Vector fields on π-manifolds

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1. Introduction

Let M^n be an n-dimensional differentiable manifold. As in [8], we denote by $\sigma(M)$ (the "span" of M) the maximal number of linearly independent vector fields on M, and we also put $\sigma_n = \sigma(S^n)$.

The number σ_n has been determined by J. F. Adams in [1]. In this paper we show how to determine the span of an arbitrary π-manifold. This is done by two theorems, below, the first of which states that only two values are possible for the span of a π-manifold of given dimension, and the second provides an easy way to decide which value is the case.

THEOREM 1. *If M^n is a π-manifold, then either $\sigma(M^n)=\sigma_n$ or $\sigma(M^n)=n$.*

The invariant which separates these cases is the semi-characteristic χ^* defined as follows: if n is even, then $\chi^*(M^n) = \frac{1}{2}\chi(M^n)$ where χ denotes the Euler characteristic; if $n = 2r+1$, then $\chi^*(M^n)$ is the mod 2 congruence class of $\sum_{i=0}^{r} \operatorname{rank} H_i(M; Z_2)$. We also define the *reduced* semi-characteristic $\hat{\chi}$ by $\hat{\chi}(M) = 1 - \chi^*(M)$, and note that it is additive with respect to connected sum of manifolds.

THEOREM 2. *$\hat{\chi}$ is a homomorphism of the semi-group of connected oriented n-dimensional π-manifolds onto Z for n even, and onto Z_2 for n odd, such that if $n \neq 1, 3, 7$, then $\sigma(M^n) = n$ if and only if $\hat{\chi}(M^n) = 1$.*

It is clear that Theorem 2 reduces to the theorems of Kervaire [4], [5]. The proof given here differs from Kervaire's. Theorem 1 was first proved for $n \neq 4k-1$ by E. Thomas. Later the authors and Thomas produced independent proofs of this for general n. Thomas' treatment of the subject is substantially different from that of the present paper and is presented in [8].

We wish to thank E. Thomas for bringing this problem to our attention, and for many fruitful discussions concerning it.

An interesting example of applications of Theorems 1 and 2 is provided by Stiefel manifolds (real, complex, or quaternionic). It is known that they are π-manifolds [2]. We prove

COROLLARY. *Stiefel manifolds $V_{n,k}$ are parallelizable if $k > 1$.*

PROOF. Compare [7]. One could of course apply Theorem 2. The following

* Supported respectively by grants NSF GP 1610 and NSF GP 2497.

method simplifies computations and applies in many situations. Since $V_{n,k}$ is fibered over $V_{n,k-1}$, it follows that $\sigma(V_{n,k}) \geqq \sigma(V_{n,k-1})$. Using the elementary fact that dim $V_{n,k-1} > (2/3)$ dim $V_{n,k}$ for $k > 2$, together with the inequality $\sigma_m < 2m/3$ (which follows easily from [1]) for $m \neq 1, 3, 7$, we deduce from Theorem 1 that, if $V_{n,2}$ is parallelizable, then so are all the $V_{n,k}$ for $k > 1$. To check that $V_{n,2}$ is parallelizable, we use Theorem 2.

Later on in this paper we shall need a sharper inequality for σ_n, namely: $2\sigma_n < n - 1$ for $n \neq 1, 3, 7, 15$. This is also easily checked from the explicit formula for σ_n in [1].

2. The Gauss map

We recall that a π-manifold is a manifold which can be embedded in some euclidean space with trivial normal bundle. It is known [6] that M is a π-manifold if and only if M is stably parallelizable, that is, if and only if $\tau(M) + \varepsilon$ is trivial, where $\tau(M)$ is the tangent bundle of M, and ε is a trivial line bundle on M.

Let M^n be a closed oriented π-manifold, and let F be a framing of the stable tangent bundle $\tau + \varepsilon$ of M compatible with the given orientation on $\tau + \varepsilon$. Referring a vector $v \in \tau + \varepsilon$ to the coordinate system based on the frame F (at the base point of v), we obtain a point $F^*(v) \in R^{n+1}$. If $x \longrightarrow \varepsilon(x)$ denotes the canonical cross-section of the trivial line bundle ε, then $x \longrightarrow F^*(\varepsilon(x))$ is the "Gauss map"

$$\nu_F: M^n \longrightarrow S^n .$$

Clearly ν_F is covered by a bundle map of $\tau(M^n)$ into $\tau(S^n)$. Thus $\nu_F^*(\tau(S^n)) = \tau(M^n)$ which implies $\sigma(M^n) \geqq \sigma_n$.

The degree of the map ν_F will be denoted by $d(M, F)$. If $-F$ is the framing (of $\tau(-M) + \varepsilon$) obtained from F by reversing the first vector, then

$$d(-M, -F) = d(M, F) , \tag{2.1}$$

since ν_{-F} differs from ν_F by a reflection through a hyperplane, but the orientation of M has also been changed.

We note that $\sigma(M) \geqq k$ if and only if for each framing F there is a map $f_k: M^n \longrightarrow V_{n+1,k+1}$ such that $\nu_F = \pi \circ f_k$ where $\pi: V_{n+1,k+1} \longrightarrow V_{n+1,1} = S^n$ is the canonical projection.

If F and G are two framings of $\tau + \varepsilon$, then there is a map $g: M^n \longrightarrow \mathrm{SO}(n+1)$ such that, for each $x \in M$, $F_x = g(x) \cdot G_x$. Thus $\nu_G(x) = g(x)(\nu_F(x))$. Let $\pi: \mathrm{SO}(n + 1) \longrightarrow S^n$ be the canonical projection, and note that ν_F is homotopic to a map ν_F' taking the complement of some n-cell U onto the base point $\pi(e) \in S^n$, while g is homotopic to a map taking U into e. Thus ν_G is homotopic to the composition

$$M^n \xrightarrow{g' \times \nu_F'} \mathrm{SO}(n + 1) \vee S^n \xrightarrow{\pi \vee 1} S^n \vee S^n \longrightarrow S^n ,$$

the last map being of degree (1,1). It follows that

$$d(M, G) = \deg(\pi \circ g) + d(M, F) . \tag{2.2}$$

Notice that for n odd, $\deg(\pi \circ g)$ can be any even integer (at least), since there is a map $f: S^n \to SO(n+1)$ with $\deg(\pi \circ f) = 2$.

3. Two lemmas

3.1. LEMMA. *Let W^n be a π-manifold with boundary S^{n-1} (n odd). Let K be a $(k-1)$-connected complex with $2k \geqq n$. Then, for any map $f: W \to K$, $f \mid \partial W$ is null-homotopic.*

PROOF. By [6; Th. 6.6] W can be submitted to a sequence of spherical modifications (away from ∂W) ending with a contractible manifold. We must only check that at each stage there is a map into K extending the given map f on ∂W. But, by the proof of [6; Th. 6.6] surgery need only be performed on embedded m-spheres for $m \leqq (n-1)/2 < k$. Thus, at any stage, the map into K may be assumed to be constant in the neighborhood of this embedded m-sphere, and hence the map into K can be defined in the obvious way on the manifold resulting from this spherical modification.

3.2. LEMMA. *Let M^n be a π-manifold with n odd. Let $\nu: M^n \to S^n$ be any map of degree one. Then ν can be lifted to $f: M^n \to V_{n+1,k+1}$ with $\nu = \pi \circ f$ if and only if $k \leqq \sigma_n$.*

PROOF. The lemma is trivial for $n = 1, 3, 7$. Assume for the moment that $n \neq 1, 3, 7, 15$. Suppose that $k = \sigma_n + 1$, and that $f: M^n \to V_{n+1,k+1}$ exists such that $\pi \circ f = \nu$ is of degree one. Let W be the complement of an open cell in M. We may assume that ν is one-one on $M - W$ and maps W to a point. Then f maps W into a fibre $V_{n,k}$ of π. Since $V_{n,k}$ is $(n-k-1)$-connected and (for $n \neq 1, 3, 7, 15$) $2(n-k) = 2(n - \sigma_n - 1) > n - 1$ (that is, $2\sigma_n < n - 1$, see [1]), it follows from 3.1 that $f \mid \partial W: \partial W \to V_{n,k}$ is null-homotopic. However, $f \mid \partial W$ is just the characteristic map of the fibering $\pi: V_{n+1,k+1} \to S^n$ since ν is one-one on $M - W$. Thus $f \mid \partial W$ is not null-homotopic in $V_{n,k}$, for otherwise $\pi: V_{n+1,k+1} \to S^n$ would have a cross-section, and S^n would have a field of k-frames, $k = \sigma_n + 1$.

It remains to consider the case $n = 15$. More generally, let σ'_n be the largest integer k for which there exists a π-manifold M^n, and a map $f: M^n \to V_{n+1,k+1}$ with $\pi \circ f$ of degree one. We have shown that $\sigma'_n = \sigma_n$ for $n \neq 15$, and we wish to show this for $n = 15$. We claim that, if $k \leqq \sigma'_n, \sigma'_m$, then $k \leqq \sigma'_{n+m+1}$. To see this, let $f_1: M_1^n \to V_{n+1,k+1}$ and $f_2: M_2^m \to V_{m+1,k+1}$ be as above. Let $g: M_1 \times M_2 \times S^1 \to M_1 * M_2$ (the join of M_1 and M_2) be a map of degree one (that is, inducing isomorphism on homology in dimension $n + m + 1$). (g can be taken to be the natural map into the reduced join of M_1 and M_2 followed by a homotopy equivalence with $M_1 * M_2$.) Consider the diagram

$$
\begin{array}{ccccc}
 & & V_{n+1,k+1} * V_{m+1,k+1} & \longrightarrow & V_{n+m+2,k+1} \\
 & \nearrow & \downarrow & & \downarrow \\
M_1 \times M_2 \times S^1 \longrightarrow M_1 * M_2 & \searrow & S^n * S^m & \xrightarrow{\approx} & S^{n+m+1}
\end{array}
$$

where the map $V_{n+1,k+1} * V_{m+1,k+1} \to V_{n+m+2,k+1}$ is the map defined by James [3]. The bottom row of this diagram consists of maps of degree one, so that $k \leqq \sigma'_{n+m+1}$ by definition.

Now we have, by induction, $\sigma'_n \leqq \sigma'_{pn+p-1}$. For $n = 15$, $p = 3$ this yields

$$\sigma'_{15} \leqq \sigma'_{47} = \sigma_{47} = 8 = \sigma_{15} .$$

Consequently $\sigma'_n = \sigma_n$ for all n, and the lemma follows.

4. Proof of Theorem 1

4.1. *Suppose that either n is even and, for some F, $d(M^n, F) = 0$; or n is odd, and $d(M^n, F) \equiv 0 \bmod 2$. Then $\sigma(M^n) = n$.*

PROOF. The assumptions imply, by 2.2 and the remark following it, that in either case there is a framing F such that $d(M, F) = 0$. But then ν_F is null-homotopic and can be lifted to $f_n : M \to \mathrm{SO}(n+1) = V_{n+1,n+1}$. This gives the desired framing of $\tau(M)$.

4.2. *Let n be even, and suppose that for some F, $d(M, F) \neq 0$. Then $\sigma(M) = \sigma_n$.*

PROOF. For n even, $\sigma_n = 0$. Suppose $\sigma(M) \neq 0$. Then there is, for each F, a map $f_1 : M \to V_{n+1,2}$ such that $\nu_F = \pi \circ f_1$. Since $H_n(V_{n+1,2}) = Z_2$, it follows that $d(M, F) = \deg(\pi \circ f_1) = 0$, which proves 4.2.

4.3. *Let n be odd $\neq 1, 3, 7$ and suppose that, for some F, $d(M, F) \equiv 1 \bmod 2$. Then $\sigma(M) = \sigma_n$.*

PROOF. By 2.2 we can find a framing F such that $d(M, F) = 1$. Therefore 4.3 follows from 3.2.

Now, 4.1, 4.2 and 4.3 imply Theorem 1.

5. Proof of Theorem 2

We first prove

5.1. *If n is even, then $d(M, F)$ does not depend on F. If n is odd $\neq 1,3,7$, then the* mod 2 *congruence class of $d(M, F)$ does not depend on F.*

PROOF. Suppose first that n is even. Then, in 2.2, we have $\deg(\pi \circ g) = 0$, for π factors through $V_{n+1,2}$, and $H_n(V_{n+1,2}) = Z_2$ in this case. Therefore $d(M, F)$ does not depend on F.

Suppose now that n is odd. By 4.1 and 4.3, the mod 2 congruence class of $d(M, F)$ is determined by the span of M, and therefore does not depend on F.

We now define $d(M)$ to be the mod 2 congruence class of $d(M, F)$ if n is odd, and to be $d(M, F)$ if n is even.

To prove Theorem 2, it is now sufficient to prove the following theorem:

THEOREM 3. *If M^n is a π-manifold and $n \neq 1, 3, 7$, then $d(M^n) = \chi^*(M^n)$.*
The proof is preceded by a lemma.

5.2. LEMMA. *If M^n is the boundary of an oriented π-manifold W^{n+1}, and if F is a framing of $\tau(W)$, then $d(M, F) = \chi(W)$.*

PROOF. Here we have used F to denote also the restriction of F to a framing of $\tau(W) \mid M = \tau(M) + \varepsilon$. Also, we select our orientation conventions so that the canonical section (orientation) of ε is the *outward* normal vector. Let W be riemannian and let f be a real-valued differentiable function on W, which has only a finite number of non-degenerate critical points, and which is constant on M. We may assume that $(\operatorname{grad} f) \mid M$ is the canonical section of ε (the outward normal to M in W) and that $\| \operatorname{grad} f \| \leqq 1$ everywhere on W.

Let μ_F denote the map $W^{n+1} \to D^{n+1}$ defined by $x \to F^*((\operatorname{grad} f)_x)$, where D^{n+1} is the unit disk in R^{n+1}. Clearly $\nu_F = \mu_F \mid M$, and thus $\deg \nu_F = \deg \mu_F$ (considering μ_F as a map of pairs $(W, M) \to (D^{n+1}, S^n)$), as follows easily from the homology sequences of the pairs (W, M) and (D^{n+1}, S^n). Since f is non-degenerate, 0 is a regular value of μ_F, and thus $\deg \mu_F$ is the sum of the "local degrees" at the critical points of f; that is, $\deg \mu_F$ is the sum of the indices of the vector field $\operatorname{grad} f$. However, it is well-known that the latter is also the Euler characteristic of W.

PROOF OF THEOREM 3. Suppose first that n is even, and consider $M \times I$. By 2.1, 2.4, and 5.1, we have

$$\chi(M) = \chi(M \times I) = d(M) + d(-M) = 2d(M) ,$$

which proves the theorem in this case.

Assume now that n is odd and different from 1, 3, and 7. By [6, Th. 6.6], M^n is frame-cobordant to a homotopy sphere $-\Sigma^n$, that is, there exists an $(n+1)$-manifold W^{n+1}, and a framing F of $\tau(W)$, such that $\partial W = M \cup \Sigma$, and $F \mid M$ is the given framing of $\tau(M) + \varepsilon$.

Recall that for any even dimensional π-manifold V^{2r}, $\chi(V) \equiv \chi^*(\partial V) \pmod 2$. This follows from the fact that $\chi(V) = \chi^*(\partial V) + \rho$ where ρ is the rank of the intersection pairing $H_r(V, Z_2) \otimes H_r(V, Z_2) \to Z_2$ [6, Lem. 5.9], and the fact that

ρ is even since V is a π-manifold [6, p. 525].

Thus, in the present situation,

$$\chi^*(M) + \chi^*(\Sigma) \equiv \chi^*(\partial W) \equiv \chi(W) \equiv d(\partial W, F \mid \partial W) \equiv d(M) + d(\Sigma) \pmod 2.$$

Now $\chi^*(\Sigma) = 1$, and $d(\Sigma)$ must be odd; for if $d(\Sigma)$ were even, then 4.1 would imply that Σ is parallelizable contrary to the assumption that $n \neq 1, 3, 7$. This concludes the proof of Theorem 3, and hence also of Theorem 2.

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(Received September 2, 1965)