k \eta$ on P_k and thus a bundle on P_n/P_k. This construction then extends to a homomorphism

$$M_k \to \widetilde{KO}(P_n/P_k)$$

and our result, which we proved by a double induction and a product formula yields the theorem:

THEOREM 19.2. The sequence

$$M_n \to M_k \to \widetilde{KO}(P_n/P_k) \to 0$$

where the first homomorphism is the restriction, is exact.

The same result holds over the complex numbers if M_n *is defined as the subgroup generated by the complex Spin representations in* $RU\{Spin(n)\}$.

The details of either of these proofs are a little too long to be given here. Adams' account of these computations can be found in "Vector fields on spheres", Ann. of Math. (2) 75 (1962), 603-632.

Noteworthy corollaries are:

COROLLARY 19.1. *Consider the sequence*

$$0 \leftarrow \widetilde{KO}(P_n) \leftarrow \widetilde{KO}(P_{n+1}) \leftarrow \widetilde{KO}(S_n) \quad .$$

Then the generator of $\widetilde{KO}(S_n)$ *is mapped onto* $a_n \cdot \xi \in \widetilde{KO}(P_{n+1})$. *In particular* $\widetilde{KO}(S_n)$, $n \equiv 1, 2\ (8)$ *is injected into* $\widetilde{KO}(P_{n+1})$.

COROLLARY 19.2. *The operation of* ψ_k *on* $\widetilde{KO}(P_n)$, *and hence on* $\widetilde{KO}(S_n)$, $n \equiv 1, 2\ (8)$ *is given by*:

$$\psi_{2k+1} = \text{identity}$$

$$\psi_{2k} = 0 \quad .$$

Proof: Recall that η is the sub-bundle of P_n. Hence, in particular, a line bundle. Thus $\lambda_t\eta = 1 + t\eta$, whence $\psi_t\eta = \frac{\eta}{1 - t\eta}$, so that $\psi_{2k+1}\eta = \eta$ and $\psi_{2k}\eta = 1$. Now $\xi = 1 - \eta$ generates $\widetilde{KO}(P_n)$. Q.E.D.

The following gives the crucial result in the Adams solution of the vector-field-problem.

THEOREM 19.3. $\widetilde{KO}(P_n) \cong J(P_n)$.

Proof: We have to show that if $b \cdot \eta$ is J-equivalent to zero, then b is a multiple of a_n. For $n = 1, \ldots, 9$, the Whitney class gives the correct result. Indeed for $b \cdot \eta$ to be J-trivial $w(\eta)^b$ has to equal 1. Further, because $w(\eta) = 1 + x$ where x generates $H^1(P_n)$ we may check explicitly that the lowest power of b which will solve the equation $(1 + x)^b = 1$ is precisely a_n.

Consider the case $n > 9$ next. As $J(P_n)$ is a quotient of $J(P_{n+m})$ a possible value of b will have to be a multiple of 8, say $8m$. Now $8m\eta$ admits a Spin-reduction, so that the cocycle $\theta_k(8m\eta)$ is well defined. In fact we have already computed this cocycle in Section 13 and found that

$$\theta_k(8_m\eta) = \begin{cases} k^{4m} - \frac{k}{2}^{4m}(1 - \eta) & k \text{ even} \\ k^{4m} - \left\{\frac{k^{4m} - 1}{2}\right\}(1 - \eta) & k \text{ odd} . \end{cases}$$

Now by Corollary 2 of Theorem B in Section 13 we obtain as a necessary condition for the J-triviality of $8m\,\eta$ that

$$\theta_k(8m\,\eta) = k^{4m}\psi_k u/u$$

where u is an invertible element of $KO(P_n)$. But for k odd, we have seen that ψ_k acts as the identity on $KO(P_n)$ so that the condition reduces to

$$k^{4m} - \left\{\frac{k^{4m}-1}{2}\right\}\xi = k^{4m} \qquad k \text{ odd} .$$

Hence we must have $\frac{k^{4m}-1}{2} \equiv 0 \mod a_n$ for odd k. Now a little number theory shows that this condition implies that 4m is divisible by $a_n/2$, i.e., that 8m is divisible by a_n. However this is also the condition for stable J-triviality, which reads as follows:

$$k^s\,\theta_k(8m\,\eta) = k^{4m+s}\psi_k u/u \qquad \text{for some } s.$$

Hence for odd k one still has $\frac{k^{4m}-1}{2} \equiv 0 \mod a_n$. Q.E.D.

COROLLARY 19.3. $\widetilde{KO}(S_n) \cong J(S_n)$, $n \equiv 1, 2 \mod 8$.

Proof: We have $0 \to \widetilde{KO}(S_n) \to \widetilde{KO}(P_{n+1}) \approx (P_{n+1})$ whence $J(S_n) \neq 0$. Q.E.D.

Let me conclude by sketching the path, à la James, Atiyah, from this theorem to the vector-field problem on the spheres. The theorem of Adams [1], [2] may be stated

as follows:

THEOREM: *Let $O_{n,k}$ denote the space of orthonormal k-frames in E_n, and let $O_{n,k} \to O_{n,1}$ be the projection. Then this fibering has a section, if and only if n is a multiple of the Hurwitz-Radon number a_k.*

One considers the fibering:

$$O_{n-1,k-1} \longrightarrow O_{n,k} \xrightarrow{\pi} O_{n,1} .$$

Also let $P_n \subset O_n$ be the projective space imbedded in O_n by a assigning to a 1-space, e, in E_n the reflection in the corresponding orthogonal hyperplane. The sequence above then gives rise to a sequence

$$P_{n-1}/P_{n-k} \longrightarrow P_n/P_{n-k} \xrightarrow{\pi'} P_n/P_{n-1}$$

and one checks that in the stable range π has a section if and only if π' has a section. Now $P_n/P_{n-k} = P_k^{(n-k)\eta}$ as is easily checked. Hence if $P_k^{(n-k)\eta} \to S_{n-1}$ has a section s, the S-dual of this map will determine a map $S_m \to P_k^{(n\eta+n'1)}$, $n + n' = m$, which yields a coreduction of $P_k^{(n\eta+n'1)}$ — or, quite equivalently, a J-trivialization of $n\eta$. (One here uses the duality theorem [3] which asserts that if X is a manifold with normal bundle N in some imbedding of

$X \subset E_n$, and if E is any bundle over X, then $X^{(E+N)}$ represents the dual of X^E in the Spanier-Whitehead sense.) The pertinent references here are [3], [12], [13] .

§20. The difference element. Although I have avoided the "difference" construction of bundles in these notes, it is such a useful device that a short discussion of it seems advisable. The situation is as follows:

Let E and F be bundles over X, and let ϕ be an isomorphism of their restriction to a subcomplex $A \subset X$. Thus

$$\phi : E|A \to F|A \ .$$

We wish to construct an element $d(E, F) \in K(X, A)$ which is the analogue of the difference cocycle. For this purpose let $Y = X_1 \cup_A X_2$ be the space obtained from the disjoint union of two copies of X, say X_1 and X_2, by gluing them together along $A \subset X_i$. We now construct a bundle $E \underset{\phi}{\cup} F$ over Y in the plausible manner: We take E over X_1, F over X_2 and glue them together via ϕ over A.

Note that we have a natural projection $Y \xrightarrow{\pi} X$ given by the identity on each factor, also that we have two inclusions $X \xrightarrow{s_i} Y$ onto the two factors $X_i \subset Y$, $i = 1$,

2, and finally that

$$\begin{tikzcd} X \arrow[r, shift left, "s_i"] & Y \arrow[l, shift left, "\pi"] \arrow[r, "j"] & Y/X_i \cong X/A \end{tikzcd}$$

exhibits X/A as a quotient of an exact sequence which splits. Thus we may identify $\tilde{K}(X/A)$ with the kernel of $s_2^!$ in $K(Y)$ and this will be done in the subsequent discussion.

With this understood one defines $d_\phi(E, F) \in \tilde{K}(X/A)$ as the class of $E \cup_\phi F - \pi^! F$ in $K(Y)$. This element is in the kernel of $s_2^!$ as $s_2^!(E \cup_\phi F) = F$ and $s_2^! \pi^! F = F$. To simplify the notation we consider $K(Y)$ as a module over $K(X)$ — i.e., suppress the $\pi^!$ — so that $d_\phi(E, F) = E \cup_\phi F - F$ in $\tilde{K}(Y/A) \subset K(Y)$.

The following proposition is easily verified by an explicit check:

PROPOSITION 20.1. The construction $E \cup_\phi F$ has the following properties:

(20.1) $$E \underset{1}{\cup} E = E$$

(20.2) $$E \underset{-\phi}{\cup} F = E \underset{\phi}{\cup} F$$

(20.3) $$E \underset{\phi}{\cup} F + E' \underset{\phi}{\cup} F' = (E + E') \underset{\phi+\phi'}{\cup} (F + F')$$

(20.4) $$(E \underset{\phi}{\cup} F)(E' \underset{\phi'}{\cup} F') = E \cdot E' \underset{\phi\otimes\phi'}{\cup} F \cdot F'$$

(20.5) $$\lambda^i E \underset{\lambda^i\phi}{\cup} \lambda^i F = \lambda^i(E \underset{\phi}{\cup} F) \quad .$$

Recalling that K(X) is defined by homotopy classes of maps of X into $\underline{\underline{K}}$, we see further that:

(20.6) $E \underset{\phi}{\cup} F$ depends only on the homotopy class of ϕ .

An immediate application of this formula is:

(20.7) $$E \underset{\phi}{\cup} F + F \underset{\phi^{-1}}{\cup} E = E + F \quad .$$

Indeed the LHS is given by $E + F \underset{\phi+\phi^{-1}}{\cup} F + E$ while the RHS is given by $E + F \underset{1+1}{\cup} E + F$. However $\phi + \phi^{-1}$ can be deformed through isomorphisms into $1 + (-1)$ whence by (19.2), the relation (19.7) follows. As another application we cite the formula:

(20.8) $$d(e, F)(E - F) = d(E, F)^2$$

which may be derived similarily.

With the aid of the difference construction one may get at the Thom-complex of a bundle directly. In fact consider the following general situation envisaged in Section 11: $H \xrightarrow{i} G$, the inclusion of a closed subgroup; E a principal G-bundle over X ; $\mathbb{P} = \mathbb{P}(E) = E/H$; and finally $M \xrightarrow{\pi} X$ the mapping cylinder of $\mathbb{P}$. We then have the diagram

$$\begin{tikzcd}
K(\mathbb{P}) & K(M) \arrow[l] & \tilde{K}(M/\mathbb{P}) \arrow[l] \\
R(H) \arrow[u] & R(G) \arrow[l, "i^!"'] \arrow[u] & R(G,H) \arrow[l]
\end{tikzcd}$$

where R(G, H) denotes the kernel of $i^!$ and the vertical homomorphisms are $\alpha_{\hat{E}}$ and $\pi^! \circ \alpha_E$ respectively. Now by the use of the difference construction we may complete this diagram with a compatible λ^i - homomorphism $d : R(G, H) \to \tilde{K}(M/\mathbb{P})$, at least for the KU-theory. Indeed let A and B be two complex G-modules. Then by the permanence formula $\pi^! \circ \alpha_E A$ and $\pi^! \alpha_E B$, when restricted to $\mathbb{P}$, become canonically isomorphic to $\alpha_{\hat{E}}(i^! A)$ and $\alpha_{\hat{E}}(i^! B)$ respectively. Suppose now that $i^! A \cong i^! B$ and that ϕ is an H-isomorphism of these two H-modules. Then $d_\phi(A, B) = d_\phi(\pi^! \circ \alpha_E A, \pi^! \alpha_E B)$ is a well defined element of $\tilde{K}(M/\mathbb{P})$. Now if we are working with complex modules it is easily seen that the set of possible H-isomorphisms $\phi : i^! A \to i^! B$ is connected. (The group of H-automorphisms of an H-module is just a product of full linear groups. Q.E.D.)

Hence $d_\emptyset(A, B)$ is in this case seen to depend only on A and B. In fact $d_\emptyset(A, B) = d_\emptyset(A', B')$ if $A - B = A' - B'$ in $RU(G)$. This follows from: $A + B' = A' + B$, as G-modules, $\Rightarrow d(A + B', A' + B) = 0 \Rightarrow d(A, B) + d(B', A') = 0$ $\Rightarrow d(A, B) = d(A', B')$. Q.E.D.

(Here we have suppressed the $\emptyset$ because it is unique.)

Every element $x \in R(G, H)$ may be written in the form $A - B$ where A and B are G-modules which have isomorphic restrictions to $R(H)$, and one defines $d(x)$ as $d(A, B)$.

Over the real numbers the construction of a canonical $d : RO(G, H) \to KO(M/\mathbb{P})$ is not so clear. In this case the group of H-automorphisms of an H-module may have several components and it is not quite clear to me that the consequent choices may be constructed compatibly. However in simple cases --- such as $G = \mathrm{Spin}(2n)$, $H = \mathrm{Spin}(2n\text{-}1)$ there is no difficulty in the real case either.

Exercise 1. Obtain the formulae of Theorem C', Section 14, directly by using the difference construction.

Exercise 2. Let $f : (G', H') \to (G, H)$ be a homo-

G-extension of E', and set $\mathbb{P}' = E'/H'$, $\mathbb{P} = E/H$. In this situation we therefore have the commutative diagram:

$$\begin{tikzcd}
RU(G') & RU(G) \arrow[l] \arrow[d] \\
RU(H') \arrow[from=u, d] & RU(H) \arrow[l]
\end{tikzcd}$$

Construct d so as to complete the following commutative diagram:

$$\begin{tikzcd}
KU(\mathbb{P}') & KU(\mathbb{P}) \arrow[l] & \widetilde{KU}(\mathbb{P}/\mathbb{P}') \arrow[l] & \\
R(H') \arrow[u, "\alpha_{\hat{E}'}"] & R(G') \underset{R(G)}{\otimes} R(H) \arrow[l] \arrow[u, "\pi^! \alpha_{E'} \otimes \alpha_{\hat{E}}"'] & R(G, G'; H, H') \arrow[l] \arrow[u, "d"'] & 0 . \arrow[l]
\end{tikzcd}$$

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APPENDIX I

ON THE PERIODICITY THEOREM FOR COMPLEX VECTOR BUNDLES

By

M. Atiyah and R. Bott

§ 1. Introduction. The periodicity theorem for the infinite unitary group [2], is most usefully expressed by the Kunneth formula:

$$K(X \times S^2) \cong K(X) \otimes K(S^2) \tag{1.1}$$

where $K(X)$ denotes the group of virtual complex vector bundles over X. In this formula X is a finite complex, and S^2 denotes the Gauss sphere.

This note is devoted to a direct proof of (1.1) using only the quite elementary properties of the functor K.

Our proof arose out of a proposition which we needed in the study of well posed boundary conditions for elliptic operators, and its basic principle is that the polynomial approximation which leads to the determination of $K(S^2)$ can be modified so as to determine $K(X \times S^2)$ over $K(X)$.

§2. Preliminaries. We assume familiarity with the elementary theory of vector bundles and the definition and elementary properties of the functor $K(X)$ on the category, $\mathfrak{A}$, of finite CW-complexes, see for example [1]. In particular, we will need the following "clutching" construction of vector bundles on the union of two spaces.

Let $X = X_1 \cup X_2$, with $A = X_1 \cap X_2$, where the X_i, X and A are all objects of $\mathfrak{A}$. Assume also that E_i are vector bundles over X_i, and that $\varphi : E_1|A \to E_2|A$ is an isomorphism of the bundles E_i restricted to A. These data then define a bundle $E_1 \cup_\varphi E_2$ on X which is obtained by gluing E_1 and E_2 together via φ on A. Elementary properties of this construction are the following:

(2.1) If E is a bundle over X and $E_i = E|X_i$, then the identity defines an isomorphism $1_A : E_1|A \to E_2|A$, and

$$E_1 \underset{A}{\cup} E_2 \cong E .$$

(2.2) If $\beta_i : E_i \to E_i'$ are isomorphisms on X_i then

$$E_1 \underset{\varphi}{\cup} E_2 \cong E_1' \underset{\varphi'}{\cup} E_2' \quad \text{with} \quad \varphi' = \beta_2 \circ \varphi \circ \beta_1^{-1} .$$

(2.3) If (E_i, φ) and (E'_i, φ') are two "clutching data" on the X_i then:

$$E_1 \underset{\varphi}{\cup} E_2 \oplus E'_1 \underset{\varphi'}{\cup} E'_2 \cong (E_1 \oplus E'_1) \underset{\varphi\oplus\varphi'}{\cup} (E_2 \oplus E'_2)$$

$$(E_1 \underset{\varphi}{\cup} E_2) \otimes (E'_1 \underset{\varphi'}{\cup} E'_2) \cong (E_1 \otimes E'_1) \underset{\varphi\otimes\varphi'}{\cup} (E_2 \otimes E'_2) \quad .$$

These properties are immediate consequences of the definitions and the notion of isomorphism of bundles. From the fact that homotopic maps induce isomorphic bundles, it follows further that:

(2.4) $E_1 \underset{\varphi}{\cup} E_2$ depends only on the homotopy class of the isomorphism $\varphi : E_1|A \to E_2|A$.

If E and F are bundles over X and Y , then $E \hat{\otimes} F$ — their exterior product — is a bundle over $X \times Y$. This is the operation which induces the homomorphism

$$\mu : K(X) \otimes K(S^2) \to K(X \times S^2)$$

which is to be shown to be an isomorphism. This is of course the basic tensor-product, in the sense that the "interior" tensor product of two bundles E and F on the same space, that is, $E \otimes F$, is defined by: $E \otimes F = \Delta^*(E \hat{\otimes} F)$, with $\Delta : X \to X \times X$ the diagonal inclusion.

§3. *Bundles over* $X \times S^2$. Let S^2 be thought of as the compactification of the complex numbers $\mathbb{C}$ and let D^+ denote the disc $|z| \leq 1$, while D^- shall stand for the opposite disc $|z| \geq 1$.

We set $X_1 = X \times D^+$ and $X_2 = X \times D^-$; $A = X \times S$ where $S = D^+ \cap D^-$ is the unit circle. The natural projections of these spaces on X are denoted by π_1 , π_2 and π_A respectively, while the map $X \to A$ sending X into $(X, 1)$ will be denoted by s .

PROPOSITION 3.1. *Let* E *be a bundle over* $X \times S^2$ *and let* $F = s^*E$ *be the bundle on* X *induced by the map* s *from* E . *Then there is an automorphism* f $f : \pi_A^*F \to \pi_A^*F$ *unique up to homotopy, such that*

(3.2) $$E \cong \pi_1^*F \underset{f}{\cup} \pi_2^*F \ , \qquad \text{and}$$

(3.3) $f|X \times 1$ *is homotopic to the identity.*

Proof: We consider s as a map of X into X_1 . Then $s \circ \pi_1 : X_1 \to X_1$ is a homotopy equivalence. Hence the natural isomorphism $E|X \times 1 \approx \pi_A^*F|X \times 1$, may be extended to an isomorphism $f_1 : E|X_1 \cong \pi_1^*F$. Further, any two such extensions differ by an automorphism α of

π_1^*F which is the identity on $\pi_1^*F|X \times 1$ and therefore homotopic to the identity on all of X_1. Thus the homotopy class of f_1 is well determined. Similarly one defines an isomorphism $f_2 : E|X_2 \cong \pi_2^*F$ and now the proposition follows by taking $f = f_2 \circ f_1^{-1}$. The clutching function f satisfying (3.2) and (3.3) is called a normalized clutching function for E.

We next describe an especially simple class of clutching data for $X \times S^2$. Suppose then that F is a bundle over X, and consider an automorphism φ of π_A^*F. Clearly such a φ amounts to a function which in a continuous fashion assigns to each pair (x, z), $x \in X$, $z \in S$, an automorphism:

$$\varphi(x, z) : F_x \to F_x \quad .$$

Now given a sequence a_i, $i \in \mathbb{Z}$ of endomorphisms of F (i.e., continuous sections of the bundle $\mathrm{Hom}(F, F)$) consider the expression:

$$f = \sum_{|i| \leq N} a_i z^i \quad .$$

For each $x \in X$ and $z \in \mathbb{C}$, $f(x, z) = \Sigma_{|i| \leq N} a_i(x) z^i$ is then an endomorphism of F_x. Hence if $f(x, z)$ is an isomorphism for each x, and $z \in S$, then f defines an automorphism - also denoted by f - of π_A^*F, and therefore a bundle $\pi_1^*F \cup_f \pi_2^*F$ on $X \times S^2$.

For obvious reasons we call an expression of the type (3.3) a <u>Laurent series</u> of endomorphisms over F, and call such a Laurent series proper if $f(x, z)$ is nonsingular for $z \in S$. If no negative powers of z occur in f, then f is called a <u>polynomial</u>. Finally, if f is a proper Laurent series over F then the bundle $\pi_1^*F \cup_f \pi_2^*F$ on $X \times S^2$, will be denoted by: (F, f, F), and will be said to have been obtained from F by a <u>Laurent construction.</u>

As an example consider the finite proper Laurent-series $f(z) = z^{-n} \times$ (Identity). This "universal" series applies to all bundles F over X. In particular if X is a point, and F is the trivial bundle, then (F, z^{-n}, F) is a bundle on S^2 which we denote by H^n. For $n = 1$ one obtains the "hyperplane" bundle H and it is clear by (2.3) that $H^k \otimes H^s = H^{k+s}$. More generally it follows from (2.3) that for any bundle E over X, the bundle $E \hat{\otimes} H^n$ on $X \times S^2$ is described by (E, z^{-n}, E).

Our first step towards a proof of (1.1) is the following proposition:

PROPOSITION 3.4. *Let* E *be a bundle over* $X \times S^2$, *and let* $s : X \to X \times S^2$ *be the constant map* $x \to (x, 1)$. *Then* E *is obtained from the bundle* $F = s^*E$ *by a Laurent construction.*

Proof: By Proposition 3.1 there is a clutching function f for F, so that $E = \pi_1^* F \cup_f \pi_2^* F$. Consider now the Fourier series of $f : \Sigma_{-\infty}^{\infty} a_k z^k$, where a_k is the section of $\mathrm{Hom}(F, F)$ defined by the integral:

$$a_k(x) = \frac{1}{2\pi i} \int_S z^{-k} f(x, z)\, dz/z \quad .$$

We set $S_k = \Sigma_{-k}^{k} a_i z^i$, and $f_n = (1/n+1)\, \Sigma_0^n S_k$. Thus f_n is the n'th partial Cesaro-sum of the Fourier series, and so by an easy extension of Fejer's theorem, f_n is seen to converge to f uniformly in z, and in X - the latter because f is uniformly continuous on $X \times S^2$.

It follows that for n large enough f_n will be arbitrarily close to f and hence, in particular, proper. Finally because close maps are homotopic, it follows that $E \cong (F, f_n, F)$ for n large enough. Q.E.D.

Our next aim is to classify the Laurent bundles over $X \times S^2$. Because every Laurent series is of the form $z^{-n} p$ where p is a polynomial, the essential complications

of the Laurent construction already occur in the polynomials. Using an operation analogous to the one which transforms an n'th order differential equations into a number of first order ones, we first present a linearization procedure.

Consider a polynomial $p(z) = \Sigma_{i=0}^{n} a_i z_i$, of degree $\leq n$, over F. One then defines $L^n(p)$ as the linear polynomial over $L^n(F) = F \oplus \cdots \oplus F$ (n + 1, copies) given by:

$$L^n(p)(z) \cdot \{f_0, \ldots, f_n\} = \left\{ \sum_{i=0}^{n} a_i f_i,\ -zf_0 + f_1,\ -zf_1 + f_2, \ldots, -zf_{n-1} + f_n \right\} . \tag{3.5}$$

In matrix-notion, $L^n(p): \underbrace{F \oplus \cdots \oplus F}_{n+1} \to \underbrace{F \oplus \cdots \oplus F}_{n+1}$ is therefore described by the matrix

$$L^n(p) = \left\{ \begin{array}{cccc} a_0, & a_1 & \cdots & a_n \\ -z, & 1, 0 & & 0 \\ & -z, 1 & & \\ 0 & & & 0 \\ & & & -z, 1 \end{array} \right\} \tag{3.5}$$

PROPOSITION 3.7. *Let* p *be a proper polynomial of degree* $\leq n$ *over* F. *Then* $L^n(p)$ *is a proper linear polynomial on* $L^n(F)$, and

$$(3.8) \qquad (F, p, F) + (L^{n-1}(F), 1, L^{n-1}(F)) \cong (L^n(F), L^n(p), L^n(F)).$$

Proof: Let $\hat{p} : L^n(F) \to L^n(F)$ be given by $\hat{p}(z)\{f_0, \cdots, f_n\} = \{p(z)f_0, f_1, \cdots, f_n\}$. Then the LHS of (3.8) is clearly isomorphic to $(L^n(F), \hat{p}, L^n(F))$. Hence we will be done once it can be shown that $\hat{p}$ and $L^n(p)$ can be deformed into each other through *proper* polynomials.

For this purpose define $L_t^n(p)$ by the formula:

$$L_t^n(p) = \begin{pmatrix} p - t^{n+1}(p - a_0), & t^n a_1, & t^{n-1} a_2, & & ta_n \\ & -tz, & 1, & & \\ & & -tz, & 1, & 0 \\ 0 & & & -tz & 1 - \end{pmatrix}$$

and observe the identity:

$$L_t^n(p) = \begin{pmatrix} p, & t^n p_1, & t^{n-1} p_2, \cdots tp_n \\ & 1, & \cdots & \\ & & 1, & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1, & & & \\ -tz, & 1, & & \\ & -tz, & 1, & \\ & & & 1 \end{pmatrix}$$

where $p_r(z)$ are polynomials defined inductively by $p_r(z) = \Sigma_{i=r}^n a_i z^{i-r}$.

It is then clear that if p is proper then $L_t^n(p)$ will be proper for all t, so that this family furnishes a canonical homotopy from $\hat{p}$ at $t = 0$, to $L^n(p)$ at $t = 1$. Q.E.D.

From (3.6) some easy homotopies of proper linear polynomials lead one to:

LEMMA 3.9. *Let* p *be a proper polynomial of degree* $\leq n$ *on* F. *Also write* $L^m(F, p, F)$ *for* $\{L^m(F), L^m(p), L^m(F)\}$. *Then,*

$$(3.10) \qquad L^{n+1}(F, p, F) \cong L^n(F, p, F) + (F, 1, F)$$

$$(3.11) \qquad L^{n+1}(F, zp, F) \cong L^n(F, p, F) + (F, z, F) \quad .$$

For example the family of matrices

$$\begin{pmatrix} a_0, \cdots & a_{n-1}, & a_n, & 0 \\ -z, 1, & & & \\ & -z, & 1, & \\ & -(1-t)z, & & 1 \end{pmatrix}$$

proves (3.10).

As explicit instances of these identities we have: $L^2(1, z^2, 1) \cong L^1(1, z, 1) + (1, z, 1)$ by (3.11), whence by (3.8), $(1, z^2, 1) + 2(1, 1, 1) \cong (1, z, 1) + (1, 1, 1) + (1, z, 1)$. Thus

(3.12) $$H^{-2} + 2 \cong 2H^{-1} + 1 \ .$$

We note that this is the basic relation of the hyperplane bundle.

PROPOSITION 3.13. *Let* p *be a proper linear polynomial on* F . *Then* F *decomposes into a direct sum:* $F = F_+ \oplus F_-$, *such that on* $X \times S^2$,

(3.14) $$(F, p, F) = (F_+, z, F_+) + (F_-, 1, F_-) \ .$$

The bundles F_+ *and* F_- *are called the* + *and* - *bundles of* p *on* F .

The decomposition of F which we need here is given by the following theorem in linear algebra.

LEMMA. *Let* a *and* b *be endomorphisms of a vector space* V, *and let* Γ *be a closed curve in the com complex plane for which* $p(z) = az + b$; $z \in \Gamma$, *is non-singular*. *Then the following holds:*

(3.15) The operators $P = \frac{1}{2\pi i} \int_\Gamma p(z)^{-1} dp(z)$ and $Q = \frac{1}{2\pi i} \int_\Gamma dp(z) p(z)^{-1}$ *are projection operators which satisfy the identity*

(3.16) $$p(\lambda) \cdot P = Qp(\lambda) \quad \textit{for all} \quad \lambda \in \mathbb{C} \ .$$

(3.17) *For* λ *outside* Γ, p(λ) *maps* PV *onto* QV *isomorphically*
For λ *inside* Γ, p(λ) *maps* (1-P)V *onto* (1-Q)V *isomorphically*.

This lemma clearly applies to each fiber of our situation, with Γ the unit circle, and so defines two continuous projection operators P and Q on F . In terms of these define:

(3.18) $$p_t(z) = Q(az + tb)P + (1 - Q)(taz + b)(1 - P) \ .$$

It then follows directly from (3.15) and (3.16) that $p_1(z) = p(z)$; while (3.17) implies that, in addition, p_t is proper for each t . Hence (3.18) deforms p into the clutching function

(3.19) $$p_0 = zQaP + (1 - Q)b(1 - P) \ .$$

Thus:

(3.20) $$(F, p, F) \approx (PF, za, QF) + ((1 - P)F, b, (1 - Q)F) \ .$$

Now, define F_+ as PF, and F_- as (1 - P)F . Then applying the isomorphism $a^{-1} : QF \to PF$ and $b^{-1} : (1 - Q)F \to (1 - P)F$ in the second factors of these clutching formulae, yields the desired isomorphism:

$$(F, p, F) \cong (F_+, z, F_+) + (F_-, 1, F_-) \quad . \tag{3.21}$$

We finally combine (3.9) with (3.21) in a straightforward way to obtain the following:

PROPOSITION 3.22. *Let* p *be a proper polynomial over* F *of degree* $\leq n$, *and let* $L^n(F, p, F)_\pm$ *be the* $\pm$ *bundles of* $L^n(p)$ *on* $L^n(F)$. *Then:*

$$\begin{aligned} L^{n+1}(F, p, F)_+ &= L^n(F, p, F)_+ \ , \\ L^{n+1}(F, p, F)_- &= L^n(F, p, F)_- + F \end{aligned} \tag{3.23}$$

while

$$\begin{aligned} L^{n+1}(F, zp, F)_+ &= L^n(F, p, F)_+ + F \\ L^{n+1}(F, zp, F)_- &= L^n(F, p, F)_- \quad . \end{aligned} \tag{3.24}$$

§ 4. *The proof of* $K(X \times S^2) = K(X) \otimes K(S^2)$.

The proposition of the last section may be assembled to construct a homomorphism

$$\nu : K(X \times S^2) \to K(X) \otimes K(S^2) \tag{4.1}$$

which will turn out to be an inverse to μ and so establish (1.1).

First let f be an arbitrary clutching function f over F on X . Let f_n be the Cesaro means of its Fourier series, and put $p_n = z^n f_n$. Then for n large enough, p_n is a polynomial clutching function (of degree $\leq 2n$) over F. Consider now the element $\nu_n(f)$ in $K(X) \otimes K(S^2)$ defined by:

$$\nu_n(f) = [L_+^{2n}(F, p_n, F)] \otimes (h^{n-1} - h^n) + [F] \otimes h^n, \quad h = [H] \tag{4.2}$$

where [E] denotes the element of K(X) determined by the bundle E .

We assert first of all that $\nu_n(f) = \nu_{n+1}(f)$ for large enough n . Indeed if n is large enough, the linear segment joining p_{n+1} to $z \cdot p_n$ provides a homotopy of polynomial clutching functions of degree $\leq 2(n+1)$. Hence, by the continuous dependance of $L_+^n(F, p, F)$ on p, we have:

$$\begin{aligned} L_+^{2n+2}(F, p_{n+1}, F) &\cong L_+^{2n+2}(F, zp_n, F) \\ &\cong L_+^{2n+1}(F, zp_n, F) && \text{by (3.23)} \\ &\cong L_+^{2n}(F, p_n, F) + F && \text{by (3.24).} \end{aligned}$$

Thus

$$\begin{aligned} \nu_{n+1}(f) &= [L_+^{2n} + F] \otimes \{h^n - h^{n-1}\} + [F] \otimes h^{n+1} \\ &= [L_+^{2n}] \otimes \{h^n - h^{n-1}\} + [F] \otimes h^n \\ &= \nu (f) . \end{aligned}$$

Hence for large n, $\nu_n(f)$ is independent of n and so depends only on f. We write it as $\nu(f)$. Now if g is a clutching function over F sufficiently close to f and n is sufficiently large, then the linear segment joining f_n to g_n provides a proper polynomial homotopy and shows that $\nu(f) = \nu(g)$. Thus $\nu(f)$ is a locally constant function of f and so depends only on the homotopy class of f. Hence if E is any bundle over $X \times S^2$ and f is a normalized clutching function for E as given in (3.2), then we can define

$$\nu(E) = \nu(f)$$

and $\nu(E)$ will depend only on the isomorphism class of E. Since $\nu(E)$ is clearly additive for direct sums, ν induces a homomorphism $\nu : K(X \times S^2) \to K(X) \otimes K(S^2)$.

This is the desired inverse to μ. Indeed the isomorphisms

$$E = (F, f, F) = (F, f_n, F) = (F, p_n, F) \otimes (1, z^{-n}, 1)$$

show by (3.8) and (3.14) that $\mu\nu$ is the identity on $K(X \times S^2)$: By (3.8) we have

$$[(F, p_n, F)] = [L^{2n}(F, p_n, F)] - 2n[F] \otimes 1$$

and by (3.14) we have, after eliminating $L_-^{2n}(F, p_n, F)$:

$$[(L^{2n}(F, p_n, F)] = [L_+^{2n}(F, p_n, F)] \otimes (h^{-1} - 1) + (2n+1)[F] \otimes 1 .$$

so that adding these two expressions one obtains $[E] = \mu\nu[E]$.

Finally the composition $\nu \cdot \mu$ is quite directly seen to be the identity on elements of the form $[F] \otimes [H]$ or $[F] \otimes [1]$. Further, taking X to be a point, we see from the identity $\mu\nu = 1$ that every K-class over S^2 is representable in the form $a[H] + b[1]$. Hence $\nu \cdot \mu$ is also 1.

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CLIFFORD MODULES

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INTRODUCTION

This paper developed in part from an earlier version by the last two authors. It is presented here, in its revised form, by the first two authors in memory of their friend and collaborator Arnold Shapiro.

The purpose of the paper is to undertake a detailed investigation of the role of Clifford algebras and spinors in the *KO*-theory of real vector bundles. On the one hand the use of Clifford algebras throws considerable light on the periodicity theorem for the stable orthogonal group. On the other hand the use of spinors seems essential in some of the finer points of the *KO*-theory which centre round the Thom isomorphism. As far as possible we have endeavoured to make this paper self-contained, assuming only a knowledge of the basic facts of *K*- and *KO*-theory, such as can be found in [3]. In particular we develop the theory of Clifford algebras from scratch. The paper is divided into three parts.

Part I is entirely algebraic and is the study of Clifford algebras. This contains nothing essentially new, though we formulate the results in a novel way. Moreover the treatment given in §§ 1–3 differs slightly from the standard approach: our Clifford group (Definition (3.1)) is defined via a 'twisted' adjoint representation. This twisting, which is a natural consequence of our emphasis on the *grading*, leads, we believe, to a simplification of the algebra. On the group level our definitions give rise in a natural way to a group† Pin(k) which double covers $O(k)$ and whose connected component Spin(k) double covers $SO(k)$. This group is very convenient for the topological considerations of §§ 13 and 14. In § 4 we determine the structure of the Clifford algebras and express the results in Table 1. The basic algebraic periodicity (8 in the real case, 2 in the complex case) appears at this stage. In § 5 we study Clifford modules, i.e. representations of the Clifford algebras. We introduce certain groups A_k, defined in terms of Grothendieck groups of Clifford modules, and tabulate the results in Table 2. In § 6, using tensor products, we turn $A_* = \sum_{k\geqslant 0} \mathrm{A}_k$ into a graded ring and determine its structure. These groups A_k are an algebraic counterpart of the homotopy groups of the stable orthogonal group, as will be shown in Part III.

Part II, which is independent of Part I, is concerned essentially with the 'difference bundle' construction in *K*-theory. We give a new and more complete treatment of this topic

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† This joke is due to J-P. Serre.

(see [4] and [7] for earlier versions) which includes a Grothendieck-type definition of the relative groups $K(X, Y)$ (Proposition (9.1)) and a product formula for difference bundles (Propositions (10.3) and (10.4)).

In Part III we combine the algebra of Part I with the topology of Part II. We define in § 11 a basic homomorphism

$$\alpha_P : A_k \to \widetilde{KO}(X^V)$$

where P is a principal Spin(k)-bundle over X, $V = P \times_{\text{Spin}(k)} R^k$, and X^V is the Thom complex of V. One of our main results is a product formula for α_P (Proposition (11.3)). Applying this in the case when X is a point gives rise to a ring homomorphism

$$\alpha : A_* \to \sum_{k \geqslant 0} KO^{-k}(\text{point}).$$

Using the periodicity theorem for the stable orthogonal group, as refined in [6], we then verify that α is an isomorphism (Theorem (11.5)). It is this theorem which shows the significance of Clifford algebras in K-theory and it strongly suggests that one should look for a *proof* of the periodicity theorem using Clifford algebras. Since this paper was written a proof on these lines has in fact been found by R. Wood†. It is to be hoped that Theorem (11.5) can be given a more natural and less computational proof.

Using α_P for general X gives us the Thom isomorphism (Theorem (12.3)) in a very precise form. Moreover the product formula for α_P asserts that the 'fundamental class' is multiplicative—just as in ordinary cohomology theory. Developing such a Thom isomorphism with all the good properties was one of our main aims. The treatment we have given is, we claim, more elementary, as well as more complete, than earlier versions which involved heavy use of characteristic classes.

In [7] another approach to the Thom isomorphism is given which has certain advantages over that given here. On the other hand the multiplicative property of the fundamental class does not come out of the method in [7]. To be able to use the advantages of both methods it is therefore necessary to identify the fundamental classes given in the two cases. This is done in §§ 13 and 14.

Finally in § 15 we discuss some other geometrical interpretations of Clifford modules. These throw considerable light on the vector-field problem for spheres.

Although the main interest in this paper lies in the KO-theory, most of what we do applies equally well in the complex case. It is one of the features of the Clifford module approach that the real and complex cases can be treated simultaneously.

PART I

§1. Notation

Let k be a commutative field and let Q be a quadratic form on the k-module E. Let $T(E) = \sum_{i=0}^{\infty} T^iE = k \oplus E \oplus E \otimes E \oplus \ldots$ be the tensor algebra over E, and let $I(Q)$ be the two-sided ideal generated by the elements $x \otimes x - Q(x) \cdot 1$ in $T(E)$. The quotient algebra

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† See also the proof given in: J. MILNOR: Morse Theory, *Ann. Math. Stud.* **51**, (1963).

$T(E)/I(Q)$ is called the Clifford algebra of Q and is denoted by $C(Q)$. We also define $i_Q : E \to C(Q)$ to be the canonical map given by the composition $E \to T(E) \to C(Q)$. Then the following propositions relative to $C(Q)$ are not difficult to verify:

(1.1) $i_Q : E \to C(Q)$ is an injection.

(1.2) Let $\phi : E \to A$ be a linear map of E into a k-algebra with unit A, such that for all $x \in E$, the identity $\phi(x)^2 = Q(x)1$ is valid. Then there exists a unique homomorphism $\tilde{\phi} : C(Q) \to B$, such that $\tilde{\phi} \cdot i_Q = \phi$. (We refer to $\tilde{\phi}$ as the 'extension' of ϕ.)

(1.3) $C(Q)$ is the universal algebra with respect to maps of the type described in (1.2).

(1.4) Let $F^qT(E) = \sum_{i \leqslant q} T^iE$ be the filtered structure in $T(E)$. This filtering induces a filtering in $C(E)$, whose associated graded algebra is isomorphic to the exterior algebra ΛE, on E. Thus $\dim_k C(Q) = 2^{\dim E}$, and if $\{e_i\}$ $(i = 1, \ldots, n)$ is a base for $i_Q(E)$, then 1 together with the products $e_{i_1} \cdot e_{i_2} \ldots e_{i_k}$, $i_1 < i_2 < \ldots < i_k$, form a base for $C(Q)$.

(1.5) Let $C^0(Q)$ be the image of $\sum_{i=0}^{i=\infty} T^{2i}(E)$ in $C(Q)$ and set $C^1(Q)$ equal to the image of $\sum_0^\infty T^{2i+1}(E)$ in $C(Q)$. Then this decomposition defines $C(Q)$ as a Z_2-graded algebra. That is:

(*a*) $C(Q) = \sum_{i=0,1} C^i(Q)$;

(*b*) If $x_i \in C^i(Q)$, $y_j \in C^j(Q)$, then

$$x_iy_j \in C^k(Q), \qquad k \equiv i + j \bmod 2.$$

That the graded structure of $C(Q)$ should not be disregarded is maybe best brought out by the following:

PROPOSITION (1.6). *Suppose that $E = E_1 \oplus E_2$ is an orthogonal decomposition of E relative to Q, and let Q_i denote the restriction of Q to E_i. Then there is an isomorphism*

$$\psi : C(Q) \cong C(Q_1) \underset{k}{\hat{\otimes}} C(Q_2)$$

of the graded tensor-product of $C(Q_1)$ and $C(Q_2)$ with $C(Q)$.

Recall first, that the graded tensor product of two graded algebras $A = \sum_{\alpha=0,1} A^\alpha$, $B = \sum_{\alpha=0,1} B^\alpha$, is by definition the algebra whose underlying vector space is $\sum_{\alpha,\beta=0,1} A^\alpha \otimes B^\beta$, with multiplication defined by:

$$(u \otimes x_i) \cdot (y_j \otimes v) = (-1)^{ij} uy_j \otimes x_iv, \; x_i \in C^i(Q), \; y_j \in C^j(Q).$$

This graded tensor product is denoted by $A \hat{\otimes} B$; and is again a graded algebra:

$$(A \hat{\otimes} B)^k = \sum A^i \otimes B^j \quad (i + j \equiv k(2)).$$

Proof of the proposition. Define $\psi :\; E \to C(Q_1) \underset{k}{\hat{\otimes}} C(Q_2)$ by the formula, $\psi(e) = e_1 \otimes 1 + 1 \otimes e_2$, where e_1 and e_2 are the orthogonal projections of e on E_1 and E_2. Then

$$\psi(e)^2 = (e_1 \otimes 1 + 1 \otimes e_2)^2 = \{Q_1(e_1) + Q_2(e_2)\}(1 \otimes 1) = Q(e)(1 \otimes 1).$$

Hence ψ extends to an algebra homomorphism $\psi : C(Q) \to C(Q_1) \hat{\otimes} C(Q_2)$, by (1.2). Checking the behavior of ψ on basis elements now shows that ψ is a bijection. Note that the graded structure entered through the formula $(e_1 \otimes 1 + 1 \otimes e_2)^2 = e_1^2 \otimes 1 + 1 \otimes e_2^2$ which is valid as $e_i \in C^1(Q_i)$.

The algebra $C(Q)$ also inherits a canonical antiautomorphism from the tensor algebra $T(E)$. Namely if $x = x_1 \otimes x_2 \dots \otimes x_k \in T^k(E)$, then the map $x \to x^t$, given by

$$x_1 \otimes x_2 \otimes \dots \otimes x_k \to x_k \otimes \dots \otimes x_2 \otimes x_1$$

clearly defines an antiautomorphism of $T(E)$, which preserves $I(Q)$ because $\{x \otimes x - Q(x) \cdot 1\}^t = x \otimes x - Q(x) \cdot 1$. Hence this operation induces a well defined antiautomorphism on $C(Q)$ which we also denote by $x \to x^t$ and refer to as the transpose. The transpose is the identity map on $i_Q(E) \subset C(Q)$.

The following two operations on $C(Q)$ will also be useful:

Definition (1.7). *The canonical automorphism of $C(Q)$ is defined as the 'extension' of the map $\alpha : E \to C(Q)$, given by $\alpha(x) = -i_Q(x)$.* (It is clear that $\{\alpha(x)\}^2 = Q(x)1$ and so α is well-defined by (1.1)). We denote this automorphism by α.

Definition (1.8). *Let $x \to \bar{x}$ be defined by the formula $x \to \alpha(x^t)$. This 'bar operation' is then an antiautomorphism of $C(Q)$.*

Note. (1) The identity $\alpha(x^t) = \{\alpha(x)\}^t$ holds as both are antiautomorphisms which extend the map $E \to C(Q)$ given by $x \to -i_Q(x)$;

(2) The grading on $C(Q)$ may be defined in terms of α: $C^i(Q) = \{x \in C(Q) | \alpha(x) = (-1)^i x\}$, $i = 0, 1$.

§2. The algebras C_k

We are interested in the algebras $C(Q_k)$, where Q_k is a negative definite form on k-space over the real numbers. Quite specifically, we let $\mathbf{R}^k$ denote the space of k-tuples of real numbers, and define $Q_k(x_1, \dots, x_k) = -\sum x_i^2$. Then we define C_k as the algebra $C(Q_k)$ and identify $\mathbf{R}^k$ with $i_{Q_k}\mathbf{R}^k \subset C_k$ and $\mathbf{R}$ with $\mathbf{R} \cdot 1 \subset C_k$. For $k = 0$, $C_k = \mathbf{R}$.

Proposition (2.1). *The algebra C_1 is isomorphic to $\mathbf{C}$ (the complex numbers) considered as an algebra over $\mathbf{R}$. Further*

$$C_k \cong C_1 \hat{\otimes} C_1 \hat{\otimes} \dots \hat{\otimes} C_1 \qquad (k \textit{ factors}).$$

Clearly C_1 is generated by 1 and e_1, where 1 denotes the real number 1 in $\mathbf{R}^1$. Hence $e_1^2 = -1$. The formula $C_k \cong C_1 \hat{\otimes} \dots \hat{\otimes} C_1$ now follows from repeated application of Proposition (1.6).

We will denote the k-tuple, $(0, \dots, 1, \dots, 0)$ with 1 in the ith position by e_i. The e_i, $i \leqslant k$ then form a base of $\mathbf{R}^k \subset C_k$.

Corollary (2.2). *The e_i, $i = 1, \dots, k$, generate C_k multiplicatively and satisfy the relations*

$$e_j^2 = -1, \qquad e_i e_j + e_j e_i = 0, \qquad i \neq j. \tag{2.3}$$

C_k may be identified with the universal algebra generated over $\mathbf{R}$ by a unit, 1, and the symbols e_i, $i = 1, \dots, k$, subject to the relations (2.3).

§3. The groups, Γ_k, Pin(k), and Spin (k)

Let C_k^* denote the multiplicative group of invertible elements in C_k.

Definition (3.1). *The Clifford group Γ_k is the subgroup of those elements $x \in C_k^*$ for which $y \in \mathbf{R}^k$ implies $\alpha(x)yx^{-1} \in \mathbf{R}^k$.*

It is clear enough that Γ_k is a subgroup of C_k, because α is an automorphism. We also write $\alpha(x)\mathbf{R}^k x^{-1} \subset \mathbf{R}^k$ for the condition defining Γ_k. As α and the transpose map $\mathbf{R}^k$ into itself, it is then also evident that we have:

Proposition (3.2). *The maps $x \to \alpha(x)$, $x \to x^t$ preserve Γ_k, and respectively induce an automorphism and an antiautomorphism of Γ_k. Hence $x \to \bar{x}$ is also an antiautomorphism of Γ_k.*

The group Γ_k comes to us with a ready-made homomorphism $\rho : \Gamma_k \to \mathrm{Aut}(\mathbf{R}^k)$. By definition $\rho(x)$, for $x \in \Gamma_k$, is the linear map $\mathbf{R}^k \to \mathbf{R}^k$ given by $\rho(x) \cdot y = \alpha(x)yx^{-1}$. We refer to ρ as the *twisted adjoint representation* of Γ_k on $\mathbf{R}^k$. This representation ρ turns out to be nearly faithful.

Proposition (3.3). *The kernel of $\rho : \Gamma_k \to \mathrm{Aut}(\mathbf{R}^k)$ is precisely $\mathbf{R}^*$, the multiplicative group of nonzero multiples of $1 \in C_k$.*

Proof. Suppose $x \in \mathrm{Ker}(\rho)$. This implies

$$\alpha(x)y = yx \qquad \text{for all } y \in \mathbf{R}^k. \tag{3.4}$$

Write $x = x^0 + x^1$, $x^i \in C_k^i$. Then (3.4) becomes

$$x^0 y = yx^0 \tag{3.5}$$

$$x^1 y = -yx^1. \tag{3.6}$$

Let $e_1, \dots, e_k$ be our orthonormal base for $\mathbf{R}^k$, and write $x^0 = a^0 + e_1 b^1$ in terms of this basis. Here $a^0 \in C_k^0$ does not involve e_1 and $b^1 \in C_k^1$ does not involve e_1. By setting $y = e_1$ in (3.5) we get $a^0 + e_1 b^1 = e_1 a^0 e_1^{-1} + e_1^2 b^1 e_1^{-1} = a_0 - e_1 b^1$. Hence $b^1 = 0$. That is, the expansion of x^0 does not involve e_1. Applying the same argument with the other basis elements we see that x^0 does not involve any of them. Hence x^0 is a multiple of 1. Next we write x^1 in the same form: $x^1 = a^1 + e_1 b^0$ and set $y = e_1$. We then obtain $a^1 + e_1 b^0 = -\{e_1 a^1 e_1^{-1} + e_1^2 b^0 e_1^{-1}\} = a^1 - e_1 b^0$. We again conclude that x^1 does not involve the e_i. Hence x^1 is a multiple of 1. On the other hand $x^1 \in C_k^1$ whence $x^1 = 0$. This proves that $x = x_0 \in \mathbf{R}$ and as x is invertible $x \in \mathbf{R}^*$.

Q.E.D.

Consider now the function $N : C_k \to C_k$ defined by

$$N(x) = x \cdot \bar{x}. \tag{3.7}$$

If $x \in \mathbf{R}^k$, then $N(x) = x(-x) = -x^2 = -Q_k(x)$. Thus $N(x)$ is the square of the length in $\mathbf{R}^k$ relative to the positive definite form $-Q_k$.

Proposition (3.8). *If $x \in \Gamma_k$ then $N(x) \in \mathbf{R}^*$.*

Proof. We show that $N(x)$ is in the kernel of ρ. Let then $x \in \Gamma_k$, whence for every $y \in \mathbf{R}^k$ we have

$$\alpha(x)yx^{-1} = y', \qquad y' = \rho(x)y \in \mathbf{R}^k.$$

Applying the transpose we obtain: (as $y^t = y$)

$$(x^t)^{-1} y \alpha(x)^t = \alpha(x)yx^{-1}$$

whence $y\alpha(x^t)x = x^t\alpha(x)y$. This implies that $\alpha(x^t)x$ is in the kernel of ρ, and hence in $\mathbf{R}^*$ by (3.3). It follows that $x^t\alpha(x) \in \mathbf{R}^*$, whence $N(x^t) \in \mathbf{R}^*$. However $x \to x^t$ is an antiautomorphism of Γ_k, by (3.2). Hence $N(\Gamma_k) \subset \mathbf{R}^*$.

PROPOSITION (3.9). *$N : \Gamma_k \to \mathbf{R}^*$ is a homomorphism. Moreover $N(\alpha x) = N(x)$.*

Proof. $N(xy) = xy\,\bar{y}\bar{x} = xN(y)\bar{x} = N(x)\cdot N(y)$, $N(\alpha(x)) = \alpha(x)x^t = \alpha N(x) = N(x)$.

PROPOSITION (3.10). *$\rho(\Gamma_k)$ is contained in the group of isometries of $\mathbf{R}^k$.*

Proof. Using (3.9) and the fact that $\mathbf{R}^k - \{0\} \subset \Gamma_k$ we have

$$N(\rho(x)\cdot y) = N(\alpha(x)y\,x^{-1}) = N(\alpha(x))N(y)N(x^{-1}) = N(y).$$

Q.E.D.

THEOREM (3.11). *Let* Pin(k) *be the kernel of $N : \Gamma_k \to \mathbf{R}^*$, $k \geqslant 1$, and let $O(k)$ denote the group of isometries of $\mathbf{R}^k$. Then $\rho|$*Pin(k) *is a surjection of* Pin(k) *onto $O(k)$ with kernel Z_2, generated by $-1 \in \Gamma_k$. We thus have the exact sequence*

$$1 \to Z_2 \to \mathrm{Pin}(k) \xrightarrow{\rho} O(k) \to 1.$$

Proof. We show first that ρ is onto. For this purpose consider $e_1 \in R^k$. We have $N(e_1) = -e_1e_1 = +1$, and

$$\alpha(e_1)e_ie_1^{-1} = \begin{cases} -e_i & \text{if } \ i = 1 \\ \ \ e_i & \text{if } \ i \neq 1. \end{cases}$$

Thus $e_1 \in$ Pin(k), and $\rho(e_1)$ is the reflection in the hyperplane perpendicular to e_1. Applying the same argument to any orthonormal base $\{e_i\}$ in $\mathbf{R}^k$, we see that the unit sphere

$$\{x \in \mathbf{R}^k | N(x) = 1\}$$

is in Pin(k) whence all the orthogonal reflections in hyperplanes of $\mathbf{R}^k$ are in $\rho\{\mathrm{Pin}(k)\}$. But these are well known to generate $O(k)$. Thus ρ maps Pin(k) onto $O(k)$. Consider next the kernel of this map, which clearly consists of the intersection Ker $\rho \cap \{N(x) = 1\}$. Thus the kernel of $\rho|$Pin(k) consists of the multiples $\lambda\cdot 1$, with $N(\lambda 1) = 1$. Thus $\lambda^2 = +1$ which implies $\lambda = \pm 1$.

DEFINITION (3.12). *For $k \geqslant 1$ let* Spin(k) *be the subgroup of* Pin(k) *which maps onto $SO(k)$ under ρ.*

The groups Pin(k) and Spin(k) are double coverings of $O(K)$ and $SO(k)$ respectively. As such they inherit the Lie-structure of the latter groups. One may also show that these groups are closed subgroups of C_k^* and get at their Lie structure in this way.

PROPOSITION (3.13). *Let* $\mathrm{Pin}(k)^i = \mathrm{Pin}(k) \cap C_k^i$. *Then* $\mathrm{Pin}(k) = \cup_{i=0,1} \mathrm{Pin}(k)^i$, *and* $\mathrm{Spin}(k) = \mathrm{Pin}(k)^0$.

Proof. Let $x \in$ Pin(k). Then $\rho(x)$ is equal to the composition of a certain number of reflections in hyperplanes: $\rho(x) = R_1 \circ \dots \circ R_n$. We may choose elements $x_i \in \mathbf{R}^k$, such that $\rho(x_i) = R_i$. Hence, by (3.11), $x = \pm x_1x_2 \dots x_n$ and is therefore either in C_k^0 or in C_k^1. Finally x is in Spin(k) if and only if the number n in the above decomposition of $\rho(x)$ is even, i.e. if and only if $x \in \mathrm{Pin}(k)^0$.

PROPOSITION (3.14). *When $k \geqslant 2$, the restriction of ρ to* Spin(k) *is the nontrivial double covering of $SO(k)$.*

Proof. It is sufficient to show that $+1$, -1, the kernel of $\rho|$Spin(k), can be connected by an arc in Spin(k). Such an arc is given by:

$$\lambda : t \to \cos t + \sin t\cdot e_1e_2 \qquad 0 \leqslant t \leqslant \pi.$$

COROLLARY (3.15). *When $k \geqslant 2$,* Spin(k) *is connected and, when $k \geqslant 3$, simply-connected.*

This is clear from the fact that $SO(k)$ is connected for $k \geqslant 2$, and that $\pi_1\{SO(k)\} = Z_2$ if $k \geqslant 3$.

We note finally that Spin$(1) = Z_2$, while Pin$(1) = Z_4$.

All the preceding discussion can be extended to the complex case. We define α, t on $C_k \otimes_{\mathbf{R}} \mathbf{C}$ by

$$\alpha(x \otimes z) = \alpha(x) \otimes z$$

$$(x \otimes z)^t = x^t \otimes \bar{z}$$

and we take the bar operation and N to be defined in terms of α, t as before.

DEFINITION (3.16). *Γ_k^c is the subgroup of invertible elements $x \in C_k \otimes_{\mathbf{R}} \mathbf{C}$ for which $y \in \mathbf{R}^k$ implies $\alpha(x)yx^{-1} \in \mathbf{R}^k$.*

Propositions (3.2)–(3.10) go through with $\mathbf{R}^*$ replaced by $\mathbf{C}^*$ and (3.11) becomes:

THEOREM (3.17). *Let* Pin$^c(k)$ *be the kernel of $N : \Gamma_k^c \to \mathbf{C}^*$, $k \geqslant 1$, then we have an exact sequence:*

(3.18) $$1 \to U(1) \to \mathrm{Pin}^c(k) \to O(k) \to 1$$

where $U(1)$ is the subgroup consisting of elements $1 \otimes z \in C_k \otimes_{\mathbf{R}} \mathbf{C}$ with $|z| = 1$.

COROLLARY (3.19). *We have a natural isomorphism*

$$\mathrm{Pin}(k) \times_{Z_2} U(1) \to \mathrm{Pin}^c(k),$$

where Z_2 acts on Pin(k) *and $U(1)$ as $\{\pm 1\}$.*

Proof. The inclusions Pin$(k) \subset C_k$, $U(1) \subset \mathbf{C}$ induce an inclusion

$$\mathrm{Pin}(k) \times_{Z_2} U(1) \to C_k \otimes_{\mathbf{R}} \mathbf{C},$$

and it follows from the definitions that this factors through a homomorphism:

$$\psi : \mathrm{Pin}(k) \times_{Z_2} U(1) \to \mathrm{Pin}^c(k).$$

Now we have an obvious exact sequence

(3.20) $$0 \to U(1) \to \mathrm{Pin}(k) \times_{Z_2} U(1) \to \mathrm{Pin}(k)/_{Z_2} \to 1$$

and ψ induces a homomorphism of (3.20) into (3.18). The 5-lemma and (3.11) now complete the proof.

We define Spin$^c(k)$ as the inverse image of $SO(k)$ in the homomorphism

$$\mathrm{Pin}^c(k) \to O(k).$$

Then from (3.19) we have

$$\mathrm{Spin}^c(k) \cong \mathrm{Spin}(k) \times_{Z_2} U(1).$$

The groups Spin$^c(k)$ are particularly relevant to an understanding of the relationship between spinors and complex structure, as we proceed to explain. The natural homomorphism

$$j : U(k) \to SO(2k)$$

does not lift to Spin$(2k)$, as one easily verifies. However the homomorphism

$$l : U(k) \to SO(2k) \times U(1)$$

defined by

$$l(T) = j(T) \times \det T$$

does lift to Spin$^c(2k)$. This follows at once from elementary topological considerations and the fact that

$$\det : U(k) \to U(1)$$

induces an isomorphism of fundamental groups.

Explicitly the lifted map

$$\tilde{l} : U(k) \to \mathrm{Spin}^c(2k)$$

is given as follows. Let $T \in U(k)$ be expressed, relative to an orthonormal base $f_1, \ldots, f_k$ of $\mathbf{C}^k$, by the diagonal matrix

$$\begin{pmatrix} \exp it_1 & & & \\ & \exp it_2 & & \\ & & \ddots & \\ & & & \exp it_k \end{pmatrix}$$

Let $e_1, \ldots, e_{2k}$ be the corresponding base of $\mathbf{R}^{2k}$, so that

$$e_{2j-1} = f_j \qquad e_{2j} = if_j$$

Then

$$\tilde{l}(T) = \prod_{j=1}^{k} \left(\cos t_j/2 + \sin t_j/2 \cdot e_{2j-1}e_{2j}\right) \times \exp\left(\frac{i\Sigma t_j}{2}\right).$$

§4. Determination of the algebras C_k

In the following we will write $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ respectively for the real, complex and quarternion number-fields. If F is any one of these fields, $F(n)$ will be the full $n \times n$ matrix algebra over F. The following are well known identities among these:

$$(4.1) \qquad \begin{cases} F(n) \cong \mathbf{R}(n) \otimes_{\mathbf{R}} F,\ \mathbf{R}(n) \otimes_{\mathbf{R}} \mathbf{R}(m) \cong \mathbf{R}(nm) \\ \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C} \\ \mathbf{H} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C}(2) \\ \mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathbf{R}(4). \end{cases}$$

To compute the algebras C_k one now proceeds as follows: Let C'_k be the universal $\mathbf{R}$-algebra generated by a unit and the symbols e'_i $(i = 1, \ldots, k)$ subject to the relations $(e'_i)^2 = +1$; $e'_ie'_j + e'_je'_i = 0$, $i \neq j$. Thus C'_k may be identified with $C(-Q_k)$.

Proposition (4.2). *There exist isomorphisms:*

$$(4.3) \qquad \begin{aligned} C_k \otimes_{\mathbf{R}} C'_2 &\cong C'_{k+2} \\ C'_k \otimes_{\mathbf{R}} C_2 &\cong C_{k+2}. \end{aligned}$$

Proof. Denote by R'^k the space spanned by the e'_i in C'_k.

Consider the linear map $\psi : R'^{k+2} \to C_k \otimes C'_2$ defined by

$$\psi(e'_i) = \begin{cases} e_{i-2} \otimes e'_1e'_2 & 2 \leqslant i \leqslant k \\ 1 \otimes e'_i & 1 \leqslant i \leqslant 2. \end{cases}$$

Then it is easily seen that ψ satisfies the universal property (1.1) for C'_k and hence extends to an algebra homomorphism $\psi : C'_{k+2} \to C_k \otimes C'_2$. As the map takes basis elements into basis elements and the spaces in question have equal dimension, it follows that ψ is a bijection. If we now replace the dashed symbols by the undashed ones and apply the same argument we obtain the second isomorphism.

Now it is clear that

$$\begin{aligned} C_1 &\cong \mathbf{C}, \qquad & C'_1 &\cong \mathbf{R} \oplus \mathbf{R} \\ C_2 &\cong \mathbf{H}, \qquad & C'_2 &\cong \mathbf{R}(2). \end{aligned}$$

Hence (4.1) and repeated application of (4.3) yields the following table:

Table 1

k	C_k	C'_k	$C_k \otimes_{\mathbf{R}} \mathbf{C} = C'_k \otimes_{\mathbf{R}} \mathbf{C}$
1	$\mathbf{C}$	$\mathbf{R} \oplus \mathbf{R}$	$\mathbf{C} \oplus \mathbf{C}$
2	$\mathbf{H}$	$\mathbf{R}(2)$	$\mathbf{C}(2)$
3	$\mathbf{H} \oplus \mathbf{H}$	$\mathbf{C}(2)$	$\mathbf{C}(2) \oplus \mathbf{C}(2)$
4	$\mathbf{H}(2)$	$\mathbf{H}(2)$	$\mathbf{C}(4)$
5	$\mathbf{C}(4)$	$\mathbf{H}(2) \oplus \mathbf{H}(2)$	$\mathbf{C}(4) \oplus \mathbf{C}(4)$
6	$\mathbf{R}(8)$	$\mathbf{H}(4)$	$\mathbf{C}(8)$
7	$\mathbf{R}(8) \oplus \mathbf{R}(8)$	$\mathbf{C}(8)$	$\mathbf{C}(8) \oplus \mathbf{C}(8)$
8	$\mathbf{R}(16)$	$\mathbf{R}(16)$	$\mathbf{C}(16)$

Note that (4.2) implies $C_4 \cong C'_4$; $C_{k+4} \cong C_k \otimes C_4$; $C_{k+8} \cong C_k \otimes C_8$; further $C_8 \cong \mathbf{R}(16)$, whence if $C_k \cong F(m)$ then, $C_{k+8} \cong F(16m)$. Thus both columns are in a quite definite sense of period 8. If we move up eight steps, the field is left unaltered, while the dimension is multiplied by 16. Note also the considerably simpler behavior of the complexifications of these algebras, which of course can be interpreted as the Clifford algebra of Q_k over the complex-numbers. Over the complex field, the period is 2.

§5. Clifford modules

We will now describe the set of $\mathbf{R}$– and $\mathbf{C}$– modules for the algebras C_k. We write $M(C_k)$ for the free abelian group generated by the irreducible Z_2-graded C_k-modules, and $N(C_k^0)$ for the corresponding group generated by the (ungraded) C_k^0-modules. The corresponding objects for the complex algebras $C_k \otimes_{\mathbf{R}} \mathbf{C}$ are denoted by $M^c(C_k)$ and $N^c(C_k^0)$.

Proposition (5.1). *Let $R : M \mapsto M^0$ be the functor which assigns to a graded C_k-module $M = M^0 \oplus M^1$ the C_k^0-module M^0. Then R induces isomorphisms*

$$(5.2) \qquad M(C_k) \cong N(C_k^0).$$

Proof. If M^0 is a C_k^0-module, let

$$S(M^0) = C_k \otimes_{C_k^0} M^0.$$

The left action of C_k on C_k then defines $S(M^0)$ as a graded C_k-module. We now assert that $S \circ R$ and $R \circ S$ are naturally isomorphic to the identity. In the first case the isomorphism is induced by the 'module-map' $C_k \otimes M^0 \to M$, while in the second case the map $M^0 \to 1 \otimes M^0$ induces the isomorphism.

We of course also have the corresponding formula:

$$M^c(C_k) \cong N^c(C_k^0). \tag{5.3}$$

PROPOSITION (5.4). *Let $\phi : \mathbf{R}^k \to C_{k+1}^0$ be defined by $\phi(e_i) = e_i e_{k+1}$, $i = 1, \ldots, k$. Then ϕ extends to yield an isomorphism $C_k \cong C_{k+1}^0$.*

Proof. $\phi(e_i)^2 = e_i e_{k+1} e_i e_{k+1} = -1$. Hence ϕ extends. As it maps distinct basis elements onto distinct basis elements the extension is an isomorphism.

In view of these two propositions and Table 1, we may now write down the group $M(C_k)$ etc., explicitly. This is done in Table 2, where we also tabulate the following quantities:

Let $i : C_k \to C_{k+1}$ be the inclusion which extends the inclusion $\mathbf{R}^k \to \mathbf{R}^{k+1}$, let $i^* : M(C_{k+1}) \to M(C_k)$ be the induced homomorphism, and set $A_k =$ cokernel of i^*. Similarly define A_k^c as $M^c(C_k)/i^*\{M^c(C_{k+1})\}$ and finally define $a_k[a_k^c]$ as the $\mathbf{R}[\mathbf{C}]$-dimension of M^0 when M is an irreducible graded module for $C_k[C_k \otimes_{\mathbf{R}} \mathbf{C}]$.

TABLE 2

k	C_k	$M(C_k)$	A_k	a_k	$M^c(C_k)$	A_k^c	a_k^c
1	$\mathbf{C}(1)$	$\mathbf{Z}$	Z_2	1	$\mathbf{Z}$	0	1
2	$\mathbf{H}(1)$	$\mathbf{Z}$	Z_2	2	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	1
3	$\mathbf{H}(1) \oplus \mathbf{H}(1)$	$\mathbf{Z}$	0	4	$\mathbf{Z}$	0	2
4	$\mathbf{H}(2)$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	4	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	2
5	$\mathbf{C}(4)$	$\mathbf{Z}$	0	8	$\mathbf{Z}$	0	4
6	$\mathbf{R}(8)$	$\mathbf{Z}$	0	8	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	4
7	$\mathbf{R}(8) \oplus \mathbf{R}(8)$	$\mathbf{Z}$	0	8	$\mathbf{Z}$	0	8
8	$\mathbf{R}(16)$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	8	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	8

$$M_{k+8} \cong M_k, \qquad A_{k+8} \cong A_k, \qquad a_{k+8} = 16a_k$$
$$M_{k+2}^c \cong M_k^c, \qquad A_{k+2}^c \cong A_k^c, \qquad a_{k+2}^c = 2a_k^c.$$

Most of the entries in Table 2 follow directly from Table 1, because the algebras $F(n)$ are simple and hence have only one class of irreducible modules, the one given by the action of $F(n)$ on the n-tuples of elements in F. The only entries which still need clarification are therefore A_{4n} and A_{2n}^c.

Before explaining these entries observe that if $M = M^0 \oplus M^1$, then $M^* = M^1 \oplus M^0$. i.e. the module obtained from M by merely interchanging labels, is again a graded module. This operation therefore induces an involution on $M(C_k)$ and $M^c(C_k)$ which we again denote by $*$.

PROPOSITION (5.5). *Let x and y be the classes of the two distinct irreducible graded modules in $M(C_{4n})$. Then*

$$x^* = y, \qquad y^* = x. \tag{5.6}$$

COROLLARY (5.7). $A_{4n} \cong \mathbf{Z}$.

Indeed if z generates $M(C_{4n+1})$, then $z^* = z$ as there is only one irreducible graded module for C_{4n+1}. Hence as $(i^*z)^* = i^*(z^*)$ we see that $i^*z = x + y$, by a dimension count.

To prove (5.5) we require the following lemma which is quite straight-forward and will be left to the reader.

LEMMA (5.8). *Let $y \in \mathbf{R}^k$, $y \neq 0$ and denote by $A(y)$ the inner automorphism of C_k induced by y. Thus $A(y) \cdot w = ywy^{-1}$. We also write $A(y)$ for the induced automorphism on $M(C_k)$. Similarly $A^0(y)$ denotes the restriction of $A(y)$ to C_k^0, as well as the induced automorphism on $N(C_k^0)$. Then we have*

$$\begin{aligned} A(y) \cdot x &= x^* \\ A^0(y) \cdot R(x) &= R(x^*), \qquad x \in M(C_k) \\ A^0(e_k)\phi(w) &= \phi\{\alpha(w)\}. \end{aligned} \tag{5.9}$$

Here $R : M(C_k) \mapsto N(C_k^0)$ is the functor introduced earlier, and $\phi : C_{k-1} \to C_k$, the map introduced in (5.4), while α is the canonical automorphism of C_k.

It now follows from these isomorphisms, that $*$ on $M(C_{4n})$ corresponds to the action of α on the ungraded modules of C_{4n-1}. Now the centre of C_{4n-1} is spanned by 1 and $w = e_1 e_2 \ldots e_{4n-1}$. Further $w^2 = +1$. Hence the projections of C_{4n-1} on the two ideals which make up C_{4n-1} are $(1 + w)/2$ and $(1 - w)/2$. Hence α interchanges these, and therefore clearly interchanges the two irreducible C_{4n-1} modules.

Finally, the evaluation $A_{2n}^c \cong \mathbf{Z}$ proceeds in an entirely analogous fashion.

Actually in the complex case there is a relation with Grassmann algebras which we shall now describe. Give $\mathbf{C}^k$ the standard Hermitian metric. Then the complex Grassmann algebra

$$\Lambda(\mathbf{C}^k) = \sum_{j=0}^{k} \Lambda^j(\mathbf{C}^k)$$

inherits a natural metric. In terms of an orthonormal basis $f_1, \ldots, f_k$ of $\mathbf{C}^k$ the elements

$$f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_k} \qquad i_1 < i_2 < \ldots < i_k$$

form an orthonormal basis of $\Lambda(\mathbf{C}^k)$. For each $v \in \mathbf{C}^k$ let d_v denote the (vector space) endomorphism of $\Lambda(\mathbf{C}^k)$ given by the exterior product:

$$d_v(w) = v \wedge w,$$

and let δ_v denote its adjoint with respect to the metric. We now define a pairing

$$\mathbf{C}^k \otimes_{\mathbf{R}} \Lambda(\mathbf{C}^k) \to \Lambda(\mathbf{C}^k) \tag{5.10}$$

by

$$v \otimes w \to d_v(w) - \delta_v(w).$$

One verifies that

$$(d_v - \delta_v)^2 w = -\|v\|^2 w$$

so that (5.10) makes $\Lambda(\mathbf{C}^k)$ into a complex module for the Clifford algebra C_{2k} (identifying $\mathbf{C}^k$ with $\mathbf{R}^{2k}$ as usual) i.e. into a module for $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$. Moreover $\Lambda(\mathbf{C}^k)$ has a natural Z_2-grading

$$\Lambda^0 = \sum \Lambda^{2r}$$
$$\Lambda^1 = \sum \Lambda^{2r+1}$$

compatible with (5.10). A dimension count then shows that $\Lambda(\mathbf{C}^k)$ must be one of the two irreducible Z_2-graded modules for $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$. Now if $u = iv$ we see that

$$(d_v - \delta_v)(d_u - \delta_u)(1) = -i\|v\|^2(1) \qquad 1 \in \Lambda^0(\mathbf{C}^k).$$

Hence $\Lambda(\mathbf{C}^k)$ is a $(-i)^k$-module, and so we get

PROPOSITION (5.11). *$\Lambda(\mathbf{C}^k)$ is a graded $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$-module defining the class*

$$(-1)^k(\mu^c)^k \in A_k^c.$$

Remark. Using the explicit formula for $\bar{l} : U(k) \to \mathrm{Spin}^c(2k)$ given in §3 it is easy to verify the commutativity of the following diagram

$$\begin{tikzcd} U(k) \arrow[r, "\bar{l}"] \arrow[d, "i"'] & \mathrm{Spin}^c(2k) \arrow[d, "\sigma"] \\ \mathrm{End}(\mathbf{C}^k) \arrow[r, "\Lambda"] & \mathrm{End}(\Lambda(\mathbf{C}^k)) \end{tikzcd}$$

Here Λ is the functorial homomorphism, i is the inclusion and σ is the homomorphism induced by the action of $C_{2k} \otimes_{\mathbf{R}} \mathbf{C}$ on $\Lambda(\mathbf{C}^k)$ defined above.

§6. The multiplicative properties of the Clifford modules

If M and N are graded C_k and C_l modules, respectively, then their graded tensor product $M \hat{\otimes} N$ is in a natural way a graded module over $C_k \hat{\otimes} C_l$. By definition $(M \hat{\otimes} N)^0 = M^0 \otimes N^0 \oplus M^1 \otimes N^1$ and $(M \hat{\otimes} N)^1 = M^0 \otimes N^1 \oplus M^1 \otimes N^0$, the action of $C_k \hat{\otimes} C_l$ on $M \hat{\oplus} N$ being given by:

(6.1) $\quad (x \otimes y) \cdot (m \otimes n) = (-1)^{qi}(x \cdot m) \otimes (y \cdot n), \qquad y \in C_l^q, \qquad m \in M^i (q, i = 0, 1).$

We also have the isomorphism $\phi_{k,l} : C_{k+l} \to C_k \hat{\otimes} C_l$ defined by the linear extension of the map

$$\phi_{k,l}(e_i) = \begin{cases} e_i \otimes 1 & 1 \leqslant i \leqslant k \\ 1 \otimes e_{k+i} & k < i \leqslant k + l. \end{cases}$$

The operation $(M, N) \mapsto M \hat{\otimes} N \mapsto \phi_{k,l}^*(M \hat{\otimes} N)$ is easily seen to give rise to a pairing

$$M(C_k) \otimes_{\mathbf{Z}} M(C_l) \to M(C_{k+l})$$

and thus induces a $\mathbf{Z}$-graded ring structure on the direct sum $M_* = \sum_0^\infty M(C_k)$. We denote this product by $(u, v) \to u \cdot v$. It is clearly associative.

PROPOSITION (6.2). *The following formulae are valid for $u \in M(C_k)$, $v \in M(C_l)$*

(6.3) $$(u \cdot v)^* = u \cdot v^*$$

(6.4) $$u \cdot v = \begin{cases} v \cdot u & \text{if } kl \text{ is even} \\ (v \cdot u)^* & \text{if } kl \text{ is odd.} \end{cases}$$

(6.5) *If $i^* : M(C_k) \to M(C_{k-l})$ is the restriction homomorphism, as defined in §5, then*

$$u \cdot i^* v = i^*(u \cdot v) \qquad k \geqslant 1.$$

The formulae (6.3) and (6.5) follow immediately from the definitions.

Proof of (6.4). We have the diagram:

$$\begin{tikzcd} C_k \hat{\otimes} C_l \arrow[d, "T"'] & C_{k+l} \arrow[l, "\phi_{k,l}"'] \arrow[dl, "\phi_{l,k}"] \\ C_l \hat{\otimes} C_k & \end{tikzcd}$$

where T is the isomorphism $x \otimes y \to (-1)^{pq} y \otimes x$, $x \in C_k^p$, $y \in C_l^q$. Now the composition $\phi_{l,k}^{-1} \circ T \circ \phi_{k,l} : C_{k+l} \to C_{k+l}$ is an automorphism σ of C_{k+l}, which clearly is the linear extension of the map which permutes the first k elements of the basis $\{e_i\}$ with the last l elements

$$\sigma(e_i) = \begin{cases} e_{i+l} & 1 \leqslant i \leqslant k \\ e_{i-k} & k < i \leqslant k + l \end{cases}$$

Thus σ is the composition of inner automorphisms by elements in $\mathbf{R}^k - \{0\}$. It follows therefore from (5.9) that the effect of σ on $M(C_k)$ is equal to the effect of the operation (*) applied kl times. If we combine this with the fact that $T^*(N \hat{\otimes} M) \cong M \hat{\otimes} N$, whence

$$\phi_{k,l}^*(N \hat{\otimes} M) \cong \sigma^* \circ \phi_{l,k}^* \cdot (M \hat{\otimes} N),$$

we obtain the desired formula.

COROLLARY (6.6). *Let $\lambda \in M(C_8)$ be the class of an irreducible module of C_8. Then multiplication by λ induces an isomorphism: $M(C_k) \cong M(C_{k+8})$.*

Proof. This follows from our table of the a_k, in all cases except when $k = 4n$. In that case let x, y be the generators corresponding to the two irreducible graded modules of C_k. Then we know that $x^* = y$. Now $\lambda \cdot x \in M(C_{k+8})$ is the class of one of the irreducible graded modules of C_{k+8} by a dimension count. Hence by (6.4) $\lambda \cdot y = \lambda(x^*) = (\lambda x)^*$ corresponds to the other generator.

COROLLARY (6.7). *The image of $i^* : M_* \to M_*$ is an ideal, and hence the quotient ring $A_* = \sum_0^\infty A_k$ inherits a ring structure from M_*.*

This follows from (6.5). The element λ above projects into a class—again called λ—in A_8, and we clearly have:

PROPOSITION (6.8) *Multiplication by λ induces an isomorphism $A_k \cong A_{k+8}$, $k \geqslant 0$.*

The complete ring-structure of A_* is given by:

THEOREM (6.9). *A_* is the anticommutative graded ring generated by a unit $1 \in A_0$, and by elements $\xi \in A_1$, $\mu \in A_4$, $\lambda \in A_8$ with relations: $2\xi = 0$, $\xi^3 = 0$, $\mu^2 = 4\lambda$.*

Proof. As $A_1 \cong Z_2$, it is clear that $2\xi = 0$. From the fact that $a_1 = 1$, and $a_2 = 2$, we conclude that ξ_1^2 generates A_2. There remains the computation of μ^2. To settle this case we introduce a notion which will be of use later in any case. Let $k = 4n$, and let $\omega = e_1 \dots e_{4n}$. Then as we have already remarked, the centre of C_k^0 is generated by 1 and ω, whence, as $\omega^2 = +1$, the projection of C_k^0 on its two ideals is given by $(1 \mp \omega)/2$. It follows that if M is an irreducible graded C_k-module, then ω acts on M^0 as the scalar $\varepsilon = \pm 1$. In general we call a graded module for C_k an ε-module, ($\varepsilon = \pm 1$) if ω acts as ε on M^0. Now because $e_i\omega = -\omega e_i$, it follows immediately that if M is an ε-module, then M^* is a $(-\varepsilon)$-module, i.e., ω acts as $-\varepsilon$ on M^1, and finally, that if M is an ε-module and M' an ε'-module for C_k then $M \hat{\otimes} M'$ is an $\varepsilon\varepsilon'$-module for C_{2k}.

With this understood, let μ be the class of an irreducible C_4-module M in A_4. Then M is of type ε. Hence $M \hat{\otimes} M$ is of type $\varepsilon^2 = +1$ in C_8. Now if $\lambda \in A_8$ is chosen as the class of the irreducible $(+1)$-module W of C_8 it follows that $M \hat{\otimes} M \cong 4W$ by a dimension count, and so finally that $\mu^2 = 4\lambda$.

The corresponding propositions for the complex modules are clearly also valid. Thus we may define M_*^c and A_*^c, and now already the generator μ^c corresponding to an irreducible $C_2 \otimes_{\mathbf{R}} \mathbf{C}$-module yields periodicity. In fact the following is checked readily.

THEOREM (6.10). *The ring A_*^c is isomorphic to the polynomial ring $\mathbf{Z}[\mu^c]$.*

We consider again the element $\omega = e_1 \dots e_k \in C_k$. For $k = 2l$ we have $\omega^2 = (-1)^l$. Hence if M is an irreducible complex graded C_k-module then ω acts on M^0 as the complex scalar $\varepsilon = \pm i^l$. We call a complex graded C_k-module an ε-module if ω acts as ε on M^0. Let $\mu_l^c \in M^c(C_{2l})$ denote the generator given by an irreducible i^l-module. Then $\mu_l^c = (\mu^c)^l$ where $\mu_1^c = \mu^c$.

Comparing our conventions in the real and complex cases we see that if M is a real ε-module for C_{4n} then $M \otimes_{\mathbf{R}} \mathbf{C}$ is a complex $(-1)^n$ ε-module for C_{4n}. Now we choose $\mu \in A_4$ to be the class of an irreducible (-1)-module. Then in the homomorphism $A_* \to A_*^c$ given by complexification $\mu \to 2(\mu^c)^2$. From (6.9) and (6.10) we then deduce

$$\lambda \to (\mu^c)^4 \tag{6.11}$$

under complexification.

PART II

§7. Sequences of bundles

In this and succeeding sections we shall show how one can give a Grothendieck-type definition for the relative groups $K(X, Y)$. This will apply equally to real or complex vector bundles and we will just refer to vector bundles. For simplicity we shall work in the category of finite CW-complexes (and pairs of complexes).

For $Y \subset X$ we shall consider the set $\mathscr{C}_n(X, Y)$ of sequences

$$E = (0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \dots \longrightarrow E_1 \xrightarrow{\sigma_1} E_0 \longrightarrow 0)$$

where the E_i are vector bundles on X, the σ_i are homomorphisms defined on Y and the sequence is exact on Y. An isomorphism $E \to E'$ in $\mathscr{C}_n$ will mean a diagram

$$\begin{tikzcd}
{} \arrow[r] & E_i \arrow[r, "\sigma_i"] \arrow[d] & E_{i-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & E'_i \arrow[r, "\sigma'_i"] & E'_{i-1} \arrow[r] & {}
\end{tikzcd}$$

in which the vertical arrows are isomorphisms on X and the squares commute on Y.

An *elementary* sequence in $\mathscr{C}_n$ is one in which

$$\begin{array}{lll} E_i = E_{i-1}, & \sigma_i = 1 & \text{for some } i \\ E_j = 0 & & \text{for } j \neq i,\ i-1. \end{array}$$

The direct sum $E \oplus F$ of two sequences is defined in the obvious way. We consider now the following equivalence relation:

DEFINITION (7.1). *$E \sim F \Leftrightarrow$ there exist elementary sequences P^i, $Q^j \in \mathscr{C}_n$ so that*

$$E \oplus P^1 \oplus \dots \oplus P^r \cong F \oplus Q^1 \oplus \dots \oplus Q^s.$$

In other words this is the equivalence relation generated by isomorphism and addition of elementary sequences. The set of equivalence classes will be denoted by $L_n(X, Y)$. The operation $\oplus$ induces on L_n an abelian semi-group structure. If $Y = \varnothing$ we write $L_n(X) = L_n(X, \varnothing)$.

If $E \in \mathscr{C}_n$ then we can consider the sequence in $\mathscr{C}_{n+1}$ obtained from E by just defining $E_{n+1} = 0$. In this way we get inclusions

$$\mathscr{C}_1 \to \mathscr{C}_2 \to \dots \to \mathscr{C}_n \to$$

and we put $\mathscr{C} = \mathscr{C}_\infty = \lim \mathscr{C}_n$. These induce homomorphisms

$$L_1 \to L_2 \to \dots \to L_n \to$$

and it is clear that

$$L = L_\infty = \varinjlim L_n$$

is obtained from $\mathscr{C}$ by an equivalence relation as above applied now to sequences of finite but unbounded length.

LEMMA (7.2). *Let E, F be vector bundles on X and $f: E \to F$ a monomorphism on Y. Then if $\dim F > \dim E + \dim X$, f can be extended to a monomorphism on X and any two such extensions are homotopic* rel. *Y.*

Proof. Consider the fibre bundle $\mathrm{Mon}(E, F)$ on X whose fibre at $x \in X$ is the space of all monomorphisms $E_x \to F_x$. This fibre is homeomorphic to $GL(n)/GL(n-m)$ where $n = \dim F$, $m = \dim E$, and so it is $(n - m - 1)$-connected. Hence cross-sections can be extended and are all homotopic if

$$\dim X \leqslant n - m - 1 = \dim F - \dim E - 1.$$

But a cross-section of $\mathrm{Mon}(E, F)$ is just a global monomorphism $E \to F$.

LEMMA (7.3). *$L_n(X, Y) \to L_{n+1}(X, Y)$ is an isomorphism for $n \geqslant 1$.*

Proof. Let $\tilde{\mathscr{C}}_{n+1}$ denote the subset of $\mathscr{C}_{n+1}$ consisting of sequences E such that

$$\dim E_n > \dim E_{n+1} + \dim X. \tag{1}$$

If $n \geqslant 1$ then given any $E \in \mathscr{C}_{n+1}$ we can add an elementary sequence to it so that it will satisfy (1). Hence $\tilde{\mathscr{C}}_{n+1} \to L_{n+1}$ is surjective. Now let $E \in \bar{\mathscr{C}}_{n+1}$, then by (7.2) σ_{n+1} can be extended to a monomorphism σ'_{n+1} on the whole of X. Put $E'_n = \text{Coker } \sigma'_{n+1}$, let P denote the elementary sequence with $P_{n+1} = P_n = E_{n+1}$, and let

$$E' = (0 \longrightarrow E'_n \xrightarrow{\rho'_n} E_{n-1} \xrightarrow{\sigma_{n-1}} E_{n-2} \longrightarrow \dots \xrightarrow{\sigma_1} E_0 \longrightarrow 0),$$

where ρ'_n is defined by the commutative diagram on Y:

$$\begin{tikzcd}
E_n \arrow[r] \arrow[dr, "\sigma_n"'] & E'_n \arrow[d, "\rho'_n"] \\
 & E_{n-1}
\end{tikzcd}$$

A splitting of the exact sequence on X

$$0 \longrightarrow E_{n+1} \xrightarrow{\sigma'_{n+1}} E_n \longrightarrow E'_n \longrightarrow 0$$

then defines an isomorphism in $\mathscr{C}_{n+1}$

$$P \oplus E' \cong E.$$

If σ''_{n+1} is another extension of σ_{n+1} leading to a sequence E'', then by (7.2) $E'_n \cong E''_n$ and this isomorphism can be taken to extend the given one on Y, i.e., the diagram

$$\begin{tikzcd}
E'_n \arrow[r, "\rho'_n"] \arrow[d] & E_{n-1} \arrow[d, "1"] \\
E''_n \arrow[r, "\rho''_n"] & E_{n-1}
\end{tikzcd}$$

commutes on Y. Hence $E' \cong E''$ in $\mathscr{C}_n$ and so we have a well-defined map $E \mapsto E'$ from the isomorphism classes in $\tilde{\mathscr{C}}_{n+1}$ to the isomorphism classes in $\mathscr{C}_n$. Moreover, if

$$Q = (0 \longrightarrow Q_{n+1} \longrightarrow Q_n \longrightarrow 0), \qquad R = (0 \longrightarrow R_i \longrightarrow R_{i-1} \longrightarrow 0) \qquad (i \leqslant n)$$

are elementary sequences, then

$$(E \oplus Q)' \cong E', \qquad (E \oplus R)' \cong E' \oplus R.$$

Hence the class of E' in L_n depends only on the class of E in L_{n+1}. Since $\tilde{\mathscr{C}}_{n+1} \to L_{n+1}$ is surjective it follows that $E \to E'$ induces a map $L_{n+1} \to L_n$. From its construction it is immediate that its composition in either direction with $L_n \to L_{n+1}$ is the identity, and this completes the proof.

From (7.3) we deduce, by induction on n, and then passing to the limit:

PROPOSITION (7.4). *The homomorphisms* $L_1(X, Y) \to L_n(X, Y)$ *are isomorphisms for* $1 \leqslant n \leqslant \infty$.

§8. Euler characteristics

DEFINITION (8.1) *An Euler characteristic for* $\mathscr{C}_n$ *is a natural homomorphism* (i.e. *a natural transformation of functors*)

$$\chi : L_n(X, Y) \to K(X, Y)$$

which for $Y = \varnothing$ *is given by*

$$\chi(E) = \sum_{i=0}^{n} (-1)^i E_i.$$

Remark. It is clear that, if $Y = \varnothing$, $E \mapsto \sum(-1)^i E_i$ gives a well-defined map $L_n(X) \to K(X)$.

LEMMA (8.2). *Let* χ *be an Euler characteristic for* $\mathscr{C}_1$ *then*

$$\chi : L_1(X) \longrightarrow K(X)$$

is an isomorphism.

Proof. χ is an epimorphism by definition of $K(X)$. Suppose $\chi(E) = 0$, then $E_1 \oplus F \cong E_0 \oplus F$ for some F (in fact F can be taken trivial). Hence if

$$P : 0 \longrightarrow F \longrightarrow F \longrightarrow 0$$

is the elementary sequence defined by F, $E \oplus P$ is isomorphic to the elementary sequence defined by $E_1 \oplus F$. Hence $E \sim 0$ in $\mathscr{C}_1(X)$ and so $E = 0$ in $L_1(X)$. To conclude we need the following elementary lemma:

LEMMA (8.3). *Let* A *be a semi-group with an identity element* 1, B *a group*, $\phi : A \to B$ *an epimorphism with* $\phi^{-1}(1) = 1$. *Then* ϕ *is an isomorphism.*

Proof. It is sufficient to prove that A is a group, i.e., has inverses. Let $a \in A$, then from the hypotheses there exists $a' \in A$ so that

$$\phi(a') = \phi(a)^{-1}.$$

Hence

$$\phi(a \cdot a') = \phi(a) \cdot \phi(a') = 1,$$

and so $aa' = 1$ as required.

LEMMA (8.4). *Let* χ *be an Euler characteristic for* $\mathscr{C}_1$, *and let* Y *be a point. Then*

$$\chi : L_1(X, Y) \to K(X, Y)$$

is an isomorphism.

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & L_1(X, Y) \arrow[r, "\alpha"] \arrow[d, "\chi"] & L_1(X) \arrow[r, "\beta"] \arrow[d, "\chi"] & L_1(Y) \arrow[d, "\chi"] \\
0 \arrow[r] & K(X, Y) \arrow[r] & K(X) \arrow[r] & K(Y).
\end{tikzcd}$$

By (8.2) and (8.3) and the exactness of the bottom line it will be sufficient to show the exactness of the top line. Now $\beta\alpha = 0$ obviously and so we have to show

(i) $\alpha^{-1}(0) = 0$;

(ii) if $\beta(E) = 0$ then $E \in \operatorname{Im} \alpha$.

We consider (ii) first. Since Y is a point, and $\chi : L_1(Y) \cong K(Y)$, $\beta(E) = 0$ is equivalent to

$$\dim E_1 | Y = \dim E_0 | Y.$$

But then we can certainly find an isomorphism

$$\sigma : E_1 | Y \longrightarrow E_0 | Y,$$

showing that $E \in \operatorname{Im}(\alpha)$. Finally we consider (i). Thus let

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0)$$

be an element of $\mathscr{C}_1(X, Y)$ and suppose $\alpha(E) = 0$ in $L_1(X)$. Then $\chi\alpha(E) = 0$ in $K(X)$, and hence, if we suppose $\dim E_i > \dim X$ (as we may), there is an isomorphism

$$\tau : E_1 \longrightarrow E_0$$

on the whole of X. Then $\sigma\tau^{-1} \in \operatorname{Aut}(E_0 | Y)$. Since Y is a point this automorphism is homotopic to the identity† and hence can be extended to an element $\rho \in \operatorname{Aut}(E_0)$. Then $\rho\tau : E_1 \to E_0$ is an isomorphism extending σ. This shows that E represents 0 in $L_1(X, Y)$ as required.

LEMMA (8.5). *Let χ be an Euler characteristic for $\mathscr{C}_1$, then χ is an equivalence of functors $L_1 \to K$.*

Proof. Consider, for any pair (X, Y), the commutative diagram

$$\begin{tikzcd}
L_1(X/Y, Y/Y) \arrow[r, "\chi"] \arrow[d, "\phi"] & K(X/Y, Y/Y) \arrow[d, "\psi"] \\
L_1(X, Y) \arrow[r, "\chi"] & K(X, Y).
\end{tikzcd}$$

Since ψ is an isomorphism (by definition) and χ on the top line is an isomorphism by (8.4) it will be sufficient (by (8.3)) to prove that ϕ is an epimorphism. Now any element ξ of $L_1(X, Y)$ can be represented by a sequence

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0)$$

where E_0 is a product bundle. But then we can define a 'collapsed bundle' $E_1' = E_1/\sigma$ over X/Y and a collapsed sequence $E' \in \mathscr{C}_1(X/Y, Y/Y)$ defining an element $\xi' \in L_1(X/Y, Y/Y)$. Then $\xi = \phi(\xi')$ and so ϕ is an epimorphism.

LEMMA (8.6). *Let χ, χ' be two Euler characteristics for $\mathscr{C}_1$. Then $\chi = \chi'$.*

Proof. Let $T = \chi'\chi^{-1}$ (which is well-defined by (8.5)). This is a natural automorphism of $K(X, Y)$ which is the identity when $Y = \varnothing$. Replacing X by X/Y and considering the exact sequence for $(X/Y, Y/Y)$ we deduce that $T = 1$, i.e., that $\chi' = \chi$.

---

† This argument needs modification in the real case since $GL(n, R)$ is not connected: we replace E_i by $E_i \oplus 1$ and σ, τ by $\sigma \oplus 1$, $\tau \oplus (-1)$.

From (8.6) and (7.4) we deduce

LEMMA (8.7). *There is a bijective correspondence ($\chi_1 \longleftrightarrow \chi_n$) between Euler characteristics for $\mathscr{C}_1$ and $\mathscr{C}_n$ such that the diagram*

$$\begin{tikzcd}
L_1 \arrow[r] \arrow[d, "\chi_1"'] & L_n \arrow[dl, "\chi_n"] \\
K &
\end{tikzcd}$$

commutes.

These lemmas show that there is at most one Euler characteristic. In the next section we shall prove that it exists by giving a direct construction.

§9. The difference bundle

Given a pair (X, Y) define $X_i = X \times \{i\} i = 0, 1$, $A = X_0 \cup_Y X_1$ (obtained by identifying $y \times \{0\}$ and $y \times \{1\}$ for all $y \in Y$). Then we have retractions

$$\pi_i : A \to X_i$$

so that we get split exact sequences:

$$0 \longrightarrow K(A, X_i) \xrightarrow{\rho^*_i} K(A) \underset{j^*_i}{\overset{\pi^*_i}{\leftrightarrows}} K(X_i) \longrightarrow 0$$

Also, if we regard the index $i \in Z_2$, the natural map $X \to X_i$ gives an inclusion

$$\phi_i : (X, Y) \to (A, X_{i+1}),$$

which induces an isomorphism

$$\phi_i^* : K(A, X_{i+1}) \to K(X, Y).$$

Now let $E \in \mathscr{C}_1(X, Y)$,

$$E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0),$$

and construct the vector bundle F on A by putting E_i on X_i and identifying on Y by σ. It is clear that the isomorphism class of F depends only on the isomorphism class of E in $\mathscr{C}_1(X, Y)$. Let $F_i = \pi_i^*(E_i)$. Then $F | X_i \cong F_i$ and so $F - F_i \in \operatorname{Ker} j_i^*$. We define an element $d(E) \in K(X, Y)$ by

$$\rho_1^*(\phi_0^*)^{-1}\, d(E) = F - F_1.$$

It is clear that d is additive:

$$d(E \oplus E') = d(E) + d(E').$$

Also if E is elementary $F \cong F_1$ so that $d(E) = 0$. Hence d induces a homomorphism

$$d : L_1(X, Y) \to K(X, Y)$$

which is clearly natural. Moreover if $Y = \varnothing$, $A = X_0 + X_1$, $F = E_0 \times \{0\} + E_1 \times \{1\}$ (disjoint sum), $F_i = E_i \times \{0\} + E_i \times \{1\}$ and so

$$d(E) = E_0 - E_1.$$

Thus d is an Euler Characteristic in the sense of §8. The existence of this d together with the lemmas of §8 lead to the following proposition:

PROPOSITION (9.1). *For any integer n with $1 \leqslant n \leqslant \infty$ there exists a unique natural homomorphism*

$$\chi : L_n(X, Y) \to K(X, Y)$$

which, for $Y = \varnothing$, is given by

$$\chi(E) = \sum_{i=0}^{n} (-1)^i E_i.$$

Moreover χ is an isomorphism.

The unique χ given by (9.1) will be referred to as the *Euler characteristic.* From (8.6) we see that we may effectively identify the χ for different n.

Two elements $E, F \in \mathscr{C}_n(X, Y)$ are called *homotopic* if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of an element in $\mathscr{C}_n(X \times I, Y \times I)$.

PROPOSITION (9.2). *Homotopic elements in $\mathscr{C}_n(X, Y)$ define the same elements in $L_n(X, Y)$.*

Proof. This follows at once from (9.1) and the homotopy invariance of $K(X, Y)$.

Proposition (9.1) shows that we could take $L_n(X, Y)$ (for any $n \geqslant 1$) as a *definition* of $K(X, Y)$. This would be a Grothendieck-type definition.

We shall now give a method for constructing the inverse of $j : L_1(X, Y) \to L_n(X, Y)$. If $E \in \mathscr{C}_n(X, Y)$, then by introducing metrics we can define the adjoint sequence E^* with maps $\sigma_i^* : E_{i-1} \to E_i$. Consider the sequence

$$F = (0 \to F_1 \xrightarrow{\tau} F_0 \to 0)$$

where $F_0 = \bigoplus_i E_{2i}$, $F_1 = \bigoplus_i E_{2i+1}$ and

$$\tau(e_1, e_3, e_5, \dots) = (\sigma_1 e_1, \sigma_2^* e_2 + \sigma_3 e_3, \sigma_4^* e_3 + \sigma_5 e_5, \dots).$$

Since, on Y, we have the decomposition

$$E_{2i} = \sigma_{2i+1}(E_{2i+1}) \oplus \sigma_{2i}^*(E_{2i-1})$$

it follows that $F \in \mathscr{C}_1(X, Y)$. If $E \in \mathscr{C}_1$ then $E = F$. Since two choices of metric in E are homotopic it follows by (9.2) that F will be a representative for $j^{-1}(E)$.

§10. Products

In this section we shall consider complexes of vector bundles, i.e., sequences

$$0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \dots \longrightarrow E_0 \longrightarrow 0$$

in which $\sigma_{i-1}\sigma_i = 0$ for all i.

LEMMA (10.1). *Let $E_0, \dots, E_n$ be vector bundles on X,*

$$0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \longrightarrow \dots \longrightarrow E_0 \longrightarrow 0$$

a complex on Y. Then the σ_i can be extended so that this becomes a complex on X.

Proof. By induction on the cells of $X - Y$ it is sufficient to consider the case when X is obtained from Y by attaching one cell. Thus let

$$X = Y \cup_f e^k$$

where $f : S^{k-1} \to Y$ is the attaching map. If B^k denotes the unit ball in $\mathbf{R}^k$, with boundary S^{k-1}, then X is the quotient of $Y + B^k$ by an identification map π induced by f. The bundle $\pi^* E_i$ is then the disjoint sum of $E_i | Y$ and a trivial bundle $B^k \times V_i$. The homomorphism $\sigma_i : E_i \to E_{i-1}$ on Y lifts to give a homomorphism $\tau_i : S^{k-1} \times V_i \to S^{k-1} \times V_{i-1}$, i.e. a map $S^{k-1} \to \mathrm{Hom}(V_i, V_{i-1})$. Extend each τ_i to B^k by defining

$$\tau_i(u) = \|u\| \sigma_i(u) \qquad u \in B^k.$$

This induces an extension of the σ_i to X preserving the relations $\sigma_{i-1}\sigma_i = 0$, as required.

We now introduce the set $\mathscr{D}_n(X, Y)$ of complexes of length n on X acyclic (i.e. exact) on Y. Two such complexes are *homotopic* if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of an element in $\mathscr{D}_n(X \times I, Y \times I)$. By restricting the homomorphisms to Y we get a natural map

$$\Phi : \mathscr{D}_n(X, Y) \to \mathscr{C}_n(X, Y).$$

LEMMA (10.2). *$\Phi : \mathscr{D}_n \to \mathscr{C}_n$ induced a bijective map of homotopy classes.*

Proof. Applying (10.1) we see that Φ itself is surjective. Next, applying (10.1) to the pair

$$(X \times I, X \times \{0\} \cup X \times \{1\} \cup Y \times I)$$

we see that

$$\Phi(E) \text{ homotopic to } \Phi(F) \Rightarrow E \text{ homotopic to } F$$

which completes the proof.

If $E \in \mathscr{D}_n(X, Y)$, $F \in \mathscr{D}_m(X', Y')$ then $E \otimes F$ is a complex on $X \times X'$ acyclic on $X \times Y' \cup Y \times X'$ so that

$$E \otimes F \in \mathscr{D}_{n+m}(X \times X', X \times Y' \cup Y \times X').$$

This product is additive and compatible with homotopies. Hence it induces a bilinear product on the homotopy classes. From (10.2) and (9.2) it follows that it induces a natural product

$$L_n(X, Y) \otimes L_m(X', Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X').$$

PROPOSITION (10.3). *The tensor product of complexes induces a natural product*

$$L_n(X, Y) \otimes L_m(X', Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X')$$

and

$$\chi(ab) = \chi(a)\chi(b) \tag{1}$$

where χ is the Euler characteristic.

Proof. The formula (1) is certainly true when $Y = Y' = \varnothing$. On the other hand there is a unique natural extension of the product $K(X) \otimes K(X') \to K(X \times X')$ to the relative case (cf. [3]). Hence, by (9.1), formula (1) is also true in the general case.

Remark. This result is essentially due to Douady (*Séminaire Bourbaki* (1961) No. 223).

PROPOSITION (10.4). *Let*

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0) \in \mathscr{D}_1(X, Y)$$

$$E' = (0 \longrightarrow E'_1 \xrightarrow{\sigma'} E'_0 \longrightarrow 0) \in \mathscr{D}_1(X', Y')$$

and choose metrics in all the bundles. Let

$$F = (0 \longrightarrow F_1 \xrightarrow{\tau} F_0 \longrightarrow 0) \in \mathscr{D}_1(X \times X', X \times Y' \cup Y \times X')$$

be defined by

$$F_1 = E_0 \otimes E'_1 \oplus E_1 \otimes E'_0$$

$$F_0 = E_0 \otimes E'_0 \oplus E_1 \otimes E'_1$$

$$\tau = \begin{pmatrix} 1 \otimes \sigma', & \sigma \otimes 1 \\ \sigma^* \otimes 1, & -1 \otimes \sigma'^* \end{pmatrix}$$

where σ^*, σ'^* *denote the adjoints of* σ, σ'. *Then*

$$\chi(F) = \chi(E) \cdot \chi(E').$$

Proof. By (10.3) $\chi(E) \cdot \chi(E') = \chi(E \otimes E')$. Now the construction of §9 for the inverse of $j_2 : L_1 \rightarrow L_2$ turns $E \otimes E'$ into F and so $\chi(E \otimes E') = \chi(F)$.

PART III

§11. Clifford bundles

In this section and the next we shall consider the Thom complex of a vector bundle. If V is a (real) Euclidean vector bundle over X (i.e. the fibres have a positive definite inner product) we denote by X^V the one-point compactification of V and refer to it as the Thom complex of V. It inherits a natural structure of CW-complex (with base point) from that of X. An alternative description which is also useful is the following. Let $B(V)$, $S(V)$ denote the unit ball and unit sphere bundles of V, then X^V may be identified with $B(V)/S(V)$. A technical point which arises here is that $(B(V), S(V))$ is not obviously a CW-pair. However the following remarks show that there is no real loss of generality in assuming that $(B(V), S(V))$ is a CW-pair.

1. If X is a differentiable manifold then $(B(V), S(V))$ is a manifold with boundary and hence triangulable.
2. Every vector bundle over a finite complex is induced by a map of the base space into a differentiable manifold (namely a Grassmannian).

There are of course more satisfactory ways of dealing with this point but a lengthy discussion would be out of place in this context.

With our assumption therefore we have the isomorphism

$$\tilde{K}(X^V) \cong K(B(V), S(V))$$

where $\tilde{K}$ denotes K modulo the base point.

Since each fibre V_x of V is a vector space with a positive definite quadratic form Q_x, we can form the *Clifford bundle* $C(V)$ of V. This will be a bundle of algebras whose fibre at x is the Clifford algebra $C(-Q_x)$. Contained in $C(V)$ are bundles of groups, $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$. All these bundles are associated to the principal $O(k)$-bundle of V by the natural action of $O(k)$ on C_k, $\mathrm{Pin}(k)$, $\mathrm{Spin}(k)$.

By a graded Clifford module of V we shall mean a Z_2-graded vector bundle E (real or complex) over X which is a graded $C(V)$-module. In other words $E = E^0 \oplus E^1$ and we have vector bundle homomorphisms

$$V \otimes_{\mathbf{R}} E^0 \rightarrow E^1, \qquad V \otimes_{\mathbf{R}} E^1 \rightarrow E^0$$

(denoted simply by $v \otimes e \rightarrow v(e)$) such that

$$v(v(e)) = -\|v\|^2 e \tag{1}$$

For notational convenience we shall consider real modules only. The complex case is entirely parallel.

Let $E = E^0 \oplus E^1$ be a graded $C(V)$-module. Then E^0 is a $\mathrm{Spin}(V)$-module and by integration over the fibres of $\mathrm{Spin}(V)$ we can give E^0 a metric invariant under $\mathrm{Spin}(V)$. This can then be extended to a metric on E invariant under $\mathrm{Pin}(V)$ and such that E^0 and E^1 are orthogonal complements. If now $v \in V_x$ and $v \neq 0$ then $v/\|v\| \in \mathrm{Pin}(V_x)$. Hence we deduce, for all $v \in V_x$ and $e \in E_x$,

$$\|ve\| = \|v\| \cdot \|e\|.$$

This, together with (1), implies that the adjoint of

$$v : E_x^0 \rightarrow E_x^1 \qquad \text{is} \qquad -v : E_x^1 \rightarrow E_x^0.$$

Let $\pi : B(V) \rightarrow X$ be the projection map and let

$$\sigma(E) : \pi^* E^1 \longrightarrow \pi^* E^0$$

be given by multiplication by $-v$, i.e.

$$\sigma(E)_v(e) = -ve.$$

Then

$$0 \longrightarrow \pi^* E^1 \xrightarrow{\sigma(E)} \pi^* E^0 \longrightarrow 0 \tag{2}$$

is an element of $\mathscr{D}_1(B(V), S(V))$ and hence defines an element $\chi_V(E)$ of $KO(B(V), S(V))$, or equivalently an element of $\widetilde{KO}(X^V)$. If the $C(V)$-module structure of E extends to a $C(V \oplus 1)$-module structure (1 denoting the trivial line-bundle) then the isomorphism $\sigma(E)$ extends from $S(V)$ to $S^+(V \oplus 1)$ the 'upper hemisphere' of $S(V \oplus 1)$. Since the pairs $(B(V), S(V))$ and $(S^+(V \oplus 1), S(V))$ are clearly equivalent it follows that $\chi_V(E)$ will, in this case, be zero.

Following §5, which is the special case $X =$ point, we now define $M(V)$ as the Grothendieck group of graded $C(V)$-modules, and we let $A(V)$ denote the cokernel of the natural homomorphism

$$M(V \oplus 1) \rightarrow M(V).$$

Then the construction described above gives rise to a homomorphism

$$\chi_V : A(V) \rightarrow \widetilde{KO}(X^V).$$

This homomorphism is of fundamental importance in the theory, and our next step is to discuss its multiplicative properties.

Let V, W be Euclidean vector bundles over X, Y respectively. Then we have a natural homeomorphism

$$X^V \divideontimes Y^W \approx X \times Y^{V \oplus W}$$

which induces a homomorphism (or 'cup-product')

$$\widetilde{KO}(X^V) \otimes \widetilde{KO}(Y^W) \to \widetilde{KO}(X \times Y^{V \oplus W}).$$

If $a \in \widetilde{KO}(X^V)$, $b \in \widetilde{KO}(Y^W)$ the image of $a \otimes b$ will simply be written as ab.

PROPOSITION (11.1). *The following diagram commutes*

$$\begin{tikzcd}
A(V) \otimes A(W) \arrow[r, "\mu"] \arrow[d, "\chi_V \otimes \chi_W"] & A(V \oplus W) \arrow[d, "\chi_{V \oplus W}"] \\
\widetilde{KO}(X^V) \otimes \widetilde{KO}(Y^W) \arrow[r] & \widetilde{KO}(X \times Y^{V \oplus W})
\end{tikzcd}$$

where μ is induced by the graded tensor product of Clifford modules. Thus

$$\chi_{V \oplus W}(E \hat{\otimes} F) = \chi_V(E)\chi_W(F).$$

Proof. Let E, F be graded $C(V)$- and $C(W)$-modules and let them both be given invariant metrics as above. Applying Proposition (10.2) it follows that

$$\chi_V(E) \cdot \chi_W(F) \in KO(B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W))$$

is equal to $\chi(G)$ where

$$G \in \mathscr{D}_1(B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W))$$

is defined by

$$G_1 = \pi^*(E^0 \otimes F^1 \oplus E^1 \otimes F^0)$$
$$G_0 = \pi^*(E^0 \otimes F^0 \oplus E^1 \otimes F^1)$$

and $\tau : G_1 \to G_0$ is given by

$$\tau = \begin{pmatrix} 1 \otimes \sigma(F), & \sigma(E) \otimes 1 \\ -\sigma(E) \otimes 1, & 1 \otimes \sigma(F) \end{pmatrix}$$

(since $\sigma(E)^* = -\sigma(E)$, $\sigma(F)^* = -\sigma(F)$). Thus, at a point $v \oplus w \in V \oplus W$, τ is given by the matrix

$$\tau_{v \oplus w} = \begin{pmatrix} 1 \otimes -w, & -v \otimes 1 \\ v \otimes 1, & 1 \otimes -w \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \otimes w, & v \otimes 1 \\ v \otimes 1, & -1 \otimes w \end{pmatrix}$$

where v, w denote module multiplication by v, w. Hence

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sigma(E \hat{\otimes} F) \tag{3}$$

On the other hand let $B'(V \oplus W)$ denote the ball of radius 2 and let

$$S'(V \oplus W) = \overline{B'(V \oplus W) - B(V \oplus W)},$$

so that the inclusions

$$i : B(V \oplus W), S(V \oplus W) \to B'(V \oplus W), S'(V \oplus W)$$

$$j : B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W) \to B'(V \oplus W), S'(V \oplus W)$$

are both homotopy equivalences. Let

$$H \in \mathscr{D}_1(B'(V \oplus W), S'(V \oplus W))$$

be defined by $\sigma(E \hat{\otimes} F)$. Then $i^*(H)$ defines the element $\chi_{V \oplus W}(E \hat{\otimes} F)$, while (3) shows that $j^*(H)$ and G define the same element of $KO(B(V) \times B(W), B(V) \times S(W) \cup S(V) \times B(W))$. Hence we have

$$\chi_V(E) \cdot \chi_W(F) = \chi_{V \oplus W}(E \otimes F)$$

as required.

Suppose now that P is a principal Spin(k)-bundle over X, $V = P \times_{\text{Spin}(k)} \mathbf{R}^k$ the associated vector bundle. If M is a graded C_k-module then $E = P \times_{\text{Spin}(k)} M$ will be a graded $C(V)$-module. In this way we obtain a homomorphism of groups

$$\beta_P : A_k \to A(V).$$

Similarly in the complex case we obtain

$$\beta_P^c : A_k^c \to A^c(V).$$

PROPOSITION (11.2). *Let* P, P' *be* Spin (k), Spin (l) *bundles over* X, X' *and let* $V = P \times_{\text{Spin}(k)} \mathbf{R}^k$, $V' = P' \times_{\text{Spin}(l)} \mathbf{R}^l$. *Let* P'' *be the* Spin($k + l$)*-bundle over* $X \times X'$ *induced from* $P \times P'$ *by the standard homomorphism*

$$\text{Spin}\,(k) \times \text{Spin}\,(l) \to \text{Spin}(k + l).$$

Then if $a \in A_k$, $b \in A_l$, *we have*

$$\beta_{P''}(ab) = \beta_P(a)\beta_{P'}(b).$$

A similar formula holds for β_P^c.

The verification of this result is straightforward and is left to the reader.

Let $\alpha_P : A_k \to \widetilde{KO}(X^V)$ be defined by $\alpha_P = \chi_V \beta_P$.

Then from Propositions (11.1) and (11.2) we deduce

PROPOSITION (11.3). *With the notation of* (11.2) *we have*

$$\alpha_{P''}(ab) = \alpha_P(a)\alpha_{P'}(b),$$

and a similar formula for α_P^c.

If we apply all the preceding discussion to the case when X is a point (and P denotes the trivial Spin(k)-bundle) we get maps

$\alpha : A_k \to \widetilde{KO}(S^k)$ in the real case
$\alpha^c : A_k^c \to \widetilde{K}(S^k)$ in the complex case.

Proposition (11.3) then yields the following corollary, as a special case:

COROLLARY (11.4). *The maps*

$$\alpha : A_* \to \sum_{k \geqslant 0} KO^{-k}(\text{point})$$

$$\alpha^c : A_*^c \to \sum_{k \geqslant 0} K^{-k}(\text{point})$$

are ring homomorphisms.

Now the rings A_* and A^c_* were explicitly determined in §6 (Theorems (6.9) and (6.10)). On the other hand the additive structure of $B_* = \sum KO^{-k}(\text{point})$ and $B^c_* = \sum K^{-k}(\text{point})$ was determined in [5], while their multiplicative structure was (essentially) given in [6]. These results may be summarized as follows:

(i) B^c_* is the polynomial ring generated by an element $x \in B^c_2$ corresponding to the reduced Hopf bundle on $P_1(\mathbf{C}) = S^2$;

(ii) B_* contains a polynomial ring $\mathbf{Z}[y]$ with $y \in B_8$, and $y \to x^4$ under the complexification map $B_* \to B^c_*$;

(iii) As a module over $\mathbf{Z}[y]$, B_* is freely generated by elements 1, a, b, z where $a \in B_1$, $b \in B_2$, $z \in B_4$, subject to the relations $2a = 0$, $2b = 0$.

If we use Stiefel-Whitney classes then a simple calculation shows that

$$w_2(a^2) \neq 0$$

where we regard $a^2 \in \tilde{K}(S^2)$. Thus we must have $a^2 = b$.

Consider now the ring homomorphism

$$\alpha^c : A^c_* \to B^c_*.$$

It is immediate from the definition of α^c that $\alpha^c(\mu^c)$ gives the reduced Hopf bundle on S^2. Hence from (6.10) we deduce that α^c is an isomorphism.

Consider next the ring homomorphism

$$\alpha : A_* \to B_*.$$

Because of the commutative diagram

$$\begin{tikzcd} A_* \arrow[r, "\alpha"] \arrow[d] & B_* \arrow[d] \\ A^c_* \arrow[r, "\alpha^c"] & B^c_* \end{tikzcd}$$

the results on α^c together with (6.11) and (ii) above imply that

$$\alpha(\lambda) = y.$$

Similarly using (6.9) and (iii) above we get

$$\alpha(\mu) = z.$$

It remains to consider $\alpha(\xi)$ and $\alpha(\xi^2)$. But as in the complex case it is immediate that $\alpha(\xi)$ is the reduced Hopf bundle on $P_1(\mathbf{R}) = S^1$. Since a is the unique non-zero element of B_1 we must therefore have

$$\alpha(\xi) = a.$$

Using (6.9) and (ii), (iii) above it follows that α is an isomorphism. Thus we have established:

THEOREM (11.5). *The maps*

$$\alpha : A_* \to \sum_{k \geqslant 0} KO^{-k}(\text{point})$$

and

$$\alpha^c : A^c_* \to \sum_{k \geqslant 0} K^{-k}(\text{point})$$

are ring isomorphisms.

As remarked in the introduction this theorem shows clearly the intimate relation between Clifford algebras and the periodicity theorems. It is to be hoped that a less computational proof of (11.5) will eventually be found and that the theorem will then appear as the foundation stone of K-theory.

We shall conclude this section by taking up again the relation between Clifford and Grassmann algebras mentioned in §3. Let V be a complex vector bundle over X, $\Lambda(V)$ its Grassmann bundle, i.e. the bundle whose fibre at $x \in X$ is the Grassmann algebra $\Lambda(V_x)$. Let $\pi : V \to X$ be the projection and consider the complex

$$\Lambda_V \quad : \longrightarrow \pi^*(\Lambda^r(V)) \xrightarrow{d} \pi^*(\Lambda^{r+1}(V)) \longrightarrow$$

where d is given by the exterior product:

$$d_v(w) = v \wedge w \qquad\qquad v \in V_x,\ w \in \Lambda(V_x).$$

This is acyclic outside the zero-section and hence defines an element

$$\chi(\Lambda_V) \in \tilde{K}(X^V)$$

On the other hand, if we give V a Hermitian metric, and use the homomorphism

$$\tilde{l} : U(k) \to \mathrm{Spin}^c(2k) \qquad\qquad k = \dim_C V$$

we obtain a principal $\mathrm{Spin}^c(2k)$-bundle P over X, and hence a homomorphism

$$\alpha^c_P : A^c_{2k} \to \tilde{K}(X^V).$$

The relation between α^c_P and $\chi(\Lambda_V)$ is then given by:

PROPOSITION (11.6). $\chi(\Lambda_V) = \alpha^c_P((\mu^c)^k)$.

Proof. Applying the construction at the end of §9 for the inverse of

$$j_k : L_1 \to L_k$$

to the complex Λ_V, we obtain a sequence

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0)$$

where

$$\begin{aligned} E_0 &= \pi^*\Lambda^k \oplus \pi^*\Lambda^{k-2} \oplus \ldots \\ E_1 &= \pi^*\Lambda^{k-1} \oplus \pi^*\Lambda^{k-3} \oplus \ldots \\ \sigma_v &= d_v + \delta_v. \end{aligned}$$

In fact we could equally well have taken

$$\sigma_v = d_v - \delta_v$$

in §5. In view of (5.10), (5.11) and the final remark of §5 this shows that

$$\chi(\Lambda_V) = \alpha^c_P((\mu^c)^k)$$

as required.

Remark. The multiplicative property of Grassmann algebras:

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W)$$

can be used directly to establish a product formula for $\chi(\Lambda_V)$. This corresponds of course to (11.3).

§12. The Thom isomorphism

We begin with some brief remarks on the Thom isomorphism for general cohomology theories.

Let F be a generalized cohomology theory with products. Thus $F^{\#}(X) = \sum F^q(X)$ is a graded anti-commutative ring with identity and $F^{\#}(X, Y)$ is a graded $F^{\#}(X)$-module. Moreover the product must be compatible with the coboundary in the sense that

$$\delta(ab) = \delta(a)\cdot b + (-1)^{\alpha} a\delta b$$

where $\alpha = \deg a$ and a, b belong to suitable F-groups.

In $\tilde{F}^n(S^n)$ we have a canonical element σ^n which corresponds to the identity element $1 = \sigma^0 \in F^0(\text{point}) = \tilde{F}^0(S^0)$ under suspension. $\tilde{F}^{\#}(S^n)$ is then a free module over $F^{\#}(\text{point})$ generated by σ^n.

Suppose now that V is a real vector bundle of dimension n over X. We choose a metric in V and introduce the pair $(B(V), S(V))$ (or the Thom complex X^V). For each point $P \in X$ we consider the inclusion

$$i_P : P^V \to X^V$$

and the induced homomorphism

$$i_P^* : \tilde{F}^n(X^V) \to \tilde{F}^n(P^V).$$

Suppose now that V is *oriented*, then for each $P \in X$ we have a well-defined suspension isomorphism

$$S_P : F^0(P) \to \tilde{F}^n(P^V).$$

We let $\sigma_P^n = S_P(1)$. We shall say that V is *F-orientable* if there exists an element $\mu_V \in \tilde{F}^n(X^V)$ such that, for all $P \in X$,

$$i_P^*(\mu_V) = \sigma_P^n.$$

A definite choice of such a μ_V will be called an *F-orientation* of V. Then we have the following general Thom isomorphism theorem:

THEOREM (12.1). *Let V be an F-oriented bundle over X with orientation class μ_V. Then $\tilde{F}^{\#}(X^V)$ is a free $F^{\#}(X)$-module with generator μ_V.*

Proof. Multiplication by μ_V defines a homomorphism of the F-spectral sequence of X into the $\tilde{F}$-spectral sequence of X^V which is an isomorphism on E_2 (the Thom isomorphism for cohomology) and hence on E_∞. Hence

$$a \to \mu_V a$$

gives an isomorphism $F^{\#}(X) \to \tilde{F}^{\#}(X^V)$ as stated.†

Applying (12.1) to the special theories K, KO we obtain††:

THEOREM (12.2). *Let V be an oriented real vector bundle of dimension n over X. Then*

(i) *if $n \equiv 0 \bmod 2$ and there is an element $\mu_V \in \tilde{K}(X^V)$ whose restriction to $\tilde{K}(P^V)$ for each $P \in X$ is the generator, then $\tilde{K}^*(X^V)$ is a free $K^*(X)$-module generated by μ_V;*

---

† One can also use the Mayer–Victoris sequence instead of the spectral sequence.

†† We use K^*, KO^* to denote the sum of K^q, KO^q over the period (2, or 8) in distinction with $K^{\#}$ which is the sum over all integers.

(ii) *if $n \equiv 0 \bmod 8$ and there is an element $\mu_V \in \widetilde{KO}(X^V)$ whose restriction to each $\widetilde{KO}(P^V)$ for each $P \in X$ is the generator, then $\widetilde{KO}^*(X^V)$ is a free $\widetilde{KO}^*(X)$-module generated by μ_V.*

Remark. Since K^0 (point) $\cong KO^0(\text{point}) \cong \mathbf{Z}$ these groups are generated by the identity element of the ring. This element and its suspensions are what we mean by *the generator.*

Suppose now that V has a Spin-structure, i.e., that we are given a principal Spin(n)-bundle P and an isomorphism

$$V \cong P \times_{\mathrm{Spin}(n)} \mathbf{R}^n.$$

Then from §11 we have a homomorphism

$$\alpha_P : A_n \to \widetilde{KO}(X^V).$$

Similarly if V has a Spin^c-structure, i.e. we are given a principal Spin^c(n)-bundle P and an isomorphism

$$V \cong P \times_{\mathrm{Spin}^c(n)} \mathbf{R}^n$$

then we get a homomorphism

$$\alpha_P^c : A_n^c \to \tilde{K}(X^V).$$

In the real case assume $n = 8k$ and in the complex case $n = 2k$, and put

$$\mu_V = \alpha_P(\lambda^k)$$
$$\mu_V^c = \alpha_P^c((\mu^c)^k).$$

Then by the naturality of α_P, α_P^c and Theorem (11.1) we see that μ_V, μ_V^c define KO and K orientations of V and hence (12.2) gives:

THEOREM (12.3). (i) *Let P be a* Spin(8k)*-bundle $V = P \times_{\mathrm{Spin}(8k)} \mathbf{R}^{8k}$. Then $\widetilde{KO}^*(X^V)$ is a free $KO^*(X)$-module generated by μ_V;* (ii) *Let P be a* Spin^c(2k)*-bundle, $V = P \times_{\mathrm{Spin}^c(2k)} \mathbf{R}^{2k}$. Then $\tilde{K}^*(X^V)$ is a free $K^*(X)$-module generated by μ_V^c.*

Remark. It is easy to see, by considering the first differentials in the spectral sequence, that the existence of a Spin (Spin^c)-structure is necessary for $KO(K)$-orientability. Theorem (12.3) shows that these conditions are also sufficient.

(12.3) together with (11.3) shows that, for Spin bundles, we have a Thom isomorphism for KO and K with all the good formal properties. It is then easy to show that for Spin-manifolds one can define a functorial homomorphism

$$f_! : KO^*(Y) \to KO^*(X) \qquad \text{for maps} \qquad f : Y \to X,$$

and similarly for Spin^c-manifolds in K-theory. This improves the results of [2].

§13. The sphere

The purpose of these next sections is to identify the generator of $\widetilde{KO}(X^V)$ (for a V with Spinor structure and dim $\equiv 0 \bmod 8$) given in §12 with that given in [7]. Essentially we have to study the sphere as a homogeneous space of the spinor group. This actually leads to simpler formulae (Proposition (13.2)) for the characteristic map of the tangent bundle than one gets from using the orthogonal group.

We recall first the existence of an isomorphism $\phi : C_k \to C^0_{k+1}$ (Proposition (5.2)) and we note that, on C^0_k, ϕ coincides with the standard inclusion $C_k \to C_{k+1}$. We introduce the following notation: $K = \mathrm{Spin}(k+1)$, $H = \phi(\mathrm{Pin}(k)) = H^0 + H^1$. $H^0 = \phi(\mathrm{Spin}(k))$ (where + here denotes *disjoint* sums of the two components).

$$\begin{aligned} S^k &= \text{unit sphere in } \mathbf{R}^{k+1} \\ S_+ &= S^k \cap \{x_{k+1} \geqslant 0\}, \qquad S_- = S^k \cap \{x_{k+1} \leqslant 0\} \\ S^{k-1} &= S^+ \cap S^-. \end{aligned}$$

We consider S^k as the orbit space of e_{k+1} for the group K operating on $\mathbf{R}^{k+1}$ by the representation ρ. Thus $K/H^0 = S^k$ and we have the principal H^0-bundle

$$K \xrightarrow{\pi} K/H^0.$$

Let $K_+ = \pi^{-1}(S_+)$, $K_- = \pi^{-1}(S_-)$. We shall give explicit trivializations of K_+ and K_-, and the identification will then give the 'characteristic map' of the sphere.

We parametrize S_+ by use of 'polar co-ordinates':

$$(x, t) = \mathrm{Cos}\, t \,.\, e_{k+1} + \mathrm{Sin}\, t \,.\, x \qquad x \in S_{k-1}, \qquad 0 \leqslant t \leqslant \pi/2.$$

Now define a map $\beta_+ : S_+ \times H^0 \to K_+$ by

$$\beta_+(x, t, h^0) = (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . x\, e_{k+1})h^0.$$

Since

$$\begin{aligned} &\rho((-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})h^0)e_{k+1} \\ &\quad = (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})e_{k+1}(-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})^{-1} \\ &\quad = (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})^2 e_{k+1} \\ &\quad = \mathrm{Cos}\, t . e_{k+1} + \mathrm{Sin}\, t . x = (x, t), \end{aligned}$$

it follows that β_+ is an H^0-bundle isomorphism.

Similarly we parametrize S_- by

$$(x, t) = -\mathrm{Cos}\, t . e_{k+1} + \mathrm{Sin}\, t . x \qquad x \in S_{k-1}, \qquad 0 \leqslant t \leqslant \pi/2.$$

Note that for points of S_{k-1} the two parametrizations agree (putting $t = \pi/2$). Now define a map $\beta_- : S_- \times H^1 \to K_-$ by

$$\beta_-(x, t, h^1) = (\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})h^1.$$

Since

$$\begin{aligned} &\rho((\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})h^1)e_{k+1} \\ &\quad = (\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})(-e_{k+1})(\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})^{-1} \\ &\quad = -(\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . xe_{k+1})^2 e_{k+1} = -\mathrm{Cos}\, t . e_{k+1} + \mathrm{Sin}\, t . x, \end{aligned}$$

it follows that β_- is an H^0-bundle isomorphism.

Putting $t = \pi/2$ above we get

$$\begin{aligned} \beta_+(x, \pi/2, h^0) &= (-\mathrm{Cos}\, \pi/4 + \mathrm{Sin}\, \pi/4 . xe_{k+1})h^0 \\ \beta_-(x, \pi/2, h^1) &= (\mathrm{Cos}\, \pi/4 + \mathrm{Sin}\, \pi/4 . xe_{k+1})h^1. \end{aligned}$$

These are the same point of $K_+ \cap K_-$ if

$$\begin{aligned} h^1 &= -(\mathrm{Cos}\, \pi/4 - \mathrm{Sin}\, \pi/4 . xe_{k+1})^2 h^0 \\ &= xe_{k+1}h^0. \end{aligned}$$

Thus we have a commutative diagram

$$\begin{tikzcd} S_{k-1} \times H^0 \arrow[r, "\beta_+"] \arrow[d, "\delta"] & K_+ \cap K_- \arrow[d, "1"] \\ S_{k-1} \times H^1 \arrow[r, "\beta_-"] & K_+ \cap K_- \end{tikzcd}$$

where

$$\delta(x, h^0) = (x, xe_{k+1}h^0). \tag{1}$$

LEMMA (13.1). *If we regard H^0 as (left) operating on both factors of $S_+ \times H^0$ and $S_- \times H^1$, then β_+ and β_- are compatible with left operation.*

Proof (i) $\beta_+ g(x, t, h^0) = \beta_+(g(x), t, gh^0)$

$$\begin{aligned} &= (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . gxg^{-1}e_{k+1})gh^0 \\ &= g\beta_+(x, t, h^0) \end{aligned}$$

where $g \in H^0$ and $g(x) = \rho_{k+1}(g) . x = gxg^{-1}$.

(ii) $\beta_- g(x, t, h^1) = \beta_-(\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2 . gxg^{-1}e_{k+1})gh^1$

$$= g\beta_-(x, t, h^1).$$

Since $\phi(x) = xe_{k+1}$ for $x \in \mathbf{R}^k$ formula (1) above can be rewritten

$$\delta(x, g) = (x, xg) \qquad x \in \mathbf{R}^k, g \in \mathrm{Spin}(k).$$

Summarizing our results therefore we get:

PROPOSITION (13.2). *The principal* Spin(*k*)*-bundle* Spin(k + 1) → S^k *is isomorphic to the bundle obtained from the two bundles*

$$\begin{aligned} S_+ \times \mathrm{Pin}^0(k) &\to S_+ \\ S_- \times \mathrm{Pin}^1(k) &\to S_- \end{aligned}$$

by the identification

$$(x, g) \leftrightarrow (x, xg) \qquad \textit{for } x \in S^{k-1}, g \in \mathrm{Pin}^0(k).$$

Moreover this isomorphism is compatible with left multiplication by Spin(*k*).

Here $\mathrm{Pin}^0(k) = \mathrm{Spin}(k)$ and $\mathrm{Pin}^1(k)$ are the two components of $\mathrm{Pin}(k)$.

§14. Spinor bundles

Let P^0 be a principal Spin(k)-bundle over X and put

$$\begin{aligned} &P^1 = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Pin}^1(k), \qquad Q = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Spin}(k+1) \\ &T^k = P^0 \times_{\mathrm{Spin}(k)} S^k = T_+ \cup T_-, \text{ where} \\ &T_+ = P^0 \times_{\mathrm{Spin}(k)} S_+, \qquad T_- = P^0 \times_{\mathrm{Spin}(k)} S_- \\ &\pi_+ : T_+ \to X, \qquad \pi_- : T_- \to X \text{ the projections.} \end{aligned}$$

Consider now the two commutative diagrams

$$\begin{tikzcd}
P^0 \times_{\mathrm{Spin}(k)}(S_+ \times \mathrm{Pin}^0(k)) \arrow[r, "\lambda^0"] \arrow[d] & P^0 \arrow[d] \\
T_+ \arrow[r, "\pi_+"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
P^0 \times_{\mathrm{Spin}(k)}(S_- \times \mathrm{Pin}^1(k)) \arrow[r, "\lambda^1"] \arrow[d] & P^1 \arrow[d] \\
T_- \arrow[r, "\pi_-"] & X.
\end{tikzcd}$$

where $\lambda^i(p, s, g) = pg$, $p \in P^0$, $s \in S_\pm$, $g \in \mathrm{Pin}^i(k)$, $i = 0, 1$.

These allow us to identify the two Spin(k) bundles occurring in the first column with $\pi_+^*(P^0)$ and $\pi_-^*(P^1)$ respectively. Now because of the left compatibility in (13.2) we immediately get

PROPOSITION (14.1). *The principal* Spin(k)-*bundle* $Q \to T^k$ *is isomorphic to the bundle obtained from the two bundles*

$$\pi_+^*(P^0) \longrightarrow T_+, \qquad \pi_-^*(P^1) \longrightarrow T_-$$

by the identification

$$(p, s, g) \longleftrightarrow (p, s, sg)$$

for $s \in S^{k-1}$, $g \in \mathrm{Spin}(k)$ *and* $p \in P^0$.

Now suppose that $M = M^0 \oplus M^1$ is a graded C_k-module. Then we have a natural isomorphism

$$M^1 \cong \mathrm{Pin}^1(k) \times_{\mathrm{Spin}(k)} M^0.$$

Hence

$$\begin{aligned} P^1 \times_{\mathrm{Spin}(k)} M^0 &= P^0 \times_{\mathrm{Spin}(k)} \mathrm{Pin}^1(k) \times_{\mathrm{Spin}(k)} M^0 \\ &\cong P^0 \times_{\mathrm{Spin}(k)} M^1. \end{aligned}$$

From (14.1) and this isomorphism we obtain:

PROPOSITION (14.2). *The vector bundle* $Q \times_{\mathrm{Spin}(k)} M^0$ *over* T^k *is isomorphic to the bundle obtained from the two bundles*

$$\pi_+^*(P^0 \times_{\mathrm{Spin}(k)} M^0) \to T_+, \qquad \pi_-^*(P^0 \times_{\mathrm{Spin}(k)} M^1) \to T_-$$

by the identification

$$(p, s, m) \longleftrightarrow (p, s, sm) \qquad \text{for} \qquad p \in P^0, s \in S^{k-1}, m \in M^0.$$

Note. Here we have identified $\pi_+^*(P^0)$ with $P^0 \times S_+$, and $\pi_+^*(P^0 \times_{\mathrm{Spin}(k)} M^0)$ with $\pi_+^*(P^0) \times_{\mathrm{Spin}(k)} M^0$ etc.

Let us consider now the construction of §11 which assigned to any graded C_k-module M and any Spin(k)-bundle P^0 an element $\alpha_{P^0}(M) \in KO(B(V), S(V))$ where $V = P^0 \times_{\mathrm{Spin}(k)} \mathbf{R}^k$. This construction depended on the 'difference bundle' of §9. In our present case the spaces A, X_0, X_1 of §9 can be effectively replaced by T^k, T_+, T_- and we see from (14.2) (and the fact that $s^2 = -1$ for $s \in S_{k-1}$) that the bundle F of §9 is isomorphic to the bundle $Q \times_{\mathrm{Spin}(k)} M^0$. Now from the split exact sequence of the pair (T^k, T_-) and the isomorphisms

$$KO(T^k, T_-) \cong KO(T_+, T^{k-1}) \cong KO(B(V), S(V))$$

we obtain a natural projection

$$KO(T^k) \to KO(B(V), S(V)).$$

Then what we have shown may be stated as follows:

THEOREM (14.3). *Let* P^0 *be a principal* Spin(k)-bundle, M *a graded* C_k-*module*, $Q = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Spin}(k+1)$, $V = P^0 \times_{\mathrm{Spin}(k)} \mathbf{R}^k$, $T^k = Q/\mathrm{Spin}(k)$, $E^0 = Q \times_{\mathrm{Spin}(k)} M^0$, $p : KO(T^k) \to KO(B(V))$, $S(V)$ *the natural projection, then*

$$\alpha_{P^0}(M) = p(E^0).$$

If $k \equiv 0 \bmod 8$ and M is an irreducible (+1)-module then $p(E^0)$ is the element of $KO(B(V), S(V))$ used in [7] as the fundamental class. Thus (14.3) implies that this class coincides with our class μ_V. For some purposes, such as the behaviour under our definition of μ_V is more convenient. For others, such as computing the effect of representations, the definition in [7] is better. (14.3) enables us to switch from one to the other.

The proof of (14.3) carries over without change to the complex case, Spin being replaced by Spin^c throughout.

§15. Geometric interpretation of Clifford modules

Consider the data of §11. Thus V is a vector-bundle over X, $C(V)$ the corresponding Clifford bundle, and E a graded real Clifford module for V. The construction of χ_V in that section then depended on a particular geometric interpretation of the pairing

$$V \otimes E^1 \to E^0 \tag{15.1}$$

induced by the $C(V)$-structure on E. More precisely we passed from (15.1) to the family of maps

$$S(V_x) \times E_x^1 \to E_x^0 \qquad x \in X, \tag{15.2}$$

which describe a definite isomorphism along $S(V)$, of E^0 and E^1 lifted to $B(V)$, and so by the difference construction a definite element $\chi_V(E) \in KO(B(V), S(V))$.

There are two other geometric interpretations of (15.2) which we will discuss here briefly. The first one leads to a rather uniform description of the bundles on stunted projective spaces, while the second one explains the relation between Clifford modules and the vector field problem.

A. *The generalized* χ_V.

Let V be a Euclidean (real) vector bundle over X, $S(V)$ its unit sphere bundle. The group Z_2 then acts on $S(V)$ by the antipodal map, and we denote the projective bundle $S(V)/Z_2$ by $P(V)$. The projection $P(V) \to X$ will be denoted by π, and $\xi(V)$ shall stand for the line bundle induced over $P(V)$ by the nontrivial representation of Z_2 on $\mathbf{R}^1$:

$$\xi(V) = S(V) \times_{Z_2} \mathbf{R}^1$$

Consider now the data at the beginning of this section, in particular the induced family of maps:

$$S(V_x) \times E_x^1 \to E_x^0 \qquad x \in X.$$

We can clearly divide by Z_2 on the left due to the bilinearity of the inducing map. Thus we obtain maps

$$S(V_x) \times_{Z_2} E_x^1 \to E_x^0 \qquad x \in X, \tag{15.3}$$

which may be interpreted directly as an explicit isomorphism

$$\phi(V, E) : \xi(V) \otimes \pi^*(E^1) \to \pi^*(E^0).$$

We now let $W \subset V$ be a sub-bundle, and consider a graded $C(W)$-module E. The bundles $\xi(V) \otimes \pi^* E^1$ and $\pi^* E^0$ then become explicitly isomorphic along $P(W) \subset P(V)$ by means of $\phi(W, E)$, and so determine a well-defined difference element $\chi(V, W)E \in KO(P(V), P(W))$.

The linear extension of this construction now leads to a homomorphism,

$$\chi(V, W) : M(W) \to KO(P(V), P(W)), \tag{15.4}$$

and an analogous homomorphism

$$\chi^c(V, W) : M^c(W) \to K(P(V), P(W))$$

in the complex case. (15.4) is the desired generalization of the χ_W in §11. Before justifying this assertion, we remark that $\chi(V, W)$ clearly vanishes on those $C(W)$-modules which are restrictions of $C(V)$-modules. Hence if we set $A(V, W)$ equal to the cokernel of the restriction map $M(V) \xrightarrow{i^*} M(W)$, then $\chi(V, W)$ induces a homomorphism

$$A(V, W) \to KO(P(V), P(W)). \tag{15.5}$$

To see that the operation $\chi(V, W)$ indeed generalizes our earlier χ, one may proceed as follows: Let $V = W \oplus 1$, and let $f : B(W) \to P(V)$ be the fibre map which sends $w \in W_x$, into the line spanned by $(w, (1 - \|w\|^2))$ in $P(V)$. Thus f induces an isomorphism of $B(W)/S(W)$ with $P(V)/P(W)$. Now one just checks that the following diagram is commutative:

(15.6)
$$\begin{tikzcd}
M(W) \arrow[r, "\chi(V,W)"] \arrow[d, phantom, "\wr"] & KO\{P(V), P(W)\} \arrow[d, phantom, "f^{*}\wr"] \\
M(W) \arrow[r, "\chi_W"] & KO(B(W), S(W)).
\end{tikzcd}$$

It would be possible to extend a considerable portion of our work on χ_W to $\chi(W, V)$, but this does not seem justified by any application at present. However we wish to draw attention to the following property of $\chi(V, W)$.

PROPOSITION (15.7). *Let X be a point. Then the sequence*

$$M(V) \xrightarrow{i^*} M(W) \xrightarrow{\chi(V,W)} KO(P(V)P(W)) \to 0 \tag{15.8}$$

is exact. A similar result holds in the complex case.

In other words, over a point, the relation $A(V, W) \cong KO(P(V)/P(W))$ holds. As we gave a complete survey of the groups M_k and their inclusions in §5, this proposition gives the desired uniform description of the KO (and K) of a stunted real projective space. For example, taking dim $V = k$, dim $W = 1$, we obtain

$$\widetilde{KO}((P_{k+1}) \cong KO(P_{k-1}, P_0)) \cong Z_{a_k},$$

where a_k is the kth. Radon–Hurwitz number.

We know of no really satisfactory proof of proposition (15.7), primarily because we know of no good algebraic description of the higher KO^i of these spaces. On the other hand it is easy to show that $A(V, W) \to KO(P(V), P(W))$ is onto. For this purpose consider the diagram associated with a triple of vector-spaces $W \subset V' \subset V$

(15.9)
$$\begin{tikzcd}
 & 0 & & 0 \\
 & KO(P(V'), P(W)) \arrow[u] & KO(P(V), P(W)) \arrow[l] & KO(P(V), P(V')) \arrow[l] \arrow[u] \\
0 & A(V', W) \arrow[l] \arrow[u] & A(V, W) \arrow[l] \arrow[u] & A(V, V') \arrow[l] \arrow[u]
\end{tikzcd}$$

whose horizontal rows are exact; the upper one by the exact sequence of a triple, the lower one by the definition of the A-groups. We know, by (15.6), that $\chi(V, W)$ is a bijection if dim $V -$ dim $W \leqslant 1$. Hence, arguing by induction on dim $V -$ dim W we may assume that the vertical homomorphisms of (15.9) are also exact. But then the middle homomorphism must be onto, proving the assertion for the next higher value of dim $W -$ dim V.

The proof of proposition (15.7) may now be completed either by obtaining a lower bound for the groups in question from the spectral sequence of KO-theory, or by a detailed analysis of the sequence (15.9), which unfortunately involves several special cases. In view of the fact that a computation of $KO(P(k)/P(l))$ is now already in the literature [1] we will not pursue this argument further here.

B. *Relation with the vector-field problem*

We again consider the pairing

$$V \times E^0 \to E^1$$

of §11, but now focus our attention on the induced maps:

$$V_x \times_{Z_2} S(E_x^0) \to E_x^1 \qquad x \in X. \tag{15.10}$$

Note that this is only relevant if E is a *real* module.

The geometric interpretation of (15.10) is clear: if $\pi : P(E^0) \to X$ is the projective bundle of E^0 over X, and ξ is the canonical line bundle over $P(E^0)$, then (15.10) describes a definite injection:

$$\omega(V, E) : \pi^* V \otimes \xi \to \pi^* E^1. \tag{15.11}$$

It is possible to give (15.11) a more geometric setting if $S(V)$ admits a section, s. One may then use $w(V, E)$ to 'trivialize' a certain part of the 'tangent bundle along the fibres' of $P(E^0)$. Recall first that this bundle, which we will denote by $\mathscr{T}_F(E^0)$, is described in the following manner. The bundle $\xi = \xi(E^0)$ is canonically embedded in $\pi^*(E^0)$, whence $\pi^*(E^0)/\xi$ is well defined. Then we have

$$\mathscr{T}_F(E^0) = (\pi^*(E^0)/\xi) \otimes \xi. \tag{15.12}$$

With this understood, let V' be the quotient of V by the line bundle determined by s:

$$0 \to 1 \xrightarrow{s} V \to V' \to 0$$

and let $s_* : E^0 \to E^1$ be the isomorphism induced by multiplication by $s(x)$ in E^0_x. It is then quite easy to check that the homomorphism $s_*^{-1} \cdot \omega(V, E) : \pi^* V \otimes \xi \to \pi^* E^0$ induces an injection

$$\pi^* V' \otimes \xi \to \pi^* E^0/\xi .$$

Tensoring this homomorphism with ξ, we obtain the desired injection:

$$\omega(s, V, E) : \pi^* V' \to \mathscr{T}_F(E^0). \tag{15.13}$$

Let us now again restrict the whole situation to a point. Then if $\dim V = k$, $\dim E^0 = m$, V' will be a trivial bundle of dimension $k - 1$, and $\mathscr{T}_F(E^0)$ will be the tangent-bundle of projective $(m - 1)$-space P_{m-1}.

Applying the results of §5 we conclude that the following proposition is valid:

PROPOSITION (15.14). *Let $m = \lambda a_k$ where a_k is the* kth. *Radon–Hurwitz number. Then the tangent bundle of P_{m-1} (and hence of S_{m-1}) contains a $(k - 1)$-dimensional trivial bundle.*

The work of Adams [1], gives the converse of this proposition: if the tangent bundle of S_{m-1} contains a trivial $(n - 1)$-bundle, then $m = \lambda a_n$.

We remark in closing that on the other hand the generalized vector-field question is still open. This question is: let ξ be the line bundle over P_n, then what is the maximum dimension of a trivial bundle in $m\xi$, $m \geq n$. Thus the vector field problem solves this question for $m = n$. The general solution would, by virtue of the work of M. Hirsch, give the most economical immersions of P_n in Euclidean space.

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THE STABLE HOMOTOPY OF THE CLASSICAL GROUPS

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1. Introduction

Throughout this paper M shall denote a compact connected Riemann manifold of class C^∞. Let $\nu = (P, Q; h)$ be the triple consisting of two points P and Q on M together with a homotopy class h of curves joining P to Q. We will refer to such triples as *base points* on M.

Corresponding to $\nu = (P, Q; h)$ we define M^ν to be the set of all geodesics of *minimal* length which join P to Q and are contained in h.

There is an obvious map of the suspension of M^ν into M: one merely assigns to the pair (s, t), $s \in M^\nu$; $t \in [0, 1]$, the point on s which divides s in the ratio t to $1 - t$. (For fixed small $t > 0$, this map is 1 to 1 on M^ν and serves to define a topology on M^ν.) The induced homomorphism of $\pi_k(M^\nu)$ into $\pi_{k+1}(M)$ will be denoted by ν_*.

Let s be an arbitrary geodesic on M from P to Q. The index of s, denoted by $\lambda(s)$, is the properly counted sum of the conjugate points of P in the interior of s. We write $|\nu|$ for the first *positive* integer which occurs as the index of some geodesic from P to Q in the class h. In terms of these notions our principal result is the following theorem.

THEOREM I. *Let M be a symmetric space. Then for any base point ν on M, M^ν is again a symmetric space. Further, ν_* is onto in positive dimensions less than $|\nu|$ and is one to one in positive dimensions less than $|\nu| - 1$. Thus:*

$$\pi_k(M^\nu) = \pi_{k+1}(M) \qquad 0 < k < |\nu| - 1 . \tag{1.1}$$

As an example, let M be the n-sphere, $n \geqq 2$, and let $\nu = (P, Q)$ consist of two antipodes. (Because S^n is simply connected the class h is unique.) Then M^ν is the $(n - 1)$-sphere, and $\nu_* : \pi_k(S^{n-1}) \to \pi_{k+1}(S^n)$ coincides with the usual suspension homomorphism. The integers which occur as indexes of geodesics joining P to Q, are seen to form the set 0, $2(n-1)$, $4(n-1)$, etc. Hence $|\nu| = 2(n - 1)$, and (1.1) yields the Freudenthal suspension theorem. If $\nu = (P, Q)$ with Q not the antipode of P, then M^ν is a single point, while $|\nu|$ is seen to be $(n - 1)$. In that case (1.1) merely implies that $\pi_k(S^n) = 0$ for $0 < k \leqq n - 2$.

At first glance the evaluation of $|\nu|$ may seem a formidable task.

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However on a symmetric space (see section 5) every pair of points (P, Q) is contained in a maximal flat geodesic torus T, and every index $\lambda(s)$ already occurs as the index of a geodesic joining P to Q on T. Further, for such a geodesic, $\lambda(s)$ is equal to the number of times s crosses the "singular" subtori of T. The disposition of these singular tori is well known. The computation of $|\nu|$ is therefore a routine matter.

Theorem I yields new results in the following manner: In view of the fact that with M the space M^{ν} is again symmetric, one may repeat the procedure of passing from M to M^{ν}. To facilitate the use of this iteration we will agree to call a sequence of symmetric spaces $\cdots M_1 \to M_2 \to M_3 \cdots$ a ν-sequence if at each step $M_i = M^{\nu}_{i+1}$ for some appropriate base point ν in M_{i+1}. For example, the sequence $\cdots S^n \to S^{n+1} \to S^{n+2} \cdots$ is a ν-sequence,

THEOREM II. *The following are three ν-sequences with the value of $|\nu|$ indicated at each step.*

(1.2) $$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n) \xrightarrow{2n+2} \mathrm{U}(2n)$$

(1.3) $$\mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n) \xrightarrow{n+1} \mathrm{U}(2n)/\mathrm{O}(2n)$$
$$\xrightarrow{2n+1} \mathrm{Sp}(2n)/\mathrm{U}(2n) \xrightarrow{4n+2} \mathrm{Sp}(2n)$$

(1.4) $$\mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n) \xrightarrow{4n+1} \mathrm{U}(4n)/\mathrm{Sp}(2n)$$
$$\xrightarrow{8n-2} \mathrm{SO}(8n)/\mathrm{U}(4n) \xrightarrow{8n-2} \mathrm{SO}(8n)$$

Here we have used the standard notations and inclusions.

Notice that $|\nu|$ tends to ∞ with n at each step of these sequences. On the other hand it is well known that for each of the symmetric spaces involved, π_k becomes independent of $n \gg k$. (We will indicate these stable values of π_k by dropping the subscript n and using bold face type. For example, $\pi_k(\mathbf{U}/\mathbf{O}) = \pi_k\{\mathrm{U}(n)/\mathrm{O}(n)\}$ for $n \gg k$.) Finally, recall that in this notation $\pi_k(\mathbf{U}) = \pi_{k+1}(\mathbf{U}/\mathbf{U}\times\mathbf{U})$, $\pi_k(\mathbf{O}) = \pi_{+1}(\mathbf{O}/\mathbf{O}\times\mathbf{O})$ and $\pi_k(\mathbf{Sp}) = \pi_{k+1}(\mathbf{Sp}/\mathbf{Sp}\times\mathbf{Sp})$ $(k = 0, 1, \cdots)$, because in each instance the space on the right hand side represents the universal base space of the group in question. Combining these three observations with Theorem I, we obtain the following corollary to Theorem II.

COROLLARY. *The stable homotopy of the classical groups is periodic:*

(1.5) $$\begin{aligned} \pi_k(\mathbf{U}) &= \pi_{k+2}(\mathbf{U}) \\ \pi_k(\mathbf{O}) &= \pi_{k+4}(\mathbf{Sp}) \\ \pi_k(\mathbf{Sp}) &= \pi_{k+4}(\mathbf{O}) \qquad k = 0, 1, \cdots \end{aligned}$$

The groups $\pi_k(\mathbf{U})$ are $0, Z$ for $k = 0, 1$. Hence $0, Z$ is the period of $\pi_*(\mathbf{U})$. In the case of $\mathbf{Sp}$, one has the groups $0, 0, 0, Z$, for $k = 0, 1, 2, 3$ respectively. For $\mathbf{O}$ these first four groups are Z_2, Z_2 0, Z. Hence the period of $\pi_*(\mathbf{O})$ is $Z_2, Z_2, 0, Z, 0, 0, 0, Z$. Applying (1.3) and (1.4) one also obtains the stable homotopy of the other symmetric spaces. Thus:

(1.6) $$\begin{aligned} \pi_k(\mathbf{Sp}/\mathbf{U}) &= \pi_{k+1}(\mathbf{Sp}) \qquad k = 0, 1, 2\cdots \\ \pi_k(\mathbf{U}/\mathbf{O}) &= \pi_{k+2}(\mathbf{Sp}) \qquad k = 0, 1, 2\cdots \end{aligned}$$

while

(1.7) $$\begin{aligned} \pi_k(\mathbf{O}/\mathbf{U}) &= \pi_{k+1}(\mathbf{O}) \qquad k = 0, 1, 2\cdots \\ \pi_k(\mathbf{U}/\mathbf{Sp}) &= \pi_{k+2}(\mathbf{O}) \qquad k = 0, 1, 2\cdots \end{aligned}$$

(In the third formula we have replaced $\mathbf{SO}/\mathbf{U}$ by $\mathbf{O}/\mathbf{U}$ to obtain the correct value of π_0.)

The formulas (1.5) to (1.7) were already announced in [4]. The unitary groups were discussed by a different method in [5], where the unstable group $\pi_{2n}\{\mathrm{U}(n)\}$ was also evaluated as $Z/n!Z$.

The proof of Theorem I is summarized in this fashion: Let $\nu = (P, Q; h)$ be a base point, and let $\Omega_\nu M$ be the space of path from P to Q on M in the class h. We then construct a CW-model for $\Omega_\nu M$ which is of the form $K = M^{\nu} \cup e_1 \cup e_2$ etc., where the e_i are cells of dimension greater than or equal to $|\mu|$.

The existence of such a K follows readily from the Morse theory. For instance the deformations given in Seifert-Threlfall [10, pp. 34, 35] and can be interpreted us follows: Suppose that a smooth function f defined on a compact manifold N has a single nondegenerate critical point p, of index k in the range $a \leqq f \leqq b$, $a < f(p) < b$. Let N^a respectively N^b be the sets $f \leqq a$ and $f \leqq b$ on N. The assertion is, that then N^b is obtained from N^a by attaching a k-cell, e_k, to N^a. In symbols, $N^b = N^a \cup e_k$. (This point of view is also emphasized in notes by Pitcher [9], and R. Thom [12].)

To prove our theorem this interpretation of the Morse theory is first extended in two ways:

(A) The loopspace problem is reduced to the manifold problem.
(B) The notion of nondegeneracy is extended.

Thereafter it is shown that on a symmetric space the critical sets in the loopspace are nondegenerate for every choice of a base point.

The step (A) is already essentially contained in Morse [8]; while the notion of a nondegenerate critical manifold (step B) was introduced in [2][1].

The final step follows easily from the results of [6].

It is clear from this rough plan of the proof that considerable reviewing of more or less known material will be necessary to make the account intelligible. Because the theory of a nondegenerate function on a smooth manifold is by now well known, while some mystery still seems to hang over Morse's extension of this theory to loop spaces, we will review step (A) in greater detail than the other two steps.

2. Review of the Morse theory. A reduction theorem

Let $\mu = (P, Q)$ be any two points of M. The space of paths from P to Q on M is denoted by $\Omega_\mu M$ and is defined as follows:

DEFINITION 2.1. *The points of $\Omega_\mu M$ are the piecewise differentiable maps c: $[0,1] \to M$ which are parametrized proportionally to arc length, take 0 into P, and map 1 onto Q. The distance between two points c and c' in $\Omega_\mu M$ is given by:*

$$\rho_1(c, c') = \max_{t \in [0,1]} \rho\{c(t),\ c'(t)\} + |J(c) - J(c')|$$

where ρ is the metric on M, and J denotes the length function on $\Omega_\mu M$.

The advantage of this definition of $\Omega_\mu M$ is that $J(c)$, the length of c, is a continuous function of $\Omega_\mu M$. On the other hand $\Omega_\mu M$ is not complete.

If a is a real number, the subset of $\Omega_\mu M$ on which $J \leqq a$, is denoted by $\Omega_\mu^a M$, and is referred to as a half space of $\Omega_\mu M$. Such a half space is called regular if $\Omega_\mu^a M$ contains no geodesic of length a.

Let F be a continuous real valued function on a compact manifold N. The set $\{x \in N;\ F(x) \leqq a\}$ will be denoted by $F^a N$, or just N^a if the function is understood, and is also called a half-space for F on N. The half-space is called regular if F is of class C^∞ in some neighborhood of $F^a N$, and if F has no critical points at the level a. (In other words $dF(x) \neq 0$ if $F(x) = a$.)

The aim of this section is to show that every regular half space of $\Omega_\mu M$, is of the same homotopy type as a regular half-space of a manifold.

It turns out that if one steers a middle course between Morse and Seifert and Threlfall such a "model" for $\Omega_\mu^a M$ is easily constructed. We have just defined $\Omega_\mu M$ according to Seifert and Threlfall; for the rest we follow, in spirit at least, Morse's account of thirty years ago.

Let φ_n: $M^n \to R$, be the function from the n^{th} cartesian product of M with itself, which assigns to $(x) = (x_1, \cdots, x_n)$ the number:

$$\varphi_n(x) = \rho^2(P, x_1) + \rho^2(x_1, x_2) + \cdots \rho^2(x_n, Q) .$$

were $\rho(x, y)$ denotes the distance between x and y on M, as before.

REDUCTION THEOREM I. *Let a be a positive number. Then there exists an integer n such that $\Omega_\mu^a M$ is of the same homotopy type as the half space $\varphi_n^b M^n$ of φ_n on M^n, where $b = a^2/n + 1$. Thus,*

$$\Omega_\mu^a M \approx \varphi_\mu^b M^n . \tag{2.1}$$

The statement (2.1) is new, although quite implicit in Morse's account. He, of course, did not have a definition of $\Omega_\mu M$ on which the length function was continuous. A slightly surprising technical phenomenon is that the function φ_n alone suffices to define a model for $\Omega_\mu^a M$. In Morse's original account, he essentially shows that $\Omega_\mu^a M$ is of the same homotopy type as the subset of M^n characterized by $\rho(x_i, x_{i+1}) < \bar{\rho}$; $\sum_{i=0}^{i=n} \rho(x , x_{i+1}) \leqq a$. (Here $x_0 = P$; $x_{n+1} = Q$).

PROOF OF (2.1). There exists a number $\bar{\rho} > 0$ such that two points of M with distance less than $\bar{\rho}$ have a unique shortest geodesic joining them. This shortest geodesic then varies smoothly with the end points, in particular $\rho^2(x, y)$ is a C^∞ function of x and y as long as $\rho(x, y) < \bar{\rho}$. Suppose now that n is chosen so large that:

$$a/\sqrt{n+1} < \bar{\rho} . \tag{2.4}$$

Under this condition on n we define maps α: $\Omega_\mu^a M \to \varphi_n^b M$ and β: $\varphi_n^b M^n \to \Omega_\mu^a M$ which constitute a homotopy equivalence. (For convenience we write φ for φ_n and denote $\varphi_n^b M^n$ by M_*^b in the sequel.)

DEFINITION OF α. Let $c \in \Omega_\mu^a M$. Then $\alpha(c) \in M^n$ is to be the point:

$$\alpha(c) = \{c(t_1), c(t_2), \cdots, c(t_n)\} ; \qquad t_i = i/n + 1 ;$$

Clearly α is a continuous function from $\Omega_\mu^a M$ to M^n. Next, $\varphi(\alpha c) = \sum_{i=0}^{i=n} \rho^2\{c(t_i), c(t_{i+1})\}$. Each term of this sum is $\leqq (a/n+1)^2$ because c is parametrized proportionately to arc-length. Hence $\varphi(\alpha c) \leqq (a^2/n+1) = b$. The map α therefore take values in M_*^b.

DEFINITION OF β. If $x = (x_1, \cdots, x_n)$ is a point of $M_*^b = \varphi^b M^n$, then each of the numbers, $\{\rho(P, x_1),\ \rho(x_1, x_2) \cdots \rho(x_{n+1}, Q)\}$ is less than $a/\sqrt{n+1}$, hence less than $\bar{\rho}$. The unique geodesics joining consecutive points of the

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[1] The applications given in [2] are false, as was pointed out to me by A. S. Schwartz [11]. A distressingly simple example shows that the assertion [2, p. 253] to the effect that $\tilde{V}_{l,k}$ is a manifold is wrong. This mistake invalidates the computations for the circular connectivities of the n-sphere.

array $P, x_1, \cdots, x_n, Q$ are therefore well defined and combine to yield a curve, c, in $\Omega_\mu M$. By the Cauchy inequality the length of c does not exceed a. The correspondence $x \to c$ defines the map β.

LEMMA 2.1. *There exists a homotopy* D_t, $0 \leqq t \leqq 1$ *of* $\Omega_\mu^a M$ *on itself such that* D_0 *is the identity, and* $D_1 = \beta \circ a$.

The needed deformation is given explicitly in [10, p. 51]. One deforms the segment of c between t_i and t_{i+1}, into the geodesic chord joining $c(t_i)$ to $c(t_{i+1})$. The intermediate curves are geodesic segments from $c(t_i)$. $c(t_i + \varepsilon)$ followed by the original curve from $t_i + \varepsilon$ to t_{i+1}.

LEMMA 2.2. *There exists a homotopy* Δ_t, $0 \leqq t \leqq 1$, *of* M_*^b *on itself, such that* Δ_0 *is the identity, and,* $\Delta_1 = \alpha \circ \beta$.

This homotopy is to be found in Morse [8, p. 217]. If $x \in M_*^b$, $\beta(x)$ is a polygonal curve joining P to Q. Let c: $[0,1] \to M$ the parametrization of $\beta(x)$ which is proportional to arc length. Let $0 \leqq a_1, < \cdots, a_n \leqq 1$, be the pre-images under c of the points $x = \{x_1, \cdots, x_n\}$ on $\beta(x)$. The $\{a_i\}$ then correspond to the parameter values of the original vertices on $\beta(x)$. The composition $\alpha \circ \beta$ takes x into $\{c(t_1), c(t_2), \cdots, c(t_n)\}$ where $t_i = i/n+1$. Hence if $a_i = t_i$, then the $\alpha \circ \beta(x) = x$, and what is needed is a "universal" homotopy which takes the points a_i into the points t_i. The natural way of constructing this homotopy is to dispatch a_i on its way to t_i at a linear speed proportional to the distance to be traversed. In formulas, let

$$\begin{aligned} a_0^\tau &= 0 \\ a_0^\tau &= a_i(1-\tau) + \tau\, t_i\,, \qquad 0 \leqq \tau \leqq 1;\ i = 1, \cdots, n \\ a_{n+1}^\tau &= 1 \end{aligned}$$

The homotpy Δ_τ assigns to x the point $\{c(a_i^\tau)\}$ where $c = \beta(x)$. Clearly the a_i vary continuously with x for $x \in M_*^b$, so that Δ_τ is a proper homotopy. It remains to be checked that Δ_τ keeps M_*^b invariant. For this purpose it is sufficient to prove that $\varphi(\Delta_\tau x) \leqq \varphi(x)$; $0 \leqq \tau \leqq 1$.

Let $J(x)$ be the length of $\beta(x)$, and set $\delta_i = J(x)(a_i - a_{i-1})$. Thus $\sum_{i=1}^{i=n+1} \delta_i = J(x)$, while $\sum_{i=1}^{i=n+1} \delta_i^2 = \varphi(x)$. We also write $\{x_i^\tau\}$ for the coordinates $\Delta_\tau x$. Then:

$$\rho(x_i^\tau, x_{i+1}^\tau) \leqq \delta_{i+1}(1-\tau) + \tau(J(x)/n+1)\,,$$

because $\beta(x)$ is parametrized proportionally to arc length. Hence:

$$\varphi(\Delta_\tau x) \leqq \sum_{i=1}^{i=n+1} [\delta_i(1-\tau) + \tau(J(x)/n+1)]^2\,.$$

After expanding, the right hand side is seen to equal

$$\varphi(x) - 2\tau(\varphi(x) - \{J^2(x)/n+1\}) + \tau^2(\varphi(x) - \{J^2(x)/n+1\})$$

By the Cauchy inequality $\varphi(x) - J^2(x)/n + 1 \geqq 0$. Hence in the range $0 \leqq \tau \leqq 1$, $\varphi(\Delta_\tau x) \leqq \varphi(x)$. This completes the proof of the lemma, and hence of (2.1).

The statement (2.1) has a refinement which will be formulated next. Its purpose is to relate certain geometric properties of the geodesics in $\Omega_\mu^a M$ with the critical points of φ on M_*^b. Recall first the notion of the *index of a critical point.* If p is a critical point of the smooth function φ on the manifold N, the Hessian of φ, denoted by $H_p\varphi$, is the bilinear symmetric function on the tangent space N_p of N at p, which in terms of local coordinates is defined by $H_p\varphi(\partial/\partial x_\alpha, \partial/\partial x_\beta) = \partial^2\varphi/\partial x_\alpha \partial x_\beta$. The index of p as a critical point of φ is by definition the dimension of a maximal subspace of N_p on which the Hessian is negative definite. This integer is denoted by $\lambda_\varphi(p)$. Finally we briefly review the notion of a conjugate point on a geodesic. For details the reader is referred to [8] and [6].

If $s(\alpha, t)$ is a smooth family of geodesics, depending on a parameter α, then the vector field $\partial s(\alpha, t)/\partial\alpha|_{\alpha=0}$ along $s(0, t)$ is called a J-field along $s = s(0, t)$. The totality of such vector fields along s, forms a vector space J_s over the real numbers. If the length of s is less than ρ, every V in J_s is uniquely determined by its values at the end-points of s. In general, if P and Q are two points of s, Q is called a conjugate point of P (along s) of multiplicity k if the subspace of J_s, consisting of the fields which vanish at both P and Q, is of dimension precisely k.

REDUCTION THEOREM II. *The homotopy equivalence* α: $\Omega_\mu^a M \to M_*^b$ *constructed in the proof of* (2.1) *has the following properties*:

(2.2) *Under* α *the geodesics of* $\Omega_\mu^a M$ *are mapped one to one onto the critical points of* φ *on* M_*^b.

(2.3) *If* s *is a geodesics of* $\Omega_\mu^a M$ *and* p *is its image under* α, *then*:

The dimension of the nullspace of $H_p\varphi$ *equals the multiplicity of* Q *as a conjugate point of* P *along* s.

The index $\lambda_\varphi(p)$ *is equal to the number (counted with multiplicities) of conjugate points of* P *in the interior of* s.

Except for a minor technicality, (2.2) and (2.3) are the content of Morse's index theorem. See [8, p. 91]. The technicality in question is the following one. Let ψ be the function $\rho(P, x_1) + \rho(x_1, x_2) + \cdots + \rho(x_n, Q)$. This function is smooth provided that no two consecutive coordinates coincide. Thus, except in a trivial case, the function ψ is smooth near the point p of (2.3), and, as will be shown in a moment, p is also a critical point of ψ. If in (2.3) we replace $\lambda_\varphi(p)$ by $\lambda_\psi(p)$ we obtain the statement of Morse. Note however that (2.2) with φ replaced by ψ is not true. Indeed, the critical sets of ψ are cells obtained by sliding the vertices along a given geodesic.

To prove our theorem it is therefore sufficient to establish (2.2) and the equality of $\lambda_\varphi(p)$ with $\lambda_\psi(p)$.

PROOF OF (2.2). If s is a geodesic segment of $\Omega^a_\mu M$ then $\beta \circ \alpha(s) = s$. Hence α imbeds this set of curves in M^b_*, and it remains to identify the critical points of φ on this set. Let $x \in M^n$, let X be a tangent vector to M^n at x, and consider the derivative $X\varphi$ of φ in the direction X. The point x is critical if and only if $X\varphi = 0$ for all X in the tangent space at x. Suppose that x has the coordinates $(x_1, \cdots, x_n)$ and that X has the corresponding components $(X_1, \cdots, X_n)$ in the natural product structure of the tangent space to M^n at x. Let s_i denote the geodesic segment from x_i to x_{i+1}, where we now set $x_0 = P$, $x_{n+1} = Q$, and let $\dot{s}^1_i$, respectively $\dot{s}^0_i$, be the unit tangent vector of s_i at x_{i+1} and x_i. By the well known first variation formula:

$$X \cdot \rho^2(x_i, x_{i+1}) = 2\,|\,s_i\,|\,\{\langle \dot{s}^1_i, X_{i+1}\rangle - \langle \dot{s}^0_i X_i\rangle\}\ ,$$

where $<, >$ denotes the inner product of the Riemannian structure, and $|\,s_i\,|$ denotes the length of s_i one obtains the expression:

$$X\varphi = 2\sum_{i=0}^{i=n-1} \langle\,|\,s_i\,|\,\dot{s}^1_i - |\,s_{i+1}\,|\,\dot{s}^0_{i+1} X_{i+1}\,\rangle\ .$$

The components X_i of X are independent. Hence $X\varphi = 0$ for all X if and only if $\dot{s}^1_i = \dot{s}^0_{i+1}$; $|\,s_i\,| = |\,s_{i+1}\,|$; $i = 1, \cdots, n-1$. In other words x is a critical point if and only if $\beta(x)$ is a geodesic, and $\alpha \circ \beta(x) = x$. This completes the proof of (2.2).

PROOF OF (2.3) Let A be the tangent space M^n_p. By varying the vertices of p along s, we single out a subspace $A^{\natural}$ of A on which $H_p\varphi$ is clearly positive definite. It therefore suffices to study the restriction of $H_p\varphi$ to a suitable complement of $A^{\natural}$ in A. Such a complement is furnished by the elements $X = \{X_i\}$ in A with each X_i perpendicular to s. Let this complement be denoted by A^{*}, and suppose $X, Y \in A^{*}$. For each segment s_i choose J-fields U_i and V_i. so that at the end points s_i, U_i coincides with X_{i-1} and X_i, while V_i coincides with Y_{i-1} and Y_i. We write this condition in the form $U^+_i = X_{i+1}$; $U^-_i = X_i$, etc. Because $|\,s_i\,| < \rho$, the U_i, V_i are uniquely determined by X and Y. Now by the second variation formula,

$$H_p\varphi(X, Y) = k\sum \langle \Delta U^+_i - \Delta U^-_{i+1}, V^+_i\rangle$$

where ΔU_i denotes the covariant derivative of U_i along s, and k is equal

to $(2/n+1) \times$ length of $\beta(x)$. For the function ψ we obtain similiarly the expression

$$H_p\psi(X, Y) = \sum \langle \Delta U^+_i - \Delta U^-_{i+1}, V^+_i\rangle$$

Thus on A^{*} these two Hessians differ only by a positive factor. On the complementary subspace $H_p\psi$ vanishes. Hence $\lambda_\varphi(p) = \lambda_\psi(p)$ as was to be shown.

REMARK. These formulas immediately prove the first part of (2.3). Indeed, a vector X is in the null space of $H_p\varphi$ if and only if the J-fields U_i along s_i fit together to form a global J-field along s which vanishes at both P and Q. In this manner Morse obtains the formula for the null space of $H_p\varphi$. Concerning the index formula, let me just remark that Morse obtains it by deforming Q along s into P, and observing that the index form $H_p\psi$ does not change during this deformation except when Q passes through conjugate points of P. At such points the index is shown to decrease by precisely the multiplicity of the conjugate point.

The two reduction theorems complete our original program of assigning to every regular half space of $\Omega^a_\mu M$ a regular half space of a compact manifold which is of the same homotopy type. (The fact that regularity is preserved under α follows from (2.2)). We will call the set M^b_* constructed in this section a model for $\Omega^a_\mu M$. If $\nu = (P, Q\colon h)$ is a base point, $\Omega_\nu M$ denotes the component of h in $\Omega_\mu M$ and the image of $\Omega^a_\nu M$ under α will be called a model for $\Omega^a_\nu M$. It is clear that the reduction theorem holds equally well in this new setting.

3. Review of the Morse Theory. The nondegenerate case

The classification of critical points according to index and nullity has topological implications which are usually expressed by the Morse inequalities. Actually however this "homology formulation" is proved by homotopy arguments. It is better therefore to state these implications in the language of CW-complexes [13]. In this manner homology consequences are easily accessible while the homotopy implications are not lost. (See [9] and [12].)

DEFINITION 3.1. (See [2].) *Let V be a smooth connected submanifold of the regular half space $N^a = f^aN$. Such a manifold is called a nondegenerate critical manifold of f on N^a if:*

(3.1) *Each point of V is a critical point of f.*

(3.2) *For any $p \in V$, the nullspace of H_pf is the tangent space of V at p.*

An immediate consequence of (3.2) is that $\lambda_f(p)$ is a constant on V.

This integer is the index of V, and is written $\lambda_f(V)$. If V reduces to a point, H_pf is non-singular by the condition (3.2). The present notion therefore generalizes the classical definition of a nondegenerate critical point.

Let V be a nondegenerate critical manifold of f on N^a. We define the *negative bundle*, ξ_V, over V in the following manner.

Let a Riemannian structure be defined on N. At each point $p \in V$ the form H_pf then uniquely determines a linear self-adjoint transformation T_p on the tangent space of N at p, by the formula,

$$\langle T_pX, Y\rangle = H_pf(X, Y) \qquad X, Y \in N_p\,. \tag{3.3}$$

These transformations combine to define a linear endomorphism, T, of the tangent space to N along V. By condition (3.2) the kernel of T is precisely the tangent space to V. Thus T is an automorphism of the normal bundle of V in N.

Now let ξ_V be the subbundle of this normal bundle which is spanned by the negative eigendirections of T. Thus the fiber of ξ_V at $p \in V$ is spanned by the normal vectors to V at p, for which $T_p \cdot Y = \lambda Y, \lambda < 0$. The fiber of ξ_V therefore has dimension $\lambda_f(V)$. If $\lambda_f(V)=0$, we set ξ_V equal to V. The bundle ξ_V is independent of the Riemannian structure used.

Finally, recall the notion of attaching a vector bundle ξ, to a space Y to form the space $Y \cup \xi$.

In general if α: $A \rightarrow Y$ is a map of a subset $A \subset X$ one forms the space $Y \cup_\alpha X$ by identfying $a \in A \subset X$ with $\alpha(a) \in Y$ in the disjoint union Y with X.

This attaching construction has the following elementary properties:

(3.4) The homotopy type of $Y \cup_\alpha X$ depends only on the homotopy type of α.

(3.5) If (X_1, A_1) is a deformation retract of (X, A) and if $\alpha_1 = \alpha \mid A_1$, then $Y \cup_{\alpha_1} X_1$ is of the same homotopy type as $Y \cup_\alpha X$.

When X is an n-cell e_n, and A is the bounding sphere of e_n, $Y \cup_\alpha e_n$ is referred to as Y with the cell e_n attached. If ξ is an orthogonal n-plane bundle, we form the space $Y \cup \xi$, by taking, in the above procedure, X equal to the set D_ξ of vectors of length $\leqq 1$ and setting A equal to $S_\xi = \partial D_\xi$. In this case we speak of Y with ξ attached, and if α is not explicitly in evidence just use the notation $Y \cup \xi$. If ξ is a 0-dimensional vector-bundle $Y \cup \xi$ stands for the disjoint union of Y with the base-space of ξ.

With this notation and terminology understood, the principal result of the nondegenerate Morse theory can be stated as follows:

THEOREM III. *Suppose that $N^a \subset N^b$ are two regular half-spaces of the function f on the compact manifold N.*

(3.6) *If f has no critical point in the range $a \leqq f \leqq b$ then N^a is a deformation retract of N^b.*

(3.7) *If f has a single nondegenerate critical manifold V in the range $a \leqq f \leqq b$, then N^b is of the same homotopy type as N^a with the negative bundle of f along V attached:*

$$N^b = N^a \cup \xi_V$$

where ξ_V is the negative bundle of f along V.

Immediate consequences in homotopy, [13], are:

COROLLARY 1. *Under the assumptions of* (3.7):

$$N^b = N^a \cup e_1 \cup \cdots \cup e_s \tag{3.8}$$

where the cells e_i, $i = 1, \cdots, s$, have dimension $\geqq \lambda_f(V)$. In particular.

$$\pi_r(N^b, N^a) = 0 \qquad \text{for } 0 \leqq r < \lambda(V)\,. \tag{3.9}$$

Using excision and Poincaré duality (3.2) implies:

COROLLARY 2. *Under the assumptions of* (3.7)

$$H^r(N^b, N^a; G) \approx H_c^r(\xi_V; G) = H^{r-\lambda}(V; G') \qquad \lambda = \lambda_f(V)\,. \tag{3.10}$$

Here the subscript c denotes compact cohomology, and by G' we mean the tensor of the coefficients G by the orientation sheaf of ξ_V.

REMARKS. In [2] we derived (3.10) with G specialized to Z_2. In this paper we will need only (3.9) but it seemed to me that (3.7) summarizes the situation better than any of the other versions. Remark that (3.10) implies (3.9) *if N^a is assumed to be simply connected.* On the other hand (3.8) yields (3.9) without this troublesome hypothesis.

The restriction that V be the only critical set of f in the range from a to b is not essential. If all the critical sets are nondegenerate, they are necessarily finite in number, so that if we denote them by V^i: $i = 1 \cdots s$; then Theorem III is easily modified to yield the formula

$$N^b = N^a \cup \xi_{V_1} \cup \cdots \cup \xi_{V_s}\,.$$

If N^a is triangulated, the attaching map of cell e_k can be deformed into the $(\dim e_k - 1)$-skeleton of N^a. In this way N^b becomes a CW-complex.

The case when V is a point, p, is completely treated in [10]. The present extension is best summarized by saying that what is done for a neighborhood of p in [10] can equally well be done in a normal neighborhood of V in the present case. On each fiber of such a neighborhood one encounters the nondegenerate critical point problem.

PROOF OF 3.6. Let N^b be endowed with a Riemann structure and denote the gradient of f corresponding to this structure by ∇f. If $p \in \overline{N^b - N^a}$, L_p shall denote the integral curve of $-\nabla f$ through p in its natural parameter. Because $df \neq 0$ on this set L_p is well defined. Further because $\overline{N^b - N^a}$ is compact, $|\nabla f| > \varepsilon_0 > 0$ on this set. Hence each L_p intersects $f^{-1}(a)$ at some point, say $h(p)$, and the function $p \to h(p)$ defines $f^{-1}(a)$ as a retract of $\overline{N^b - N^a}$. By assigning to p the point $h_t(p)$ on L_p which divides the segment from p to $h(p)$ in the ratio $1 : 1-t$, $f^{-1}(a)$ is seen to be a deformation retract $\overline{N^b - N^a}$. Hence (3.6) is true.

NOTE. The critical values of f form a closed set. Hence $N^{a-\varepsilon}$ is again a regular half-space of f when $\varepsilon > 0$ is small enough. Using this additional space it is easily seen that under the conditions of (3.6) N^b and N^a are in fact homeomorphic.

PROOF OF 3.7. We may assume that $f(V)=0$, and that f has no critical points in the range $[(-\varepsilon_0, 0); (0, \varepsilon_0)]$. It is also sufficient to prove that under these conditions $N^\varepsilon = N^{-\varepsilon} \cup \xi_V$ for some $0 < \varepsilon < \varepsilon_0$.

We have already defined $\xi = \xi_V$ as the negative bundle of f along V. Let ξ^+ be the negative bundle of function $-f$ along V. Then, clearly, the normal bundle η of V in N is the direct sum ξ^+ with ξ.

$$\eta = \xi^+ \oplus \xi \,. \tag{3.11}$$

We let $\pi: \eta \to \xi$ be the natural projection. The length of a vector $X \in \eta$ is denoted by $|X|$ and the function $X \to |X|^2$ is denoted by φ.

Let $\rho: \eta \to N$ be the exponential map. This map is a homeomorphism in the vicinity of V included in η as the zero cross-section. Thus ρ induced a Riemann structure $(\, , \,)$ on this vicinity. The function $f \circ \rho$ will be donoted by f_*.

The condition that V is a nondegnerate critical manifold of f clearly implies that the function f_* restricted to any fiber of η has a nondegenerate critical point. More precisely the following is true:

(3.12) The function f_*, restricted to any fiber of ξ^+, $[\xi]$, has a nondegenerate minimum [maximum] at 0.

An easy computation now yields the following consequence:

(3.13) The function $(df_*, d\varphi)$, restricted to any fiber of $\xi^+[\xi]$ has a non-degenerate minimum [maximum] at 0.

The geometric interpretation of this remark is in turn :

(3.14) If $\varepsilon > 0$ is small enough the set $f_* \leqq \varepsilon$ on a fiber of ξ^+ is star-shaped with respect to 0, and therefore linearly contractible.

(3.15) If $\mu > 0$ is small enough, the gradient of $-f_*$ points out of the set $\varphi(X) \leqq \mu$, at points with $\varphi(X) = \mu$, on any fiber of ξ^-.

Now, let X^ε_μ be the subset defined by:

$$X^\varepsilon_\mu = \{X \in \eta \,|\, f_*(X) \leqq \varepsilon;\ \varphi \circ \pi(X) \leqq \mu\} \,. \tag{3.16}$$

Then we can as a consequence of (3.14) and (3.15), find positive numbers ε and μ with the following properties:

(a) We have $\varepsilon < \varepsilon_0$.

(b) The map ρ is a homeomorphism on X^ε_μ.

(c) If $A^\varepsilon_\mu \subset X^\varepsilon_\mu$ is the subset of X^ε_μ on which $\varphi \circ \pi(X) = \mu$, then the pair $(X^\varepsilon_\mu \cap \xi^-, A^\varepsilon_\mu \cap \xi)$ is a deformation retract of $(X^\varepsilon_\mu, A^\varepsilon_\mu)$.

(d) The gradient of $-f$ points out of the set $\rho(X^\varepsilon_\mu)$ at the points of $\rho(A^\varepsilon_\mu)$.

Assume in the sequel that ε, μ have been chosen in the above manner. Also let $Y^\varepsilon_\mu = \overline{N - \rho(X^\varepsilon_\mu)}$. From (b) we conclude that $N^\varepsilon = Y^\varepsilon_\mu \cup_\alpha X^\varepsilon_\mu$ with attaching map $\alpha = \rho \,|\, A^\varepsilon_\mu$. From (c) it follows that $N^\varepsilon = Y^\varepsilon_\mu \cup \xi$. (Clearly the pair (D_ξ, S_ξ) is equivalent to the pair $(X^\xi_\mu \cap \xi, A^\varepsilon_\mu \cap \xi)$.) Finally, from (d) we conclude that at the boundary points of Y^ε_μ the gradient $-\nabla f$ points inward. Further there are no points with $\nabla f = 0$ on this set in the range $-\varepsilon \leqq f$, in view of (a). Hence $N^{-\varepsilon}$ is a deformation retract of Y^ε_μ by the argument used in the proof of (3.6). Thus N^ε is of the same homotopy type as $N^{-\varepsilon} \cup \xi$ as was to be shown.

REMARKS ON (3.8). This result follows from (3.7). One triangulates V and uses the preimages of these cells under the map $D_\xi \to V$ as the cells e_i.

The following is a different argument which proves (3.8) under the weaker hypothesis that (3.7) holds if V is a point. Let g be a function on V which has only nondegenerate critical points on V. Extend g to a function $\hat{g}$ on a normal neighborhood, B, of V in N by making $\hat{g}$ constant along the fibers, F, of B. Finally smooth $\hat{g}$ out to 0 inside a slightly bigger normal neighborhood. There results a C^∞ function $\hat{g}$ on M. Now consider the function $\hat{f} = f + \varepsilon\hat{g}$, with $\varepsilon > 0$. For ε sufficiently small $\hat{f}$ will have only nondegenerate critical points in the range $a \leqq f \leqq b$, and these will be precisely the critical points of g on V. Note that this part of the argument holds without the nondegeneracy hypothesis. All that is needed is that V be an isolated critical manifold. However, under such a general condition nothing can be said *a priori* about the indexes of the critical points of f. Under the nondegeneracy condition, $H_p f$ and $H_p g$ have complementary nullspaces at all critical points of f. Hence the indexes add, and are therefore $\geqq \lambda_f(V)$.

We close this section with the following easy corollary of Theorem I, corresponding to the case $\lambda_f(V) = 0$, i.e.. when $\xi_V = V$.

COROLLARY 3. *Let f be a smooth function on the compact manifold M. Assume that the critical set of f consists entirely of nondegenerate critical manifolds. Let M_* be the set on which f takes on its absolute minimum, and let $|f|$ denote the smallest index of the critical points of f on $M - M_*$. Then M is obtained from M_* by successively attaching cells of dimension no less then $|f|$. Thus: $M = M_* \cup e_1 \cup \cdots \cup e_i$;* $\dim e_i \geqq |f|$.

4. The suspension theorem

Le ν be a base point on M. The space $\Omega_\nu M$ is called nondegenerate if the set of geodesics in $\Omega_\nu M$ is the union of nondegenerate critical manifolds. Precisely, this condition should be formulated as follows: $\Omega_\nu M$ is nondegenerate if, given any regular half-space $\Omega_\nu^a M$, with model M_*^a, then the critical set of φ on M_*^a is the (necessarily) disjoint union of nondegenerate critical manifolds.

Combining the reduction Theorem III the following proposition becomes evident:

SUSPENSION THEOREM. *Let $\Omega_\nu M$ be nondegenerate. Let $\mathcal{V} = \mathcal{V}_\nu(M)$ be the collection of critical manifolds in $\Omega_\nu M$.*

Let $\mathcal{V}$ be well ordered, $\mathcal{V}=\{V_1, V_2, \cdots\}$, compatibly with the partial order defined on V by the length of the geodesics, and let $\xi_{V_i} = \xi_i$ be the negative bundle of V_i. Then $\Omega_\nu M$ has the same homotopy groups as the CW-*complex*:

$$K = \xi_1 \cup \xi_2 \cup \xi_3 \cup \cdots \tag{4.1}$$

We call this the suspension theorem because (1.1) follows from it trivially. Indeed, if $|\nu| > 1$, then only one of the critical manifolds V_i can have index 0, because $\Omega_\nu M$ is connected, (whence K is connected) and attaching a vector bundle of fiber dimension >1 does not change the number of components. Hence in this case V_1 has index 0 while all other V_i have index $\geqq |\nu|$. It follows that $M^\nu = V_1$. Thus going over to the corollary of Theorem III, K is of the form:

$$K = M^\nu \cup e_1 \cup e_2 \cup \cdots \qquad \dim e_i \geqq |\nu| . \tag{4.2}$$

Let $i : M^\nu \to \Omega_\nu M$ be the inclusion and let σ_* denote the suspension (in homotopy) from $\Omega_\nu M$ to M. Then $\sigma_* \circ i_*;\ \pi_k(M^\nu) \to \pi_{k+1}(M)$ agrees with

the definition of ν_* given in the introduction. Hence by (4.2) we obtain the corollary:

COROLLARY (4.1). *Under the hypothesis of the suspension theorem,*

$$\nu_*: \pi_r(M^\nu) \to \pi_{r+1}(M) \qquad 0 < r < |\nu| - 1 \tag{4.3}$$

is an isomorphism onto.

For completeness, we state an immediate cohomology consequence of (4.1):

COROLLARY (4.2). *Under the hypothesis of the suspension theorem, $H^*(\Omega_\nu M; G)$ admits a spectral sequence E_r which converges to a graded group of $H^*(\Omega_\nu M; G)$ and whose E_1 term is given by:*

$$E_1 = \sum H_c^*(\xi_i; G) \tag{4.4}$$

where ξ_i ranges over the negative bundles ξ_V; $V \subset \mathcal{V}$. (The subscript c denotes cohomology with compact supports.)

By Poincarè duality one has further that (in the notation of (3.10)):

$$H_c^r(\xi_V; G) = H^{r-\lambda}(V; G'), \qquad \lambda = \lambda(V) . \tag{4.5}$$

REMARKS. Recall that nondegerate $\Omega_\nu M$ exist for every manifold M of the type we are considering. In fact nearly every base point, ν gives rise to an $\Omega_\nu M$ in which the geodesics are nondegenerate *critical points*. In that case (4.3) is quite uninteresting, however (4.4) is still useful; in particular, E_1 will then be free if G is taken as the integers. For instance, if M is a compact group, $E_1 = E_\infty$ is was shown in [3], while for compact symmetric spaces, in general, $E_1 = E_\infty$ at least mod 2. [6].

5. The proof of Theorem I

Theorem I follows from the suspension theorem of the last section once it is proved that:

(5.1) *If M is a symmetric space then $\Omega_\mu M$ is nondegenerate for every base point ν on M.*

(5.2) *With M, M^ν is again a symmetric space for every base point ν on M.*

Recall that the manifold M is called symmetric if the following condition is satisfied:

(5.3) *For every $P \in M$, there exists an isometry L_P, of M which keeps P fixed and reverses the geodesics through P.*

From the second condition it follows that $I_P^2 =$identity for every $P \in M$.

Another equivalent definition can be given in terms of the group of isometries of M. This group, which is known to be a compact Lie-group, will be denoted by G in the sequel. Using the fact that any two points of M can be joined by a geodesic one easily derives the following consequences of (5.3).

(5.4) The e-component G' of G acts transitively on M.

(5.5) If $K_P \in G$ is the stability of $P \in M$, then K_P is pointwise fixed under the automorphism A_P: $k \to I_P k I_P^{-1}$ of G.

(5.6) The e-component K'_P coincides with the e-component of the fixed point set of A_P in G.

The converse of (5.6) yields the alternate definition of symmetric spaces:

(5.7) If G is a compact group, and A is an involution of G, then in an invariant Riemannian structure, the coset space G/K is called a symmetric space if K' coincides with the e-component of the fixed point group of A.

In the sequel we assume M is a symmetric space with K_P the stability group of $P \in M$. The e-components of groups will be denoted by a dash, e.g., K'_P.

The action of K_P on M was discussed in [6], and was shown to be variationally complete.

As a consequence the following is true: (see [6, chapter II].)

PROPOSITION 5.1. *Let s be a nontrivial geodesic on M starting at P. Let Q be any point of s, and set K_{PQ} respectively K_s, equal to the subgroup of K'_P which keeps Q, respectively s, pointwise fixed. Then the multiplicity of Q as a conjugate point of P is equal to* $\dim K_{PQ}/K_s$.

The statement (5.1) is an immediate corollary of this proposition. Indeed, let $\nu = (P, Q; h)$ and let the set of geodesics in $\Omega_\nu M$ be denoted by $S_\nu M$. Clearly K'_{PQ} acts on $S_\nu M$, the orbit of $s \in S_\nu M$, being homeomorphic to K'_{PQ}/K'_s. In any model, M^ν_*, for $\Omega^a_\nu M$ these orbits are certainly imbedded as smooth submanifolds. Now we see by Proposition 5.1 and (2.3) that the nullity of any point on such an orbit is equal to the dimension of the orbit.. This is precisely the second condition for nondegeneracy. (see (3.2)).

There remains the statement (5.2). To prove it, we show that each orbit of K'_{PQ} on M^ν is a symmetric space. Let then V be the orbit of $s \in M^\nu$. We may assume that s does not degenerate, for then M^ν reduces to a point. Thus $V = K'_{PQ}/K_s$ and we have to produce an involution A of K_{PQ} whose fixed point set contains K'_s as e-component. Because s is a minimal geodesic in the $\Omega_\nu M$, no conjugate point of P occurs in the interior of s. In particular, the midpoint R of s is not conjugate to P along s. Hence $K'_s = K'_{PR}$ by Proposition 5.1.

Now $I_R P = Q$, and $I_R Q = P$. Hence if $k \in K_{PQ}$, then $I_R k I_R^{-1} \subset K_{PQ}$. Thus A: $K_{PQ} \to K_{PQ}$ defined by $A(k) = I_R k I_R^{-1}$ is an involution of K_{PQ}. On the other hand, the e-component of the fixed point set of A is precisely K'_{PR}. This proves (5.2) and completes the proof of Theorem I.

For future reference we close this section with the following theorem, which is a straightforward generalization of Theorem I of [6].

THEOREM IV. *Let ν be any base point on the symmetric space M. Then the spectral sequence,* (4.2), *attached to $\Omega_\nu M$ by the decomposition* (4.1), *is trivial over the integers* mod 2. *Thus:*

$$H^*(\Omega_\nu M; Z_2) = \sum H^*_c(\xi_V; Z_2) \qquad V \in \mathcal{V}_\nu(M) \; . \tag{5.8}$$

In the group case (5.8) *holds with integer coefficients.*

NOTE ON THE PROOF. The spectral sequence (4.2) is derived from the filtering of $K = \xi_0 \cup \xi_1 \cup \cdots$, by the subcomplexes $K_i = \xi_0 \cup \cdots \cup \xi_i$. Let α: $S_{\xi_i} \to K_{i-1}$ be the attaching map of ξ_i. The problem is to show that α induces a trivial homomorphism in homology. Let $s \in V_i$ and consider the K cycle Γ_s as defined in [6]. This is a manifold fibered over V with a section σ: $V \to \Gamma$. One has a map of $\Gamma \to K_i$, which transforms ξ_i into the normal bundle of $\sigma(V)$ in Γ. Thus $\Gamma = \Gamma' \cup \xi_i$ corresponds to $K_i = K_{i-1} \cup \xi_i$ and in Γ' the attaching map α_* is always homologically trivial mod 2 (because ξ_i is the normal bundle of a section). If the fiber of Γ over V is orientable α_* will also be trivial over the integers.

The simplest application of Theorem IV is obtained by considering (5.8) in dimension 0. Because $\Omega_\nu M$ is always connected for any base point ν on M, (5.8) implies that M^ν is connected. This fact will also be apparent in the explicit computations of sections 7 and 8 which evaluate the integers $|\nu|$ of Theorem II.

Before proceeding to the proof of this theorem we have to review the basic conjugacy theorems for symmetric spaces which make the explicit computations possible. This is done in the next section.

6. The roots of a symmetric space

In this section G is to be a compact connected Lie group, in a left and right invariant metric, which an involution A. The full fixed point set under A is denoted by K, while the e-component of K is written K'. (Note that K thus plays the role of K_P in section 5.)

Let $\mathfrak{g}$ be the Lie algebra of G, and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

be the decomposition of $\mathfrak{g}$ into the fixed point set of A, (this is $\mathfrak{k}$, the Lie algebra of K) and its orthogonal complement. Let $\mathfrak{h}_\mathfrak{m}$ be a maximal abelian subalgebra of $\mathfrak{m}$, and let $\mathfrak{h} \supset \mathfrak{h}_\mathfrak{m}$ be a Cartan subalgebra of $\mathfrak{g}$.

Let $\eta: G \to G$ be defined by: $\eta(g) = g \cdot A(g^{-1})$. Then $\eta(gk) = \eta(g)$ so that η is constant along the left cosets of K and in this manner defines a map $\eta_*: G/K \to G$. We also let M be the image of $\mathfrak{m}$ under the exponential map. Thus $M = e^{\mathfrak{m}}$. Then it is known [1], [7], that η_* is a homeomorphism of G/K onto M. Further the natural action of K on G/K now translates into the adjoint action of K on G restricted to M. In the sequel we will therefore always think of the symmetric space G/K as the subset $M \subset G$.

Let $T_\mathfrak{m}$ be the image of $\mathfrak{h}_\mathfrak{m}$ under the exponential map. This is a torus in M which is geodesically imbedded. Any torus of this form is called a maximal torus of M, and its dimension is the *rank of* M.

We write $W(G, K)$ or $W(M)$ for the group of automorphisms of $T_\mathfrak{m}$ which are induced by inner automorphisms of K'. The following are basic properties of maximal tori: (see [1], [6], [7])

(6.1) If T and T' are two maximal tori of M, then there exists a $k \in K'$ so that $T = kT'k^{-1}$.

(6.2) If X is a subset of $T_\mathfrak{m}$ and $k \in K$ has the property $kXk^{-1} \subset T_\mathfrak{m}$, then there exists an element σ of $W(G, K)$ so that $\sigma(x) = kxk^{-1}$, for all $x \in X$.

(6.3) Every point of M lies on a maximal torus of M.

We also have:

(6.4) The geodesics of M through e coincide with the one-parameter groups of G which lie in M.

(6.5) If $x \in m$, then the index of the geodesic segment:

$$\bar{x}(t) = e^{tx} \qquad 0 \leqq t \leqq 1.$$

in M is computed as follws:

Let $\Sigma(G) = \{\theta_i\}$, $i = 1, \cdots, m$, be a system of positive roots of G on $\mathfrak{h}$. Also if a is any real number, let $\| a \|$ denote the number 0 if $a = 0$, otherwise let $\| a \|$ be the greatest integer $< | a |$. With this understood, the index in question is given by:

$$\text{(6.6)} \qquad \lambda(\bar{x}) = \sum_1^m \| \theta_i(x) \|$$

REMARKS.

(1) The formula (6.6) is to be found in [6], except for a factor 2 in the definition of the exponential map. This discrepancy is explained by the fact that the inverse of $\eta_*: G/K \to M$, is not given by the projection $M \to G/K$ induced by the natural map $\pi: G \to G/K$. Rather, one has $\eta_*^{-1}(p) = \pi(\sqrt{p})$ where for $p \in M$, $\sqrt{p}$ is any point of M with $(\sqrt{p})^2 = p$. That this factor 2 could be done away with by considering M rather than G/K was pointed out to me by A. Borel.

(2) We can find distinct non-trivial forms $\{\varphi_i\}$, $i = 1, \cdots m'$, on $\mathfrak{h}_\mathfrak{m}$ such that each $\theta \in \Sigma(G)$ restricts to some $\pm \varphi_i$ on $\mathfrak{h}_\mathfrak{m}$. Such a system of forms is called a root system for M, and is denoted by $\Sigma(M)$. For each $\varphi \in \Sigma(M)$ let n_φ be the number of forms in $\Sigma(G)$ which restrict to $\pm \varphi$ on $\mathfrak{h}_\mathfrak{m}$. These integers are the multiplicities of the root forms of M. In terms of them, (6.6) is expressed by:

$$\text{(6.7)} \qquad \lambda(\bar{x}) = \sum n_\varphi \| \varphi(x) \| \qquad \varphi \in \Sigma(M).$$

This formula has the following geometric interpretation: Consider the set of planes on which one of the root-forms $\varphi \in \Sigma(G/K)$ has an integral value. Then $\lambda(\bar{x})$ counts how many of these planes the line-segment tx, $0 \leqq t \leqq 1$, *crosses*, each crossing being counted by the appropriate multiplicity.

Finally, we recall the following facts:

(6.8) Let Λ_* be the lattice of those $x \in \mathfrak{h}_\mathfrak{m}$, for which the segment $\bar{x}(t) = e^{tx}$, $0 \leqq t \leqq 1$, represents a closed curve which is homotopic to zero in M. Then Λ_* is generated by elements $\mathfrak{h}_\varphi$, $\varphi \in \Sigma(M)$, characterized by:

$$\mathfrak{h}_\varphi \text{ is perpendicular to the plane } \varphi = 0, \text{ and } \varphi(\mathfrak{h}_\varphi) = 2 .$$

(6.9) The representation of $W(M)$ on $\mathfrak{h}_\mathfrak{m}$ is generated by the reflections in the planes $\varphi = 0$ for $\varphi \in \Sigma(M)$.

These propositions enable us to survey the possible indexes of elements in $S_\nu M$ entirely in terms of the roots of G on $\mathfrak{h}$. Indeed, by (6.3) no generality is lost if we assume that the base-point $\nu = (P, Q; h)$ is of the form $P = e$; $Q \in T_\mathfrak{m}$. According to (5.1) the set $S_\nu M$ will consist of the cllection $\mathcal{C}V_\nu M$ of nondegenerate critical manifolds. If s is a geodesic of $V \in \mathcal{C}V M$, then V consists precisely of the set of geodesics ksk^{-1} where k is in the subgroup of K' keeping Q fixed. Hence, by (6.1), (6.2) and (6.4), each V contains geodesics which lie on $T_\mathfrak{m}$, and join e to Q. Further two such geodesics lie in the same V precisely if they are conjugate under $W(G, K)$.

We will adhere to the convention that if $x \in \mathfrak{h}_\mathfrak{m}$, then $\bar{x}$ represents the geodesic e^{tx}, $0 \leqq t \leqq 1$, in M. Because the geodesics on $T_\mathfrak{m}$ can be lifted into $\mathfrak{h}_\mathfrak{m}$ in the obvious fashion, our earlier conclusions can be summarized as follows:

PROPOSITION 6.1. *Let* $x_\nu \in \mathfrak{h}_{\mathfrak{m}}$ *be any point with* $\bar{x}_\nu \in \Omega_\nu M$. *Then if* $x \in x_\nu + \Lambda_*$ *there is a unique critical manifold* $V_x \subset S_\nu M$ *which contains* $\bar{x}$. *This manifold is homeomorphic to* K'/K_x, *where* K_x *is the centralizer of* x *in* K'.

The function $x \to V_x$ *maps* $x_\nu + \Lambda_*$ *onto the set* $^{C}V_\nu M$, *and if* $V_x = V_y$, $x, y, \in x_\nu + \Lambda_*$, *then* x *and* y *are conjugate under the action of* $W(G, K)$ *on* $\mathfrak{h}_{\mathfrak{m}}$.

COROLLARY. *The set of indexes* $\lambda(s)$, $s \in S_\nu M$, *consists of the integers* $\lambda(\bar{x})$, *computed according to* (6.7) *as* x *ranges over the points of* $x_\nu + \Lambda_*$.

In the next sections this proposition is applied to compute the values of $|\nu|$ given in Theorem II, case by case.

7. Computations when M is a group

If the compact connected group G is to be considered as a symmetric space, M, we must, to follow our general procedure, consider M as the subset (g, g^{-1}), $g \in G$, in $G \times G$. Then $M = G$, while $\mathfrak{h}_{\mathfrak{m}}$ corresponds to the anti-diagonal in $h \times h$. Thus in this case $\sum(M)$ is a positive root system for G each root being counted with multiplicity 2. The group K then corresponds to G acting on M by the adjoint action.

In each case to be considered, we will choose orthogonal coordinates in $\mathfrak{h}_{\mathfrak{m}}$, and so identify $\mathfrak{h}_{\mathfrak{m}}$ with R^l, the space of l-tuples of real numbers with the usual inner product $((x, y) = \sum x_i \cdot y_i$, where x_i, y_i are the coordinates of x and y respectively). The form which assigns to $x \in R^l$ its α^{th} coordinate will always be denoted by ω_α. The exponential map then gives rise to a map $R^l \to M$, which will be denoted by ρ. We will define this map in each case, and then give the root-system of M as it is expressed by the forms ω_α.

(7.1) *The unitary groups,* $M = \mathrm{U}(2n)$. Let d_α be the diagonal $2n \times 2n$ matrix with α^{th} entry $2\pi\sqrt{-1}$, and all other entries 0. Then $\rho: R^{2n} \to \mathrm{U}(2n)$ is given by:

$$\rho(x) = \exp\{\textstyle\sum \omega_\alpha(x)\, d_\alpha\} \qquad x \in R^{2n}.$$

and the root-forms of $M = \mathrm{U}(2n)$ are:

$$\Sigma(M)\colon \omega_\beta - \omega_\alpha \qquad 1 \leqq \alpha < \beta \leqq 2n.$$

It follows that $W(M)$ is permutation group of the coordinates in R^{2n}, and that Λ_* is generated by $\{1, -1, 0, 0, \cdots, 0\}$ and its transforms under $W(M)$.

Let $x_\nu \in R^{2n}$ be the element:

$$x_\nu = \{0, 0, \cdots, 0; 1, 1, \cdots, 1\} \quad (n \text{ entries } 0,\ n \text{ entries } 1)$$

and let $\nu = (P, Q; h)$ be the unique base point containing the curve $\bar{x}_\nu$ (Note that then $P = Q =$ identity). Thus K_{PQ} (in the sense of section 5) is equal to $\mathrm{U}(2n)$ and $K_{x_\nu} = \mathrm{U}(n) \times \mathrm{U}(n)$, whence $V_{x_\nu} = \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$.

The points of $x_\nu + \Lambda_*$ are of the form: $x = \{a_1, \cdots, a_{2n}\}$ with $a_\alpha \in Z$; $\sum a_\alpha = n$. Let $b_1 < b_2 \cdots < b_k$ be the different integers which occur among the $\{a_i\}$, and assume that b_k occurs n_k times. Then according to (6.7):

$$\lambda(\bar{x}) = 2 \textstyle\sum_{\beta > \alpha} n_\alpha n_\beta (a_\beta - b_\alpha - 1)\,.$$

We conclude:

(1) If $x \in x_\nu + \Lambda_*$, with $\lambda(\bar{x}) = 0$ then x is conjugate to x_ν under $W(M)$.

(2) The next lowest value of λ on $x_\nu + \Lambda_*$ is $2(n + 1)$. Up to conjugation by elements of $W(M)$ this value is taken on only at the points:

$$\{0, \cdots, 0; 0, 1, 1, \cdots, 1, 2\} \text{ and } \{-1, 0, 0, \cdots, 0, 1; 1, 1, \cdots, 1\}\,.$$

Hence:

(7.2) *In this case,* $M^\nu = V_{x_\nu} = \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, *while* $|\nu| = 2(n + 1)$.

COROLLARY. *The sequence* (1.2) *is a* ν*-sequence.*

(7.3) *The orthogonal groups,* $M = \mathrm{SO}(2n)$. Let O_k be the $2n \times 2n$ matrix with only entry the diagonal box $2\pi\sqrt{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ at the k^{th} level. Now $\rho: R^n \to \mathrm{SO}(2n)$ is given by: $\rho(x) = \exp\{\sum \omega_\alpha(x) O_\alpha\}$, and we have:

$$\Sigma(M)\colon \omega_\beta \pm \omega_\alpha\,; \qquad 1 \leqq \alpha < \beta \leqq n.$$

Further $W(M)$ is generated by the permutations $\omega_\alpha \to \omega_\beta$, and $\omega_\alpha \to -\omega_\beta$, $\alpha < \beta$; and Λ_* is generated by the element $\{1, -1, 0, \cdots, 0\}$ as a $W(M)$ module.

Let $x_\nu = \{1/2, 1/2, \cdots, 1/2\}$, and let ν be the base point determined by $\bar{x}_\nu$. Then $V_{x_\nu} = \mathrm{SO}(2n)/\mathrm{U}(n)$. By, (6.7) we see that $\lambda(\bar{x}) = 0$, x in $x_\nu + \Lambda_*$ implies x conjugate to x_ν under $W(M)$, while $|\nu|$ is given by $2(n - 1)$. In fact the index of $\{\pm 1/2, 1/2, 1/2, \cdots, 3/2\}$ is precisely $2(n - 1)$. Thus,

(7.4) *In this case* $M^\nu = \mathrm{SO}(2n)/\mathrm{U}(n)$, *while* $|\nu| = 2(n - 1)$.

(7.5) *The symplectic groups,* $M = \mathrm{Sp}(n)$. Let $\mathrm{U}(n) \subset \mathrm{Sp}(n)$ be a standard inclusion, and let $\rho: R^n \to \mathrm{Sp}(n)$ be defined by the map $R^n \to \mathrm{U}(n)$ as in (7.1), (with n replaced by $2n$) followed by the inclusion. Then:

$$\Sigma(M)\colon \omega_\beta \pm \omega_\alpha;\ 2\omega_\alpha\,, \qquad 1 \leqq \alpha < \beta \leqq n.$$

$W(M)$: All signed permutaions.

Λ_*: Generated by $\{1, -1, 0, \cdots, 0\}$ as a $W(M)$-module.

Again, we choose $x_\nu = \{1/2, \cdots, 1/2\}$. Then $V_{x_\nu} = \mathrm{Sp}(n)/\mathrm{U}(n)$ as is easily seen. As before $V_{x_\nu} = M^\nu$. However now $\lambda(\{1/2, 1/2, \cdots, 3/2\}) = 2(n+1)$, and this is the value of $|\nu|$. Thus:

(7.6) *In this case,* $M^\nu = \mathrm{Sp}(n)/\mathrm{U}(n)$ *with* $|\nu| = 2(n+1)$.

8. The remaining computations. Proof of Theorem II

(8.1) *The space* $M = \mathrm{SO}(4n)/\mathrm{U}(2n)$. Let Q be the field of quaternions $x_0 \cdot 1 + x_1 \cdot i + x_2 \cdot j + x_3 \cdot k$; $x_s \in R^1$, where the $1, i, j, k$ are the usual quaternion units. We define the following endomorphisms of R^{4n}: E_0 the identity; E_1 is to take the α^{th} coordinate into minus the $(\alpha + 2n)^{\text{th}}$ coordinate, while it takes the $(\alpha + 2n)^{\text{th}}$ coordinate into the α^{th} one $(1 \leqq \alpha \leqq 2n)$. The endomorphism E_2 is to be represented by the matrix

$$\{O_1 + \cdots + O_n - O_{n+1} - \cdots - O_{2n}\}$$

where O_α is as defined in (7.3). The assignment $1 \to E_0$, $i \to E_1$, $j \to E_2$, defines a representation of Q on R^{4n}. Because $1, i$ generate a field isomorphic to the complex numbers, we see that the elements of $\mathrm{SO}(4n)$ which commute with E_1 form a subgroup $\mathrm{U}(2n) \subset \mathrm{SO}(4n)$. The elements of this subgroup which commute with E_2 in turn define $\mathrm{Sp}(n) \subset \mathrm{U}(2n)$. Hence if we set $G = \mathrm{SO}(4n)$, and let A be the inner automorphism by E_1, then A^2 is the identity and the fixed point set, K, of A is $\mathrm{U}(n)$. Thus $G/K = M$ is a symmetric space.

Let $R^{2n} \to \mathrm{SO}(4n)$ be defined as in (7.3) with n replaced by $2n$. Then R^{2n} corresponds to the Cartan algebra, $\mathfrak{h}$, of section (6), and we have to determine the inclusion $\mathfrak{h}_{\mathrm{III}} \subset \mathfrak{h}$. It is not hard to see that this inclusion corresponds to a map $R^n \to R^{2n}$ given by

$$(x_1, \cdots, x_n) \to (x_1, \cdots, x_n; -x_1, \cdots, -x_n) .$$

Restricting the forms of (7.3) to this subspace, we obtain the following set of forms for $\Sigma(M)$: $\omega_\beta \pm \omega_\alpha$; $(1 \leqq \alpha < \beta \leqq n)$; $2\omega_\alpha$ $(1 \leqq \alpha \leqq n)$ Further the multiplicity of $\omega_\beta \pm \omega_\alpha$; $(\alpha \neq \beta)$ is 4, while that of $2\omega_\alpha$ is 1. Schematically we denote this set of forms by:

$$\begin{array}{llll} \Sigma(M): & \omega_\beta + \omega_\alpha & 2\omega_\alpha & \qquad 1 \leqq \alpha < \beta < n \\ & \quad 4 & \ 1 & \end{array}$$

(Thus the integer below the form denotes its multiplicity. This notation will be used throughout the sequel.) $W(M)$ and $\Lambda_*(M)$ are therefore the same as in (7.3)

Choose $x_\nu = \{1/2, \cdots, 1/2\}$, and let ν be the determined by $\bar{x}_\nu$. Note that $\bar{x}_\nu(t) = \exp(\pi\sqrt{-1}\, t\, E_2)$. It follows that in this case $K_{PQ} = \mathrm{U}(2n)$,

while $K_{x_\nu} = \mathrm{Sp}(n)$. Thus $V_{x_\nu} = \mathrm{U}(2n)/\mathrm{Sp}(n)$. Just as previously, V_{x_ν} is actually M^ν, while $|\nu|$ is the index of $\{1/2, \cdots, 1/2, 3/2\}$, and thus given by $4n - 2$. We conclude:

(8.2) *In this case* $M^\nu = \mathrm{U}(2n)/\mathrm{Sp}(n)$ *with* $|\nu| = 4n - 2$.

(8.3) *The space* $M = \mathrm{U}(4n)/\mathrm{Sp}(2n)$. Let E_1 be the matrix described in the last section. Then it is well known that the subgroup of $\mathrm{U}(4n)$ whose elements satisfy the identity $U^t E_1 U = E_1$, form the linear symplectic group $\mathrm{Sp}(2n) \subset \mathrm{U}(4n)$. Let A be the automorphism of $\mathrm{U}(2n)$ which takes U into $E_1 \bar{U} E_1^{-1}$. (Here the bar denotes complex conjugation.) Then A^2 is the identity, and because $\bar{U}^t = U^{-1}$, the subgroup of $\mathrm{U}(2n)$ fixed under A is precisely $\mathrm{Sp}(n)$. Let $R^{2n} \to R^{4n}$ be the map:

$$(x_1, \cdots, x_{2n}) \to (x_1, \cdots, x_{2n}, x_1, \cdots, x_{2n}) . \tag{8.4}$$

Then this map followed by the map $R^{4n} \to \mathrm{U}(4n)$ described in (7.1) describes ρ in this case. Restricting the forms of $\mathrm{U}(4n)$ according to (8.4) we obtain the following array for $\Sigma(M)$:

$$\begin{array}{lll} \Sigma(M): & \omega_\beta - \omega_\alpha & \qquad 1 \leqq \alpha < \beta \leqq 2n. \\ & \quad 4 & \end{array}$$

Hence $W(M)$ and Λ_* are as described in (7.1). Accordingly choose $x_\nu = \{0, \cdots, 0, 1, \cdots, 1\}$, just as in (7.1), and let ν be determined by $\bar{x}_\nu$. This is then a closed curve in M. Thus K_{PQ} is represented by $\mathrm{Sp}(2n)$. The centralizer of $\bar{x}_\nu$ in $\mathrm{U}(4n)$ is clearly $\mathrm{U}(2n) \times \mathrm{U}(2n)$. Hence the centralizer in $\mathrm{Sp}(2n)$ is precisely $\mathrm{Sp}(n) \times \mathrm{Sp}(n)$. Thus V_{x_ν} is homeomorphic to $\mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n)$. Just as in (7.1) we see that $M^\nu = V_{x_\nu}$. However $|\nu|$ is now given by $4(n+1)$, because each root has weight 4 instead of 2. To summarize:

(8.5) *In this case* $M^\nu = \mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n)$ *while* $|\nu| = 4(n+1)$.

If we combine (7.4) with (8.2) and (8.5) we obtain the

COROLLARY. *The sequence* (1.4) *is a* ν*-sequence.*

(8.6) *The space* $M = \mathrm{Sp}(n)/\mathrm{U}(n)$. We will now interpret $\mathrm{Sp}(2n)$ as the group of $n \times n$ nonsingular matrixes with entries from Q which keep the symplectic product invariant. We also write $i[j]$ for the diagonal matrix $i \times$ Identity $[j \times$ Identity$]$. Consider the subgroup of $\mathrm{Sp}(n)$ which commutes with j. Because the elements of Q which commute with $j \in Q$ form a field isomorphic to C, this subgroup will be isomorphic to $\mathrm{U}(2n)$. Hence if A denotes the inner automorphism with j, then the fixed-point set of A is $\mathrm{U}(n)$. By a similar argument, the subgroup commuting with both i and j is the group $\mathrm{O}(n) \subset \mathrm{U}(n)$.

Let ρ: $R^n \to \mathrm{Sp}(n)$ be defined as in (7.1), except that $\sqrt{-1}$ is to be replaced by $i \in Q$, and $2n$ is to be replaced by n. Then $A\rho(x) = \rho(-x)$. Further the image of ρ is a maximal torus of Sp(n) as is seen from (7.5). This is therefore a case when $\mathfrak{h}_{\mathfrak{m}} = \mathfrak{h}$. If follows that the root system, $\Sigma(M)$, identical with $\Sigma(\mathrm{Sp}(n))$, except that each root has multiplity 1. Thus

$$\Sigma(M): \quad \underset{1}{\omega_\beta \pm \omega_\alpha} \quad \underset{1}{2\omega_\alpha} \qquad\qquad 1 \leqq \alpha < \beta \leqq n.$$

We chose x_ν as in (7.5), and ν correspondingly. If follows that the endpoint of $\bar{x}_\nu$ is minus the identity, whence $K_{PQ} = \mathrm{U}(n)$. The centralizer of x_ν must commute with j. Hence $K_{x_\nu} = \mathrm{O}(n)$. Thus $V_{x_\nu} = U(n)/\mathrm{O}(n)$. Using the results of (7.5) it follows that:

(8.7) *In this case* $M^{\nu} = \mathrm{U}(n)/\mathrm{O}(n)$ with $|\nu| = (n+1)$.

(8.8) *The space* $M = \mathrm{U}(2n)/\mathrm{O}(2n)$. It is clear that here the automorphism in question is the complex conjugation. We let $\rho: R^{2n} \to \mathrm{U}(2n)$ be defined precisely as in (7.1). We then see that this is again where $\mathfrak{h}_{\mathfrak{m}} = \mathfrak{h}$. Thus

$$\Sigma(M): \quad \underset{1}{\omega_\beta - \omega_\alpha} \qquad\qquad 1 \leqq \alpha < \beta \leqq 2n.$$

We choose x_ν just as in (7.1), whence $V_{x_\nu} = \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n)$. By dividing the answer in (7.1) by 2, we finally obtain for $|\nu|$ the integer $(n+1)$. Thus:

(8.9) *In this case* $M^{\nu} = \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n)$, *and* $|\nu| = (n+1)$.

Now combining (7.6) with (8.7) and (8.9) we obtain the

COROLLARY. *The sequence* (1.3) *is a ν-sequence.*

This then completes the proof of Theorem II. It might be useful for later reference, to summarize the computations of the last two sections in terms of the suspension theorem of section 4. In this summary, the symbol $X = Y \cup e_k \cdots$ will be interpreted to mean that X is obtained from Y by attaching cells of dimension $\geqq k$. With this understood we have shown that:

$$\begin{aligned} \Omega_\nu \mathrm{U}(2n) &\cong \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n) \cup e_{2n+2} \cdots . \\ \Omega_\nu \mathrm{SO}(2n) &\cong \mathrm{SO}(2n)/\mathrm{U}(n) \cup e_{2n-2} \cdots . \\ \Omega_\nu \mathrm{Sp}(n) &\cong \mathrm{Sp}(n)/\mathrm{U}(n) \cup e_{2n+2} \cdots . \end{aligned} \tag{8.10}$$

Further,

$$\begin{aligned} \Omega_\nu \mathrm{Sp}(n)/\mathrm{U}(n) &\cong \mathrm{U}(n)/\mathrm{O}(n) \cup e_{n+1} \cdots \\ \Omega_\nu \mathrm{U}(2n)/\mathrm{O}(2n) &\cong \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n) \cup e_{n+1} \cdots \end{aligned} \tag{8.11}$$

and

$$\begin{aligned} \Omega_\nu \mathrm{SO}(4n)/\mathrm{U}(2n) &\cong \mathrm{U}(2n)/\mathrm{Sp}(n) \cup e_{4n-2} \cdots \\ \Omega_\nu \mathrm{U}(4n)/\mathrm{Sp}(2n) &\cong \mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n) \cup e_{4n+4} \cdots \end{aligned} \tag{8.12}$$

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