HOMOTOPICAL LOCALIZATIONS OF SPACES

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Abstract. For a map f of spaces, Dror Farjoun and the author have constructed an f-localization functor, where a space Y is called f-local when map(f, Y) is an equivalence. This very general construction gives all known idempotent homotopy functors of spaces. The main theorem of this paper shows that f-localization functors always preserve fiber sequences of connected H-spaces up to small error terms. For instance, the localization with respect to the nth Morava K-theory preserves such fiber sequences up to error terms with at most three nontrivial homotopy groups. This implies, for example, that a $K(1)$-homology equivalence of H-spaces must induce an isomorphism of v_1-periodic homotopy groups. Results are also obtained on the A-nullification or A-periodization functors, which are just the f-localization functors for the maps f from spaces A to points. Two spaces are said to have the same nullity when they give the same nullification functors, and it is shown that arbitrary sets of nullity classes have both least upper bounds and greatest lower bounds. The A-nullifications of nilpotent Postnikov spaces are completely determined.

1. Introduction. During the past two decades, great progress has been made toward a global understanding of stable homotopy theory, showing that some major features arise "chromatically" from an interplay of periodic phenomena arranged in a hierarchy (see [Ra]). These phenomena have been quite effectively exposed using localizations of spectra with respect to periodic homology theories such as the Morava K-theories $K(n)_*$. We would like to similarly expose periodic phenomena in unstable homotopy theory using localizations of spaces. Some encouraging progress in this direction has been made by Mahowald and Thompson ([MT]), Dror Farjoun and Smith ([DFS]), the author ([Bo 7]), and others, and a powerful general theory of unstable homotopical localizations has begun to emerge. In this paper, we investigate that theory and prove a general fibration theorem showing, for instance, that $K(n)_*$-localizations preserve fiber sequences of connected H-spaces up to error terms with at most three nontrivial homotopy groups.

To describe our results, we first recall the very general notion of an f-localization of spaces, and we initially work in the pointed homotopy category Ho_* of CW-complexes. For a fixed map $f\colon A \to B$ of spaces, we say that a space Y is *f-local* when $f^*\colon \text{map}(B, Y) \simeq \text{map}(A, Y)$. As shown by Dror Farjoun ([DF 1]) and the author ([Bo 3, Corollary 7.2]), each space X has a natural f-localization $X \to L_f X$. The f-localization generalizes both the E_*-localization $X \to X_E$ for a homology theory E_* and the A-nullification $X \to P_A X$ for a space A, which may

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respectively be obtained using a suitable huge E_*-equivalence f and the trivial map $f\colon A \to *$. We give a brief general account of f-localizations in Section 2, and refer the reader to [Bo 8], [Ca 1], and particularly to Dror Farjoun's book ([DF 3]) for additional background information. We remark that Dror Farjoun's earlier paper ([DF 1]) helped to stimulate widespread interest in f-localizations.

For a map f and space A, we consider the localization class $\langle f \rangle$ consisting of all maps giving the same local spaces as f, and the nullity class $\langle A \rangle$ consisting of all spaces giving the same null spaces as A. The collections *Locs* of localization classes and *Nuls* of nullity classes have obvious partial orderings (see [Bo 7], [DF 2], [DF 3]), and we prove that they are actually small-complete large lattices in the sense that their (small) subsets have greatest lower bounds and least upper bounds (see 4.3 and 4.5). Moreover, we prove that each localization class $\langle f \rangle$ has a best possible approximation by a nullity class $\langle A(f) \rangle$ such that $P_{A(f)}$ and L_f have the same acyclic spaces (Theorem 4.4). Thus $P_{A(f)}$ is related to L_f in the same way as Quillen's plus-construction is related to the $H_*(\Leftrightarrow; Z)$-localization functor. For each map f, we also obtain a closed model category structure for spaces, where the "weak equivalences" are the L_f-equivalences (Theorem 4.6). Thus each localization class $\langle f \rangle$ determines its own brand of homotopy theory. Versions of this result have been obtained by Dror Farjoun ([DF 3]), Hirschhorn ([Hi]), Smith, the author ([Bo 2, Appendix]), and others. Our present approach actually shows the existence of $\mathcal{F}$-localizations and associated homotopy theories for many large classes of maps $\mathcal{F}$.

Our main result is a general fibration theorem. For a map f of connected spaces, we prove that the localization functors $L_{\Sigma f}$ and $L_f\Omega$ preserve homotopy fiber sequences up to error terms whose p-completions have at most three non-trivial homotopy groups for each prime p (Main Theorem 9.7). This generalizes a fibration theorem of Dror Farjoun and Smith ([DFS]) for the localization functors $L_{\Sigma^2 f}$ and $L_f\Omega^2$, which in turn partially generalizes the fibration theorems of [Bo 7] and [DFS] for the nullification functors $P_{\Sigma A}$ and $P_A\Omega$. These results all depend on a key lemma (Lemma 5.3) which we originally proved in [Bo 7, 6.9]. The crux of our present proof is in Section 6, where we show that the homotopy fiber of an $L_{\Sigma f}$-equivalence of spaces is "almost" $L_{\Sigma f}$-acyclic (Theorem 6.2), and where we find a very convenient expression for the $L_{\Sigma f}$-error term of a homotopy fiber sequence (Theorem 6.4). Our main proof is completed in Section 9 after we have determined the nullifications of nilpotent Postnikov spaces and of other nilpotent "generalized polyGEMs" (see Theorems 7.5 and 8.8).

In Section 10, we develop general homological consequences of the preceding work and show that $K(n)_*$-localizations, and other E_*-localizations, "almost" preserve homotopy fiber sequences of H-spaces (Theorem 10.10). This also applies to various cohomological localizations including those with respect to stable cohomotopy theory (see 2.6 and 10.12). Finally, in Section 11, we introduce the notion of a *virtual E_*-equivalence,* defined as a map of spaces $\phi\colon X \to Y$ such that $\phi_*\colon \pi_i(\Omega X)_E \to \pi_i(\Omega Y)_E$ is an isomorphism for all sufficiently large i. We

find that the virtual E_*-equivalences are much more "durable" than ordinary E_*-equivalences. For instance, in a map of fiber sequences, if any two of the component maps is a virtual E_*-equivalence, then so is the third (Theorem 11.4). We show that each E_*-equivalence of H-spaces is a virtual E_*-equivalence (Theorem 11.3), and each virtual E_*-equivalence of spaces induces an E_*-equivalence of sufficiently highly connected covers (Theorem 11.7). We deduce that if an H-space is E_*-acyclic, then so are all of its connected covers and all of its Postnikov sections (Theorem 11.9). Turning to K-theory, we show that the virtual K/p_*-equivalences (or virtual $K(1)_*$-equivalences) of spaces are the same as the $v_1^{-1}\pi_*(\Leftrightarrow; Z/p)$-equivalences (Theorem 11.11), and conclude, for instance, that each K/p_*-equivalence of H-spaces is a $v_1^{-1}\pi_*(\Leftrightarrow; Z/p)$-equivalence (Corollary 11.12). This generalizes a result of Thompson ([Th], [Bo 7, 11.9]), and is needed for the author's subsequent work on K/p_*-localizations and v_1-periodizations. We also obtain some results on virtual $K(n)_*$-equivalences of spaces for $n > 1$. We show that if a map of spaces is a v_j-periodic homotopy equivalence for $1 \leq j \leq n$, then it is a virtual $K(n)_*$-equivalence (Theorem 11.13). This implies, for example, that if X is a space with trivial v_j-periodic homotopy groups for $1 \leq j \leq n$, then the Postnikov map $X \to P^{n+1}X$ is a $K(n)_*$-equivalence (Corollary 11.14). This should help to make $K(n)_*X$ more accessible, and extends a similar result of Hopkins, Ravenel, and Wilson ([HRW]) for infinite loop spaces.

This paper generalizes the fundamental results of Dror Farjoun and Smith ([DFS]), and we have benefited from their ideas.

We work simplicially so that "space" means "simplicial set." However, to make the presentation more accessible, we frequently work in the pointed homotopy category Ho_*.

2. The basic homotopical localization theory. In this section, we recall the basic theory of f-localizations of spaces and discuss some general examples. We refer the reader to [Bo 8], [Ca 1], and [DF 3] for additional background information and results. A thorough account of the basic theory is being developed by Hirschhorn ([Hi]) in a general model category setting.

For pointed spaces X, $Y \in Ho_*$, let $\mathrm{map}_*(X, Y) \in Ho_*$ and $\mathrm{map}(X, Y) \in Ho_*$ respectively denote the pointed and unpointed mapping spaces from X to a fibrant representative for Y, and recall that $\pi_0\mathrm{map}_*(X, Y) \cong [X, Y]$. For a map $f\colon A \to B$ and space Y in Ho_*, we consider the orthogonality conditions:

(H1) $f^*\colon [B, Y] \cong [A, Y]$;

(H2) $f^*\colon \mathrm{map}_*(B, Y) \simeq \mathrm{map}_*(A, Y)$;

(H3) $f^*\colon \mathrm{map}(B, Y) \simeq \mathrm{map}(A, Y)$.

It is easy to show

LEMMA 2.1. (H3) $\Rightarrow$ (H2) $\Rightarrow$ (H1) *and, when Y is connected,* (H2) $\Leftrightarrow$ (H3).

We adopt (H3) as our main orthogonality condition in Ho_*. For a fixed map $f\colon A \to B$ in Ho_*, a space $Y \in Ho_*$ is called *f-local* when $f^*\colon \mathrm{map}(B,Y) \cong \mathrm{map}(A,Y)$; a map $u\colon X \to X'$ in Ho_* is called an *f-local equivalence* when $u^*\colon \mathrm{map}(X',Y) \cong \mathrm{map}(X,Y)$ for each f-local space Y; and a map $u\colon X \to X'$ is called an *f-localization* when it is an f-local equivalence to an f-local space X'. By Lemma 2.1, an f-localization $u\colon X \to X'$ is an initial example of a map from X to an f-local space in Ho_*, and is a terminal example of an f-local equivalence out of X in Ho_*. Thus the f-localizations are unique up to equivalence in Ho_*, and by [Bo 3, Cor. 7.2] or [DF 1], we have

THEOREM 2.2. *For each map $f\colon A \to B$ and space X in Ho_*, there exists an f-localization of X.*

Hence, there is an idempotent functor $L_f\colon Ho_* \to Ho_*$ giving a natural f-localization $u\colon X \to L_f X$ for $X \in Ho_*$.

2.3. The functor L_f on spaces. The idempotent functor $L_f\colon Ho_* \to Ho_*$ is actually induced from a coaugmented functor $L_f\colon \mathcal{S} \to \mathcal{S}$ on the category $\mathcal{S}$ of spaces (i.e. simplicial sets). Roughly speaking, for $X \in \mathcal{S}$, $L_f X$ may be constructed from X by expressing f as an inclusion of spaces $A \subset B$ and taking all possible pushouts from the pairs $(\Delta^n, \dot{\Delta}^n) \times (B,A)$ with $n \geq 0$ and the pairs (Δ^n, Δ^n_k) with $0 \leq k \leq n > 0$, where Δ^n denotes the standard n-simplex with boundary $\dot{\Delta}^n$ and kth horn Δ^n_k. This construction is continued over an appropriate section of ordinals to achieve the extension property with respect to the above pairs and to create $L_f X$ as a colimit. More elaborate versions of this construction in [Bo 5], [Bo 7], and [DF 3] produce a functor L_f which is simplicial in the sense of Quillen ([Qu, II.1]) with $L_f(*) = *$ for a point $*$.

We refer the reader to Casacuberta and Peschke ([CP]) for an analysis of the f-localization in the illuminating basic case where f is a self-map of S^1; we now turn to some other important general examples.

2.4. Nullifications. For a space $A \in Ho_*$, the localization with respect to the trivial map $f\colon A \to *$ is called the *A-nullification* or *A-periodization;* the functor L_f is denoted by P_A; the f-local spaces are called *A-null* or *A-periodic;* and the f-local equivalences are called *A-periodic equivalences* or *P_A-equivalences*. For connected spaces $A, Y \in Ho_*$, note that Y is A-null if and only if $\mathrm{map}_*(A,Y) \cong *$. Thus the S^{n+1}-nullification functor on Ho_* is equivalent to the nth Postnikov functor. Many other interesting nullifications are discussed in [Bo 7], [Bo 8], [Ca 2], [Ch], [DF 3], and [Ne].

2.5. Homological localizations. For a spectrum E, the E_*-localization functor $(\quad)_E\colon Ho_* \to Ho_*$ of [Bo 2] may be viewed as an f-localization for a huge E_*-equivalence f. For instance, we may use the map $f\colon \vee_\alpha A_\alpha \to \vee_\alpha B_\alpha$ obtained by wedging representatives $\{A_\alpha \subset B_\alpha\}_\alpha$ of all isomorphism classes of inclusions

of pointed spaces with $E_*(B_\alpha, A_\alpha) = 0$ and with cardinality $\#B_\alpha \leq \#E_*(pt)$ where $\#B_\alpha$ denotes the number of nondegenerate simplices of B_α. This follows since the f-localization map $u\colon X \to L_f X$ is an E_*-equivalence by its construction and since $L_f X$ is E_*-local by [Bo 2, Lemma 11.3].

2.6. Cohomological localizations. Let E be a spectrum whose mod p homotopy groups $\pi_*(E/p)$ are all finite for each prime p. Then, following [Bo 4] or [Ho] as explained below, there exists a spectrum G such that the G_*-equivalences are the same as the E^*-equivalences for spectra and hence for spaces. Thus by 2.5, there is an E^*-localization functor of the form L_f for spaces. To construct G, recall that the $(E/p)^*$-equivalences are the same as the $c(E/p)_*$-equivalences where $c(E/p)$ is the Brown-Comenetz ([BC]) dual of E/p. Thus when the groups $\pi_* E$ are all Ext-complete (i.e. when $\mathrm{Hom}(Q, \pi_* E) = 0 = \mathrm{Ext}(Q, \pi_* E)$), we may use $G = \vee_p c(E/p)$; and when the groups $\pi_* E$ are not all Ext-complete, we may use $G = HQ \vee \vee_p c(E/p)$ where HQ is the rational Eilenberg-MacLane spectrum. We do not know whether localizations of spaces exist with respect to arbitrary cohomology theories, although they do for all ordinary cohomology theories by [Bo 1].

3. Localizations with respect to classes of maps. The notion of an f-localization of spaces for a single map f can obviously be extended to that of an $\mathcal{F}$-localization for a class $\mathcal{F} = \{f_\alpha\colon A_\alpha \to B_\alpha\}_\alpha$ of maps in Ho_*. In particular, a space $Y \in Ho_*$ is called *$\mathcal{F}$-local* when $f_\alpha^*\colon \mathrm{map}(B_\alpha, Y) \cong \mathrm{map}(A_\alpha, Y)$ for each $f_\alpha \in \mathcal{F}$; a map $u\colon X \to X'$ in Ho_* is called an *$\mathcal{F}$-local equivalence* when $u^*\colon \mathrm{map}(X', Y) \cong \mathrm{map}(X, Y)$ for each $\mathcal{F}$-local space Y; and a map $u\colon X \to X'$ is called an *$\mathcal{F}$-localization* when it is an $\mathcal{F}$-local equivalence to an $\mathcal{F}$-local space X'. Let $\mathcal{L}(\mathcal{F})$ denote the class of $\mathcal{F}$-local spaces in Ho_*. When $\mathcal{F} = \{f_\alpha\}_\alpha$ is a (small) set of maps in Ho_*, there is a single map $f = \vee_\alpha f_\alpha$ such that $\mathcal{L}(f) = \mathcal{L}(\mathcal{F})$, and the f-localizations in Ho_* immediately give $\mathcal{F}$-localizations. In this section, we develop machinery showing that many large classes of maps $\mathcal{F} = \{f_\alpha\}_\alpha$ can similarly be replaced by single maps f with $\mathcal{L}(f) = \mathcal{L}(\mathcal{F})$. This will generalize the prototypical example of E_*-localizations (2.5) where the class of all E_*-equivalences is replaced by a single huge E_*-equivalence. Our main applications of this machinery will be given in Section 4. We remark that in every case where we are able to show the existence of $\mathcal{F}$-localizations in Ho_*, we are also able to replace $\mathcal{F}$ by a single map f. We shall need

3.1. Coherent functors. Let $Sets_*$ be the category of pointed sets; let $\mathcal{S}_*$ be the category of pointed spaces (i.e. pointed simplicial sets); and let $\vec{\mathcal{S}}_* = \mathcal{S}_*^2$ be the usual category of maps in $\mathcal{S}_*$. For such a map $f\colon A \to B$, write $\#f$ for the number of nondegenerate simplices of $A \vee B$, and call $f'\colon A' \to B'$ a *submap* of f (denoted by $f' \subset f$) when $A' \subset A$, $B' \subset B$, and $f' = f \mid A'$. Call a functor

T: $\vec{\mathcal{S}}_* \to$ *Sets*$_*$ *b-coherent* for an infinite cardinal number b when each $f \in \vec{\mathcal{S}}_*$ has $T(f) = \operatorname{colim}_\phi T(\phi)$ for ϕ ranging over the submaps of f with $\#\phi \leq b$. This is equivalent to saying that T preserves colimits of diagrams in $\vec{\mathcal{S}}$ indexed by directed sets having upper bounds for their subsets of cardinality $\leq b$. Note that if T is b-coherent, then it is b'-coherent for all $b' \geq b$. Call a functor T: $\vec{\mathcal{S}}_* \to$ *Sets*$_*$ *coherent* when it is b-coherent for sufficiently large b. For such a T, call a map $f \in \vec{\mathcal{S}}_*$ *T-acyclic* when $T(f) = *$. For instance, for a spectrum E, the relative homology functor E_*: $\vec{\mathcal{S}}_* \to$ *Sets*$_*$ is coherent and the E_*-acyclic maps are the E_*-equivalences.

LEMMA 3.2. *If T: $\vec{\mathcal{S}}_* \to$ Sets$_*$ is a coherent functor, then there exists an infinite cardinal number d such that for each T-acyclic $g \in \vec{\mathcal{S}}_*$ and each $\theta \subset g$ with $\#\theta \leq 2^d$, there exists a T-acyclic $\overline{\theta} \in \vec{\mathcal{S}}_*$ with $\theta \subset \overline{\theta} \subset g$ and $\#\overline{\theta} \leq 2^d$.*

Proof. Assume that T is b-coherent and let d be a cardinal such that $b \leq d$ and $\#T(\phi) \leq d$ for all $\phi \in \vec{\mathcal{S}}_*$ with $\#\phi \leq b$. Then for each $f \in \vec{\mathcal{S}}_*$ with $\#f \leq 2^d$, there are at most $(2^d)^b = 2^d$ submaps $\phi \subset f$ with $\#\phi \leq b$, and hence $\#T(f) \leq d \cdot 2^d = 2^d$. Given a T-acyclic map $g \in \vec{\mathcal{S}}_*$ and $\theta \subset g$ with $\#\theta \leq 2^d$, each element $x \in T(\theta)$ maps trivially to $T(\theta')$ for some $\theta' \subset g$ with $\theta \subset \theta'$ and $\#\theta' \leq 2^d$. Hence, there is a transfinite increasing sequence

$$\theta = \theta_0 \subset \theta_1 \subset \cdots \subset \theta_\lambda \subset \theta_{\lambda+1} \subset \cdots \subset \theta_\gamma = \overline{\theta}$$

of submaps of g indexed through the first ordinal γ of cardinality greater than b, where each $T(\theta_\lambda) \to T(\theta_{\lambda+1})$ is trivial, where $\#\theta_\lambda \leq 2^d$ for each $\lambda \leq \gamma$, and where $\theta_\beta = \cup_{\alpha<\beta}\theta_\alpha$ for each limit ordinal $\beta \leq \gamma$. Since T is b-coherent, we deduce that $T(\theta_\gamma) = \operatorname{colim}_{\lambda<\gamma} T(\theta_\lambda) = *$ and take $\overline{\theta} = \theta_\gamma$. □

For a coherent functor T: $\vec{\mathcal{S}}_* \to$ *Sets*$_*$, let $\mathcal{A}(T)$ denote the class of all maps in *Ho*$_*$ represented by T-acyclic maps in $\mathcal{S}_*$.

THEOREM 3.3. *If T: $\vec{\mathcal{S}}_* \to$ Sets$_*$ is a coherent functor, then $\mathcal{L}(\mathcal{A}(T)) = \mathcal{L}(f)$ for some wedge f of T-acyclic maps. Hence, $\mathcal{A}(T)$-localizatons exist in Ho$_*$ and are given by f-localizations.*

Proof. Let d be an infinite cardinal given by Lemma 3.2. Then each T-acyclic map ϕ in $\mathcal{S}_*$ is the colimit of the directed system of all T-acyclic submaps $\overline{\theta} \subset \phi$ with $\#\overline{\theta} \leq 2^d$, and thus ϕ is weakly equivalent to a homotopy colimit of these submaps by [BK, p. 332]. Hence, $\mathcal{L}(\mathcal{A}(T)) = \mathcal{L}(\mathcal{W})$ where $\mathcal{W}$ is a set containing a representative of each isomorphism class of T-acyclic maps ϕ in $\mathcal{S}_*$ with $\#\phi \leq 2^d$. Thus we may let f be the wedge of all maps in $\mathcal{W}$. □

Note that this theorem gives another proof of the existence of E_*-localizations in Ho_* (see 2.5) using the relative homology functor $E_*\colon \vec{\mathcal{S}}_* \to Sets_*$. Before turning to our main applications, we must formulate a nonconnected Whitehead theorem (Lemma 3.4) and derive a partial converse (Theorem 3.5) to the above theorem. Let $Ho\vec{\mathcal{S}}_*$ be the homotopy category obtained by inverting the termwise weak equivalence in $\vec{\mathcal{S}}_*$ (see e.g. [BF, A.3]).

LEMMA 3.4. *A map of pointed spaces $\phi\colon X \to Y$ is a weak equivalence if and only if the natural function $h_n\colon \pi_0 X \to [i_n, \phi]$ is onto for $n \geq 0$ where $[i_n, \phi]$ consists of the morphisms from $i_n\colon \dot{\Delta}^n \cup * \subset \Delta^n \cup *$ to ϕ in $Ho\vec{\mathcal{S}}_*$.*

Proof. We can assume that ϕ is a fibration of the fibrant spaces. Then the surjectivity of the functions h_n is equivalent to the right lifting property of ϕ with respect to the map i_n, and this is equivalent to the weak equivalence property. □

THEOREM 3.5. *For each map of pointed spaces $f\colon A \to B$, there exists a coherent functor $T_f\colon \vec{\mathcal{S}}_* \to \mathrm{Sets}_*$ whose acyclic maps are the f-local equivalences in $\mathcal{S}_*$.*

Proof. By Lemma 3.4, a map of pointed spaces $\phi\colon X \to Y$ is an f-local equivalence if and only if $h_n\colon \pi_0 L_f X \to [i_n, L_f\phi]$ is onto for each $n \geq 0$. For an infinite cardinal number $b \geq \#f$, each space X has $L_f X = \mathrm{colim}_\alpha L_f X_\alpha$ where $\{X_\alpha\}_\alpha$ are the subspaces of X of cardinality $\leq b$. Thus a suitable functor T_f is

$$T_f(\phi) = \bigvee_{n=0}^{\infty} [i_n, L_f\phi]/\mathrm{im}\ h_n. \qquad \square$$

Note. The definitions and results of this section have obvious versions for unpointed spaces with $\mathcal{S}$ in place of $\mathcal{S}_*$.

4. The lattice of localization functors and closed model category structures. Using the preceding machinery, we now prove several fundamental results on homotopical localizations: that the possible localization functors form a small-complete large lattice; that each localization functor has a best possible approximation by a nullification; that the nullity classes also form a small-complete large lattice; and that each localization functor determines a closed model category structure for spaces, and thus determines its own brand of homotopy theory.

For a class $\mathcal{F}$ of maps in Ho_*, we let $\mathcal{E}(\mathcal{F})$ denote the class of all $\mathcal{F}$-local equivalences in Ho_*. In general, $\mathcal{F} \subset \mathcal{E}(\mathcal{F})$ and

LEMMA 4.1. *If the following conditions are satisfied, then* $\mathcal{F} = \mathcal{E}(\mathcal{F})$*:*

(i) *for each space* $X \in Ho_*$*, there exists a map* $X \to X'$ *in* $\mathcal{F}$ *such that* X' *is* $\mathcal{F}$*-local;*

(ii) *each equivalence in* Ho_* *belongs to* $\mathcal{F}$*;*

(iii) *if a composition* gf *is defined in* Ho_* *and if any two of* f*,* g*,* gf *are in* $\mathcal{F}$*, then so is the third.*

Proof. A map $f\colon X \to Y$ in Ho_* induces a map $f'\colon X' \to Y'$, where $X \to X'$ and $Y \to Y'$ are $\mathcal{F}$-localizations given by (i). If $f \in \mathcal{E}(\mathcal{F})$, then f' is an equivalence in Ho_* and hence $f \in \mathcal{F}$ by (ii) and (iii). □

THEOREM 4.2. *For a set* $\{f_\alpha\}_\alpha$ *of maps in* Ho_**, there exists a map* f *such that* $\mathcal{E}(f) = \bigcap_\alpha \mathcal{E}(f_\alpha)$*.*

Proof. By Theorem 3.5, for each α, there is a coherent functor $T_{f_\alpha}\colon \vec{\mathcal{S}}_* \to Sets_*$ with $\mathcal{A}(T_{f_\alpha}) = \mathcal{E}(f_\alpha)$. These combine to give a coherent functor $T\colon \vec{\mathcal{S}}_* \to Sets_*$ with $T(\phi) = \vee_\alpha T_{f_\alpha}(\phi)$, where $\mathcal{A}(T) = \bigcap_\alpha \mathcal{E}(f_\alpha)$. By Theorem 3.3, there is a map $f \in \bigcap_\alpha \mathcal{E}(f_\alpha)$ with $\mathcal{E}(f) = \mathcal{E}\big(\bigcap_\alpha \mathcal{E}(f_\alpha)\big)$. Since f is an f_α-equivalence for each α, so are the f-localization maps, and $\mathcal{E}\big(\bigcap_\alpha \mathcal{E}(f_\alpha)\big) = \bigcap_\alpha \mathcal{E}(f_\alpha)$ by Lemma 4.1. □

4.3. The lattice of localization functors. Two maps f and g in Ho_* give equivalent functors $L_f \simeq L_g$ if and only if $\mathcal{L}(f) = \mathcal{L}(g)$. The resulting equivalence classes $\langle f \rangle$ form a partially ordered collection *Locs*, where $\langle f \rangle \leq \langle g \rangle$ means $\mathcal{L}(f) \supset \mathcal{L}(g)$ or equivalently $\mathcal{E}(f) \subset \mathcal{E}(g)$. Each (small) set $\{\langle f_\alpha \rangle\}_\alpha$ in *Locs* has a least upper bound $\langle \vee_\alpha f_\alpha \rangle$ and has a greatest lower bound $\langle f \rangle$ given by Theorem 4.2. Hence, *Locs* is a small-complete large lattice. For $\langle f \rangle \leq \langle g \rangle$ in *Locs*, the idempotent localization functors L_f and L_g on Ho_* are related by a canonical transformation $L_f \to L_g$ giving $L_g L_f \simeq L_g$.

A space X is called L_f*-acyclic* or f*-acyclic* when $L_f X \simeq *$, and X is called P_A*-acyclic* or *killed by* A when $P_A X \simeq *$.

THEOREM 4.4. *For each map* f *in* Ho_**, there exists a space* $A(f) \in Ho_*$ *such that* $P_{A(f)}$ *and* L_f *have the same acyclic spaces.*

Proof. Let $T_f\colon \vec{\mathcal{S}}_* \to Sets_*$ be a coherent functor whose acyclic maps are the f-equivalences. Then by Lemma 3.2, there exists an infinite cardinal 2^d such that each f-acyclic space X is the colimit of a directed system of f-acyclic subspaces of cardinality $\leq 2^d$. Thus $A(f)$ exists as a wedge of representatives of isomorphism classes of pointed f-acyclic spaces of cardinality $\leq 2^d$. □

For example, if L_f is the $H_*(\leftrightrightarrows Z)$-localization functor, then $P_{A(f)}$ is the Quillen plus-construction functor by [Ca 1] or [DF 3].

4.5. The lattice of nullity classes. Two pointed spaces X, $Y \in Ho_*$ give equivalent functors $P_X \simeq P_Y$ if and only if the X-null spaces are the same as the Y-null spaces. The resulting equivalence classes $\langle X \rangle$ are called *nullity classes* (see [Bo 7, §9], [Bo 8], [Ch], [DF 2], or [DF 3]), and form a partially ordered collection *Nuls*, where $\langle X \rangle \leq \langle Y \rangle$ means that the Y-null spaces are X-null or equivalently that X is killed by Y. There is an inclusion *Nuls* $\subset$ *Locs* where $\langle X \rangle$ is identified with $\langle X \to * \rangle$. For each $\langle f \rangle \in$ *Locs*, Theorem 4.4 gives a greatest member $\langle A(f) \rangle \in$ *Nuls* with $\langle A(f) \rangle \leq \langle f \rangle$. Thus each set $\{\langle X_\alpha \rangle\}_\alpha$ in *Nuls* has a least upper bound $\langle \vee_\alpha X_\alpha \rangle$ and has a greatest lower bound given by 4.3 and 4.4. Hence, *Nuls* is a small-complete large lattice. In addition, *Nuls* has the obvious finite smash products. For $\langle V \rangle \leq \langle W \rangle$ in *Nuls*, the idempotent nullification functors P_V and P_W on Ho_* are related by a canonical transformation $P_V \to P_W$ giving $P_W P_V \simeq P_W$.

Finally, we show that each map f in Ho_* determines a closed simplicial model category structure on $\mathcal{S}_*$, and thus determines a homotopy theory. Versions of this result have been obtained by Dror Farjoun ([DF 3]), Hirschhorn ([Hi]), Smith, the author ([Bo 2, Appendix]), and others. We call a map ϕ in S_* an *f-trivial cofibration* when it is both an f-local equivalence and a cofibration, and we call ϕ an *f-fibration* when it has the right lifting property for the f-trivial cofibrations.

THEOREM 4.6. *For each map f in Ho_*, the simplicial category $\mathcal{S}_*$ of pointed spaces has a closed simplicial model category structure with "weak equivalences," "fibrations," and "cofibrations" respectively defined as f-local equivalences, f-fibrations, and ordinary cofibrations.*

Proof. First note that a map ϕ is an ordinary trivial fibration if and only if it is both an f-local equivalence and f-fibration, where the "if" part follows by factoring ϕ as ji for a cofibration i and trivial fibration j, then deducing that i is an f-trivial cofibration, and concluding that ϕ is a retract of j. The theorem now follows from Lemma 4.7 below and a direct check of Quillen's condition SM7(b) ([Qu]). □

LEMMA 4.7. *Each map $\phi\colon X \to Y$ in $\mathcal{S}_*$ can be factored as $\phi = ji$ for an f-local equivalence i and f-fibration j.*

Proof. Let $T_f\colon \vec{\mathcal{S}}_* \to Sets_*$ be a coherent functor whose acyclic maps are the f-local equivalences. Then by Lemma 3.2, there exists an infinite cardinal 2^d such that each f-trivial cofibration α is the colimit of a directed system of f-trivial subcofibrations of cardinality $\leq 2^d$, and hence α is equivalent to the homotopy colimit of these subcofibrations. Consequently, an ordinary fibration θ of fibrant spaces is an f-fibration if and only if θ has the right lifting property for the f-trivial cofibrations of cordinality $\leq 2^d$. Thus, by a transfinite inductive construction, we may factor the composite of $\phi\colon X \to Y$ with $e\colon Y \subset \mathrm{Ex}^\infty Y$ to give $e\phi = j'i'$ for an f-local equivalence i' and f-fibration j', where e is Kan's

weak equivalence to a fibrant space $\mathrm{Ex}^\infty Y$. A pullback now gives the required factorization of ϕ. □

Note. Theorem 4.6 and its proof can immediately be modified to show that a map f actually determines a closed simplicial model category structure on the category $\mathcal{S}$ of unpointed spaces.

5. Acyclic spaces and their loop spaces. We now let $f\colon A \to B$ be a fixed map in Ho_* and consider the f-localization $u\colon X \to L_f X$ of a space X. The reader should keep in mind the case of a (co)homological localization (see 2.5 and 2.6) which will be studied more fully in Sections 10 and 11. We say that a space X is *f-acyclic* when $L_f X \simeq *$, or equivalently by Theorem 4.4 when $P_{A(f)}X \simeq *$.

The f-acyclic pointed spaces are closed under homotopy colimits and under fiber extensions, but are not closed under most homotopy inverse limits. However, the following key theorem will show that the loopspace of an f-acyclic H-space is "almost" f-acyclic. A space $M \in Ho_*$ is called a *GEM* when M is connected and $M \simeq \prod_{n=1}^{\infty} K(\pi_n M, n)$ with $\pi_1 M$ abelian.

THEOREM 5.1. *If $Y \in Ho_*$ is a connected f-acyclic H-space, then $L_{\Sigma f}Y$ is a GEM, as are the components of $L_f \Omega Y$.*

This will be proved in 5.6 after some preliminaries, and will imply the related results of [Bo 7], [Bo 8], and [DF 5]. For each space $Y \in Ho_*$, the loop space $\Omega L_{\Sigma f}Y$ is f-local, and thus $\Omega u\colon \Omega Y \to \Omega L_{\Sigma f}Y$ induces a map $\lambda\colon L_f \Omega Y \to \Omega L_{\Sigma f}Y$. A fundamental result of Dror Farjoun ([DF 3]) and the author ([Bo 7]) is

THEOREM 5.2. *If $Y \in Ho_*$ is a connected space, then $\lambda\colon L_f \Omega Y \simeq \Omega L_{\Sigma f}Y$.*

Thus, in Theorem 5.1, it suffices to show that $L_{\Sigma f}Y$ is a GEM. Our main tool from [Bo 7, Cor. 6.9] will be

KEY LEMMA 5.3. *For connected spaces X, $Y \in Ho_*$, if $\mathrm{map}_*(X, Y)$ is homotopically discrete and if $\pi_1 Y$ acts trivially on $[X, Y]$, then the inclusion $X \subset SP^\infty X$ induces an equivalence $\mathrm{map}_*(SP^\infty X, Y) \simeq \mathrm{map}_*(X, Y)$.*

The infinite symmetric product $SP^\infty X$ is a GEM with $\pi_* SP^\infty X \cong \widetilde{H}_*(X; Z)$ by Dold-Thom, and we may obtain other GEMs by

LEMMA 5.4. *If $M \in Ho_*$ is a GEM, then so are its homotopy retracts.*

Proof. For a homotopy retraction $r\colon M \to N$ with homotopy fiber $i\colon F \to M$, choose a map $h\colon M \to \prod_n K(\pi_n F, n)$ such that $h_* i_*\colon \pi_* F \cong \pi_* F$, and deduce that F is a GEM with $M \simeq F \times N$. Then reverse the roles of F and N to conclude that N is a GEM. □

To show that $\mathrm{map}_*(X,Y)$ is homotopically discrete, it is generally not sufficient to check that its base component is contractible (see e.g. [DM]). However, this difficulty disappears when Y is an H-space.

LEMMA 5.5. *For connected spaces $X, Y \in Ho_*$, if $\mathrm{map}_*(X,\Omega Y) \simeq *$, and if Y is an H-space, then $\mathrm{map}_*(X,Y)$ is homotopically discrete and $\pi_1 Y$ acts trivially on $[X,Y]$.*

Proof. Since Y is an H-space, there is a Hopf fibration $Y \to Y*Y \to \Sigma Y$ whose fiber inclusion is nullhomotopic (see e.g. [St, p. 5]). Hence, $\pi_1\mathrm{map}_*(X,\Sigma Y)$ acts transitively on the components of $\mathrm{map}_*(X,Y)$, and they must all be contractible since the base component is. □

5.6. Proof of Theorem 5.1. $L_{\Sigma f}Y$ is an H-space since $L_{\Sigma f}$ preserves products in Ho_*, and $L_{\Sigma f}Y$ is L_f-acyclic since $L_f L_{\Sigma f}Y \simeq L_f Y \simeq *$. Thus $\mathrm{map}_*(L_{\Sigma f}Y, \Omega L_{\Sigma f}Y) \simeq *$ since $\Omega L_{\Sigma f}Y \simeq L_f \Omega Y$ is f-local, and hence $L_{\Sigma f}Y$ is a retract of $SP^\infty L_{\Sigma f}Y$ by Lemmas 5.5 and 5.3. Now $L_{\Sigma F}Y$ is a GEM by Lemma 5.4. □

The GEMs in Theorem 5.1 have a special "transitory" property which we now introduce. Recall that an abelian group G is called Ext-*complete* when $\mathrm{Hom}(Q,G) = 0 = \mathrm{Ext}(Q,G)$, and that such a group decomposes as

$$G \cong \mathrm{Ext}(Q/Z, G) \cong \prod_p \mathrm{Ext}(Z_{p^\infty}, G)$$

where p ranges over all primes ([BK]). For a set J of primes, G is called Ext-J-*complete* when it is Ext-complete with $\mathrm{Ext}(Z_{p^\infty}, G) = 0$ for each $p \notin J$.

LEMMA 5.7. *For abelian groups G and H, the condition* $\mathrm{Hom}(G,H) = 0 = \mathrm{Ext}(G,H)$ *holds if and only if there exist complementary sets of primes J and J' such that:* (i) *G is J'-torsion and H is J-local; or* (ii) *G is J'-local and H is* Ext-J-*complete.*

Proof. This follows by [Bo 7, 5.5] and [Bo 1, 2.3]. □

COROLLARY 5.8. *For abelian groups G and H, if* $\mathrm{Hom}(G,H) = 0 = \mathrm{Ext}(G,H)$, *then $K(H,n)$ is $K(G,m)$-null for all $m,n \geq 1$. Conversely, if $K(H,n)$ is $K(G,m)$-null for some $m,n \geq 1$ with $n \geq m+1$, then* $\mathrm{Hom}(G,H) = 0 = \mathrm{Ext}(G,H)$.

Proof. By 5.7 and [Bo 2, 4.3], there exists a set J of primes such that: (i) $K(G,m)$ is $HZ_{(J)*}$-acyclic and $K(H,n)$ is $HZ_{(J)*}$-local; or (ii) $K(G,m)$ is $H(\bigoplus_{p\in J} Z/p)_*$-acyclic and $K(H,n)$ is $H(\bigoplus_{p\in J} Z/p)_*$-local. Hence, $K(H,n)$ is $K(G,n)$-null by [Bo 2, 12.2]. The converse follows since $K(H,m+1)$ must be $K(G,m)$-null. □

A graded abelian group G_* will be called *transitory* when $\mathrm{Hom}(G_m, G_n) = 0$ for $n \geq m+1$ and $\mathrm{Ext}(G_m, G_n) = 0$ for $n \geq m+2$.

PROPOSITION 5.9. *For a GEM M, the homotopy π_*M is transitory if and only if* $\mathrm{map}_*(M, \Omega M) \simeq *$.

Proof. This follows from Corollary 5.8 since $\mathrm{map}_*\big(K(\pi_m M, m), K(\pi_n M, n)\big) \simeq *$ for all $m > n$ if and only if $\mathrm{map}_*(M, \Omega M) \simeq *$. □

A GEM M will be called *transitory* when π_*M is transitory

Remark 5.10. By Lemma 5.7 and [BK], if M is a transitory GEM, then the homotopy groups $\pi_*\mathbf{F}_{p\infty}M$ of its p-completion are trivial except in two successive dimensions m_p and m_p+1 with $\pi_{m_p+1}\mathbf{F}_{p\infty}M$ torsion-free. In general, we may view the transitory GEMs as "small abelian spaces," using the language of [DFS].

We now obtain a stronger version of Theorem 5.1.

THEOREM 5.11. *If $Y \in Ho_*$ is a connected f-acyclic H-space, then $L_{\Sigma f}Y$ is a transitory GEM, as are the components of $L_f\Omega Y$.*

Proof. This follows for $L_{\Sigma f}Y$ by 5.1 and 5.6, and then follows for $L_f\Omega Y$ by 5.2 and 5.9. □

A space $X \in Ho_*$ is called *simple* when X is connected with abelian π_1X acting trivially on π_nX for $n \geq 2$. We need

PROPOSITION 5.12. *If $X \in Ho_*$ is a simple space with P^2X a GEM and with π_*X transitory, then X is a GEM.*

Proof. Assume inductively that the nth Postnikov section P^nX is a GEM where $n \geq 2$. Then $\widetilde{H}^*(P^{n-1}X; \pi_{n+1}X) = 0$ by Corollary 5.8 and $\widetilde{H}^i\big(K(\pi_nX, n); \pi_{n+1}X\big) = 0$ for $i \leq n$ and $i = n+2$, where the vanishing of

$$H^4\big(K(\pi_2X, 2); \pi_3X\big) \cong \mathrm{Hom}(\Gamma\pi_2X, \pi_3X)$$

follows using the exact sequence

$$\pi_2X \otimes \pi_2X \leftrightarrows \Gamma\pi_2X \leftrightarrows \pi_2X \otimes Z/2 \leftrightarrows 0.$$

Since $H^1(P^{n-1}X; \pi_{n+1}X) = 0$, the Serre spectral sequence now shows that $H^{n+2}(P^nX; \pi_{n+1}X) = 0$, and thus $P^{n+1}X$ is a GEM. Hence, X is a GEM by induction. □

This leads to an easy proof of the following slightly enhanced theorem of Dror Farjoun and Smith ([DFS]).

THEOREM 5.13. *If $X \in Ho_*$ is a space with $L_{\Sigma f}X \simeq *$, then $L_{\Sigma^2 f}X$ is a transitory GEM.*

Proof. Since $L_{\Sigma f}X \simeq *$, X must be simply connected with $L_f\Omega X \simeq *$ by Theorem 5.2. Hence, $L_{\Sigma f}\Omega X$ is a transitory GEM by Theorem 5.11, and its classifying space $L_{\Sigma^2 f}X$ is also a transitory GEM by Proposition 5.12. □

Finally, for later use, we need

PROPOSITION 5.14. *If F is the homotopy fiber of a map of simply connected spaces $X \to Y$ and if $\pi_* F$ is transitory, then F is a transitory GEM.*

Proof. By Proposition 5.12, it suffices to show P^2F is a GEM. Let $Y \to \overline{Y}$ be a map such that: $\pi_i Y \cong \pi_i \overline{Y}$ for $i \leq 2$; $\pi_3 \overline{Y}$ is the cokernel of $\pi_3 X \to \pi_3 Y$; and $\pi_n \overline{Y} = 0$ for $n > 3$. Construct a fiber sequence

$$P^2F \to K(\pi_2 X, 2) \to \overline{Y}$$

by taking a Moore-Postnikov section of $X \to \overline{Y}$. The k-invariant of $\overline{Y}$ corresponds to a homomorphism ϕ: $\Gamma\pi_2\overline{Y} \to \pi_3\overline{Y}$ vanishing on the image of $\Gamma\pi_2 X \to \Gamma\pi_2\overline{Y}$. Using the exact sequence

$$\pi_2\overline{Y} \otimes \pi_1 F \twoheadrightarrow \Gamma\pi_2\overline{Y}/\Gamma\pi_2 X \twoheadrightarrow \Gamma\pi_1 F \twoheadrightarrow 0,$$

we find that $\phi = 0$ since $\pi_3\overline{Y} \subset \pi_2 F$ and $\mathrm{Hom}(\pi_1 F, \pi_2 F) = 0$. Hence $\overline{Y}$ and P^2F are GEMs. □

6. General fibration theorems. For a map f: $A \to B$ of pointed spaces, we shall show in Section 9 that the functors $L_{\Sigma f}$ and $L_f\Omega$ "almost" preserve homotopy fiber sequences. In preparation, we now develop a series of general fibration theorems.

THEOREM 6.1. *For a homotopy fiber sequence $F \to X \to B$ of pointed connected spaces, there is a natural homotopy fiber sequence $L_{\Sigma f}F \to L_{\Sigma f}X \to Y'$ and a natural diagram*

$$\begin{array}{ccccc} F & \twoheadrightarrow & X & \twoheadrightarrow & Y \\ \downarrow u & & \downarrow u & & \downarrow u' \\ L_{\Sigma f}F & \twoheadrightarrow & L_{\Sigma f}X & \twoheadrightarrow & Y' \end{array}$$

such that u': $Y \twoheadrightarrow Y'$ is a Σf-local equivalence and Y' is $\Sigma^2 f$-local.

The proof is in §6.8. We view u' as a "mixture" of the Σf-localization and the $\Sigma^2 f$-localization of Y. Generalizing a result of Dror Farjoun and Smith ([DFS, Theorem A]), we now obtain

THEOREM 6.2. *For a homotopy fiber sequence $F \to X \to Y$ of pointed connected spaces, if $X \to Y$ is a Σf-local equivalence, then $L_{\Sigma f}F$ is an f-acyclic transitory GEM.*

Proof. In the homotopy fiber sequence $L_{\Sigma f}F \to L_{\Sigma f}X \to Y'$, the first map is nullhomotopic since the second composes with $Y' \to L_{\Sigma f}Y' \simeq L_{\Sigma f}Y$ to give an equivalence. Thus $L_{\Sigma f}F$ is an H-space by a theorem of Sugawara (since $L_{\Sigma f}F$ is a homotopy retract of $\Omega Y'$), and is f-acyclic by [Bo 7, 4.8(ii) and 4.12] since $X \to Y$ is a Σf-local equivalence. Hence, $L_{\Sigma f}F$ is a transitory GEM by Theorem 5.11. □

The above theorem may be viewed as a partial converse to the following result of [Bo 7, 4.8 and 4.12] or [DF 3].

THEOREM 6.3. *For a homotopy fiber sequence $F \to X \to Y$ of pointed connected spaces, if F is f-acyclic, then $X \to Y$ is an f-local equivalence.*

For a homotopy fiber sequence $F \to X \to Y$ and localization functor L, the homotopy fiber of the map

$$LF \to \text{fiber } (LX \to LY)$$

is called the *L-error term.* It measures the failure of $LF \to LX \to LY$ to be a homotopy fiber sequence. Note that the L_f-error term of $\Omega F \to \Omega X \to \Omega Y$ is the loop space of the $L_{\Sigma f}$-error term of $F \to X \to Y$. The following theorem will help to make these $L_{\Sigma f}$-error terms accessible.

THEOREM 6.4. *For a homotopy fiber sequence $F \to X \to Y$ of pointed connected spaces, the $L_{\Sigma f}$-error term is naturally equivalent to* Ω fiber $(L_{\Sigma^2 f}Y' \to L_{\Sigma f}Y')$, *where Y' is given by Theorem* 6.1.

Proof. This follows by taking vertical fibers in the diagram of homotopy fiber sequences

$$\begin{array}{ccccc} L_{\Sigma f}F & \twoheadrightarrow & L_{\Sigma f}X & \twoheadrightarrow & Y' \simeq L_{\Sigma^2 f}Y' \\ \downarrow & & \downarrow & & \downarrow \\ \text{fiber} & \twoheadrightarrow & L_{\Sigma f}X & \twoheadrightarrow & L_{\Sigma f}Y \simeq L_{\Sigma f}Y'. \end{array}$$

□

We now turn to the proof of Theorem 6.1, and start by recalling

THEOREM 6.5. *For a homotopy fiber sequence $F \to X \to Y$ of pointed spaces with X and Y connected, there is a natural homotopy fiber sequence $L_f F \to \overline{X} \to L_{\Sigma f}Y$ and a natural diagram*

$$\begin{array}{ccccc} F & \twoheadrightarrow & X & \twoheadrightarrow & Y \\ \downarrow u & & \downarrow \overline{u} & & \downarrow u \\ L_f F & \twoheadrightarrow & \overline{X} & \twoheadrightarrow & L_{\Sigma f}Y \end{array}$$

such that $\overline{u}$: $X \to \overline{X}$ is an f-local equivalence and $\overline{X}$ is Σf-local.

Proof. This follows by [Bo 7, 4.1 and 4.12] using $\Omega\overline{X} \simeq \text{fiber}\,(L_f\Omega Y \to L_f F)$ to show that $\overline{X}$ is Σf-local. □

We view $\overline{u}$ as a "mixture" of the f-localization and the Σf-localization. It takes the latter form in the following case.

LEMMA 6.6. *If $F \to X$ is nullhomotopic in the above homotopy fiber sequence, then $\overline{u}$: $L_{\Sigma f}X \simeq \overline{X}$.*

Proof. In the diagram of homotopy fiber sequences

$$\begin{array}{ccccc} \Omega X & \twoheadrightarrow & \Omega Y & \twoheadrightarrow & F \\ \downarrow{\scriptstyle \Omega\overline{u}} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ \Omega\overline{X} & \twoheadrightarrow & L_f\Omega Y & \twoheadrightarrow & L_f F, \end{array}$$

it suffices to show that $\Omega\overline{u}$ is an f-local equivalence. This follows since L_f preserves products and since the principal fiber sequence $\Omega X \to \Omega Y \to F$ is equivalent to a projection sequence $\Omega X \to \Omega X \times F \to F$ because it has a cross-section. □

For a fibration $X \to Y$, we let $\text{Pow}(X/Y)$ denote the simplicial space (i.e. bisimplicial set) with

$$\text{Pow}(X/Y)_{m\bullet} = X \times_Y \cdots \times_Y X$$

given by the fiber product of $m+1$ copies of X over Y, and with horizontal simplicial operators given by the usual formulae ([Ma, 1.4]). Applying the diagonal functor $\text{diag}(\twoheadrightarrow)$ to the natural augmentation map $\text{Pow}(X/Y)_{0\bullet} \to Y$, we obtain

LEMMA 6.7. *For a fibration $X \to Y$, there is a natural weak equivalence* $\text{diag Pow}\,(X/Y) \simeq Y$.

Proof. This follows by [BK, p. 335] or [BF, B.2] since the augmentation map $\text{Pow}(X/Y)_{\bullet n} \to Y_n$ is a weak equivalence in each vertical dimension n. □

6.8. Proof of Theorem 6.1. For the homotopy fiber sequence $F \to X \to Y$, there is a natural square

$$\begin{array}{ccc} \text{Pow}(PF/F) & \twoheadrightarrow & \text{Pow}(X/X) \\ \downarrow & & \downarrow \\ \text{Pow}(PF/*) & \twoheadrightarrow & \text{Pow}(X/Y) \end{array}$$

of pointed simplicial spaces, which restricts to a homotopy fiber square in each horizontal dimension m, where $PF \to F$ is the path fibration on F. Since

$\mathrm{Pow}(PF/*)$ is weakly equivalent to a simplicial point, we may identify $\mathrm{Pow}(PF/F)$ with the homotopy fiber of $\mathrm{Pow}(X/X) \to \mathrm{Pow}(X/Y)$ and obtain a diagram of homotopy fiber sequences

$$\begin{array}{ccccc} \mathrm{diag}\ \mathrm{Pow}(PF/F) & \twoheadleftarrow\!\!\!\!\!\rightarrow & \mathrm{diag}\ \mathrm{Pow}(X/X) & \twoheadleftarrow\!\!\!\!\!\rightarrow & \mathrm{diag}\ \mathrm{Pow}(X/Y) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{diag}\ L_f\ \mathrm{Pow}(PF/F) & \twoheadleftarrow\!\!\!\!\!\rightarrow & \mathrm{diag}\ \overline{\mathrm{Pow}}(X/X) & \twoheadleftarrow\!\!\!\!\!\rightarrow & \mathrm{diag} L_{\Sigma f}\mathrm{Pow}(X/Y) \end{array}$$

by Theorem 6.5 and [BF, B.4]. The top sequence is weakly equivalent to $F \to X \to Y$ by Lemma 6.7, and β is equivalent to $u\colon X \to L_{\Sigma f}X$ by Lemma 6.7 since $\mathrm{Pow}(X/X)$ is constant at X. We claim that $\mathrm{Pow}(PF/F)$ is equivalent to the pointed simplicial space $\Omega_{\mathrm{bis}}F$ with

$$(\Omega_{\mathrm{bis}}F)_{m\bullet} = \mathrm{map}_*(\Delta^m/\mathrm{sk}^0\Delta^m, F)$$

for $m \geq 0$ as in [Bo 7, 3.3]. This follows since the weak equivalences of pointed cosimplicial spaces

$$\{C\,\mathrm{sk}^0\Delta^m/\mathrm{sk}^0\Delta^m\}_m \leftarrow\!\!\!\!\!\twoheadleftarrow \{\Delta^m \cup C\,\mathrm{sk}^0\Delta^m\}_m \twoheadrightarrow\!\!\!\!\!\rightarrow \{\Delta^m/\mathrm{sk}^0\Delta^m\}_m$$

are carried by $\mathrm{map}_*(\twoheadrightarrow, F)$ to weak equivalences relating $\mathrm{Pow}(PF/F)$ and $\Omega_{\mathrm{bis}}F$, where C is the unreduced cone functor. Since $\mathrm{diag}\Omega_{\mathrm{bis}}F \to \mathrm{diag}L_f\Omega_{\mathrm{bis}}F$ is weakly equivalent to $u\colon F \to L_{\Sigma f}F$ by [Bo 7, 3.4 and 3.6], so is α in our diagram. We now let $u'\colon Y \to Y'$ correspond to γ. It is an $L_{\Sigma f}$-equivalence since it is a homotopy colimit of $L_{\Sigma f}$-equivalences by [BK, p. 335]. Finally, Y' is $\Sigma^2 f$-local since $\Omega Y' \simeq$ fiber $(L_{\Sigma f}F \to L_{\Sigma f}X)$. □

7. Nullifications with respect to Moore spaces. Before continuing our study of fibration theorems for $L_{\Sigma f}$, we must determine the A-nullification of a nilpotent space X in two important cases: (i) when A is a wedge of Moore spaces; and (ii) when X is a Postnikov space or other "generalized polyGEM." These nullifications act very much like classical localizations and completions, transforming homotopy groups in an elementary arithmetic way. Our results extend those of [Bo 7], [Bo 8], and [Ca 2].

7.1. Nullifications of groups. A group M is called *G-null* or *G-reduced* for a group G when $\mathrm{Hom}(G, M) = \{1\}$. As explained in [Bo 7, 5.1] and more generally in [Ca 2], each group M has a maximal G-null quotient group $M//G$ called the *G-nullification* or *G-reduction* of M. It is the initial example of a homomorphism from M to a G-null group. The kernel of the quotient map $M \to M//G$ is denoted by $T_G M$ and called the *G-radical* of M. Its G-nullification $T_G M//G$ must be trivial, since the G-null groups are closed under extensions.

The following general examples are easily verified:

(i) For a set I of primes, if G is an I-torsion abelian group with $G/pG \neq 0$ for each $p \in I$, and if M is a nilpotent group, then $M//G$ is the image of the localization $M \to M[I^{-1}]$, and $I_G M$ is the maximal I-torsion subgroup of M.

(ii) For a set J of primes, if G is a uniquely J-divisible abelian group having $Z[J^{-1}]$ as a direct summand, and if M is a nilpotent group, then $M//G$ is the image of the Ext-J-completion $M \to \prod_{p\in J} \mathrm{Ext}(Z_{p^\infty}, G)$, and $T_G M$ is the maximal J-divisible subgroup of M (see [BK, p. 177]).

The next three propositions will help to reduce the nullification theory of nilpotent groups to that of abelian groups. Let ab$G$ denote the abelianization of G.

PROPOSITION 7.2. *For a group G and nilpotent group M with center Z^1M, the following are equivalent:*

(i) *M is G-null*

(ii) *Z^1M is G-null;*

(iii) *M is* ab*G-null.*

Proof. Let $\{Z^nM\}_{n\geq 1}$ be the upper central series of M. By [Wa, 2.1], $\mathrm{Hom}(Z^{n+1}M/Z^nM, Z^1M)$ separates points of $Z^{n+1}M/Z^nM$ for $n \geq 1$, and thus $Z^{n+1}M/Z^nM$ embeds as a subgroup of a product of copies of Z^1M. Hence, M is G-null if and only if Z^1M is G-null, and the result follows since Z^1G is abelian. □

This implies

PROPOSITION 7.3. *For a group G and nilpotent group M, $M//G$ equals $M//$ab$G$.*

Let $\{\Gamma_s M\}_{s\geq 1}$ denote the lower central series of M.

PROPOSITION 7.4. *For a group G and nilpotent group M, the following are equivalent:*

(i) *$M//G = 0$;*

(ii) *$(\mathrm{ab}M)//G = 0$;*

(iii) *$(\Gamma_s M/\Gamma_{s+1}M)//G = 0$ for $s \geq 1$.*

Proof. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are straightforward, and (ii) $\Rightarrow$ (iii) follows using the natural Lie bracket epimorphism

$$\mathrm{ab}M \otimes \cdots \otimes \mathrm{ab}M \twoheadrightarrow \Gamma_s M/\Gamma_{s+1}M. \qquad \square$$

By a *Moore space* $M(G,n) \in Ho_*$ for G abelian and $n \geq 1$, we mean a space equivalent to a pointed CW-complex whose only nonbasepoint cells are

in dimensions n and $n+1$, and whose only reduced integral homology is G in dimension n. Thus we allow $M(G,1)$ to range over many different homotopy types for a given abelian group G. For a sequence $\{G_i\}_{i\geq 1}$ of abelian groups, let $MG = \vee_{i=1}^{\infty} M(G_i, i)$ be a corresponding wedge of Moore spaces; let $MG(n) = \vee_{i=1}^{n} M(G_i, i)$ be the subwedge for $0 \leq n < \infty$; and let J_n denote the set of all primes p such that $G_1 \oplus \cdots \oplus G_n$ is uniquely p-divisible. Generalizing results of [Bo 7], [Bo 8], and [Ca 2], we have

THEOREM 7.5. *For a nilpotent space $Y \in Ho_*$, the nullification $P_{MG}Y$ is nilpotent with natural isomorphism*

$$\pi_{n+1}P_{MG}Y \cong (\pi_{n+1}P_{MG(n)}Y)//(G_1 \oplus \cdots \oplus G_{n+1})$$

for $n \geq 0$. Moreover, for $i \geq n+1$, there is a natural isomorphism

$$\pi_i P_{MG(n)}Y \cong \pi_i Y \otimes Z_{(J_n)}$$

when $G_1, \ldots, G_n$ are all torsion, and otherwise there is a splittable natural short exact sequence

$$0 \Leftrightarrow \prod_{p\in J_n} \mathrm{Ext}(Z_{p^\infty}, \pi_i Y) \Leftrightarrow \pi_i P_{MG(n)}Y \Leftrightarrow \prod_{p\in J_n} \mathrm{Hom}(Z_{p^\infty}, \pi_{i-1}Y) \Leftrightarrow 0.$$

The proof will depend on the following fundamental result of Dror Farjoun which may be deduced from Theorem 6.5.

THEOREM 7.6. *For $A \in Ho_*$, P_A preserves each homotopy fiber sequence $F \to X \to Y$ of pointed spaces such that Y is A-null and connected.*

7.7. Proof of Theorem 7.5. To determine $P_{M(G,1)}Y$, we use the approach of Casacuberta ([Ca 2]). Let $\widetilde{Y}$ be the homotopy fiber of the Postnikov map $Y \to K(\pi_1 Y//G_1, 1)$, and apply Proposition 7.2 and Theorem 7.6 to give a homotopy fiber sequence

$$P_{M(G_1,1)}\widetilde{Y} \Leftrightarrow P_{M(G_1,1)}Y \Leftrightarrow K(\pi_1 Y//G_1, 1).$$

Then $\pi_1 P_{M(G_1,1)}\widetilde{Y} = (\pi_1\widetilde{Y})//G_1 = *$ by Proposition 7.3, and $P_{M(G_1,1)}\widetilde{Y}$ is the HR_*-localization of $\widetilde{Y}$ by Lemma 5.7 and [Bo 2, 4.3], where $R = Z_{(J_1)}$ for G_1 torsion and $R = \bigoplus_{p\in J_1} Z/p$ otherwise. Thus, $\pi_1 P_{M(G_1,1)}Y \cong \pi_1 Y//G_1$ and $\pi_i P_{M(G_1,1)}Y$ is given by the arithmetic expressions of Theorem 7.5 for $i \geq 2$. Hence, $\pi_i P_{M(G_1,1)}Y$ is a nilpotent $\pi_1 P_{M(G_1,1)}Y$-module, and $P_{M(G_1,1)}Y$ is nilpotent. The theorem now follows as in [Bo 7, §5] or [Bo 8, §4]. □

We easily deduce

COROLLARY 7.8. *A nilpotent space $Y \in Ho_*$ is MG-null if and only if $K(Z^1\pi_1 Y, 1)$ and $K(\pi_n Y, n)$ are MG-null for each $n \geq 2$.*

A space A is said to *kill* a space X when $P_A X \simeq *$.

COROLLARY 7.9. *For a nilpotent space $Y \in Ho_*$, MG kills Y if and only if it kills $K(\Gamma_s\pi_1 Y/\Gamma_{s+1}\pi_1 Y, 1)$ and $K(\pi_n Y, n)$ for each $s \geq 1$ and $n \geq 2$.*

Proof. By Theorem 7.5, *MG* kills Y if and only if it kills $K(\pi_n Y, n)$ for each $n \geq 1$. Given that *MG* kills $K(\pi_1 Y, 1)$, let $\Gamma_t\pi_1 Y$ be the smallest nontrivial term in the lower central series of $\pi_1 Y$. Then $\mathrm{Hom}(Z_{p^\infty}, \Gamma_t\pi_1 Y)$ is a direct summand of $\mathrm{Hom}(Z_{p^\infty}, \pi_1 Y)$ for each prime p, since it is Ext-complete and the quotient is contained in the torsion-free group $\mathrm{Hom}(Z_{p^\infty}, \pi_1 Y/\Gamma_t\pi_1 Y)$. Thus *MG* kills $K(\Gamma_t\pi_1 Y, 1)$ by Proposition 7.4 and Theorem 7.5. Hence, *MG* kills each $K(\Gamma_s\pi_1 Y/\Gamma_{s+1}\pi_1 Y, 1)$ by an inductive argument using Theorem 6.3, and the result follows easily. □

Note 7.10. We cannot replace "$s \geq 1$" by "$s = 1$" in the above result. For instance, $M(Q, 1)$ kills $K(\mathrm{ab}N, 1)$ but does not kill $K(N, 1)$, where N is the nilpotent group of [Wa, 5.2] with $\mathrm{ab}N = Q \oplus Q$, $\Gamma_2 N = Z_{p^\infty}$, and $\Gamma_3 N = 0$.

8. Generalized polyGEMs and their nullifications. In [DFS], Dror Farjoun and Smith introduced the notion of a polyGEM to describe the $L_{\Sigma^2 f}$-error terms of fiber sequences. Roughly speaking, a polyGEM is built from GEMs in the same way as a Postnikov section is built from Eilenberg-MacLane spaces. We now introduce a class of generalized polyGEMs and determine their nullifications. This will be used in our study of $L_{\Sigma f}$-error terms and may be of independent interest. Our results extend those of [Bo 8, §4].

Definition 8.1. A space $X \in Ho_*$ is *Z/p-Postnikov* for a prime p, when $\pi_i X$ is uniquely p-divisible for sufficiently large i. A space $X \in Ho_*$ is a *generalized polyGEM* when $P_{\Sigma K(Z/p,n)}X$ is Z/p-Postnikov for each $n \geq 1$ and prime p.

PROPOSITION 8.2. *If a space X is a GEM, then it is a generalized polyGEM.*

Proof. This follows since $P_{\Sigma K(Z/p,n)}X$ is a $\Sigma K(Z/p, n)$-null GEM by [Bo 8, 2.11]. □

PROPOSITION 8.3. *In a homotopy fiber sequence $F \to X \to Y$ of pointed connected spaces, if any two of the spaces are generalized polyGEMs, then so is the third.*

Proof. This follows since the $P_{\Sigma K(Z/p,n)}$-error term of the fiber sequence is of the form $K(G, n)$ by [Bo 7, 8.1]. □

The following four propositions on generalized polyGEMs now follow easily.

PROPOSITION 8.4. *The generalized polyGEMs are closed under homotopy retraction.*

PROPOSITION 8.5. *A connected Postnikov space is a generalized polyGEM.*

PROPOSITION 8.6. *For $i \geq 0$, a connected space $X \in Ho_*$ is a generalized polyGEM if and only if its i-connected section $X\langle i\rangle$ is a generalized polyGEM.*

For a nilpotent space $Y \in Ho_*$, the homotopy fiber of the localization map $Y \to Y[1/p]$ is called the *p-torsion component* of Y and denoted by $\tau_p Y$.

PROPOSITION 8.7. *For $i \geq 1$, a connected space $X \in Ho_*$ is a generalized polyGEM if and only if $\tau_p(X\langle i\rangle)$ is a generalized polyGEM for each prime p.*

We now determine the A-nullification of a nilpotent generalized polyGEM X, for an arbitrary connected space A. Let $MHA = \vee_{i=1}^{\infty} M(H_i A, i)$ be a corresponding wedge of Moore spaces where $M(H_1 A, 1)$ is chosen so that there is an H_1-equivalence $M(H_1 A, 1) \to A$ mapping π_1 surjectively. Then MHA kills A, since it successively kills the homology groups of A. Thus there is a natural transformation $P_A \to P_{MHA}$ by 4.5.

THEOREM 8.8. *For a connected space A and a nilpotent generalized polyGEM X, there is a natural equivalence $P_A X \simeq P_{MHA} X$, and $P_A X$ is a nilpotent generalized polyGEM.*

The proof is in §8.16. This theorem shows that $\pi_* P_A X \cong \pi_* P_{MHA} X$ is given by the arithmetic expressions of Theorem 7.5. Moreover, it combines with Corollaries 7.8 and 7.9 to give

COROLLARY 8.9. *A nilpotent generalized polyGEM X is A-null for a space A if and only if $K(Z^1\pi_1 X, 1)$ and $K(\pi_n X, n)$ are A-null for $n \geq 2$.*

COROLLARY 8.10. *A space A kills a nilpotent generalized polyGEM X if and only if A kills $K(\Gamma_s \pi_1 X/\Gamma_{s+1}\pi_1 X, 1)$ and $K(\pi_n X, n)$ for each $s \geq 1$ and $n \geq 2$.*

By Theorem 4.4, this implies

COROLLARY 8.11. *A nilpotent generalized polyGEM X is f-acyclic for a map $f\colon A \to B$ if and only if $K(\Gamma_s \pi_1 X/\Gamma_{s+1}\pi_1 X, 1)$ and $K(\pi_n X, n)$ are f-acyclic for each $s \geq 1$ and $n \geq 2$.*

For instance, this shows that a nilpotent Postnikov space X is E_*-acyclic for a generalized homology theory E_* if and only if $K(\Gamma_s \pi_1 X/\Gamma_{s+1}\pi_1 X, 1)$ and $K(\pi_n X, 1)$ are E_*-acyclic for each $s \geq 1$ and $n \geq 2$.

To prove Theorem 8.8, we need four lemmas.

LEMMA 8.12. *If X is a nilpotent generalized polyGEM, then so is $P_{MHA} X$.*

Proof. Since $P_{MHA}X$ is nilpotent by Theorem 7.5, it suffices by Proposition 8.7 to show that $\tau_p P_{MHA}X\langle 1\rangle$ is a generalized polyGEM for each prime p. If $\widetilde{H}_*A$ is uniquely p-divisible, then the nullification map $X \to P_{MHA}X$ is an HZ/p_*-equivalence of nilpotent spaces and consequently induces an isomorphism $\pi_i\tau_p X\langle 1\rangle \cong \pi_i\tau_p P_{MHA}X\langle 1\rangle$ for $i \geq 2$. In this case, $\tau_p P_{MHA}X\langle 1\rangle$ must be a generalized polyGEM because $\tau_p X\langle 1\rangle$ is. If $\widetilde{H}_*A$ is not uniquely p-divisible, then $\pi_i\tau_p P_{MHA}X\langle 1\rangle = 0$ for sufficiently large i by Theorem 7.5. In this case, $\tau_p P_{MHA}X\langle 1\rangle$ is a generalized polyGEM by Proposition 8.5. □

LEMMA 8.13. *For $n \geq 2$ and a nonzero p-torsion abelian group G, $K(G,n)$ kills each $(n \Leftrightarrow 1)$-connected generalized polyGEM X with p-torsion homotopy groups and with $\pi_n X/p = 0$ when $G/p = 0$.*

Proof. Since $P_{\Sigma K(Z/p,n+1)}X$ is a Postnikov space with p-torsion homotopy groups and with the same $(n+1)$-Postnikov section as X, $K(G,n)$ kills $P_{\Sigma K(Z/p,n+1)}X$. Since $K(G,n)$ also kills $K(Z/p,n+1)$ and $\Sigma K(Z/p,n+1)$, it must kill X. □

LEMMA 8.14. *For $n \geq 2$ and an abelian group G with $G \otimes Q \neq 0$, $K(G,n)$ kills each $(n \Leftrightarrow 1)$-connected rational space X.*

Proof. Since $K(G,n)$ kills $K(Q,i)$ for $i \geq n$, it kills the rational sphere S^i_Q for $i \geq n$, and thus kills the Moore spaces $M(H_iX,i)$ for $i \geq n$. Hence, $K(G,n)$ kills X. □

LEMMA 8.15. *If X is a generalized polyGEM, and if a space W kills $K(\pi_iX,i)$ for each i, then W kills X.*

Proof. Choose $n \geq 2$ such that either $\pi_iX \otimes Q = 0$ for each $i > n$ or $\pi_kX \otimes Q \neq 0$ for some k with $2 \leq k \leq n$. Since W kills the Postnikov section P^nX, it suffices by Theorem 6.3 to show that W kills the n-connected section $X\langle n\rangle$. By Lemma 8.14, W kills $\Omega K(\pi_iX,i)_Q$ for $i > n$ and kills $X\langle n\rangle_Q$. Hence, for each prime p, W kills $\tau_p K(\pi_iX,i)$ for $i > n$, and we must show that it kills the generalized polyGEM $\tau_p X\langle n\rangle$. This follows easily from Lemma 8.13. □

8.16. Proof of Theorem 8.8. The homotopy fiber F of the nullification map $X \to P_{MHA}X$ is a nilpotent generalized polyGEM by Lemma 8.12 and Proposition 8.3. Since *MHA* kills F by Theorem 7.6, it kills $K(\Gamma_s\pi_1F/\Gamma_{s+1}\pi_1F,1)$ and $K(\pi_nF,n)$ for each $s \geq 1$ and $n \geq 2$ by Corollary 7.9. Since $SP^\infty A \simeq SP^\infty MHA$, the space A must also kill these abelian Eilenberg-MacLane spaces by [Bo 8, 3.2]. Hence, A kills F by Lemma 8.15, and $X \to P_{MHA}X$ is a P_A-equivalence by Theorem 6.3. Since *MHA* kills A, we conclude that $P_{MHA}X$ is A-null and that $X \to P_{MHA}X$ is the A-nullification. □

9. The main fibration theorem. Our Main Theorem 9.7 will show that the functors $L_{\Sigma f}$ and $L_f\Omega$ "almost" preserve homotopy fiber sequences. To understand the error terms, we require some preliminaries.

A graded abelian group G_* will be called *k-transitory* for a positive integer k when $\mathrm{Hom}(G_m, G_n) = 0$ for $n \Leftrightarrow m \geq k$ and $\mathrm{Ext}(G_m, G_n) = 0$ for $n \Leftrightarrow m \geq k+1$. Note that "1-transitory" means "transitory" in the sense of Proposition 5.9.

PROPOSITION 9.1. *For a simple space X, the homotopy π_*X is k-transitory if and only if X is a generalized polyGEM with* $\mathrm{map}_*(X, \Omega^k X) \simeq *$.

Proof. If π_*X is k-transitory, then X is a generalized polyGEM by Lemma 5.7 and Proposition 8.7, and $\mathrm{map}_*(X, \Omega^k X) \simeq *$ by an inductive argument using Corollaries 8.9 and 5.8. Conversely, if X is a generalized polyGEM with $\mathrm{map}_*(X, \Omega^k X) \simeq *$, then $K(\pi_{i+k}X, i)$ is X-null for $i \geq 1$ by Corollary 8.9. Since X kills itself, it also kills each $K(\pi_j X, j)$ by Corollary 8.10. Consequently, $K(\pi_{i+k}X, i)$ is $K(\pi_j X, j)$-null for $i, j \geq 1$, and π_*X is k-transitory by Corollary 5.8. □

Definition 9.2. A space $X \in Ho_*$ is a *k-transitory polyGEM* if:

(i) X is the homotopy fiber of a map of simply connected spaces; and

(ii) π_*X is k-transitory.

Condition (i) implies that X is simple, and (i) holds when X is simply connected or when X is a connected H-space (and thus has a Hopf fiber sequence $X \to X * X \to \Sigma X$). By Proposition 9.1, a k-transitory polyGEM is a generalized polyGEM, and by Proposition 5.14 we have

PROPOSITION 9.3. *A space $X \in Ho_*$ is a 1-transitory polyGEM if and only if X is a transitory GEM.*

By Proposition 9.1, a k-transitory polyGEM X is $\Sigma^k X$-null and has a Postnikov-like decomposition

$$X \simeq P_{\Sigma^k X}X \Leftrightarrow\!\!\to P_{\Sigma^{k-1}X}X \Leftrightarrow\!\!\to \cdots \Leftrightarrow\!\!\to P_{\Sigma X}X \Leftrightarrow\!\!\to P_X X \simeq *.$$

PROPOSITION 9.4. *If X is a k-transitory polyGEM, then the homotopy fiber F_i of $P_{\Sigma^i X}X \to P_{\Sigma^{i-1}X}X$ is a transitory GEM for $1 \leq i \leq k$.*

Proof. Since ΩF_i is $\Sigma^{i-1}X$-null and since $\Sigma^{i-1}X$ kills F_i by Theorem 7.6, we have $\mathrm{map}_*(F_i, \Omega F_i) \simeq *$. Since X is a homotopy fiber of a map of simply connected spaces, so is F_1 by Theorem 6.5. Thus, since F_i is $(i \Leftrightarrow 1)$-connected, Propositions 9.1 and 9.3 show that F_i is a transitory GEM. □

In 9.16 we shall prove

THEOREM 9.5. *For a map $f\colon A \to B$ with $f_*\colon \pi_0 A \cong \pi_0 B$ and for a connected space X in Ho_*, the homotopy fiber of $L_{\Sigma^2 f}X \to L_{\Sigma f}X$ is a 2-transitory polyGEM.*

Remark 9.6. By Lemma 5.7 and [BK], if M is a 2-transitory polyGEM, then the homotopy groups of its p-completion $\pi_*\mathbf{F}_{p^\infty}M$ are trivial except in three successive dimensions m_p, $m_p + 1$, and $m_p + 2$ with $\pi_{m_p+2}\mathbf{F}_{p^\infty}M$ torsion-free.

Theorem 9.5 immediately combines with Theorems 6.4 and 5.2 to give

MAIN THEOREM 9.7. *For a map $f\colon A \to B$ with $f_*\colon \pi_0 A \cong \pi_0 B$ and for a homotopy fiber sequence $F \to X \to Y$ of pointed connected spaces, the $L_{\Sigma f}$-error term is the loop space of a 2-transitory polyGEM, and the $L_f\Omega$-error term is the double loop space of a 2-transitory polyGEM. In particular, the components of these error terms are 2-transitory polyGEMs.*

Recall that the $L_{\Sigma f}$-error and $L_f\Omega$-error terms are the homotopy fibers of the maps

$$L_{\Sigma f}F \twoheadrightarrow \text{fiber}(L_{\Sigma f}X \twoheadrightarrow L_{\Sigma f}Y)$$
$$L_f\Omega F \twoheadrightarrow \text{fiber}(L_f\Omega X \twoheadrightarrow L_f\Omega Y).$$

The theorem shows that $L_{\Sigma f}$ almost preserves a homotopy fiber sequence $F \to X \to B$, while L_f almost preserves $\Omega F \to \Omega X \to \Omega B$. Before turning to the proof of Theorem 9.5, we shall refine our main theorem to show that the components of the above error terms are often GEMs.

LEMMA 9.8. *A connected H-space X is a GEM if and only if the p-completion $\mathbf{F}_{p\infty}X$ is a GEM for each prime p.*

Proof. This follows by a straightforward arithmetic square argument ([DFDK]). □

LEMMA 9.9. *For a map $f\colon A \to B$ with $f_*\colon \pi_0 A \cong \pi_0 B$ and for a connected space X, let F be the homotopy fiber of $u\colon L_{\Sigma^3 f}X \to L_{\Sigma^2 f}X$. Then the Hopf map $\eta\colon \Omega^2 F \to \Omega^3 F$ is null-homotopic.*

Proof. This follows since the f-equivalence $\Omega^2 u\colon \Omega^2 L_{\Sigma^3 f}X \to \Omega^2 L_{\Sigma^2 f}X$ has the left lifting property with respect to the map $\Omega^3 u\colon \Omega^3 L_{\Sigma^3 f}X \to \Omega^3 L_{\Sigma^2 f}X$ of f-local spaces. □

9.10. Refinements of Main Theorem 9.7. For a map $f\colon A \to B$ with $f_*\colon \pi_0 A \cong \pi_0 B$ and for a homotopy fiber sequence $F \to X \to Y$ of pointed connected spaces, it is now straightforward to show that the following spaces have GEM components: the $L_{\Sigma f}$-error term localized at odd primes; the $L_f\Omega$-error term localized at odd primes; and the $L_f\Omega^2$-error term. However, we do not know whether the $L_{\Sigma f}$-error term and the $L_f\Omega$-error term always have GEM components.

We devote the rest of this section to proving Theorem 9.5. Let $f\colon A \to B$ be a fixed map with $f_*\colon \pi_0 A \cong \pi_0 B$.

LEMMA 9.11. *For a connected space X, the homotopy fiber F of $L_{\Sigma^2 f}X \to L_{\Sigma f}X$ has the following properties:* (i) *F is the homotopy fiber of a map of simply connected spaces;* (ii) *F is f-acyclic;* (iii) *$L_{\Sigma f}F$ is a GEM; and (iv) F is $\Sigma^2 f$-local.*

Proof. Property (i) follows since $\pi_1 L_{\Sigma^2 f} X \cong \pi_1 L_{\Sigma f} X$; properties (ii) and (iii) follow from Theorem 6.2; and property (iv) follows since $L_{\Sigma^2 f} X$ and $L_{\Sigma f} X$ are $\Sigma^2 f$-local. □

Our main task will be to show that a space F with the above properties is a generalized polyGEM. We shall use

LEMMA 9.12. *Let $F' \to F \to F''$ be a homotopy fiber sequence of connected spaces such that F is $\Sigma^2 f$-local and F' is Σf-acyclic. If F'' is a generalized polyGEM, then so is F.*

Proof. In the fiberwise localization diagram

$$\begin{array}{ccccc} F' & \twoheadrightarrow & F & \twoheadrightarrow & F'' \\ \downarrow & & \downarrow & & \downarrow \\ L_{\Sigma^2 f} F' & \twoheadrightarrow & \overline{F} & \twoheadrightarrow & F'' \end{array}$$

(see [Bo 7] or [DF 3]), $L_{\Sigma^2 f} F'$ is a GEM by Theorem 5.13 since F' is Σf-acyclic. Hence, $\overline{F}$ is a generalized polyGEM by Proposition 8.3. Since $F \to \overline{F}$ is a $\Sigma^2 f$-equivalence and F is $\Sigma^2 f$-local, F is a retract of $\overline{F}$, and the lemma follows by Proposition 8.4. □

LEMMA 9.13. *Each $K(Z/p, n)$ is either f-acyclic or f-local.*

Proof. Since $K(Z/p, n)$ is the infinite loop space of an HZ/p-module spectrum, Corollary 2.11 of [Bo 8] shows that $L_f K(Z/p, n) \simeq K(V, n)$ for a Z/p-module V. If $V \neq 0$, then $K(Z/p, n)$ is a retract of $K(V, n)$ and is f-local. □

9.14. The natural map $F \to JF$. For p prime, let $n_p \geq 0$ be the largest integer such that $K(Z/p, n_p)$ is f-local, or let $n_p = \infty$ when $K(Z/p, i)$ is f-local for all i. We call n_p the *mod p transitional dimension* of L_f. For a simple space F, we let $J_p F$ be the modified $(n_p + 2)$-Postnikov section of $\mathbf{F}_{p\infty} F$ with

$$\pi_i J_p F = \begin{cases} \pi_i \mathbf{F}_{p\infty} F & \text{for } i < n_p + 2 \\ \overline{\pi}_i \mathbf{F}_{p\infty} F & \text{for } i = n_p + 2 \\ 0 & \text{for } i > n_p + 2 \end{cases}$$

where $\overline{\pi}_i \mathbf{F}_{p\infty} F$ is the maximal torsion-free Ext-p-complete quotient of $\pi_i \mathbf{F}_{p\infty} F$ (see [BF, p. 182]). We construct JF as a homotopy pullback in the square

$$\begin{array}{ccc} JF & \twoheadrightarrow & \prod_p J_p F \\ \downarrow & & \downarrow \\ F_Q & \twoheadrightarrow & (\prod_p J_p F)_Q \end{array}$$

and we let $F \to JF$ be the induced map.

LEMMA 9.15. *If F is an f-acyclic simple space, then JF is a generalized polyGEM and $\Omega_0 JF$ is a GEM, where Ω_0 is the connected loop functor.*

Proof. Since F is f-acyclic and $K(Z/p, i)$ is f-local for $i \leq n_p$, $\widetilde{H}^i(F; \mathbf{F}_p) = 0$ for $i \leq n_p$. Thus $\mathbf{F}_{p\infty}F$ is n_p-connected, and the conclusions follow from Proposition 8.7 and Lemma 9.8. □

LEMMA 9.16. *If F is an f-acyclic simple space such that $L_{\Sigma f}F$ is a GEM, then $F \to JF$ is a Σf-local equivalence whose homotopy fiber $\widetilde{F}$ is Σf-acyclic.*

Proof. A comparison with the arithmetic square of F shows that $\pi_i F \cong \pi_i JF$ for $i \leq 1$. Thus, to show that $F \to JF$ is a Σf-local equivalence, it suffices by Theorem 5.2 to show that $\Omega_0 F \to \Omega_0 JF$ is an f-local equivalence. $\Omega_0 JF$ is a GEM by Lemma 9.15. Since $K(Z/p, n_p + 1)$ is f-acyclic, if M is an f-local GEM, then $\pi_i M$ is p-torsion-free for $i = n_p + 1$ and uniquely p-divisible for $i \geq n_p + 2$ by Corollary 5.8. Hence, $\Omega_0 F \to \Omega_0 JF$ is a $\mathrm{map}_*(\Leftrightarrow, \mathbf{F}_{p\infty}M)$-equivalence as well as a rational equivalence and is therefore a $\mathrm{map}_*(\Leftrightarrow, M)$-equivalence. Thus $\Omega_0 F \to L_f \Omega_0 JF$ is the universal map in Ho_* from $\Omega_0 F$ to an f-local GEM, since $L_f \Omega_0 JF$ is a GEM by [Bo 8, 2.11] or [DF 3]. Since $L_{\Sigma f}F$ is a GEM, the map $\Omega_0 F \to L_f \Omega_0 JF$ has the same universal property, and therefore $L_f \Omega_0 F \simeq L_f \Omega_0 JF$. Consequently, $F \to JF$ is an $L_{\Sigma f}$-equivalence, and $L_{\Sigma f}\widetilde{F}$ is a GEM by Theorem 6.2. Since $K(Z/p, n_p + 2)$ is Σf-acyclic, $\pi_i L_{\Sigma f}\widetilde{F}$ is p-torsion-free for $i = n_p + 2$ and uniquely p-divisible for $i \geq n_p + 3$, and hence $JL_{\Sigma f}\widetilde{F} \simeq L_{\Sigma f}\widetilde{F}$. Since $J\widetilde{F} \simeq *$, we find that $\widetilde{F} \to L_{\Sigma f}\widetilde{F}$ is nullhomotopic and therefore $L_{\Sigma f}\widetilde{F} \simeq *$. □

9.17. Proof of Theorem 9.5. It will suffice to show that a space F, satisfying the conditions of Lemma 9.11 (i)-(iv), is a 2-transitory polyGEM. In the homotopy fiber sequence $\widetilde{F} \to F \to JF$, the space $\widetilde{F}$ is Σf-acyclic by Lemma 9.16, and JF is a generalized polyGEM by Lemma 9.15. Thus, F is a generalized polyGEM by Lemma 9.12 since it is $\Sigma^2 f$-local. Moreover, since F is f-acyclic and $\Omega^2 F$ is f-local, $\mathrm{map}_*(F, \Omega^2 F) \simeq *$ and thus $\pi_* F$ is 2-transitory by Proposition 9.1. Hence F is a 2-transitory polyGEM. □

10. On E_*-equivalences and E_*-localizations of spaces. For a spectrum E, the E_*-equivalences and E_*-localizations of spaces may be viewed as f-local equivalences and f-localizations where f is a huge E_*-equivalence as explained in 2.5. In this section, we develop some general homological consequences of the preceding work, culminating in the result that the E_*-localization functor "almost" preserves fiber sequences of connected H-spaces (Theorem 10.10). We then briefly discuss the examples of Morava K-theories and stable cohomotopy theory. We begin with an elementary lemma which permits us to study E_*-equivalences and E_*-acyclicity at individual primes.

LEMMA 10.1. *For a spectrum E, a map of spaces is an E_*-equivalence if and only if it is an EQ_*-equivalence and an E/p_*-equivalence for each prime p.*

Let $\mathrm{tran}_p E$ denote the mod p transitional dimension of E (see 9.14). Thus $\mathrm{tran}_p E$ is the largest integer i such that $\widetilde{E}_* K(Z/p, i) \neq 0$, or is ∞ when $\widetilde{E}_* K(Z/p, i) \neq 0$ for all i. In [Bo 6] we proved

THEOREM 10.2. *For a spectrum E and prime p:*

(i) *if* $\mathrm{tran}_p E = 0$, *then* $E/p \simeq 0$;

(ii) *if* $\mathrm{tran}_p E = \infty$, *then the E/p_*-equivalences of spaces are the same as the $H_*(\Leftrightarrow; Z/p)$-equivalences.*

We now prove complementary results.

LEMMA 10.3. *Let E be a spectrum with* $\mathrm{tran}_p E = n$ *and* $\widetilde{E/p}_* K(Z, n+1) \neq 0$, *where p is a prime and $0 < n < \infty$. Then a nilpotent generalized polyGEM X is E/p_*-acyclic if and only if $\pi_{n+1} X/\mathrm{torsion}$ and $\pi_i X$ are uniquely p-divisible for $i \leq n$.*

Proof. By Corollary 8.11, X is E/p_*-acyclic if and only if $K(\Gamma_s \pi_1 X/\Gamma_{s+1}\pi_1 X, 1)$ and $K(\pi_i X, i)$ are E/p_*-acyclic for each $s \geq 1$ and $i \geq 2$. The lemma now follows by applying the E/p_*-acyclicity criteria of [Bo 6, 4.3] to these Eilenberg-MacLane spaces. □

Similarly,

LEMMA 10.4. *Let E be a spectrum with* $\mathrm{tran}_p E = n$ *and* $\widetilde{E/p}_* K(Z, n+1) = 0$, *where p is a prime and $0 < n < \infty$. Then a nilpotent generalized polyGEM X is E/p_*-acyclic if and only if $\pi_n X$ is p-divisible and $\pi_i X$ is uniquely p-divisible for $i \leq n \Leftrightarrow 1$.*

The preceding results combine to give necessary and sufficient conditions for the E_*-acyclicity of nilpotent generalized polyGEMs. We do not know of any spectrum E satisfying the hypotheses of Lemma 10.4, and our main examples will be covered by the following theorem. For a spectrum E, we let $\mathcal{P}E$ denote the set of primes p such that E/p is nontrivial. For $0 < n < \infty$, we say that E has *acyclicity level* n if $\mathrm{tran}_p E = n$ and $\widetilde{E/p}_* K(Z, n+1) \neq 0$ (or equivalently $\widetilde{E}_* K(Z_{p\infty}, n) \neq 0$) for each $p \in \mathcal{P}E$. For instance, the nth Morava K-theory spectrum $K(n)$ has acyclicity level n by [RW], and other examples are discussed in Sections 10.11 and 10.12.

THEOREM 10.5. *Let E be a spectrum of acyclicity level n where $0 < n < \infty$. Then a nilpotent generalized polyGEM X is E_*-acyclic if and only if $\pi_{n+1} X/\mathrm{torsion}$ and $\pi_i X$ are uniquely $\mathcal{P}E$-divisible for $i \leq n$, and $\pi_* X$ is torsion when $\pi_* E$ is not torsion.*

For a spectrum E, we next derive necessary conditions for a nilpotent generalized polyGEM to be E_*-local. Let $R^E = \bigoplus_{p \in \mathcal{P}E} Z/p$ when $\pi_* E$ is torsion, and let $R^E = Z_{(\mathcal{P}E)}$ otherwise. As in [Bo 6] and [Bo 2], we have

LEMMA 10.6. *Each HR^E_*-equivalence of spaces is an E_*-equivalence, and each E_*-local space is HR^E_*-local. A nilpotent space X is HR^E_*-local if and only if π_*X is* Ext-$\mathcal{P}E$*-complete when π_*E is torsion and is $\mathcal{P}E$-local otherwise.*

THEOREM 10.7. *If E is a spectrum of acyclicity level n where $0 < n < \infty$, and if X is an E_*-local nilpotent generalized polyGEM, then: π_*X is* Ext-$\mathcal{P}E$*-complete when π_*E is torsion and is $\mathcal{P}E$-local otherwise; $\pi_{n+1}X$ is torsion-free; and, for $i \geq n+2$, π_iX is zero when π_*E is torsion and is rational otherwise.*

Proof. Since X is E_*-local and $K(Z/p, j)$ is E_*-acyclic for $j \geq n+1$, $K(\pi_iX, i)$ is $K(Z/p, j)$-null for each i by Corollary 8.9. Hence, the theorem follows using Lemma 10.6. □

We now adapt our Main Theorem 9.7 to show that the E_*-localization functor preserves a homotopy fiber sequence of loop spaces up to an error term with at most three nontrivial homotopy groups. This extends a similar result of Dror Farjoun and Smith ([DFS]) for a fiber sequence of double loop spaces.

THEOREM 10.8. *If E is a spectrum of acyclicity level n where $0 < n < \infty$, and if $F \to X \to Y$ is a homotopy fiber sequence of simply connected spaces, then the E_*-localization functor preserves $\Omega F \to \Omega X \to \Omega Y$ up to an error term $\Omega^2 D$ with $\pi_iD = 0$ unless $n+1 \leq i \leq n+3$. This conclusion holds more generally when $F \to X \to Y$ is a fiber sequence of connected spaces such that the kernel of $\pi_1F \to \pi_1X$ is* Ext-$\mathcal{P}E$*-complete when π_*E is torsion and is $\mathcal{P}E$-local otherwise.*

Proof. Let f be an E_*-equivalence such that L_f is the E_*-localization functor, and let $L_{\Sigma f}F \to L_{\Sigma f}X \to Y'$ be the fiber sequence of Theorem 6.1. By Theorems 5.2 and 6.4, the L_f-error term of $\Omega F \to \Omega X \to \Omega Y$ is $\Omega^2 D$ where D is the homotopy fiber of $Y' \simeq L_{\Sigma^2 f}Y' \to L_{\Sigma f}Y'$. Since the kernel of $\pi_1L_{\Sigma f}F \to \pi_1L_{\Sigma f}X$ equals the kernel of $\pi_1F \to \pi_1X$, we find that π_2Y' is Ext-$\mathcal{P}E$-complete when π_*E is torsion and is $\mathcal{P}E$-local otherwise. Hence, D is an HR^E_*-local space. Moreover, by Theorem 9.5 and Lemma 9.11, D is an E_*-acyclic 2-transitory polyGEM and $\Omega^2 D$ is E_*-local. The desired results now follow from Theorems 10.5 and 10.7. □

Note. We may easily obtain more detailed information on D. If π_*E is torsion, then π_*D is Ext-$\mathcal{P}E$-complete with $\pi_{n+1}D$ adjusted (i.e. torsion-by-rational) and with $\pi_{n+3}D$ torsion-free. If π_*E is not torsion, then π_*D is $\mathcal{P}E$-torsion with $\pi_{n+3}D = 0$. Moreover, by 9.10, if $\mathcal{P}E$ consists of odd primes, then ΩD is a GEM, and if $F \to X \to Y$ is a fiber sequence of loop spaces and loop maps, then $\Omega^2 D$ has GEM components. For simplicity, we shall henceforth omit detailed descriptions of error terms.

COROLLARY 10.9. *If E is a spectrum of acyclicity level n where $0 < n < \infty$, and if X is a simply connected H-space, then the homotopy fiber of the natural*

map $B(\Omega X)_E \to X_E$ is an H-space Δ with $\pi_i\Delta = 0$ unless $n \le i \le n+2$. This conclusion holds more generally when X is a connected H-space such that $\pi_1 X$ is Ext-$\mathcal{P}E$*-complete when $\pi_* E$ is torsion and is $\mathcal{P}E$-local otherwise.*

Proof. Since X is a retract of a loop space $\Omega\Sigma X$, it suffices to assume that $X = \Omega Y$ for a simply connected space Y. The map $B(\Omega X)_E \to X_E$ is now $L_{\Sigma f}\Omega Y \to L_f \Omega Y$ for f as above, and its homotopy fiber is now $\Delta = \Omega D$ where D is the homotopy fiber of $L_{\Sigma^2 f} Y \to L_{\Sigma f} Y$. As in Theorem 10.8, $\pi_i D = 0$ unless $n+1 \le i \le n+3$, and the corollary follows. □

Using the above results, we finally show that the E_*-localization functor "almost" preserves fiber sequences of H-spaces.

THEOREM 10.10. *If E is a spectrum of acyclicity level n where $0 < n < \infty$, then the E_*-localization functor preserves a homotopy fiber sequence of connected H-spaces $F \to X \to Y$ up to an error term Δ with $\pi_i\Delta = 0$ unless $n \Leftrightarrow 1 \le i \le n+1$.*

Proof. Since the HR^E_*-localization functor preserves the fiber sequence and does not affect Δ, we may assume that F, X, and Y are HR^E_*-local. By Corollary 10.9, the homotopy fibers of the maps $B(\Omega F)_E \to F_E$, $B(\Omega X)_E \to X_E$, and $B(\Omega Y)_E \to Y_E$ have trivial π_i-groups unless $n \le i \le n+2$. By Theorem 10.8, the functor $B(\Omega \Leftrightarrow)_E$ preserves the fiber sequence up to an error term with trivial π_i groups unless $n \le i \le n+2$. Thus, the original error term Δ has $\pi_i\Delta = 0$ unless $n \Leftrightarrow 1 \le i \le n+3$, and has $\pi_*\Delta$ torsion when $\pi_* E$ is not torsion. Hence, the Postnikov map $\Delta \to P^{n+1}\Delta$ is an E_*-equivalence, since its fiber is E_*-acyclic by Theorem 10.5. This implies $\Delta \simeq P^{n+1}\Delta$ since Δ is E_*-local. □

The preceding results apply to many important (co)homology theories, and we conclude with some examples.

10.11. The Morava K-theories. For the nth Morava K-theory $K(n)_*$ at a prime p with $n \ge 1$, Ravenel and Wilson ([RW]) have shown that $K(Z/p, i)$ is $K(n)_*$-acyclic if and only if $i > n$, while $K(Z, i)$ is $K(n)_*$-acyclic if and only if $i > n+1$. (See [JW, Appendix] or [HRW, 4.4] for an explanation of the case $p = 2$.) Hence, $K(n)$ has acyclicity level n, where $\pi_* K(n)$ is torsion and $\mathcal{P}K(n) = \{p\}$.

For the BP-related spectrum $E(n)$ with $\pi_* E(n) = Z_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]$, a map is an $E(n)_*$-equivalence if and only if it is a $K(i)_*$-equivalence for $0 \le i \le n$ where $K(0) = HQ$. Hence, $E(n)$ has acyclicity level n, where $\pi_* E(n)$ is not torsion and $\mathcal{P}E(n) = \{p\}$. Likewise, the spectrum K of periodic complex K-theory has acyclicity level 1, where $\pi_* K$ is not torsion and $\mathcal{P}K = \{\text{all primes}\}$.

10.12. Stable cohomotopy theory. Since stable cohomotopy theory $\pi^* = S^*$ is the cohomology theory represented by the sphere spectrum S, it has the same equivalences as some homology theory ∇S_* by 2.6, and hence π^*-localizations of spaces and spectra always exist. Chun-Nip Lee ([Le]) has shown that the space

$K(Z/p,2)$ is π^*-acyclic for each prime p, and he has implicitly shown that each π^*-equivalence of spaces is a K_*-equivalence (see Lemma 10.13 below). This implies that $K(Z_{p^\infty},1)$ is not π^*-acyclic. Thus ∇S has acyclicily level 1, where $\pi_*\nabla S$ is not torsion and $\mathcal{P}\nabla S = \{\text{all primes}\}$. From this standpoint, π^* resembles K_*, but there are many examples of K_*-equivalences, such as the Adams maps of mod p Moore spaces, which are not π^*-equivalences.

We have used the following result shown implicitly by Chun-Nip Lee ([Le, 3.4]).

LEMMA 10.13. *Each π^*-equivalence of spaces is a K_*-equivalence.*

Proof. This depends on Miller's ([Mi]) stable splitting

$$\Sigma^\infty U \simeq \bigvee_{k\geq 1} BU_k^{AdU_k} \simeq \prod_{k\geq 1} BU_k^{AdU_k}.$$

By the Segal conjecture, $(BU_k^{AdU_k})^\wedge_p$ is a stable summand of the π^*-local spectrum $(DBU_k)^\wedge_p$, and thus $BU_k^{AdU_k}$ is π^*-local by an arithmetic square argument. Hence, the spectrum $\Sigma^\infty U$ and the spaces $\Omega^\infty\Sigma^\infty U$ and U are all π^*-local. Consequently, each π^*-equivalence of spaces is a K^*-equivalence, and hence a K_*-equivalence. □

11. Virtual E_*-equivalences of spaces. We say that a spectrum E has *n-elevated acyclicity* for an integer $n \geq 0$ if $\text{tran}_pE \leq n$ for each prime p, or equivalently if $K(Z/p, n+1)$ is E_*-acyclic for each prime p. This holds, for instance, when E has acyclicity level i for some $i \leq n$, and thus holds for the Morava K-theory spectrum $K(i)$ when $i \leq n$. More generally, we say that a spectrum E has *elevated acyclicity* if it has n-elevated acyclicity for some integer $n \geq 0$. In view of Lemma 10.1 and Theorem 10.2, we have

LEMMA 11.1. *A p-local spectrum E must have elevated acyclicity if it does not have the same homology equivalences of spaces as $H_*(\Leftrightarrows; Z/p)$ or $H_*(\Leftrightarrows; Z_{(p)})$.*

For a spectrum E of elevated acyclicity, a map $\phi\colon X \to Y$ is Ho_* is called a *virtual E_*-equivalence* if $\phi_*\colon \pi_i(\Omega X)_E \to \pi_i(\Omega Y)_E$ is an isomorphism for all sufficiently large i, and a space $X \in Ho_*$ is called *virtually E_*-acyclic* if $\pi_i(\Omega X)_E$ is zero for all sufficiently large i. We shall see that the virtual E_*-equivalences are closely related to ordinary E_*-equivalences, but have some better homotopy theoretic properties, and we shall give various applications. For instance, generalizing work of Thompson and the author ([Bo 7]), we show that each K/p_*-equivalence of H-spaces is a v_1-periodic homotopy equivalence (Corollary 11.12). We shall need the main results of Section 10 in the following partially generalized, but similarly proved, form.

THEOREM 11.2. *For a spectrum E of n-elevated acyclicity, we have:*

(i) *If X is an $(n+1)$-connected generalized polyGEM such that π_*X or π_*E is torsion, then X is E_*-acyclic.*

(ii) *If X is an E_*-local nilpotent generalized polyGEM, then: $\pi_{n+1}X$ is torsion-free; π_iX is rational for $i > n+1$; and $\pi_iX = 0$ for $i > n+1$ when π_*E is torsion.*

(iii) *For a fiber sequence of connected spaces $F \to X \to Y$, the E_*-localization functor preserves the homotopy fiber sequence $\Omega F \to \Omega X \to \Omega Y$ up to an error term $\Omega^2 D$ with $\pi_i\Omega^2 D = 0$ for $i > n+1$.*

(iv) *For a connected H-space X, the homotopy fiber of the natural map $B(\Omega X)_E \to X_E$ is an H-space Δ with $\pi_i\Delta = 0$ for $i > n+2$.*

(v) *The E_*-localization functor preserves a fiber sequence of connected H-spaces $F \to X \to Y$ up to an error term Δ with $\pi_i\Delta = 0$ for $i > n+1$.*

Theorem 11.2 (iv) implies

THEOREM 11.3. *If E is a spectrum of elevated acyclicity, and if $\phi\colon X \to Y$ is an E_*-equivalence of connected H-spaces, then ϕ is a virtual E_*-equivalence.*

Many other examples of virtual E_*-equivalences follow from

THEOREM 11.4. *For a spectrum E of elevated acyclicity, and for a map of fiber sequences*

$$\begin{array}{ccccc} F & \twoheadrightarrow & X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ F' & \twoheadrightarrow & X' & \twoheadrightarrow & Y', \end{array}$$

if any two of $F \to F'$, $X \to X'$, and $Y \to Y'$ are virtual E_*-equivalences, then so is the third.

Proof. Note that a map of pointed spaces is a virtual E_*-equivalence if and only if the induced map of universal covers is such. Thus we may assume that the given spaces are connected and may apply Theorem 11.2 (iii). □

The next two corollaries show the "durability" of virtual E_*-equivalences.

COROLLARY 11.5. *For a spectrum E of elevated acyclicity and for $i \geq 0$, a map $\phi\colon Y \to Y'$ in Ho_* is a virtual E_*-equivalence if and only if $\Omega^i\phi\colon \Omega^iY \to \Omega^iY'$ is such.*

Proof. The case $i = 1$ follows by Theorem 11.4, and the other cases follows inductively. □

COROLLARY 11.6. *For a spectrum E of elevated acyclicity and for $i \geq 0$, a map $\phi\colon Y \to Y'$ in Ho_* is a virtual E_*-equivalence if and only if $\phi\langle i\rangle\colon Y\langle i\rangle \to Y'\langle i\rangle$ is such.*

Proof. In the fiber sequence $Y\langle i\rangle \to Y \to P^iY$, the ith Postnikov section P^iY is virtually E_*-acyclic by Corollary 11.5 since $\Omega^{i+1}P^iY \simeq *$, and thus $Y\langle i\rangle \to Y$ is a virtual E_*-equivalence by Theorem 11.4. □

To compare virtual E_*-equivalences with ordinary E_*-equivalences, we use

THEOREM 11.7. *If E is a spectrum of n-elevated acyclicity with π_*E torsion, and if $\phi\colon X \to Y$ is a virtual E_*-equivalence in Ho_*, then $\phi\langle i\rangle\colon X\langle i\rangle \to Y\langle i\rangle$ is an E_*-equivalence for $i \geq n+2$.*

Proof. Since the maps $X\langle n+3\rangle \to X\langle n+2\rangle$ and $Y\langle n+3\rangle \to Y\langle n+2\rangle$ are E_*-equivalences, we may assume $i \geq n+3$. The homotopy fiber F of $\phi\langle i\rangle\colon X\langle i\rangle \to Y\langle i\rangle$ is virtually E_*-acyclic by Theorem 11.4, and thus $(\Omega F)_E$ is Postnikov. Hence, the map $\Omega F \to (\Omega F)_E$ is nullhomotopic by Theorem 11.2 (ii), and ΩF is E_*-acyclic. Therefore, F is E_*-acyclic and $\phi\langle i\rangle$ is an E_*-equivalence. □

When we wish to compute the E_*-homology of a virtually E_*-acyclic space, we may replace the space by a suitable Postnikov section.

COROLLARY 11.8. *If E is a spectrum of n-elevated acyclicity with π_*E torsion, and if $X \in Ho_*$ is a virtually E_*-acyclic space, then $X\langle n+1\rangle$ is E_*-acyclic and the Postnikov map $X \to P^{n+1}X$ is an E_*-equivalance.*

Proof. Since $X\langle n+2\rangle$ is virtually E_*-acyclic by Corollary 11.6, it is E_*-acyclic by Theorem 11.7. The result now follows since $K(\pi_{n+2}X, n+2)$ is E_*-acyclic by Theorem 11.2 (i). □

Using this corollary, we deduce

THEOREM 11.9. *For an arbitrary spectrum E, if $X \in Ho_*$ is an E_*-acyclic H-space, then $X\langle i\rangle$ and P^iX are E_*-acyclic for all i.*

Proof. It suffices by Lemma 10.1 to prove the corresponding result for each E/p_*, since it is clear for EQ_*. Let X be E/p_*-acyclic for a prime p. If $\mathrm{tran}_pE = \infty$, then the E/p_*-acyclic spaces are the same as the $H_*(\Leftrightarrow; Z/p)$-acyclic spaces by Theorem 10.2, and the E/p_*-acyclic H-spaces are those with uniquely p-divisible homotopy groups. Hence, each $X\langle i\rangle$ and P^iX is E/p_*-acyclic. If $\mathrm{tran}_pE < \infty$, we may assume that E/p has n-elevated acyclicity. Since X is virtually E/p_*-acyclic by Theorem 11.3, $X\langle i\rangle$ and P^iX are E/p_*-acyclic for $i \geq n+1$ by Corollary 11.8. Since the H-space $P^{n+1}X$ is E/p_*-acyclic, so is $K(\pi_iX, i)$ for $i \leq n+1$ by Corollary 8.11, and hence each $X\langle i\rangle$ and P^iX is E/p_*-acyclic. □

Our next result characterizes the virtual E_*-equivalences in terms of ordinary E_*-equivalences, strengthening Theorem 11.7.

THEOREM 11.10. *For a spectrum E of n-elevated acyclicity with π_*E torsion, and for integers $k \geq 1$ and $i \geq n+2$, a map $\phi\colon X \to Y$ in Ho_* is a virtual E_*-*

equivalence if and only if $(\Omega^k\phi)\langle i\rangle\colon (\Omega^k X)\langle i\rangle \to (\Omega^k Y)\langle i\rangle$ *is an* E_**-equivalence. The "only if" statement also holds for* $k = 0$ *and* $i \geq n+2$.

Proof. This follows easily by 11.3–11.7. □

We conclude with some results relating the virtual $K(n)_*$-equivalences of spaces to the v_j-periodic homotopy equivalences for $j \leq n$ at a prime p, where the v_j-periodic homotopy groups $v_j^{-1}\pi_*(X; V_{j-1})$ of a space X are defined as in [Bo 7, 11.9] and [Bo 8, 5.2]. Note that the virtual $K(1)_*$-equivalences are the same as the virtual K/p_*-equivalences.

THEOREM 11.11. *A map* $\phi\colon X \to Y$ *in* Ho_* *is a virtual* K/p_**-equivalence if and only if* $\phi_*\colon v_1^{-1}\pi_*(X; Z/p) \cong v_1^{-1}\pi_*(Y; Z/p)$.

Proof. By work of Thompson ([Th]) and the author ([Bo 7, 14.7]), ϕ induces a K/p_*-equivalence $(\Omega^2 X)\langle 3\rangle \to (\Omega^2 Y)\langle 3\rangle$ if and only if $\phi_*\colon v_1^{-1}\pi_*(X; Z/p) \cong v_1^{-1}\pi_*(Y; Z/p)$. Our result now follows from Theorem 11.10. □

Generalizing a result of Thompson for iterated loop spaces ([Th], [Bo 7, 14.4]), we now obtain

COROLLARY 11.12. *If* $\phi\colon X \to Y$ *is a* K/p_**-equivalence of connected H-spaces, then* $\phi_*\colon v_1^{-1}\pi_*(X; Z/p) \cong v_1^{-1}\pi_*(Y; Z/p)$.

Proof. This follows by combining Theorems 11.3 and 11.9. □

The "if" part of Theorem 11.11 can be generalized to

THEOREM 11.13. *If* $\phi\colon X \to Y$ *is a map in* Ho_* *with* $\phi_*\colon v_j^{-1}\pi_*(X; V_{j-1}) \cong v_j^{-1}\pi_*(Y; V_{j-1})$ *for* $j = 1, 2, \ldots, n$ *where* $n \geq 1$, *then* ϕ *is a virtual* $K(n)_*$*-equivalence and hence* $(\Omega^k\phi)\langle i\rangle$ *is a* $K(n)_*$*-equivalence for each* $k \geq 0$ *and* $i \geq n+2$.

Proof. By [Bo 7, 13.3], the map of p-torsion components $\tau_p\Omega X\langle m\rangle \to \tau_p\Omega Y\langle m\rangle$ is a P_{v_n}-equivalence for sufficiently large m. Thus, by [Bo 7, 12.1], the map $\Omega X\langle m\rangle \to \Omega Y\langle m\rangle$ is a $K(n)_*$-equivalence for sufficiently large m, and hence ϕ is a virtual $K(n)_*$-equivalence by Theorem 11.3, Corollary 11.5, and Corollary 11.6. The last statement follows by Theorem 11.10. □

The condition "$j = 1, 2, \ldots, n$" in this theorem cannot be reduced to "$j = n$." For instance, as noted at the end of [Bo 8], $v_2^{-1}\pi_*(U\langle 4\rangle; V_1) = 0$ while $\widetilde{K(2)}_*U\langle 4\rangle \neq 0$. We do not know whether the converse to this theorem holds. However, as a consequence of this theorem and Corollary 11.8, we have

COROLLARY 11.14. *If* $X \in Ho_*$ *is a space with* $v_j^{-1}\pi_*(X; V_{j-1}) = 0$ *for* $1 \leq j \leq n$ *where* $n \geq 1$, *then* X *is virtually* $K(n)_*$*-acyclic. Hence,* $X\langle n+1\rangle$ *is* $K(n)_*$*-acyclic and the Postnikov map* $X \to P^{n+1}X$ *is a* $K(n)_*$*-equivalence.*

Hopkins, Ravenel, and Wilson ([HRW]) have proved a similar result when X is an infinite loop space and have applied it to calculate $K(n)_*X$ in some interesting cases. Corollary 11.14 should help to extend that method.

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