

Morse-Bott theory and equivariant cohomology

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1 Introduction

Critical points of functions and gradient lines between them form a cornerstone of physical thinking. In Morse theory the topology of a manifold is investigated in terms of these notions with equally profound success: Smale proved the h-cobordism and generalized Poincaré conjectures using surgery cobordisms, see [Mi2].

Morse's original application (see [Mo] and [Mi1]) illustrates an important theme. He chose to investigate the energy functional on the infinite dimensional space of loops in a manifold (see [Mo] and [Mi1]). Not only is this physically relevant, for it describes geodesic motion, but it also led to the Bott periodicity theorem, see Bott [B4] and Bott and Samuelson [BS]. The condition of non-degenerate critical points was relaxed by Bott to allow for non-degenerate manifolds of critical points. This was more or less necessary to deal with the natural group actions on the loop space of a symmetric space.

This theme is borne out in a later application of Morse theory by Atiyah and Bott [AB2]. They consider the Yang-Mills functional as a function on the space of connections over a Riemann surface which is invariant under the group of gauge transformations. Morse-theoretic ideas applied in the setting of equivariant cohomology enable the computation of cohomological information about the space of holomorphic bundles on the surface, which coincide with the minima of the Yang-Mills functional. The same ideas were subsequently used by Frances Kirwan [K] to enable the computation of the cohomology of geometric quotients in a very general setting.

Acting from the perspective of quantum field theory, Witten [Wi] renewed interest in Morse's complex. Morse's complex is the free module generated by the critical points and graded by their index. A differential on this complex is given by counting the number of gradient lines between critical points differing in index by 1. Inspired by this, Floer made his dramatic contributions by building a Morse complex both for the symplectic action functional on the loop space of a symplectic manifold (Floer [F2]) and for the Chern-Simons functional on the space of connections over a homology 3-sphere, [F1]. Not only is the configuration

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space infinite dimensional, so is the index of each critical point. These theories had important applications: they solved (part of) the Arnold conjecture and established an important gluing relation for Donaldson polynomial invariants.

Floer's homology groups for homology 3-spheres asked for many generalizations to describe more general gluing laws governing Donaldson's polynomial invariants, both through incorporating other auxiliary information (see Fukaya [Fu] and Braam and Donaldson [BD]) as well as by considering general 3-manifolds [AuB]. In the course of this work, new techniques were developed in these infinite dimensional cases relating to equivariant cohomology, cup products and various alternative approaches to problems of classical Morse theory. The purpose of this paper is to give a self-contained finite dimensional description of these new aspects.

When a function has nondegenerate critical manifolds, we say it is Morse-Bott. A complex which computes the homology of the manifold in terms of the critical submanifolds and gradient flows was missing. In this paper, we build such a Morse-like complex which computes the deRham cohomology of the manifold. The chains of the complex are differential forms on the critical submanifolds and the boundary map is defined in terms of integration over the flow lines of the gradient vector field. In the case when the function is truly Morse, our complex reduces to the standard Morse complex. The existence of our complex easily implies the Morse inequalities of Bott in the same way that the standard Morse inequalities follow from the existence of a complex computing homology. Furthermore, in the spirit of Atiyah and Bott, we show how to compute the equivariant cohomology of a manifold with group action in terms of an invariant function on the manifold. We do so in various ways, using a Morse-Bott function on approximations of the homotopy quotient as well as through equivariant differential forms.

The isomorphism between the homologies of the deRham complex and ours is obtained at the chain level by integrating differential forms over the unstable manifolds of critical submanifolds. This clearly illuminates the fundamental role played by the Thom isomorphism in Morse theory. Furthermore, it leads to a detailed investigation of the structure of the unstable manifolds near their frontiers. With this in hand, additional algebraic topological invariants, such as the cup product structure and Poincaré duality pairing, can be easily obtained from the Morse complex. When studying the cup product, we come particularly close to Witten's idea of deforming the exterior derivative using the Morse function; instead we use a classical analogue and pull back differential forms under the gradient flow for large time. Using these tools we also find isomorphisms at the chain level between the Morse and Morse-Bott complexes. These considerations should be intimately connected with the results of Cohen, Jones and Segal, [CJS].

This paper is organized as follows: §2 contains a review of finite dimensional Morse theory. Since so many of these ideas are applicable to the later Morse-Bott case, we have attempted to be as thorough as possible. A comprehensive discussion of the asymptotics of the gradient line spaces is found in an appendix. A

new feature included within this section is an explanation of the cup product in terms of the Morse complex. Next we revise standard Morse theory for Morse-Bott functions in §3 by developing a complex to compute deRham cohomology in the case when the function has nondegenerate critical submanifolds. Cup products are again explained. §4 is a brief introduction to some tools from equivariant cohomology which we apply in §5 where it is shown how to compute the equivariant cohomology of a manifold with group action in terms of an invariant function. This is an extension of the Morse-Bott ideas in §3. Finally, §6 contains a more detailed look at the module structure for equivariant cohomology. This will be useful in [AuB] to explain the «u»-map on Floer homology.

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2 Review of classical Morse theory

This section presents a review of classical finite dimensional Morse theory. The fundamental observation is that there is an intimate relationship between the critical points of a function on a manifold and the topology of the manifold. Succinctly stated, this says that critical points correspond to handles added to the manifold. As such, one can build a complex (C_*, ∂) whose homology equals the singular homology of the manifold — a purely topological invariant. We will make this relationship clear in this review. For additional background, we refer the reader to the following: Smale [Sm1], Milnor [Mi1], Milnor [Mi2], Witten [Wi], and Floer [F3]. Bott [B1] gives an interesting historical perspective.

While this section is primarily a review, it is also designed to motivate our techniques in the more general Morse-Bott case. For this reason, the discussion is self-contained and many details are given. Several of the more technical results are proven in Appendix A. We also give a proof here that the homology of this complex is independent of the Morse function and metric on the manifold. This is useful for infinite dimensional applications, such as Floer homology, where the complex is not computing any known homology of the space. Finally in §2.4, a new feature of Morse theory is described: the cup product structure on the cohomology ring is accessible from a Morse function in terms of critical points and gradient flow information.

2.1 The complex (C_*, ∂)

Suppose that B is a smooth closed, oriented Riemannian manifold of dimension n and $f : B \rightarrow \mathbb{R}$ is a smooth function. A point $\alpha \in B$ is called a critical point of f if $df_\alpha = 0$. At a critical point, the Hessian d^2f_α is well defined as a quadratic form on $T_\alpha B$, and we call a critical point non-degenerate if its Hessian is a non-degenerate symmetric bilinear form. We will henceforth assume that f is a Morse function meaning that all the critical points of f are non-degenerate.

For a critical point α , let H_α be a maximal negative definite subspace of $T_\alpha B$ with respect to $d^2 f_\alpha$. The index is defined by

$$i(\alpha) = \dim H_\alpha.$$

Non-degenerate critical points are necessarily isolated, and in suitable local coordinates the function is a quadratic form. To such a critical point α , we can assign an unstable manifold \mathcal{U}_α defined by

$$\mathcal{U}_\alpha = \{b \in B; \phi_t(b) \rightarrow \alpha \text{ as } t \rightarrow -\infty\}$$

where ϕ_t is the gradient flow for the vector field $-\text{grad } f$ (see Smale [Sm2]). It is well known that \mathcal{U}_α is an embedded $\mathbb{R}^{i(\alpha)}$. Similarly, one defines a stable manifold \mathcal{S}_α for α using the flow as $t \rightarrow \infty$. Again, \mathcal{S}_α is an embedded $\mathbb{R}^{n-i(\alpha)}$. The *moduli spaces of gradient lines* between critical points α and β are of central interest in our discussion. These spaces have an \mathbb{R} -action so we define:

$$\begin{aligned}\mathcal{M}(\alpha, \beta) &= \mathcal{S}_\beta \cap \mathcal{U}_\alpha \\ \tilde{\mathcal{M}}(\alpha, \beta) &= (\mathcal{S}_\beta \cap \mathcal{U}_\alpha) / \mathbb{R}.\end{aligned}$$

We begin by stating that «most» functions are Morse:

Proposition 2.1 *For a Baire set of functions $f \in C^\infty(B)$, all critical points are nondegenerate and \mathcal{U}_α and \mathcal{S}_β intersect transversely for all α and β . Such an f is called a generic Morse function.*

Proof. See Appendix B.

To begin to build a complex, we arbitrarily choose an orientation of all unstable manifolds. At a critical point α , the tangent space splits into the stable and unstable parts so that $T_\alpha B = T_\alpha \mathcal{U}_\alpha \oplus T_\alpha \mathcal{S}_\alpha$. Since B is oriented, the orientations of the \mathcal{U}_α orient all stable manifolds. Because the intersections of stable and unstable manifolds are assumed to be transverse. All moduli spaces of gradient lines are now oriented and in particular if $i(\alpha) = i$ and $i(\beta) = i - 1$, then $\mathcal{U}_\alpha \cap \mathcal{S}_\beta$ is smooth, oriented and of dimension 1. Moreover it is invariant under the flow ϕ_t so that

$$(\mathcal{U}_\alpha \cap \mathcal{S}_\beta) / \mathbb{R}$$

is simply a collection of oriented points.

Now form a complex with chain groups

$$C_i = \text{free } \mathbb{Z}\text{-module generated by critical points of index } i.$$

The boundary operator $\partial : C_i \rightarrow C_{i-1}$ is described by

$$\partial(\alpha) = \sum m_k \beta_k,$$

where α is a critical point of index i , the β_k are the critical points of index $i - 1$ and

$$m_k = \#(\mathcal{G}_{\beta_k} \cap \mathcal{U}_\alpha) / \mathbb{R} \in \mathbb{Z},$$

where $\#$ denotes the algebraic count of the oriented points. The finiteness of the intersection is guaranteed by the compactness of B .

It is also useful to consider the dual complex (C^*, ∂^*) . Here

C^i = free \mathbb{Z} -module generated by critical points of index i

and the coboundary operator $\partial^* : C^i \rightarrow C^{i+1}$ is given by

$$\partial^*(\alpha) = \sum \# \tilde{\mathcal{M}}(\gamma_j, \alpha) \gamma_j$$

where the sum is taken over the critical points γ_j of index $i + 1$.

The fact that (C_*, ∂) and (C^*, ∂^*) form complexes will be discussed in some detail and proven as Proposition 2.8 since the ideas are relevant for our discussion in later sections. Assuming this fact for now, we state the fundamental theorem concerning this complex:

Theorem 2.2 *The homology of the complex (C_*, ∂) ((C^*, ∂^*)) is naturally isomorphic to the singular (co)homology of B ; that is,*

$$H_j(B, \mathbb{Z}) \cong H_j(C_*, \partial) \quad H^j(B, \mathbb{Z}) \cong H^j(C^*, \partial^*).$$

Of course, this theorem holds for general coefficient groups. It certainly was known to Milnor [Mi2], although he did not define the boundary operator equally explicitly. For a self contained proof see, for example, Floer [F3]; in Section 3, a new proof will be given over the real numbers.

There are two interesting corollaries of this theorem. That the critical points form a basis of a complex which computes the cohomology of B immediately implies the well-known Morse inequalities. Let $c_i = \text{rank } C_i$, $h_i = \text{rank } H_i(B)$ and $z_i = \text{rank ker } (\partial : C_i \rightarrow C_{i-1}) \leq c_i$. Defining $k_i = c_i - z_i \geq 0$ gives

Corollary 2.3 [Morse inequalities] *There are $k_i \geq 0$ such that*

$$\sum_i (c_i - h_i) t^i = (1 + t) \sum_i k_i t^i.$$

Secondly, we may consider the Morse function $-f$ which gives a new complex $(\bar{C}_*, \bar{\partial})$; by Theorem 2.2, the homology of $(\bar{C}_*, \bar{\partial})$ also computes the homology of B . The critical points of f and $-f$ agree; however, a critical point of f of index i is a critical point of $-f$ of index $n - i$. Hence

\bar{C}_i = free \mathbb{Z} -module generated by critical points of f of index $n - i$.

Moreover, $\bar{\partial}(\alpha) = \sum \# \tilde{\mathcal{M}}(\gamma_j, \alpha) \gamma_j$. This shows that the two complexes, (C^*, ∂^*) and $(\bar{C}_*, \bar{\partial})$, are isomorphic by the natural map taking

$$C^i \rightarrow \bar{C}_{n-i}.$$

This gives the well known isomorphism

Corollary 2.4 [Poincaré Duality] $H^*(B; \mathbb{Z}) \cong H_{n-*}(B; \mathbb{Z})$.

Next we will discuss why C_* is a complex and why its cohomology is independent of the generic Morse function f . These are the key ideas in our paper.

2.2 (C_*, ∂) is a complex

The basic idea is quite simple. If $\partial \circ \partial(\alpha) \neq 0$ then there must be gradient lines from α to some β and from β to some γ with $i(\alpha) = i(\beta) + 1 = i(\gamma) + 2$. This means that the space parametrizing gradient lines from α to γ (which is of dimension 1) has a boundary component equal to the factorization through β . But boundary points of 1-dimensional manifolds come in pairs, so there must be another factorization $\alpha \rightarrow \beta' \rightarrow \gamma$ (as in figure 1).

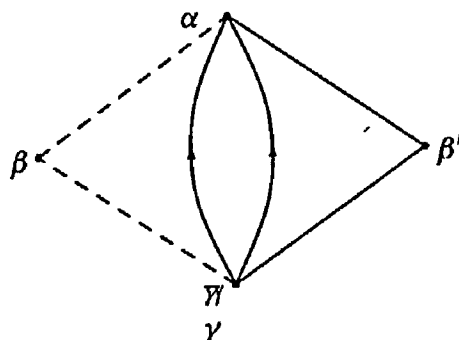


Fig. 1

The proof consists of showing that this is indeed so and that the orientations cause these two factorizations to cancel out. For this reason, we begin a careful analysis of the asymptotic structure of the space of gradient lines referring to Appendix A for the more technical proofs. Our attention to detail here will ease the more general Morse-Bott case in later sections. The discussion here owes much to Floer [F1].

The assumption that f is a generic Morse function — that is, a Morse function whose stable and unstable manifolds intersect transversely — implies that for $\alpha \neq \beta$, $\tilde{\mathcal{M}}(\alpha, \beta)$ is a smooth manifold of dimension $i(\alpha) - i(\beta) - 1$. In particular, this says that if $i(\alpha) \leq i(\beta)$, then $\tilde{\mathcal{M}}(\alpha, \beta) = \emptyset$. Notice that, as a set, $\tilde{\mathcal{M}}(\alpha, \beta)$ can also be regarded as a submanifold of $f^{-1}(c)$ for any $c \in (f(\beta), f(\alpha))$. However, the topology of pointwise convergence on the space of gradient trajectories can be much stronger than the induced topology from B . This will be important for our study of the asymptotic structure of the $\tilde{\mathcal{M}}(\alpha, \beta)$.

First of all, we have a weak compactness property which states that limits of a sequence of gradient lines are broken gradient lines; the breaking points are intermediate critical points.

Lemma 2.5 *If α, β are two critical points and $\{\gamma_i\}$ is a sequence of gradient lines in $\tilde{\mathcal{M}}(\alpha, \beta)$, then there are*

- 1) *a subsequence $\{\gamma_j\}$*
- 2) *a finite set of critical points $\alpha = \alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1} = \beta$*
- 3) *a finite set of strictly decreasing real numbers r_1, \dots, r_k with the property that $r_i \in (f(\alpha_{i+1}), f(\alpha_i))$*

such that

- 1) *the points $x_{i,j} = \gamma_j(s)$ such that $f(x_{i,j}) = r_i$ converge to a regular point on B which lies on a gradient line in $\tilde{\mathcal{M}}(\alpha_i, \alpha_{i+1})$*
- 2) *the indices of the α_j are strictly decreasing with j ; that is,*

$$i(\alpha_1) > i(\alpha_2) > \dots > i(\alpha_{k+1}).$$

Pictorially, this may be seen as in figure 2.

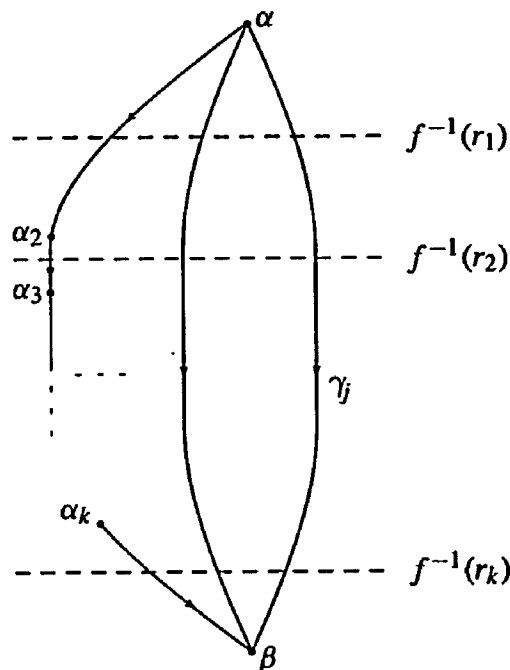


Fig. 2

Proof. The proof is in Appendix A (A.1.5).

Notice that for $i(\alpha) - i(\beta) = 1$ the lemma states that $\tilde{\mathcal{M}}(\alpha, \beta)$ is compact. We can now describe a local model of the boundary of the moduli of gradients.

Lemma 2.6 Let $\alpha = \alpha_0, \alpha_1, \dots, \alpha_{k-1}, \alpha_k = \beta$ be a finite set of critical points, the indices of which are strictly decreasing with j . For sufficiently small ϵ , there is a diffeomorphism, called the gluing map,

$$G : \prod_{j=0}^k \tilde{\mathcal{M}}(\alpha_j, \alpha_{j+1}) \times \prod_{j=1}^k (0, \epsilon) \rightarrow \tilde{\mathcal{M}}(\alpha, \beta)$$

mapping onto an open set in $\tilde{\mathcal{M}}(\alpha, \beta)$. Moreover, if γ_i is a sequence of gradient lines as in Lemma 2.5, then $\gamma_i \in \text{image } G$ for all large i . The compactification in the sense of Lemma 2.5 proceeds by letting the real parameters in $(0, \epsilon)$ approach zero.

Proof. See A.2.2. Figure 3 indicates geometrically what the parameters

$$(\{\gamma_i\}_{j=0}^k, \{t_i\}_{j=1}^k) \in \prod \tilde{\mathcal{M}}(\alpha_i, \alpha_{i+1}) \times \prod [0, \epsilon)$$

signify. □

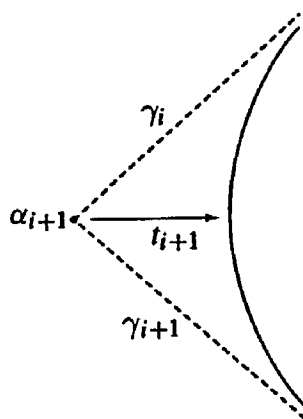


Fig. 3

Together the two lemmas prove that $\tilde{\mathcal{M}}(\alpha, \beta)$ has a compactification

$$\bar{\tilde{\mathcal{M}}}(\alpha, \beta) = \tilde{\mathcal{M}}(\alpha, \beta) \cup \bigcup_{k \in \mathbb{Z}} \bigcup_{\alpha = \alpha_0, \dots, \alpha_k = \beta} \left(\prod_{j=0}^k \tilde{\mathcal{M}}(\alpha_j, \alpha_{j+1}) \times \prod_{j=1}^k [0, \epsilon) \right)$$

which is a smooth manifold with corners.

In showing that (C_*, ∂) is a complex, we need to understand how the orientations of the various moduli spaces of gradient lines relate to one another through the gluing map. We now develop a calculus for discussing orientations of submanifolds and their intersections.

Recall that an oriented manifold with boundary induces an orientation on its boundary. This orientation is defined by the rule that the outward pointing normal followed by the boundary orientation must add up to the orientation of the full manifold; with this convention, Stokes' theorem is true. If R is an oriented submanifold of B then the normal bundle is oriented by the rule $[R][\mathcal{N}_R] = [B]$, where the notation means juxtaposition of oriented frames. Transverse intersections of oriented submanifolds R and T are oriented by the rule that the sum of the oriented normal bundles $\mathcal{N}_R \oplus \mathcal{N}_T$ is the oriented normal bundle of $R \cap T$. These rules are more conveniently expressed by noting that the juxtaposition of frames can be expressed as Clifford multiplication in $C(T^*B)$, which in terms of an orthonormal framing e_i is defined by $e_i^2 = 1$ and $e_i e_j = -e_j e_i$ if $i \neq j$. At different points in a submanifold of B we can compare the two framings by using parallel transport.

Starting from a choice of orientations of stable manifolds \mathcal{S}_α for critical points α we orient the unstable ones by

$$[\mathcal{S}_\alpha][\mathcal{U}_\alpha] = [B].$$

For oriented intersections $\mathcal{M}(\alpha, \beta) = \mathcal{S}_\beta \cap \mathcal{U}_\alpha$, we find

$$[\mathcal{M}(\alpha, \beta)] = [\mathcal{U}_\alpha][B]^{-1}[\mathcal{S}_\beta].$$

Orient the gradient line space $\tilde{\mathcal{M}}(\alpha, \beta)$ in a level set by $[\mathcal{M}(\alpha, \beta)] = [\tilde{\mathcal{M}}(\alpha, \beta)][\nabla f]$.

Proposition 2.7 *The gluing map*

$$G : \tilde{\mathcal{M}}(\alpha, \beta) \times (0, \epsilon) \times \tilde{\mathcal{M}}(\beta, \gamma) \rightarrow \tilde{\mathcal{M}}(\alpha, \gamma)$$

is orientation reversing.

Proof. Consider the image of G in the level set $f^{-1}(\delta)$ where δ is in $(f(\beta), f(\alpha))$ (as in the figure 4).

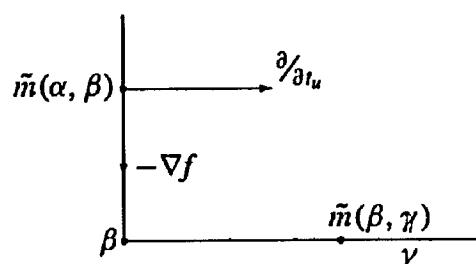


Fig. 4

The rules above give:

$$[\tilde{\mathcal{M}}(\alpha, \gamma)] = -[\tilde{\mathcal{M}}(\alpha, \beta)][t_u][\tilde{\mathcal{M}}(\beta, \gamma)]. \quad \square$$

To complete the discussion that (C_*, ∂) is a complex, notice that

$$\begin{aligned} \partial \circ \partial(\alpha) &= \sum_{i(\gamma)=i(\alpha)-2} \sum_{i(\beta)=i(\alpha)-1} \#\tilde{\mathcal{M}}(\alpha, \beta) \#\tilde{\mathcal{M}}(\beta, \gamma) \gamma \\ &= \sum_{\gamma} \partial \tilde{\mathcal{M}}(\alpha, \gamma) \gamma \\ &= 0 \end{aligned}$$

since the boundary of a 1-manifold has zero points counted with orientation. This gives our result

Proposition 2.8 (Complex) $\partial \circ \partial = 0$.

The only fact used in the proof is that the oriented boundary of a 1-manifold has zero points when counted with sign. These arguments, and the analogous ones in the following sections, call on a detailed knowledge of the asymptotics: limits of a sequence of trajectories should be understood in terms of gluing trajectories which connect intermediary critical points.

2.3 Independence of function and metric

The construction above is at the heart of the definition of Floer homology. To show that Floer homology is a topological invariant, one cannot proceed by comparing it to a known topological invariant. We therefore present the usual alternative proof in the finite dimensional setting which shows that the homology groups computed are independent of the choices. This proceeds along the lines of Floer [F1] and Fukaya [Fu]. It makes use of time dependent gradient flows on $B \times \mathbb{R}$.

The construction of the complex (C_*, ∂) depended on a Morse function and metric on the manifold. Of course, since the cohomology of this complex is just the singular cohomology of the manifold, the cohomology is completely independent of the choice of Morse function and metric. However, in many modern applications, the complex is defined for an infinite dimensional manifold and there is no known cohomology theory with which to compare. For this reason, we will give an alternative proof, in finite dimensions, that the homology groups computed from the complex (C_*, ∂) are independent of the Morse function and metric, following Floer [F1] and Fukaya [Fu]. This will also be useful in the definition of equivariant Floer cohomology (see [AuB]).

If two Morse functions f_i and metrics g_i on B are given, we obtain two complexes, (C_*^1, ∂^1) and (C_*^2, ∂^2) , as above. We now follow a three step procedure. First we construct chain homomorphisms from C^1 to C^2 , and vice versa, using a time dependent flow. Secondly we show that their composition is a chain homomorphism which also arises from a time dependent flow. Finally we construct

a homotopy operator showing that the composition induces the same map as the identity (which arises from a constant family). The three together show that the homology is canonical up to isomorphism.

To construct a map of complexes we use a family of functions f_t and metrics g_t on B parametrized by \mathbb{R} and equal to f_1 (resp. g_1) for $t < -1$ and to f_2 (resp. g_2) for $t > 1$. This pair is conveniently denoted by $F = (f_t, g_t)$, and defines a vector field

$$-\text{grad}_{g_t} f_t + \frac{\partial}{\partial t}$$

on $B \times \mathbb{R}$ equal to $-\text{grad}_{g_1} f_1 + \frac{\partial}{\partial t}$ for $t < -1$ and to $-\text{grad}_{g_2} f_2 + \frac{\partial}{\partial t}$ when $t > 1$. By abuse of notation we will also denote the vector field by F or F_{21} if we want to indicate that it interpolates between the $\text{grad} f_i$, $i = 1, 2$.

For such vector fields F with flow ϕ_s , one can easily define submanifolds

$$\mathcal{U}_\alpha = \{(x, t) \in B \times \mathbb{R}; \phi_s(x, t) \rightarrow (\alpha, -\infty) \text{ as } s \rightarrow -\infty\},$$

of dimension $i(\alpha) + 1$ when α is a critical point of f_1 , and similarly manifolds \mathcal{S}_β for critical points of f_2 . They will inherit orientations from the stable and unstable manifolds in B and intersect transversely for a Baire set of vector fields F on $B \times \mathbb{R}$ equal to the $\text{grad} f_i$ in the prescribed areas (see the transversality discussion in Appendix B). Assume that we have chosen such an F on $B \times \mathbb{R}$ and let

$$\tilde{\mathcal{M}}_F(\alpha, \beta) = (\mathcal{S}_\beta \cap \mathcal{U}_\alpha) / \mathbb{R}$$

of dimension $i(\alpha) - i(\beta)$ denote the oriented intersection.

A chain map $F_{21,*} : C_*^1 \rightarrow C_*^2$ can now easily be defined: for basis elements $\alpha \in C_i^1, \beta \in C_i^2$, we define

$$F_{21,*}(\alpha) = \sum_{\beta} \# \tilde{\mathcal{M}}_F(\alpha, \beta) \beta$$

using an algebraic count of oriented points.

Before explaining why this is a chain map, we observe that if F_{21} interpolates between f_1 and f_2 , and F_{32} between f_2 and a third Morse function f_3 on B , then

Lemma 2.9 $(F_{31})_* = (F_{32})_*(F_{21})_*$ where

$$F_{31,t} = F_{32,t-T}, \quad t > \frac{1}{2}, \text{ and } F_{31,t} = F_{21,t+T}, \quad t < -\frac{1}{2},$$

for large T .

Proof. This is a minor variant of the proof of Proposition 2.6 describing the gluing of gradient lines and it can be found in the appendix (see A.2.4). \square

Lemma 2.10 $F_* : (C_*^1; \partial^1) \rightarrow (C_*^2; \partial^2)$ is a chain map; that is, $F_*\partial^1 + \partial^2 F_* = 0$.

Proof. Consider α a critical point of f_1 of index i and β a critical point of f_2 of index $i-1$. This space of gradient lines, $\tilde{\mathcal{M}}_F(\alpha, \beta)$, has dimension 1 and boundary consisting of factorizations:

$$(\cup_{\gamma} \tilde{\mathcal{M}}_{f_1}(\alpha, \gamma) \times \tilde{\mathcal{M}}_F(\gamma, \beta)) \cup (\cup_{\delta} \tilde{\mathcal{M}}_F(\alpha, \delta) \times \tilde{\mathcal{M}}_{f_2}(\delta, \beta))$$

where γ runs over critical points of f_1 of index $i-1$ and δ runs over critical points of f_2 of index i (see A.2.3). This is shown as before keeping in mind that due to the presence of the time factor in the flow, only one of the factorizing trajectories can go through the time dependent area. The gluing and orientation proceeds as before. Then $F_*\partial^1(\alpha) = \sum_{\beta, \gamma} \# \tilde{\mathcal{M}}_{f_1}(\alpha, \gamma) \# \tilde{\mathcal{M}}_F(\gamma, \beta) \beta$ and $\partial^2 F_*(\alpha) = \sum_{\beta, \delta} \# \tilde{\mathcal{M}}_F(\alpha, \delta) \# \tilde{\mathcal{M}}_{f_2}(\delta, \beta) \beta$. Thus the coefficient of β in $F_*\partial^1 + \partial^2 F_*$ is precisely the number of oriented boundary points of $\tilde{\mathcal{M}}_F(\alpha, \beta)$ and hence zero. \square

Let F^i , $i = 1, 2$, be two families interpolating between f_1, g_1 and f_2, g_2 . To complete the proof that the Morse complex defines a topological invariant, we construct a chain homotopy to show that the induced maps F_*^i on homology are the same. To do so, we use an appropriate family of interpolating functions and metrics F_u on $B \times \mathbb{R}$, with $u \in [0, 1]$. We require that $F_u = F^1$ when $u < \frac{1}{4}$ and $F_u = F^2$ when $u > \frac{3}{4}$. Furthermore for $t > T$ we have $F_u(t) = (f_1, g_1)$ for all $u \in [0, 1]$, and similarly for $t < -T$ we want (f_2, g_2) .

We shall show on the level of complexes:

Lemma 2.11 The maps $(F_1)_*$ and $(F_2)_*$ are chain homotopic; that is, there is a map $H : C_i^1 \rightarrow C_{i+1}^2$ such that $(F_1)_* - (F_2)_* = -H\partial^1 - \partial^2 H$.

Proof. To find the homotopy and the relation we once more consider the boundary of a space of trajectories: this time trajectories for the flow of the vector field G on $B \times \mathbb{R} \times [0, 1]$ which is tangent to the $B \times \mathbb{R}$ for every $u \in [0, 1]$, and agrees with the vector field F_u considered above on $B \times \mathbb{R} \times \{u\}$. We pick critical points α of f_1 of and β of f_2 of index j . After perturbing our vector field slightly, we can assume that

$$\tilde{\mathcal{M}}_G(\alpha \times [0, 1], \beta \times [0, 1])$$

is a smooth manifold of dimension $i-j+1$ (see A.3.3). There are endpoint maps defined on this space mapping it to $\alpha \times [0, 1]$ and $\beta \times [0, 1]$.

When the index of α is i and that of β is $i+1$, then $\tilde{\mathcal{M}}_G(\alpha \times [0, 1], \beta \times [0, 1])$ is a compact zero-dimensional manifold. The fibers of the endpoint maps are non-empty for only finitely many $u \in [0, 1]$. Now define the chain homotopy $H : C_i^1 \rightarrow C_{i+1}^2$ as

$$H(\alpha) = \sum_{\beta} \# \tilde{\mathcal{M}}_G(\alpha \times [0, 1], \beta \times [0, 1]) \beta.$$

For δ a critical point of f_2 of index i , $\tilde{\mathcal{M}}_G(\alpha \times [0, 1], \delta \times [0, 1])$ is a 1-dimensional manifold whose boundary, when studied as above, equals:

$$\begin{aligned} & - \tilde{\mathcal{M}}_{F_0}(\alpha, \delta) \cup \tilde{\mathcal{M}}_{F_1}(\alpha, \delta) \cup \\ & - \{ \{ \cup_{\gamma} (\tilde{\mathcal{M}}_{f_1}(\alpha, \gamma) \times \tilde{\mathcal{M}}_G(\gamma \times [0, 1], \delta \times [0, 1])) \} \cup \\ & - \{ \cup_{\delta} (\tilde{\mathcal{M}}_G(\alpha \times [0, 1], \beta \times [0, 1]) \times \tilde{\mathcal{M}}_{f_2}(\beta, \delta)) \} \}. \end{aligned}$$

Here γ, β range over the critical points of f_1, f_2 of index $i-1, i+1$ respectively. Counting boundary points gives the desired relation. \square

Observe that the homotopy measures which gradient lines cease or begin to exist under variation of the parameter u . Putting everything together we obtain:

Proposition 2.12 *The assignment $(M, f, g) \rightarrow H_*(C_*, \partial)$ defines topological invariants of M up to canonical isomorphism, i.e. changing the metric or Morse function induces a canonical isomorphism between the corresponding homology groups.*

Proof. If $(f_1, g_1), (f_2, g_2)$ are as above, then the crucial observation to be made is that the composition $F_{12} \circ F_{21}$ can be realized by the constant family $F_t = (f_1, g_1)$. By 2.11, this induces the identity on homology. This shows that $F_{12} = F_{21}^{-1}$. \square

Observe that the compactness assumption on B can be relaxed in the following way. Let \mathcal{C} be a class of functions and metrics such that:

- 1) for any $(f, g) \in \mathcal{C}$ the critical point set of f is compact, and families of gradient lines connecting critical points of at most index +2 apart can only degenerate into factorizations into gradient lines connecting intermediary critical points;
- 2) the family of critical points and the moduli of gradients between critical points of index at most two apart in $\mathcal{C} \times B$ are cut out transversally;
- 3) for generic families $F_t = (f_t, g_t) : \mathbb{R} \rightarrow \mathcal{C}$ connecting two elements in \mathcal{C} , the stable and unstable manifolds intersect transversally, and the boundary of the $\tilde{\mathcal{M}}_F(\alpha, \beta)$ for $i(\alpha) = i(\beta)$ or $i(\alpha) = i(\beta) + 1$ is as described in the proof of Lemma 2.11;
- 4) between any two F_1, F_2 there exist families $G = F_u$ with smooth moduli space $\tilde{\mathcal{M}}_G(\alpha \times [0, 1], \beta \times [0, 1])$ which has a boundary as in the proof of Lemma 2.11 if $i(\alpha) = i(\beta)$.

Observe that the last two conditions exclude the possibility of critical points walking off to infinity in families. The homology is now an invariant of the manifold and the class of functions \mathcal{C} .

Example

Let $f : B \rightarrow \mathbb{R}$ be a function such that its critical points lie in a compact subset equal to $K \equiv f^{-1}([-N, N])$ of B . Let $O \subset B$ be a relatively compact neighborhood of K , and let \mathcal{C} be the affine space of functions consisting of functions $f+g$ where g has support in O . The metric may be arbitrary. Apart from the generic intersection properties, which are proved in Appendix B, it is obvious that all conditions are met.

The importance of existence, orientation and asymptotics of the stable and unstable manifolds must be stressed. All this found its origin in the fundamental work of Smale in dynamical systems and in the transversality theory of R. Thom.

2.4 Cup products

The complex (C_*, ∂) derived from a Morse function computes the homology of B using only information about the one-dimensional gradient line moduli spaces. It is natural to ask if additional information is obtained by studying the higher dimensional moduli spaces. In this section, we indicate that these spaces determine the cap and cup product structures on homology and cohomology. As we will cover this in more generality in §3, we delay some of the proofs until then. Braam and Donaldson use this construction in a gauge-theoretic application (see [BD]).

Let (C_*, ∂) denote the Morse complex defined by f with coefficient group \mathbb{R} . Given a differential form $\omega \in \Omega^k(B)$, define

$$c(\omega) : C_i \rightarrow C_{i-k} : \alpha \mapsto \sum \left(\int_{\mathcal{M}(\alpha, \beta_l)} \omega \right) \beta_l \quad (2.1)$$

where the sum runs over the critical points β_l of index $i - k$.

A careful study of the asymptotics of the gradient line spaces (as is becoming routine) shows that

$$(c(d\omega)(\alpha)) = c(\omega)(\partial\alpha) + \partial(c(\omega)(\alpha)). \quad (2.2)$$

This follows from

$$\begin{aligned} c(d\omega)(\alpha) &= \sum \left(\int_{\mathcal{M}(\alpha, \beta)} d\omega \right) \beta \\ &= \sum \left(\int_{\partial\mathcal{M}(\alpha, \beta)} \omega \right) \beta. \end{aligned}$$

By Proposition 2.6, it follows that the relevant part of $\partial\mathcal{M}(\alpha, \beta)$ is

$$\cup_{i(\gamma)=i-1} \tilde{\mathcal{M}}(\alpha, \gamma) \times \mathcal{M}(\gamma, \beta) \cup \cup_{i(\delta)=i-k} \mathcal{M}(\alpha, \delta) \times \tilde{\mathcal{M}}(\delta, \beta).$$

Then

$$\begin{aligned} c(d\omega)(\alpha) &= \sum_{\beta} \left(\sum_{\gamma} \# \tilde{\mathcal{M}}(\alpha, \gamma) \left(\int_{\mathcal{M}(\gamma, \beta)} \omega \right) \beta + \sum_{\delta} \# \tilde{\mathcal{M}}(\delta, \beta) \left(\int_{\mathcal{M}(\alpha, \delta)} \omega \right) \beta \right) \\ &= \partial(c(\omega)(\alpha)) + c(\omega)(\partial\alpha) \end{aligned}$$

which proves (2.2).

There are induced maps:

$$\begin{aligned} H_{dR}^k(B) \otimes H_i(C_*; \mathbb{R}) &\rightarrow H_{i-k}(C_*; \mathbb{R}) \\ H_{dR}^k(B) \otimes H^i(C^*; \mathbb{R}) &\rightarrow H^{i+k}(C^*; \mathbb{R}). \end{aligned} \quad (2.3)$$

Theorem 2.13 *The maps 2.3 induced by 2.1 agree with the usual cap and cup products, under the isomorphism between the (co)homology of the Morse complex with singular (co)homology.*

A simple consequence of this theorem is the following. We may define a pairing

$$C^i \otimes \overline{C}^{n-i} \rightarrow \mathbb{R}$$

$$\text{by } \sum m_\alpha \alpha \otimes \sum n_\alpha \alpha \mapsto \sum m_\alpha n_\alpha.$$

Corollary 2.14 *This induces the Poincaré duality pairing*

$$H^i(B; \mathbb{R}) \otimes H^{n-i}(B; \mathbb{R}) \rightarrow \mathbb{R}$$

on cohomology.

Perhaps the most important point we encountered here is that the spaces $\mathcal{M}(\alpha, \beta)$ contain definite homological information. To obtain a product structure on $H^*(C^*, f)$ without invoking differential forms on B , one would have to relate the cycles defined using these moduli spaces to Morse cycles. This is done by taking intersection numbers with stable (or unstable) manifolds of complementary dimension, and warrants further study.

We will see how to generalize this to equivariant cohomology in §5.

3 Morse-Bott theory

The theory outlined in §2 has two deficiencies. Proposition 2.1 tells us that «most» functions on a manifold are Morse. However, often one is presented with a natural function which is not Morse. Moreover, this theory is ill-suited for the study of equivariant functions, for in this case, the critical point set will necessarily contain orbits of the group action and so the critical points are generally not isolated.

In this section, we will rectify these deficiencies by studying so-called Morse-Bott functions. Here we allow the critical point sets to occur as positive dimensional submanifolds while requiring that the Hessian be non-degenerate on the normal bundle. Additionally, the equivariant case suggests some natural assumptions on the function which will amount to assuming that the stable and unstable manifolds intersect transversely — this is not the generic case for equivariant functions. Our methods give a complex, defined in terms of the deRham complex of the critical point sets and gradient line spaces, which computes the deRham

cohomology of the manifold. As such, this theory will generalize the usual Morse theory of §2. A lengthy overview will describe the complex with the technical details supplied afterwards. Later, we show how cup products may be computed and demonstrate the theory with a few examples. For the sake of exposition, we will often refer to the proofs of the special cases given in §2 when no new ideas are involved.

3.1 Overview

Let $f : B \rightarrow \mathbb{R}$ be a function on the closed, oriented Riemannian manifold B , non-degenerate in the sense of Bott; that is, the critical points are parametrized by a submanifold, S , and on S , the Hessian of f defines a fiberwise non-degenerate pairing on the normal bundle. For a component of S , S_α , we use the metric to decompose the normal bundle into the positive and negative eigenspaces of the Hessian, viz.

$$\nu(S_\alpha) = \nu_\alpha^+ \oplus \nu_\alpha^-.$$

As in the special case of §2, we define an index

$$i(\alpha) = \text{rank } \nu_\alpha^-.$$

Grouping the components of equal index defines $S_i = \cup_{i(\alpha)=i} S_\alpha$. Denote by $\mathcal{M}(S_i, S_j)$ the collection of points in B which are connected by gradient lines to S_i and S_j . The flow ϕ_t induces an \mathbb{R} action on this space whose quotient, denoted $\tilde{\mathcal{M}}(S_i, S_j)$, parametrizes the space of gradient lines beginning at S_i and ending at S_j . Following the flow as $t \rightarrow \pm\infty$ gives lower and upper endpoint maps

$$\begin{aligned} l_j^i : \tilde{\mathcal{M}}(S_i, S_j) &\rightarrow S_j \\ u_j^i : \tilde{\mathcal{M}}(S_i, S_j) &\rightarrow S_i. \end{aligned} \quad (3.1)$$

We will now state our assumptions on f and the metric g . First of all, we require that f be «weakly» self-indexing so that $\tilde{\mathcal{M}}(S_i, S_j) = \emptyset$ if $j \geq i$ (this is in contrast to f being «strictly» self-indexing meaning $f(S_i) = i$). Secondly, a transversality assumption on the stable and unstable manifolds defined by f enables us to conclude that $\tilde{\mathcal{M}}(S_i, S_j)$ is a smooth manifold and that the endpoint maps are locally trivial fiber bundles with oriented fibers. We will be more precise momentarily.

An example to keep in mind is the following: Consider S^2 as the unit sphere in \mathbb{R}^3 and let $f : S^2 \rightarrow \mathbb{R}$ be given by the function $f(x, y, z) = z^2$. The critical points of index 2 are the north and south pole — $S_2 = \{(0, 0, 1), (0, 0, -1)\}$ — while the minima of f are the points on the equator $\{z = 0\} = S_0$. Then the moduli space of gradient lines $\tilde{\mathcal{M}}(S_2, S_0)$ is diffeomorphic to the disjoint union of two copies of S^1 .

We now define our filtered complex. Let $C^{i,j} = \Omega^j(S_i)$ with the operators

$$\partial_r : C^{i,j} \rightarrow C^{i+r, j-r+1} : \omega \mapsto \begin{cases} d\omega & \text{when } r = 0 \\ (-1)^j (u_i^{i+r})_* (l_i^{i+r})^* (\omega) & \text{otherwise} \end{cases}$$

where the map $(u_i^{i+r})_*$ is integration along the fiber of the bundle in (3.1). Let

$$C^p = \oplus_{i+j=p} C^{i,j} = \oplus_{i+j=p} \Omega^j(S_i)$$

with boundary operator $\partial = \sum \partial_i$. The complex is filtered by

$$0 \subset C_p^p \subset C_{p-1}^p \subset \cdots \subset C_0^p = C^p$$

where

$$C_k^p = \oplus_{i+j=p, i \geq k} C^{i,j}.$$

The main theorem of this section gives

Theorem 3.1 *The cohomology of the complex (C^*, ∂) is isomorphic to the de Rham cohomology of B ; that is,*

$$H^*(C^*; \partial) = H_{\text{dR}}^*(B).$$

Notice that the theory of §2 is contained within this more general framework. In the case when f is a generic Morse function, the critical points are isolated. Hence $C^i = \Omega^0(S_i)$ is the vector space generated by the critical points of index i . Furthermore, $\partial_r = 0$ for all $r \neq 1$ for dimensional reasons. As before, $\tilde{\mathcal{M}}(S_{i+1}, S_i)$ is an oriented 0-manifold — that is, a collection of signed points. Integrating over the upper endpoint map simply counts these points with orientations. In this way, we recover the ordinary Morse complex.

3.2 The Morse-Bott complex

We will now make this more precise. First, we introduce the stable and unstable manifolds of the gradient flow and describe the assumptions needed to insure that $\mathcal{M}(S_i, S_j)$ and $\tilde{\mathcal{M}}(S_i, S_j)$ are smooth and that the endpoint maps are fibrations. As our definition of the boundary operator ∂ involves integration along the fiber of the upper endpoint maps $u_j^i : \tilde{\mathcal{M}}(S_i, S_j) \rightarrow S_i$, we will carefully study the structure of the boundary of $\tilde{\mathcal{M}}(S_i, S_j)$ (Lemmata 3.3 and 3.4). Stokes' Theorem leads naturally to the fact that $\partial \circ \partial = 0$ (Proposition 3.5). Next we set up a chain map between the ordinary de Rham complex and (C^*, ∂) (Lemma 3.6). Knowledge of the boundary structure of the unstable manifolds enables us to show that this induces a map of filtered complexes inducing an isomorphism of filtered cohomology groups (Theorem 3.8). This proves Theorem 3.1.

Proposition 3.2 *The unstable manifold, \mathcal{U}_i , of S_i which is defined as*

$$\mathcal{U}_i = \{b \in B \mid \lim_{t \rightarrow -\infty} \phi_t(b) \in S_i\}.$$

is the image of a one-to-one immersion of ν_i^- into B . There is a smooth endpoint map $u^i : \mathcal{U}_i \rightarrow S_i$ given by $u^i(b) = \lim_{t \rightarrow -\infty} \phi_t(b)$ which when restricted to a neighborhood of S_i has the structure of a locally trivial fiber bundle.

Proof. This is shown in the appendix (A.9). \square

Define $\mathcal{U}_i^s = (\mathcal{U}^i)^{-1}(s)$ for $s \in S_i$. Similarly, the stable manifold \mathcal{S}_i is defined using the flow as $t \rightarrow \infty$ and leads to the map $l_i : \mathcal{S}_i \rightarrow S_i$. Of course, the moduli space of gradient lines is $\tilde{\mathcal{M}}(S_i, S_j) = (\mathcal{U}_i \cap \mathcal{S}_j)/\mathbb{R}$. We will make the following assumptions in the sequel:

Assumptions:

- 1) $\tilde{\mathcal{M}}(S_i, S_j) = \emptyset$ if $i \leq j$. That is, f is weakly self-indexing.
- 2) For all i, j and all $s \in S_i$, \mathcal{U}_i^s intersects \mathcal{S}_j transversally.
- 3) Both the critical submanifolds S_i and their negative normal bundle ν_i^- are orientable for all i .

By the transversality assumption (2) above, the following spaces are manifolds:

$$\mathcal{M}(S_i, S_j) = \mathcal{S}_j \cap \mathcal{U}_i$$

$$\tilde{\mathcal{M}}(S_i, S_j) = \mathcal{M}(S_i, S_j)/\mathbb{R}$$

where the \mathbb{R} action is given by translation along the flow lines. We have smooth endpoint maps

$$\begin{aligned} l_j^i : \tilde{\mathcal{M}}(S_i, S_j) &\rightarrow S_j \\ \mathcal{U}_j^i : \tilde{\mathcal{M}}(S_i, S_j) &\rightarrow S_i. \end{aligned} \quad (3.2)$$

Furthermore, the transversality assumption (2) implies that $\mathcal{U}_j^i : \tilde{\mathcal{M}}(S_i, S_j) \rightarrow S_i$ has the structure of a locally trivial fiber bundle.

To define and study our filtered complex, we must first describe the structure of the boundary of the spaces $\tilde{\mathcal{M}}(S_i, S_j)$ and \mathcal{U}_i . This is entirely analogous to §2.

For a sequence $j = i_0 < i_1 < \dots < i_m = i$, using the fact that the \mathcal{U}_j^i induce fibrations, define

$$X_{i_0, i_1, \dots, i_m} = (l_{i_{m-1}}^{i_m})^* \dots (l_{i_1}^{i_2})^* \tilde{\mathcal{M}}(S_{i_1}, S_{i_0})$$

and

$$Y_{i_0, i_1, \dots, i_m} = (l_{i_{m-1}}^{i_m})^* \dots (l_{i_0}^{i_1})^* \mathcal{U}_{i_0}.$$

Using the usual notation for fibered products, we may more explicitly write,

$$X_{i_0, i_1, \dots, i_m} = \tilde{\mathcal{M}}(S_{i_m}, S_{i_{m-1}}) \times_{S_{i_{m-1}}} \tilde{\mathcal{M}}(S_{i_{m-1}}, S_{i_{m-2}}) \times_{S_{i_{m-2}}} \dots \times_{S_{i_1}} \tilde{\mathcal{M}}(S_{i_1}, S_{i_0})$$

$$Y_{i_0, i_1, \dots, i_m} = \tilde{\mathcal{M}}(S_{i_m}, S_{i_{m-1}}) \times_{S_{i_{m-1}}} \tilde{\mathcal{M}}(S_{i_{m-1}}, S_{i_{m-2}}) \times_{S_{i_{m-2}}} \dots \times_{S_{i_0}} \mathcal{U}_{i_0}.$$

One thinks of the X_{i_0, i_1, \dots, i_m} as describing the space of «broken» gradient lines from S_i to S_j factorizing through intermediate critical submanifolds as in Lemma 2.5. We may topologize the disjoint unions

$$\tilde{\mathcal{M}}(S_i, S_j) \bigcup_{j=i_0 < i_1 < \dots < i_m=i} X_{i_0, i_1, \dots, i_m}$$

and

$$\mathcal{U}_i = \bigcup_{i_0 < i_1 < \dots < i_m = i} Y_{i_0, i_1, \dots, i_m}$$

by requiring that sequences converge as in Lemma 2.5. As in §2, the following lemma holds.

Lemma 3.3 (Compactification) *The spaces of gradient lines $\tilde{\mathcal{M}}(S_i, S_j)$ and the unstable manifolds \mathcal{U}_i may be compactified so that*

$$\begin{aligned} \partial \tilde{\mathcal{M}}(S_i, S_j) &= \bigcup_{j=i_0 < i_1 < \dots < i_m = i} X_{i_0, i_1, \dots, i_m} \\ \text{and } \partial \tilde{\mathcal{U}}_i &= \bigcup_{i_0 < i_1 < \dots < i_m = i} Y_{i_0, i_1, \dots, i_m}. \end{aligned}$$

These compactified spaces both have the structure of a manifold with corners — that is, they are locally diffeomorphic to an open set in $U^N = \{(x_i \in \mathbb{R}^N | x_i \geq 0 \text{ for all } i)\}$. There is a natural injective immersion of the $\tilde{\mathcal{U}}_i$ into B , which is not usually proper.

Proof. See Appendix §A.3. It should be noted that the transversality assumption (2) above is necessary. \square

The maps $u_j^i : \tilde{\mathcal{M}}(S_i, S_j) \rightarrow S_i$ extend smoothly to X_{i_0, i_1, \dots, i_m} and agree with $u_{i_{m-1}}^i$. Similarly for l_j^i . Furthermore, $u_j^i : \partial \tilde{\mathcal{M}}(S_i, S_j) = \bigcup X_{i_0, i_1, \dots, i_m} \rightarrow S_i$ and has the structure of a bundle map which we denote $u_{j,0}^i$ and similarly for $u^i : \partial \tilde{\mathcal{U}}_i \rightarrow S_i$.

Before defining our filtered complex we need to choose orientations of the S_i and their normal bundles. This orients all the spaces of gradient lines. Let $(\tilde{\mathcal{M}}(S_{i+l}, S_i))^s = (u_l^{i+l})^{-1}(s)$ for $s \in S_{i+l}$.

Lemma 3.4 (Boundary orientation) *The codimension 1 stratum*

$$(l_{i+l}^{i+k})^* \tilde{\mathcal{M}}(S_{i+l}, S_i) \rightarrow (\tilde{\mathcal{M}}(S_{i+k}, S_{i+l}))^x$$

of $(\tilde{\mathcal{M}}(S_{i+k}, S_i))^x$ has boundary orientation

$$(-1)^{k-l-1} [(\tilde{\mathcal{M}}(S_{i+k}, S_{i+l}))^x] [(\tilde{\mathcal{M}}(S_{i+l}, S_i))^y].$$

The codimension one stratum

$$(l_l^k)^* \mathcal{U}_l \rightarrow (\tilde{\mathcal{M}}(S_k, S_l))^x$$

of $\tilde{\mathcal{U}}_k^x$ has boundary orientation:

$$(-1)^{k-l-1} [(\tilde{\mathcal{M}}(S_k, S_l))^x] [\mathcal{U}_l^y].$$

Proof. As in Lemma 2.7. \square

We digress briefly to discuss integration along the fiber of a bundle $\pi : E \rightarrow C$ with fiber F . We furthermore assume that C is a closed manifold and that the typical fiber is an oriented compact d -dimensional manifold with corners so that $\pi_{\partial} : \partial E \rightarrow C$ is also a bundle with fiber ∂F . Take coordinates x on C and t on F . A form on the total space E may locally be written as a linear combination of forms $\pi^*(\phi)f(x,t)dt_{i_1} \wedge \dots \wedge dt_{i_r}$, where ϕ is a form on C . Then integration along the fiber, $\pi_* : \Omega^j(E) \rightarrow \Omega^{j-d}(C)$ is defined by

$$\begin{aligned}\pi_*(\pi^*(\phi)f(x,t)dvol_F) &= \phi \int f(x,t)dvol_F \\ \pi_*(\pi^*(\phi)f(x,t)dt_{i_1} \dots dt_{i_r}) &= 0 \text{ for } r < d.\end{aligned}$$

It is important to consider how exterior differentiation is related to π_* . Since Stokes' theorem holds for manifolds with corners, we have for $\omega \in \Omega^j(C)$

$$\pi_*(d\omega) = d\pi_*(\omega) + (-1)^{j-d+1}(\pi_{\partial})_*(\omega|_{\partial E}). \quad (3.3)$$

To see that we have defined a complex, we must first show that $\partial^2 = (\sum_r \partial_r)^2 = 0$. This is equivalent to the following proposition.

Proposition 3.5 (Complex) For each k , $\sum_{l=0}^{l=k} \partial_{k-l} \partial_l = 0$.

Proof. The fact that $d^2 = 0$ on $\Omega^j(S_i)$ proves the case $k = 0$.

For $k > 0$, we are interested in studying $\partial_k \partial_0 \omega$ for a form $\omega \in \Omega^j(S_i)$. Using (3.3) above and the study of the asymptotics of the space of gradient lines, we arrive at

$$\begin{aligned}\partial_k \partial_0 \omega &= (-1)^{j+1} (u_i^{i+k})_* (l_i^{i+k})^* d\omega \\ &= (-1)^{j+1} (u_i^{i+k})_* d(l_i^{i+k})^* \omega \\ &= (-1)^{j+1} d(u_i^{i+k})_* (l_i^{i+k})^* \omega + (-1)^{j+1} (-1)^{j-k} (u_{i,\partial}^{i+k})_* (l_i^{i+k})^* \omega \\ &= -\partial_0 \partial_k \omega + (-1)^{k-1} (u_{i,\partial}^{i+k})_* (l_i^{i+k})^* \omega.\end{aligned} \quad (3.4)$$

We now study the boundary integral by considering the codimension one strata of the boundary. These strata fit into the following commutative diagram:

$$\begin{array}{ccccc} (l_{i+l}^{i+k})^* \tilde{\mathcal{M}}(S_{i+l}, S_i) & \rightarrow & \tilde{\mathcal{M}}(S_{i+l}, S_i) & \xrightarrow{l_i^{i+l}} & S_i \\ \swarrow & & \downarrow & & \downarrow u_i^{i+l} \\ S_{i+k} & \xleftarrow{u_{i+l}^{i+k}} & \tilde{\mathcal{M}}(S_{i+k}, S_{i+l}) & \xrightarrow{l_{i+l}^{i+k}} & S_{i+l} \end{array}$$

Integration over the fiber commutes with the pullback of bundles; hence,

$$\begin{aligned} (u_{i,\partial}^{i+k})_*(l_i^{i+k})^*\omega &= \sum_{0 \leq l < k} (-1)^{k-l-1} (u_{i+l}^{i+k})_*(l_{i+l}^{i+k})^* (u_i^{i+l})_*(l_i^{i+l})^*\omega \\ &= \sum_{0 \leq l < k} (-1)^k \partial_{k-l} \partial_l. \end{aligned} \quad (3.5)$$

These two expressions, (3.4) and (3.5), together prove the lemma. \square

3.3 Comparison with the de Rham complex

We now claim that this complex computes the deRham cohomology of the manifold B . To do this, we will assume that f is strictly self-indexing ($f(S_i) = i$) and define the sets $B_k = f^{-1}((k - \frac{1}{2}, \infty))$. The essential feature of B_k is that it contains all critical submanifolds of index less than k and their unstable manifolds while containing no critical submanifolds of higher index. When f is not strictly self-indexing, it is possible, using the transversality assumption, to define sets B_k also having this property. This is a simple extension of an idea of Smale [Sm2]. Thus, the self-indexing assumption only simplifies our notation without restricting the generality of the conclusion.

Consider the standard deRham complex $(\Omega^j(B), d)$ filtered by

$$0 \subset \Omega_c^j(B_n) \subset \Omega_c^j(B_{n-1}) \subset \cdots \subset \Omega_c^j(B_{-1}) = \Omega^j(B)$$

where $B_k = f^{-1}((k - \frac{1}{2}, \infty))$ and the subscript denotes compactly supported forms. With $C'^j = \Omega^j(B)$ and $C_k'^j = \Omega_c^j(B_k)$, we will describe a map of filtered complexes

$$\Phi : C'^j \rightarrow C^j$$

inducing a map on the filtered cohomology groups. To define $\Phi_k : C'^j \rightarrow C^{k,j-k}$, recall that $u^k : \mathcal{U}_k \rightarrow S_k$ has the structure of a locally trivial bundle. For $\omega \in C'^j = \Omega^j(B)$, let

$$\Phi_k(\omega) = (u^k)_*(\omega|_{\mathcal{U}_k}).$$

Then $\Phi = \oplus \Phi_k$.

Lemma 3.6 (Chain map) Φ is a map of filtered complexes — that is, $\Phi \circ d = \partial \circ \Phi$.

Proof. The statement of the lemma is equivalent to

$$\Phi_k(d\omega) = \sum_{l=0}^{l=k} \partial_{k-l} \Phi_l(\omega).$$

The codimension one stratum of the boundary $\partial\mathcal{U}_k$ is given by $Y_{l,k}$ with $l < k$. We rewrite $Y_{l,k} = (l_l^k)^*\mathcal{U}_l$. Then

$$\begin{aligned}\Phi_k(d\omega) &= (u^k)_*(d\omega) \\ &= d(u^k)_*(\omega) + (-1)^{j-k-1}(u_0^k)_*(\omega) \\ &= d(u^k)_*(\omega) + (-1)^{j-k-1} \sum_{l < k} (-1)^{k-l-1} (u_l^k)_*(l_l^k)^*(u^l)_*(\omega) \\ &= \partial_0 \Phi_k(\omega) + (-1)^{j-k-1} \sum_{l < k} (-1)^{k-l-1} (-1)^{j-l} \partial_{k-l} \Phi_l(\omega).\end{aligned}$$

Furthermore $\Phi_{k-1}(\omega) = 0$ if $\omega \in \Omega_c^j(B_k)$ so that $\Phi : C_k^j \rightarrow C_k^j$. This shows that Φ defines a map of filtered complexes. \square

The previous lemma enables us to say that Φ induces a map on the filtered cohomology groups of the complexes. In fact, this map is an isomorphism as will be proven in the next theorem. First, we review how a filtered complex induces a spectral sequence (see [BT]). Let K be a complex filtered by

$$\dots \subset K_n \subset \dots \subset K_1 \subset K_0 = K = K_{-1} = \dots$$

and let $GK_n = K_n/K_{n+1}$ be the associated graded complex. The short exact sequence

$$0 \rightarrow K_{n+1} \rightarrow K_n \rightarrow GK_n \rightarrow 0$$

induces the long exact sequence in cohomology

$$\dots \rightarrow H^p(K_{n+1}) \rightarrow H^p(K_n) \rightarrow H^p(GK_n) \rightarrow \dots$$

This exact couple results in a spectral sequence whose E_1 term is $E_1^{n,p} = H^p(GK_n)$. The naturality of this construction implies that a map between filtered complexes induces a map on the spectral sequences. The following algebraic lemma will be useful in what follows.

Lemma 3.7 *Let $f : K^1 \rightarrow K^2$ be a chain map of filtered complexes. If f induces an isomorphism of the E_1 term of the associated spectral sequences, then f induces an isomorphism on homology.*

Proof. Since f induces an isomorphism of the E_1 terms of the spectral sequences, it must also induce an isomorphism of the E_∞ terms. However, the E_∞ term is the associated graded group of the homology of the complex. The result follows by noting that a homomorphism of filtered groups which is an isomorphism on the associated graded groups must be an isomorphism. \square

We are now ready to prove our main theorem (Theorem 3.1). For the remainder of this section, we will use real coefficients without explicit notation. Let $N_k = f^{-1}((-\infty, k - \frac{1}{2}])$ so that the cohomology of the complex $H^p(\Omega_c^*(B_k), d) = H^p(B, N_k)$. Thus the filtration on the deRham complex induces a filtration on $H^p(B)$ given by the image subgroups $(H^p(B, N_k) \rightarrow H^p(B))$. The spectral sequence obtained by filtering forms on B by level was studied by Fary [Fa] and is described in [Bo].

Theorem 3.8 Φ induces an isomorphism of filtered cohomology groups; hence

$$\Phi : H^p(C_k^*, \partial) \simeq \text{image } (H^p(B, N_k) \rightarrow H^p(B)).$$

Theorem 3.1 appears as a special case.

Proof. We begin by showing that Φ induces an isomorphism on the E_1 term of the spectral sequences. The associated graded complexes of the two sequences are given by

$$\begin{aligned} GC_k^p &= \Omega^{p-k}(S_k) \\ \text{and} \quad GC_k'^p &= \Omega_c^p(B_k) / \Omega_c^p(B_{k+1}). \end{aligned}$$

On the E_1 term, Φ induces $\Phi : H^p(GC_k') \rightarrow H^p(GC_k) = H^{p-k}(S_k)$. Recall that this map is defined by $\Phi(\omega) = (u_k)_*(\omega|_{\mathcal{U}_k})$ for $\omega \in \Omega_c^p(B_k)$.

Let $F_k = B_k \cap \mathcal{U}_k$ be a neighborhood of S_k in \mathcal{U}_k . By the stable manifold theorem A.9, F_k may be identified with the disk bundle D_k^- in the negative normal bundle ν_k^- . The map Φ factors as

$$H^p(GC_k') \rightarrow H_c^p(F_k) \rightarrow H^{p-k}(S_k)$$

where the first map is induced by restriction of forms $\Omega_c^p(B_k) \rightarrow \Omega_c^p(F_k)$ and the second map is the Thom isomorphism given by integrating over the fibration $F_k \rightarrow S_k$. To show that Φ is an isomorphism, it suffices to show that restriction of forms induces an isomorphism $H^p(GC_k') \rightarrow H_c^p(F_k)$ as follows.

The short exact sequence

$$0 \rightarrow \Omega_c^p(B_{k+1}) \rightarrow \Omega_c^p(B_k) \rightarrow GC_k'^p \rightarrow 0$$

induces a long exact sequence in cohomology related to the exact sequence of the triple (B, N_{k+1}, N_k) as

$$\begin{array}{ccccccc} \dots & H_c^p(B_{k+1}) & \rightarrow & H_c^p(B_k) & \rightarrow & H^p(GC_k') & \rightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \dots & H^p(B, N_{k+1}) & \rightarrow & H^p(B, N_k) & \rightarrow & H^p(N_{k+1}, N_k) & \rightarrow \dots \end{array}$$

By the Five Lemma, $H^p(GC_k') \rightarrow H^p(N_{k+1}, N_k)$ is an isomorphism. Furthermore, the gradient flow defines a deformation retract of the pair (N_{k+1}, N_k)

onto $(F_k \cup N_k, N_k)$ (as in [Mi1] for the case of nondegenerate critical points). Hence, $H^p(N_{k+1}, N_k) \rightarrow H^p(F_k, \partial F_k) = H_c^p(F_k)$ is an isomorphism by excision. Putting these together implies that restriction of forms gives an isomorphism $H^p(GC'_k) \rightarrow H_c^p(F_k)$ as claimed.

Now that Φ induces an isomorphism on the E_1 term of the spectral sequences, it must induce an isomorphism on cohomology by Lemma 3.7. \square

Without the transversality and self-indexing assumptions, the unstable manifolds $\{\mathcal{U}_i\}$ still form a stratification of B . From this, we could compute the cohomology of B resulting in a theory over the integers. However, as this method does not generalize to infinite dimensions — the stable manifolds need not form a stratification — we will not pursue this course.

As in §2, the existence of the Morse-Bott complex implies the more general Morse inequalities of [B3] and [Bi]:

Corollary 3.9 *There is a polynomial $Q(t)$ with non-negative coefficients such that*

$$\sum_{i,j} \dim H^j(S_i; \mathbb{R}) t^{i+j} = \sum_k \dim H^k(B; \mathbb{R}) t^k + (1+t)Q(t).$$

A Morse-Bott function f is called *perfect* if $Q(t) = 0$. If either the higher differentials vanish in the spectral sequence or the components of the boundary operator ∂_k for $k > 0$ are zero, then f is perfect. This is the case in Bott [B4].

3.4 Comparison with the Morse complex

Remember that Proposition 2.1 states that a Morse function necessarily lies arbitrarily close to our given Morse-Bott function f . It is instructive to prove how the two complexes are related. This is in fact not much different from the comparison with the de Rham complex.

Choose tubular neighborhoods T_i of the critical submanifolds S_i and generic Morse functions f_i on S_i . Let ρ_i be bump functions which are identically 1 near S_i and identically zero outside of T_i . Define

$$f_\epsilon = f + \epsilon \sum_i \rho_i f_i.$$

For sufficiently small ϵ , f_ϵ is a nondegenerate Morse function. We denote the Morse complex that it defines by $(C_\epsilon^*, \partial_\epsilon^*)$. We will describe a chain map between (C^*, ∂) and $(C_\epsilon^*, \partial_\epsilon^*)$ which induces an isomorphism on cohomology. Fintushel and Stern [FS] have used this idea in computing Floer homology groups for Seifert fibered homology 3-spheres.

The critical points of f_ϵ are exactly the critical points of f_i ; moreover, if α is a critical point of index j for f_i , it is critical of index $i+j$ for f_ϵ . Denote the unstable manifold of α in S_i by \mathcal{U}_α^i .

Proposition 3.10 For small ϵ , the map $F : (C^*, \partial) \rightarrow (C_\epsilon^*, \partial_\epsilon^*)$ defined by

$$F(\omega) = \left(\int_{\mathcal{Q}_\alpha^i} \omega \right) \alpha,$$

where $\omega \in \Omega^j(S_i)$, is a chain morphism inducing an isomorphism on cohomology.

Proof. Given critical points α and β of f_ϵ , denote the moduli space of gradient lines of f_ϵ by $\tilde{\mathcal{M}}_\epsilon(\alpha, \beta)$. Showing that F is a chain map is equivalent to checking

$$F(\partial_k \omega) = \sum \# \tilde{\mathcal{M}}_\epsilon(\beta, \alpha) \beta$$

where the sum is taken over the critical points β of f_{i+k} of index $j - k + 1$. For $k = 0$, this is just Proposition 3.6. The proof for $k > 0$ involves the asymptotics of the gradient line moduli spaces. Let p denote the minima of f_i so that

$$\mathcal{M}_\epsilon(\beta, p) = u_{i+j}^{-1}(\mathcal{Q}_\beta^{i+j}) \cap l_i^{-1}(\mathcal{G}_p^i).$$

Then the boundary of $\mathcal{M}_\epsilon(\beta, p)$ contains factorizations

$$u_{i+j}^{-1}(\mathcal{Q}_\beta^{i+j}) \cap \tilde{\mathcal{M}}(S_{i+j}, S_i) \cup \tilde{\mathcal{M}}_\epsilon(\beta, \alpha) \times \mathcal{Q}_\alpha^i.$$

Moreover the integral $\int_{\mathcal{M}_\epsilon(\beta, p)} l_i^* d\omega$ vanishes since the form has no component along the flow lines. Putting this together implies that

$$\begin{aligned} 0 &= \int_{\mathcal{M}_\epsilon(\beta, p)} l_i^* d\omega \\ &= \int_{\partial \mathcal{M}_\epsilon(\beta, p)} \omega \\ &= \int_{\mathcal{Q}_\beta^{i+j}} ((u_{i+j})_* l_i^* \omega)|_{\mathcal{Q}_\beta^{i+j}} - \# \tilde{\mathcal{M}}_\epsilon(\beta, \alpha) \int_{\mathcal{Q}_\alpha^i} \omega|_{\mathcal{Q}_\alpha^i} \\ &= F(\partial_k \omega) - \partial^*(F(\omega)). \end{aligned}$$

This implies that F is a chain map.

Moreover, both complexes are filtered by the index of the critical submanifolds of the original Morse-Bott function f and F is a chain map of filtered complexes. It follows that the E_1 of both spectral sequences is $H^j(S_i)$ and by Proposition 3.1, F induces an isomorphism on E_1 and hence on the homology of the complexes (Lemma 3.7). \square

3.5 Cup products

Here we will generalize the discussion on cup products in §2 to include the Morse-Bott case and a proof will be given for Theorem 2.13. The idea is that the cup product can be realized by integrating over moduli spaces of gradient lines.

For $\omega \in \Omega^k(B)$, define

$$c(\omega) : C^l \rightarrow C^{l+k}$$

in terms of its decomposition $c(\omega) = \oplus_{m \geq 0} c(\omega)_m$ where

$$c(\omega)_m : \Omega^j(S_i) \rightarrow \Omega^{j-m+k}(S_{i+m}).$$

For $m = 0$, $c(\omega)_0(\alpha) = \omega|_{S_i} \wedge \alpha$. For $m > 0$, define

$$c(\omega)_m(\alpha) = (u^{i+m})_*(\omega|_{\mathcal{M}(S_{i+m}, S_i)} \wedge l_i^* \alpha).$$

Studying the gradient line spaces, exactly as in §2.4, gives the chain relation

$$c(d\omega)(\alpha) = \partial(c(\omega)(\alpha)) + c(\omega)(\partial\alpha).$$

Then there is an induced map on cohomology such that the generalization of Theorem 2.13 holds.

Theorem 3.11 *The map*

$$c : H_{dR}^k(B) \otimes H^l(B; \mathbb{R}) \rightarrow H^{l+k}(B; \mathbb{R})$$

is the cup product map.

Proof. We will prove this in the case in which f is a nondegenerate Morse function as in §2. The general Morse-Bott case is no more complicated.

The result is proven once we construct a commutative diagram

$$\begin{array}{ccc} H_{dR}^k(B) \otimes H_{dR}^l(B) & \xrightarrow{\wedge} & H_{dR}^{k+l}(B) \\ \downarrow & & \downarrow \\ H_{dR}^k(B) \otimes H^l(C^*, \partial) & \xrightarrow{c} & H^{k+l}(C^*, \partial) \end{array}$$

where the vertical arrows are induced by the isomorphism in Theorem 3.1 — that is, by integration over the unstable manifolds.

Let $\omega \in \Omega^k(B)$, $\eta \in \Omega^l(B)$ and α, β be critical points of f of index l and $l+k$ respectively. To demonstrate the commutative diagram above, we wish to show that the chains $\int_{\mathcal{U}_\beta} \omega \wedge \eta$ and $\int_{\mathcal{M}(\beta, \alpha)} \omega \int_{\mathcal{U}_\alpha} \eta$ define the same cohomology class.

In the spirit of Witten's deformation of the deRham complex [Wi], we consider the integral $\int_{\mathcal{U}_\beta} \phi_{-t}^* \omega \wedge \phi_t^* \eta$. Since we have pulled the forms back by a diffeomorphism isotopic to the identity, they still define the same cohomology classes and hence the integral over the unstable manifold \mathcal{U}_β must be cohomologous to $\int_{\mathcal{U}_\beta} \omega \wedge \eta$. In the limit that $t \rightarrow \infty$, the support of the forms becomes concentrated around the critical points. Let γ be a critical point of index i and denote the eigenvalues of the Hessian of f at γ by λ_j where $\lambda_j > 0$ for $j \leq i$ and $\lambda_j < 0$ for $j > i$. Set $\rho = \min |\lambda_j|$ and $R = \max |\lambda_j|$. For convenience, assume that around γ there is a coordinate chart $(x_1, \dots, x_i, y_{i+1}, \dots, y_n)$ such that the vector field $-\text{grad } f$ is linear so that the gradient flow may be written

$$\phi_t(x, y) = (e^{\lambda_1 t} x_1, \dots, e^{\lambda_i t} x_i, e^{\lambda_{i+1} t} y_{i+1}, \dots, e^{\lambda_n t} y_n).$$

Near γ , we write $\omega(x, y) = \omega_{IJ} dx^I \wedge dy^J$ and similarly for η . Then using multi-index notation,

$$\phi_{-t}^* \omega(x, y) = \omega_{IJ}(\phi_{-t}(x, y)) e^{-\lambda_I t} e^{\lambda_J t} dx^I dy^J$$

where $\lambda_K = \sum_{k \in K} \lambda_k$. The form $\phi_t^* \eta$ admits a similar expression. Near the critical point γ , the unstable manifold \mathcal{U}_β factorizes as $\mathcal{M}(\beta, \gamma) \times \mathcal{U}_\gamma$. We can then apply Fubini's theorem to estimate

$$\begin{aligned} \left| \int_{\mathcal{U}_\beta} \phi_{-t}^* \omega \wedge \phi_t^* \eta \right| &\leq \int_{\mathcal{M}(\beta, \gamma)} \int_{\mathcal{U}_\gamma} |\omega_{IJ}(\phi_{-t}(x, y)) \eta_{I'J'}(\phi_t(x, y))| e^{(\lambda_{I'} - \lambda_I)t} dx^I dx^{I'} \\ &\quad e^{(\lambda_J - \lambda_{J'})t} dy^J dy^{J'} \\ &\leq \int_{\mathcal{M}(\beta, \gamma)} \int_{\mathcal{U}_\gamma} |\omega_{IJ}(\phi_{-t}(x, y)) \eta_{I'J'}(\phi_t(x, y))| e^{(R|I'| - \rho|I|)t} dx^I dx^{I'} \\ &\quad e^{(\lambda_J - \lambda_{J'})t} dy^J dy^{J'}. \end{aligned}$$

Counting dimensions shows that any non-zero terms must satisfy

$$\begin{aligned} |I| + |I'| &= i \\ |J| + |J'| &= l + k - i \\ |I| + |J| &= k \\ |I'| + |J'| &= l. \end{aligned}$$

Making the coordinate change

$$g(x, y) = (e^{\lambda_1 t} x_1, \dots, e^{\lambda_i t} x_i, y_{i+1}, \dots, y_n),$$

we may rewrite the inner integral over the unstable manifold \mathcal{U}_γ . The contribution to each term from the inner integral then becomes bounded by

$$\int_{\mathcal{U}_\gamma} |\omega_{IJ}(g(\phi_{-t}(x, y))) \eta_{I'J'}(g(\phi_t(x, y)))| e^{-(\rho+R)|I|t} dx^I dx^{I'}.$$

As t grows, the contributions from terms with $|I| > 0$ become increasingly small and can hence be neglected. For this reason, we can suppose that $|I| = 0$. A similar argument shows that the only significant contribution is from the term with $|J'| = 0$. Putting together the equalities above, we conclude that the index of γ must be l and the integral becomes

$$\int_{\mathcal{M}(\beta, \gamma)} \omega \int_{\mathcal{U}_\gamma} \eta.$$

This proves the result when the vector field is linear about a critical point. In general, this is not so; however, we may use the estimates in Lemma A.3 to bound the corrections arising from nonlinearity of the vector field. The same argument then holds. \square

Furthermore, the Poincaré pairing can be understood by reversing the gradient flow as in Corollary 2.14. First suppose that $-f$ is a Morse-Bott function satisfying the assumptions above. Then $-f$ defines a Morse-Bott complex $(\bar{C}^*, \bar{\partial})$ which again computes the cohomology of B . Then we have

Corollary 3.12 *The Poincaré duality pairing $H^k(B; \mathbb{R}) \otimes H^{n-k}(B; \mathbb{R}) \rightarrow \mathbb{R}$ can be expressed in terms of the Morse-Bott complex*

$$C^k \otimes \bar{C}^{n-k} \rightarrow \mathbb{R}$$

$$\text{by } \omega_i \otimes \eta_j \rightarrow \int_{S_i} \omega_i \wedge \eta_j.$$

Proof. The key observation is that the unstable manifold, $\bar{\mathcal{U}}_i$, of the critical submanifold S_i under the function $-f$ is (up to factorizations) just the gradient line space $\mathcal{M}(S_k, S_i)$ where S_k runs over all the local maxima for f . Then, arguing as in Theorem 3.11 shows that

$$\begin{aligned} \int_B \omega \wedge \eta &= \int_{\mathcal{M}(S_k, S_i)} \omega \int_{\mathcal{U}_i} \eta \\ &= \int_{\bar{\mathcal{U}}_i} \omega \int_{\mathcal{U}_i} \eta. \end{aligned}$$

\square

3.6 Examples

Though the Morse-Bott complex may appear somewhat daunting, it is often quite easy to compute. In this section, we will present some examples to bolster this claim.

1. Consider the two-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ with the Morse-Bott function $f(x, y, z) = z^2$. There are two critical points, p and q , of index 2, while the equatorial S^1 is critical of index 0. Hence the E_1 term of the spectral sequence is

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \mathbb{R} & 0 & 0 & 0 \\ \mathbb{R} & 0 & \mathbb{R}^2 & 0 \end{array}$$

The differential $d_2 : \mathbb{R} \rightarrow \mathbb{R}^2$ is given by integrating the volume form on S^1 along the moduli spaces $\tilde{\mathcal{M}}(p, S^1)$ and $\tilde{\mathcal{M}}(q, S^1)$ which are both circles. Then $d_2(\omega) = (1, -1)$ and the E_∞ term of the spectral sequence looks like

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbb{R} & 0 & \mathbb{R} & 0 \end{array}$$

2. Consider $\mathbb{CP}^n = S^{2n+1}/S^1$ with coordinates $[z_0 : z_1 : \dots : z_n]$. Define the Morse-Bott function

$$f([z_0 : z_1 : \dots : z_n]) = |z_n|^2.$$

There is a critical point $p = [0 : 0 : \dots : 1]$ of index $2n$ and a critical submanifold $\mathbb{CP}^{n-1} = [z_0 : z_1 : \dots : z_{n-1} : 0]$ of index 0. Then the E_1 term of the spectral sequence looks like

$$\begin{array}{ccccccc} 2n-2 & \mathbb{R} & 0 & \dots & & & \\ 2n-3 & 0 & 0 & \dots & & & \\ 2n-4 & \mathbb{R} & 0 & \dots & & & \\ \vdots & \vdots & & & & & \\ 0 & \mathbb{R} & 0 & \dots & \mathbb{R} & & \\ & 0 & 1 & \dots & 2n & & \end{array}$$

Notice that all higher differentials vanish for dimension reasons. This gives a simple way to inductively compute the cohomology of complex projective spaces. By changing f to $-f$, one can easily obtain the cup product structure.

3. Let $\pi : E \rightarrow B$ be a fiber bundle with fiber F and let f be a nondegenerate Morse function on B . Pulling back to E gives a Morse-Bott function $\tilde{f} = \pi^* f$ on E . The critical submanifolds of \tilde{f} are the fibers over the critical points of f and hence all diffeomorphic to F . Moreover, the index of a fiber is exactly the index of the critical point over which it lies. The Morse-Bott complex then gives a spectral sequence whose E_2 term is the cohomology $H^*(F) \otimes H^*(B)$. The resulting spectral sequence is the Leray spectral sequence for the fibration (see [BT] and [Bo]).

As an example, consider the Hopf fibration $S^3 \rightarrow S^2$ with fibers S^1 . Let f be the height function on S^2 with two critical points, one at the north pole of index 2 and one at the south pole of index 0. Now the E_2 term of the spectral sequence given by \bar{f} is

$$\begin{array}{ccc} \mathbb{R} & 0 & \mathbb{R} \\ \mathbb{R} & 0 & \mathbb{R} \end{array}$$

The differential $d_2 : \mathbb{R} \rightarrow \mathbb{R}$ precisely measures the Euler class of the bundle. This is a general phenomena; for k -sphere bundles, the differential d_{k+1} will measure the Euler class.

4. There is one example of a fiber bundle which is relevant to our study of equivariant cohomology in the next sections. Let X be a manifold with G -action and $EG \rightarrow BG$ the classifying bundle for G . As we will see in the next section, the homotopy quotient, defined as $X_G = EG \times_G X$, forms a fiber bundle $\pi : X_G \rightarrow BG$ with fiber X . The cohomology of X_G is called the equivariant cohomology of X . If f is a Morse function on BG , then π^*f is a Morse-Bott function on X_G as in example 3 whose critical submanifolds are all diffeomorphic to X . The Morse-Bott complex then computes the equivariant cohomology of X .

For example, suppose that $G = SU(2)$. In this case, the classifying bundle is $S^\infty \rightarrow \mathbb{H}\mathbb{P}^\infty$. On $\mathbb{H}\mathbb{P}^\infty$, we may take a perfect Morse function whose critical points all have index divisible by 4. The only non-zero components of the boundary operator are ∂_0 and ∂_4 . Of course, ∂_0 is just the usual exterior derivative on $\Omega^*(X)$. The component $\partial_4 : \Omega^{j+3}(X) \rightarrow \Omega^j(X)$ is easily described. Given critical submanifolds X_{i+4} and X_i , the moduli space of gradient lines is

$$\tilde{M}(X_{i+4}, X_i) = X \times SU(2).$$

The upper endpoint map is simply projection onto X while the lower endpoint map is given by the group action: $l_i(x, g) = g \cdot x$. Then if $\omega \in \Omega^{j+3}(X)$ and M is a submanifold of X of dimension j ,

$$\int_M \partial_4(\omega) = \int_{G \cdot M} \omega$$

where $G \cdot M = \{g \cdot m | g \in G, m \in M\}$. In particular, if $\omega \in \Omega^3(X)$, then $\partial_4(\omega)(x) = \int_{G \cdot x} \omega$.

Many properties of equivariant cohomology can be understood geometrically in terms of this picture. In the next sections, we will build a different complex to compute equivariant cohomology.

4 Equivariant cohomology

Here we review the basic definitions and standard results about equivariant cohomology. This will be useful in the next section where we discuss the computation

of equivariant cohomology using critical submanifolds and gradient flows as in the previous sections.

Let G a compact Lie group and denote the universal G -fibration by $EG \rightarrow BG$. If X is a space with a G -action, the space $X_G \equiv EG \times_G X$, called the homotopy quotient, is a fiber bundle over BG with fiber X . The G -equivariant cohomology of X is defined by

$$H_G(X; \Gamma) = H^*(X_G; \Gamma).$$

Moreover, pullback by the fibration $X_G \rightarrow BG$ gives $H_G(X; \Gamma)$ the structure of a module over the ring $H^*(BG; \Gamma)$. This module structure will be investigated more fully in §6.

The following examples illustrate some properties of equivariant cohomology. First of all $H_G(\{\text{point}\}; \mathbb{Z}) = H^*(BG; \mathbb{Z})$. If the G -action on X is free, then $H_G(X; \mathbb{Z}) = H^*(X/G; \mathbb{Z})$ and the equivariant cohomology of an orbit is given by $H_G(G/K; \mathbb{Z}) = H^*(BK; \mathbb{Z})$ for a closed subgroup K .

The equivariant cohomology of a space intertwines cohomological information from the quotient with the subgroups appearing as stabilizers of points. When G acts freely on X , the module structure is easily understood.

Proposition 4.1 *Suppose G acts freely on X , then the $H^*(BG; \mathbb{Z})$ module structure of $H_G(X; \mathbb{Z}) = H^*(X/G; \mathbb{Z})$ is described as follows: if $\phi \in H^*(BG; \mathbb{Z})$ and $\omega \in H^*(X/G)$, then*

$$\phi \cdot \omega \mapsto c^*(\phi) \cup \omega,$$

where $c : X/G \rightarrow BG$ is the classifying map for the principal G -bundle $X \rightarrow X/G$.

Observe that the image of $c^* \subset H^*(X/G; \mathbb{Z})$ consists of the characteristic classes of the G -bundle $X \rightarrow X/G$.

There is a convenient model with which we will consider equivariant cohomology: the equivariant cohomology with real coefficients can be computed from a deRham like complex of differential forms. The version which we discuss is due to Cartan [C] and we refer to the excellent book by Berline, Getzler and Vergne [BGV]. Denote the Lie algebra of G by \mathfrak{g} . Define the so-called equivariant differential forms

$$\Omega_G(X) = (\Omega^*(X) \otimes S^*(\mathfrak{g}^*))^G,$$

where G acts on its Lie algebra by conjugation. Linear polynomials on \mathfrak{g} have degree two, and this turns $\Omega_G(X)$ into a bigraded complex. Pick a basis X_α of \mathfrak{g} with dual basis ϕ_α . The differential is a graded derivation defined on the generators by

$$d_G \omega = d\omega - \sum_{\alpha} \phi_\alpha i_{X_\alpha} \omega, \quad d_G \phi^\alpha = 0$$

for $\omega \in \Omega^*(X)$ and \tilde{X}_α the vector field on X induced by $X_\alpha \in \mathfrak{g}$. Cartan shows that this defines a complex and:

Proposition 4.2 $H_G(X; \mathbb{R}) = H^*(\Omega_G(X), d_G)$.

Proof. The isomorphism is induced by the Chern-Weil homomorphism which we now describe. Let $\pi : P \rightarrow M$ be a principal G -bundle. The pullback map π^* identifies differential forms on M , $\Omega^*(M)$, with *basic* differential forms on P ,

$$\Omega_{\text{basic}}^*(P) = \{\eta \in \Omega^*(P) \mid \eta \text{ is horizontal and } G\text{-invariant}\}.$$

Now consider the classifying G -bundle $\pi : EG \rightarrow BG$ with a connection whose curvature is $\Omega \in \Omega^2(EG; \mathfrak{g})$. For a G -manifold X , define a map

$$\chi : \Omega_G^*(X) \rightarrow \Omega_{\text{basic}}^*(EG \times X) : \omega \otimes \phi \mapsto \phi(\Omega) \otimes \omega.$$

Using the identification of basic forms on $EG \times X$ with forms on X_G , we have an induced map $\chi : \Omega_G^*(X) \rightarrow \Omega^*X_G$. The differential d_G was defined so that χ is a chain map. The induced map on cohomology is the isomorphism which gives the proposition. \square

When X is a point, we see the well known fact $H^*(BG; \mathbb{R}) = S^*(\mathfrak{g}^*)^G$, which elegantly demonstrates the module structure of $H_G^*(X; \mathbb{R})$.

It is important to notice that the Chern-Weil homomorphism is natural with respect to pullback under equivariant maps and integration over equivariant fibrations. Suppose $f : X \rightarrow Y$ is a G -equivariant map. Then there is an induced map $f^* : \Omega_G^*(Y) \rightarrow \Omega_G^*(X)$ by $f^*(\omega \otimes \phi) = f^*(\omega) \otimes \phi$ which commutes with the Chern-Weil homomorphism

$$\begin{array}{ccc} \Omega_G^*(Y) & \rightarrow & \Omega^*(Y_G) \\ \downarrow & & \downarrow \\ \Omega_G^*(X) & \rightarrow & \Omega^*(X_G). \end{array}$$

We digress briefly to discuss integration along the fiber of an equivariant fiber bundle. Let $E \rightarrow B$ be a G -equivariant fiber bundle with fiber F . Furthermore, let \tilde{X} denote the vector fields on E and B induced by an element of the Lie algebra \mathfrak{g} . We claim that integration along the fiber commutes with contraction by \tilde{X} ; that is,

$$\pi_*(i_{\tilde{X}}\omega) = i_{\tilde{X}}\pi_*(\omega). \quad (4.1)$$

This follows since if $\omega = \pi^*\phi f(x, t) d\text{vol}_F$, then

$$i_{\tilde{X}}(\pi^*(\phi)f(x, t)d\text{vol}_F) = \pi^*(i_{\tilde{X}}\phi)f(x, t)d\text{vol}_F + (-1)^{\deg \phi} \pi^*(\phi)f(x, t)i_{\tilde{X}}d\text{vol}_F.$$

Integrating along the fiber kills the second term while the first term integrates to $i_{\tilde{X}}\pi_*(\omega)$. There is a map $h_* : \Omega_G^*(E) \rightarrow \Omega_G^{*-n}(B)$, where n is the dimension of the fiber F , induced by

$$h_*(\omega \otimes \phi) = h_*(\omega) \otimes \phi.$$

By (4.1), h_* induces a map, integration over the fibers, on equivariant cohomology. Notice that $\chi(h_*(\omega \otimes \phi)) = \phi(\Omega) \otimes h_*(\omega) = h_*(\chi(\omega \otimes \phi))$ so that h_* commutes with the Chern-Weil homomorphism.

In the case when $\pi : E \rightarrow B$ is an oriented vector bundle over a compact manifold of rank k , the Thom isomorphism $H_c^{*+k}(E; \mathbb{R}) \rightarrow H^*(B; \mathbb{R})$ can be realized by integration over the fiber — that is, $\omega \rightarrow \pi_*(\omega)$. An inverse is constructed using the Thom class $u \in H_c^k(E)$ to define $\omega \rightarrow u \wedge \pi^*\omega$. The class u is characterized by the fact that it is closed and has integral 1 over each fiber of π . Beautiful representatives for the Thom class were found by Matthai and Quillen [MQ] (see also [BGV]) in which the Thom form is acquired by substitution of the curvature of a connection on E into a standard formula.

The Thom isomorphism also holds in the more general setting of equivariant cohomology. Let $E \rightarrow B$ be an oriented G -equivariant vector bundle of rank k . We define $\Omega_{G,c}^*(E) = (\Omega_c^*(E) \otimes S^*(\mathfrak{g}^*))^G$ and a map $\pi_{G,*} : \Omega_{G,c}^{*+k}(E) \rightarrow \Omega_G^*(B)$ by $\pi_{G,*}(\omega \otimes \phi) = \pi_*(\omega) \otimes \phi$. As before, this induces a map on equivariant cohomology $H_{G,c}^{*+k}(E) \rightarrow H_G^*(B)$. Constructing an equivariant Thom form will give the inverse map as above. Let $P \rightarrow B$ be the G -equivariant principal K -bundle associated to E and choose a G -invariant connection A . The equivariant curvature $F_G^A \in \Omega_G^2(P, \text{Lie}(K))$ is defined as follows (see [BGV]). Fixing a point $p \in P$, an element $\psi_p \in \mathfrak{g}^* \otimes \text{Lie}(K)$ can be produced by assigning to a vector $X \in \mathfrak{g}$ the vector

$$A_p(\tilde{X}) \in \text{Lie}(K)$$

where \tilde{X} is the infinitesimal action of X on P . The equivariant curvature is now:

$$F_G^A = F^A - \psi.$$

The usual substitution into the formula of [MQ] produces a Thom form and proves:

Proposition 4.3 (The Thom Isomorphism) *Given a G -equivariant vector bundle $E \rightarrow B$ of rank k , integration over the fiber defines an isomorphism $H_{G,c}^{*+k}(E; \mathbb{R}) \rightarrow H_G^*(B; \mathbb{R})$.*

5 Equivariant cohomology and Morse-Bott theory

Many examples of Morse-Bott functions arise as invariant functions on a manifold with group action. Here the critical point sets are invariant under the group action and so must necessarily contain orbits. The theory in §3 tells us how to build a complex which computes the deRham cohomology of the manifold. It is but a slight generalization, using equivariant differential forms rather than ordinary differential forms, to compute the equivariant cohomology of the manifold. We take up this question in this section whose outline closely follows that of §3.

5.1 The equivariant Morse complex

Let X be a compact G -manifold. An interesting theorem of Wasserman [Wa] asserts that the G -invariant functions on X with isolated non-degenerate critical orbits form a Baire set. Let $f : X \rightarrow \mathbb{R}$ be a G -invariant function with nondegenerate critical orbits. The critical point set of index i of such a function is of the form

$$S_i = \cup_{\alpha} G/K_{\alpha}^i$$

for certain subgroups K_{α}^i which are the stabilizers of these critical orbits. Following the notation of §3, $\mathcal{M}(S_i, S_j)$ denotes the set of points connected by a gradient line to S_i and S_j , $\tilde{\mathcal{M}}(S_i, S_j) = \mathcal{M}(S_i, S_j)/\mathbb{R}$, and the lower (upper) endpoint maps are l_j^i (u_j^i) : $\mathcal{M}(S_i, S_j) \rightarrow S_j$ (S_i). Notice that these maps are equivariant with respect to the G -action.

We shall now assume that the stable and unstable manifolds intersect transversally. This is not an assumption which will be satisfied generically for equivariant non-degenerate functions (see [P]), but it is often satisfied by interesting group actions. For instance, if the complement of the critical orbits contains only one orbit type, then the usual obstructions to equivariant transversality disappear and we can make this assumption. Observe that due to equivariance, $\mathcal{M}(S_i, S_j) = \emptyset$ for $j > i$, i.e. our weakly self-indexing assumption is implied by the transversality assumption. Similarly the assumption that the endpoint maps induce fibrations is an immediate consequence of the presence of a transitive G -action on the components of the critical point set.

We will form a complex which computes the equivariant cohomology of X , again using equivariant forms on the critical submanifolds. Using the Cartan model for equivariant cohomology, we define a filtered complex (C^*, ∂_G) with

$$C^p = \oplus_{i+l=p} \Omega_G^l(S_i) = \oplus_{i+j+2k=p} (\Omega^j(S_i) \otimes S^k(\mathfrak{g}^*))^G.$$

The filtration is given by

$$C_n^p = \oplus_{i+l=p, i \geq n} \Omega_G^l(S_i)$$

so that

$$C^p = C_0^p \subset C_1^p \subset \dots \subset C_n^p \subset \dots$$

As before, the boundary operator $\partial_G : C_n^p \rightarrow C_n^{p+1}$ splits into $\partial_G = \sum_k (\partial_G)_k$ where $(\partial_G)_k : C_n^p \rightarrow C_{n+k}^{p-k+1}$ is a map of $S^*(\mathfrak{g}^*)^G$ modules as follows: for $\omega \otimes \phi \in (\Omega^j(S_i) \otimes S^j(\mathfrak{g}^*))^G$, let $(\partial_G)_0(\omega \otimes \phi) = d_G(\omega \otimes \phi)$ and for $k > 0$, let

$$(\partial_G)_k(\omega \otimes \phi) = \partial_k \omega \otimes \phi.$$

The following theorem shows that the filtered complex introduced above is in fact a complex.

Theorem 5.1 $\partial_G \circ \partial_G = 0$.

Proof. The proof closely follows that of Theorem 3.5. Again we must show that

$$\sum_l (\partial_G)_l (\partial_G)_{k-l} = 0$$

for all fixed k . The case that $k = 0$ is trivial so we concentrate on $k > 0$. From our definitions,

$$\begin{aligned} \sum_l (\partial_G)_l (\partial_G)_{k-l} (\omega \otimes \phi) &= \sum_l (\partial_l \partial_{k-l} \omega) \otimes \phi \\ &\quad - \sum i_{\tilde{X}_\alpha} \partial_k \omega \otimes \phi \phi^\alpha - \sum \partial_k i_{\tilde{X}_\alpha} \omega \otimes \phi \phi^\alpha. \end{aligned}$$

As the first sum vanishes by Proposition 3.5, our investigation focuses on the remaining terms. The theorem is proven if we have $i_{\tilde{X}_\alpha} \partial_k \omega + \partial_k i_{\tilde{X}_\alpha} \omega = 0$ which is equivalent to

$$i_{\tilde{X}_\alpha} (u_{i+k})_* l_i^*(\omega) = (u_{i+k})_* l_i^* i_{\tilde{X}_\alpha}(\omega).$$

This is true precisely because integration along the fiber and pullback commute with contraction by \tilde{X} for equivariant maps. \square

Remark 5.2 If \mathcal{I} is an ideal in $S^*(\mathfrak{g})^G$, then we can also consider the complex $(C^*/\mathcal{I}, \partial_G)$. This has a variety of interesting applications, one of which we shall discuss below.

Still following the outline of §3, we now show that the cohomology of our filtered complex is the equivariant cohomology. Again we filter the equivariant forms on X . Because the transversality assumption is satisfied, there is no loss of generality in assuming the invariant function to be strictly self-indexing (that is, $f(S_i = i)$), and so let $X_n = f^{-1}((n - \frac{1}{2}, \infty))$. Notice that X_n is a G -space since f is G -invariant. Define

$$C'^* = \Omega_G^*(X) = (\Omega^*(X) \otimes S^*(\mathfrak{g}^*))^G$$

with the filtration

$$C'_n{}^* = \Omega_{G,c}^*(X_n).$$

With the usual boundary operator d_G , (C'^*, d_G) forms a filtered complex.

We define a map of filtered complexes

$$\Psi : (C'^*, d_G) \rightarrow (C^*, \partial_G) : \Psi(\omega \otimes \phi) \mapsto \Phi(\omega) \otimes \phi.$$

Recall the chain morphism $\Phi = \oplus \Phi_k$ where $\Phi_k : \Omega^p(X) \rightarrow \Omega^{p-k}(S_k) : \omega \mapsto (u_k)_*(\omega|_{u_k})$, integration along the unstable manifold of S_k . To show that Ψ is a map of complexes, we must show $\Psi \circ d_G = \partial_G \circ \Psi$. Applying Lemma 3.6, this is equivalent to

$$\Phi_k(i_{\tilde{X}} \omega) = i_{\tilde{X}} \Phi_k(\omega)$$

which follows easily from the fact that contraction by \tilde{X} commutes with integration along the fiber of an equivariant bundle. Notice that Ψ is a map of $S^*(\mathfrak{g}^*)$ modules.

In analogy with Theorem 3.1, the main theorem of this section is

Theorem 5.3 *The map Ψ induces an isomorphism of filtered complexes. In particular, $\Psi : H_G^*(X; \mathbb{C}) \rightarrow H^*(C^*, \partial_G)$ is an isomorphism of $S^*(\mathfrak{g}^*)$ modules.*

Proof. The proof closely follows that of Theorem 3.8. We first consider the E_1 terms of the spectral sequences induced by the two filtered complexes. As before, the associated graded complexes are

$$GC_k^p = \Omega_G^{p-k}(S_k) \quad \text{and} \quad GC_k'^p = \Omega_{G,c}^p(X_k) / \Omega_{G,c}^p(X_{k+1})$$

and the E_1 terms give the cohomology of the associated graded complexes. Hence Ψ induces a map on the E_1 term, $H^p(GC_k') \rightarrow H^p(GC_k) = H_G^p(S_k)$.

Recalling the notation of §3, we define $F_k = X_k \cap \mathcal{U}_k$ as a neighborhood of S_k in the unstable manifold \mathcal{U}_k . Again, Ψ factors through the restriction of forms and integration over the fibration $F_k \rightarrow S_k$. The same argument as in the proof of Theorem 3.8 shows that restriction of equivariant forms gives an isomorphism $H^p(GC_k') = H_{G,c}^p(F_k)$. Furthermore, the Thom isomorphism holds for equivariant cohomology so that $\Psi : H^p(GC_k') \rightarrow H^p(GC_k) = H_G^{p-k}(S_k)$ is an isomorphism of real vector spaces. Then Ψ induces an isomorphism on the E_1 terms of the spectral sequences and hence by Lemma 3.7, Ψ induces an isomorphism of $S^*(\mathfrak{g}^*)$ -modules in cohomology. \square

As in §3, the filtered complex gives rise to a spectral sequence whose E_1 term equals

$$H_G^*(S_i; \mathbb{C}) = \oplus_{\alpha} H^*(BK_{\alpha}^i; \mathbb{C}).$$

Since we are assuming that the manifold X is finite dimensional, convergence of the spectral sequence is guaranteed. However, in infinite dimensional applications, this may not hold (see [Au] and [AuB]).

As always, the existence of the complex leads to equivariant Morse inequalities and a notion of perfection.

Corollary 5.4 *There is a polynomial $Q(t)$ with non-negative coefficients such that*

$$\sum_{i,j} \dim H_G^j(S_i; \mathbb{R}) t^{i+j} = \sum_k \dim H_G^k(B; \mathbb{R}) t^k + (1+t)Q(t).$$

The function is called *equivariantly perfect* if $Q(t) = 0$. The Yang-Mills function on the space of connections over a Riemann surface is equivariantly perfect (see [AB2]), and more generally, the norm squared of the moment map for a algebraic group action is perfect, Kirwan [K].

At this point it would be logical to insert a discussion which shows that the cohomology of the equivariant Morse complex is independent of metric and function. This issue is of importance in equivariant Floer cohomology and will be taken up in [AuB].

5.2 Comparison with the Morse-Bott complex

The invariant function f on X defines a Morse-Bott function on the homotopy quotient. Thus, we have two complexes which compute equivariant cohomology: the equivariant complex above and the Morse-Bott complex on the homotopy quotient. We will use the Chern-Weil homomorphism to construct a chain map between the two complexes which induces an isomorphism on cohomology.

Without too much work, it would be possible to apply our Morse-Bott theory directly to the infinite dimensional homotopy quotient. Instead, we choose to work with finite dimensional approximations as follows. Let $\{EG_n \rightarrow BG_n\}$ be a sequence of finite dimensional principal G -bundles approximating $EG \rightarrow BG$; that is, the homotopy groups $\pi_i(EG_n) = 0$ for $i < n$ and EG_{n+1} is constructed from EG_n by adding cells of sufficiently large dimension. Then we may form the approximate homotopy quotients to X , $X_{G,n} = EG_n \times_G X$. It is a standard fact that for a fixed j , $H_G^j(X) = H^j(X_{G,n})$ for all $n \gg 0$. For example, when $G = SU(2)$, we may take $S^{4n+3} \rightarrow \mathbb{H}\mathbb{P}^n$ as an approximating sequence. Define the ideal $\mathcal{I}_n = \ker\{H^*(BG; \mathbb{C}) \rightarrow H^*(BG_n; \mathbb{C})\}$.

Now a G -invariant function on X with non-degenerate critical orbits gives a Morse-Bott function on $X_{G,n}$. Hence the method of §3 is applicable to the computation of $H^*(X_{G,n}; \mathbb{R})$ and thus $H_G^*(X)$. This produces a Morse-Bott complex which we denote by (C_n^*, ∂') . We will show that there is a natural chain map $\chi: (C^*, \partial_G) \rightarrow (C_n^*, \partial')$.

Notice that the critical submanifolds of index i on $X_{G,n}$ are precisely $(S_i)_{G,n}$ while the gradient line moduli spaces are $\tilde{\mathcal{M}}((S_i)_{G,n}, (S_j)_{G,n}) = \tilde{\mathcal{M}}(S_i, S_j)_{G,n}$. The Chern-Weil homomorphism (see §4) defines

$$\chi: \Omega_G^*(S_i) \rightarrow \Omega^*((S_i)_{G,n}).$$

To check that this defines a chain map, one must show that

$$\chi((\partial_G)_k(\omega \otimes \phi)) = \partial'_k \chi(\omega \otimes \phi).$$

For $k = 0$, this is just the fact that the Chern-Weil homomorphism induces a chain map between equivariant differential forms and forms on the homotopy quotient. For $k > 0$, this follows from the fact that the Chern-Weil homomorphism commutes with pullback and integration over equivariant fiber bundles.

Now note that we also have a chain morphism:

$$\chi: C^*/\mathcal{I}_n \rightarrow C_n^*/\chi(\mathcal{I}_n).$$

and that the cohomology of $C_n^*/\chi(\mathcal{I}_n)$ equals that of C_n^* , because $\chi(\mathcal{I}_n) \subset \text{im } \partial'$. This gives:

Proposition 5.5 *The cohomology groups $H^*(C^*/\mathcal{I}_n, \partial_G)$ and $H^*(X_{G,n}; \mathbb{C})$ are isomorphic through χ .*

Proof. As always, both complexes are filtered by index and the chain map χ is a map of filtered complexes. By Proposition 4.2, χ induces an isomorphism on the E_1 terms and hence by Lemma 3.7, χ is an isomorphism on homology. \square

Notice that \mathcal{F}_n is zero in degrees small compared to n . It is now quite clear that using equivariant differential forms is an effective way to deal only with low dimensional moduli spaces of gradient lines. This is important in gauge theoretic applications where higher dimensional spaces may exhibit new forms of non-compactness (bubbling-off).

5.3 Cup products

The cup product structure on $H_G^*(X)$ can be recovered as in §3. Given $\omega \otimes \phi \in \Omega_G^*(X)$, define

$$c_G(\omega \otimes \phi) : C^l \rightarrow C^{l+k} : (\eta_i \otimes \psi_i) \mapsto c(\omega)(\eta_i) \otimes \phi \psi_i.$$

Again, we have

Theorem 5.6 *The map c_G induces a map on cohomology*

$$c_G : H_G^*(X) \otimes H^*(C^*, \partial_G) \rightarrow H^*(C^*, \partial_G)$$

which agrees with the cup product map.

The analog to the Poincaré duality pairing in equivariant cohomology is integration over the fiber $X_G \rightarrow BG$, assuming that X is oriented and closed:

$$H_G^k(X) \otimes H_G^l(X) \rightarrow H^{l+k-n}(BG),$$

where n is the dimension of X . In terms of the equivariant complex, this is given by reversing the flow (or considering the invariant function $-f$) to obtain an equivariant complex $(\overline{C}^*, \overline{\partial}_G)$.

Corollary 5.7 *The pairing $H_G^k(X) \otimes H_G^l(X) \rightarrow H^{l+k-n}(BG)$ is described by the equivariant complexes as a map*

$$C^* \otimes \overline{C}^* \rightarrow S^*(\mathfrak{g}^*)$$

given by $(\omega_i \otimes \phi_i) \otimes (\eta_j \otimes \psi_j) \mapsto (\int_{S_j} \omega_i \wedge \eta_j) \phi_i \psi_j$.

5.4 Examples

1. The simplest non-trivial example of equivariant cohomology is $X = S^2$ where S^1 acts by equatorial rotations. It is well known that

$$H_{S^1}^*(S^2; \mathbb{R}) = \mathbb{R}[u] \oplus \mathbb{R}[v]/(c, -c)$$

where $(c, -c)$ is the subgroup of constant polynomials. Consider S^2 as the unit sphere in \mathbb{R}^3 with coordinates (x, y, z) . We will study the following three S^1 -invariant functions:

- 1) $f(x, y, z) = z$ with critical points of index 0 at the south pole and of index 2 at the north pole. Then $(\partial_G)_l = 0$ for $l > 0$. Then $E_\infty^{i,j} = E_1^{i,j} = \mathbb{R}$ if $i = 0, 2$ and j is even and is 0 otherwise.
- 2) $f(x, y, z) = 1 - z^2$ takes on minima at the poles and a maximum at the equator. Then

$$E_1^{i,j} = \begin{cases} \mathbb{R} & \text{if } i = 1, j = 0 \\ \mathbb{R}^2 & \text{if } i = 0, j \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

The differential $d_1(a, b) = (\partial_G)_1(a, b) = a - b$ by counting gradient lines.

- 3) $f(x, y, z) = z^2$ takes on maxima at the poles and a minimum at the equator. Then

$$E_1^{i,j} = E_\infty^{i,j} = \begin{cases} \mathbb{R} & \text{if } i = 0, j = 0 \\ \mathbb{R}^2 & \text{if } i = 2, j \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

2. Let (B, ω) be a symplectic manifold with an S^1 action preserving the symplectic form. A moment map is a function $\mu : B \rightarrow \mathbb{R}$ such that

$$d\mu = i \frac{\partial}{\partial \theta} \omega$$

where $\frac{\partial}{\partial \theta}$ is the vector field on B generating the S^1 action. In this example, we will consider μ as a Morse function. Since ω is a nondegenerate 2-form, the critical point set of μ equals the fixed point set of the S^1 action; that is

$$\{b \in B \mid d\mu_b = 0\} = B^{S^1}.$$

It is well known that the critical point set is nondegenerate in the sense of Bott. The fibers of normal bundle of the fixed point set are S^1 representations with no trivial factors; hence the critical point sets are of even index. Denote the components of index i by $B_i^{S^1}$. We will furthermore suppose that the gradient flow is Morse-Smale so that our theory applies.

For this example, $(\partial_G)_k = 0$ for $k > 0$ for the following reason. The moduli spaces $\tilde{\mathcal{M}}(B_{j+k}^{S^1}, B_j^{S^1})$ have an S^1 action commuting with the endpoint maps. However, the S^1 action fixes the endpoints. Then for $\eta \in \Omega^*(B_j^{S^1})$,

$$i \frac{\partial}{\partial \theta} (l_j^{j+k})^* \eta = (l_j^{j+k})^* i \frac{\partial}{\partial \theta} \eta = 0.$$

However, $\frac{\partial}{\partial \theta}$ is tangent of the fiber of $u_j^{j+k} : \tilde{\mathcal{M}}(B_{j+k}^{S^1}, B_j^{S^1}) \rightarrow B_i^{S^1}$ and hence

$$\partial_k \eta = (u_j^{j+i})_* (l_j^{i+j})^* \eta = 0.$$

Then $\partial_G = (\partial_G)_0$ which implies that μ is equivariantly perfect; that is,

$$H_{S^1}^*(B; \mathbb{R}) = \oplus_i H_{S^1}^{*-i}(B_i^{S^1}; \mathbb{R}).$$

This is a special case of a general theory developed by Atiyah-Bott [AB2] and Kirwan [K] in which it is seen that for reductive groups acting algebraically and Kählerian on projective varieties, the length of the moment map squared is a perfect 'Morse' function; it is known in general if the function is non-degenerate, but techniques with stratifications establish perfection. Ginzburg [G] has similar results for a torus action on a Kähler manifold.

3. We will consider the case when $G = SU(2)$. Additionally, we impose the constraint that any stabilizer K is the centralizer of a subgroup of $SU(2)$ and that the action is of a single orbit type away from the critical points, equal to $SU(2)/(\mathbb{Z}/2)$. In the sequel [AuB], we are interested in precisely this situation when studying the $SU(2)$ -action on the space of based $SU(2)$ -connections over a 3-manifold. We will see that the possible orbit types and the possible nonzero boundary operators are quite restricted.

There are only three orbit types possible:

- 1) For $K = \mathbb{Z}/2$, the orbits are $SO(3)$'s and called irreducible.
- 2) For $K = U(1)$, the orbits are S^2 's and called $U(1)$ -orbits.
- 3) For $K = SU(2)$, the orbits are points and called $\mathbb{Z}/2$ -orbits.

The terminology is borrowed from the situation with connections on principal $SU(2)$ -bundles. Notice that any irreducible critical orbit contributes $H^*(B\mathbb{Z}_2; \mathbb{R}) = H^*(pt; \mathbb{R})$ to the E_1 -term of the spectral sequence, a critical $U(1)$ -orbit contributes a $H^*(\mathbb{C}P^\infty; \mathbb{R})$ and a critical $\mathbb{Z}/2$ -orbit a $H^*(BSU(2); \mathbb{R}) = H^*(\mathbb{H}P^\infty; \mathbb{R})$.

For dimension reasons, many differentials vanish. For example, the differential d_1 is a count of gradient lines. The second differential d_2 always vanishes since all of the cohomology is in even vertical dimensions. For $k = 3$, only $d_3 : H^2(\mathbb{C}P^\infty; \mathbb{R}) \rightarrow H^0(\mathbb{R}P^\infty; \mathbb{R})$ is nonzero. Furthermore, $d_4 \equiv 0$.

In the next section, we will consider this example further in relation to the $H^*(BG)$ -module structure.

6 The $H^*(BG)$ module structure

The G -equivariant cohomology of a G -space X has a rich algebraic structure: $H_G^*(X)$ appears as an $H^*(BG)$ -module. In this section, we will investigate this module structure in terms of the complex introduced in §5 paying special attention

to the case $G = SU(2)$. This material explains the «u»-map on Floer homology and is relevant to the equivariant Floer cohomology developed in [AuB].

First, we say a few words about the origins of this structure and its expression in the Cartan model. Recall that the homotopy quotient X_G is a fibration over the classifying space BG , $\pi : X_G \rightarrow BG$. Given an element of $u \in H^*(BG)$, define $u : H_G^*(X) \rightarrow H_G^*(X)$ by cupping with $\pi^*(u)$. Using the Cartan model for equivariant cohomology and identifying $H^*(BG) = (S^*(\mathfrak{g}^*))^G$, the map $u : H_G^*(X) \rightarrow H_G^*(X)$ is described by multiplying a G -equivariant form on X with an invariant symmetric polynomial; that is,

$$u(\omega \otimes \phi) = \omega \otimes u\phi.$$

In the case that X admits a G -invariant Morse-Bott function with nondegenerate isolated critical orbits, we have the complex described in §5; namely,

$$C^p = \oplus_{i+l=p} \Omega_G^l(S_i) = \oplus_{i+j+2k=p} (\Omega^j(S_i) \otimes S^k(\mathfrak{g}^*))^G.$$

Furthermore, we have the chain map $\Psi : \Omega_G^*(X) \rightarrow C^*$, given by integrating forms along the unstable manifolds, which induces an isomorphism on cohomology. As Ψ does not act on the polynomial part of an equivariant form, we see that the module structure on $H_G^*(X)$ in terms of the equivariant complex is simply

$$u(\omega \otimes \phi) = \omega \otimes u\phi$$

for $\omega \otimes \phi \in \Omega_G^*(S_i)$. That is, in the array of cochains $\Omega_G^j(S_i)$, multiplying by elements of $H^*(BG)$ translates an element vertically.

It is interesting to investigate the situation when $G = SU(2)$ as in the final example from §5; that is, the stabilizers which occur are always the centralizer of some subgroup and the stabilizer away from the critical orbits is exactly $\mathbb{Z}/2$. For instance, let $S_i^k = SO(3)$ be a critical orbit of index i and with $\mathbb{Z}/2$ stabilizer. Choose a cohomology class represented by $\alpha \in \Omega_G^0(S_i^k)$ so that $\partial_G(\alpha) = 0$. Let $u \in H^4(BSU(2))$ be the universal second Chern class. We wish to study the cohomology class $u(\alpha)$. Since $H_G^*(S_i^k) = H^*(pt)$, it follows that $u(\alpha) = (\partial_G)_0(\beta) = d_G(\beta)$ for some $\beta \in \Omega_G^3(S_i^k)$. Then in cohomology, $u(\alpha)$ is represented by $\sum_{k=1}^4 (\partial_G)_k(\beta)$. We will compute these components of the boundary operator and see that they contain interesting information about the configuration of gradient lines. Notice that this discussion only applies to orbits with trivial stabilizer.

Choosing an orthonormal basis $\{\phi_i\}$ for \mathfrak{g}^* , u is represented as an element of $S^*(\mathfrak{g}^*)$ by $\sum \phi_i^2$. Let X_i be the left-invariant vector fields on S_i^k generated by ϕ_i^* , and ω_i left-invariant 1-forms dual to X_i . Denoting the G -equivariant volume form on S_i^k by $dvol_{SO(3)}$, one checks that defining $\gamma = dvol_{SO(3)} + \sum \omega_i \otimes \phi_i$ leads to $d_G(\gamma) = \sum \phi_i^2$. Hence, $\beta = \alpha \wedge \gamma$. We will discuss the components of

$\partial_G(\beta)$. As α is simply a constant function, it is insignificant in the computations that follow.

First, we study the component $(\partial_G)_2(\gamma) \in \Omega_G^2(S_{i+2})$. Let S_{i+2}^l be a component of S_{i+2} . Notice that if the stabilizer of S_{i+2}^l is not $U(1)$, the $(\partial_G)_0$ cohomology vanishes in dimension 2. We then assume that S_{i+2}^l is stabilized by $U(1)$ and hence $S_{i+2}^l = S^2$. For convenience, assume that $\tilde{\mathcal{M}}(S_{i+2}^l, S_i^k)$ is compact. Then $\tilde{\mathcal{M}}(S_{i+2}^l, S_i^k) = \bigcup SO(3)$ and $\tilde{\mathcal{M}}(S_{i+2}^l, S_i^k)/SO(3)$ is a collection of oriented points. Denote the algebraic count of these points by n . From the Gysin sequence, we see that integrating over the fiber in $SO(3) \rightarrow S^2$ takes the G -equivariant volume form on $SO(3)$ to the G -equivariant volume form on S^2 ; that is $\partial_2(dvol_{SO(3)}) = n dvol_{S^2}$. Furthermore, notice that the infinitesimal action of ϕ_i^* on $S_{i+2}^l = S^2$ defines two poles and hence height functions h_i taking the values ± 1 on the poles. One checks that $\partial_2(\omega_i) = nh_i$. Then the component of $(\partial_G)_2(\gamma)$ in $\Omega_G^2(S_{i+2}^l)$ is $n(dvol_{S^2} + \sum h_i \otimes \phi_i)$. Notice that $dvol_{S^2} + \sum h_i \otimes \phi_i$ generates $H_G^2(S_{i+2}^l)$.

Consider now the component $(\partial_G)_4(\gamma) \in \Omega_G^0(S_{i+4})$. By counting dimensions, we see that $(\partial_G)_4(\gamma) = \partial_4(dvol_{SO(3)})$. For convenience, we will assume that $\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k)$ is compact in the sequel. Suppose the stabilizer of S_{i+4}^l is:

- 1) $SU(2)$. As above, $\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k) = \bigcup SO(3)$ so again let n denote the algebraic count of points in the quotient $\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k)/SO(3)$. The component of $(\partial_G)_4(\gamma)$ in $\Omega_G^0(S_{i+4}^l)$ is n .
- 2) $U(1)$. Then $\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k)$ is a principal $SO(3)$ bundle over a compact surface Σ . In fact, a fiber of the lower endpoint map trivializes this bundle so that $\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k) = \Sigma \times SO(3)$. Meanwhile, a fiber of the upper endpoint map has the form, $u_{i+4}^{-1}(pt) = \Sigma \times S^1$ and as such the component of $\partial_G(\gamma)$ in $\Omega_G^0(S_{i+4}^l)$ must vanish.
- 3) $\mathbb{Z}/2$. Thinking of $\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k)$ as a 6-dimensional submanifold of X , form

$$P = \bigcup_{-\infty \leq t \leq \infty} \phi_t(\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k))$$

where ϕ_t is the flow of the gradient vector field of our Bott-Morse function. Then $P \rightarrow P/SO(3)$ is a principal $SO(3)$ fibration. We claim that the component of $(\partial_G)_4(\gamma)$ in $\Omega_G^0(S_{i+4}^l)$ is related to the first Pontrjagin number $p_1(P)[P/SO(3)]$. Over $\bigcup_{-\infty \leq t \leq 0} \phi_t(\tilde{\mathcal{M}}(S_{i+4}^l, S_i^k))/SO(3)$, a fiber of the upper endpoint map $u_{i+4}^{-1}(pt)$ provides a trivialization. Likewise a fiber of the lower endpoint map gives a trivialization over the rest. The lower endpoint map on $u_{i+4}^{-1}(pt)$ gives the transition function and hence $\partial_4(dvol_{SO(3)})$ measures its degree. The Pontrjagin number is 4 times this degree. In [AuB], we will show that this explains « u »-map on Floer homology as arising from the

$H^*(BSO(3))$ -module structure on the equivariant cohomology of the space of based connections.

To summarize,

Proposition 6.1 *When $\alpha \in \Omega_G^*(S_i^k)$ with S_i^k is an irreducible orbit, $u(\alpha)$ measures the number of gradient lines to $U(1)$ -orbits of index $i+2$ and $\mathbb{Z}/2$ -orbits of index $i+4$ as well as the Pontrjagin number of the framed moduli space connecting irreducibles of index $i+4$. For a cohomology class represented by $\alpha \in \Omega_G^*(S_i^k)$ with $S_i^k = pt$ or S^2 , the module structure simply carries α vertically.*

Appendix A

Asymptotics of gradient line moduli spaces

The methods of this paper depend in a crucial way on a detailed understanding of the ends of the moduli spaces of gradient lines. In this appendix, we will discuss the techniques necessary for a careful examination of these spaces as required in the body of the paper. We begin by assuming that the critical points of the flow are isolated before attacking the more general case of manifolds of critical points. Furthermore, these methods enable us to prove a local stable manifold theorem for the flow of the Chern-Simons functional about a nondegenerate critical point.

In this introduction, we explain a special case of the motivating boundary value problem. Our first concern is the structure of solution curves near nondegenerate critical points of vector fields. To begin, consider the special case of the linear vector field on $\mathbb{R}^k \oplus \mathbb{R}^l$ given by $v(x, y) = (-x, y)$. Trajectories of this vector field are solutions to the differential equation

$$\frac{d}{dt} \begin{pmatrix} x^s(t) \\ x^u(t) \end{pmatrix} (t) = \begin{pmatrix} -x^s(t) \\ x^u(t) \end{pmatrix}.$$

For all $T > 0, p \in \mathbb{R}^k, q \in \mathbb{R}^l$ with $\|p\| = \|q\| = 1$, we have a unique solution curve

$$[-T, T] \rightarrow \mathbb{R}^k \oplus \mathbb{R}^l : t \rightarrow (x^s(t), x^u(t))$$

such that

$$\begin{aligned} x^s(-T) &= p \\ x^u(+T) &= q. \end{aligned}$$

This solution curve is explicitly given by $t \rightarrow (e^{-(T+t)}p, e^{-(T-t)}q)$.

We call the set of points $(x, 0)$ the stable manifold since the trajectories through these points are bounded. Likewise, the points $(0, y)$ comprise the unstable manifold. Notice that as $T \rightarrow \infty$, the trajectory described above approaches a trajectory in the stable manifold on $[-T, 0]$ and a trajectory in the unstable manifold on $[0, T]$.

To change our perspective, we can consider this family of trajectories as the result of *gluing* trajectories in the stable and unstable manifolds in the following sense. Let $a(t)$ be a trajectory in the stable manifold, $b(t)$ in the unstable manifold. As $t \rightarrow \pm\infty$ we get the graphs in figure 5 for the length of the vector field $\|v(a(t))\|$ and $\|v(b(t))\|$.

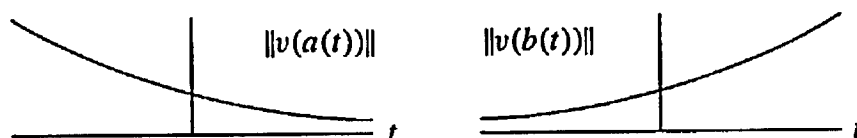


Fig. 5

We now assert that if we chop off the trajectories at $t_0 > 0, t_1 < 0$, respectively, then we can find a unique trajectory $c(t)$ such that figure 6 results.

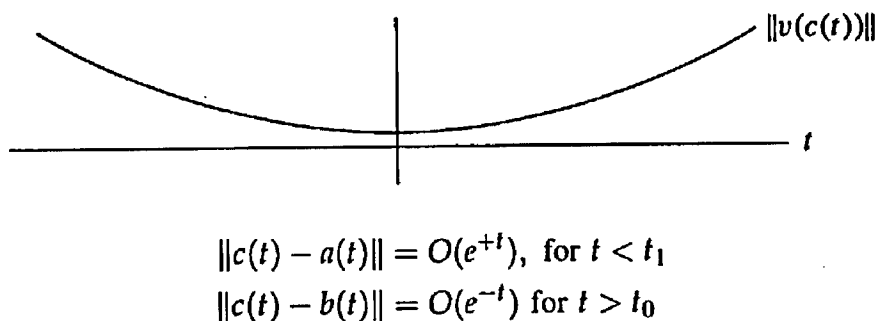


Fig. 6

We have glued the trajectory $a(t)$ to $b(t)$. It is this sort of graphical picture that appears in Floer's work [F1] on trajectories for the anti-self-duality equation. This property was recognized in the context of gradient like systems under the name of transitivity (Smale [Sm2]). In fact it holds in considerable generality as we shall see.

A.1 Gluing trajectories near isolated critical points

Consider a bundle of Banach spaces

$$\pi : E \rightarrow \mathbb{R}$$

with coordinates x in the fiber and t along the base and a «linear» vector field $(x, t) \rightarrow v(x, t)$ such that

$$\begin{aligned} \pi_*(v(x, t)) &= \frac{\partial}{\partial t} \\ v(0, t) &= 0 \end{aligned}$$

for $t \in \mathbb{R}$, $x \in E_t = \pi^{-1}(t)$. We assume that the flow of v is defined by linear maps $\Phi(t, \tau) : E_\tau \rightarrow E_t$ satisfying the following hyperbolicity assumptions. The bundle E is a direct sum:

$$E = E^s \oplus E^u$$

of Banach subbundles and $\Phi(t, \tau)$ preserves the splitting. Furthermore, there are constants $C > 0, \rho > 0$ such that

$$\|\Phi(t, \tau)x^s\| \leq Ce^{-\rho(t-\tau)}\|x^s\| \quad \text{for } t \geq \tau \text{ and } x^s \in E_\tau^s$$

$$\|\Phi(t, \tau)x^u\| \leq Ce^{\rho(t-\tau)}\|x^u\| \quad \text{for } t \leq \tau \text{ and } x^u \in E_\tau^u.$$

We also assume that the projections

$$\pi_t^i : E_t \rightarrow E_t^i \quad \text{for } i = s, u$$

along E_t^u and E_t^s have bounded norm

$$\|\pi_t^i\| \leq A$$

uniformly in t .

This is a general situation: it is easy to check that the flow of the linear part of a vector field in the neighborhood of a hyperbolic critical point can easily be cast in this form. Furthermore, nondegenerate periodic orbits and more generally orbits in a hyperbolic set satisfy such estimates.

Let g be a vector field on E , to be considered as a vertical perturbation of v , such that

$$\pi_*(g(x, t)) = 0.$$

We shall be interested in the flow of

$$w(x, t) = v(x, t) + g(x, t).$$

Before stating our gluing theorem, let us first recall how the stable manifold theory applies in this situation (see [Du]).

Theorem A.1 (i) *Assume that*

$$\|g(0, t)\| \leq \alpha,$$

$$\|D_x g(x, t)\| \leq \beta$$

for all $(x, t) \in E$. If $\gamma\beta < 1$ with $\gamma = \frac{A}{\rho}(C + \frac{1}{C})$, then for each

$$a \in E_0^s,$$

there is a unique $\xi(a) \in E_0$ with, $\pi^s(\xi(a)) = a$, such that the trajectory $(x(t), t)$ satisfying

$$\begin{cases} \frac{d(x(t), t)}{dt} = w(x, t) \\ x(0) = \xi(a) \end{cases}$$

is bounded for all t . It satisfies the a priori bound:

$$\|x(t)\|_t \leq (1 - \gamma\beta)^{-1}(\|a\| + \gamma\alpha)$$

for all $t \geq 0$. Further the assignment $a \rightarrow \xi(a)$ is as smooth as $(x, t) \rightarrow g(x, t)$.

(ii) Assume that

$$\|g(0, t)\| = 0 \text{ and } \|D_x g(x, t)\| < \beta$$

and that for ρ' such that $0 < \rho' < \rho$, we have

$$\beta\gamma' < 1 \text{ with } \gamma' = \frac{A}{(\rho - \rho')} \left(C + \frac{1}{C}\right).$$

Then the bounded solution above satisfies:

$$\|x(t)\|_t \leq (1 - \gamma\beta)^{-1} C e^{-\rho' t} \|a\|.$$

This theorem can be applied *locally* to vector fields $w(x, t) = v(x, t) + h(x, t)$ when $h(x, t)$ is quadratic in x and satisfies an estimate

$$\|D_x h(x, t)\| \leq B\|x\|$$

for all t . One shows easily that if $\psi_\delta(x, t)$ is a cutoff function equal to one on a ball of radius δ and zero outside radius 2δ , then for δ sufficiently small

$$g(x, t) = \psi_\delta(x, t)h(x, t)$$

is such that the estimates of Proposition 2.1 apply.

Henceforth we assume that

$$w(x, t) = v(x, t) + g(x, t)$$

satisfies

$$g(0, t) = 0 \text{ for all } t$$

$$\|D_x g(x, t)\| \leq B\|x\| \text{ for all } (x, t) \in E$$

for fixed $B > 0$. In this case, E_0^s is tangent to the stable manifold at O . Now we restrict ourselves to a δ neighborhood of zero section in E . Using Theorem A.1 we may assume that E is the fibred product of E^u, E^s the unstable and stable manifold for w ; that is, we have used a coordinate transformation $(x, t) \rightarrow (\xi_t(x), t)$ for $x \in E^s$ and similarly for $x \in E^u$.

In analogy with the situation described in the introduction, we can now state the gluing theorem:

Theorem A.2 Let $S_{\epsilon,t}^s$ be the sphere of radius ϵ in E_t^s and $S_{\epsilon,t}^u$ and sphere of radius ϵ in E_t^u . For $T_1 < T_2$, ϵ sufficiently small and $p \in S_{\epsilon,T_1}^s$, $q \in S_{\epsilon,T_2}^u$, there is a unique solution $x(t)$ of

$$\frac{dx}{dt} = w(x(t), t)$$

such that

$$\pi^s(x(T_1)) = p$$

$$\pi^u(x(T_2)) = q$$

$$\|x(t)\| \leq \epsilon.$$

The solution depends smoothly on the parameters p, q, T_1 , and T_2 and will be denoted $x(t, p, q, T_1, T_2)$.

Proof. Denote by $x^s(t), x^u(t), w^s(x, t), w^u(x, t)$ the images of $x(t), w(x, t)$ under the projections π^s, π^u . A solution $x(t)$ to the initial value problem for

$$\frac{dx}{dt} = v(x(t), t) + g(x(t), t)$$

is generally described by the integral equation:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, s)g(x(s), s)ds.$$

This is the Lagrange formula for solutions to inhomogeneous linear equations.

More generally we can write flow lines of w as

$$\begin{aligned} x(t) = & \Phi(t, T_1)p + \int_{T_1}^t \Phi(t, s)g^s(x(s), s)ds \\ & + \Phi(t, T_2)q - \int_t^{T_2} \Phi(t, s)g^u(x(s), s)ds \end{aligned} \quad (\text{A.1})$$

Notice that a solution to (A.1) satisfies the 'initial' conditions

$$x^s(T_1) = p$$

$$x^u(T_2) = q.$$

We can write (A.1) as a fixed point equation:

$$x = F(x, p, q, T_1, T_2)$$

where F is the right hand side of (A.1) and x lies in the Banach space

$$C^0([T_1, T_2], E)$$

equipped with the supremum norm. Now the set of $t \rightarrow x(t)$ with $\|x\| < \eta$ is mapped into itself since

$$\begin{aligned}\|F(x, p, q, T_1, T_2)\| &\leq (C + \frac{1}{C})\epsilon + \frac{A}{\rho}(C + \frac{1}{C}) \sup_{\|x\| < \delta} \|g(x, t)\| \\ &\leq (C + \frac{1}{C})\epsilon + \frac{A}{\rho}(C + \frac{1}{C})B\eta^2 \leq \eta\end{aligned}$$

provided that ϵ and η are sufficiently small. In the same way, one sees that F is a contraction when η is sufficiently small.

The theorem now follows from the contraction principle. Applying the implicit function theorem shows that solutions depend smoothly on s, u, T_1, T_2 since

$$\|\frac{\partial F}{\partial x}\| \leq \frac{A}{\rho}(C + \frac{1}{C}) \sup_{\|x\| < \eta} \|\frac{\partial g}{\partial x}\| < 1$$

for η sufficiently small. □

Lemma A.3 For all $0 < \rho' < \rho$, we may choose ϵ sufficiently small and D sufficiently large so that the following a priori estimates hold:

$$\begin{aligned}\|x^s(t, p, q, T_1, T_2)\| &\leq \epsilon D e^{-\rho'(t-T_1)} \\ \|\frac{\partial x^s}{\partial p}(t, p, q, T_1, T_2)\| &\leq \epsilon D e^{-\rho'(t-T_1)} \\ \|\frac{\partial x^s}{\partial q}(t, p, q, T_1, T_2)\| &\leq \epsilon D e^{-\rho'(t-T_1)}.\end{aligned}$$

Similarly, for x^u we have

$$\begin{aligned}\|x^u(t, p, q, T_1, T_2)\| &\leq \epsilon D e^{-\rho'(T_2-t)} \\ \|\frac{\partial x^u}{\partial p}(t, p, q, T_1, T_2)\| &\leq \epsilon D e^{-\rho'(T_2-t)} \\ \|\frac{\partial x^u}{\partial q}(t, p, q, T_1, T_2)\| &\leq \epsilon D e^{-\rho'(T_2-t)}.\end{aligned}$$

All estimates hold uniformly in p, q , and T_1, T_2 .

Proof. The estimates hold by considering the integral equation (A.1) and applying the method of continuity. We will demonstrate the ideas by showing

$$\|x^s(t, p, q, T_1, T_2)\| \leq \epsilon D e^{-\rho'(t-T_1)}.$$

Define the open set $\mathcal{E} = \{t \in [T_1, T_2] : \|x^s(t, p, q, T_1, T_2)\| < \epsilon D e^{-\rho'(t-T_1)}\}$. If $D > 1$, then $T_1 \in \mathcal{E}$. So suppose that $[T_1, t) \subset \mathcal{E}$ where

$$x^s(t) = \Phi(t, T_1)p + \int_{T_1}^t \Phi(t, \sigma)g^s(x(\sigma), \sigma)d\sigma.$$

Estimating the norm

$$\begin{aligned}\|x^s(t)\| &\leq Ce^{-\rho(t-T_1)}\|p\| + C \int_{T_1}^t e^{-\rho(t-\sigma)} \|g^s(x(\sigma), \sigma)\| d\sigma \\ &< Ce^{-\rho(t-T_1)}\epsilon + ABCD\epsilon^2 \int_{T_1}^t e^{-\rho(t-\sigma)} e^{-\rho'(\sigma-T_1)} \\ &< (C + \frac{ABC\epsilon}{\rho-\rho'} D)\epsilon e^{-\rho'(t+T_1)}.\end{aligned}$$

Choose ϵ to be small so that the quantity

$$\frac{ABC\epsilon}{\rho-\rho'} < 1.$$

Then require D to satisfy

$$A + \frac{ABC\epsilon}{\rho-\rho'} D < D.$$

Now $t \in \mathcal{E}$ showing that \mathcal{E} is closed and hence $\mathcal{E} = [T_1, T_2]$. \square

The previous lemma conveniently leads to the following compactness theorem. For a large value of T , let $T_1 = -T$ and $T_2 = T$. Then denote $x(t, p, q, T) = x(t, p, q, -T, T)$.

Theorem A.4 *Let $t \rightarrow x(t, p, q, T)$ be as in Theorem A.2. For $T \rightarrow \infty$, the trajectories*

$$t \rightarrow x(t - T, p, q, T) : [0, T] \rightarrow E$$

approach the solution to

$$\frac{dx(t)}{dt} = w(x(t), t), \quad x(0) = (s, 0)$$

uniformly in T and on compact sets parameters p and q . Similarly, the trajectories

$$t \rightarrow x(T - t, p, q, T) : [0, T] \rightarrow E$$

approach the unstable manifold solutions.

Proof. The initial value for $x(t - T, p, q, T)$ is the point $(s, x''(-T, p, q, T))$. But by Lemma A.3,

$$\|s''(-T, p, q, T)\| < \epsilon D e^{-2\rho'T}$$

so that $x(-T, p, q, T) \rightarrow (s, 0)$ as $T \rightarrow \infty$. \square

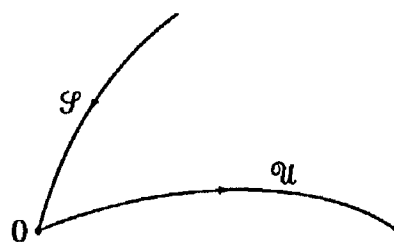


Fig. 7

The implication of this theorem is that the limit of the trajectories is really the «broken» trajectory where the first piece lies in the stable manifold and the second in the unstable (see figure 7).

These preliminaries allow us to proceed to our main gluing theorem.

A.2 Gluing theorems

Let v be a vector field on a manifold X and suppose that α, β, γ are nondegenerate critical points of v such that the intersections $\mathcal{U}_\alpha \cap \mathcal{S}_\beta$ and $\mathcal{U}_\beta \cap \mathcal{S}_\gamma$ are transverse. Denote the space of trajectories from α to β by

$$\tilde{\mathcal{M}}(\alpha, \beta) = \mathcal{U}_\alpha \cap \mathcal{S}_\beta / \mathbb{R}$$

where the \mathbb{R} action is given by the flow. Similarly, define $\tilde{\mathcal{M}}(\beta, \gamma)$ and $\tilde{\mathcal{M}}(\alpha, \gamma)$. We will study the relationship between the ends of $\tilde{\mathcal{M}}(\alpha, \gamma)$ and the spaces $\tilde{\mathcal{M}}(\alpha, \beta)$ and $\tilde{\mathcal{M}}(\beta, \gamma)$.

We use the framework of §A.1. Consider a small ball $B \subset X$ around the critical point β and let $E \rightarrow \mathbb{R}$ be a bundle of Banach spaces such that the unit disk subbundle $E_1 = B \times \mathbb{R}$. There is a vector field w on E induced by v which has the form $v + \frac{\partial}{\partial t}$. Consider $\tilde{\mathcal{M}}(\alpha, \beta) \subset S_{\epsilon, T_1}^s$ and $\tilde{\mathcal{M}}(\beta, \gamma) \subset S_{\epsilon, T_2}^u$. Due to the assumption on the transversality of the intersections, there are maps

$$\begin{aligned} u : \tilde{\mathcal{M}}(\alpha, \beta) \times B_{\epsilon, T_1}^u &\rightarrow B_{\epsilon, T_1}^s \\ s : \tilde{\mathcal{M}}(\beta, \gamma) \times B_{\epsilon, T_2}^s &\rightarrow B_{\epsilon, T_2}^u \end{aligned}$$

with

$$\begin{aligned} u(\phi, 0) &= \phi \\ s(\psi, 0) &= \psi \end{aligned}$$

such that a slice to the \mathbb{R} -action on \mathcal{U}_α is locally described by the graph $(u(\phi, \sigma), \sigma)$. Likewise, a slice to the \mathbb{R} -action on \mathcal{S}_γ is given by $(\tau, s(\psi, \tau))$.

Theorem A.5 For $T = T_2 - T_1$ large, there is a smooth injection

$$G_T : \tilde{\mathcal{M}}(\alpha, \beta) \times \tilde{\mathcal{M}}(\beta, \gamma) \rightarrow \tilde{\mathcal{M}}(\alpha, \gamma)$$

depending smoothly on T .

Technically, the following proof works for compact subsets $K_{\alpha,\beta} \subset \tilde{\mathcal{M}}(\alpha,\beta)$ and $K_{\beta,\gamma} \subset \tilde{\mathcal{M}}(\beta,\gamma)$. The generalization to the statement given above is simple. In the gauge theory case this generalization cannot be made so easily.

Proof. First we describe the boundary value problem. For trajectories $\phi \in \tilde{\mathcal{M}}(\alpha,\beta)$ and $\psi \in \tilde{\mathcal{M}}(\beta,\gamma)$, we wish to perturb the broken trajectory slightly to produce a trajectory from α to γ . This is done by finding small $\sigma \in B_{\epsilon,T_1}^u$ and $\tau \in B_{\epsilon,T_2}^s$ and a trajectory $x(t)$ so that

$$\begin{aligned} x(T_1) &= (u(\phi, \sigma), \sigma) \in \mathcal{U}_\alpha \\ x(T_2) &= (\tau, s(\psi, \tau)) \in \mathcal{S}_\gamma. \end{aligned} \quad (\text{A.2})$$

Notice that this amounts to specifying the two components at the two endpoints T_1 and T_2 .

Our principal tool for accomplishing this perturbation is Theorem A.2. This says that for gradient lines $\gamma \in \tilde{\mathcal{M}}(\alpha,\beta)$ and $\psi \in \tilde{\mathcal{M}}(\beta,\gamma)$ and $\sigma \in B_{\epsilon,T_1}^u, \tau \in B_{\epsilon,T_2}^s$, there is a gradient line $x(t)$ so that $x^s(T_1) = u(\gamma, \sigma)$ and $x^u(T_2) = s(\psi, \tau)$. Moreover, $x(t)$ depends smoothly on all the parameters. This gives one of the components at each endpoint. The other is given by considering the map

$$F_{(\gamma,\psi,T)} : B_{\epsilon,T_1}^u \times B_{\epsilon,T_2}^s \rightarrow B_{\epsilon,T_1}^u \times B_{\epsilon,T_2}^s$$

by

$$F_{(\gamma,\psi,T)}(\sigma, \tau) = (x^u(T_1), x^s(T_2)).$$

Notice that a fixed point (σ, τ) will satisfy (A.2) and hence the accompanying trajectory $x(t)$ is a gradient line from α to γ . Fixed points exist for large T since F is a contraction as we demonstrate now.

We have

$$|F(v_1, w_1) - F(v_2, w_2)| < \eta_v |v_1 - v_2| + \eta_w |w_1 - w_2|$$

where

$$\begin{aligned} \eta_v &< (|\frac{\partial x^u(T_1)}{\partial p}| + |\frac{\partial x^s(T_2)}{\partial p}|) |\frac{\partial u}{\partial \sigma}| \\ \eta_w &< (|\frac{\partial x^u(T_1)}{\partial q}| + |\frac{\partial x^s(T_2)}{\partial q}|) |\frac{\partial s}{\partial \tau}|. \end{aligned}$$

The partials $|\frac{\partial u}{\partial \sigma}|$ and $|\frac{\partial s}{\partial \tau}|$ are bounded and Lemma A.3 shows that

$$|\frac{\partial x^u(T_1)}{\partial p}|, |\frac{\partial x^u(T_1)}{\partial q}|, |\frac{\partial x^s(T_2)}{\partial q}|, |\frac{\partial x^s(T_2)}{\partial p}|$$

decay exponentially with T . Hence for large T , F is a contraction and has a unique fixed point denoted

$$(\sigma(\gamma, \psi, T), \tau(\gamma, \psi, T)).$$

In the same way, Lemma A.3 shows that the derivative $\frac{dF}{dz}$ is small for large T . The implicit function theorem implies that the gluing map

$$G_T : \tilde{\mathcal{M}}(\alpha, \beta) \times \tilde{\mathcal{M}}(\beta, \gamma) \rightarrow \tilde{\mathcal{M}}(\alpha, \gamma)$$

is smooth. \square

This theorem and its proof have several important consequences for us.

Theorem A.6 *For large T , there is an injective local diffeomorphism*

$$G : \tilde{\mathcal{M}}(\alpha, \beta) \times \tilde{\mathcal{M}}(\beta, \gamma) \times [T, \infty) \rightarrow \tilde{\mathcal{M}}(\alpha, \gamma)$$

onto an end of $\tilde{\mathcal{M}}(\alpha, \gamma)$. For $A_1 \in \mathcal{M}(\alpha, \beta)$, $A_2 \in \mathcal{M}(\beta, \gamma)$, the paths $G(A_1, A_2, T)$ converge to the broken trajectory formed by A_1 and A_2 at the rate $\epsilon De^{-\rho'T}$.

Proof. The proof of the first part mimics the proof of Theorem A.5 while the second part follows from the estimates of Lemma A.3. \square

Given two Morse functions f_1 and f_2 , consider the manifold $X \times \mathbb{R}$ with a vector field F equal to

$$\begin{aligned} & -\nabla f_1 + \frac{\partial}{\partial t} \text{ for } t < -1 \\ & -\nabla f_2 + \frac{\partial}{\partial t} \text{ for } t > 1. \end{aligned}$$

For critical points α, β of f_1 and γ of f_2 , we have the following

Theorem A.7 *For large T , there is an injective local diffeomorphism*

$$G : \tilde{\mathcal{M}}_{f_1}(\alpha, \beta) \times \tilde{\mathcal{M}}_F(\beta, \gamma) \times (T, \infty) \rightarrow \tilde{\mathcal{M}}_F(\alpha, \gamma)$$

onto an end of $\tilde{\mathcal{M}}_F(\alpha, \gamma)$.

Proof. Proceed as in Theorem A.5 considering the stable manifold of γ , $\mathcal{S}_\gamma \subset B_{\epsilon, -1}$. \square

Consider two vector fields F_1, F_2 on $X \times \mathbb{R}$ with

$$F_1 = \begin{cases} -\nabla f_1 + \frac{\partial}{\partial t} & \text{for } t < -1 \\ -\nabla f_2 + \frac{\partial}{\partial t} & \text{for } t > 1 \end{cases}$$

and

$$F_2 = \begin{cases} -\nabla f_2 + \frac{\partial}{\partial t} & \text{for } t < -1 \\ -\nabla f_3 + \frac{\partial}{\partial t} & \text{for } t > 1 \end{cases}$$

Define $F_{3,T}$ to be

$$F_{3,T}(t) = \begin{cases} F_1(t+T) & \text{for } t < 0 \\ F_2(t-T) & \text{for } t > 0. \end{cases}$$

Let α, β, γ be critical points for f_1, f_2, f_3 respectively.

Theorem A.8 For large T , there is an injective local diffeomorphism

$$G_T : \tilde{\mathcal{M}}_{F_1}(\alpha, \beta) \times \tilde{\mathcal{M}}_{F_2}(\beta, \gamma) \rightarrow \tilde{\mathcal{M}}_{F_3}(\alpha, \gamma)$$

onto an end of $\tilde{\mathcal{M}}_{F_3}(\alpha, \gamma)$.

Proof. Consider the flow $-\nabla f_2 + \frac{\partial}{\partial t}$ on $X \times [-T, T]$ and apply Theorem A.5 where we regard $\mathcal{U}_\alpha \subset E_{-T}$ and $\mathcal{V}_\gamma \subset E_T$. \square

A.3 Manifolds of critical points

We now consider the case in which the critical points are no longer isolated. Let w be a vector field on a manifold M and assume that S is a submanifold consisting of zeroes of w . Denote the normal bundle of S by $N_S \equiv TM|_S/TS$. Suppose that for all $s \in S$,

$$dw_s : N_s \rightarrow N_s$$

has no eigenvalues with real part 0. When $w = \nabla f$, we say that f is a Morse-Bott function.

In this situation, N splits smoothly into subbundles

$$N = N^+ \oplus N^-$$

such that the real parts of the eigenvalues of dw_s on N_s^\pm have sign \pm . Around a base point $s \in S$, there are coordinates in a neighborhood $U \subset M$ such that

$$U \rightarrow S \times N_0^- \times N_0^+ : m \mapsto (m^0, m^-, m^+).$$

In these coordinates, the spaces $\{0\} \times N_0^- \times \{0\}$ and $\{0\} \times \{0\} \times N_0^+$ are tangent to the strictly stable and strictly unstable manifolds of $(0, 0, 0)$ for w . We shall first show that this family \mathcal{V}_s of stable manifolds of $(s, 0, 0)$ is a smooth fibration isomorphic to $N^- \rightarrow S$, when intersected with a small neighborhood of S . This is the analogue of Theorem A.1.

Choose a small coordinate neighborhood U in which to work and denote the coordinates $z = (s, x, y)$. Write $L = dw_{(0,0,0)}$ for the linear part of the vector field. Locally $w(z) = Lz + g(z)$ where $g(z)$ may be decomposed into its components $g(z) = (g^0(z), g^-(z), g^+(z))$ which are at most quadratic in z . More specifically, notice that $g(s, 0, 0) = 0$ and that $g^-(s, 0, y) = 0$ and $g^+(s, x, 0) = 0$. This implies that the leading order terms of g are

$$\begin{aligned} g^0(s, x, y) &\sim xy \\ g^-(s, x, y) &\sim xy + xs + x^2 \\ g^+(s, x, y) &\sim xy + ys + y^2. \end{aligned} \tag{A.3}$$

With this in mind, we prove the local stable manifold theorem for families of critical points.

Theorem A.9 Given $s \in S$, $n^- \in N^-$, there is a unique bounded trajectory of w , $x : [0, \infty) \rightarrow U$ such that $x^-(0) = n^-$ and $\lim_{t \rightarrow \infty} x(t) = (s, 0, 0)$. Moreover, for varying (s, n^-) the map $s \mapsto x^0(0)$ defines a diffeomorphism between a neighborhood of the zero section of N^- and a neighborhood of S in \mathcal{P}_S , intertwining the projection and endpoint map.

This theorem is not new: it can be seen as a consequence of results in [HPS].

Proof. In the same way, we set up the boundary value problem. Define the complete metric space

$$\mathcal{F} = \{y : [0, \infty) \rightarrow U \mid \begin{aligned} \|x^-(t)\| &\leq \epsilon D e^{-\rho' t} \\ \|x^+(t)\| &\leq \epsilon \\ \|x^0(t)\| &\leq \epsilon \end{aligned}\}$$

for constants D and small ϵ and ρ' . Let $\Phi(t, \sigma)$ denote the flow of L from time σ to t and $g^{+,0}(z) = g^+(z) + g^0(z)$. Then define the operator

$$\begin{aligned} F(y)(t) &= \phi(t, 0)n^- + \int_0^t \Phi(t, \sigma)g^-(y(\sigma))d\sigma \\ &\quad + s - \int_t^\infty \Phi(t, \sigma)g^{+,0}(y(\sigma))d\sigma \end{aligned}$$

A fixed point of F describes a bounded trajectory with the appropriate boundary data: $x^-(0) = n^-$ and $x^0(t) \rightarrow s$ as $t \rightarrow \infty$. Using the same reasoning as before, we conclude that $F : \mathcal{F} \rightarrow \mathcal{F}$ and that F is a contraction.

The fact that $s \mapsto x^0(0) = s - \int_0^\infty g^0(x(\sigma))d\sigma$ is a submersion follows from estimating

$$\frac{\partial}{\partial s} \int_0^\infty g^0(x(\sigma))d\sigma.$$

Using the information about the leading order terms in (A.3), it follows that

$$\left| \frac{\partial}{\partial s} \int_0^\infty g^0(x(\sigma))d\sigma \right| < C \int_0^\infty |x^+(\sigma) \frac{\partial x^-(\sigma)}{\partial s} + x^-(\sigma) \frac{\partial x^+(\sigma)}{\partial s}| d\sigma.$$

The integrand decays exponentially with σ and hence the variation of the integral with s can be made sufficiently small. \square

We continue by demonstrating the generalization of Theorem A.2: incoming and outgoing gradient lines can be glued provided they converge to the same point on S .

Theorem A.10 Let $(s, n^-, n^+) \in U$ and T be sufficiently large. There is a trajectory $x : [-T, T] \rightarrow U$ so that $x^-(-T) = n^-$, $x^+(T) = n^+$, and $x^0(0) = s$. Furthermore, as $T \rightarrow \infty$ the trajectories converge to a broken trajectory consisting of a solution in the stable manifold followed by one in the unstable.

Proof. We define the complete metric space

$$\mathcal{F} = \{y : [-T, T] \rightarrow U \mid \|y^-(t)\| \leq \epsilon D e^{-\rho(t+T)} \\ \|y^+(t)\| \leq \epsilon D e^{-\rho(T-t)} \\ \|y^0(t)\| \leq \epsilon\}$$

for suitable constants D, ϵ, ρ . Define the operator

$$F(y)(t) = \Phi(t, -T)n^- + \int_{-T}^t \Phi(t, \sigma)g^-(y(\sigma))d\sigma \\ + \Phi(t, T)n^+ - \int_t^T \Phi(t, \sigma)g^+(y(\sigma))d\sigma \\ + s - \int_t^0 g^0(y(\sigma))d\sigma.$$

As before, $F : \mathcal{F} \rightarrow \mathcal{F}$ and is a contraction. The fixed point is a trajectory satisfying the hypotheses of the theorem. One sees that as $T \rightarrow \infty$, the integral equation breaks into the integral equations for the stable and unstable manifold solutions. \square

Suppose that $S_\alpha, S_\beta, S_\gamma$ are critical submanifolds so that the intersections of stable and unstable manifolds are transverse. Suppose, in addition, that \mathcal{P}_γ intersects $(\mathcal{U}_\beta)_s$ transversely for all s . Applying the reasoning of Theorem A.5 leads to

Theorem A.11 *For T large, there is an injective local diffeomorphism*

$$G : \tilde{\mathcal{M}}(S_\alpha, S_\beta) \times_{S_\beta} \tilde{\mathcal{M}}(S_\beta, S_\gamma) \times (T, \infty) \rightarrow \tilde{\mathcal{M}}(S_\alpha, S_\gamma)$$

onto an end of $\tilde{\mathcal{M}}(S_\alpha, S_\gamma)$ where $\tilde{\mathcal{M}}(S_\alpha, S_\beta) \times_{S_\beta} \tilde{\mathcal{M}}(S_\beta, S_\gamma)$ denotes the fibered product over the endpoint maps.

A.4 The stable manifold theorem for the Chern-Simons function

The discussion in section A.1 lends itself to the study of the gradient flow of the Chern-Simons function. In particular, it follows that the local stable manifold theorem, as in Theorem A.1, holds in this setting.

Let A be a nondegenerate critical point of the Chern-Simons function on \mathcal{B} , the space of gauge equivalence classes of connections on the compact 3-manifold M . A neighborhood of A in \mathcal{B} , and also the tangent space $T_A\mathcal{B}$, may be identified with $\ker d_A^* : \Omega^1(M, su(2)) \rightarrow \Omega^0(M, su(2))$. We will work with the Sobolev norm L_1^2 without explicitly denoting it. The gradient flow of the Chern-Simons function, at a connection $\nabla = A + A'$, is $-*F^\nabla$, which decomposes into linear and quadratic pieces:

$$-*F^\nabla = -*d_AA' - *[A', A'].$$

The Hessian at the critical point A is given by

$$*d_A : \ker d_A^* \rightarrow \ker d_A^*.$$

This is a self-adjoint elliptic operator and as such has a discrete real spectrum. Decompose $\ker d_A^* = \oplus_\lambda H^\lambda$ into its eigenspaces and define $H^\pm = \oplus_{\lambda < (>) 0} H^\lambda$. The nondegeneracy of A implies that $\ker *d_A = 0$ so that $\ker *d_A = H^- \oplus H^+$.

Notice that the flow of the linear part of the vector field, $-*d_A$, is defined by

$$\Phi(t, \tau)\psi = e^{(t-\tau)*d_A}\psi.$$

Therefore, the flow is continuous on H^- only for positive time, where it is a smoothing operator, and similarly the flow is continuous on H^+ only for negative time. Remember that the vector field is not continuous on L_1^2 connections.

Reflecting on the methods of §A.1, we see that this is sufficient to prove the stable manifold theorem for the Chern-Simons flow. Consider a ball B of radius ϵ in $L_1^2(M, su(2))$. For $a \in H^-$ and $x \in C^0([0, \infty), B)$, define

$$F(x, a)(t) = \Phi(t, 0)a + \int_0^t \Phi(t, \sigma)g^-(x(\sigma))d\sigma - \int_t^\infty \Phi(t, \sigma)g^+(x(\sigma))d\sigma$$

where $g^\pm(A) = [A, A]^\pm$, the decomposition of the quadratic term given by H^\pm . Notice that this is well defined since we use the forward flow on H^- and the backwards flow on H^+ . Also $\Phi(t, \sigma)$ satisfies the estimates of section A.1 so that the techniques used in the proof of the stable manifold theorem apply. This leads to:

Theorem A.12 *Let $a \in H^-$ be sufficiently small in L_1^2 . There is $\xi(a) \in H^+$ such that the trajectory of the flow $-*F^\nabla$ through $(a, \xi(a))$ is bounded. Moreover, $a \mapsto \xi(a)$ is a smooth map.*

The fact that $a \mapsto \xi(a)$ follows from the implicit function theorem for Banach spaces. It is important here to use L_1^2 connections since ξ is as smooth as g is. Using L_1^2 implies that g is smooth.

Appendix B Transversality

In the interest of completeness, we present in this appendix a discussion of transversality theory, a principal tool throughout this paper, and a proof of Proposition 2.1. This presentation proceeds as in Abraham and Robbin [AR].

We recall the basic definitions of transversality. Let X, Y be manifolds, $W \subset Y$ a submanifold and $f : X \rightarrow Y$ a smooth map. We say that f is *transverse* to W at $x \in X$ if either $f(x) \notin W$ or $f(x) \in W$ and

$$T_{f(x)}W + (f_*)(T_xX) = T_{f(x)}Y.$$

Globally, f is transverse to W if it is transverse for all $x \in X$. When f is transverse to W , the implicit function theorem implies that $f^{-1}(W)$ is a submanifold of X .

An important application comes from considering a family of maps parametrized by a manifold \mathcal{C} , or equivalently a map

$$F : X \times \mathcal{C} \rightarrow Y.$$

The following theorem shows that transverse maps are «generic».

Theorem B.1 *Suppose that $F : X \times \mathcal{C} \rightarrow Y$ is transverse to W . Then there is a Baire set $\mathcal{C}_g \subset \mathcal{C}$ so that for all $c \in \mathcal{C}_g$, $f_c : X \rightarrow Y$ is transverse to W .*

The importance of this theorem, as we shall see, is that it can sometimes be quite easy to show that a family of maps is transverse. It is easy to see that if

$$F : X \times \mathcal{C} \rightarrow Y$$

is transverse to W , then the canonically defined one-parameter version of F ,

$$\mathcal{F} : (X \times [0, 1]) \times \text{Maps}([0, 1], \mathcal{C}) \rightarrow Y,$$

is also transverse to W . Theorem B.1 implies that if $F : X \times \mathcal{C} \rightarrow Y$ is transverse to W , then for a «generic» one parameter family, $\gamma : [0, 1] \rightarrow \mathcal{C}$, the map \mathcal{F}_γ is transverse to W and hence the space

$$\mathcal{F}_\gamma^{-1}(W) \subset X \times [0, 1]$$

is a submanifold of $X \times [0, 1]$. The set of $s \in [0, 1]$ for which $F_{\gamma(s)}^{-1}(W)$ is not cut out transversely is the bifurcation locus of the path. In the case that W is compact and F is proper then the bifurcation points are an isolated closed subset of $[0, 1]$.

For submanifolds $R, S \subset Y$, we say that R intersects S transversally at $y \in R \cap S$ if

$$T_y R + T_y S = T_y Y.$$

Furthermore, R intersects S transversally if this is true at every point in the intersection. There are equivalent definitions that are useful. Let i_R (resp i_S) : R (resp S) $\rightarrow Y$ be the inclusions of R and S . Then R intersects S transversally if i_R is transverse to the submanifold S . Again, this is equivalent to

$$i_R \times i_S : R \times S \rightarrow Y \times Y$$

being transverse to the diagonal submanifold. A theory for families of submanifolds parametrized by \mathcal{C} holds as above.

Now we are ready to discuss the genericity properties of the gradient flow of Morse functions. Recall that a function is Morse if its critical points are non-degenerate and that the gradient flow of a Morse function is Morse-Smale if all stable and unstable manifolds intersect transversely. The following is then a variant of the celebrated Kupka-Smale theorem and we shall just outline the interesting elements. For details we refer to Abraham and Robbin [AR].

Proposition B.2 *Given a metric g on B , there is a Baire set $\mathcal{C}_g \subset C^\infty(B)$ so that $f \in \mathcal{C}_g$ is Morse with Morse-Smale gradient flow.*

Proof. The fact that the set \mathcal{C}_M of Morse functions in $C^\infty(B)$ is a Baire set follows from the discussion above as follows. Notice that a function f is Morse precisely when the differential form df , considered as a section of T^*B , is transverse to the 0-section. It is easy to verify that the map

$$B \times C^\infty(B) \rightarrow T^*B$$

given by $(b, f) \mapsto df_b$ is transverse to the 0-section. Theorem B.1 then implies that the set of Morse functions \mathcal{C}_M is a Baire set. Notice that \mathcal{C}_M is independent of the metric. When B is compact, we can also say that \mathcal{C}_M is open in $C^\infty(B)$. However, this open set is not connected since critical points can appear and disappear as we vary the function.

When B is compact, there is a neighborhood \mathcal{O}_f of $f \in \mathcal{C}_M$ in $C^\infty(B)$ consisting entirely of Morse functions and such that the number of critical points of a given index on this neighborhood is constant. This gives a universal parametrization of the stable and unstable manifolds for the critical points α_j of all functions in \mathcal{O}_f :

$$\begin{aligned} u : \cup_j \mathbb{R}^{i(\alpha_j)} \times \mathcal{O}_f &\rightarrow B \\ s : \cup_j \mathbb{R}^{n-i(\alpha_j)} \times \mathcal{O}_f &\rightarrow B \end{aligned}$$

In fact one can cover \mathcal{C}_g with a countable collection of such \mathcal{O}_f 's, such that the resulting cover is locally finite ([§33.2][AR]).

The important step is to show that

$$u \times s : \cup_{j,k} \mathbb{R}^{i(\alpha_j)} \times \mathbb{R}^{n-i(\alpha_k)} \times \mathcal{O}_f \rightarrow B$$

is transversal to the diagonal. For this, it is necessary to know how the gradient flow is perturbed by varying the function. As a first step, consider perturbing the gradient flow by an arbitrary vector field ([§32][AR]): let $X + sY$ be a one parameter family of vector fields and let Φ^s denote the flow of $X + sY$. Choose an arbitrary point $b \in B$ and let $t \mapsto \gamma(t) = \Phi_t^0(b)$ be the trajectory of X through b . Then we compute

$$\frac{d\Phi_{t_0}^s}{ds}(b)|_{s=0} = \int_0^{t_0} (\Phi_t^0)_*(Y_{\gamma(t_0-t)})dt. \quad (\text{B.1})$$

It is important to notice that this derivative only depends on the perturbing vector field Y along the trajectory $\gamma(t)$.

The heart of the proof of the Kupka-Smale theorem is then to show that the vector fields (in our case, the gradient vector fields) are rich enough to make the universal stable and unstable manifolds transversal. To this end, suppose that there

is $(x, y, f) \in \mathbb{R}^i \times \mathbb{R}^{n-i} \times \mathcal{O}_f$ so that $(u \times s)(x, y, f) = (b, b)$. Denote the derivative of u and s along \mathcal{O}_f by $\frac{\partial u}{\partial X}$ and $\frac{\partial s}{\partial X}$ respectively. Then $u \times s$ is transverse to the diagonal if for every $v \in T_b B$, there is a vector field Y supported away from the critical points, such that either

$$\begin{aligned} \frac{\partial u}{\partial X}(x, y, f)(0, 0, Y) &= v \\ \frac{\partial s}{\partial X}(x, y, f)(0, 0, Y) &= 0, \end{aligned} \quad (\text{B.2})$$

or

$$\begin{aligned} \frac{\partial s}{\partial X}(x, y, f)(0, 0, Y) &= v \\ \frac{\partial u}{\partial X}(x, y, f)(0, 0, Y) &= 0, \end{aligned} \quad (\text{B.4})$$

a relation which in fact is considerably stronger than transversality. We need to perturb by vector fields whose support is disjoint from an open set containing the critical points. Suppose that γ connects critical points α and β . When b lies on γ and is close to β , we choose Y to satisfy (B.2). Otherwise when b is close to α , we find Y satisfying (B.4).

We construct the vector field Y as follows. Choose $h \in C^\infty(\mathbb{R})$ with support in $[-t_0, 0]$ so that $\int_{\mathbb{R}} h dt = 1$. Let $\gamma(t)$ be the gradient line through b and define Y along $\gamma(t)$ by

$$Y(\gamma(t)) = h(t)(\Phi_t)_*(v).$$

Using (B.1), it is clear that

$$\begin{aligned} \frac{\partial u}{\partial X}(x, y, f)(0, 0, Y) &= v \\ \frac{\partial s}{\partial X}(x, y, f)(0, 0, Y) &= 0. \end{aligned}$$

It is elementary to further verify that Y may be extended to B with support in a neighborhood of $\gamma(t)$ so that $Y = -\nabla \tilde{f}$ for some function \tilde{f} . \square

The proof for genericity of time dependent flows is slightly easier, as one may now use time dependent functions to perturb the given vector field. The property of the flow used in the homotopy equivalence of complexes is the one parameter version of the transversality of time dependent gradient flows.

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