Note added in proof

Concerning Problem 11.1, J. C. SU has given an example with $F=CP^3$ elsewhere in these Proceedings. In a recent paper entitled "Representations at fixed points of smooth actions of compact groups", I have shown that $F=\mathrm{pt}+CP^2$ cannot occur with the possible exception of $X=S^3\times S^4$. Similarly, $F=\mathrm{pt}+QP^2$ cannot occur except in six possible cases. $F=\mathrm{pt}+$Cayley plane cannot occur at all.

Concerning Conjecture 11.5, I have shown (loc. cit.) that, for $n=7$, $F=\mathrm{pt}+S^6$ or F is 3 points. Thus the conjecture has been reduced to the question of whether or not F can consist of exactly 3 points.

I have also proved conjecture 11.6 except for the case $r=-1$ (loc. cit.).

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Equivariant Homotopy

GLEN E. BREDON*

This report concerns the homotopy classification of equivariant maps between spheres with involutions. Some of the results we shall describe are also mentioned in the research announcement [3]. For the main part the proofs of the results stated here are much too complicated by details to be given here. A full treatment of the subject will eventually appear in another publication.

Let us begin by considering the simplest case, that of antipodal maps. The classical theorem of BORSUK and ULAM states that if $f: S^n\to S^m$ is equivariant with respect to the antipodal map (i.e. if $f(-x)=-f(x)$), then $n\leqslant m$. The best known and easiest proof of this fact is obtained by considering the induced map $\bar{f}: P^n\to P^m$ on the orbit spaces since it is easily deduced that $\bar{f}^*: H^*(P^m;Z_2)\to H^*(P^n;Z_2)$ is an epimorphism.

Another proof of this fact may be obtained by the use of the P. A. SMITH theory. From this point of view it is technically simpler to accomplish the proof if the involutions in question have fixed points. But this situation is easily achieved merely by suspending f to obtain Sf: $S^{n+1}\to S^{m+1}$ where both involutions now have two fixed points. Actually it is desirable to suspend twice so as to obtain connected fixed point sets (circles). Note that the induced map between fixed point sets is then the identity.

Before indicating the Smith theory proof let us introduce some convenient notation. We let $S^n(r)$ denote the space-with-involution whose underlying space is S^n and whose involution is given by the matrix

$$\begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r+1} \end{pmatrix}.$$

Note that the fixed point set of $S^n(r)$ is S^{n-r} so that the argument "r" refers to the codimension of the fixed point set. Also note that this codimension is unchanged by suspension.

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Some further convenient notation is the use of X^G to denote the stationary point set of a G-space X (for us, $G=Z_2$) and f^G for the induced map $X^G \to Y^G$ from an (equivariant) map $f: X \to Y$.

To return to the question at hand, let $X = S^{r+k}(r)$ and $Y = S^{t+k}(t)$ ($r=n+1$ and $t=m+1$ in the former notation). Then we are given a map (equivariant)

$$f: X \to Y$$

such that the induced map $f^G: S^k = X^G = Y^G = S^k$ is the identity. We may also assume $k \geqslant 1$. Then Smith theory provides the following commutative diagram

$$\begin{array}{ccccccccc}
H^k(Y^G) & \xrightarrow{\approx} & {}_\sigma H^{k+1}(Y) & \xrightarrow{\approx} & {}_\sigma H^{k+2}(Y) & \xrightarrow{\approx} \cdots \xrightarrow{\approx} & {}_\sigma H^{k+t}(Y) & \xleftarrow{\approx} & H^{k+t}(Y) \\
\approx \Big\downarrow f^{G*} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & 0 \Big\downarrow f^* \\
H^k(X^G) & \xrightarrow{\approx} & {}_\sigma H^{k+1}(X) & \xrightarrow{\approx} & {}_\sigma H^{k+2}(X) & \xrightarrow{\approx} \cdots \xrightarrow{\approx} & {}_\sigma H^{k+t}(X) & \longleftarrow & H^{k+t}(X)
\end{array}$$

with coefficients in Z_2. (For background on this see [4] and [5]. The dual proof in homology works equally well.) The isomorphisms on the bottom row follow from the assumption that $t<r$ as does the fact that $f^*=0$. But the diagram is clearly self-contradictory. This proof gives more information than we have claimed since it clearly shows, more generally, that if

$$(1) \qquad f: S^{r+k}(r) \to S^{t+k}(t)$$

and if $r>t$ then $f^G: S^k \to S^k$ has *even degree* (i.e. $(f^G)^*=0$ on $H^k(S^k; Z_2)$).

The question arises as to what extent this latter result is best possible. One can, in fact, achieve any even degree for f^G.

Although such maps can be written down quite explicitly, we shall give a suggestive general procedure for constructing them. First, we note that there is a map $g: S^{t+k}(t) \to S^{t+k}(t)$ which has degree -1 (forgetting the action) and such that g^G has degree $+1$. (If t is odd, then the involution itself is such a map g.) Second, note that $S^{t+k-1}(t) \subset S^{t+k}(t)$ (the latter is the suspension of the former) and that the quotient space is the one point union $S^{t+k}(t) \vee S^{t+k}(t)$. Consider the resulting composition

$$h: S^{t+k}(t) \longrightarrow S^{t+k}(t) \vee S^{t+k}(t) \xrightarrow{1 \vee g} S^{t+k}(t).$$

This map h has degree zero and h^G has degree 2. Now consider $S^{t+k}(t) \subset S^{t+k+1}(t+1)$. This divides S^{t+1+k} into two hemispheres *which are interchanged by the involution*. Thus h may be extended to one hemisphere (as a map) and then to the other hemisphere by equivariance. This defines a map

$$f: S^{t+1+k}(t+1) \to S^{t+k}(t)$$

and $f^G = h^G$ has degree two. Such maps of any even degree can now be constructed by using the group structure (see below).

We have found maps

$$f: S^{r+k}(r) \to S^{t+k}(t)$$

with f^G of any even degree *when* $r=t+1$. It is natural to ask whether this can be done when $r=t+2$. The construction outlined above will provide such maps with f^G of degree *four* (and hence of any degree divisible by four). It turns out however that this is the best that can be done. Unfortunately the proof of this fact, and of similar ones to follow, is much to difficult to give here and we will confine ourselves to a very sketchy discussion of it. A general result of this nature is given by the following theorem:

Theorem A. *Let $f: S^{r+k}(r) \to S^{t+k}(t)$ and put $d=r-t$. Then $\deg f^G$ is divisible by $2^{\Phi(d-1)+1}$, where $\Phi(n)$ is the number of integers i with $0<i \leqslant n$ and $i \equiv 0,1,2,$ or $4 \pmod 8$. This result is best possible when $d \not\equiv 0 \pmod 4$. For $d=4$, 8, or 12 we have that $\deg f^G$ is divisible by $2^{\Phi(d-1)+2}$ ($=16$, 32, and 256 respectively), which is also best possible.*

We conjecture that for $d \equiv 0 \pmod 4$, $\deg f^G$ is divisible by $2^{\Phi(d-1)+2}$ (which would be best possible). [Added in proof: P. LANDWEBER has found a proof of this conjecture using operations in equivariant K-theory.]

A small part of Theorem A was also stated, in a different form, in our research announcement [2], which contains more detailed information in the cases $d \leqslant 8$.

So far we have dealt only with the case in which the fixed point sets have the same dimension. Let us now consider equivariant maps

$$f: S^{r+k+j}(r) \to S^{t+k}(t)$$

so that

$$f^G: S^{k+j} \to S^k.$$

We are able to obtain results here only for $j=1,2,3$ and also only for the stable class $\{f^G\} \in \pi_j = \lim_i \pi_{j+i}(S^i)$ of f^G. As in Theorem A the method of Smith theory is completely inadequate for this situation. Our computations have yielded the following information:

Theorem B. *Let $f: S^{r+k+j}(r) \to S^{t+k}(t)$ and put $d=r+j-t$.*

If $j=1$ and $d \geqslant 4$ then $\{f^G\}=0 \in \pi_1 \approx Z_2$.
If $j=2$ and $d \geqslant 7$ then $\{f^G\}=0 \in \pi_2 \approx Z_2$.

$$\text{If } j=3 \text{ and } d \begin{Bmatrix} =8 \\ =9 \\ \geqslant 10 \end{Bmatrix} \text{ then } \{f^G\} \in \begin{Bmatrix} 2\pi_3 \\ 4\pi_3 \\ 8\pi_3 \end{Bmatrix} \subset \pi_3 \approx Z_{24}.$$

This result is best possible (stably) in the sense, for example, that if $j=1$ and $d<4$ then any value of $\{f^G\} \in \pi_1$ is possible.

There is an elementary general result along the lines of Theorem B which says that if $d \leqslant 2j$ then any element of π_j can be achieved by $\{f^G\}$. In fact, if $g: S^n \to S^m$ is any map then

$$g \wedge g: S^n \wedge S^n \to S^m \wedge S^m$$

is equivariant with respect to switching factors and hence is a map $f: S^{2n}(n) \to S^{2m}(m)$ with $f^G = g$. Here $j = n - m$ and $d = 2n - 2m = 2j$ and any smaller value of d is obtained by restriction of f to a subspace. As Theorem B shows, this result is not best possible. It does seem reasonable to us, nevertheless, that in some asymptotic sense this result is best possible.

Up to now we have discussed the question of which maps f^G can arise from a map $f: S^n(r) \to S^k(t)$. We shall now ask which maps $\bar{f}: S^n \to S^k$ can arise from a map $f: S^n(r) \to S^k(t)$ by forgetting equivariance. This question can be rather thoroughly answered and we shall list a few examples here:

Theorem C. *Suppose $f: S^{n+j}(t+k) \to S^n(t)$ so that the map $\bar{f}: S^{n+j} \to S^n$ on the underlying spaces defines the class $\{\bar{f}\} \in \pi_j$. Then*

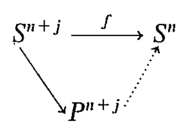
$$\begin{array}{ll}
j=1, \; k \neq 1 \quad \text{and } k \not\equiv 0(4) & \Rightarrow \{\bar{f}\} = 0, \\
j=2, \; k \neq 2 \quad \text{and } k \not\equiv 0,1(4) & \Rightarrow \{\bar{f}\} = 0, \\
j=3 \begin{cases} k \text{ even } \neq 2 \text{ and } k \not\equiv 0(8) \\ k \text{ odd } \begin{cases} \neq 3 \text{ and } k \not\equiv 1(4) \\ \text{otherwise} \end{cases} \end{cases} & \begin{array}{l} \Rightarrow \{\bar{f}\} \in 2\pi_3, \\ \Rightarrow \{\bar{f}\} = 0, \\ \Rightarrow \{\bar{f}\} \in 12\pi_3, \end{array} \\
j=6, \; k \neq 4 \text{ and } k \not\equiv 0,2,3(8) & \Rightarrow \{\bar{f}\} = 0.
\end{array}$$

This result is best possible in the sense that, for example, if $j=1$ and $k=1$ or $k \equiv 0(4)$ then such maps f exist (when n and t are sufficiently large; see [3]) with $\{\bar{f}\} \neq 0$.

Similar results are known to us for $j \leqslant 9$ at least and can probably be deduced in the range $j \leqslant 13$ without great trouble.

The exceptional cases ($j=1$, $k=1$, $j=2$, $k=2$; $j=3$, $k=2,3$; $j=6$, $k=4$) arise because of the possibility of non-trivial f^G. If one required that f^G be inessential then the same theorem would hold with the absence of these exceptions.

Let us state as a corollary the most interesting special case of Theorem C, that in which $t=0$ (i.e. no action on S^n) and $t+k=n+j+1$ (i.e. the antipodal action on S^{n+j}). This can be reformulated as follows. Consider the diagram



Corollary. *If f factors as shown above, then*

$$\begin{array}{ll}
j=1 \text{ and } n \not\equiv 2(4) & \Rightarrow \{f\} = 0, \\
j=2 \text{ and } n \not\equiv 1,2(4) & \Rightarrow \{f\} = 0, \\
j=3 \begin{cases} n \text{ even } \not\equiv 4(8) \\ n \text{ odd } \begin{cases} \not\equiv 1(4) \\ \equiv 1(4) \end{cases} \end{cases} & \begin{array}{l} \Rightarrow \{f\} \in 2\pi_3, \\ \Rightarrow \{f\} = 0, \\ \Rightarrow \{f\} \in 12\pi_3, \end{array} \\
j=6 \text{ and } n \not\equiv 1,3,4(8) & \Rightarrow \{f\} = 0.
\end{array}$$

Actually this is a "stable" result and in the stable category it is best possible. As stated, however, some of the possibilities for f consistent with the theorem would not actually exist. We don't know the best possible results in the non-stable case.

Again similar results are known for $j \leqslant 13$. The case $j=1$ of this corollary was originally proved by J. H. C. WHITEHEAD in 1941 and was rediscovered by CONNER and FLOYD in 1962. The cases $j=2,3$ (at least) and some similar results have been recently obtained independently by E. REES, a student of D. B. A. EPSTEIN, by the use of K-theory. He also has some results in the non-stable case.

We shall outline very briefly some of the methods by which these results, and many others along these lines were obtained. Since the proofs of the results involve a considerable amount of detailed "calculations" the full account of these matters will have to await publication elsewhere.

The object of interest to us is

(2) $$[S^n(r); S^k(t)]$$

the equivariant homotopy classes of maps $S^n(r) \to S^k(t)$. For technical reasons we consider only base point preserving maps and homotopies, where the base points are fixed under the involutions.

There is the fixed point set morphism

$$\varphi: [S^n(r); S^k(t)] \to [S^{n-r}; S^{k-t}]$$

and the forgetful morphism

$$\psi: [S^n(r); S^k(t)] \to [S^n; S^k].$$

Of course, Theorems A and C are concerned with the images of these morphisms.

This object (2) can be generalized in two ways by replacing the image or the domain by a general G-space ($G = Z_2$). We shall briefly discuss both of these generalizations.

In the case of equivariant homotopy one considers the set

(3) $$\pi_{n,r}(X) = [S^n(r); X],$$

where X is some given G-space with base point in X^G. These sets were first considered by LEVINE in [8], with somewhat different notation. $\pi_{n,r}(X)$ is a group if $n-r\geqslant 1$ and is abelian when $n-r\geqslant 2$ since $S^n(r)$ is a suspension or double suspension (as a G-space) in these cases.

Let us describe some important homomorphisms associated with these groups. First, as above, there is the fixed point set functor

$$\varphi:[S^n(r);X]\to[S^{n-r};X^G];$$

that is,

$$\varphi:\pi_{n,r}(X)\to\pi_{n-r}(X^G).$$

There is also the forgetful functor

$$\psi:[S^n(r);X]\to[S^n;X]$$

(forgetting the actions); that is,

$$\psi:\pi_{n,r}(X)\to\pi_n(X).$$

Restriction to $S^{n-1}(r-1)\subset S^n(r)$ yields a homomorphism

$$\beta:\pi_{n,r}(X)\to\pi_{n-1,r-1}(X).$$

Also there is a homomorphism

$$\alpha:\pi_n(X)\to\pi_{n,r}(X)$$

defined by assigning to a map $f:S^n\to X$ the map

$$S^n(r)\to\frac{S^n(r)}{S^{n-1}(r-1)}=S^n\vee S^n\xrightarrow{f\vee f}X\vee X\xrightarrow{1\vee T}X.$$

Here $S^n\vee S^n$ has the involution interchanging factors and T denotes the involution on X.

It turns out to be quite important to consider also the groups obtained by requiring the fixed point set $S^{n-r}=S^{n-r}(0)$ of $S^n(r)$ to be sent to the base point of X by all maps and homotopies. We denote these groups by $\pi^*_{n,r}(X)$. Thus

$$\pi^*_{n,r}(X)=[S^n(r)/S^{n-r}(0);X].$$

One has the natural homomorphism

$$i:\pi^*_{n,r}(X)\to\pi_{n,r}(X).$$

As in the non-equivariant case the quotient $S^n(r)/S^{n-r}(0)$ is (equivariantly) homotopically equivalent to the mapping cone of the inclusion $S^{n-r}(0)\to S^n(r)$. Hence there is a canonical map $S^n(r)/S^{n-r}(0)\to S\,S^{n-r}(0)$ $=S^{n-r+1}(0)$. This induces the homomorphism

$$\Delta:\pi_{n-r+1}(X^G)\to\pi^*_{n,r}(X)$$

since the equivariant maps $S^{n-r+1}(0)\to X$ are just the maps $S^{n-r+1}\to X^G$.

All the homomorphisms we have described, in fact, come from Puppe sequences associated with mapping sequences in the diagram

$$\begin{array}{ccccc}
S^0(0) & \to & S^{r-1}(r-1) & \to & S^{r-1}(r-1)/S^0(0)\\
\downarrow & & \downarrow & & \downarrow\\
S^0(0) & \to & S^r(r) & \longrightarrow & S^r(r)/S^0(0)\\
\downarrow & & \downarrow & & \downarrow\\
* & \longrightarrow & S^r\vee S^r & \longrightarrow & S^r\vee S^r
\end{array}$$

This also contains analogues of α, β and ψ for the $\pi^*_{n,r}$ groups.

It is not hard to see then that this diagram induces the following commutative "braid" diagram (Fig.) in which all four "sine curves" are exact sequences:

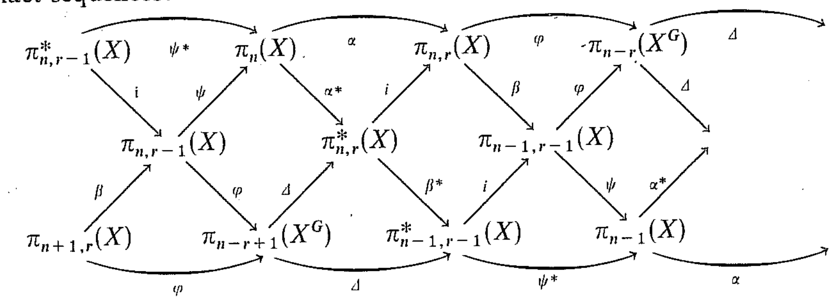


Fig.

The sequence consisting of $\alpha^*_,\ \beta^*$and $\psi^*$was previously considered by LEVINE [8]. Note that this sequence is an exact couple and hence induces a spectral sequence (there are some sticky technical difficulties here). It is this spectral sequence, or rather one closely associated with it, that we use in our calculations.

Let us now discuss the second method of approach which was, in fact, my original point of view when first attacking these questions. In the second approach we consider the set $[X,S^m(t)]$, or rather we stabilize this situation and consider the stable equivariant cohomotopy groups

$$\{X;S^m(t)\}=\lim_{\to k}[S^kX;S^{m+k}(t)].$$

These groups form a (generalized) equivariant cohomology theory, the theory associated with the G-spectrum $\mathbf{S}(t)=\{S^n(t)\}$. That is

$$\{X;S^m(t)\}=\tilde{H}^m_G(X;\mathbf{S}(t)). \qquad (4)$$

We are interested in calculating these groups. According to [2] there is a spectral sequence

$$E_2^{p,q}=\tilde{H}^p_G(X;s^q(t))\Rightarrow\tilde{H}^{p+q}_G(X;\mathbf{S}(t)).$$

Here $s^q(t)$ refers to the "coefficients" of the cohomology theory on the right. The E_2-term refers to the "equivariant classical comohology theory" defined in [2]. It is closely akin to the Steenrod equivariant cohomology theory but is somewhat more general. It is, however, easy to calculate this E_2-term. Moreover for $X = S^n(r)$ we can find a large number of the differentials in this spectral sequence and hence derive much information about the groups of interest to us. In general, however, the investigation of the differentials is a difficult problem. For example, the non-triviality of certain of the differentials turns out to be equivalent to the vector field problem on spheres.

We shall now turn our attention to the description of a remarkable general result concerning equivariant homotopy classes of maps of spheres. Let us use the notation

$$\begin{aligned}\pi_n(r;t) &= \lim_{\to k} [S^{n+k}(r); S^k(t)] \\ &= \lim_{\to k} \pi_{n+k,r}(S^k(t)) = \tilde{H}_G^{r-n}(S^r(r); \mathbf{S}(t)).\end{aligned}$$

Similarly we let $\pi_n^*(r,t)$ denote the analogous group where we require fixed point sets to be sent to the base point. That is

$$\begin{aligned}\pi_n^*(r;t) &= \lim_{\to k} [S^{n+k}(r)/S^{n+k-r}(0); S^k(t)] \\ &= \lim_{\to k} \pi_{n+k,r}^*(S^k(t)) \\ &= \tilde{H}_G^{r-n}(S^r(r)/S^0(0); \mathbf{S}(t)).\end{aligned}$$

When we originally computed some of the groups $\pi_n(r,t)$ it was discovered that there was a periodicity of these groups in t but with some exceptional cases. It became clear that the exceptions resulted because of possibly non-trivial maps on the fixed point sets. This even affected the kernel of $\varphi: \pi_n(r;t) \to \pi_{n-r+t}$, the fixed point homomorphism, which is even closer to being actually periodic in t. This was the reason for introducing the groups $\pi_n^*(r;t)$ which exhibit an exact periodicity.

The periodicity result is that

(5) $$\pi_n^*(r;t) \approx \pi_n^*(r; t+2^{\Phi(r-1)})$$

where Φ is the well-known function

$$\Phi(p) = \#\{k \mid 0 < k \leqslant p \quad \text{and} \quad k \equiv 0,1,2 \text{ or } 4(8)\}.$$

Let us outline the proof of this fact. First it is necessary to generalize these groups and consider

$$\begin{aligned}\pi_n(r,q;t) &= \lim_{\to k} [S^{n+k}(r)/S^{n+k-r+q}(q); S^k(t)] \\ &= \tilde{H}_G^{r-n}(S^r(r)/S^q(q); \mathbf{S}(t)).\end{aligned}$$

(Note that $\pi_n^*(r;t) = \pi_n(r,0;t)$.) Then one shows that the suspension with action (i.e. the involution also interchanges the vertices of the suspension) induces an isomorphism

$$\Sigma: \pi_n(r,q;t) \xrightarrow{\approx} \pi_n(r+1, q+1; t+1).$$

Now one constructs, by an inductive procedure, a map

$$\lambda: S^{p-1} \to O(k); \qquad k = 2^{\Phi(p-1)}$$

such that $\lambda(-x) = -\lambda(x)$. [Proceeding one step further in the inductive construction requires that λ be inessential, forgetting equivariance. When $\pi_{p-1}(O(k)) \neq 0$ this can be achieved (via a certain trick) by replacing $O(k)$ by $O(2k)$ and, according to the known stable homotopy groups of the orthogonal groups, this produces the function Φ.] By projection on the equator, λ can be extended to an equivariant map

$$\lambda: S^p(p) - S^0(0) \to O(k)$$

the involution on $O(k)$ being $A \to -A$.

Now suppose that X and Y are G-spaces with base points and let

$$f: (S^p(p) \wedge X, S^0(0) \wedge X) \to (Y, *)$$

be any equivariant map. Define

$$f^\lambda: (S^p(p) \wedge X \wedge S^k(0), S^0(0) \wedge X \wedge S^k(0)) \to (S^k(k) \wedge Y, *)$$

where $k = 2^{\Phi(p-1)}$ by $f^\lambda(a \wedge x \wedge b) = (\lambda(a) \cdot b) \wedge f(a \wedge x)$.

We specialize to the case $Y = (S^p(p) \wedge X)/(S^0(0) \wedge X)$ with X compact and $f: S^p(p) \wedge X \to Y$ the canonical projection. Then it is easily seen that f^λ induces an *equivariant homeomorphism*

$$\frac{S^p(p) \wedge X \wedge S^k(0)}{S^0(0) \wedge X \wedge S^k(0)} \xrightarrow{\approx} \frac{S^k(k) \wedge S^p(p) \wedge X}{S^k(k) \wedge S^0(0) \wedge X}.$$

Now taking $X = S^r(r)$ we obtain

$$\frac{S^{p+k+r}(p+r)}{S^{k+r}(r)} \xrightarrow{\approx} \frac{S^{p+k+r}(p+k+r)}{S^{k+r}(k+r)}.$$

By composition this induces an isomorphism

$$\pi_n(p+k+r, k+r; t+k) \xrightarrow{\approx} \pi_n(p+r, r; t+k).$$

Preceding this by the isomorphism Σ^k yields

(6) $$\pi_n(p+r, r; t) \approx \pi_n(p+r, r; t+2^{\Phi(p-1)})$$

which clearly generalizes (5).

This periodicity leads to the following remarkable isomorphism:

Theorem D. *If* $t > n+r+1$ *then*

$$\pi_n^*(r;t) \approx \pi_{n-r+t}(V_{t,r})$$

where $V_{t,r}$ *is the Stiefel manifold of r-frames in euclidean t-space.*

We shall briefly indicate the proof. Let us denote $P_{t,r} = P^{t-1}/P^{t-r-1}$. Then, according to JAMES [7], we have

$$\pi_{n-r+t}(V_{t,r}) \approx \pi_{n-r+t}(P_{t,r})$$

for $t > n+r$ and the latter group is *stable* when $t > n+r+1$. According to ATIYAH [1] $P_{t,r}$ is S-dual to $SP_{r-t,r}$ (a stable "object") so that

$$\pi_{n-r+t}(V_{t,r}) \approx \{S^0; P_{t,r}\}_{n-r+t} \approx \{SP_{r-t,r}; S^0\}_{n-r+t}.$$

The latter means

$$\{P^{aj+r-t-1}/P^{aj-t-1}; S^{aj+r-t-1-n}\}$$

where $j = 2^{\phi(r-1)}$, a is a certain integer whose value is immaterial, and $aj > t$. This group is just $\pi_n(aj+r-t, aj-t; 0)$ and by periodicity (6) (a times), this is $\pi_n(aj+r-t, aj-t; aj)$. By the isomorphism Σ^{t-aj}, this is the same as $\pi_n(r,0;t) = \pi_n^*(r;t)$, which proves Theorem D.

For most purposes it is sufficient to consider the groups resulting from a further stabilization. This is obtained via the suspension with action which yields a homomorphism

$$\Sigma : \pi_n(r;t) \to \pi_n(r+1; t+1)$$

which turns out to be an epimorphism if $r \geqslant n+1$ and an isomorphism for $r \geqslant n+2$. We define

(7) $$\pi_{n,k} = \lim_{\to t} \pi_n(t+k, t)$$

and similarly for the "starred" groups.

The $\pi_{n,k}^*$ are periodic in k with period $2^{\Phi(n+1)}$ and moreover Theorem D implies that

(8) $$\pi_{n,-k}^* \approx \pi_{k,r}^n \quad \text{for} \quad n < k-1 \quad \text{and} \quad n < r-1.$$

Here $\pi_{k,r}^n$ is the established shorthand for $\pi_{k+n}(V_{k+r,r})$.

Formula (8) together with the periodicity and HOO and MAHOWALD's calculations of $\pi_{k,r}^n$ in [6] suffice to compute $\pi_{n,-k}^*$ for $n \leqslant 13$. We shall give some further information which allows the calculation of $\pi_{n,k}$ in many cases.

Consider Fig. If we delete the arguments X and X^G, then this diagram is also valid for these doubly stable groups $\pi_{n,k}$ and yields a large amount of information.

Let us use the notation

$$\tilde{\pi}_{n,k} = \ker\{\varphi : \pi_{n,k} \to \pi_{n-k}\}$$

so that we have the exact sequence

(9) $$0 \longrightarrow \tilde{\pi}_{n,k} \longrightarrow \pi_{n,k} \xrightarrow{\varphi} \pi_{n-k}$$

and the exact sequence

(10) $$\pi_{n+1,k} \xrightarrow{\varphi} \pi_{n-k+1} \xrightarrow{\Delta} \pi_{n,k}^* \longrightarrow \tilde{\pi}_{n,k} \longrightarrow 0.$$

For $k \leqslant 0$ it can be seen that φ is onto and splits so that (9) yields

(11) $$\pi_{n,k} \approx \tilde{\pi}_{n,k} \oplus \pi_{n-k} \quad \text{for} \quad k \leqslant 0.$$

From the argument given below Theorem B it can be seen that

(12) $$\varphi : \pi_{n,k} \to \pi_{n-k} \quad \text{is onto for} \quad n \geqslant 2k \quad \text{or for} \quad k \leqslant 0.$$

From (10) it follows that

(13) $$\pi_{n,k}^* \xrightarrow{\approx} \tilde{\pi}_{n,k} \quad \text{for} \quad k \geqslant n+2 \quad \text{or} \quad n \geqslant 2k-1.$$

This suffices to compute $\tilde{\pi}_{n,k}$ ($n \leqslant 13$) except in the range

(14) $$k-1 \leqslant n \leqslant 2k-2.$$

Although more detailed information is available we content ourselves here with listing these "exceptional" values of $\tilde{\pi}_{n,k}$ in the range (14) for $n \leqslant 6$. In fact the only non-zero groups in this range are

$$\tilde{\pi}_{3,4} \approx Z_{12},$$
$$\tilde{\pi}_{6,6} \approx Z_2,$$
$$\tilde{\pi}_{6,7} \approx Z_2.$$

For $n \geqslant 2k$ or $k \leqslant 0$ $\varphi : \pi_{n,k} \to \pi_{n-k}$ is onto by (12) and in the other cases (for small n) the image of φ can be deduced from Theorems A and B. By (9) this reduces the computation of $\pi_{n,k}$ to an extension problem. In general, however, we do not know how to determine this extension. A case in which this extension does not split is given by

$$0 \longrightarrow \tilde{\pi}_{3,2} \longrightarrow \pi_{3,2} \xrightarrow{\varphi} \pi_1 \longrightarrow 0.$$

Here $\tilde{\pi}_{3,2} = \pi_{3,2}^* \approx Z_{12}$, but it can be shown that $\psi : \pi_{3,2} \to \pi_3$ is an isomorphism so that $\pi_{3,2} \approx Z_{24}$. For all other cases with $n \leqslant 6$ one can see that the extension (9) does split but this must be regarded as accidental.

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Gaps in the Dimensions of Compact Transformation Groups

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Let (X, G) denote a G-space where G is a compact connected Lie group, X a connected n-dimensional manifold and the action of G on X is effective. A well-known result of MONTGOMERY and ZIPPIN [8] states that

$$\dim G \leqslant \frac{r(r+1)}{2} \leqslant \frac{n(n+1)}{2}$$

where r is the maximal dimension of the orbits of G on X. In the extreme case where $\dim G = n(n+1)/2$ it is known that G is locally isomorphic to $SO(n+1)$ and X is homeomorphic to either the n-sphere S^n or real projective n-space $P^n(R)$ [1], [3, p. 239]. Below this maximum case there is a gap of $n-2$ dimensions, at least for $n \neq 4$ and $n \geqslant 1$. In fact, we have the following result [11, p. 63], [10].

Theorem (WANG). *If* $\frac{(n-1)n}{2} + 1 < \dim G < \frac{n(n+1)}{2}$, *then* $n = 4$.

Proof. We outline a proof as follows. By (1) G acts transitively on X. Hence G acts differentiably on $X = G/H$, where H is the isotropy or stability subgroup of G at a point x in X. By BOCHNER's theorem on local linearity about x [8, p. 243],

$$H^0 \subset SO(n)$$

where H^0 denotes the identity component of H. Now

$$\dim G = \dim H^0 + n$$

and for $n \neq 4$, $H^0 = SO(n)$ or

$$\dim H^0 \leqslant \dim SO(n-1) = \frac{(n-2)(n-1)}{2}$$

(see, for example, [7]). The result follows. For $n = 4$, there is an effective action of $SU(3)/Z$ of dimension 8, Z denoting the center of $SU(3)$,

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