## **Characteristic Polynomials of Special Matrices**

## By

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ROTH showed [4] that if the rank of A - B is small enough, it is possible to exhibit a formula for the characteristic polynomial of AB in terms of the characteristic polynomials of A and B. GODDARD [2] extended the result, with a new method of proof, to a product  $A_1A_2A_3$  of three matrices; and PARKER [3] found the characteristic polynomial not only of AB but also of AB - k(A - B).

By the use of GODDARD's arguments we are able to extend his results to find the characteristic polynomial of a product of any number of matrices; and we are able to give another proof, and with it an extension, of PARKER's results. In section 4 we clarify the circumstances under which the matrices  $A_1, A_2, \ldots$  will be "sufficiently equal" to satisfy the hypotheses of the various theorems.

2. Theorem. Let  $A_1, A_2, A_3$  be  $n \times n$  matrices; let their characteristic polynomials be

$$\det (x I - A_i) = \alpha_{0i}(x^3) + x \alpha_{1i}(x^3) + x^2 \alpha_{2i}(x^3), \quad i = 1, 2, 3.$$

Write

$$H = A_1 + A_2 + A_3; K = A_1A_2 + A_1A_3 + A_2A_3.$$

Then the characteristic polynomial of  $A_1A_2A_3$  is given by the formula of GODDARD

$$det (x I - A_1 A_2 A_3) = \alpha_{01} \alpha_{02} \alpha_{03} + x \alpha_{11} (\alpha_{02} \alpha_{23} + \alpha_{03} \alpha_{22}) + + x \alpha_{12} (\alpha_{01} \alpha_{23} + \alpha_{03} \alpha_{21}) + x \alpha_{13} (\alpha_{01} \alpha_{22} + \alpha_{02} \alpha_{21}) + + x^2 \alpha_{21} \alpha_{22} \alpha_{23} \quad (\alpha_{ij} \equiv \alpha_{ij}(x))$$

in any one of the following cases.

i. (GODDARD) xH - K has rank 1 or 0.

ii. H = 0; rank K is 2. iii. K = 0; rank H is 2.

This theorem can be extended so as to find the characteristic polynomial of a product of more than 3 matrices. For example, let us write

(1) det 
$$(x I - A_i) = \alpha_{0i}(x^4) + x \alpha_{1i}(x^4) + x^2 \alpha_{2i}(x^4) + x^3 \alpha_{3i}(x^4), i = 1, 2, 3, 4;$$
  
 $H_1 = A_1 + A_2 + A_3 + A_4;$   
 $H_2 = A_1 A_2 + A_1 A_3 + A_1 A_4 + A_2 A_3 + A_2 A_4 + A_3 A_4;$   
 $H_3 = A_1 A_2 A_3 + A_1 A_2 A_4 + A_1 A_3 A_4 + A_2 A_3 A_4,$ 

and suppose that we have one of the following additional hypotheses.

ii.  $H_1 = H_2 = 0$ ;  $H_3$  has rank 2 or 3.

iii.  $H_2 = H_3 = 0$ ;  $H_1$  has rank 2.

Then the characteristic polynomial of  $A_1A_2A_3A_4$  can be found explicitly in terms of the  $\alpha_{ij}$ , and is the rational part of that function of y which results when  $y^{1/4}$  is substituted for x in each of the four formulas (1), and the four formulas are multiplied.

3. PARKER's result is the following. Let A, B have characteristic polynomials given by the formulas

$$det (x I - A) = \alpha_0(x^2) - x \alpha_1(x^2),$$
  
$$det (x I - B) = \beta_0(x^2) - x \beta_1(x^2).$$

Let the rank of A - B be 1 or 0. Then the characteristic polynomial of AB + k(A - B) is

$$(-)^n (\alpha_0 \beta_0 - x \alpha_1 \beta_1 - k \alpha_0 \beta_1 + k \alpha_1 \beta_0), \alpha_1 \equiv \alpha_1(x), \text{ etc.}$$

By using a different method of proof (essentially a variation of GODDARD's), it is possible to extend this result, in other words to obtain further results of the same type. We begin with the identity

$$(x I - A) (x I + B) = x^2 I - A B - x(A - B),$$

take determinants of both sides, and equate the odd and even parts. As GODDARD points out, the determinant of the right member is equal to det  $(x^2 I - A B) - x \sum \Delta_i$ , where  $\Delta_i$  is the determinant of the matrix obtained by replacing the *i*th column of  $x^2 I - A B$  by the *i*th column of A - B. This gives  $\sum \Delta_i = (-1)^n \{\alpha_1 \beta_0 - \alpha_0 \beta_1\}$ . In the next step of the argument, we use the identity

$$(x I - A) (x I + B) = x^2 I - A B + k(A - B) - (x + k) (A - B).$$

The determinant of the right side can be written

$$\det \left(x^2 I - A B + k (A - B)\right) - (x + k) \sum \Theta_i,$$

where  $\Theta_i$  is the determinant of the matrix obtained by replacing the *i*th column of  $x^2I - AB + k(A - B)$  by the *i*th column of A - B. By an elementary theorem on determinants, this must be equal to  $\Delta_i$ . The last step is to take determinants of both sides of the last displayed identity, save the even powers of x, and replace  $x^2$  by y; this gives PARKER's result.

To illustrate the fact that PARKER's ideas can be used to extend the results of the second section of this paper still further, we take the example of three matrices, and use the notation of theorem 1. Suppose that H is 0 and K has rank 1. The result

$$\det (x I - A_1 A_2 A_3 + k K) = \det (x I - A_1 A_2 A_3) + \alpha_{01} \alpha_{12} \alpha_{03} + \alpha_{01} \alpha_{02} \alpha_{13} + \alpha_{11} \alpha_{02} \alpha_{03} + x \alpha_{01} \alpha_{22} \alpha_{23} + x \alpha_{21} \alpha_{22} \alpha_{03} + x \alpha_{21} \alpha_{02} \alpha_{23}$$

is obtained by the use of arguments similar to the above.

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4. The question when xH - K has rank 1 for indeterminate x is settled by GODDARD, although the information is not really needed for his proof. The following discussion gives further insight into the situation.

If M is an  $n \times m$  matrix of rank 1 with elements in the ring extension F[x] of any field, then two matrices R, S in F[x] can be found, of dimensions  $1 \times n$ ,  $1 \times m$  respectively, such that the relation  $M = RS^*$  holds. This follows from a suitable application of the euclidean algorithm. In the case at hand we have the relation

$$xH - K = -R(x)S^*(x).$$

But since every element of xH - K has degree 1 at most, either R(x) or S(x) (or both) is constant. Moreover, the relations  $K = R(0)S^*(0)$ ,  $H = R(0)S^*(0) - R(1)S^*(1)$  obviously hold. But if R(x) is constant, this last relation can be written

$$H = R(0) [S(0) - S(1)]^*$$

Similarly, necessary and sufficient conditions that  $x^2H_1 - xH_2 + H_3$  have rank 1 are that at least one of the subjoined sets of relations hold for some constant  $1 \times n$  matrices  $R_i, S_i$ .

(1) 
$$H_i = R_1 S_i^*, \quad i = 1, 2, 3.$$

(2) 
$$H_i = R_i S_1^*, \quad i = 1, 2, 3.$$

(3) 
$$H_1 = R_1 S_1^*, \quad H_2 = R_1 S_2^* + R_2 S_1^*, \quad H_3 = R_2 S_2^*.$$

5. Finally, we remark that the hypotheses on the ranks are really needed, and not imposed by the method of proof. Indeed if A, B are  $2 \times 2$  matrices, and if  $\alpha_0$ ,  $\alpha_1$ ,  $\beta_0$ ,  $\beta_1$  are defined as in section 3, the characteristic polynomial of A B is always given by the formula

$$\det (y I - A B) = \alpha_0 \beta_0 - y \alpha_1 \beta_1 - y \det (A - B),$$

the last term of which cannot be neglected if the rank of A - B is as great as 2.

## References

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Eingegangen am 12. 4. 1961

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