SYSTEMS OF CIRCUITS ON TWO-DIMENSIONAL MANIFOLDS.

By H. R. Brahana.

1. In this paper we first give a method of reducing any two-dimensional manifold to one of the known polygonal normal forms. The method used is one by which a polygon on which the manifold is represented is subjected to a series of transformations by cutting it apart in a simple manner and then joining it together again so as to obtain a new polygon representing the same manifold.

We next (§§ 11 to 18) apply the same series of transformations to the problem of reducing a system of curves on the manifold to a normal form.* We then introduce certain matrices of separation by means of which the relations among the pairs of sides of the polygon are described and study the effect on these matrices of the transformation of cutting. By this means we obtain a number of theorems on systems of curves which follow closely along the lines of the theory indicated in Poincaré's "Cinquième Complement à l'Analysis Situs."†

We shall use the terms *manifold, cell, circuit, orientable, one-sided,* etc., as they are defined by Professor Veblen in his Cambridge Colloquium lectures on Analysis Situs. It is there shown (Chapt. II, § 65) that any two-dimensional manifold can be imaged on a planar polygon in such a way that any point of the manifold has for its image an interior point, a pair of "conjugate points" (cf. § 3 below), or a "conjugate set of vertices" of the polygon.

I take this opportunity to acknowledge my indebtedness to Dr. J. W. Alexander for suggestions and to Professor O. Veblen for proposing the problem and for advice in working it out.

2. **Conjugate Points and Sides of a Polygon.** Consider a polygon of an even number, $2n$, of sides in a Euclidean plane. Let P_1, P_2, P_3 be three distinct points taken in the order $P_1P_2P_3$ on the side a_i of the polygon. These three points determine a sense of description of the boundary of the polygon. A (1–1) continuous correspondence may be set up between the points of a_i and the points of any other side a_j of the polygon. Let such a correspondence be established and let the points which correspond to $P_1P_2P_3$ be $P_1'P_2'P_3'$ respectively. In case the three points $P_1'P_2'P_3'$ determine the same sense on the boundary of the polygon as is determined

* This question was first considered by Jordan, Journal de math., (2) 11, pp. 105, 110.

† Rendiconti del Circolo Matematico di Palermo, vol. 18 (1904), p. 45.

by the points $P_1P_2P_3$, the correspondence will be called *direct;* in case the two senses are not the same, the correspondence will be called *opposite.*

Suppose the sides of the polygon have been paired arbitrarily and denote the members of a pair by a_i and a_i'. Let a_i be called the side *conjugate* to the side a_i', and a_i' the side conjugate to a_i. Let a correspondence, direct or opposite, be established between the members of each pair. Two corresponding points P_1 and P_1', interior to a_i and a_i' respectively, will be called a *conjugate pair* of points.

3. Choose $4n$ points on the boundary of the polygon in the following manner: Take two arbitrary distinct points on each of the n sides a_i; then take the two points conjugate to them on each of the n sides a_i'. (Fig. 1.) Let the two points nearest to the vertex P_i, one on each of the

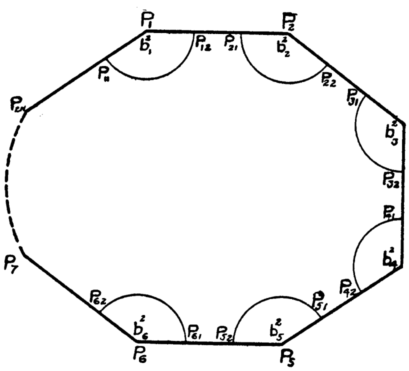

FIG. 1.

sides that has an end at P_i, be called P_{i1} and P_{i2}. Join P_{i1} to P_{i2} by a 1-cell p_i on the polygon. Do the same for each vertex, choosing the 1-cells p_i so that no two intersect. Let the 2-cell whose boundary is made up of the segments $P_{i1}\ P_i$ and $P_{i2}\ P_i$, the 1-cell p_i, and the points P_{i1}, P_{i2}, and P_i be called b_i^2. Consider the side $P_{i1}\ P_i$ of the 2-cell b_i^2. There is a unique 2-cell b_j^2 one of whose sides $P_{j1}\ P_j$ (or $P_{j2}\ P_j$) is a segment conjugate to the segment $P_{i1}\ P_i$. Join together these 2-cells by matching up conjugate points on their boundaries. Then there exists a unique 2-cell b_k^2 one of whose sides is conjugate to $P_{j2}\ P_j$ (or $P_{j1}\ P_j$). Join b_k^2 to b_j^2 in the same manner. This may be continued until a 2-cell b_l^2 is reached one of whose sides is the conjugate of the side $P_{i2}\ P_i$ of the 2-cell b_i^2. The vertices $P_i\ P_j\ P_k\ \cdots\ P_l$ of the polygon which are on the boundaries of such a set of 2-cells will be called a *conjugate set of vertices.*

4. If the 2-cells b_i^2, b_j^2, $\cdots$, b_l^2 which determine a conjugate set of vertices be fitted together at their edges in such a way that conjugate pairs of points coincide, it is evident that they will constitute a single

2-cell. Hence it is evident that, for any polygon of $2n$ sides on which conjugate pairs of points and sets of vertices have been defined, there can be found a two-dimensional manifold such that there is a continuous correspondence in which each point of the polygon corresponds to one, and only one, point of the manifold and each point of the manifold corresponds either to one, and only one, point interior to the polygon, or to a pair of conjugate points on the boundary, or to a set of conjugate vertices. Conversely, for any two-dimensional manifold a polygon of $2n$ sides can be found (cf. the reference above) which is its image in the manner just described.

5. We shall assume that a sense has been arbitrarily assigned to each of the sides a_i. This sense may be denoted by the order of any three distinct points on a_i. The three conjugate points on a_i' determine a definite sense on a_i'. In case the senses of a_i and a_i' for all values of i are such that one of them agrees and the other disagrees with a fixed sense of description of the boundary of the polygon, it is obvious that the manifold represented by the polygon is *orientable* or *two-sided.* In case there is one pair of sides a_i and a_i' the senses of which both agree with a fixed sense of description of the boundary of the polygon, it is equally obvious that the manifold represented is *one-sided.*

6. **Transformations of the Polygon.** A 1-cell x on the polygon with its ends on the boundary divides the polygon into two 2-cells α and β (see Fig. 3). Suppose the side b_2 is on the boundary of α and the side b_2' is on the boundary of β. By cutting the polygon along x and joining the two 2-cells by matching up conjugate points of the two sides b_2 and b_2' a new polygon is obtained (see Fig. 4) which is in the same relation to the manifold as was the original polygon. If c is the image on the manifold of the 1-cell x, then on the new polygon the image of c will be two conjugate sides; the image of a point interior to c will be a pair of conjugate points.

This transformation will be referred to as the *method of cutting.* The 1-cell x will be called a *cut.* The method of cutting will now be used to reduce the polygon to a normal form.* We shall first reduce to one the number of points a_i^0 of the manifold which correspond to vertices of the polygon, and secondly shall obtain a definite arrangement of pairs of conjugate sides of the polygon.

7. **Reduction to a Single Conjugate Set of Vertices.** A sense may be assigned arbitrarily to each of the edges a_i and denoted by the order of any three distinct points on it. The three conjugate points on a_i' deter-

* The application of the method of cutting to the normalization of a polygon is due to Professor Veblen; it was first given by him in a seminar on Analysis Situs in 1915.

mine a sense on a_i'. The sense of any side determines a sense of description of the boundary of the polygon.

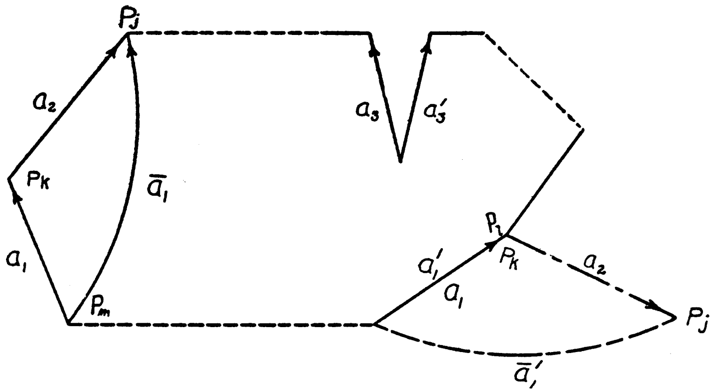

FIG. 2.

REDUCTION 1. If vertices of the polygon correspond to more than one point of the manifold, there will be some side, say a_2, whose ends, P_j and P_k, correspond to distinct points of the manifold; let a_1 be a side with one end at P_k and the other at a vertex P_m (Fig. 2). First let us suppose that the side a_2 is not a_1'. Let P_l be the end of a_1' which corresponds to the same point of the manifold as P_k. Draw a cut $\bar{a}_1$ joining P_m to P_j and join the two parts of the polygon along the sides a_1 and a_1'. This gives a polygon on which the number of vertices in the conjugate set to which P_k and P_l belong has been reduced by one; the number of sides of the polygon has not been changed.

REDUCTION 2. In case a side, say a_3 (Fig. 2), joins two vertices which correspond to different points of the manifold and has an end in common with its conjugate side a_3', we have the case excluded in Reduction 1. From the way in which points of a_3 and a_3' correspond it follows that a_3 and a_3' must be oppositely sensed. Hence by coalescing the pairs of conjugate points of a_3 and a_3' a polygon can be formed from which the two sides a_3 and a_3' and their common vertex have been removed. The number of points of the manifold to which vertices of the polygon correspond has been reduced by one.

8. These reductions may be continued so long as there is more than one point of the manifold to which vertices of the polygon correspond. By each step either a conjugate set of vertices is removed, or the number of vertices in one conjugate set is increased while the number of vertices in another conjugate set is reduced by one (Reduction 1); also the conjugate set of which the number of vertices is to be increased can be chosen arbitrarily, because the rôles of P_j and P_k may be interchanged in Reduc-

tion 1. Hence by a finite number of steps a polygon may be obtained whose vertices constitute a single conjugate set corresponding to an arbitrarily chosen 0-cell a_i^0 of the manifold, or else a polygon of two sides may be obtained whose vertices constitute two conjugate sets, and whose sides are oppositely sensed. The manifold defined by the latter polygon is a sphere.

Hereafter we shall call the 0-cell a_i^0 the point A. Each pair of conjugate sides of the polygon will be imaged on a 1-cell on the manifold whose ends coincide with A. In other words, each pair of conjugate sides of the polygon will correspond to a simple circuit on the manifold through the point A.*

9. **Normalization of the Two-Sided Polygon.** Let us first consider the two-sided case and show how to obtain a group $x\ y\ x'\ y'$ of four consecutive sides on the boundary of the polygon. Draw a cut x joining the two forward ends of a_i and a_i' (a_2 and a_2' in Fig. 3). Let the two parts of the

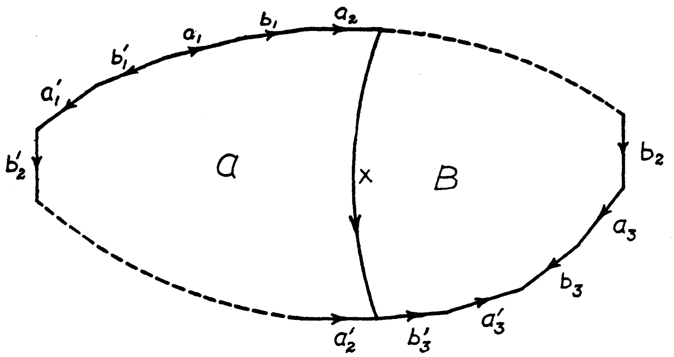

FIG. 3

polygon be α and β where a_i and a_i' are on α. There must be some side a_j (b_2' in Fig. 3) on α whose conjugate a_j' is on β, otherwise the vertices of β together with the two vertices of α at the forward ends of a_i and a_i' would constitute a conjugate set without including all the vertices of the polygon. Join α and β along the sides a_j and a_j'. On the resulting polygon the three sides $a_i\ x\ a_i'$ will be consecutive (Fig. 4). Draw a cut y joining the forward ends of x and x'. Join the two parts of the polygon along the sides a_i and a_i' (Fig. 5). The four sides $y'\ x\ y\ x'$ are consecutive and in that order.

This process may be repeated for any other pair of conjugate sides a_k and a_k' without disturbing the arrangement of the sides $x\ y\ x'\ y'$ for no cut will be drawn from a vertex at which two of these sides abut.

* The same result could be obtained by shrinking to points 1-cells joining distinct 0-cells of the manifold.

From the above reasoning it follows that the number of sides of the polygon of a two-sided manifold, if the polygon has a single conjugate set of

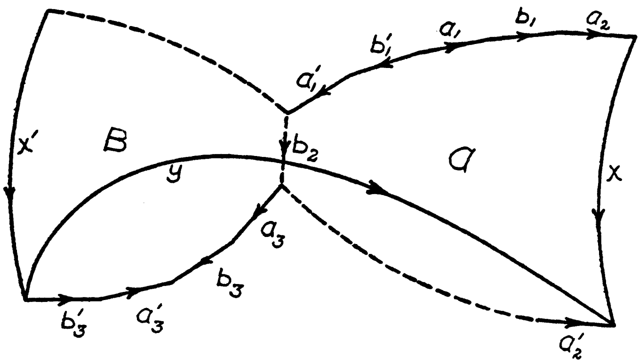

FIG. 4

vertices, is a multiple of four. Completing the reduction and changing the notation we get the following arrangement of the sides of the polygon:

$$a_1\ b_1\ a_1'\ b_1'\ a_2\ b_2\ a_2'\ b_2'\ \cdots\ a_p\ b_p\ a_p'\ b_p'.$$

This is the normal form of the polygon. The number p is called the *genus* of the manifold. The *connectivity* R_1 of the manifold is $2p + 1$.

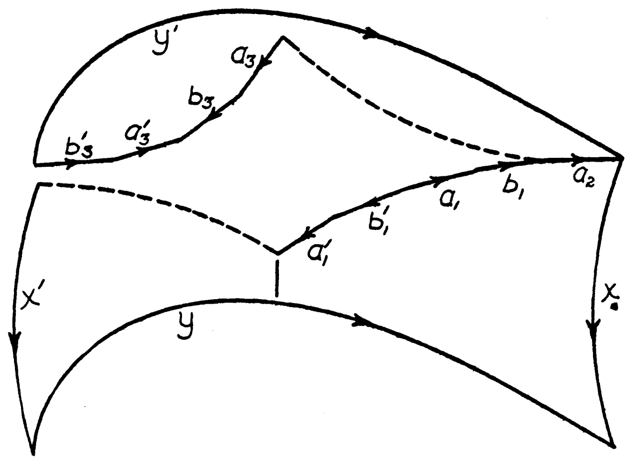

FIG. 5.

10. **Normalization of the One-Sided Polygon.** In the consideration of the one-sided case we make use of the transformation just described for the two-sided case if there exists a group of four sides having the same relations among themselves that the sides a_i, a_i', a_j, and a_j' had above. Thus we obtain on the boundary of the polygon a certain number of groups of four consecutive sides in the order $a_i\ b_i\ a_i'\ b_i'$.

Let a_k and a_k' be a conjugate pair of sides which have the same sense (a_3 and a_3' in Fig. 6). Draw a cut x joining the forward ends of a_k and

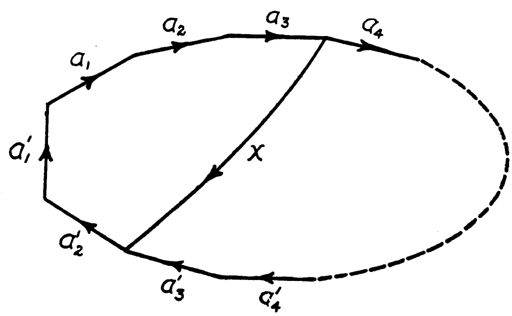

FIG. 6.

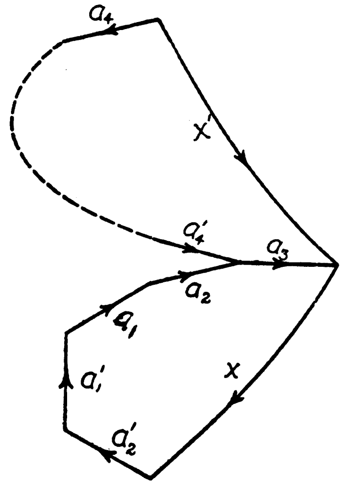

FIG. 7.

a_k', and join the parts of the polygon along a_k and a_k'. This replaces the pair $a_k\, a_k'$ by the pair $x\, x'$ (Fig. 7) which is a pair of consecutive conjugate sides having the same sense. By application of the two transformations the sides of the polygon may be arranged in groups of four of the form $a_i\, b_i\, a_i'\, b_i'$ and groups of two of the form $c_j\, c_j'$.*

A group of six sides of the form $a_i\, b_i\, a_i'\, b_i'\, c_j\, c_j'$ may be replaced by three groups of two of the form $c_k\, c_k'\, c_l\, c_l'\, c_m\, c_m'$. Draw a cut x joining the forward ends of a_i and c_j (Fig. 8). Join the two parts of the polygon

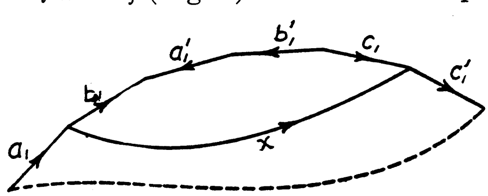

FIG. 8.

along the sides c_j and c_j'. This gives six consecutive sides $a_i\, x\, b_i'\, a_i'\, b_i\, x'$ (Fig. 9). Draw a cut y joining the backward end of a_i to the forward

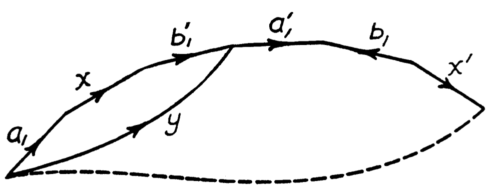

FIG. 9.

* Attention is called to the fact that the members of a pair $a_i\, a_i'$, or $b_i\, b_i'$, are oppositely sensed, and that members of a pair $c_j\, c_j'$ have the same sense.

end of b_i', and join the two parts of the polygon along the sides a_i and a_i'. This gives the six consecutive sides $y\ y'\ b_i'\ x\ b_i\ x'$ (Fig. 10). Draw a cut

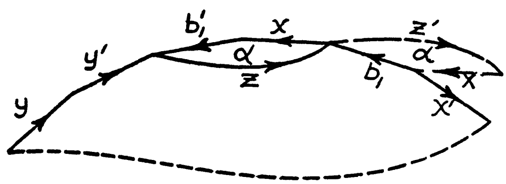

FIG. 10.

z joining the forward ends of b_i and b_i', join the two parts of the polygon along the sides b_i and b_i'. This gives the six consecutive sides $y\ y'\ z\ z'\ x\ x'$, which is the desired form (Fig. 10).

From the above it follows that the polygon of a one-sided manifold may be put in the form:

(1) $$a_1\, a_1'\, a_2\, a_2'\, a_3\, a_3' \cdots a_{R_1-1}\, a_{R_1-1}'.$$

The number R_1 is the *connectivity* of the manifold.

By applying the inverse of the reduction just described to a set of three consecutive pairs the polygon of a one-sided manifold may be put in one of the two forms:

(*a*) $$a_1\, b_1\, a_1'\, b_1'\, a_2\, b_2\, a_2'\, b_2' \cdots a_p\, b_p\, a_p'\, b_p'\, c_1\, c_1';$$

or

(*b*) $$a_1\, b_1\, a_1'\, b_1'\, a_2\, b_2\, a_2'\, b_2' \cdots a_p\, b_p\, a_p'\, b_p'\, c_1\, c_1'\, c_2\, c_2',$$

according as $R_1 - 1$ is odd or even.

11. **Fundamental Sets of Circuits.** When the polygon has been so transformed that the vertices constitute a single conjugate set the image on the manifold of a pair of conjugate sides of the polygon is a simple circuit through the point A. No two of these circuits have any other point in common. The circuits constitute the complete boundary of a 2-cell which contains all the points of the manifold which are not on the circuits. Such a set of circuits has been called by Poincaré a *fundamental set.*

The discussion in the first part of this paper proves the existence of a fundamental set. We shall now prove that a fundamental set can be obtained with an arbitrary point A_1 of the manifold as the point A. If the image P_1 of A_1 is interior to the polygon, draw an arc p connecting P_1 with some vertex P of the polygon. Cut the polygon along the arc p. This gives a polygon with two more sides than the original polygon and with two conjugate sets of vertices. Now apply Reduction 1 of § 7 in such a way that the number of vertices in the conjugate set which corresponds to A_1 is increased. This may be continued until by application of Reduction 2 (§ 7) the conjugate set which corresponds to A is removed, and the number of sides of the polygon is reduced by two. This gives a polygon of the same number of sides as the original one and with a single conjugate set of vertices. Consequently we have a new fundamental set of circuits, each passing through A_1, and the number of circuits in this set is the same as in the original set.

If the point P_1 were on a side of the polygon, the number of sides would be increased by two if we considered P_1 and its conjugate point P_1' as vertices. The above procedure could then be carried out giving the same result.

12. In considering a simple circuit $\overline{C}$ on the manifold we may assume, as a result of what has just been proved, that the point A of a fundamental set F is on the circuit. Let us consider the polygon whose conjugate pairs of sides are imaged on the circuits of F, and let us suppose that $\overline{C}$ has a finite number of points in common with circuits of F.* The image of $\overline{C}$ on the polygon will be a set of arcs $[C_i']$. If $\overline{C}$ has no point in common with F other than A, this set will consist of a single arc having its ends at two vertices of the polygon; these two vertices will be distinct unless $\overline{C}$ divides the manifold into two parts. If $\overline{C}$ has points other than A in common with F, two of the arcs $[C_i']$ will have one end each at a vertex of the polygon; the other ends of arcs of $[C_i']$ will be at points interior to the sides of the polygon. The second case may be reduced to the first by a proper choice of the fundamental set F; this may be done by the method of cutting.

For, let C_1' be an arc with one end at the vertex P_i and the other at a point P interior to the side a_i (Fig. 11). Draw a cut x joining P_i to

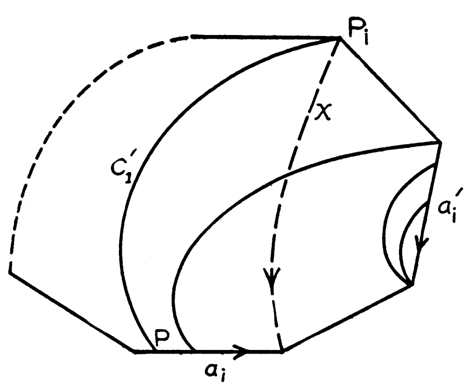

FIG. 11.

* A fundamental set may always be chosen so that the above condition is satisfied.

an end of a_i such that a_i and a_i' are on different parts of the polygon. The cut x can be drawn so that it has no intersections with C_1', so that it has no intersections with any arc joining two boundary points neither of which is interior to a_i, and so that it has no more than one intersection with any arc having an end on a_i. By joining the two parts of the polygon along the sides a_i and a_i' a new polygon is obtained (Fig. 12) such that

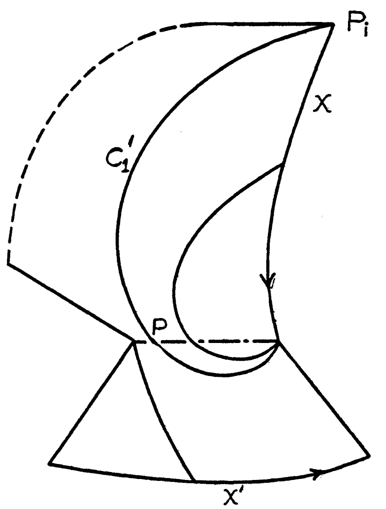

FIG. 12.

the number of ends of arcs at points interior to the sides is at least one less than on the original polygon. This process may be continued until this number is zero, i.e., until a polygon is obtained on which the image of $\bar{C}$ is a single arc C_i joining two vertices.

13. Let us suppose that the circuit $\bar{C}$ is not homologous to zero, i.e., that it does not divide the manifold into two parts. Then the two ends of the arc C_i are distinct; and if α and β are the two parts of the polygon determined by C_i, there must be some side a_i on the boundary of α whose conjugate side a_i' is on the boundary of β. Hence, if we cut the polygon along C_i and join the two parts along a_i and a_i', a polygon is obtained on which the image of $\bar{C}$ is an arc joining two consecutive vertices. Hence, *any simple circuit which is not homologous to zero may be made a member of a fundamental set.*

14. **Relations between Two Fundamental Sets.** To compare two fundamental sets F and F_1 we may assume that the points A and A_1 coincide. Let conjugate pairs of sides of the polygon be images of circuits of F. No circuit or set of circuits of F_1 divides the manifold into two parts. The image of F_1 on the polygon will be a set of non-intersecting arcs

$[C_i']$ having their ends on the boundary. By the method of § 12 we may obtain a polygon on which one of the circuits of F_1 is imaged on an arc C_i joining two vertices. Also since no two of the arcs $[C_i']$ intersect, we may obtain by the same method a polygon on which a second circuit of F_1 is imaged on an arc C_j joining two vertices; for since neither of the ends of C_i is interior to any side of the polygon, none of the required cuts will cross C_i. Continuing this process a polygon is obtained on which the image of F_1 is a set of arcs $[C_i]$ each having its ends at vertices of the polygon.

We will next see how a polygon may be obtained which is such that each conjugate pair of sides corresponds to a circuit of F_1, and which is such that every circuit of F_1 corresponds to a pair of conjugate sides. It will also be seen that the number of sides of this polygon is the same as the number of sides of the original polygon.

If C_i is an arc joining the two ends of a_i, cut the polygon along C_i and join the two parts along the sides a_i and a_i'. This gives a conjugate pair of sides whose image on the manifold is a circuit of F_1. There exists no arc C_j joining the ends of a side of this conjugate pair, for if there were such an arc, it and C_i would divide the manifold into two regions.

Let the transformation described in the last paragraph be carried out for each of the arcs of $[C_i]$ which joins two consecutive vertices of the polygon. If C_j is an arc which joins two vertices of the polygon which are not consecutive, it divides the polygon into two parts, α and β, and there must exist a conjugate pair of sides a_i and a_i' of which one is on the boundary of α and the other is on the boundary of β, and which is not the image of any of the circuits of F_1; otherwise any arc on the manifold joining two points $\overline{P}_\alpha$ and $\overline{P}_\beta$ would intersect one of the circuits of F_1. Cutting the polygon along the arc C_j and joining the two parts along the sides a_i and a_i', a polygon is obtained which has a conjugate pair of sides whose image on the manifold is a circuit of F_1. By the above methods a polygon may be obtained which has a pair of conjugate sides for every circuit of F_1. It remains to be seen that every pair of conjugate sides of this polygon is imaged on a circuit of F_1. If this were not so, F_1 would not bound a 2-cell and so would not be a fundamental set.

15. **Invariance of the Connectivity.** Since none of the transformations used changes the number of sides of the polygon and since the normalization of a polygon whose vertices constitute a single conjugate set does not change the number of sides, it follows that the values of the connectivity determined by the two fundamental sets are the same. The connectivity is independent of the particular fundamental set in terms of which it was defined.

16. **Equivalences and Homologies.** The transformations involved in

what we have called the method of cutting amount in every case to replacing one of a set of circuits by a new circuit which is related to the circuits of the original set by an equivalence in the sense of Poincaré.* For example, in Fig. 8 the cut x joins the rear end of b_1 to the front end of c_1 and we have

$$x^1 \equiv b_1^1 - a_1^1 - b_1^1 + c_1^1$$

because b_1^1, $-a_1^1$, $-b_1^1$, c_1^1 and $-x^1$ taken in order bound a 2-cell. The two parts of the polygon are joined together along c_1 and c_1' (see Fig. 9) so that the set of circuts $a_1^1, b_1^1, c_1^1, \cdots$ has been converted into $a_1^1\ b_1^1\ x_1^1 \cdots$ where the two sets of curves are related by the set of equivalences:

$$\begin{aligned} a_1^1 &\equiv a_1^1 \\ b_1^1 &\equiv b_1^1 \\ x^1 &\equiv b_1^1 - a_1^1 - b_1^1 + c_1^1 \\ d_1^1 &\equiv d_1^1 \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

In case the vertices of the polygon are all in one conjugate set, these equivalences are what Poincaré calls *proper* equivalences because all the 1-cells in question begin and end at the same point. In case they are not all in one set, the equivalences are what he calls *improper* equivalences.

In the general case it is clear that, if we pass by the method of cutting from a polygon whose sides represent a set of 1-cells $a_1, a_2, \cdots, a_m$ to one whose sides represent a set of 1-cells $b_1, b_2, \cdots, b_m$, we have a set of equivalences of the form

$$(1)\quad \begin{aligned} b_1 &\equiv \epsilon_1^{11}a_1 + \epsilon_1^{12}a_2 + \cdots + \epsilon_1^{1m}a_m + \epsilon_1^{21}a_1 + \epsilon_1^{22}a_2 + \cdots + \epsilon_1^{km}a_m \\ b_2 &\equiv \epsilon_2^{11}a_1 + \epsilon_2^{12}a_2 + \cdots + \epsilon_2^{1m}a_m + \epsilon_2^{21}a_1 + \epsilon_2^{22}a_2 + \cdots + \epsilon_2^{km}a_m \\ &\cdots \\ b_n &\equiv \epsilon_n^{11}a_1 + \epsilon_n^{12}a_2 + \cdots + \epsilon_n^{1m}a_m + \epsilon_n^{21}a_1 + \epsilon_n^{22}a_2 + \cdots + \epsilon_n^{km}a_m \end{aligned}$$

in which the ϵ's are $+1$, -1, or 0.

17. The terms of an equivalence are not commutative. If we treat them as if they were commutative and collect terms, the equivalences (1) reduce to the *homologies*

$$(2)\quad \begin{aligned} b_1 &\sim \eta_1^1a_1 + \eta_1^2a_2 + \eta_1^3a_3 + \cdots + \eta_1^ma_m \\ b_2 &\sim \eta_2^1a_1 + \eta_2^2a_2 + \eta_2^3a_3 + \cdots + \eta_2^ma_m \\ &\cdots \\ b_n &\sim \eta_n^1a_1 + \eta_n^2a_2 + \eta_n^3a_3 + \cdots + \eta_n^ma_m, \end{aligned}$$

* Loc. cit., p. 60; see also Veblen, loc. cit., Chap. V, § 28.

in which the η's are integers. It is easily seen that in a homology the right and left sides together constitute the boundary of an oriented two-dimensional manifold though not in general a 2-cell.

If the coefficients η of these homologies are reduced modulo 2, we obtain the following homologies:

$$
\begin{array}{l}
b_1 \sim \zeta_1{}^1 a_1 + \zeta_1{}^2 a_2 + \cdots + \zeta_1{}^m a_m \\
b_2 \sim \zeta_2{}^1 a_1 + \zeta_2{}^2 a_2 + \cdots + \zeta_2{}^m a_m \\
\cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \\
b_n \sim \zeta_n{}^1 a_1 + \zeta_n{}^2 a_2 + \cdots + \zeta_n{}^m a_m
\end{array} \qquad \pmod 2 \tag{3}
$$

in which the ζ's are all 1 or 0. It is easily seen that in a homology (mod 2) the right and left sides constitute the boundary of a two-dimensional manifold which need not be oriented. (See Veblen, loc. cit., Chap. II, § 37.)

It is obvious that the homologies (mod 2) are the simplest and easiest to work with, that the Poincaré homologies are the next simplest, and that the equivalences are the most difficult on account of their non-commutative character. We shall therefore in what follows first consider the homologies (mod 2), then the Poincaré homologies.

18. We have now seen that it is possible to pass from any fundamental set of circuits to any other by the method of cutting, and also that the number of circuits in all fundamental sets is the same. In terms of the equivalences of § 16 this means that, between any two fundamental sets $a_1\, a_2 \cdots a_\mu$ and $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_\mu$, there exist the equivalences

$$
\begin{array}{l}
\bar{a}_1 \equiv \sum_{i=1}^{m} \sum_{j=1}^{\mu} \epsilon_1{}^{ij} a_j \\
\cdot \qquad \cdot \\
\cdot \qquad \cdot \\
\cdot \qquad \cdot \\
\bar{a}_\mu \equiv \sum_{i=1}^{m} \sum_{j=1}^{\mu} \epsilon_\mu{}^{ij} a_j
\end{array}
$$

From this there follow the homologies

$$
\bar{a}_p \sim \sum_{j=1}^{\mu} \beta_p{}^j a_j \qquad (p = 1, 2, \cdots, \mu),
$$

where

$$
\beta_p{}^j = \sum_{i=1}^{m} \epsilon_p{}^{ij}.
$$

We now want to investigate the question as to what are the conditions under which two fundamental sets of circuits satisfy a set of equivalences.

$$\bar{a}_p \equiv a_p \qquad (p = 1, 2, \cdots, \mu).$$

This is related to the question as to whether they satisfy the much weaker conditions

$$\bar{a}_p \sim a_p,$$

or the still weaker condition

$$\bar{a}_p \sim a_p \pmod 2.$$

With a view to studying these questions we introduce certain matrices expressing the relations among the circuits of a fundamental set.

19. **The Separation Matrix.** Suppose that each side of the polygon has been given a sense in the manner described in § 5. A 1-cell joining the forward ends of a_i and a_i' divides the polygon into two parts α and β. If one and only one of the sides a_j and a_j' is on the boundary of α, we will say that the conjugate pair $a_j\, a_j'$ *separates* $a_i\, a_i'$. As an obvious consequence of the definition we get the following theorems:

1: *If the pair $a_j\, a_j'$ separates the pair $a_i\, a_i'$, then the pair $a_i\, a_i'$ separates the pair $a_j\, a_j'$*;

2: *If the two sensed sides a_i and a_i' determine the same sense of description of the boundary of the polygon, then the pair $a_i\, a_i'$ separates itself*; *in the opposite case the pair $a_i\, a_i'$ does not separate itself.*

20. We will now construct a square matrix of $R_1 - 1$ rows which is uniquely determined by the polygon. Let e_{ij}, the element in the ith row and the jth column, be 1 or 0 according as the pair $a_j\, a_j'$ separates or does not separate the pair $a_i\, a_i'$; this matrix will be called the *separation matrix* of the polygon.

From the first theorem of § 19 it follows that e_{ij} is equal to e_{ji}; and from the second it follows that e_{ii} is 1 or 0 according as the sides a_i and a_i' have the same or opposite senses.

21. The separation matrix of the normalized polygon of a two-sided manifold is the following:

$$\left\|\begin{matrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{matrix}\right\|$$

The separation matrix of the polygon of a one-sided manifold in the normal form (1) of § 10 is:

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{Vmatrix}$$

These two matrices are also the separation matrices of the polygons whose sides are respectively in the order:

$$a_1\, b_1\, a_2\, b_2\, a_3\, b_3\, a_3'\, b_3'\, a_2'\, b_2'\, a_1'\, b_1' \text{ (Fig. 3)};$$

and

$$a_1\, a_2\, a_3\, a_4\, a_4'\, a_3'\, a_2'\, a_1' \text{ (Fig. 6)}.$$

Thus we see that a given separation matrix corresponds in general to more than one polygon. We will return later to the relations between two polygons which have the same separation matrix.

22. Let us first consider the effect of cutting along a 1-cell $\bar{a}_1$ equivalent to $a_1 + a_2$ and joining the two parts together along the sides a_1 and a_1' (cf. Fig. 2). This amounts to changing the fundamental set by the equivalence transformation

$$\begin{aligned} \bar{a}_1 &\equiv a_1 + a_2 \\ \bar{a}_2 &\equiv a_2 \\ \cdot \quad & \quad \cdot \\ \cdot \quad & \quad \cdot \\ \cdot \quad & \quad \cdot \\ \bar{a}_\mu &\equiv a_\mu . \end{aligned}$$

We shall see that this changes the polygon π into a new polygon whose separation matrix is obtained from that of π by multiplying on the right by the matrix of the above transformation and on the left by the conjugate of that matrix, and then reducing each element modulo 2.

23. Let us consider first the case where a_1 and a_1' have opposite senses on the boundary of the polygon. On comparing the separation matrix of the new polygon with that of the old we see: (*a*) The first row and column are unchanged—i.e., the row and column corresponding to $\bar{a}_1\, \bar{a}_1'$ on the transformed matrix M_1 are the same as the row and column corresponding to $a_1\, a_1'$ on the original matrix M; (*b*) The second row and the second column of M_1 are the result of adding the first row of M to the second row, adding the first column to the second column, and reducing each element modulo 2. For if the element e_{2i} of M_1 is 1, a single side of the pair $a_i\, a_i'$ is on each part of the boundary of π between a_1' and a_2'.

Hence $a_i\, a_i'$ separates one but not both of the pairs $a_1\, a_1'$ and $a_2\, a_2'$, and hence just one of the elements e_{1i} and e_{2i} of M is 1. Conversely, if one and only one of the elements e_{1i} and e_{2i} of M is 1, the pair $a_i\, a_i'$ separates one but not both of the pairs $a_1\, a_1'$ and $a_2\, a_2'$, and hence has one side on each of the parts of the polygon π between a_1' and a_2'; hence e_{2i} of M_1 is 1. (*c*) The element e_{ij} of M_1, where $i, j \neq 1, 2$, is the same as the element e_{ij} of M, for it is obvious that the above transformation does not affect the mutual relations of two pairs neither of which is $a_1\, a_1'$ or $a_2\, a_2'$.

In the case where a_1 and a_1' have the same sense, it follows similarly that the matrix M_1 is obtained from the matrix M by adding the first row and column to the second row and column respectively and reducing each element modulo 2.

24. We shall next see that any transformation of the polygon by a single cut may be obtained as the resultant of a series of cuts of the simple kind just considered. First it is obvious that the polygon obtained by two cuts $\bar{a}_1 \equiv a_1 + a_2$ and $\bar{\bar{a}}_1 \equiv \bar{a}_1 + a_3$ is the same as the polygon obtained by the cut $\bar{\bar{a}}_1 \equiv a_1 + a_2 + a_3$, where for the first cut the parts of the polygon are joined along the sides a_1 and a_1', for the second along $\bar{a}_1$ and $\bar{a}_1'$, and for the third along a_1 and a_1'. This shows that any transformation $\bar{a}_1 \equiv \Sigma_i a_i$ where the two parts of the polygon are joined along two sides a_1 and a_1', one of which has an end in common with $\bar{a}_1$, may be obtained by a series of transformations of the type $\bar{a}_1 \equiv a_1 + a_2$. In the case where the two parts of the polygon are joined along a pair of sides neither of which has an end in common with $\bar{a}_1$, we note that such a transformation may be obtained as the resultant of two transformations of the preceding type.* Thus any transformation of the polygon by a single cut may be accomplished by a series of transformations of the type $\bar{a}_1 \equiv a_1 + a_2$, and consequently any transformation of the polygon by the method of cutting may be accomplished by a series of transformations of the same type.

25. In § 23 we saw that M_1 can be obtained from M by adding the first row to the second row, adding the first column to the second column, and reducing each element modulo 2. From the theory of matrices† it follows that the ith row of M may be added to the jth row and the ith column to the jth column by multiplying M on the left by a certain matrix A of determinant 1 and multiplying the result on the right by the conjugate matrix A'. Since by § 24 any transformation by the method

* For example, the result of the cut $\bar{a}_k \equiv a_1 + \cdots + a_j + a_k + a_l + \cdots + a_m$, where the two parts are joined along a_k and a_k', is the same as the result of the cut $\bar{a}_k \equiv a_1 + \cdots + a_j + a_k$ followed by the cut $\bar{\bar{a}}_k \equiv \bar{a}_k + a_l + \cdots + a_m$, where the parts are joined along a_k and a_k' in the first case and along $\bar{a}_k$ and $\bar{a}_k'$ in the second.

† See Veblen and Franklin, these Annals, vol. 23, pp. 1–15.

of cutting may be effected by a series of cuts of the type described in § 23, it follows that *if the polygon π_1 is obtained from the polygon π by the method of cutting, the separation matrix M_1 of π_1 may be obtained from the separation matrix M of π by multiplying M on the left by a matrix A of determinant* 1 *and on the right by the conjugate matrix A', and then reducing each element modulo* 2.

The converse of this theorem is not true; we shall return to this question in a later paragraph.

26. Let us consider a polygon to which Reduction 2 of § 7 may be applied. In the separation matrix of the polygon the row and column which correspond to the conjugate pair $a_i\, a_i'$ will be made up wholly of zeros. The separation matrix of the polygon that is obtained by carrying out Reduction 2 is the matrix obtained by striking out the row and column of zeros. Reduction 1 is an operation of the type considered in § 25. Hence, *the connectivity of the manifold is one greater than the rank of the separation matrix of the polygon.*

27. **The Normalization of the Separation Matrix.** We have seen that a polygon whose conjugate pairs of sides correspond to the circuits of a fundamental set may be reduced to normal form by the method of cutting without reducing the number of sides. The separation matrix of the normalized polygon of a two-sided manifold is a matrix in which $e_{2n-1,\,2n}$ and $e_{2n,\,2n-1}$ $(n = 1, 2, \cdots, (R_1 - 1)/2)$ are equal to 1 and every other element is 0; the separation matrix of the normalized polygon of a one-sided manifold is a matrix in which $e_{n,\,n}$ $(n = 1, 2, \cdots, (R_1 - 1))$ is 1 and every other element is 0. These matrices are normal forms for symmetric matrices (mod 2) of determinant 1.* As a result of these considerations and § 25 we have the theorem: *If M is the separation matrix of a polygon whose vertices constitute a single conjugate set, there exists a matrix A of determinant* 1 *such that the product $A\,M\,A'$ is equivalent modulo* 2 *to the normal form of a symmetric matrix of determinant* 1, *and such that A corresponds to a series of cuts on the polygon.*

28. We have seen that, when the polygon is in normal form, the separation matrix is also in normal form. The converse of this statement is, however, not true, as we saw in § 21. Instead we have the following theorem: *If the separation matrix is in normal form, the polygon may be normalized by a series of cuts of which the corresponding matrix A is the identity, modulo* 2.

Let us consider the one-sided and two-sided cases separately. In the one-sided case the polygon is in normal form or else there is a pair $a_i\, a_i'$ such that one of the two parts of the boundary between a_i and a_i' is

* See Veblen and Franklin, loc. cit., p. 14.

made up of the sides $a_j\, a_j'\, a_k\, a_k' \cdots a_l\, a_l'$ in that order. The cut joining the forward ends of a_i and a_i' gives the following transformation on the circuits of the fundamental set when the two parts are joined along a_i and a_i':

$$
\begin{aligned}
\bar{a}_1^1 &\equiv a_1^1 \\
&\cdot \\
&\cdot \\
&\cdot \\
\bar{a}_i^1 &\equiv a_i^1 + 2a_j^1 + 2a_k^1 + \cdots + 2a_l^1 \\
&\cdot \\
&\cdot \\
&\cdot \\
\bar{a}\mu^1 &\equiv a\mu^1.
\end{aligned}
$$

The matrix A corresponding to this cut has a main diagonal made up of 1's and no other elements excepting 0's and 2's. This transformation has increased by one the number of pairs of sides which are in the order $a_i\, a_i'$ on the boundary of the polygon. By repeating this process the polygon may be reduced to normal form.

In the two-sided case, if the polygon is not in normal form, there must be some group of four sides $a_i\, b_i\, a_i'\, b_i'$ such that between two elements of the group, say between b_i and a_i', there are one or more groups of four consecutive sides $a_j\, b_j\, a_j'\, b_j'$. A cut joining the forward ends of b_i and b_i' gives a matrix A which is equal mod 2 to the identity, and so does a cut joining the forward ends of the sides $\bar{a}_i$ and $\bar{a}_i'$ obtained from the first cut. This transformation increases the number of groups of four consecutive sides of the form $a_j\, b_j\, a_j'\, b_j'$ and may be continued until the polygon is normalized.

29. From these theorems we can now deduce an important theorem analogous to the theorem given by Poincaré on page 70 of the Fifth Complement. *Given two fundamental sets $a_1, a_2, \cdots, a_\mu$ and $b_1, b_2, \cdots, b_\mu$; in order that there shall exist a fundamental set $c_1, c_2, \cdots, c_\mu$, into which the a's are transformable by a homeomorphism of the manifold with itself and which are homologous (mod 2) with $b_1, b_2, \cdots, b_\mu$ respectively; it is necessary and sufficient that the separation matrix of the a's shall be the same as that of the b's.*

If the a's are transformable into the c's by a homeomorphism of the manifold, this homeomorphism determines a homeomorphism of the polygon of the a's with that of the c's. Hence the separation matrix M_a of the a's is the same as the separation matrix M_c of the c's. By § 14 it is possible to pass from the c's to the b's by the method of cutting. This determines a set of homologies connecting the c's with the b's, and if A is the matrix of this set of homologies, we have by § 25,

$$M_b = A' \cdot M_c \cdot A,$$

where M_b is the separation matrix of the b's. By hypothesis we have a set of homologies, $b_i \sim c_i$ (mod 2). But there cannot be more than one set of homologies (mod 2) connecting the b's and the c's, since otherwise there would be homologies of the form $c_i \sim c_j$ (mod 2) among the c's. Hence A is the identity matrix and $M_b = M_c$. Hence $M_a = M_b$.

Conversely, let us suppose that $M_a = M_b$. The a's and the b's respectively can be converted by the method of cutting into fundamental sets $d_1, d_2, \cdots, d_\mu$ and $f_1, f_2, \cdots, f_\mu$ respectively whose polygons are in normal form. Then, if a sequence of cuts is applied to $f_1, f_2, \cdots, f_\mu$ which is homeomorphic with a sequence of cuts which converts $d_1, d_2, \cdots, d_\mu$ back into $a_1, a_2, \cdots, a_\mu$, the f's are evidently converted into a fundamental set $c_1, c_2, \cdots, c_\mu$ which is capable of being transformed into $a_1, a_2, \cdots, a_\mu$ by a homeomorphism of the manifold with itself. Hence $M_a = M_c$ and therefore $M_c = M_b$. But the c's have been obtained from the b's by the method of cutting and so are related to them by an equation of the form $A' \cdot M_c \cdot A = M_b$. By § 28 the c's can be obtained from the b's by a series of cuts for which A is the identity. Since there cannot be more than one set of homologies (mod 2) relating the b's and the c's, it follows that

$$\begin{matrix} b_1 \sim c_1 \\ b_2 \sim c_2 \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ b_\mu \sim c_\mu \end{matrix} \quad \text{(mod 2)}.$$

30. **The Matrix of Signed Separations.** In the case of the two-sided manifold we may give an algebraic sign to the separations of pairs of sides of the polygon. First let us assign a sense arbitrarily to the boundary of the polygon. Of each conjugate pair one side agrees in sense with the boundary and the other disagrees with it; let the side which agrees in sense with the boundary be designated by a_i, and the other by a_i'. Suppose an arc drawn joining the forward ends of a_i and a_i', and let α be the part of the polygon on whose boundary the two sides a_i and a_i' appear. If $a_j\, a_j'$ separates $a_i\, a_i'$ and the side a_j is on the boundary of β, we will say that $a_j\, a_j'$ separates $a_i\, a_i'$ *positively*; if a_j is on α, we will say that $a_j\, a_j'$ separates $a_i\, a_i'$ *negatively*.

As an immediate consequence of the above definitions it follows that *if $a_j\, a_j'$ separates $a_i\, a_i'$ positively, then $a_i\, a_i'$ separates $a_j\, a_j'$ negatively.* In like manner it follows that *reversing the senses of the sides a_i and a_i' changes the sign of every separation by that pair.*

Let us give each non-zero element of the separation matrix the sign + or − according as it stands for a positive or a negative separation. The resulting matrix will be called the *matrix of signed separations*. From the last paragraph it follows that this matrix is skew-symmetric.

31. Consider a cut $\bar{a}_1 \equiv a_1 + a_2$ (cf. Fig. 2). The 1-cell $\bar{a}_1$ divides the polygon into two parts one of which has on its boundary $\bar{a}_1$, a_1, and a_2. If $\bar{a}_1$ is given a sense which disagrees with the sense of a_1 on the boundary of this part, then the signed separations by $\bar{a}_1\,\bar{a}_1'$ on π_1 will be identical with the signed separations by $a_1\,a_1'$ on π. With this convention in assigning a sense to $\bar{a}_i$, we will prove that *if π_1 is obtained from π by a cut $\bar{a}_1 \equiv a_1 + a_2$, the matrix of signed separations S_1 of π_1 may be obtained from the matrix of signed separations S of π by multiplying the first row by − 1 and adding it to the second row, and performing the same operation on columns.*

Proof: The separation matrix of any polygon can be obtained from the matrix of signed separations by reducing each element of the latter modulo 2. The matrix given by the theorem when each element is reduced modulo 2 is the separation matrix of the transformed polygon. (Cf. § 25.) Therefore the proof of the theorem reduces to the proof of the facts (1) that the matrix of the transformed polygon given by the theorem contains no element different from 0, 1, and − 1, and (2) that by the method given in the theorem the proper sign is attached to each element. To prove (1) it is sufficient to show that if e_{1i} and e_{2i} are both different from 0 they have the same sign. This means that if $a_i\,a_i'$ separates both $a_1\,a_1'$ and $a_2\,a_2'$, it separates both positively or both negatively, which follows from the fact that a_1 and a_2 have the same sense. To prove (2) consider first the case where $a_i\,a_i'$ separates $a_1\,a_1'$ but does not separate $a_2\,a_2'$ on π. We are to show that e_{2i} of S_1 is 1 or − 1 according as e_{1i} of S is − 1 or 1. This follows from the fact that a_i or a_i' is on the part of the boundary of π between a_1 and a_1' which does not contain a_2'. Finally consider the case where $a_i\,a_i'$ separates $a_2\,a_2'$ but does not separate $a_1\,a_1'$. In this case the transformation does not affect the separation of $a_2\,a_2'$ by $a_i\,a_i'$, which gives that if e_{1i} of S is 0, e_{2i} of S_1 is the same as e_{2i} of S.

32. Consider the cut $\bar{a}_1 \equiv a_1 + a_2'$. This can be reduced to the case treated in § 31 by changing the sense of each of the two sides a_2 and a_2'. This changes the sign of each element in the second row and each element in the second column (§ 30). Now carry out the transformation $\bar{a}_1 \equiv a_1 + a_2$; the corresponding transformation on S multiplies the first row and column by − 1 and adds them to the second row and column respectively. Finally reverse the senses of a_2 and a_2' again and carry out the corresponding change on the matrix. The result may be expressed

as follows: *If π_1 is obtained from π by a cut $\bar{a}_1 \equiv a_1 + a_2'$, the matrix S_1 of signed separations of π_1 may be obtained from the matrix S of signed separations of π by adding the first row to the second row and performing the same operation on columns.*

33. By omitting the phrase "modulo 2" in the theorems of § 25 and § 27 and replacing M and A by S and B respectively, we get two theorems concerning the matrix of signed separations. That these theorems are true follows easily from §§ 31, 32. Corresponding to the theorem of § 28 we have: *If the matrix of signed separations is in normal form, the polygon may be normalized by a set of cuts of which the matrix B is the identity.*

To prove the theorem we need only (cf. § 28) show that the matrix B corresponding to the cut $\bar{b}_i \equiv b_i + a_j + b_j + a_j' + b_j'$ is the identity. This cut may be effected by the following series of cuts:

$$x_1 \equiv b_i + a_j, \qquad x_2 \equiv x_1 + b_j, \qquad x_3 \equiv x_2 + a_j', \qquad \bar{b}_i \equiv x_3 + b_j'.$$

The product of the matrices of these transformations is the identity.

By proceeding as in § 29 we may now establish a theorem identical with that of § 29 with omission of the modulo 2 condition. This is equivalent to the theorem given by Poincaré (l.c., p. 70).

34. Given any series of cuts on the polygon we have seen that there corresponds to it a matrix B whose determinant is 1. As a result of the first theorem of § 33 we have that there exists more than one series of cuts corresponding to a given matrix, if there exists one. It can be shown however, by means of a simple example, that *not every matrix of determinant* 1 *corresponds to the transformation of a given polygon by a series of cuts.*

35. **Criterion for a Non-singular Circuit.** *Any simple circuit which is not homologous to zero is homologous to a linear combination, with coefficients relatively prime, of circuits of any fundamental set.**

Proof: The circuit may be deformed into one which passes through the point A of any fundamental set F. The image on π of the circuit will be a set of non-intersecting arcs. By the method of cutting we may obtain a polygon π_1 on which the image of the circuit is an arc joining two consecutive vertices.

The separation matrix M_1 of π_1 is equal to AMA' (modulo 2) where M is the separation matrix of π and A is the product of a set of matrices $A_k A_j A_i \cdots A_1$, each of which corresponds to a single cut and is therefore of determinant 1. The matrix $A_k' A_j' A_i' \cdots A_1'$ is the matrix of the homology transformation of the circuits of F into the circuits of F_1. (See § 22.) The elements of the ith row of this matrix are the coefficients of a combination of the circuits of F which is homologous to the circuit

* Poincaré proves this theorem and its converse for two-sided manifolds, l.c., page 70.

C_i' of F_1, $C_i' \sim \Sigma_j e_{ij} C_j$. Since the matrix is of determinant 1 the theorem follows.

From the foregoing it is evident that the theorem just proved is true in the case of a two-sided manifold without the restriction in the hypothesis to circuits which are not homologous to zero. It is equally evident that the restriction is necessary in the case of a one-sided manifold, for a circuit whose image on the polygon together with two sides C_i and C_i' which have the same sense bounds a part α of the polygon is equivalent to $2C_i$. However C_i, a circuit of the fundamental set, is homologous to a linear combination with coefficients relatively prime of circuits of any fundamental set. Thus we have the result that on a one-sided manifold any simple circuit is homologous to a linear combination with coefficients relatively prime of any fundamental set, or else it is homologous to a linear combination with coefficients containing 2 as a highest common factor. This factor 2 is the *coefficient of torsion* of a one-sided manifold.

Any linear combination with coefficients relatively prime of circuits of a fundamental set for a two-sided manifold is homologous to a simple circuit.

Proof: The method of proof will be to show that a matrix B with an arbitrary first row, provided the elements are relatively prime, may be built up by taking the product of a set of matrices each of which corresponds to a cut on the polygon. First reduce the polygon to normal form. Let D be the matrix to which this reduction corresponds. We shall now find a matrix C such that $B = C \cdot D$ has an arbitrary first row and such that the matrix C corresponds to a set of cuts. That B may have an arbitrary first row, it is sufficient that the first row of C may be chosen arbitrarily.

The two transformations which follow can be carried out on the normalized polygon and each transformation leaves the polygon in normal form.

$$\bar{a}_{2n-1} \sim a_{2n-1} + b_{2n}, \tag{1}$$

$$\bar{a}_{2n-1} \sim a_{2n-1} + a_{2m-1} \quad \text{followed by} \quad \bar{b}_{2m} \sim b_{2m} - b_{2n}. \tag{2}$$

The matrix B_1 corresponding to transformation (1) is (for $n = 2$) of the form:

$$B_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{Vmatrix}.$$

The matrix corresponding to the transformation (2) is (for $n = 2$, $m = 1$) of the form:

$$B_2 = \left\Vert \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array} \right\Vert .$$

By taking products of matrices of the type B we may obtain a matrix of the form:

$$B_3 = \left\Vert \begin{array}{ccccccccc} a_{11} & a_{12} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{2p-1,\,2p-1} & a_{2p-1\;2p} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{2p,\,2p-1} & a_{2p,\,2p} \end{array} \right\Vert ,$$

where $a_{i,\,i+1}$ and $a_{i,\,i}$ are any two integers relatively prime and where

$$\left| \begin{array}{cc} a_{i,\,i} & a_{i,\,i+1} \\ a_{i+1,\,i} & a_{i+1,\,i+1} \end{array} \right| = 1.$$

This follows from the fact that any two-rowed matrix of determinant 1 may be normalized by elementary transformations on the rows alone. The elements $a_{i,\,i}$ and $a_{i,\,i+1}$ may be chosen so that $e_{1,\,i}/a_{i,\,i} = e_{1,\,i+1}/a_{i,\,i+1}$, where $e_{1,\,i}$ and $e_{1,\,i+1}$ are elements of the arbitrarily given first row of C. Then, by application of matrices of type B_2 above, any odd row of B_3 may be added to the first row a sufficient number of times to give the arbitrary first row of C. This completes the proof of the theorem.

36. **Intersections of Circuits of Fundamental Set.** Consider two sensed circuits C_1 and C_2 on a two-sided manifold. Let them have a point P in common. A 2-cell a_1^2 may be constructed which contains P, and no other point common to the two circuits, as an interior point and which contains a simple arc of each of the circuits on the interior. Let one of the senses of description of the boundary be designated as positive. Let the forward end of the arc of C_i which is interior to a_1^2 be called a_{i1}^0 and the other end a_{i2}^0. If the points $a_{11}^0\, a_{12}^0$ separate the points $a_{21}^0\, a_{22}^0$, the

two circuits C_1 and C_2 will be said to *intersect* at P. If the two circuits intersect at P and the point a_{22}^0 is on the part of the boundary of a_1^2 that runs positively from a_{11}^0 to a_{12}^0, C_2 will be said to intersect C_1 *positively:* if a_{21}^0 is on that arc C_2, it will be said to intersect C_1 *negatively.* We have as an obvious theorem: *If C_2 intersects C_1 positively at the point P, then C_1 intersects C_2 negatively at the point P.*

37. Consider now two circuits C_1 and C_2 which have more than one point in common. A 2-cell may be constructed at each common point as in § 36. These 2-cells may be assigned senses in such a way that they all agree in sense. Making use of these sensed 2-cells, we may determine the number of positive and the number of negative intersections of the circuit C_2 with the circuit C_1. Let $N(C_2, C_1)$ be a positive or a negative number equal to the number of positive intersections of C_2 with C_1 minus the number of negative intersections of C_2 with C_1. As a result of this definition and the theorem of § 36 we have

$$N(C_2, C_1) = -N(C_1, C_2).$$

The following theorems may be easily proved:

If $C_1 \sim 0$, and C_2 is any circuit whatever, then $N(C_2, C_1) = 0$.

If $C_3 \equiv C_1 + C_2$, and C_4 is any circuit whatever, then

$$N(C_4, C_3) = N(C_4, C_1) + N(C_4, C_2).$$

If $C_1 \sim C_2$, and C_3 is any circuit whatever, then $N(C_3, C_1) = N(C_3, C_2)$.

38. **The Intersection Matrix.** Let us consider the intersections of pairs of circuits of a fundamental set, and let us construct a matrix of $2p$ rows and $2p$ columns by making the element e_{ij} equal the number $N(C_j, C_i)$. Since the circuits are simple circuits and no two have more than one point in common, the elements of the matrix will be 0, 1, and -1. Every element e_{ii} will be 0; the element $e_{j,i}$ will be the negative of the element e_{ij}. Thus the matrix is skew-symmetric.

39. A cut $\bar{a}_1 \equiv a_1 + a_2$ performs a certain transformation on the circuits of the fundamental set. According to § 37 the intersections by the circuit on which $\bar{a}_1$ is imaged are obtained by adding the rows corresponding to C_1 and C_2 in the intersection matrix N. Then to get the intersection matrix of the transformed fundamental set we add the second row of N to the first row and perform the same operation on columns. This may be accomplished by multiplying N on the left by the matrix which is the inverse of the conjugate of the matrix B used in § 33, and by multiplying on the right by the conjugate matrix.

From this it follows at once that *if the fundamental set F_1 is obtained from the fundamental set F by the method of cutting, the intersection matrix*

N_1 of F_1 and the intersection matrix N of F satisfy the relation $N_1 = T \cdot N \cdot T'$, where T is a matrix of determinant 1.

40. The polygon was normalized by the method of cutting. When the polygon is in normal form, the intersection matrix of the corresponding fundamental set is in normal form, as can be seen by constructing a neighborhood of the point A in the manner of § 36, and the matrix of signed separations is also in normal form. These two normal forms are the same. The matrix of signed separations of the original polygon is normalized by a matrix $B = (B_k \cdot \cdots \cdot B_2 \cdot B_1)$; the intersection matrix is normalized by a matrix $(B_k')^{-1} \cdots (B_2')^{-1}(B_1')^{-1}$. From this it follows by a simple computation with the matrices that *the intersection matrix is the negative of the reciprocal of the matrix of signed separations.*