Contemporary Mathematics
Volume 78, 1988

AUTOMORPHIC SETS AND BRAIDS AND SINGULARITIES

E. Brieskorn

INTRODUCTION

Singularities and braids are related in many ways. I decided to concentrate on one aspect of these relations, namely the Coxeter diagrams. These are invariants of singularities which are obtained by the classical theory of Picard and Lefschetz. The braid groups B_n operate on the set of diagrams with n vertices, and the invariant associated to a singularity is an orbit of diagrams.

When preparing this survey, I found an extremely simple concept unifying many investigations on this subject as well as classical results of E. Artin and A. Hurwitz and W. Magnus. This is the notion of an automorphic set. An automorphic set is a set Δ with a product such that all left translations $b \mapsto a * b$ are automorphisms. If Δ is an automorphic set, there is a canonical operation of the braid group B_n on Δ^n for any natural number n.

When I presented our results at the conference on Artin's braid group, I learned that a special type of automorphic sets, namely those with the additional property $a * a = a$, had already been introduced and investigated by D. Joyce, who calls such a set a quandle. I found the paper of D. Joyce very stimulating. I have worked some of his ideas into this paper, and there may be other possibilities of integration yet to be discovered. However there is no overlap with respect to the main point of this survey, the operation of B_n on Δ^n, which does not occur in Joyce's paper.

The beauty of braids is that they make ties between so many different parts of mathematics, combinatorial theory, number theory, group theory, algebras, topology, geometry and analysis, and, last not least, singularities. Although I am very fond of these manifold connections, I have tried to isolate the combinatorial, the algebraic and arithmetic aspects from the geometric ones as clearly as possible. I hope that this will contribute to conceptual clarity and will make the subject easily accessible to those who do not know singularities nor algebraic or analytic geometry.

The contents of the survey is as follows. § 1 is a brief introduction to some of Artin's classical results on braid groups, presented in a way fit for future developments. § 2 introduces the automorphic sets together with a few categorical definitions obviously related to this notion and gives many examples. § 3 deals with the operation of B_n and certain other groups on the cartesian products Δ^n of automorphic sets Δ and introduces invariants of the orbits, in particular pseudo Coxeter elements, Coxeter diagrams and monodromy groups. This is followed by a discussion of problems related to the operation of B_n on Δ^n and a report on classical and recent work on these problems. Much of this is related to root systems Δ, which are a particularly nice class of automorphic sets. § 4 finally explains the applications to singularities.

1980 Mathematics Subject Classification (1985 Revision). 20F36, 14B07, 14D05, 20B25, 20B27, 17B20, 32G11.

I regret that I am unable to give an account of the history of the subject. If one sees it from a sufficient distance, many great mathematicians have been involved. E. Artin, V.I. Arnold, H.S.M. Coxeter, P. Deligne, A. Grothendieck, S. Lefschetz, J. Milnor and R. Thom are some of the names. Particularly important was the work of V.I. Arnold and some of his students, especially A.M. Gabrielov. Also some work of E. Looijenga has played an important role. A fair amount of the results forming the substance of this report has been obtained by a group of young mathematicians in Bonn working on singularities: W. Ebeling, P. Kluitmann, B. Krüger and E. Voigt. I wish to thank them for letting me report on their work, some of which is not yet published. I also want to thank them and F.J. Bilitewski for their help in preparing this survey.

§1. BRAID GROUPS

There are many ways of defining the braid groups. The original geometric definition of Emil Artin is very beautiful. It appeals to our geometric intuition and it has deep roots in human culture. However I shall present these groups in another way, which is very natural and fit for the representations of the braid groups that we are going to study. This presentation is not new. In substance it is contained in Artin's foundational "Theory of Braids".

Let me begin with a few trivial remarks about groups acting on sets. If X is any set, I shall denote by $S(X)$ the group of bijective maps of X onto itself. $S(X)$ is also called the symmetric group of X. In particular $S_n = S(\{1,\dots,n\})$ is the usual symmetric group of permutations of the set $\{1,\dots,n\}$. The group $S(X)$ acts on X, and so does any subgroup $G \subset S(X)$. When I say that a group is acting on a set I always mean that it is acting from the left. Given a subgroup $G \subset S(X)$, we may consider its centralizer in $S(X)$

$$G_o(X) := \{\gamma \in S(X) \mid \forall g \in G \ \ \gamma \cdot g = g \cdot \gamma\} .$$

$G_o(X) = \mathrm{Aut}_G(X)$ is the group of automorphisms of X considered as a G-set. I shall write G_o instead of $G_o(X)$ if X is determined by the context.

PROPOSITION 1.1. If the action of $G \subset S(X)$ on X is simply transitive, the following statements hold.

(i) The action of $G_o \subset S(X)$ on X is also simply transitive.

(ii) For any $x \in X$ there is a unique antiisomorphism $\alpha_x : G_o \to G$ such that $\alpha_x(\gamma)\,(x) = \gamma(x)$ for all $\gamma \in G_o$.

(iii) $G \cap G_o = Z(G)$ is the centre of G.

(iv) $G_{oo} = G$.

In particular, if G is any group, $X = G$, and if G is identified with the group $G \subset S(X)$ of left translations, the group G_o is the group of right translations, and if we identify G_o canonically with the opposite group of G, then $\alpha_1 = \mathrm{id}_G$. The subscript "o" refers to this opposite structure.

DEFINITION. An equivalence relation $R \subset X \times X$ is a G-equivalence for $G \subset S(X)$ if $x \sim y$ implies $gx \sim gy$ for all $x,y \in X$ and $g \in G$, where we write $x \sim y$ instead of $(x,y) \in R$. Let R be a G-equivalence and $x \in X$. Then we define subgroups of G_o and G as follows (G is assumed to be simply transitive):

$$G_o(X)_R := \{\gamma \in G_o \mid \gamma x \sim x\} = \{\gamma \in G_o \mid \forall y \in X \quad \gamma y \sim y\} .$$

$$G_{R,x} := \{g \in G \mid gx \sim x\} .$$

Conversely any subgroup $H \subset G_o$ defines an equivalence relation R_H on X such that the orbits of H in X are exactly the equivalence classes of R_H.

PROPOSITION 1.2. If the action of $G \subset S(X)$ on X is simply transitive, the following statements hold.

(i) There is a bijective correspondence between the set of G-equivalences R on X and the set of subgroups H of G_o. It is given by the assignments $R \mapsto G_o(X)_R$ and $H \mapsto R_H$.

(ii) For any G-equivalence R and any $x \in X$, one has $\alpha_x(G_o(X)_R) = G_{R,x}$.

(iii) If $H \triangleleft G_o$ is normal, $\alpha_x(H) \triangleleft G$ is also normal, and this defines a canonical bijective correspondence between the sets of normal subgroups of G_o and G.

Thus we may define subgroups of G_o by specifying suitable equivalence relations on X. We shall apply this in the following situation: G is the automorphism group of a free group of finite rank F, and X is the set of all well ordered free systems of generators of F.

DEFINITION. For any free group F of finite rank n we define simply transitive group actions as follows.

$$X_F := \{(x_1,\ldots,x_n) \in F^n \mid F = \langle x_1,\ldots,x_n \rangle\} .$$

$$A(F) := \{g \in S(F) \mid \forall a,b \in F \quad g(ab) = g(a)g(b)\} .$$

$A(F)$ acts on X_F by $g(x_1,\ldots,x_n) = (g(x_1),\ldots,g(x_n))$.

$$A(F)_o := A(F)_o(X_F) .$$

Note that for $n > 1$ the centre of $A(F)$ is trivial (c.f. [81], I, 4.3), so that $A(F) \cap A(F)_o = \{1\}$ and $A(F) \times A(F)_o \subset S(X_F)$ acts effectively on X_F.

We shall now define subgroups of $A(F)_o$ by specifying equivalence relations on X_F. To begin with, define an equivalence relation for elements $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ of X_F as follows:

$$x \approx y \Leftrightarrow \exists \pi \in S_n \quad y_i = x_{\pi(i)} .$$

We denote the corresponding subgroup of $A(F)_o$ by $S(F)$. Of course $S(F)$ is canonically isomorphic to S_n , and $S(F)$ is generated by the standard transpositions $\tau_i \in S(F)$, where

$$\tau_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n) .$$

A very useful element in $S(F)$ is the element ρ_n , where $\rho_n(x_1,\dots,x_n) = (x_n,\dots,x_1)$. In order to define more interesting equivalence relations on X_F , we first introduce a very natural equivalence relation $\sim$ on F itself, namely conjugation:

$$\forall b,c \in F \quad b \sim c \Leftrightarrow \exists a \in F \quad c = aba^{-1} .$$

Moreover we define a very natural map $\chi : X_F \to F$ by $\chi(x_1,\dots,x_n) = x_1 \dots x_n$. We also denote the value of χ for $x \in X_F$ by $c_x := \chi(x)$ and call it the pseudo Coxeter element associated to x . Now we are ready to define interesting natural equivalence relations on X_F .

<u>DEFINITION.</u> The equivalence relations R'_p , R''_q and R_{pq} $p,q = 0,1,2$ for elements $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ of X_F are defined as follows.

(i) $(x,y) \in R'_o \Leftrightarrow \exists \pi \in S_n \quad y_i \sim x_{\pi(i)} \quad i = 1,\dots,n$

(ii) $(x,y) \in R'_1 \Leftrightarrow y_i \sim x_i \quad i = 1,\dots,n$

(iii) $(x,y) \in R'_2 \Leftrightarrow \exists a \in F \quad y_i = ax_ia^{-1} \quad i = 1,\dots,n$

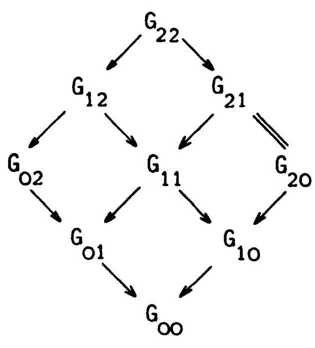
(iv) $(x,y) \in R''_o \Leftrightarrow (x,y) \in X_F \times X_F$

(v) $(x,y) \in R''_1 \Leftrightarrow c_y \sim c_x$

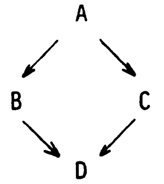
(vi) $(x,y) \in R''_2 \Leftrightarrow c_y = c_x$

(vii) $(x,y) \in R_{pq} \Leftrightarrow (x,y) \in R'_p \cap R''_q$

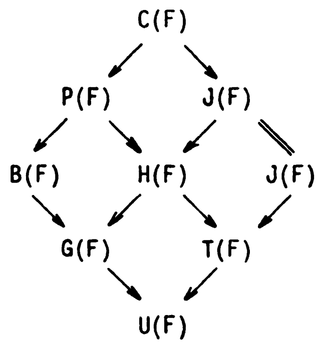
All these relations are $A(F)$-equivalences on X_F and so they define subgroups of $A(F)_o$. For the moment we denote the subgroup of $A(F)_o$ corresponding to R_{pq} by G_{pq} . One has the following diagram of inclusions:



For every subdiagram of the following form



one has $A = B \cap C$ and $D = B \cdot C$ and $A \lhd B$ and $C \lhd D$. The notation G_{pq} for these groups emphasizes the uniform logical structure of their definition. However each group is an object of its own, and some of them are classical and have been studied extensively. Therefore I shall now denote them individually by letters which partly correspond to notation used in the literature. By this change of notation, the diagram of the G_{pq} is transformed into the following diagram:



The most important group of all is the group $B(F)$, which is isomorphic to the group of braids with n strings. $P(F)$ is isomorphic to the subgroup of coloured braids, and $C(F)$ is its centre. $J(F)$ is the normal subgroup of $A(F)_o$ corresponding to the normal subgroup $I(F) \lhd A(F)$ of inner automorphisms of F. The group $T(F)$ has been studied by S.P. Humphries [54]. The role of $G(F)$ will become clear in §3.

If one wants to analyze the structure of these groups and of their representations it is very important to have a concrete description in terms of generators and relations.

<u>DEFINITION.</u> For a free group F of finite rank n the elements $\varkappa_i$, $\varkappa_{ij}$, τ_i, σ_i, $\sigma_{ij} \in A(F)_o$ are defined as follows.

(i) $\varkappa_i(x_1,\dots,x_n) = (x_i x_1 x_i^{-1},\dots,x_i x_n x_i^{-1})$ $\quad i = 1,\dots,n$

(ii) $\varkappa_{ij}(x_1,\dots,x_n) = (x_1,\dots,x_{j-1},x_i x_j x_i^{-1},x_{j+1},\dots,x_n)$ $\quad i \neq j,\ i,j = 1,\dots,n$

(iii) $\tau_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n)$ $\quad i = 1,\dots,n-1$

(iv) $\sigma_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_i x_{i+1} x_i^{-1},x_i,x_{i+2},\dots,x_n)$ $\quad i = 1,\dots,n-1$

(v) $\sigma_{ij}(x_1,\dots,x_n) = (x_1',\dots,x_n')$ $\quad 1 \le i < j \le n$

$$x_k' = \begin{cases} x_k & k < i \\ (x_i x_j)x_k(x_i x_j)^{-1} & k = i \\ [x_i,x_j]x_k[x_i,x_j]^{-1} & i < k < j \\ x_i\ x_k\ x_i^{-1} & k = j \\ x_k & k > j\ . \end{cases}$$

PROPOSITION 1.3. The following are some identities which hold for the elements $\varkappa_i,\sigma_i,\sigma_{ij} \in A(F)_o$.

(i) $\sigma_{ij} = (\sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1})^{-1}$.

(ii) $\varkappa_n\dots\varkappa_1 = (\sigma_1\dots\sigma_{n-1})^n$.

(iii) $\varkappa_n\dots\varkappa_1(x) = c_x(x)$.

(iv)
$$\sigma_i\varkappa_j\sigma_i^{-1} := \begin{cases} \varkappa_j & j \neq i,i+1 \\ \varkappa_{i+1} & j = i \\ \varkappa_{i+1}\varkappa_i\varkappa_{i+1}^{-1} & j = i+1\ . \end{cases}$$

THEOREM 1.4. The subgroups of $A(F)_o$ defined above are generated as follows:

$B(F) = \langle\sigma_1,\dots,\sigma_{n-1}\rangle$	$H(F) = \langle\varkappa_1,\dots,\varkappa_n,\sigma_{ij}\rangle$
$P(F) = \langle\sigma_{ij}\rangle$	$G(F) = \langle\varkappa_1,\dots,\varkappa_n,\sigma_1,\dots,\sigma_{n-1}\rangle$
$C(F) = \langle\varkappa_n\dots\varkappa_1\rangle$	$T(F) = \langle\varkappa_{ij}\rangle$
$J(F) = \langle\varkappa_1,\dots,\varkappa_n\rangle$	$U(F) = \langle\varkappa_{ij},\tau_i\rangle$.

The proof for $B(F)$ is classical and due to Artin (c.f. [12], theorem 16). The proof for $T(F)$ is similar and was given by Humphries [54]. The case of $P(F)$ is also classical. Nevertheless I shall indicate briefly how it can be dealt with in the present context, since this leads to presentations of $P(F)$ and $B(F)$. The remaining cases are easy consequences of the preceding ones.

In order to analyze $P(F)$ by induction on $n = \operatorname{rank} F$, we introduce two new $A(F)$-equivalences for $x,y \in X_F$.

$$x\sim\cdot y \Leftrightarrow \langle x_1,\dots,x_{n-1}\rangle = \langle y_1,\dots,y_{n-1}\rangle \text{ and } x_n = y_n \text{ and } (x,y) \in R_{12}$$
$$x\cdot\sim y \Leftrightarrow N(x_n) = N(y_n) \text{ and } x_i \equiv y_i \bmod N(x_n) \text{ and } (x,y) \in R_{12}$$

where $N(z)$ denotes the normal subgroup of F "generated" by $z \in F$. The corresponding subgroups of $A(F)_o$ are denoted as follows:

$$\cdot P(F) = \{\gamma \in A(F)_o \mid \gamma x \cdot \sim x\}$$
$$P(F)^{\cdot} = \{\gamma \in A(F)_o \mid \gamma x \sim \cdot x\} \quad .$$

Together with $P(F)$, they form a canonical short exact sequence split by inclusion:

$$(*) \qquad 1 \to {}^{\cdot}P(F) \to P(F) \rightleftarrows P(F)^{\cdot} \to 1 \quad .$$

Choose any $x = (x_1,\ldots,x_n) \in X_F$ and put $\hat{x} = (x_1,\ldots,x_{n-1})$ and $\hat{F} = \langle x_1,\ldots,x_{n-1}\rangle$. Then the exact sequence (*) identifies canonically with the following split exact sequence:

$$1 \to \hat{F} \to \hat{F} \rtimes P(\hat{F}) \rightleftarrows P(\hat{F}) \to 1 \quad ,$$

where the semidirect product is defined by the operation

$$\sigma(y) = \alpha_{\hat{x}}(\sigma^{-1})(y) \qquad \text{for } \sigma \in P(\hat{F}) \text{ and } y \in \hat{F} \quad .$$

Moreover $\sigma_{in} \in {}^{\cdot}P(F)$ is identified with $x_i \in \hat{F}$ and $\sigma_{ij} \in P(F)^{\cdot}$ is identified with $\sigma_{ij} \in P(\hat{F})$ for $j < n$. Hence one obtains $P(F) = \langle\sigma_{ij}\rangle$ by induction on $n = \operatorname{rank} F$, and moreover this argument yields directly the following presentation of $P(F)$.

<u>THEOREM 1.5.</u> For a free group F of rank n, the group $P(F)$ is presented by the generators σ_{ij}, where $1 \le i < j \le n$, with the following relations, where $1 \le i < j \le n$ and $1 \le k < m \le n$ and $j < m$.

$$\sigma_{ij}\sigma_{km}\sigma_{ij}^{-1} = \begin{cases} \sigma_{km} & k < i \\ \sigma_{jm}^{-1}\,\sigma_{km}\,\sigma_{jm} & k = i \\ [\sigma_{im}^{-1},\sigma_{jm}^{-1}]^{-1}\sigma_{km}[\sigma_{im}^{-1},\sigma_{jm}^{-1}] & i < k < j \\ (\sigma_{im}\sigma_{jm})^{-1}\sigma_{km}(\sigma_{im}\sigma_{jm}) & k = j \\ \sigma_{km} & k > j \end{cases}$$

The presentation of $P(F)$ leads to a presentation of $B(F)$ as follows.

<u>DEFINITION.</u> B_n is the group presented by generators $\sigma_1,\ldots,\sigma_{n-1}$ with the following relations:

$$\boxed{\begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \end{array}}$$

$P_n \lhd B_n$ is the kernel of the homomorphism $B_n \to S_n$ mapping σ_i onto the standard transposition τ_i.

If one applies the Reidemeister-Schreier-method to a suitable Schreier transversal for $P_n \subset B_n$, one obtains the same presentation for P_n as the one for $P(F)$ given in 1.5, where the $\sigma_{ij} \in P_n$ are now defined as in 1.3.(i) (c.f. Chow [27] or Magnus, Karrass, Solitar [82] or Birman [15]). However, we do not need the full strength of the method, nor do we have to do all the calculations involved. It is enough to prove $P_n = \langle\sigma_{ij}\rangle$ and to verify that the generators σ_{ij} of P_n satisfy the relations 1.5 of the generators σ_{ij} of $P(F)$ and the generators σ_i of $B(F)$ satisfy the defining relations of the generators σ_i of B_n. After that, one obtains immediately the following theorem and therefore a presentation of $B(F)$.

THEOREM 1.6. For any free group F of rank n, there is a canonical isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & P_n & \to & B_n & \to & S_n & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & P(F) & \to & B(F) & \to & S(F) & \to & 1 \end{array}$$

sending the generators $\tau_i, \sigma_i, \sigma_{ij}$ of S_n, B_n, P_n onto the generators $\tau_i, \sigma_i, \sigma_{ij}$ of $S(F), B(F), P(F)$.

Finally, the exact sequence (*) is also useful in proving by induction that the centre of $B(F)$ is infinite cyclic. There are several useful descriptions of a generator of this infinite cyclic group. One of them uses the fundamental element introduced by Garside [44]. The definition uses the generators σ_i, so we state it for B_n. The fundamental element is then represented by a positive word in the generators. Therefore we shall denote it by ω_n^+ and its inverse by ω_n^-.

DEFINITION. The elements $\pi, \pi', \pi'', \omega_n^+, \omega_n^- \in B_n$ are defined as follows:

$$\pi' = \prod_{i\equiv 1(2)} \sigma_i \quad \text{and} \quad \pi'' = \prod_{i\equiv 0(2)} \sigma_i \quad \text{and} \quad \pi = \pi'\pi'' .$$

$$\omega_n^+ = \begin{cases} \pi^{n/2} & \text{for} \quad n \equiv 0(2) \\ \pi^{n-1/2}\pi' & \text{for} \quad n \equiv 1(2) \end{cases} \quad \text{and} \quad \omega_n^- = (\omega_n^+)^{-1} .$$

The element ω_n^+ is called the fundamental element of B. It has the following important property: $\sigma_i\omega_n^+ = \omega_n^+\sigma_{n-i}$. A systematic treatment of the fundamental element in the more general context of Artin groups of finite type is to be found in [24].

DEFINITION. The elements $\xi_1,\ldots,\xi_n$ and $\zeta_n \in B_n$ are defined as follows: $\xi_j = \sigma_{1j}\cdots\sigma_{j-1,j}$ and $\zeta_n = \xi_n\cdots\xi_1$.

The following theorem is due to Chow [27].

THEOREM 1.7. For a free group F of rank $n \geq 2$, the centre of $P(F)$ is the following infinite cyclic group:

$$Z(P(F)) = C(F) = \langle \zeta_n \rangle \quad .$$

For $n \geq 3$ one has $Z(B(F)) = Z(P(F))$ for the centre of $B(F)$. The generator ζ_n may also be described as follows:

$$\zeta_n = \omega_n^{+2} = (\sigma_1 \ldots \sigma_{n-1})^n = \varkappa_n \ldots \varkappa_1 \quad .$$

The proof of the identity $\zeta_n = \varkappa_n \ldots \varkappa_1$ uses the fact that conjugation of an element of a group by itself is an idempotent operation. Later on, when we are dealing with automorphic sets, we shall replace conjugation by a more general operation which does not have to be idempotent. In this case, there will be no analogue of the operation of the group $G(F)$ on X_F . Instead of $G(F)$ we shall have to consider the group

$$J(F) \rtimes B(F) \quad ,$$

where the semidirect product is defined by means of the following operation of $B(F)$ on $J(F)$:

$$\sigma(\varkappa) = \sigma\varkappa\sigma^{-1} \qquad \text{for} \qquad \sigma \in B(F) \ , \ \varkappa \in J(F) \quad .$$

Because of 1.7 there is a canonical isomorphism

$$G(F) = J(F) \rtimes B(F) / \langle (\varkappa_n \ldots \varkappa_1 , \zeta_n^{-1}) \rangle \quad .$$

In view of 1.3.(iv) we are led to the definition of the following groups $J_n \rtimes B_n$ and G_n in terms of generators and relations, which are canonically isomorphic to $J(F) \rtimes B(F)$ and $G(F)$. Note that B_n and J_n are subgroups of $J_n \rtimes B_n$ and that J_n is free of rank n .

DEFINITION. $J_n \rtimes B_n$ is the group presented by generators $\varkappa_1, \ldots, \varkappa_n$ and $\sigma_1, \ldots, \sigma_{n-1}$ with the following relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| > 1 \qquad\qquad \sigma_i\varkappa_j\sigma_i^{-1} = \begin{cases} \varkappa_j & \text{if } j \neq i, i+1 \\ \varkappa_{i+1} & \text{if } j = i \\ \varkappa_{i+1}\varkappa_i\varkappa_{i+1}^{-1} & \text{if } j = i+1 \ . \end{cases}$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{if } |i-j| = 1$$

G_n is the group $J_n \rtimes B_n / \langle (\varkappa_n \ldots \varkappa_1 , \zeta_n^{-1}) \rangle$.

Now that we have presented the groups which are going to operate, our next task will be to introduce the structure which will be the substrate of the operation.

§2. AUTOMORPHIC SETS

A set with a product is a pair $(\Delta,*)$ where Δ is a set and $*$ is a map $\Delta \times \Delta \to \Delta$. The value of this map for $(a,b) \in \Delta \times \Delta$ will be denoted by $a*b$. We shall use the same symbol "$*$" for different sets with product, unless there is danger of confusion, and we shall frequently write Δ instead of $(\Delta,*)$. A morphism of sets with product $(\Delta,*) \to (\Delta',*)$ is a map $\varphi\colon \Delta \to \Delta'$ such that $\varphi(a*b) = \varphi(a) * \varphi(b)$. The sets with product form a category. In particular, any set with product has an automorphism group

$$\mathrm{Aut}(\Delta,*) = \{\varphi \in S(\Delta) \mid \varphi(a*b) = \varphi(a) * \varphi(b)\} .$$

We shall write $\mathrm{Aut}(\Delta)$ instead of $\mathrm{Aut}(\Delta,*)$, if $*$ is determined by the context. For any set with product $(\Delta,*)$ and $a \in \Delta$ the left translation λ_a is the map $\lambda_a\colon \Delta \to \Delta$ defined by

$$\lambda_a(b) = a*b .$$

We now introduce the category of automorphic sets as a full subcategory of the category of sets with products. Its objects are defined as follows.

DEFINITION. An automorphic set is a set with product such that all left translations are automorphisms. In other words: A set with product $(\Delta,*)$ is an automorphic set, if it has the following two basic properties:

(i)	$\forall a,c \in \Delta$	$\exists!\ b \in \Delta$	$a*b = c$
(ii)	$\forall a,b,c \in \Delta$		$(a*b)*(a*c) = a*(b*c)$

DEFINITION. For any automorphic set $(\Delta,*)$, the subgroup of inner automorphisms $I(\Delta) \subset \mathrm{Aut}(\Delta)$ and its centralizer $C(\Delta)$ are defined as follows:

$$I(\Delta) = \langle \lambda_a \mid a \in \Delta \rangle \subset \mathrm{Aut}(\Delta) ,$$

$$C(\Delta) = \{\varphi \in \mathrm{Aut}(\Delta) \mid \forall a \in \Delta \quad \varphi\lambda_a = \lambda_a\varphi\} .$$

Δ is called homogeneous, if $\mathrm{Aut}(\Delta)$ acts transitively on Δ. If $I(\Delta)$ acts transitively, Δ is called very homogeneous.

Note the following trivial consequence of the definitions: $\lambda_{\varphi(b)} = \varphi\lambda_b\varphi^{-1}$ for all $\varphi \in \mathrm{Aut}(\Delta)$. In particular, for $\varphi = \lambda_a$ this means

$$\lambda_{a*b} = \lambda_a\lambda_b\lambda_a^{-1} .$$

Before I give examples, let me first make some trivial statements about properties of the category of automorphic sets. The terminology concerning categories is as in [88]. The empty set $\varnothing$ is a conull object. The sets

$\Delta = \{a\}$ with $a * a = a$ are the null objects. The category has epimorphic images. It has intersections for arbitrary families of subobjects. It has products for arbitrary families of objects $(\Delta_i, *)$, $i \in I$, namely

$$\prod_{i \in I} (\Delta_i, *) = (\prod_{i \in I} \Delta_i, *) \quad \text{with} \quad (x_i) * (y_i) = (x_i * y_i) \quad .$$

The category has disjoint sums for arbitrary disjoint families of automorphic sets $(\Delta_i, *_i)$, $i \in I$. They are defined as follows:

$$\coprod_{i \in I} (\Delta_i, *_i) = (\coprod_{i \in I} \Delta_i, *)$$

$$a * b = \begin{cases} a *_i b & \text{for } a,b \in \Delta_i \\ b & \text{otherwise} \end{cases} .$$

However these disjoint sums are in general not coproducts in the sense of category theory. I call an automorphic set irreducible, if it cannot be presented in a nontrivial way as a disjoint union of automorphic sets. Every automorphic set has a unique decomposition into a disjoint sum of irreducible automorphic subsets called irreducible components. The decomposition is obtained as follows.

DEFINITION. The relations $\perp$ and $\sim$ for elements $a,b \in \Delta$ of an automorphic set $(\Delta, *)$ are defined as follows:

$$a \perp b \Leftrightarrow a * b = b \text{ and } b * a = a$$

$$a \sim b \Leftrightarrow \exists a_o, \ldots, a_r \in \Delta \text{ such that } a_o = a \text{ and } a_r = b \text{ and}$$
$$\text{for all } i = 1, \ldots, r \text{ not } a_{i-1} \perp a_i \quad .$$

Obviously $\sim$ is an equivalence relation. We also define orthogonality relation for subsets $\Delta', \Delta'' \subset \Delta$ as follows: $\Delta' \perp \Delta''$ iff $a \perp b$ for all $a \in \Delta'$ and $b \in \Delta''$.

PROPOSITION 2.1. Let Δ be an automorphic set, and $\Delta', \Delta'' \subset \Delta$ automorphic subsets. Then the following holds.

(i) $\Delta = \Delta' \amalg \Delta'' \Leftrightarrow \Delta' \cup \Delta'' = \Delta$ and $\Delta' \cap \Delta'' = \emptyset$ and $\Delta' \perp \Delta''$

(ii) The irreducible components of Δ are the $\sim$-equivalence classes.

We have several methods for constructing new automorphic sets from a given automorphic set $(\Delta, *)$. For any $\varphi \in C(\Delta)$ we get a new automorphic set ${}^{\varphi}\Delta$ with the same set Δ as underlying set and with the new product $*_{\varphi}$ defined by $a *_{\varphi} b = a * \varphi(b) = \varphi(a * b)$. Note that $\varphi \in C({}^{\varphi}\Delta)$, so that we can reconstruct Δ from ${}^{\varphi}\Delta$ as follows: $\Delta = {}^{\varphi^{-1}}({}^{\varphi}\Delta)$. An important special case is obtained by putting $\varphi = \iota_{\Delta}$, where ι_{Δ} is defined as follows.

DEFINITION. Let $(\Delta,*)$ be an automorphic set.

(i) For any $a \in \Delta$, the element $\bar{a} \in \Delta$ is defined by $a * \bar{a} = a$.

(ii) The map $\iota_\Delta : \Delta \to \Delta$ is defined by $\iota_\Delta(a) = \bar{a}$.

PROPOSITION 2.2. For any automorphic set $(\Delta,*)$ the following statements hold:

(i) $\iota_\Delta \in Z(\mathrm{Aut}(\Delta))$

(ii) $\iota_\Delta^{-1}(a) = a * a$

(iii) $\lambda_{\bar{a}} = \lambda_a$.

DEFINITION. Let $(\Delta,*)$ be an automorphic set.

(i) $(\Delta,*)$ is idempotent if $\iota_\Delta = \mathrm{id}_\Delta$

(ii) $(\Delta,*)$ is involutive if $\iota_\Delta^2 = \mathrm{id}_\Delta$

(iii) The idempotent automorphic set associated to $(\Delta,*)$ is the automorphic set $\overline{(\Delta,*)} = (\Delta,\bar{*})$, where $a \mathbin{\bar{*}} b = a * \bar{b}$. This is the automorphic set ${}^{\iota}\Delta$.

Associating $\overline{(\Delta,*)}$ to $(\Delta,*)$ defines a covariant functor from the category of automorphic sets to the subcategory of idempotent automorphic sets, and moreover we have a distinguished element in $C(\bar{\Delta})$, namely ι_Δ. Conversely, if we are given an idempotent automorphic set Δ and an element $\psi \in C(\Delta)$, then ${}^{\psi^{-1}}\Delta$ will be an automorphic set with $\iota = \psi$ and with associated idempotent automorphic set Δ. In this sense general automorphic sets are just pairs (Δ,ψ) of an idempotent automorphic set and an automorphism $\psi \in C(\Delta)$. Another simple covariant functor from the category of automorphic sets to itself is obtained by associating to any automorphic set $(\Delta,*)$ the automorphic set $\underline{(\Delta,*)} = (\Delta,\underline{*})$ with the new product defined by

$$a \,\underline{*}\, b = \lambda_a^{-1}(b) \quad .$$

One has $\underline{\underline{(\Delta,*)}} = (\Delta,*)$ for all automorphic sets and $\underline{(\Delta,*)} = (\Delta,*)$ exactly for those with $\lambda_a^2 = 1$ for all $a \in \Delta$.

DEFINITION. An automorphic set is involutory iff all left translations are involutions.

Note that "involutive" and "involutory" are completely different properties of automorphic sets. Idempotent automorphic sets were introduced and studied by D. Joyce [59]. He calls them quandles. His notation is $b \rhd a$ for our $a * b$ and $b \rhd^{-1} a$ for our $a \,\underline{*}\, b$, and he considers Δ as equipped with both products $*$ and $\underline{*}$.

It is now time to give some examples or rather classes of examples of automorphic sets. Some are obvious, some important ones were given by D. Joyce, and some come from the theory of root systems and from invariants of singularities.

<u>EXAMPLE 1.</u> Every set Δ can be given the trivial automorphic structure defined by $a * b = b$ for all $a,b \in \Delta$.

<u>EXAMPLE 2.</u> Every group Δ has a canonical idempotent automorphic structure defined by $a * b = aba^{-1}$. Every homomorphism of groups is a morphism for these automorphic structures. $\underline{\Delta}$ is the canonical automorphic structure for the opposite group of Δ . If Δ is any automorphic set and $I(\Delta)$ its inner automorphism group with its canonical automorphic structure, the map $\Delta \to I(\Delta)$ defined by $a \mapsto \lambda_a$ is a morphism of automorphic sets.

<u>EXAMPLE 3.</u> Let Δ be an abelian group. Let ψ be an automorphism of the abelian group Δ and φ an endomorphism of Δ commuting with ψ and satisfying the equation $\varphi(\varphi+\psi-1) = 0$. For example one can choose $\varphi = 1-\psi$. Then Δ is an automorphic set with the product defined by $a * b = \varphi(a) + \psi(b)$. To put it another way: If Δ is any $\mathbb{Z}[t,t^{-1}]$-module, it becomes an automorphic set with the product defined by $a * b = (1-t)a + tb$. An interesting special case is Alexander invariant of a knot.

<u>EXAMPLE 4.</u> This example is due to David Joyce. We work in the category of pairs of topological spaces (X,Y) with a base point $x_o \in X - Y$. Maps of pairs $f:(X,Y) \to (X',Y')$ have to be such that $f^{-1}(Y') = Y$ and $f(x_o) = x'_o$. We define a particular pair $(\tilde{D},\{0\})$ with base point z_o as follows. Let $z_o > 1$ be a fixed real number chosen once and for all. Let $D \subset \mathbb{C}$ be the closed unit disk in the complex plane, and $\tilde{D}$ the space $\tilde{D} = D \cup [1,z_o]$ (see figure 1).

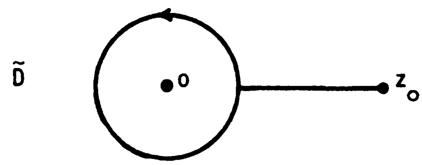


FIGURE 1.

A "noose" in X around Y is a homotopy class of maps $f:(\tilde{D},\{0\}) \to (X,Y)$. The set of nooses in X around Y with basepoint x_o is denoted by $\Delta(X,Y,x_o)$. For any noose a , restriction to the oriented boundary of $\tilde{D}$ defines a homotopy class ∂a of loops in $X - Y$ based at x_o . This defines a natural augmentation map to the fundamental group:

$$\partial:\Delta(X,Y,x_o) \to \pi_1(X-Y,x_o) \quad .$$

The fundamental group operates canonically on $\Delta(X,Y,x_o)$. For $\beta \in \pi_1(X-Y,x_o)$ and $b \in \Delta(X,Y,x_o)$ the noose $\beta(b)$ maps that half of the interval $[1,z_o]$ containing the initial point z_o to $X - Y$ by means of β , whereas $\beta(b)$ maps the rest of $\tilde{D}$ to X by means of b. Obviously $\partial(\beta(b)) = \beta(\partial b)\beta^{-1}$, and therefore $\Delta(X,Y,x_o)$ is an idempotent automorphic set with the product defined by $a * b = \partial a(b)$. This is the fundamental automorphic set of (X,Y,x_o). This is the fundamental quandle introduced by D. Joyce. To be precise: his definition is related to ours by $b \rhd a = a \underline{*} b$.

EXAMPLE 5. Let $K \subset S^3$ be a knot, $p \in S^3-K$ a base point, $\varepsilon \in H_1(S^3-K,\mathbb{Z})$ a generator and $\bar{\partial} : \Delta(S^3,K,p) \to H_1(S^3-K,\mathbb{Z})$ the composition of the augmentation and abelianization maps. Then we get an automorphic subset of $\Delta(S^3,K,p)$, namely $\Delta(K)_\varepsilon = \bar{\partial}^{-1}(\varepsilon)$. This is the "knot quandle" of David Joyce. For tame knots, a presentation of it is easily obtained from any regular projection. Joyce proves that it is a faithful invariant of unoriented equivalence classes of knots and that its abelianization is the Alexander invariant.

EXAMPLE 6. Let Δ be a Riemannian symmetric space, i.e. a connected Riemannian manifold such that for each point $a \in \Delta$ there is an involutive isometry s_a of Δ with $a \in \Delta$ as an isolated fixed point. Since s_a is necessarily unique, Δ is an automorphic set with the product defined by $a * b = s_a(b)$.

EXAMPLE 7. Let V be a vector space with an alternating bilinear form with value $\langle a,b\rangle$ for $a,b \in V$. To each $a \in V$ we associate a transvection s_a defined by $s_a(x) = x - \langle x,a\rangle a$. Then $\Delta = V$ is an automorphic set with product defined by $a * b = s_a(b)$.

EXAMPLE 8. Let V be a vector space with a symmetric bilinear form with value $\langle a,b\rangle$ for $a,b \in V$. Let Δ be the set of nonisotropic vectors, $\Delta = \{a \in V \mid \langle a,a\rangle \neq 0\}$. To each $a \in \Delta$ we associate the reflection s_a with respect to the hyperplane orthogonal to a defined as follows:

$$s_a(x) = x - 2\,\frac{\langle x,a\rangle}{\langle a,a\rangle}\,a \quad .$$

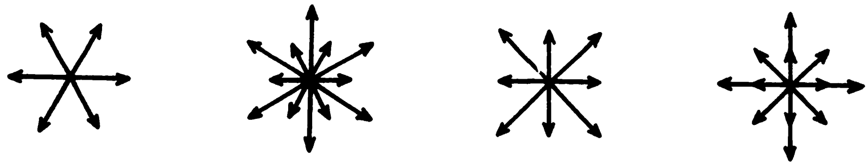
Then Δ is an automorphic set with the product defined by $a * b = s_a(b)$.

EXAMPLE 9. This example is very classical and particularly interesting in the present context. Let V be a finite dimensional euclidean vector space. $V-\{0\}$ is an automorphic set as before. Consider a subset $\Delta \subset V-\{0\}$ which has the following two properties:

(i) $\Delta \subset V-\{0\}$ is a finite automorphic subset

(ii) $\forall a,b \in \Delta \quad a * b - b \in \mathbb{Z}a$.

The finite automorphic sets defined in this way are exactly the classical root systems (c.f. Bourbaki [19], VI, §1). A root system Δ is irreducible as such iff it is irreducible as an automorphic set, and the decompositions into irreducible components for both types of structures are the same. The irreducible root systems are classified as A_n, B_n, C_n, BC_n, E_6, E_7, E_8, F_4, G_2 . Figure 2 shows those of rank 2 .



A_2 G_2 B_2 BC_2

FIGURE 2.

A root system is called homogeneous if all roots have equal length, and in this case we assume $\langle a,a\rangle = 2$ without loss of generality. The irreducible homogeneous root systems are A_n , $n \geq 1$ and D_n , $n \geq 4$, and E_6, E_7, E_8 . They are very closely related to the simple singularities of hypersurfaces.

We shall now describe a class of automorphic sets which is sufficiently large to allow for the definition of the invariants corresponding to arbitrary isolated singularities of hypersurfaces and complete intersections.

Let R be an integral domain and K its field of quotients. We shall mainly be interested in the case where R is the ring of integers, but proposition 3.7 shows that there are reasons to consider more general R . Let L be a lattice over R , that is a free R-module L of finite rank equipped with a bilinear form

$$L \times L \to R$$
$$(a,b) \mapsto \langle a,b\rangle \quad .$$

Within this common general context we shall be dealing with two different situations, which turn out to be intimately related and which we shall try to treat simultaneously whenever this is feasible. We call these situations the "symmetric case" and the "antisymmetric case".

<u>SYMMETRIC CASE.</u> In this case the bilinear form is assumed to be symmetric, i.e. $\langle a,b\rangle = \langle b,a\rangle$. Moreover in this case we always assume that the form is even, i.e. $\langle a,a\rangle \in 2R$. When $\operatorname{char} K \neq 2$, we associate to the bilinear form the quadratic form $q: L \to R$ defined by $\langle a,a\rangle = 2q(a)$.

<u>ANTISYMMETRIC CASE.</u> In this case the bilinear form is assumed to be alternating, i.e. $\langle a,a\rangle = 0$ for all $a \in L$. This implies that $\langle a,b\rangle = -\langle b,a\rangle$, and the converse is true if $\operatorname{char} K \neq 2$.

In both cases $\operatorname{Aut}(L)$ denotes the automorphism group of the lattice L .

In the symmetric case this is an orthogonal group and will also be denoted by $O(L)$, in the antisymmetric case it is a symplectic group and will also be denoted by $Sp(L)$. We now define certain automorphisms $s_{a,\varepsilon} \in Aut(L)$, where a is an element of L and $\varepsilon = \pm 1$. In the antisymmetric case a and ε can be arbitrary, but in the symmetric case they have to be such that $\varepsilon\langle a,a\rangle = 2$.

DEFINITION. The element $s_{a,\varepsilon} \in Aut(L)$ is defined as follows:

$$\boxed{s_{a,\varepsilon}(x) = x - \varepsilon\langle x,a\rangle a}$$

We shall write s_a instead of $s_{a,\varepsilon}$ if ε is determined by the context. In the symmetric case, s_a is an involution, and if $\operatorname{char} K \neq 2$, it is the unique involution s such that $s(a) = -a$ and $s(x) = x$ for x orthogonal to a. In the antisymmetric case, $s_{a,\varepsilon}$ is a symplectic transvection in the direction of a. We have $s_{a,\varepsilon}(a) = a$, and the inverse of $s_{a,\varepsilon}$ is $s_{a,-\varepsilon}$.

DEFINITION. Let L be a lattice as above and $\varepsilon = \pm 1$. The associated automorphic set $(\Delta_\varepsilon(L), \underset{\varepsilon}{*})$ is defined as follows:

$$\Delta_\varepsilon(L) = \begin{cases} \{a \in L \mid \varepsilon\langle a,a\rangle = 2\} & \text{in the symmetric case,} \\ L & \text{in the antisymmetric case .} \end{cases}$$

$$\boxed{a \underset{\varepsilon}{*} b = s_{a,\varepsilon}(b) \quad .}$$

We shall write $\Delta_\varepsilon(L)$ instead of $(\Delta_\varepsilon(L), \underset{\varepsilon}{*})$ and $*$ instead of $\underset{\varepsilon}{*}$ if there is no danger of confusion. Obviously $\bar{a} = -a$ in the symmetric case, and $\bar{a} = a$ in the antisymmetric case. Hence $\Delta_\varepsilon(L)$ is idempotent in the antisymmetric case and involutive in the symmetric. With respect to the functors defined above, the two cases compare as follows (if $\Delta_\varepsilon(L) \neq \emptyset$ and $\operatorname{char} K \neq 2$):

$$L \text{ symmetric} \Rightarrow \underline{\Delta_\varepsilon(L)} = \Delta_\varepsilon(L) \text{ and } \overline{\Delta_\varepsilon(L)} \neq \Delta_\varepsilon(L) \quad .$$

$$L \text{ antisymmetric} \Rightarrow \underline{\Delta_\varepsilon(L)} = \Delta_{-\varepsilon}(L) \text{ and } \overline{\Delta_\varepsilon(L)} = \Delta_\varepsilon(L) \quad .$$

We have chosen the underlying set $\Delta_\varepsilon(L)$ as the largest possible set for which the definition of the product given above makes sense. There are several meaningful ways of passing to smaller automorphic subsets. For instance whenever we have a property for elements $b \in L$ which is invariant under all $s_{a,\varepsilon}$, we may consider the subset of those elements in $\Delta_\varepsilon(L)$ which have this property. One property of this kind is primitivity: $b \in L$ is primitive, if it is part of some basis of L. Another one is strong primitivity, where we require in addition that there exists an $a \in L$ such that $\langle a,b\rangle = 1$, so that $\langle L,b\rangle = \mathbb{Z}$. In the symmetric case, if R is a principal ideal domain and $\operatorname{char} K \neq 2$ all elements of $\Delta_\varepsilon(L)$ will be primitive, but in general not all will be strongly primitive.

Since the automorphic sets $\Delta_\varepsilon(L)$ are imbedded into a linear structure, namely the R-module L, it is natural to introduce automorphism groups which take that linear structure into account and then to compare them with the automorphism groups defined for arbitrary automorphic sets.

DEFINITION. The subgroups $W_\varepsilon(L)$ and $C_\varepsilon(L)$ of $\mathrm{Aut}(L)$ are defined as follows:

$$W_\varepsilon(L) = \langle s_{a,\varepsilon} \mid a \in \Delta_\varepsilon(L)\rangle$$

$$C_\varepsilon(L) = \{\varphi \in \mathrm{Aut}(L) \mid \forall a \in \Delta_\varepsilon(L) \quad \varphi s_{a,\varepsilon} = s_{a,\varepsilon}\varphi\} \quad .$$

We have the following canonical diagram of group homomorphisms:

$$\begin{array}{ccccc} W_\varepsilon(L) & \longrightarrow & \mathrm{Aut}(L) & \longleftarrow & C_\varepsilon(L) \\ \downarrow\Phi & & \downarrow\Psi & & \downarrow\Psi' \\ I(\Delta_\varepsilon(L)) & \longrightarrow & \mathrm{Aut}(\Delta_\varepsilon(L)) & \longleftarrow & C(\Delta_\varepsilon(L)) \end{array}$$

The horizontal arrows are inclusions of normal subgroups, the vertical arrows are obtained by restricting the group operations to the invariant subset $\Delta_\varepsilon(L)$. The homomorphism Φ is always an isomorphism and will be used to identify the groups $W_\varepsilon(L)$ and $I(\Delta_\varepsilon(L))$. The homomorphism Ψ is obviously injective in the antisymmetric case. In the symmetric case, a sufficient condition for injectivity is that $\Delta_\varepsilon(L)$ generates $L \otimes K$. This condition is satisfied in applications in singularity theory and is assumed in most results of the following paragraph 3. However, conditions of this kind are not sufficient for surjectivity. For example, Ψ and Ψ' are not surjective for the binary integral even symmetric bilinear form with associated quadratic form $q((x,y)) = x^2 + 3xy + y^2$.

Let L be a lattice and $\mathrm{rad}\, L$ its radical, $\mathrm{rad}\, L = \{a \in L \mid \forall x \in L \ \langle x,a\rangle = 0\}$. We have a canonical morphism $\Delta_\varepsilon(L) \to I(\Delta_\varepsilon(L))$ given by $a \mapsto \lambda_a$. Note that it has the following properties:

(i) $\quad \lambda_a = 1 \quad \Leftrightarrow \quad a \in \mathrm{rad}\, L$

(ii) $\quad \lambda_a = \lambda_b \neq 1 \quad \Leftrightarrow \quad a = \pm b$ and $a,b \notin \mathrm{rad}\, L$.

In order to say something about $C_\varepsilon(L)$ and $C(\Delta_\varepsilon(L))$, we look at the decomposition of $\Delta_\varepsilon(L)$ into irreducible components Δ_j, $j \in J$. This decomposition was described in 2.1 by means of the relation $\perp$. For the automorphic sets $\Delta_\varepsilon(L)$ this is just the usual orthogonality relation: $a \perp b \Leftrightarrow \langle a,b\rangle = 0$. The sublattices generated by the Δ_j are orthogonal to each other. However, two of them may intersect in a nontrivial isotropic sublattice. This makes it difficult to determine $C_\varepsilon(L)$ and $C(\Delta_\varepsilon(L))$. All we can say is the following. Let ϑ be any map $\vartheta: J \to \{\pm 1\}$. Then ϑ defines an automorphism $\tilde{\vartheta}$ of $\Delta_\varepsilon(L)$ as follows: $\tilde{\vartheta}(a) = \vartheta(j)a$ for $a \in \Delta_j$. In the symmetric case, it is easy to prove:

$$\Psi'(C_\varepsilon(L)) \subset \{\pm 1\}^J \subset C(\Delta_\varepsilon(L)) \ .$$

Both inclusions may be strict. The binary form mentioned above is an example of that for the second inclusion. At any rate if $\Delta_\varepsilon(L)$ is irreducible, we know at least that $\Psi'(C_\varepsilon(L)) = \{\pm 1\}$. For the very special case of homogeneous root systems we can do better.

PROPOSITION 2.3. Let Δ be a homogeneous root system and $\Delta = \coprod_{j \in J} \Delta_j$ its decomposition into irreducible components. Let $A(\Delta)$ be the stabilizer of Δ in the orthogonal group $O(L)$ of the root lattice L and $W(\Delta) \subset A(\Delta)$ the Weyl group. Then the groups $I(\Delta)$, $Aut(\Delta)$ and $C(\Delta)$ of the automorphic set Δ are determined as follows:

(i) $I(\Delta) = W(\Delta)$

(ii) $Aut(\Delta) = A(\Delta) = O(L)$

(iii) $C(\Delta) = \{\pm 1\}^J$.

PROOF: Statements (i) and (iii) follow from the remarks above and from (ii), which is proved as follows. For $a,b \in \Delta$ one has $\langle a,b\rangle = \pm 2$ if $a = b$ or $a = \bar{b}$, and $\langle a,b\rangle = 0,\pm 1$ otherwise. It is trivial to prove the following equivalences:

$$\langle a,b\rangle = 0 \Leftrightarrow a * b = b$$
$$\langle a,b\rangle = 1 \Leftrightarrow (a * b) * b = a$$
$$\langle a,b\rangle = -1 \Leftrightarrow (a * b) * b = \bar{a} \ .$$

Therefore $\langle \varphi(a),\varphi(b)\rangle = \langle a,b\rangle$ for all $a,b \in \Delta$ and any $\varphi \in Aut(\Delta)$. This implies $\varphi \in A(\Delta)$.

Note that the root system Δ of type B_2 is an example where $A(\Delta) \subsetneq Aut(\Delta)$.

Let me close this paragraph by pointing out that in the symmetric case the automorphic set $\Delta_\varepsilon(L)$ may be constructed from the canonical structure of a group. The next proposition explains how this is done. For basic notions concerning Clifford algebras I refer to Bourbaki [18].

PROPOSITION 2.4. Let L be an even symmetric lattice over an integral domain of characteristic $\neq 2$. Let q be the associated quadratic form and $C(L)$ the Clifford algebra for (L,q) . Let $C(L)^*$ be the group of units of $C(L)$ with its canonical automorphic structure. Let $C(L)^*_\varepsilon$ be the automorphic subset $\{a \in C(L)^* \cap L \mid q(a) = \varepsilon\}$, where $\varepsilon = \pm 1$. This subset is invariant under the principal automorphism of $C(L)$, and its restriction φ lies in the centre of $\mathrm{Aut}(C(L)^*_\varepsilon)$. Then one has the following identities of automorphic sets:

$$\Delta_\varepsilon(L) = {}^{\varphi}C(L)^*_\varepsilon \quad \text{and} \quad \bar{\Delta}_\varepsilon(L) = C(L)^*_\varepsilon \ .$$

PROOF. The underlying sets are obviously identical. Moreover $\varphi(a) = -a = \bar{a}$ for $a \in \Delta_\varepsilon(L)$. Therefore the following chain of identities for $a,b \in \Delta_\varepsilon(L)$ proves that the products agree.

$$a \mathbin{\bar{*}} b = -b + \varepsilon\langle b,a\rangle a = -b + \langle b,a\rangle a^{-1} = (-ba+\langle b,a\rangle)a^{-1} = aba^{-1} \ .$$

§3. BRAIDS AND AUTOMORPHIC SETS

The basic fact relating braids to automorphic sets is the existence of a canonical operation of the braid group B_n on the cartesian product Δ^n of any automorphic set Δ . In addition, some other groups related to braid groups such as the groups $J_n \rtimes B_n$ or G_n introduced in §1 also operate canonically on Δ^n , and these operations are relevant to applications in singularity theory. We shall first introduce these operations for arbitrary automorphic sets and then specialize to the automorphic sets $\Delta_\varepsilon(L)$ in lattices.

Let Δ be an automorphic set and $C(\Delta)$ the centralizer of its inner automorphism group $I(\Delta)$ in $\mathrm{Aut}(\Delta)$. Then we can extend B_n by the direct product $C(\Delta)^n \times J_n$ as follows. B_n operates on the free group J_n as described in the definition of $J_n \rtimes B_n$, and B_n operates on $C(\Delta)^n$ via the canonical permutation representation $B_n \to S_n$. Explicitly $\sigma_i(\varphi_1,\dots,\varphi_n) = (\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\varphi_i,\dots,\varphi_n)$. The resulting semidirect product will be denoted by $(C(\Delta)^n \times J_n) \rtimes B_n$.

PROPOSITION 3.1. For any automorphic set $(\Delta,*)$ and any integer $n \geq 1$ there is a canonical operation

$$(C(\Delta)^n \times J_n) \rtimes B_n \times \Delta^n \longrightarrow \Delta^n$$

defined as follows:

$\sigma_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_i * x_{i+1},x_i,x_{i+2},\dots,x_n)$		$1 \leq i < n$
$\varkappa_j(x_1,\dots,x_n) = (x_j * x_1,\dots,x_j * x_n)$		$1 \leq j \leq n$
$\varphi(x_1,\dots,x_n) = (\varphi_1(x_1),\dots,\varphi_n(x_n))$		$\varphi \in C(\Delta)^n$

This operation induces an operation of the group $C(\Delta)^n \rtimes G_n$ if and only if Δ is idempotent.

The proof is easy. The last statement follows from formulas (iv) and (v) in proposition 3.2 below. Note that the group $\mathrm{Aut}(\Delta)$ also operates on Δ^n by $\varphi(x_1,\dots,x_n) = (\varphi(x_1),\dots,\varphi(x_n))$. This operation commutes with the operation of $J_n \rtimes B_n$. Note also that we may replace $C(\Delta)$ by some subgroup $C \subset C(\Delta)$. If C is in the centre of $\mathrm{Aut}(\Delta)$, the group $(C^n \times J_n) \rtimes B_n$ operates on Δ^n as a group of $\mathrm{Aut}(\Delta)$-automorphisms of Δ^n . There is no universal choice of C for arbitrary automorphic sets Δ leading always to the same nontrivial group C . However if we specialize to the class of automorphic sets $\Delta_\varepsilon(L)$, there is a natural universal choice, namely $C = \{\pm 1\}$. And in point of fact, in singularity theory the basic invariants will be orbits of $(\{\pm 1\}^n \times J_n) \rtimes B_n$ in $\Delta_\varepsilon(L)^n$.

We may also generalize some of the other group actions introduced in §1. In particular one has a canonical operation of S_n on Δ^n . We shall need the element of greatest length $\rho_n \in S_n$, which operates as follows:

$$\rho_n(x_1,\dots,x_n) = (x_n,\dots,x_1) \quad .$$

The group operations introduced in 3.1 are not new: particular cases of them occur as early as the braid groups themselves. In combinatorial group theory, the operations σ_i and σ_i^{-1} on Δ^n for an arbitrary group Δ with its canonical automorphic structure are known as Peiffer-transformations of the first kind (c.f. [81], p. 157).

It turns out that it is quite difficult to understand the operations which we have just defined so easily and so naturally, even if we restrict our investigation to very nice particular automorphic sets such as root systems. It is therefore natural to try to simplify the situation. One way of simplifying is to introduce invariants.

DEFINITION. Let Δ be an automorphic set and x any element $x \in \Delta^n$. Then we define a subgroup $\Gamma_x \subset I(\Delta)$, an element $c_x \in \Gamma_x$ and a subset $\Delta_x \subset \Delta$ as follows.

(i)	$\Gamma_x = \langle \lambda_{x_1},\dots,\lambda_{x_n} \rangle \subset I(\Delta)$
(ii)	$c_x = \lambda_{x_1} \cdots \lambda_{x_n} \in \Gamma_x$
(iii)	$\Delta_x = \Gamma_x\{x_1,\dots,x_n\} \subset \Delta$

I shall call Γ_x the monodromy group associated to x and c_x the pseudo Coxeter element. This terminology refers to applications to root systems and singularity theory. The pseudo Coxeter element plays a very important role in what follows. In particular, it is closely related to the fundamental elements $\omega_n^+,\omega_n^- \in B_n$ and to the generator of the centre $\zeta_n \in B_n$ defined in §1.

PROPOSITION 3.2. Let $(\Delta,*)$ be an automorphic set. The group $J_n \rtimes B_n$ operates on Δ^n and $\underline{\Delta}^n$. The automorphisms of Δ^n and $\underline{\Delta}^n$ corresponding to an element $\gamma \in J_n \rtimes B_n$ are denoted by γ and $\underline{\gamma}$ respectively. The following identities hold for any $x = (x_1,\ldots,x_n) \in \Delta^n$.

(i) $\sigma_{ij}(x) = (x'_1,\ldots,x'_n)$, where

$$x'_k = \begin{cases} x_k & k < i \\ \lambda_{x_i}\lambda_{x_j}\lambda_{x_i}^{-1}(x_i) & k = i \\ [\lambda_{x_i},\lambda_{x_j}](x_k) & i < k < j \\ \lambda_{x_i}(x_j) & k = j \\ x_k & k > j \end{cases}$$

(ii) $\omega_n^+(x) = (x'_1,\ldots,x'_n)$ where $x'_i = x_1 * (x_2 * (\ldots(x_{n-i} * x_{n-i+1})\ldots)$

(iii) $\rho_n\omega_n^- = \underline{\omega}_n^+\rho_n$

(iv) $\zeta_n(x) = c_x(\bar{x})$

(v) $\varkappa_n\ldots\varkappa_1(x) = c_x(x)$

(vi) $\sigma_n\ldots\sigma_1\sigma_1\ldots\sigma_n(x) = (c_x(\bar{x}_n) * x_1,\ldots,c_x(\bar{x}_n) * x_{n-1}, c_x(\bar{x}_n))$.

Our next proposition determines the extent to which c_x , Γ_x and Δ_x are invariant with respect to the action of the various groups that we consider.

PROPOSITION 3.3. The objects c_x , Γ_x , Δ_x depend on the group actions as follows.

(i) Γ_x depends only on the x-orbit of $(C(\Delta)^n \times J_n) \rtimes B_n$.

(ii) c_x depends only on the x-orbit of $C(\Delta)^n \rtimes B_n$.

(iii) The conjugacy class of c_x in Γ_x depends only on the x-orbit of $(C(\Delta)^n \times J_n) \rtimes B_n$.

(iv) Δ_x depends only on the x-orbit of $J_n \rtimes B_n$. It is an automorphic subset of Δ , and Δ_x^n contains the x-orbit of $J_n \rtimes B_n$.

(v) $\Delta'_x := C(\Delta)\Delta_x$ depends only on the x-orbit of $(C(\Delta)^n \times J_n) \rtimes B_n$, and it is an automorphic subset of Δ such that Δ'^n_x contains this orbit.

The proofs are easy. Of course we may replace $C(\Delta)$ by suitable subgroups and get similar statements. In the linear case $\Delta = \Delta_\varepsilon(L)$ we may work with $C_\varepsilon(L)$ instead of $C(\Delta)$ or with a suitable subgroup such as $\{\pm 1\}$. In fact in the linear case we define $\Delta'_x := \{\pm 1\}\Delta_x$, since this is the invariant which is meaningful in applications. Frequently one has $\Delta'_x = \Delta_x$.

In the linear case $\Delta = \Delta_\varepsilon(L)$ I want to introduce yet another invariant.

Let me begin with a trivial remark about the linear algebra of bilinear forms. Let L be a free R-module of rank n as before, and let $L^{\#} = \mathrm{Hom}(L,R)$ be its dual R-module. Then there is a canonical bijective correspondance between regular bilinear forms $\ell : L \times L \to R$ and isomorphisms $v : L^{\#} \to L$, which is defined by

$$\ell(a,b) = v^{-1}(b)(a) \quad .$$

If $x = (x_1,\dots,x_n)$ is a basis of L and if $\hat{x} = (\hat{x}_1,\dots,\hat{x}_n)$ is the dual basis of $L^{\#}$, we may specify v and hence ℓ by defining the value $v(\hat{x}) = (v(\hat{x}_1),\dots,v(\hat{x}_n)) \in L^n$.

<u>DEFINITION.</u> Let L be an R-lattice of rank n which is either alternating or symmetric and even. Let $\Delta_\varepsilon(L)^n_*$ be the following set of bases of L :

$$\Delta_\varepsilon(L)^n_* := \{(x_1,\dots,x_n) \in \Delta_\varepsilon(L)^n \mid Rx_1 + \dots + Rx_n = L\} \quad .$$

For any $x \in \Delta_\varepsilon(L)^n_*$ the isomorphisms $v^{\pm}_x : L^{\#} \to L$ and the bilinear forms $\ell^{\pm}_x : L \times L \to R$ are defined as follows:

$$\boxed{\begin{array}{lll} v^+_x(\hat{x}) = \rho_n\omega^+_n(x) & & v^-_x(\hat{x}) = \rho_n\omega^-_n(x) \\ \ell^+_x(a,b) = (v^+_x)^{-1}(b)(a) & \text{and} & \ell^-_x(a,b) = (v^-_x)^{-1}(b)(a) \end{array}}$$

We call $v^{\pm}_x$ the variation operators and $\ell^{\pm}_x$ the Seifert-forms associated to x .

<u>PROPOSITION 3.4.</u> Let L be a lattice of rank n as above and $x \in \Delta_\varepsilon(L)^n_*$. Let $v^{\pm}_x$ and $\ell^{\pm}_x$ the variation operators and Seifert forms and c_x the pseudo Coxeter element associated to x . Put $\bar{c}_x = c_x \cdot \iota_\Delta$, so that $\bar{c}_x = -c_x$ in the symmetric and $\bar{c}_x = c_x$ in the antisymmetric case. Then the following statements hold.

(i) $v^{\pm}_x$ and $\ell^{\pm}_x$ depend only on the x-orbit of $\{\pm 1\}^n \rtimes B_n$.

(ii) $\ell^+_x(a,b) = \ell^-_x(b,a)$.

(iii) $\bar{c}_x = v^+_x \cdot (v^-_x)^{-1}$.

(iv) $\varepsilon\langle a,b\rangle = \ell^+_x(a,b) + \ell^-_x(a,b)$ in the symmetric case.
$\varepsilon\langle a,b\rangle = -\ell^+_x(a,b) + \ell^-_x(a,b)$ in the antisymmetric case.

(v)
$$\ell^+_x(x_i,x_j) = \begin{cases} \varepsilon\langle x_j,x_i\rangle & \text{if } i < j \\ 1 & \text{if } i = j \\ 0 & \text{if } i > j \end{cases} \quad .$$

(vi) $\ell^{\pm}_x(a,b) = \ell^{\pm}_{\kappa_i(x)}(s_{x_i,\varepsilon}(a), s_{x_i,\varepsilon}(b))$.

The proof is not difficult if one uses our proposition 3.2 and exercise 3 in §6 of chapter V of Bourbaki [19]. We may interpret proposition 3.4 as follows. The matrix describing $\bar{c}_x$ with respect to the basis x is the product of the upper and lower triangular matrices describing v_x^+ and $(v_x^-)^{-1}$ with respect to x and $\hat{x}$. But the decomposition of a matrix as a product of an upper triangular matrix and a lower triangular matrix, both with diagonal elements equal to 1, is unique, if it exists. Therefore c_x, $v_x^{\pm}$, $\ell_x^{\pm}$ and the given bilinear form on L are all mutually equivalent invariants of the x-orbit of $\{\pm 1\}^n \rtimes B_n$ in the following sense. If we know the value of one of these invariants for one element x_o of the orbit, we know the values of the other invariants for x_o, and hence we know all the other invariants.

Proposition 3.4 motivates the definition of an operation for lattices which I shall call suspension for lack of a better name. We consider lattices L with a bilinear form $b:L \times L \to R$ which is either even and symmetric or alternating. It is convenient to distinguish both cases by an index $\eta = \pm 1$. We put $\eta(b) = +1$ in the symmetric case and $\eta(b) = -1$ in the alternating case. The suspension of a symmetric lattice will be antisymmetric and vice versa.

<u>DEFINITION.</u> Let (L,b,ε) as above, $\eta = \eta(b)$ and $x \in \Delta_\varepsilon(L)_*^n$. Then the suspension $\Sigma_x(L,b,\varepsilon) = (L',b',\varepsilon')$ is defined as follows:

$$L' = L \quad \text{and} \quad b' = \varepsilon'\ell_x^+ + \varepsilon\ell_x^- \qquad \text{and} \qquad \varepsilon' = -\varepsilon\eta \qquad \text{and} \qquad \eta' = -\eta ,$$

where $\ell_x^{\pm}$ are the Seifert forms associated to x.

<u>PROPOSITION 3.5.</u> The suspension operation for lattices has the following properties.

(i) $\Sigma_x(L,b,\varepsilon) = (L',b',\varepsilon')$ depends only on the x-orbit of $\{\pm 1\}^n \rtimes B_n$.

(ii) $x \in \Delta_{\varepsilon'}(L')_*^n$.

(iii) The $\{\pm 1\}^n \rtimes B_n$ -orbits of x in $\Delta_\varepsilon(L)_*^n$ and in $\Delta_{\varepsilon'}(L')_*^n$ are identical, and the operations of $\{\pm 1\}^n \rtimes B_n$ on these two orbits are the same.

(iv) The Seifert forms $\ell_x^{\pm}$ with respect to $\Delta_\varepsilon(L)$ and $\Delta_{\varepsilon'}(L')$ are identical.

(v) Σ_x has period 4. If we begin with (L,b,ε) with $(\varepsilon,\eta) = (1,1)$ and if b', b'', b''' are the bilinear forms of $\Sigma_x^k(L,b,\varepsilon)$, $k = 1,2,3$, we have

$$b = +\ell_x^+ + \ell_x^-$$
$$b' = -\ell_x^+ + \ell_x^-$$

$$b'' = -\ell_x^+ - \ell_x^-$$
$$b''' = +\ell_x^+ - \ell_x^-$$

The proof follows easily from proposition 3.4. The application of the suspension operation in singularity theory will be explained in §4.

Let R be an integral domain, and $M_n(R)$ the R-algebra of $n \times n$-matrices with coefficients in R. For any R-lattice (L,b) as above we have a natural map

$$\delta : L^n \to M_n(R)$$
$$\delta(x) = (b(x_i, x_j))$$

If we restrict δ to $\Delta_\varepsilon(L)^n$, the image is contained in the following set of matrices:

$$M_n(R)_{\varepsilon\eta} = \{(a_{ij}) \in M_n(R) \mid a_{ij} = \eta a_{ji} \;,\; a_{ii} = \varepsilon(1+\eta)\} \;.$$

For $n = \operatorname{rank} L$, the map δ induces an injective map

$$\delta : \Delta_\varepsilon(L)^n_* \,/\, \operatorname{Aut}(L) \to M_n(R)_{\varepsilon\eta} \;.$$

If we let (L,b) vary, the images of δ cover $M_n(R)_{\varepsilon\eta}$, and the action of $\{\pm 1\}^n \rtimes B_n$ on all $\Delta_\varepsilon(L)^n_*$ induces an action of this group on $M_n(R)_{\varepsilon\eta}$. It is easy to write this down explicitly for the generators σ_i of B_n. We have $\sigma_i(a_{jk}) = (a'_{jk})$, where a'_{jk} is the value of the linear or quadratic polynomial in the coefficients $a_{11},\dots,a_{nn}$ given in the following table.

	$k = i$	$k = i+1$	$k \neq i, i+1$
$j = i$	$\varepsilon(1+\eta)$	$-a_{i,i+1}$	$a_{i+1,k} - \varepsilon a_{i+1,i} a_{ik}$
$j = i+1$	$-a_{i+1,i}$	$\varepsilon(1+\eta)$	a_{ik}
$j \neq i, i+1$	$a_{j,i+1} - \varepsilon a_{i+1,i} a_{ji}$	a_{ji}	a_{jk}

Note that some of the polynomials are definitely quadratic and not linear! For arbitrary $\sigma \in B_n$ the coefficients of $\sigma(a_{jk})$ will be polynomials in the a_{jk} of a degree which may be arbitrarily large. We may also describe these polynomials as follows.

Let $R_n = R[t_{ij}]$ be a polynomial ring over R with generators t_{ij} algebraically independent over R, where $1 \le i < j \le n$. Let $e = (e_1,\dots,e_n)$ be the standard basis of the free R_n-module R_n^n. Define a bilinear form $b_{\varepsilon\eta}$ on R_n^n as follows:

$$b_{\varepsilon\eta}(e_i,e_j) = \begin{cases} t_{ij} & \text{if } i < j \\ \eta t_{ji} & \text{if } i > j \\ \varepsilon(1+\eta) & \text{if } i = j \end{cases} .$$

We denote the matrix $(b_{\varepsilon\eta}(e_i,e_j))$ by $T_{\varepsilon\eta}$ or T. The R_n-module R_n^n with the linear form $b_{\varepsilon\eta}$ is symmetric if $\eta = 1$ and antisymmetric if $\eta = -1$. So we can apply our theory of automorphic sets $\Delta_\varepsilon(L)$ to $L = (R_n^n, b_{\varepsilon\eta})$. We have an automorphic set $\Delta^n_{\varepsilon,\eta}(R) := \Delta_\varepsilon(R_n^n, b_{\varepsilon\eta})$, we have an element $e \in \Delta^n_{\varepsilon,\eta}(R)^n_*$, and we have its monodromy group Γ_e generated by $s_i = s_{e_i,\varepsilon}$. The braid group B_n operaties on the orbit of e in $\Delta_{\varepsilon\eta}(R)^n_*$, and to any $\sigma \in B_n$ we associate canonically the matrix

$$\Theta_\sigma := \delta(\sigma(e)) = \sigma(T) \in M_n(R_n)_{\varepsilon\eta} \quad ;$$

where our notation suppresses the dependence on (ε,η). This matrix Θ_σ with polynomial coefficients is the matrix describing the operation of B_n on $M_n(R)_{\varepsilon\eta}$. For any matrix $A \in M_n(R)_{\varepsilon\eta}$ we have $\sigma(A) = \Theta_\sigma(A)$, where $\Theta_\sigma(A)$ is the matrix in $M_n(R)$ obtained from Θ_σ by substituting the coefficients a_{ij} of A for the generators t_{ij}. The following theorem was found by Bertolt Krüger, who kindly explained to me all the essential ideas of the proof that follows.

<u>THEOREM 3.6.</u> Let R be an integral domain with infinitely many elements, $n \geq 2$ an integer and $\varepsilon,\eta = \pm 1$. The braid group B_n operates canonically on the set of matrices $M_n(R)_{\varepsilon\eta}$ by $\sigma(A) = \Theta_\sigma(A)$. The element $\zeta_n = \omega_n^{+2}$ operates trivially, and the induced action of $B_n / \langle\zeta_n\rangle$ is effective.

The proof will follow from the following proposition.

<u>PROPOSITION 3.7.</u> The operation of B_n on the orbits $B_n e \subset \Delta^n_{\varepsilon\eta}(R)^n$ and $B_n T \subset M_n(R_n)_{\varepsilon\eta}$ has the following invariants and properties of invariance.

(i) For $\eta = -1$ the monodromy group Γ_e is the free group of rank n freely generated by $s_1,\dots,s_n$.

(ii) For $\eta = +1$ the monodromy group Γ_e is the universal Coxeter group generated by $s_1,\dots,s_n$, i.e. $\Gamma_e = \langle s_1,\dots,s_n \mid s_1^2 = \dots = s_n^2 = 1\rangle$.

(iii) $\Theta_\sigma = T \Leftrightarrow \sigma \in \langle\zeta_n\rangle$.

PROOF: (i) We describe the elements of Γ_e by their matrices with respect to the basis e. The matrix of $s_i - 1$ has only one non zero line, namely the i-th one, which is $(-\varepsilon t_{1i},\dots,-\varepsilon t_{i-1,i}, 0, \varepsilon t_{i,i+1},\dots,\varepsilon t_{in})$. Multiplying this line with any integer d we get the only non zero line of $s_i^d - 1$. Using this as the beginning of an induction over the number k of factors one easily proves the following statement.

(F) For $s_{i_1}^{d_1} \dots s_{i_k}^{d_k}$ with $d_j \neq 0$ and $i_j \neq i_{j+1}$ the highest order terms of the coefficients a_{rs} of this matrix are monomials in the t_{ij} of order k with coefficient $\pm d_1 \dots d_k$. They occur exactly in the coefficients $a_{i_1,s}$ with $s \neq i_k$.

Obviously the statement (F) implies the freeness of $\Gamma_e = \langle s_1,\dots,s_n\rangle$, since $s_{i_1}^{d_1} \dots s_{i_k}^{d_k} \neq 1$ for $k > 0$.

(ii) The proof for the symmetric case is analogous to (i), since there is an analogue of (F) with all exponents $d_1 = \dots = d_k = 1$.

(iii) The statement to be proved is the following one. Let $\sigma \in B_n$ and $\sigma(e) = (e_1',\dots,e_n')$. Then the following equivalence holds:

$$\langle e_i',e_j'\rangle = t_{ij} \quad \text{for} \quad i < j \Leftrightarrow \sigma \in \langle \zeta_n \rangle .$$

The direction "$\Leftarrow$" follows from 3.2.(iv). We prove the other direction. It is enough to give a proof for the antisymmetric case, because the proof for the symmetric case follows from it by our results on suspension stated in proposition 3.5.

Let $\alpha \in \mathrm{Aut}(R_n^n, b_{\varepsilon\eta})$ be the unique automorphism such that $\alpha(e) = \sigma(e)$. We shall prove $\alpha \in \Gamma_e$. Assume that it is proved. This implies $\sigma(s_1,\dots,s_n) = i_\alpha(s_1,\dots,s_n)$, where $\sigma(s_1,\dots,s_n)$ is the result of the canonical operation of $\sigma \in B_n$ on Γ_e^n and i_α is the inner automorphism of Γ_e corresponding to $\alpha \in \Gamma_e$. But Γ_e is a free group with free system of generators $(s_1,\dots,s_n)$. Therefore $\sigma(s_1,\dots,s_n) = i_\alpha(s_1,\dots,s_n)$ implies $\sigma \in B(\Gamma_e) \cap J(\Gamma_e) = C(\Gamma_e) = \langle \zeta_n \rangle$ by the classical results of §1.

Thus it remains to prove $\alpha \in \Gamma_e$. Obviously there are $g_1,\dots,g_n \in \Gamma_e$ and a permutation $\pi \in S_n$ such that $e_i' = g_i e_{\pi(i)}$. Since $g_i e_{\pi(i)} \equiv e_{\pi(i)} \bmod (t_{jk})$, the hypothesis $\langle e_i',e_j'\rangle = t_{ij}$ implies $\pi = 1$. Put $h_i = g_1^{-1} g_i$ and $e_i'' = h_i(e_i)$. Then $\langle e_i'',e_j''\rangle = t_{ij}$ and in particular $\langle e_1, h_j e_j\rangle = t_{1j}$. From this and (F) applied to h_j one easily deduces $h_j = s_1^{c_j} s_j^{d_j}$. Hence $(e_1'',\dots,e_n'') = (e_1, s_1^{c_2} e_2,\dots,s_1^{c_n} e_n)$. By the same kind of reasoning we then deduce from $\langle e_i'',e_j''\rangle = t_{ij}$ that $c_2 = \dots = c_n =: c$. Therefore $e_i'' = s_1^c e_i'$ and $e_i' = g_1 s_1^c e_i$ for all i. Hence finally $\alpha = g_1 s_1^c \in \Gamma_e$. Q.E.D.

The proof of theorem 3.6 now follows easily from proposition 3.7. Suppose $\sigma \in B_n$ is such that $\sigma(A) = A = (a_{ij})$ for all $A \in M_n(R)_{\eta\varepsilon}$. This means $\Theta_\sigma(A) = T(A)$ for any substitution of $a_{ij} \in R$ for the t_{ij}. But this implies $\Theta_\sigma = T$ (c.f. Bourbaki [17] IV. §2, proposition 9). Thus $\sigma \in \langle \zeta_n \rangle$ by 3.7. Q.E.D.

The essential statement 3.7.(iii) may be transformed as follows. Denote the coefficient of the matrix $\Theta_\sigma = \sigma(T)$ in the j-th row and k-th column by

$\sigma(T)_{jk}$. The elements $\sigma(T)_{jk}$ with $1 \le j < k \le n$ are algebraically independent generators of the polynomial ring R_n . Therefore we get an automorphism σ of this ring when we define the values of σ for the generators t_{jk} as follows:

$$\sigma(t_{jk}) = \sigma(T)_{jk} \quad \text{for } 1 \le j < k \le n \ .$$

In this way for given (ε,η) we get an operation of B_n on the polynomial ring $R[t_{jk}]$ by ring automorphisms. For the generators σ_i the table above gives the following explicit formulas:

$$\sigma_i(t_{jk}) = \begin{cases} t_{jk} & j,k \neq i,i+1 \\ t_{ik} & j = i+1 \\ t_{ji} & j < i \text{ and } k = i+1 \\ -t_{i,i+1} & j = i \text{ and } k = i+1 \\ t_{i+1,k} - \varepsilon\eta t_{i,i+1}t_{ik} & j = i \text{ and } k > i+1 \\ t_{j,i+1} - \varepsilon\eta t_{i,i+1}t_{ji} & j < i \text{ and } k = i \ . \end{cases}$$

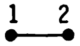
So we have just two different operations of B_n on R_n , one for $\varepsilon\eta = 1$ and one for $\varepsilon\eta = -1$. Moreover they are not essentially different, since they are conjugate by the transformation $t'_{jk} = -t_{jk}$.

<u>COROLLARY 3.8.</u> $B_n / \langle\zeta_n\rangle$ operates effectively on the polynomial ring $R[t_{ij}]$.

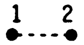
A result very similar to 3.8 was already proved in a different context by W. Magnus (c.f. [85], Corollary 4.1). Magnus has a slightly different operation of B_n on $R[t_{ij}]$. In our context it might be obtained by using the associated idempotent automorphic set $\overline{\Delta^n_{-1,1}(R)}$. Magnus does not mention a result like 3.6. On the other hand we did not know Magnus' paper when Krüger proved 3.6 and did not think of the equivalent statement 3.8 before we saw that paper. I think it is justified that I presented Krüger's proof since it is very transparent and puts the result in a new perspective.

The operation of the braid group on matrices is very difficult to understand. It has turned out that in many cases it is helpful to represent the matrices by diagrams and to interpret the generators σ_i of B_n as simple transformations of diagrams. This is done as follows. We assume that ε and η are determined by the context. Therefore a matrix $A \in M_n(R)_{\varepsilon\eta}$ is determined by its coefficients a_{ij} for $i < j$. These data are represented by a graph with vertices $1,\dots,n$ and with edges joining vertices i,j with $a_{ij} \neq 0$. Such an edge with $i < j$ is given the value εa_{ji} . For $R = \mathbb{Z}$

and small values of $|a_{ij}|$, one uses the following convention for the graphic representation of the values. The vertices $i < j$ are represented by $|a_{ij}|$ uninterrupted lines if $\varepsilon a_{ji} < 0$, and by $|a_{ij}|$ dotted lines, if $\varepsilon a_{ji} > 0$. For example if $\eta = 1$, we have the following representation of matrices by graphs:



represents $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ for $\varepsilon = 1$ and $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ for $\varepsilon = -1$.



represents $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ for $\varepsilon = 1$ and $\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ for $\varepsilon = -1$.

Note that the numbering of the vertices is essential from our point of view. It may be omitted only in those rare cases where the numbering can be changed arbitrarily by operations $\sigma \in B_n$. It follows from a theorem of P. Deligne quoted below (3.16) that this is the case for the classical diagrams representing the irreducible homogeneous root systems A_k , D_k , E_6 , E_7 , E_8 . These diagrams are shown in figure 3.

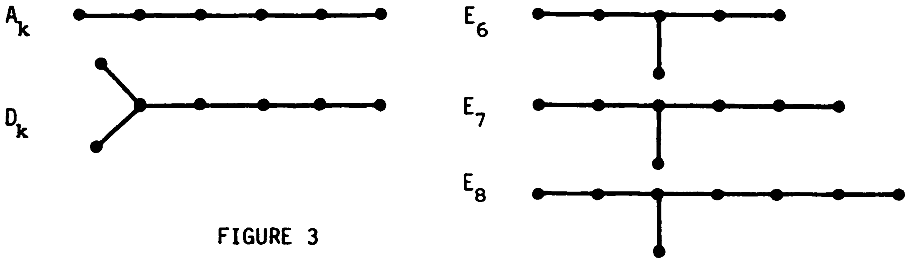


FIGURE 3

These diagrams are usually called Dynkin diagrams. However according to A.J. Coleman they first appeared in an appendix to mimeographed notes of Brauer on lectures of H. Weyl given in 1934-35 at Princeton. This appendix was written by H.S.M. Coxeter. After that they appeared in 1941 in a paper of Witt and finally also in a paper of Dynkin. Therefore I shall call the diagrams representing the matrices $\delta(x)$ the Coxeter diagrams of x .

It is easy to interpret the formulas for the operation of σ_i on matrices as transformation rules for diagrams. For instance, if $\varepsilon = \eta = 1$ and $a_{i,i+1} = 0$ or ± 1 , the subdiagram with vertices i and $i+1$ is transformed as shown in figure 4.

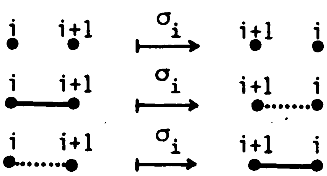
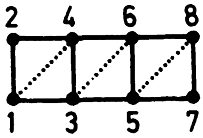


FIGURE 4

After the transformation the edges connecting $i+1$ with any $k \neq i,i+1$ are the same as those connecting i with k before the transformation. The values of edges connecting i with k after the transformation are given by $a'_{ik} = a_{i+1,k} - a_{i+1,i}a_{ik}$, and the edges between j and k for $j,k \neq i,i+1$ are unaffected.

Here are two examples for the transformation of diagrams. We start with the diagram A shown in figure 5.



We define two elements $\sigma,\sigma' \in B_8$ as follows. $\sigma' = \sigma_1^{-2}\sigma_7^2$ and

$$\sigma = \sigma_1^6\sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_4\sigma_3^{-1}\sigma_5\sigma_6\sigma_7\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5^{-1}\sigma_2^{-1}\sigma_1$$

The transformed diagrams $\sigma(A)$ and $\sigma'(A)$ are shown in figure 6.

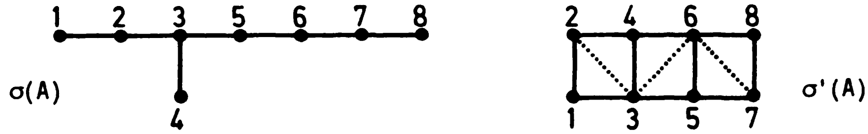


FIGURE 6

The transformation σ was found by Gabrielov [40] and I got it from Wolfgang Ebeling who is a master in the art of transforming diagrams. Both diagrams have a geometric meaning related to the famous icosahedral singularity of the surface with equation $x^2+y^3+z^5 = 0$. This will be explained in §4. At this moment I shall only show two pictures of real plane curves (figure 7). They are obtained by deformation of the plane curve with equation $y^3+z^5 = 0$. The picture on the right hand side is already to be found in a 1893 paper of Charlotte Angas Scott [102].

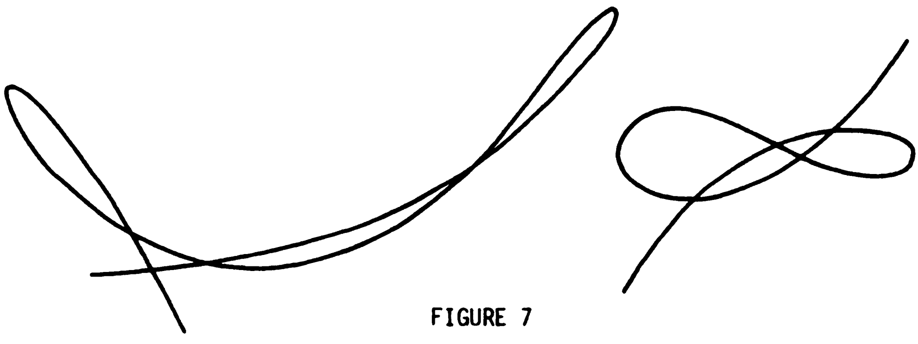


FIGURE 7

The rule by which one gets the diagrams from the curves was given by N. A'Campo [3]. Roughly speaking it is this: The points of the diagram correspond to nodes and to regions bounded by the curve. Dotted lines describe adjacencies of regions along arcs, and uninterrupted lines come from adjacencies of nodes to regions.

In the particular case of this example $\sigma(A)$ is the Coxeter diagram of type E_8 . The corresponding root lattice is positive definite, the root system Δ and the B_n-orbit of A are finite. Its cardinality was calculated by Deligne [32]:

$$\mathrm{Card}(B_8A) = 2^8 3^4 5^6 = 324\ 000\ 000 \ .$$

Within this orbit, the 8! diagrams of type E_8 (with any numbering of the vertices) are distinguished, since they are the only ones with no cycles and no dotted lines. So in this case the B_n-orbit has a class of distinguished normal forms coming from the classical Coxeter diagram. However from our point of view other diagrams in the orbit also have geometric meaning, as illustrated above.

Apart from root systems, there are a few other classes of diagrams which may be considered as natural distinguished normal forms in their B_n-orbit. These diagrams come from certain singularities of hypersurfaces. The normal forms of these diagrams T_{pqr} and S_{pqr} were obtained by Gabrielov [41] and Ebeling. Figure 8 shows S_{pqr}

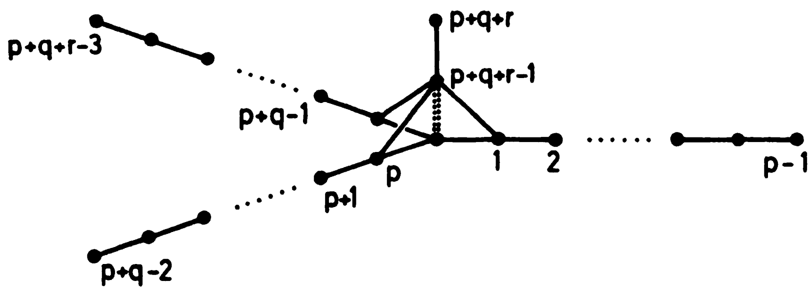


FIGURE 8

The diagram T_{pqr} is obtained from S_{pqr} by omitting the last vertex. In singularity, only the T_{pqr} with $1/p + 1/q + 1/r \leq 1$ are relevant, and of the S_{pqr} there occur only 14 triples $p \leq q \leq r$. These are the diagrams of the simplest singularities. For higher singularities there is no hope of defining a uniquely determined geometrically meaningful normal form for their diagrams. The best one can hope for is to single out a finite set of normal forms in their braid group orbit of diagrams. However at present there is no theory which does that. The state of the art is that it is a craft. The one who is best at it is Wolfgang Ebeling. He has worked out useful "normal forms" for many classes of singularities of hypersurfaces and complete intersections [36], [38]. However in order to use these diagrams in an effective way one needs answers to the problems about the braid groups which I shall state below.

Before we go into those problems we should see some examples. Let us now look at the two simplest non trivial examples for the action of $J_n \rtimes B_n$ on $\Delta_\varepsilon(L)^n_*$. The first example is the root system Δ of type A_2 . The set Δ is a configuration of 6 vectors in the euclidean plane. It is shown in figure 9 together with its Coxeter diagram.

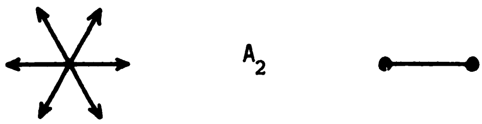


FIGURE 9

The root lattice L is the lattice generated by Δ . The inner automorphism group $I(\Delta)$ equals the Weylgroup $W(\Delta)$ and is isomorphic to S_3 . The non trivial elements are three reflections and two rotations by 120° and 240°. The automorphism group is $\mathrm{Aut}(\Delta) = A(\Delta) = I(\Delta) \times C(\Delta)$, and $C(\Delta) = \{\pm 1\}$. The set Δ^2_* of ordered bases of L consisting of root vectors has 24 elements. We may visualize the operation of $J_2 \rtimes B_2$ on Δ^2_* by means of the graph with coloured edges shown in figure 10.

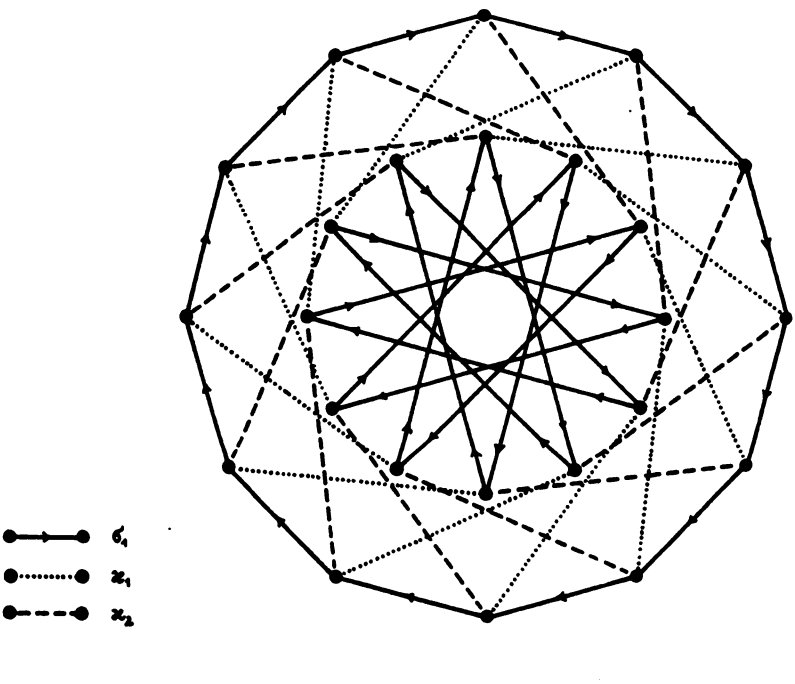


FIGURE 10

The graph is connected, so we see that $J_2 \rtimes B_2$ operates transitively on Δ^2_* . Therefore we have only one monodromy group Γ , namely the Weylgroup. The graph has two σ_1-cycles, both of length 12. These are the two orbits of B_2, and they correspond bijectively to the two Coxeter elements in the Weylgroup, which are the rotations by 120° and 240°. There are 4 orbits of J_2. Each of them is a cycle of length 6 with alternating edges of type $\varkappa_1, \varkappa_2$. Of course these four J_2-orbits may also be seen as the orbits of the Weylgroup operating on Δ^2_*. There are two Coxeter diagrams corresponding to the two possible positions of an unordered pair of roots forming a basis. To each of the two diagrams there belong to J_2-orbits, which are interchanged by $\zeta_2 = \sigma_1^2$. Finally $B_2 / \langle\zeta_2\rangle$ is the cyclic group of order 2 permuting the two Coxeter diagrams.

The second example is the antisymmetric A_2-case obtained by suspending the root lattice of type A_2 and with $\varepsilon = \eta = 1$ with respect to a basis $e = (e_1, e_2)$ with standard Coxeter diagram. The result of Σ^3 is the hyperbolic unimodular plane $\mathbb{Z}^2$ with standard basis $e = (e_1, e_2)$. The matrix of the bilinear form $(\langle e_i, e_j\rangle)$ equals

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

The transvections $s_i = s_{e_i}$ are the following matrices:

$$s_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} .$$

Therefore the monodromy group is $\Gamma_e = \langle s_1, s_2\rangle = SL(2,\mathbb{Z})$. Therefore $\sigma_1(e) = (e_2+e_1, e_1)$ implies that $(\Gamma_e \times B_2)e = \Delta^2_*$, and therefore $J_2 \rtimes B_2$ operates transitively on Δ^2_*. We identify Δ^2_* with $GL(2,\mathbb{Z})$. Then the operation of $J_2 \rtimes B_2$ is described as follows. For $x \in GL(2,\mathbb{Z})$ with determinant $|x|$, we have

$$\sigma_1(x) = x \begin{bmatrix} |x| & 1 \\ 1 & 0 \end{bmatrix} \qquad \varkappa_1(x) = x \begin{bmatrix} 1 & |x| \\ 0 & 1 \end{bmatrix} \qquad \varkappa_2(x) = x \begin{bmatrix} 1 & 0 \\ -|x| & 1 \end{bmatrix}$$

The pseudo Coxeter element for e is the matrix

$$c_e = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} ,$$

which is cyclic of order 6 with centralizer $\langle c_e\rangle \subset SL(2,\mathbb{Z})$. The set of all pseudo Coxeter elements is in bijective correspondence with $SL(2,\mathbb{Z}) / \langle c_e\rangle$. As in the first example we may describe the action of $J_2 \rtimes B_2$ on $\Delta^2_* = GL(2,\mathbb{Z})$ by means of a graph with coloured and directed edges. Since the graph is infinite, we shall build it by infinite repetition of a finite building block which is described by figure 11.

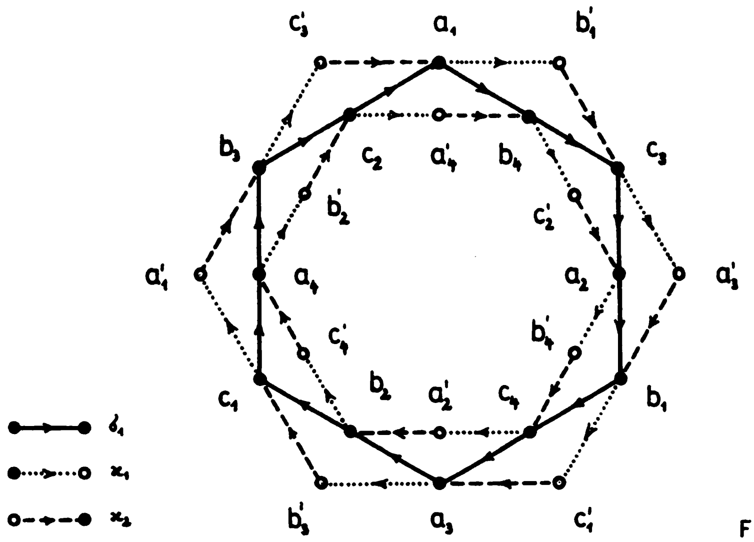


FIGURE 11

The diagram $\mathcal{D}$ shown in figure 11 has 24 vertices. These form three subsets, the "sides" of $\mathcal{D}$, namely $\partial_a(\mathcal{D}) = \{a_i, a_i'\}$ and $\partial_b(\mathcal{D}) = \{b_i, b_i'\}$ and $\partial_c(\mathcal{D}) = \{c_i, c_i'\}$. Along these three sides we attach three other copies $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ of the building block by identifying $\partial_a(\mathcal{D})$ with $\partial_a(\mathcal{D}_1)$ and $\partial_b(\mathcal{D})$ with $\partial_b(\mathcal{D}_2)$ and $\partial_c(\mathcal{D})$ with $\partial_c(\mathcal{D}_3)$. The identification is done in such a way that points a_i, b_i, c_i of $\mathcal{D}$ are identified with the corresponding points a_i', b_i', c_i' in the sides of $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and conversely points a_i', b_i', c_i' of $\mathcal{D}$ are identified with points a_i, b_i, c_i of $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$. The process is repeated with the free sides of $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and so on ad infinitum. The building blocks are connected according to the scheme of figure 12.

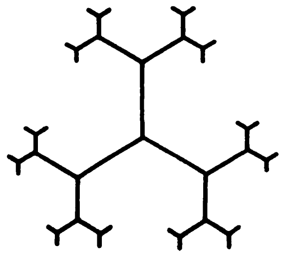


FIGURE 12

Note that each building block contains exactly one B_2-orbit, namely a σ_1-cycle of length 12. Again there is a bijective correspondence between B_2-orbits and pseudo Coxeter elements.

Let me now state some problems. All of them are related to the operation of the braid groups B_n and of the groups $J_n \rtimes B_n$ on the cartesian products Δ^n of automorphic sets Δ . For all of them there are many possible variations, such as working with a group like $(C^n \times J_n) \rtimes B_n$, where C is some subgroup of $C(\Delta)$, or such as working with suitable subsets $\Delta^n_* \subset \Delta^n$, or passing to quotients of Δ^n_* with respect to suitable group actions or some other equivalence relations. For all problems at least some work has been done, but more remains to be done. All problems make sense only if we restrict to suitable classes of automorphic sets Δ or suitable subsets $\Delta^n_* \subset \Delta^n$. It is part of the problems to find suitable restrictions so that they make sense. Now let me name the problems.

The representation problem.
The problem of invariants.
The inverse problem of invariants.
The equivalence problem.
The embedding problem.

<u>THE REPRESENTATION PROBLEM.</u> For any given automorphic set Δ and any natural number n , the braid group B_n and the group $J_n \rtimes B_n$ operate on Δ^n . We may look at this action as a representation of B_n , and the problem is to determine this representation explicitly. We may simplify the problem by considering only the actions on certain orbits in quotients of suitable subsets Δ^n_* . Even then the problem may be quite hard even for some specific simple Δ . In this way one gets many interesting representations of the braid groups. Some are old ones, and some are new.

<u>THE PROBLEM OF INVARIANTS.</u> We have defined invariants of the action of B_n , $J_n \rtimes B_n$ and other groups on Δ^n , namely the monodromy group $\Gamma_x \subset I(\Delta)$, the pseudo Coxeter element $c_x \in \Gamma_x$ and the automorphic subset $\Delta_x \subset \Delta$. The problem is to determine these invariants. To begin with, this may mean to calculate c_x, Γ_x or Δ_x explicitly for some specific $x \in \Delta^n$ and some specific Δ . This may already be a non trivial problem of arithmetical nature. More ambitiously we may ask for the determination of these invariants for classes of $x \in \Delta^n$ and for specific classes of Δ . For instance we may ask whether in the case of an automorphic set $\Delta_\varepsilon(L)$ there are only finitely many conjugacy classes of pseudo Coxeter elements $c_x \in W_\varepsilon(L)$ associated to $x \in \Delta_\varepsilon(L)^n_*$, where $n = \operatorname{rank} L$. Other refinements are to ask for specific properties of the invariants, if $x \in \Delta^n$ comes from some geometric situation. For example, for $x \in \Delta_\varepsilon(L)^n$: When is c_x quasi unipotent or even of finite order? When is $\Gamma_x \subset \operatorname{Aut}(L)$ arithmetic? When do we have $\Delta_x = \Delta_\varepsilon(L)$?

<u>THE INVERSE PROBLEM OF INVARIANTS.</u> The inverse problem is this: To which

extent are the orbits of B_n, $J_n \rtimes B_n$ and other groups determined by their invariants? It turns out that in this respect the invariants c_x, Γ_x and Δ_x behave quite differently. For instance work of W. Ebeling quoted below (c.f. 3.21) shows that for many lattices L and $x \in \Delta_\varepsilon(L)^n_*$ coming from applications in singularity theory, Γ_x and Δ_x depend only on L and do not determine the x-orbit of $(\{\pm 1\}^n \times J_n) \rtimes B_n$. On the other hand, in arithmetic situations c_x turns out to be a relatively strong invariant, so that the following problem makes sense. Let χ be the map

$$\chi : \Delta^n \to I(\Delta)$$

defined by $\chi(x) = c_x$. We shall call χ the characteristic map. B_n operates on the fibres of χ. Problem: Determine the B_n-orbits in $\chi^{-1}(c_x) \cap \Delta^n_x$ for specific $c_x = c \in I(\Delta)$ and special classes of automorphic sets Δ. In particular, answer these questions: Are there only finitely many orbits in $\chi^{-1}(c) \cap \Delta^n_x$? Is B_n even transitive on $\chi^{-1}(c)$? If one has positive answers to these questions and if there are also only finitely many pseudo Coxeter elements of specified type, this would give an answer to the finiteness problem: Is $\Delta^n_* / J_n \rtimes B_n$ finite?

<u>THE EQUIVALENCE PROBLEM.</u> This is the problem of deciding whether two given elements $x,y \in \Delta^n$ are in the same orbit of B_n or $J_n \rtimes B_n$ or some other group of this type. Of course one may try to use the invariants, and in some cases this is the only method we know. But it would be very desirable to have some kind of algorithm deciding the equivalence problem for suitable classes of automorphic sets Δ and subsets $\Delta^n_* \subset \Delta^n$ or quotients of them, or for the operation of B_n on $M_n(\mathbb{Z})_{\varepsilon\eta}$. A related problem is the problem of normal forms. This is the problem of defining a suitable set of normal forms in each orbit - hopefully finite - and of giving a reduction procedure reducing each element of Δ^n_* to one of its normal forms. For instance it would be very desirable to have such a "braid-reduction-theory" for $M_n(\mathbb{Z})_{\varepsilon\eta}$. I expect that such a theory would have to be much more subtle than the classical reduction theory of quadratic forms. For $n = 3$ the theory exists by work of B. Krüger, on which I shall report below.

<u>THE EMBEDDING PROBLEM.</u> The problems which we have stated so far refer to the action of B_n or $J_n \rtimes B_n$ on Δ^n for a fixed natural number n. However there is a natural problem which is important for applications to singularity theory in which we have to compare the orbits of $x \in \Delta^m$ and $y \in \Delta^n$ for m different from n. Note that there is a canonical inclusion

$$J_m \rtimes B_m \subset J_n \rtimes B_n \qquad \text{for } m \leq n ,$$

identifying $\varkappa_1,\dots,\varkappa_m \in J_m$ with $\varkappa_1,\dots,\varkappa_m \in J_n$ and $\sigma_1,\dots,\sigma_{m-1} \in B_m$ with $\sigma_1,\dots,\sigma_{m-1} \in B_n$. This inclusion is compatible with the operations on Δ^m and Δ^n and with the projection $\Delta^n \to \Delta^m$ mapping $(x_1,\dots,x_n)$ to $(x_1,\dots,x_m)$. This means that the following diagram is commutative:

$$\begin{array}{ccc} J_n \rtimes B_n \times \Delta^n & \to & \Delta^n \\ \cup & & \| \\ J_m \rtimes B_m \times \Delta^n & \to & \Delta^n \\ \downarrow & & \downarrow \\ J_m \rtimes B_m \times \Delta^m & \to & \Delta^m \end{array}$$

This implies that one can define a partial order relation for the disjoint union of the orbit spaces $\Delta^m / J_m \rtimes B_m$ as follows. Denote the $J_m \rtimes B_m$-orbit of $x \in \Delta^m$ by $[x]$. For $x = (x_1,\dots,x_m) \in \Delta^m$ and $x' = (x_1',\dots,x_k') \in \Delta^k$ composition gives the element $(x,x') = (x_1,\dots,x_m,x_1',\dots,x_k') \in \Delta^{m+k}$. Warning: This does not induce a composition of orbits! We define a partial order on the disjoint union $\amalg \Delta^k$ as follows: For $x \in \Delta^m$ and $y \in \Delta^n$ we have $x \le y$ iff $m \le n$ and there is an $x' \in \Delta^{n-m}$ such that $(x,x') = y$.

DEFINITION. The partial order relation $\le$ for elements $[x] \in \Delta^m / J_m \rtimes B_m$ and $[y] \in \Delta^n / J_n \rtimes B_n$ of the disjoint union $\amalg \Delta^k / J_k \rtimes B_k$ is defined as follows.

$$[x] \le [y] \Leftrightarrow \exists\, z \in [y] \quad x \le z .$$

Now we can state the embedding problem: Given $[x]$ and $[y]$, decide whether $[x] \le [y]$. Obviously the equivalence problem is the special case $m = n$ of the embedding problem. But the general embedding problem for $m \neq n$ appears to be much harder than the equivalence problem. Another version of the problem is the embedding problem for Coxeter diagrams. Define a partial order for matrices in the disjoint union $\amalg M_n(R)$ as follows. If $A \in M_m(R)$ and $B \in M_n(R)$ then $A \le B$ iff $m \le n$ and A is the submatrix of B consisting of the first m rows and columns. Also denote the B_m-orbit of $A \in M_m(R)_{\varepsilon\eta}$ by $[A]$.

DEFINITION. The partial order relation for elements $[A],[B]$ of the disjoint union of orbitspaces of matrices $\amalg M_k(R)_{\varepsilon\eta} / B_k$ is defined as follows:

$$[A] \le [B] \Leftrightarrow \exists\, C \in [B] \quad A \le C .$$

Obviously the diagram maps $\delta: \Delta_\varepsilon(L)^k \to M_k(R)_{\varepsilon\eta}$ have the following property: $[x] \le [y] \Rightarrow [\delta(x)] \le [\delta(y)]$. The embedding problem for diagrams is to decide for given matrices A, B whether $[A] \le [B]$. At present there is no theory for solving this problem. Even for specific simple matrices and

$n = m+1$ the decision can be too tough. This embedding problem is much harder than the embedding problem for quadratic forms, at least in the indefinite case, where there is a good theory (c.f. Nikulin [90]).

I shall now report on some of the work that has been done in connection with the problems which I have stated. Since the problems are all related to each other the order of subjects will not be determined by the list of problems, but rather by the nature of the objects which have been investigated.

One of the most basic objects from an algebraic point of view is a free group. Let F be a free group of finite rank $n > 1$. Consider F as an automorphic set $\Delta = F$. Since F has trivial centre, F identifies with the inner automorphism group $I(\Delta) = F$. Let $\Delta_*^n = X_F \subset \Delta^n$ be the set of well ordered systems of free generators of F as in §1. The operation of B_n on Δ^n leaves Δ_*^n invariant. Its restriction to Δ_*^n identifies with the operation of $B(F)$ on X_F defined in §1. The characteristic map $\chi:\Delta_*^n \to I(\Delta)$ identifies with the characteristic map $\chi:X_F \to F$ of §1, defined by $\chi(x_1,\ldots,x_n) = x_1\ldots x_n$. The classical results of §1 can now be stated as follows.

<u>THEOREM 3.9.</u> For $\Delta = F$ a free group of rank $n > 1$ and $x \in \Delta_*^n = X_F$, the following statements hold for the canonical operation of B_n on Δ_*^n:

(i) $\Gamma_x = F$

$c_x = x_1\ldots x_n$

$\Delta_x = \{a x_i a^{-1} \mid a \in F, \; i = 1,\ldots,n\}$

(ii) $B_n x = \chi^{-1}(c_x) \cap \Delta_x^n$

(iii) The operation of B_n on the orbit $B_n x$ is simply transitive.

Statements (i), (ii) and (iii) solve the problem of invariants, the inverse problem of invariants and the representation problem for the operation of B_n on Δ_*^n.

The basic object for geometric applications of the braid groups is the disk. Let D be a closed 2-dimensional oriented disk with boundary ∂D. Moreover let $\{z_1,\ldots,z_n\} \subset \mathring{D}$ be a set of $n > 1$ different points in the interior, and let $z_o \in \partial D$ a base point. Consider the fundamental group

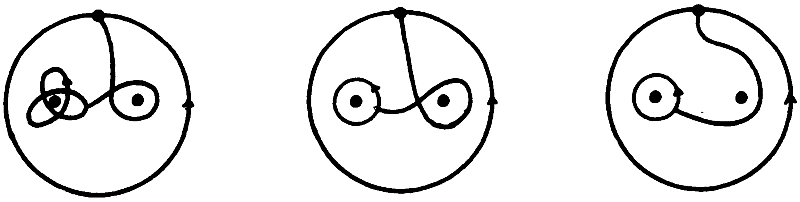
$$F = \pi_1(D - \{z_1,\ldots,z_n\},z_o) .$$

This is a free group of rank n, and we may take it as our basic automorphic set $\Delta = F$ in our algebraic description of the geometry of D. Within Δ we get additional structure coming from the topology and geometry of the disk. First of all we have the fundamental automorphic set $\Delta_f := \Delta(D,\{z_1,\ldots,z_n\},z_o)$ introduced by D. Joyce (c.f. §2, example 4). The augmentation map identifies Δ_f with an automorphic subset Δ_f of Δ. By its very definition, the pair (Δ,Δ_f) is a homotopy invariant of the pair of spaces $(D,\{z_1,\ldots,z_n\})$ with base point as an object of the category introduced in §2. But in addition to this we have a particular element $c \in F = I(\Delta)$, namely the homotopy class of the boundary circle with positive orientation. This is definitely not an invariant of the homotopy type - it is really associated to the geometry of the disk. We shall see that the triple of invariants (Δ,Δ_f,c) contains all the geometric information we need.

Let $\Delta_p \subset \Delta$ be the automorphic subset of primitive elements - an element is primitive if it can be part of a free system of generators. We define an automorphic subset $\Delta_{fp} \subset \Delta$ by $\Delta_{fp} = \Delta_f \cap \Delta_p$. Note that among other things c codes the orientation, and therefore using c we can decompose Δ_{fp} into the disjoint subsets of primitive nooses with positive and negative orientation:

$$\Delta_{fp} = \Delta^+_{fp} \cup \Delta^-_{fp} \quad .$$

Now let me describe the geometric information which we want to extract from these data. First of all we want to use a special type of positively oriented primitive nooses, namely those which can be represented by an embedding of $(\tilde{D},\{0\})$ in $(D,\{z_1,\dots,z_n\})$, where $\tilde{D}$ was defined in §2, example 4. We shall call these nooses "geometric". Figure 13 illustrates the difference between a general noose, a primitive noose and a geometric noose - all positively oriented.



general noose primitive noose geometric noose

FIGURE 13

We shall call a free system of generators $x = (x_1,\dots,x_n) \in \Delta^n_*$ geometrically distinguished, if $x_1,\dots,x_n$ can be represented by positively oriented geometric nooses meeting only at the base point $z_o \in \partial D$, and if $x_1 \dots x_n = c$. The last condition allows to determine the ordered n-tuple $(x_1,\dots,x_n)$ from the set $\{x_1,\dots,x_n\}$. Figure 14 shows an example of a geometrically distinguished system of generators for $n = 2$.

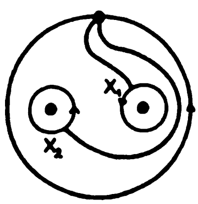


FIGURE 14

Using the Schoenflies theorem one can prove that all geometrically distinguished systems of generators are topologically equivalent. This statement can be made precise as follows.

Consider the mapping class group of $(D,\{z_1,\dots,z_n\})$. This is the group of isotopy classes of homeomorphisms of the pair $(D,\{z_1,\dots,z_n\})$ which are the identity on ∂D . I have only few letters left, so let me denote this group by $\Omega(D,\{z_1,\dots,z_n\})$ or Ω for short. Ω operates canonically on $\pi_1(D - \{z_1,\dots,z_n\},z_o)$. This operation is effective, and therefore we may identify Ω with the corresponding subgroup of $\mathrm{Aut}(F)$, thus passing from geometric topology to homotopy theory:

$$\Omega(D,\{z_1,\dots,z_n\}) \subset \mathrm{Aut}\,\pi_1(D - \{z_1,\dots,z_n\},z_o) \quad .$$

Now if we use the language of automorphic sets, we may summarize some of the classical geometric results of Artin on the braid group as follows.

<u>THEOREM 3.10.</u> Let D be a closed oriented disc, $\{z_1,\dots,z_n\} \subset \mathring{D}$ and $z_o \in \partial D$. Let $F = \pi_1(D - \{z_1,\dots,z_n\},z_o)$ and $c \in F$ the class of ∂D . Let $\Omega \subset \mathrm{Aut}\,F$ be the mapping class group. Let $\Delta = F$ as automorphic set, $\Delta_f \subset \Delta$ the fundamental automorphic set and $\Delta^+_{fp} \subset \Delta_f$ the automorphic subset of positively oriented primitive nooses. Let Δ^n_* be the set of well ordered free systems of generators of F and $\Delta^n_{**} \subset \Delta^n_*$ the subset of geometrically distinguished systems of generators. Let $\chi:\Delta^n_* \to F$ be the characteristic map. Finally, let B_n be the braid group acting on Δ^n_* . Then the following statements hold.

(i) $\Omega = \mathrm{Stab}(\Delta_f) \cap \mathrm{Stab}\{c\} \subset \mathrm{Aut}\,F$.

(ii) $\Delta^n_{**} = \Delta^{+n}_{fp} \cap \chi^{-1}(c)$

(iii) Ω and B_n both act on Δ^n_{**} . Both actions are simply transitive and
$\Omega = \mathrm{Aut}_{B_n}(\Delta^n_{**}) \qquad B_n = \mathrm{Aut}_{\Omega}(\Delta^n_{**})$

(iv) For all $x \in \Delta^n_{**}$ we have
(a) $B_n x = \Delta^n_{**} = \Omega x$ (c) $\Delta_x = \Delta^+_{fp}$
(b) $\Omega = \alpha_x(B_n)$ (d) $\Gamma_x = F$ and $c_x = c$.

So we see that the data (Δ,Δ_f,c) on one hand and the B_n-orbit $\Delta^n_{**} \subset \Delta^n$ are equivalent. They determine each other, and both determine the mapping class group $\Omega \subset \mathrm{Aut}(F)$.

Now let us come to the earliest explicit occurence of the braid groups in mathematics. They first appeared in a 1891 paper of A. Hurwitz: "Über Riemannsche Flächen mit gegebenen Verzweigungspunkten" [56] (c.f. also W. Magnus [84]). The following remarks on the subject are not intended to be historically faithful.

Let D be a closed disk $\{z_1,\dots,z_n\} \subset \mathring{D}$ and $z_o \in \partial D$. As before, let

F be the fundamental group and $\Omega \subset \mathrm{Aut}(F)$ the mapping class group. Consider m-fold ramified coverings $p:X \to D$ ramified over $\{z_1,\dots,z_n\}$. We assume $n > 1$ and $m > 2$ so that F and S_m have trivial centre. Two coverings of this kind are called equivalent if there is a homeomorphism of the covering spaces such that the following diagram commutes:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & D & \end{array}$$

We denote the set of equivalence classes by $\mathrm{Cov}_m(D,\{z_1,\dots,z_n\})$. There is a canonical action of Ω on this set. A mapping class represented by $\varphi:D \to D$ sends $p:X \to D$ to $\varphi \circ p:X \to D$. The problem is to study this action. We first transform this problem into an algebraic problem.

The equivalence classes of coverings $p:X \to D$ can be described algebraically as follows. Let $X_o = p^{-1}(z_o)$ be the fibre over the base point and $S(X_o) \cong S_m$ its symmetric group. The canonical homomorphism

$$\mu:\pi_1(D - \{z_1,\dots,z_n\},z_o) \to S(X_o)$$

is called the monodromy representation of the covering. Its image Γ_μ is called the monodromy group. Equivalent coverings define similar representations. In this way one gets a canonical bijective correspondence between $\mathrm{Cov}_m(D,\{z_1,\dots,z_n\})$ and similarity classes of permutation representations $\mu:F \to S_m$. The correspondence is compatible with the canonical actions of Ω , where $[\varphi] \in \Omega$ acts by $\mu \mapsto \mu \circ \varphi^{-1}$.

Now F is a free group of rank n . Let us choose a system of generators $x = (x_1,\dots,x_n)$. Then permutation representations $\mu:F \to S_m$ identify with n-tuples $\mu(x) = (\mu(x_1),\dots,\mu(x_n)) \in S_m^n$. Now we consider S_m as an automorphic set $\Delta = S_m$. The set of permutation representations $F \to S_m$ identifies with Δ^n . Similarity classes of such representations correspond bijectively to elements of $\Delta^n / I(\Delta) = \Delta^n / J_n$. The action of Ω may then be described as follows. Assume that x was chosen as a geometrically distinguished set of generators. By 3.10 and 1.1 there is a canonical antiisomorphism $\alpha_x:B_n \to \Omega$. We define an isomorphism $\tilde{\alpha}_x:B_n \to \Omega$ by $\tilde{\alpha}_x(\sigma) = \alpha_x(\sigma^{-1})$. Then the operation of Ω on $\mathrm{Cov}_m(D,\{z_1,\dots,z_n\})$ and on similarity classes of permutation representations identifies with the canonical operation of B_n on Δ^n / J_n . Let us summarize.

PROPOSITION 3.11. After a choice of a geometrically distinguished set of generators of $\pi_1(D - \{z_1,\dots,z_n\},z_o)$, the $\Omega(D,\{z_1,\dots,z_n\})$-set $\mathrm{Cov}_m(D,\{z_1,\dots,z_n\})$ identifies with the B_n-set Δ^n / J_n , where $\Delta = S_m$ as automorphic set.

Let us single out some types of coverings distinguished by special geometric properties. One natural property is that the covering space should be connected. This will be the case iff the monodromy group is a transitive permutation group. Another interesting property is that the covering should be "simple". This means that there is exactly one ramification point in X over each z_i and that this is a simple ramification point, i.e. of multiplicity two. Algebraically this means that the monodromy representation μ has transpositions as values $\mu(x_i)$ for all $x_1,\dots,x_n$ of any geometrically distinguished system of generators $x = (x_1,\dots,x_m)$. So the simple coverings correspond to elements of ${}_m\Delta^n$, where ${}_m\Delta \subset S_m$ is the automorphic subset of all transpositions. For $\tau = (\tau_1,\dots,\tau_n) \in {}_m\Delta^n$ the monodromy group $\Gamma_\tau = \langle\tau_1,\dots,\tau_n\rangle$ is generated by transpositions. Therefore Γ_τ is transitive iff $\Gamma_\tau = S_m$.

<u>ADDENDUM.</u> The Ω-set of equivalence classes of m-fold simple connected coverings with n ramification points identifies with the B_n-set ${}_m\Delta^n_* / J_n$, where

$${}_m\Delta^n_* := \{\tau \in {}_m\Delta^n \mid \Gamma_\tau = S_m\} .$$

P. Kluitmann has obtained very nice results concerning the operation of B_n on ${}_m\Delta^n_*$, which he presented at the Santa Cruz conference on Artin's braid group. Let me briefly quote some of them. For details, see [66]. First he solves the inverse invariant problem.

<u>THEOREM 3.12.</u> B_n operates transitively on the fibres of the characteristic map $\chi: {}_m\Delta^n_* \to S_m$.

The next result of Kluitmann is a description of ${}_m\Delta^n_* / J_n$ in terms of diagrams. Note that ${}_m\Delta$ is the quotient of the root system of type A_{m-1} by the group $\{\pm 1\}$. Diagrams for root bases are integral matrices with entries $0,\pm 1$ off the diagonal. Since we divide by $\{\pm 1\}$, it is natural to reduce to coefficients in $\mathbb{F}_2$. Explicitly: we associate to $\tau = (\tau_1,\dots,\tau_n) \in {}_m\Delta^n_*$ the matrix $\delta(\tau) = (a_{ij}) \in M_n(\mathbb{F}_2)$, where $a_{ij} = 0$ iff $\tau_i\tau_j = \tau_j\tau_i$. Let A^n_m be the image of this map $\delta: {}_m\Delta^n_* \to M_n(\mathbb{F}_2)$. There is an induced operation of B_n on A^n_m, and Kluitmann proves the following result:

<u>PROPOSITION 3.13.</u> For $m \neq 4$ the diagram map $\delta: {}_m\Delta^n_* / J_n \to A^n_m$ is an isomorphism of B_n - sets, and A^n_m can be described constructively in graph-theoretic terms.

By an ingenious analysis of the diagrams Kluitmann is able to solve the representation problem in a number of cases. Let me quote some of his results.

Consider the set of equivalence classes of (n+1)-fold connected simple coverings with n ramification points. This set identifies with ${}_{n+1}\Delta_*^n / J_n$ and has cardinality $(n+1)^{n-2}$. The image of the characteristic map is the conjugacy class of cycles of maximal length. Therefore B_n acts transitively on ${}_{n+1}\Delta_*^n / J_n$ by 3.12. But Kluitmann's solution of the representation theorem shows that much more is true!

THEOREM 3.14. For $n \equiv 0(4)$ the braid group B_n acts on the set ${}_{n+1}\Delta_*^n / J_n$ of cardinality $(n+1)^{n-2}$ as its full symmetric group. For $n \not\equiv 0(4)$, it acts as the alternating subgroup.

Kluitmann also has analogous results for the case where the root system A_n is replaced by D_n.

THEOREM 3.15. Let $\bar{\chi}: {}_3\Delta_*^n / J_n \to S_3 / I(S_3)$ be induced by the characteristic map. $S_3 / I(S_3)$ consists of three conjugacy classes: $[(1)]$, $[(1,2)]$, $[(1,2,3)]$. The permutation groups G_c of the three fibres $\bar{\chi}^{-1}(c)$ induced by the operation of B_n on the fibres can be identified with affine or projective symplectic groups as follows:

(i) $G_c = PSp(n-2,\mathbb{F}_3)$ for $c = [(1)]$ and $n \geq 4$ even,

(ii) $G_c = ASp(n-2,\mathbb{F}_3)$ for $c = [(1,2,3)]$ and $n \geq 2$ even,

(iii) $G_c = PSp(n-1,\mathbb{F}_3)$ for $c = [(1,2)]$ and $n \geq 3$ odd.

Statement (i) was proved by David Cohen [29], whereas (ii) and (iii) were proved by Kluitmann [66] using Cohen's result. Note that case (ii) for $n = 2$ just gives $G_c = \{1\}$. This should be compared with our previous discussion of the root system A_2 (example 1, figure 10). Kluitmann also identifies the kernel of the representation $B_n \to G_c$ using recent results of J. Birman and B. Wajnryb [16].

One case where the problem of invariants and the inverse problem of invariants are very well understood is the case of homogeneous root systems.

THEOREM 3.16. Let Δ be a homogeneous root system of rank n such that no irreducible component is of type A_1. Let $L(\Delta)$ be the root lattice generated by Δ and $W(\Delta)$ the Weyl group. Let Δ_*^n be the following subset of Δ^n:

$$\Delta_*^n = \{x \in \Delta^n \mid \mathbb{Z}x_1 + \ldots + \mathbb{Z}x_n = L(\Delta)\} = \{x \in \Delta^n \mid \Gamma_x = W(\Delta)\} .$$

Let $\chi: \Delta_*^n \to W(\Delta)$ be the characteristic map defined by $\chi(x_1,\ldots,x_n) = s_{x_1}\ldots s_{x_n}$. Then the braid group B_n acts canonically on Δ_*^n, and the action on the fibres $\chi^{-1}(c)$ of χ is transitive.

This theorem solves the inverse problem of invariants for homogeneous root systems. The problem of invariants is also solved. One has $\Delta_x = \Delta$ and

$r_x = W(\Delta)$ for all $x \in \Delta_*^n$. The conjugacy classes of the elements c_x in the image $\chi(\Delta_*^n)$, the so called quasi Coxeter elements, have been determined explicitly by E. Voigt [109] by using the work of R.W. Carter on conjugacy classes in the Weyl group [26]. For A_n there is only the conjugacy class of the classical Coxeter elements. But for D_n there are $[n/2]$ conjugacy classes of quasi Coxeter elements, and there are 3 for E_6, 5 for E_7 and 9 for E_8. - Theorem 3.16 was proved by stages. E. Looijenga [79] did the case A_n and P. Deligne with the help of J. Tits and D. Zagier proved the case where Δ is general, but $c \in W(\Delta)$ is a classical Coxeter element. Finally the general case was done by E. Voigt [109] (c.f.[109']for details). The cardinality of the B_n-orbits in Δ_*^n and related numbers have also been computed by P. Deligne, E. Voigt and P. Kluitmann. The list of Voigt in [109] p.189 and [109'] p.91 contains some errors. The representation problem for the operation of B_n on the fibres of χ has not been investigated, apart from Kluitmann's work on A_n and D_n quoted above. This might be an interesting and difficult problem for E_6, E_7, E_8.

The next two theorems describe another case where the inverse problem of invariants was solved by P. Kluitmann [67]. Let Δ be a root system of type E_n, $n = 6,7,8$. Choose a simple system of positive roots and add the longest root. The resulting configuration of $n+1$ roots is described by the classical extended Coxeter diagrams shown in figure 14.

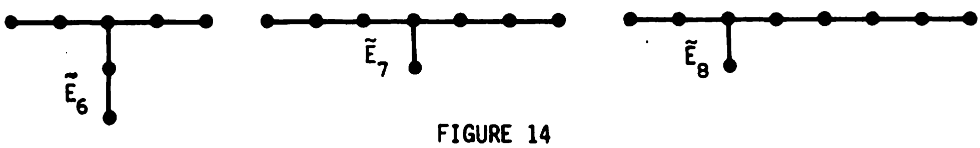


FIGURE 14

Notice that the triples (p,q,r) which give the length of the branches of these trees are just those for which $1/p + 1/q + 1/r = 1$. Now remove the central node of the graph. The resulting graphs are shown in figure 15.

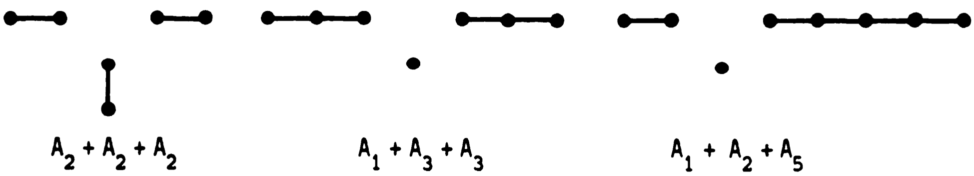


FIGURE 15

The corresponding n roots generate a sublattice of rank n in the root lattice. This sublattice is the root lattice of a root subsystem $\Delta' \subset \Delta$ of rank n and of type $A_{p-1} + A_{q-1} + A_{r-1}$. Kluitmann proves the following theorem.

THEOREM 3.17. Let Δ be a root system of type E_n, and let $\Delta' \subset \Delta$ be a root subsystem of type $A_{p-1} + A_{q-1} + A_{r-1}$ as above. Let $W(\Delta)$ be the Weyl group of Δ and let $c \in W(\Delta)$ be the element corresponding to a Coxeter element of Δ'. Let $\tilde{\Delta} \subset W(\Delta)$ be the automorphic subset of all reflections, and let $\tilde{\Delta}_*^{n+2}$ be the following set:

$$\tilde{\Delta}_*^{n+2} = \{s = (s_1,\ldots,s_{n+2}) \in \tilde{\Delta}^{n+2} \mid \Gamma_s = W(\Delta)\} .$$

Finally let $\chi:\tilde{\Delta}_*^{n+2} \to W(\Delta)$ be the characteristic map defined by $\chi(s) = s_1 \ldots s_{n+2}$. Then the braid group B_{n+2} acts transitively on the fibre $\chi^{-1}(c)$.

There is a lot of beautiful geometry associated to the pseudo Coxeter elements c of 3.17, which is related to the classical configurations of the 27 lines on a cubic surface and the 28 bitangents of a plane quartic. Theorem 3.17 is an essential step in the proof of the next theorem, which is the main result of Kluitmann's thesis [67].

Let (p,q,r) be one of the triples above and $n = p+q+r-3$. Let L_{pqr} be the integral lattice $\mathbb{Z}^{n+2}$ with that even symmetric bilinear form for which the standard basis $e_1,\ldots,e_{n+2}$ has the Coxeter diagram T_{pqr} and $\langle e_i,e_i\rangle = 2$. The reduced lattice $L_{pqr} / \operatorname{rad} L_{pqr}$ is the root lattice of type E_n. The automorphic set $\Delta_1(L_{pqr})$ and the group $W_1(L_{pqr})$ are defined as before. Let $c \in W_1(L_{pqr})$ be the product $c = s_1 \ldots s_{n+2}$ of the reflections s_i corresponding to e_i. The conjugates of c and c^{-1} in $W(L_{pqr})$ are called the Coxeter elements of T_{pqr}. Kluitmann proves that they are exactly the Coxeter elements in the sense of K. Saito's theory of extended affine root systems [98].

THEOREM 3.18. Let (p,q,r) be one of the triples $(3,3,3)$ or $(2,4,4)$ or $(2,3,6)$ and $n = p+q+r-3$.

$$\Delta_1(L_{pqr})_*^{n+2} = \{x \in \Delta_1(L_{pqr})^{n+2} \mid L_{pqr} = \mathbb{Z}x_1 + \ldots + \mathbb{Z}x_{n+2}\} .$$

Let $\chi:\Delta_1(L_{pqr})_*^{n+2} \to W(L_{pqr})$ be the characteristic map, and let $c \in W(L_{pqr})$ be a Coxeter element of T_{pqr}. Then the braid group B_{n+2} acts transitively on $\chi^{-1}(c)$.

Let me now report on Bertolt Krüger's thesis [69] dealing with braid reduction theory for even symmetric ternary forms which are minimally generated. Let us return to the operation of the braid group B_n on the set of matrices $M_n(R)_{\epsilon\eta}$ which we discussed before. By theorem 3.6 we know already that the operation of $B_n / \langle\zeta_n\rangle$ is faithful if R has infinitely many elements. In particular this applies if $R = \mathbb{R}$ or $R = \mathbb{Z}$. These are the cases which we want to study under the additional assumption $n = 3$ and $\eta = \varepsilon = 1$. Let us simplify the notation. We put $M_3^+(R) = M_3(R)_{1,1}$. So this is the space of matrices

$$\begin{bmatrix} 2 & x_1 & x_2 \\ x_1 & 2 & x_3 \\ x_2 & x_3 & 2 \end{bmatrix}$$

where $x = (x_1, x_2, x_3) \in R^3$ and $R = \mathbb{R}$ or $R = \mathbb{Z}$. We shall also denote this matrix by x . Recall that for minimally generated even symmetric ternary R-lattices L the diagram map induces an injection $\delta: \Delta_1(L)^3_* / O(L) \to M_3^+(R)$. The images of these maps are exactly the "classes" of ternary forms, where two matrices B, B' are in the same class, if there is an $A \in GL(3,R)$ such that $B' = {}^tABA$. Conversely we can associate to any $x \in M_3^+(R)$ the free R-module $L_x = R^3$ with the bilinear form defined by the matrix x for the standard basis e . The class of x will then be the image $\delta(\Delta_1(L_x)^3_* / O(L_x))$. We can now associate to x the invariants of $e \in \Delta_1(L_x)^3_*$, i.e. the monodromy group $\Gamma_x = \langle s_1, s_2, s_3 \rangle \subset O(L_x) \subset GL(3,\mathbb{Z})$ generated by the reflections s_1, s_2, s_3 corresponding to the standard basis vectors e_1, e_2, e_3 , and the pseudo Coxeter element $s_1 s_2 s_3 \in \Gamma_x$.

We have a canonical action of B_3 on $M_3^+(R)$ and we want to study the equivalence problem, the invariant problem and the inverse problem of this action. Of course the discriminant is an invariant of this action. Up to a factor 2 it is the following cubic form:

$$x_1 x_2 x_3 - x_1^2 - x_2^2 - x_3^2 + 4 \quad .$$

Such forms have already been studied. Mordell [89], page 106, treats the diophantine equation $x^2 + y^2 + z^2 - axyz = b$, and our cubic form also occurs in papers of Rosenberger [97], Horowitz [53] and Magnus [85]. Using some of their ideas, B. Krüger gave a very careful analysis of the action of B_3 on $M_3^+(R)$. Here are some of his results. He uses the square norm $Q(x) = x_1^2 + x_2^2 + x_3^2$.

<u>DEFINITION.</u> The matrix $x \in M_3^+(R)$ is a local minimum in its B_3-orbit, if $Q(x) \le Q(\sigma_i^\varepsilon(x))$ for $i = 1,2$ and $\varepsilon = \pm 1$.

It is a basic elementary fact that the set of x which are a local minimum in their orbit form a semialgebraic set in $\mathbb{R}^3$ consisting of the four octants $x_1 x_2 x_3 \le 0$ together with the subset of the other four octants described by the inequalities $2|x_i| < |x_j x_k|$, where $\{i,j,k\} = \{1,2,3\}$. In particular, if x is a local minimum, any x' obtained from x by an even number of sign charges and by permutations of the coordinates is also a local minimum. Let us call such local minima x and x' equivalent if they lie in the same orbit of B_3 .

DEFINITION. Regular and ordinary orbits of B_3 in $M_3^+(\mathbb{R})$ are defined as follows:

(i) $\mathbb{R}_* := \{c \in \mathbb{R} \mid |c| \geq 2 \text{ or } |c| = \cos(\pi/k) , \ k \geq 2\}$
$\mathbb{R}'_* := \{c \in \mathbb{R} \mid |c| > 2\}$

(ii) $B_3 x$ is regular iff $B_3 x \subset \mathbb{R}_*^3$.

(iii) $B_3 x$ is not ordinary iff $\det x = 0$ and $B_3 x \subset \mathbb{R}'^3_*$.

The orbit of an integral matrix $x \in M_3^+(\mathbb{Z})$ is always regular and ordinary. Krüger's main result is the following theorem.

THEOREM 3.19. The action of B_3 on $M_3^+(\mathbb{R})$ has the following properties:

(i) Every regular ordinary orbit contains a local minimum, and this is unique up to equivalence.

(ii) For a regular ordinary orbit $B_3 x$ the monodromy group Γ_x is a Coxeter group with Coxeter system $\{s_1, s_2, s_3\}$.

Note that (ii) solves the invariant problem, since the Coxeter matrix of the Coxeter system is easily determined from x . Note also that (i) almost solves the equivalence problem, since there is an obvious reduction process for getting from x to a local minimum x_o in finitely many steps. There are at most 24 local minima equivalent to x_o . Once they are determined, one has a well defined set of normal forms computed for x , and two elements x, x' are in the same B_3-orbit iff they have the same set of normal forms.

The inverse problem of invariants in its most naive sense does not make sense in the real case, since B_n-orbits are countable, whereas the fibres of the characteristic map χ may be real algebraic varieties of positive dimension, so that there will be uncountably many B_n-orbits in the fibres. But for integral matrices Krüger obtains the following result.

THEOREM 3.20. The action of B_3 on $M_3^+(\mathbb{Z})$ has the following properties.

(i) There are only finitely many B_3-orbits in each class of $M_3^+(\mathbb{Z})$. In other words: For each $x \in M_3^+(\mathbb{Z})$ there are only finitely many B_3-orbits in $\Delta_1(L_x)_*^3 / O(L_x)$.

(ii) Moreover, if $\det x \neq 0$, there are only finitely many B_3-orbits in the fibres of the characteristic map $\chi : \Delta_1(L_x)_*^3 \to O(L_x)$.

Krüger also deals with the representation problem. In particular he determines the isotropy groups for the B_3-actions on the regular orbits in $M_3^+(\mathbb{R})$. There are orbits on which $B_3/<\zeta_3>$ acts freely, i.e. with trivial isotropy groups.

The corresponding problems for the action of B_n on $n\times n$-matrices with $n > 3$ appear to be extremely difficult. But at least Krüger is able to deal with matrices of rank 1 . Even this very degenerate case is interesting, since it is very closely related to the known integral symplectic representations of B_n studied by V.I. Arnold [5] and Magnus - Peluso [83].

Most of the problems discussed so far are genuine combinatorial problems. On the other hand there is one problem of another character. This is the problem of calculating monodromy groups for the automorphic sets coming from lattices. This is mainly an arithmetic problem. There has been much progress with respect to this part of the problem of invariants. After important contributions of many people including A'Campo, Arnold, Chmutov, Pinkham and Wajnreb, the final results were obtained by W. Ebeling [37],[39] and W.A.M. Janssen [57],[58]. Here is an extremely condensed version of a report which I got from Wolfgang Ebeling.

Let L be an integral lattice, even symmetric or antisymmetric. $\Delta_\varepsilon(L) \subset L$ and $s_a \in \mathrm{Aut}(L)$ for $a \in \Delta_\varepsilon(L)$ are defined as before. For any subset $\Lambda \subset \Delta_\varepsilon(L)$ let $\Gamma_\Lambda \subset \mathrm{Aut}(L)$ be the group $\Gamma_\Lambda = <s_a \mid a \in \Lambda>$.

<u>DEFINITION.</u> The pair (L,Λ) is a vanishing lattice, if it satisfies the following conditions:

(i) Λ generates L .

(ii) Λ is an orbit of Γ_Λ in L .

(iii) If $\operatorname{rank} L > 1$, there exist $a,b \in \Lambda$ such that $<a,b> = 1$.

The connection with our previous theory of automorphic sets is obvious. For any $x \in \Delta_\varepsilon(L)^m$ we have the automorphic subset $\Delta_x \subset \Delta_\varepsilon(L)$ and the monodromy group Γ_x . If we put $\Lambda = \Delta_x$, we get $\Gamma_\Lambda = \Gamma_x$, and (L,Λ) is a good candidate for a vanishing lattice. Condition (i) was almost always part of our assumptions. Condition (ii) means that Δ_x is very homogeneous. In applications in singularity theory all conditions will be satisfied.

Let $j:L \to L^\#$ be the canonical map to the dual and $\mathrm{Aut}^\#(L) \subset \mathrm{Aut}(L)$ the subgroup operating trivially on $L^\#/j(L)$. Then $\Gamma_\Lambda \subset \mathrm{Aut}^\#(L)$. For symmetric L let $\nu_\varepsilon:\mathrm{Aut}(L) \to \{\pm 1\}$ the real ε-Spinornorm. We define a subgroup $O_\varepsilon^*(L) \subset O(L)$ as follows:

$$O_\varepsilon^*(L) = \mathrm{Aut}^\#(L) \cap \ker \nu_\varepsilon \quad .$$

Using a result of M. Kneser [63], W. Ebeling proved the following theorem.

THEOREM 3.21. Let (L,Λ) be an even symmetric vanishing lattice. Suppose that L contains a 6-dimensional sublattice $M \subset L$, such that M is the orthogonal direct sum of two unimodular hyperbolic planes and a lattice of type εA_2. Suppose moreover $\Lambda \supset \Delta_\varepsilon(M)$. This implies the following statements.

(i) $\Lambda = \{a \in L \mid \langle a,a\rangle = 2\varepsilon \text{ and } \langle a,L\rangle = \mathbb{Z}\} \subset \Delta_\varepsilon(L)$.

(ii) $\Gamma_\Lambda = \Gamma_{\Delta_\varepsilon(L)} = W_\varepsilon(L) = O^*_\varepsilon(L)$.

Now let (L,Λ) be an antisymmetric vanishing lattice. Choose a symplectic basis $e_1,f_1,\dots,e_m,f_m,g_1,\dots,g_k$ such that $\langle e_i,f_i\rangle = -\langle f_i,e_i\rangle = d_i \in \mathbb{N}$ with $d_i \mid d_{i+1}$ and such that all other scalar products of basis vectors are zero. Let $v_2(L)$ be the exponent of 2 in the prime factorization of d_m. If $a \in L$ is such that there exists an element $b \in L$ with $a-2b \in \Lambda$, we write $a \in \Lambda \bmod 2$. W.A.M. Janssen proved the following theorem.

THEOREM 3.22. For any antisymmetric vanishing lattice (L,Λ) the following statements hold:

(i) $\Lambda = \{a \in L \mid \langle a,L\rangle = \mathbb{Z} \text{ and } a \in \Lambda \bmod 2\}$

(ii) Γ_Λ contains the congruence subgroup modulo $2^{v_2(L)}$ of the restricted symplectic group $Sp^{\#}(L)$.

Moreover Janssen gets a complete classification of skew symmetric vanishing lattices. There is also work of S.P. Humphries [55] which is related to this subject.

Note that in general the vanishing lattice (L,Δ_x) contains less information than L together with the orbit $B_n x$ of the braid group in Δ_x^n.

Finally I have to say that very little has been done concerning the embedding problem. A few isolated results have been announced in [25].

§4. BRAIDS AND SINGULARITIES

The past thirty years have been an exciting period for those who work on singularities. There was a great confluence of ideas of outstanding mathematicians and physicists coming from algebraic geometry, complex analysis, differential topology and many other fields. It is impossible to present this rich variety of ideas and their manifold connections on a few pages. For those who want to ge a first impression I recommend reading the beautiful article of V.I. Arnold on "Critical points of smooth functions" and J. Milnor's classical work on "Singular points of complex hypersurfaces" ([7],[87]). I also recommend the books of E. Looijenga [80] and V.I. Arnold [8] and V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko [10] as well as the proceedings of the summer institute on singularities 1981 [91]. I shall now restrict myself to one approach, which has its deep roots in the mathematics of the last century.

The essence of the theory which I want to explain is the application of the "analysis situs" to analytic or algebraic entities, to objects of algebraic or analytic geometry. This way was opened by the genius of Bernhard Riemann in his thesis and in his work on Abelian integrals. After that it was followed by Picard and Simart in their treatise on algebraic functions of two variables [95], in Poincaré's "Quatrième complément à l'analysis situs" and finally in Lefschetz' famous work "L'analysis situs et la géométrie algébrique" [77]. Lefschetz' work was pioneering work using the tools of algebraic topology while or even before they were being developped. So it is not easy to read. A very readable modern exposition was given by K. Lamotke [71]. Lefschetz presented his ideas in transcendental form - the only form possible at his time. But the same ideas can be given an algebraic form using the fundamental ideas of Grothendieck. This was done and very important new results were obtained around 1967-1969 by A. Grothendieck, P. Deligne and N. Katz in the Séminaire de Géométrie Algébrique "Groupes de Monodromie en Géométrie Algébrique" [47], [31]. This work was very important in connection with P. Deligne's celebrated proof of the Weil conjectures. At about the same time Lefschetz' ideas were also applied on a much more modest scale in the analysis of isolated singularities of complex hypersurfaces. They were combined with R. Thom's ideas on the universal unfolding of singularities, work of J. Mather on stable map germs and ideas of V.I. Arnold and G.N. Tjurina. This approach was developped by F. Pham [93],[94], Lê Dũng Tráng [76], K. Lamotke [70] and A.M. Gabrielov [40],[41],[42]. I myself was also involved (see e.g. [21], Appendix). A very useful survey was given by S.M. Gusein-Zade [49].

We want to study singular points of complex analytic functions of several complex variables and of their level sets. A singular point is a point where the level set passing through this point is not smooth. The equivalent condition for the function is the vanishing of all partial derivatives at this point. A singular point is isolated iff it is the only singular point in a suitable neighbourhood. The simplest isolated singularities are the ordinary double points. These are the singular points with nondegenerate Hessian. If a function f of k variables has an ordinary double point at p and if $f(p) = 0$, one can find local coordinates $x = (x_1,\dots,x_k)$ centered at p such that

$$f(x) = x_1^2 + \dots + x_k^2 \quad .$$

Isolated singularities are a local phenomenon. In order to study them locally, we always restrict the functions to a closed ball of small radius ρ with centre the singular point, and we intersect the level sets with this ball.

If we choose ρ sufficiently small and if we then choose the level set sufficiently near the singular level, our constructions will not in an essential way depend on these choices.

Now let us look at the simplest singularity, the ordinary double point. Figure 16 should support your intuition, although it is only the picture of a real analogue of a complex situation which cannot be described properly by pictures.

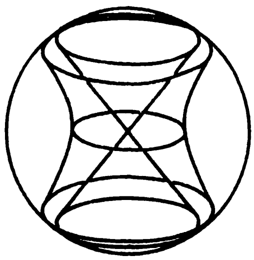


FIGURE 16

The singular level set of an ordinary double point is a cone over a quadric. The nearby nonsingular level sets are complex "spheres" with equation $x_1^2 + \ldots + x_k^2 = s$. Let us choose s real and positive. Then we have a real (k-1)-sphere of radius $\sqrt{s}$ sitting in the complex sphere, namely the subset where all coordinates are real. The complex sphere is easily identified with the tangent bundle of the real sphere so that the real sphere identifies with the 0-section. The intersection Y_s of the complex sphere with the ball gives a disc bundle in the tangent bundle. So the level set Y_s can be retracted to the real sphere. In particular its homology is the same as that of the real (k-1)-sphere. The only non zero reduced homology group is the (k-1)-th homology group, and it is generated by this real (k-1)-sphere, considered as a (k-1)-cycle in the level set Y_s . Strictly speaking we only get a cycle and hence a generator of $H_{k-1}(Y_s,\mathbb{Z})$ after choosing an orientation of the real sphere. There is no preferred choice, and this ambiguity will persist everywhere in our theory. Note that the real sphere shrinks to a point as s tends to zero. Therefore the corresponding cycles are called "vanishing cycles". This name adopted by Lefschetz goes back to Poincaré [96] p.415.

Now the basic idea for studying arbitrary isolated singular points of an analytic function f is to deform the function f slightly so that the complicated singularity decomposes into a certain number of simple ordinary double points. The configuration of the vanishing cycles corresponding to these ordinary double points can be described by a diagram, and these diagrams are Coxeter diagrams to which we can apply our theory of braids and automorphic sets.

There are several theories describing deformation processes which can be used to carry out this program. One is the semiuniversal deformation of isolated singularities of complex spaces constructed by G.N. Tjurina [107] and A. Kas - M. Schlessinger [60] for complete intersections and by H. Grauert [46] for the general case. Another possibility is the universal unfolding of functions, initiated by R. Thom [105] and worked out by G. Wassermann [110] on the basis of unpublished notes by J. Mather on right equivalence. For simplicity we choose the approach by unfoldings.

DEFINITION. An unfolding of a holomorphic germ of a function $f_o:(\mathbb{C}^k,0) \to (\mathbb{C},0)$ is a holomorphic germ of a map $F:(\mathbb{C}^k \times \mathbb{C}^p,0) \to (\mathbb{C} \times \mathbb{C}^p,0)$ of the form $F(x,t) = (f(z,t),t)$ with $f(x,0) = f_o(x)$. A morphism of unfoldings F,G of f_o is a commutative diagram of germs of maps

$$\begin{array}{ccc} (\mathbb{C}^k \times \mathbb{C}^q,0) & \xrightarrow{\Psi} & (\mathbb{C}^k \times \mathbb{C}^p,0) \\ G \downarrow & & \downarrow F \\ (\mathbb{C} \times \mathbb{C}^q,0) & \xrightarrow[\Phi]{} & (\mathbb{C} \times \mathbb{C}^p,0) \end{array}$$

where $\Psi(x,0) = (x,0)$ and where Φ is of the form $\Phi(s,t) = (s+\sigma(t),\tau(t))$. An unfolding F of f_o is versal if every unfolding G of f_o can be induced from F by a morphism from G to F . It is universal if in addition the dimension p of the parameter space $\mathbb{C}^p$ is minimal.

For germs of functions f_o with isolated singularities, a universal unfolding exists and is unique up to a non canonical isomorphism. It is constructed as follows. Consider the local Artinian algebra $\mathbb{C}\{x_1,\ldots,x_k\} / (\partial f_o/\partial x_1,\ldots,\partial f_o/\partial x_k)$. This is an important analytic invariant of f . In fact by a recent result of J. Scherk [99] this algebra determines f_o up to right equivalence. (Two germs are right equivalent if they are in the same orbit of $\mathrm{Aut}\,\mathbb{C}\{x_1,\ldots,x_k\}$) . A much simpler but basic invariant of f_o is the dimension $n(f_o)$ of this algebra, considered as a complex vector space. This number $n = n(f_o)$ is usually called the Milnor number of f_o and is denoted by μ in the literature. It equals the dimension of the target space of the universal unfolding, which we now construct as follows.

Let $g_1,\ldots,g_n \in \mathbb{C}\{x_1,\ldots,x_k\}$ be power series representing a basis of $\mathbb{C}\{x_1,\ldots,x_k\} / (\partial f_o/\partial x_1,\ldots,\partial f_o/\partial x_k)$. Assume $g_1 \equiv 1$ and define $f(x,t)$ as follows:

$$f(x_1,\ldots,x_k,t_2,\ldots,t_n) = f_o(x_1,\ldots,x_k) + \sum_{j=2}^{n} g_j(x_1,\ldots,x_k)t_j \quad .$$

Then $F(x,t) = (f(x,t),t)$ is "the" universal unfolding of f_o . For example the universal unfolding of $f_o(x) = x^{n+1}$ is $F(x,t_2,\ldots,t_n) =$

$(x^{n+1} + t_2 x^{n-1} + \ldots + t_n x, t_2, \ldots, t_n)$.

So far the universal unfolding was defined as a germ of a map. However if we want to understand the geometry hidden in this germ, we have to choose an actual map representing the germ, and we have to choose it properly. We define spaces X, S, T as follows.

$$T = \{t = (t_2, \ldots, t_n) \in \mathbb{C}^{n-1} \mid \|t\| < \tau\}$$

$$S = \{(s,t) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid |s| \leq \sigma, \quad \|t\| < \tau\}$$

$$X = \{(x,t) \in \mathbb{C}^k \times \mathbb{C}^{n-1} \mid \|x\| \leq \rho, \quad |f(x,t)| \leq \sigma, \quad \|t\| < \tau\} .$$

Here ρ, σ, τ are positive real numbers which are choosen small with respect to each other in the following order: $0 \ll \tau \ll \sigma \ll \rho \ll 1$. By this I mean the following: If we choose first ρ sufficiently small, then σ sufficiently small with respect to ρ etc., all statements which we are going to make about our representative of the universal unfolding will hold. Now we represent the universal unfolding by the map $F: X \to S$ with F defined as above. Let $p: S \to T$ be the projection $p(s,t) = t$. Let $C \subset X$ be the critical set of F . This is a smooth connected complex submanifold in X of dimension $n-1$, i.e. codimension k . Its image $D = F(C) \subset S$ is called the discriminant. It is an irreducible hypersurface in S , and $F: C \to D$ is the normalization map. The projection p induces an n-fold ramified covering map $\pi: D \to T$, which is regular exactly in the regular part D_r of D . The singular part D_s of D is of codimension 1 in D . Its image $B = \pi(D_s) \subset T$ is a hypersurface in T which is called the bifurcation set. For $t \in T$ we put $S_t = p^{-1}(t)$. This is a closed disk in the complex s-plane. We put $X_t = F^{-1}(S_t)$ and we denote the restriction of F to X_t by $F_t: X_t \to S_t$. This function depending on the parameter $t \in T$ is viewed as a deformation of the function F_o representing the given germ f_o .

<u>PROPOSITION 4.1.</u> Let $n = n(f_o)$ be the Milnor number of the germ of function f_o . Then for $t \in T - B$ the function $F_t: X_t \to S_t$ has exactly n critical points $p_1, \ldots, p_n$. These critical points are ordinary double points. The corresponding critical values $z_i = F_t(p_i)$ are n distinct points $z_1, \ldots, z_n$ in the interior of the closed disk S_t . They are the points in which S_t intersects the discriminant $D \subset S$.

Figure 17 illustrates 4.1 for the simplest non trivial case $f_o(x) = x^3$ by its real analogue. The germ of the discriminant at the origin determines the germ of the hypersurface $f_o(x_1, \ldots, x_k) = 0$ at the origin by a theorem of K. Wirthmüller [111]. Therefore the geometry of the discriminant is very important for the geometry of the singularity itself. Many investigations have been devoted to the geometry and topology of the discriminant. This is one of the

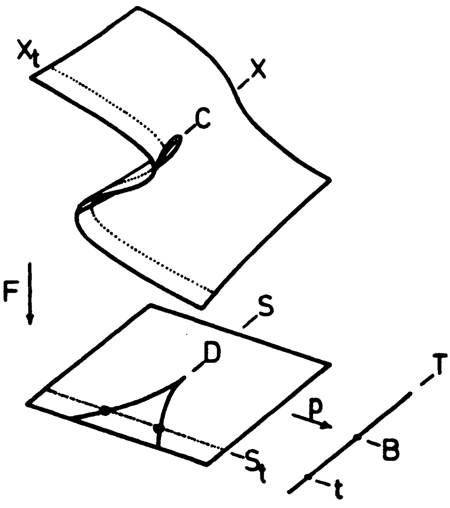


FIGURE 17

points where braid groups or generalisations of them like the Artin groups make contact with singularities. Since I cannot explain this in detail, let me just give some references: [9],[22],[23],[30],[52],[68],[108].

The next theorem is essentially due to J. Milnor [87] (c.f. also H.A. Hamm [50]).

THEOREM 4.2. Let $F: X \to S$ represent the universal unfolding of $f_o(x_1, \ldots, x_k)$ as above. Let $S' = S - D$ be the complement of the discriminant and $X' = X - F^{-1}(D)$ its inverse image. Then the restricted map $F: X' \to S'$ has the following properties.

(i) $F: X' \to S'$ is a C^∞-smooth locally trivial fibre bundle.

(ii) The typical fibre $Y_s = F^{-1}(s)$, $s \in S'$ is a compact C^∞-manifold with boundary of real dimension $2(k-1)$. This manifold which is called the Milnor fibre has the homotopy type of a bouquet of $n(f_o)$ spheres of dimension $k-1$.

(iii) The boundary ∂Y_s is a closed $(k-3)$-connected manifold of dimension $2k-3$. This holds for all fibres $Y_s = F^{-1}(s)$, $s \in S$.

(iv) The boundaries ∂Y_s, $s \in S$, are the fibres of a trivial fibre bundle over S.

(v) The fibres Y_s, $s \in \partial S$ form a locally trivial fibre bundle over the boundary $\partial S \approx S^1 \times T$ which is trivial in the direction of T. This means the following: Choose any $t \in T$, and restrict the fibre bundle $X' \to S'$ to the circle ∂S_t. The bundle over ∂S is then obtained by lifting the bundle over ∂S_t by means of the projection $\partial S \to \partial S_t$.

Consider the fibre bundle over the circle ∂S_t with fibres Y_s of statement (v). According to J. Milnor, it can be identified with the fibration of the complement of a generalized knot which is usually called the Milnor fibration and which is constructed as follows. The singular fibre $Y_o = f_o^{-1}(0) \subset \mathbb{C}^k$ intersects the sphere S_ρ bounding the ball $B_\rho \subset \mathbb{C}^k$ transversally in its boundary $K_\rho = \partial Y_o \subset S_\rho$. This is a smooth closed submanifold of real codimension 2 in the (2k-1)-sphere S_ρ. For $k = 2$ it is a link, for $k > 2$ it is connected and hence a knot. The diffeomorphism type of (S_ρ, K_ρ) does not depend on the choice of ρ and is called "the" knot of the singularity. K_ρ may be an exotic sphere ([20],[51]). The complement of the knot can be fibered over the unit circle S^1 in the complex plane by means of the map $S_\rho - K_\rho \to S^1$ defined by $x \mapsto f_o(x) / |f_o(x)|$. This fibration gives an open book structure around K_ρ. The Milnor fibration is the restriction of the fibration $S_\rho - K_\rho \to S^1$ to the complement of a small open tubular neighbourhood of K_ρ in S_ρ. We shall say that two singularities of germs of functions are topologically equivalent if their Milnor fibrations are diffeomorphic. This implies that there is a homeomorphism of neighbourhoods of the singular points, which is a C^∞-diffeomorphism on the punctured neighbourhoods and identifies the singular level sets. J. Milnor proved ([87]):

PROPOSITION 4.3. The restriction of $F_t : X_t \to S_t$ to the circle ∂S_t bounding the disk S_t can be identified with the Milnor fibration.

We can now associate several algebraic-topological invariants to the geometric situation described in 4.1, 4.2 and 4.3. First of all the Milnor fibre Y_s, $s \in S'$, is a smooth compact manifold with boundary of real dimension $2(k-1)$. Its interior is a complex manifold, so that Y_s is oriented. We consider the homology and cohomology of Y_s with integral coefficients. Composition of the natural map $H_{k-1}(Y_s) \to H_{k-1}(Y_s, \partial Y_s)$ with Poincaré duality $H_{k-1}(Y_s, \partial Y_s) \to H^{k-1}(Y_s)$ and the isomorphism $H^{k-1}(Y_s) = \mathrm{Hom}(H_{k-1}(Y_s), \mathbb{Z})$ gives a natural homomorphism $H_{k-1}(Y_s) \to \mathrm{Hom}(H_{k-1}(Y_s))$. We denote the image of $a \in H_{k-1}(Y_s)$ by a' and define a natural intersection pairing

$$H_{k-1}(Y_s, \mathbb{Z}) \times H_{k-1}(Y_s, \mathbb{Z}) \to \mathbb{Z}$$
$$(a,b) \mapsto \langle a,b\rangle := a'(b) \quad .$$

In this way $H_{k-1}(Y_s)$ becomes an integral lattice which is symmetric and even if $k-1$ is even and antisymmetric if $k-1$ is odd.

DEFINITION. The Milnor lattice L_s of a germ $f_o : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ with isolated singularity is the homology group of the Milnor fibre $H_{k-1}(Y_s, \mathbb{Z})$ with the intersection form defined above. The monodromy of f_o is the canonical homomorphism

$\mu_s : \pi_1(S',s) \to \mathrm{Aut}(L_s)$.

Its image is called the monodromy group of the singularity.

Note that μ_s is a homomorphism and not an antihomomorphism. We define the product structure of the fundamental group as usual in homotopy theory. With this definition the monodromy operation defined by most authors would be an antihomomorphism. Since we want the monodromy to be a homomorphism, our operations of elements of $\pi_1(S',s)$ are the inverse of the operations used by these authors, and the corresponding formulas differ by a sign. For the same reason, the definition of a geometrically distinguished basis given below differs from that of those authors by reversing the order of the vectors of the basis.

In addition to the invariants defined above, we get a particular element in the monodromy group, if we choose the base point s in the boundary ∂S , for it lies on the circle ∂S_t , where $t = p(s)$, and this circle with positive orientation gives us a particular homotopy class $\varkappa \in \pi_1(S',s)$. The corresponding element $\mu_s(\varkappa)$ in the monodromy group describes the monodromy of the Milnor fibration.

<u>DEFINITION.</u> The element $\mu_s(\varkappa) \in \mathrm{Aut}(L_s)$ of the monodromy group is called the classical monodromy transformation.

The classical monodromy is the automorphism h_* of the homology lattice $H_{k-1}(Y_s)$ induced by a homeomorphism $h: Y_s \to Y_s$ such that $h|\partial Y_s = \mathrm{id}$. To any such homeomorphism determined up to isotopy one can associate a well defined homomorphism $v: H_{k-1}(Y_s, \partial Y_s) \to H_{k-1}(Y_s)$. If c is a relative cycle and $[c] \in H_{k-1}(Y_s, \partial Y_s)$ its homology class, $v([c]) = [h(c) - c]$, the class of the absolute cycle $h(c) - c$. One has $v \circ j_* = h_* - \mathrm{id}$ for the homomorphism j_* induced by the inclusion $j: (Y_s, \emptyset) \to (Y_s, \partial Y_s)$. Using Poincaré-duality, we can identify v with a homomorphism $v: L_s^\# \to L_s$, where $L_s^\# = \mathrm{Hom}(L_s, \mathbb{Z})$ is the dual of L_s .

<u>DEFINITION.</u> The homomorphism $v: L_s^\# \to L_s$, $s \in \partial S$, is called the variation of the singularity.

Using the variation v , we can define a bilinear form $\ell: L_s \times L_s \to \mathbb{Z}$ by $\ell(a,b) = v^{-1}(b)(a)$. Up to a sign depending on choices of orientations and up to transposition this bilinear form can be identified with the linking form familiar from knot theory (c.f. J. Levine [78] 2.5).

<u>DEFINITION.</u> The bilinear form $\ell: L_s \times L_s \to \mathbb{Z}$ defined by $\ell(a,b) = v^{-1}(b)(a)$ is called the linking form of the singularity.

Now let me come to the decisive construction implementing the ideas of Lefschetz. This construction leads to a class of geometrically distinguished integral bases of the Milnor lattice L_s, where $s \in \partial S$. Consider again the map $F_t : X_t \to S_t$ obtained by restricting our good representative of the universal unfolding to the inverse image X_t of the complex disc S_t. Assume that $p(s) = t \in T - B$. As stated in 4.1, the function F_t has n ordinary double points $p_1, \ldots, p_n$ as its only critical points, with n distinct critical values $z_1, \ldots, z_n$ in the interior of S_t. In a suitable neighbourhood of p_i, the function F_t looks exactly like that which describes the standard ordinary double point. Therefore we can choose a closed ball B_i of small radius ρ_i with centre p_i and apply our previous analysis to $F_t | B_i$. In particular, if we choose any $s_i \in S_t$ sufficiently near to z_i but different from s_i, we find a vanishing sphere in the intersection $Y_{s_i} \cap B_i$. Hence we get a vanishing cycle in $H_{k-1}(Y_{s_i})$, which is uniquely determined up to sign. In this way, after choosing orientations, we get n vanishing cycles, one for each ordinary double point p_i. However they all lie in the homology lattices of different fibres. We have to move them into one common fibre, the fibre over the base point $s \in \partial S_t$. It is exactly at this point that the geometry of the disk and its relation to the braid groups discussed in §3 plays an important role.

Choose any geometrically distinguished system of generators $\xi = (\xi_1, \ldots, \xi_n)$ of $\pi_1(S_t - \{z_1, \ldots, z_n\}, s)$ as defined in §3. In order to simplify the notation denote the point "inside" the loops of the noose x_i by z_i. (The order of $(\xi_1, \ldots, \xi_n)$ does matter, but any ordering of $\{z_1, \ldots, z_n\}$ is irrelevant.) The loops corresponding to the nooses $\xi_1, \ldots, \xi_n$ can be made arbitrarily small by a suitable choice of the maps representing the homotopy classes ξ_i. Make them so small that the "eye" of the noose, i.e. the point s_i where the rope passes into the loop, is sufficiently near to z_i in the sense of the previous paragraph (Figure 18).

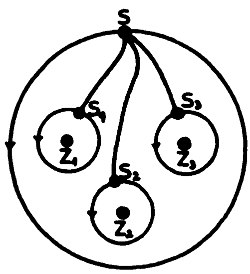


FIGURE 18

Move the vanishing cycles in the fibres over s_i into the fibre over the base point s by means of a continuous family of homeomorphisms of the fibres covering the path from s_i to s. In this way we get an n-tuple of homology classes $e = (e_1,\ldots,e_n) \in L_s^n$. It is easy to prove that e is an integral basis of the lattice L_s . Let us say that such a basis belongs to the geometrically distinguished system of generators $\xi = (\xi_1,\ldots,\xi_n)$. Because of the choices of orientations, there are exactly 2^n bases belonging to any given ξ .

DEFINITION. For any fixed choice of $t \in T - B$ and of the base point $s \in \partial S_t$, the geometrically distinguished bases of the Milnor lattice L_s are the bases $e = (e_1,\ldots,e_n)$ belonging to the geometrically distinguished systems of generators $\xi = (\xi_1,\ldots,\xi_n)$ of $\pi_1(S_t - \{z_1,\ldots,z_n\},s)$.

Figure 19 illustrates the simplest non-trivial case. This is the case where the Milnor number $n = 2$. This implies that the singularity is of type A_2 . For $k = 1$ this is the germ of the function $f(x) = x^3$. Its universal unfolding was the example considered above (Figure 17). For $k = 3$ the A_2-singularity is the germ of the function $x^3 + y^2 + z^2$. The base space S of the universal unfolding and the discriminant $D \subset S$ are the same as for $k = 1$, but the fibres of $F:X \to S$ are now complex surfaces. Figure 19 shows the real part of three of these fibres: Two surfaces are singular with an ordinary double point. They lie over the two critical values z_1, z_2 where the cuspidal cubic D intersets S_t . The third surface is nonsingular and is the real part of the Milnor fibre. On it there are two vanishing cycles. They shrink to the two singular points when we approach the singular level. The surface in the middle should be seen as lying over the one to the left and under the one to the right. Pictures like this occur already in a paper of Felix Klein [64].

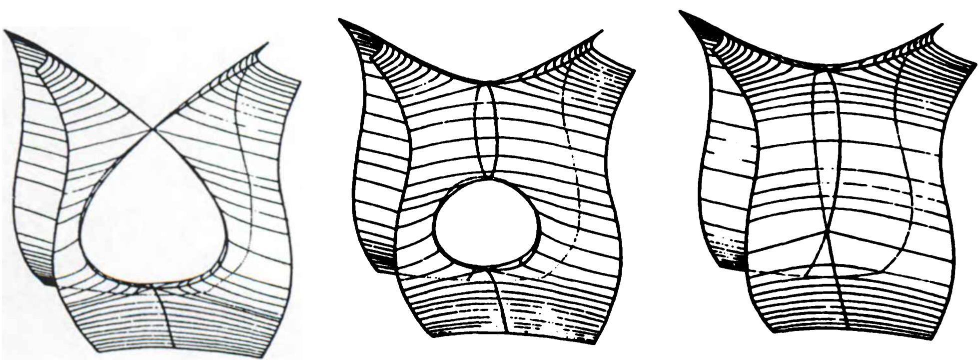


FIGURE 19

THEOREM 4.4. Let $f_o:(\mathbb{C}^k,0) \to (\mathbb{C},0)$ be a germ of a complex analytic function with an isolated singularity. Put $\varepsilon = \varepsilon(k) = (-1)^{k(k-1)/2}$. Let

$n = n(f_o)$ be the Milnor number of f_o. Choose a representative of the universal unfolding $F:X \to S$ as above. Choose a disk $S_t \subset S$ and a base point $s \in \partial S_t$ as above. Let $\{z_1,\ldots,z_n\} = S_t \cap D$ be the intersection with the discriminant. Then the topological invariants of $F:X \to S$ defined above and the invariants defined in the theory of automorphic sets are related as follows.

(0) The geometrically distinguished bases of the Milnor lattice L_s form a $\{\pm 1\}^n \rtimes B_n$-orbit in $\Delta_\varepsilon(L_s)^n$.

(1) Let $\xi = (\xi_1,\ldots,\xi_n)$ be any geometrically distinguished system of generators of $\pi_1(S_t - \{z_1,\ldots,z_n\},s)$ and let $e = (e_1,\ldots,e_n)$ be a geometrically distinguished basis belonging to ξ. Then the monodromy $\mu_s:\pi_1(S_t - \{z_1,\ldots,z_n\},s) \to \mathrm{Aut}(L_s)$ satisfies the following <u>Picard-Lefschetz-formula</u>:

$$\boxed{\mu_s(\xi_i) = s_{e_i,\varepsilon}}$$

(2) The monodromy group Γ_e in the sense of the theory of automorphic sets equals the monodromy group $\mathrm{im}\,\mu_s$.

(3) The pseudo Coxeter element c_e equals the classical monodromy $\mu_s(\varkappa)$ of the Milnor fibration.

(4) The variation operator v_e^+ and the variation operator v of the Milnor fibration are related as follows: $v_e^+ = -\varepsilon v$.

(5) The Seiffert form ℓ_e^+ and the linking form ℓ of the singularity are related by $\ell_e^+ = -\varepsilon\ell$.

The central result is the Picard-Lefschetz-formula. Qualitatively it goes back to Picard-Simart, and the exact formula was proved by Lefschetz [77], Théorème Fondamental, p.23 and p.92.

There is a suspension operation for singularities related to the suspension operation for lattices introduced in §3.

<u>DEFINITION.</u> The suspension of the germ $f:(\mathbb{C}^k,0) \to (\mathbb{C},0)$ is the germ $\Sigma f:(\mathbb{C}^{k+1},0) \to (\mathbb{C},0)$ defined by $\Sigma f(x_1,\ldots,x_{k+1}) = f(x_1+\ldots+x_k) + x_{k+1}^2$.

Conversely one might call f a desuspension of Σf. We may also introduce corresponding operations for right equivalence classes of germs of functions with isolated singularities and get well defined suspension and desuspension operations on this level. This is the contents of the generalized Morse lemma, which says that for each right equivalence class there is a unique minimal iterated desuspension, i.e. one with the minimal possible number of variables. The class of f is minimal iff the Hessian of f is zero. Germs obtained from each other by suspension or desuspension are called stably equivalent. In several respects such as their behaviour with respect to deformations they do not differ essentially. Therefore in Arnold's classification of singularities it is enough to classify them up to stable equivalence. It is clear from the construction of universal unfoldings that the universal unfoldings F of f_o and $\tilde{F}$ of Σf_o are very closely related. We can chose them so that

$$F(x_1,\dots,x_k,t) = (f_o(x_1,\dots,x_k) + \Sigma t_i g_i(x_1,\dots,x_k),t) \quad ,$$
$$\tilde{F}(x_1,\dots,x_{k+1},t) = (f_o(x_1,\dots,x_k) + x_{k+1}^2 + \Sigma t_i g_i(x_1,\dots,x_k),t) \quad .$$

Therefore representatives of these germs $F:X \to S$ and $\tilde{F}:\tilde{X} \to S$ can be chosen so that the target spaces S, the discriminants $D \subset S$, the projections $p:S \to T$ and $\pi:D \to T$ and the bifurcation sets $B \subset T$ are identical.

THEOREM 4.5. The relation between the invariants of a germ $f:(\mathbb{C}^k,0) \to (\mathbb{C},0)$ and of its suspension Σf is as follows. Put $\varepsilon = \varepsilon(k)$ and $\varepsilon' = \varepsilon(k+1)$. Choose representatives of the universal unfoldings of f and Σf with common target space S and common base point s as above. Let $L_s(f)$ and $L_s(\Sigma f)$ be the corresponding Milnor lattices, and let $e \in \Delta_\varepsilon(L_s(f))^n$ be a geometrically distinguished basis. Then

$$(L_s(\Sigma f),\varepsilon') = \Sigma_e(L_s(f),\varepsilon) \quad ,$$

where Σ_e is the suspension operation defined in §3. The $\{\pm1\}^n \rtimes B_n$-orbits of geometrically distinguished bases of $L_s(f)$ and $L_s(\Sigma f)$ identify in such a way that the relation with geometrically distinguished systems of generators of the fundamental group is preserved. The vanishing spheres of Σf are the suspensions of the vanishing spheres of f.

In the constructions above we have made many choices. One way of mitigating the dependence on the base point would be to pass from orbits of $\{\pm1\}^n \rtimes B_n$ to orbits of $(\{\pm1\}^n \times J_n) \rtimes B_n$ in $\Delta_\varepsilon(L_s)^n$. This is perhaps too generous, but the example of the root system A_2 discussed above shows that it leads to a nice combinatorial structure. On the other hand this enlarged object would still be a subset of the set $\Delta_\varepsilon(L_s)^n$, thus depending in a formal sense on the choice of s and other choices. The following definition leads to invariants which do not depend on any choice.

DEFINITION. The diagrams of a germ of a function with an isolated singularity are the Coxeter diagrams of the geometrically distinguished bases of a Milnor lattice of the germ.

The diagrams of f form a $\{\pm1\}^n \rtimes B_n$-orbit of matrices in $M_n(\mathbb{Z})_{\varepsilon\eta}$ where $\varepsilon = \varepsilon(k)$ and $\eta = (-1)^{k-1}$. This orbit is an invariant of f which does not depend on any choice. The following theorem was proved by Alan Durfee [33].

THEOREM 4.6. For $k \geq 4$ there is a one-one correspondence of isotopy classes of fibered knots in S^{2k-1} and equivalence classes of integral unimodular bilinear forms given by associating to each fibered knot its linking form.

In view of 4.4.5 we have the following corollary:

COROLLARY 4.7. For $k \neq 3$ the diagrams of a germ $f:(\mathbb{C}^k,0) \to (\mathbb{C},0)$ determine f up to topological equivalence.

There is a notion of equivalence of germs of functions which may be a bit finer than topological equivalence, although I do not know an example which would show that it is really finer. This equivalence is usually called μ-homotopy. I shall call it μ-equivalence. In order to define it, identify the set of germs of functions with the ring of convergent power series $\mathbb{C}\{x_1,\ldots,x_k\}$. Let m be its maximal ideal. $\mathbb{C}\{x_1,\ldots,x_k\}$ is given the coarsest topology such that all projections to $\mathbb{C}\{x_1,\ldots,x_k\}/m^r$ are continuous. This is the weak topology of coefficientwise convergence (c.f. [45] I, §3, Satz 6).

DEFINITION. Two germs $f_o,f_1 \in m \subset \mathbb{C}\{x_1,\ldots,x_n\}$ with isolated singularities are μ-equivalent if there is a continuous mapping $g:[0,1] \to m$ such that $g(0) = f_o$, $g(1) = f_1$ and the Milnor number $n(g(t))$ is constant. The μ-equivalence class of a germ f is denoted by $[f]$.

THEOREM 4.8. μ-equivalent germs are topologically equivalent for $k \neq 3$.

This theorem was proved by Lê Dũng Tráng - C.P. Ramanujam [75]. The case $k = 3$ has been an open problem for many years. B. Perron claimed to have a proof [92], but apparently this claim was withdrawn. For $k > 3$ the theorem is also a consequence of Durfee's result 4.7 and the following proposition (c.f. [14] 4.1.9).

PROPOSITION 4.9. μ-equivalent germs have the same diagrams.

Whenever one classifies certain objects, one wants more than just an enumeration. One also wants to see how the different classes are related to each other. One wants to compare different classes and one wants to know if one is simpler than the other. This might be expressed by the introduction of a partial order relation on the set of classes. For singularities first steps in this direction were taken by G.N. Tjurina [106]. However the problem is rather subtle and its treatment in the literature is sometimes superficial or even wrong. A notable exception is the thesis of D. Balkenborg - R. Bauer - F.J. Bilitewski [14], which leads us to the following sequence of definitions.

DEFINITION. Let $f,g \in m \subset \mathbb{C}\{x_1,\dots,x_k\}$ be germs of functions with isolated singularities. We say that g is simpler than f and write $g \preceq f$ if every neighbourhood of f in m contains elements of the μ-equivalence class $[g]$.

In the literature one can find an apparently similar definition where μ-equivalence is replaced by the much finer right equivalence. However this is not adequate. Right equivalence is too fine an equivalence relation for the description of the deformation phenomena in question. Note that the relation for germs defined above is not a partial order relation. Note also that the validity of the relation $g \preceq f$ obviously depends only on the μ-equivalence class $[g]$. But contrary to assertions in the literature (c.f. [103] 9.9) it does definitely not depend only on the equivalence class $[f]$. Counterexamples are given by Bauer - Balkenborg - Bilitewski [14]. In view of this, we have at least two possibilities of introducing a relation of this kind for μ-equivalence classes.

DEFINITION. The relations $\underset{A}{\leq}$ and $\preceq$ for μ-equivalence classes $[f_o],[g]$ are defined as follows:

(i) $[g] \underset{A}{\leq} [f_o] \Leftrightarrow \forall f \in [f_o] \quad g \preceq f$.

(ii) $[g] \preceq [f_o] \Leftrightarrow \exists f \in [f_o] \quad g \preceq f$.

Relation $\underset{A}{\leq}$ has the advantage of being a partial order relation, but it has the disadvantage of ignoring the subtle phenomenon discovered by Balkenborg - Bauer - Bilitewski. Relation $\preceq$ takes this phenomenon into account, but it is not a partial order relation. Examples show that it is not transitive. But this can be repaired by the following final definition.

DEFINITION. The partial order relation $\underset{B}{\leq}$ for μ-equivalence classes $[f]$ and $[g]$ is defined as follows. $[g] \underset{B}{\leq} [f]$ iff there exists a chain of relations of μ-equivalence classes

$$[g] = [f_o] \preceq [f_1] \preceq \dots \preceq [f_r] = [f] .$$

So we have two partial order relations. Relation $\underset{A}{\leq}$ is the one used by V.I. Arnold, and I propose $\underset{B}{\leq}$. Of course $[g] \underset{A}{\leq} [f]$ implies $[g] \underset{B}{\leq} [f]$, but the converse does not hold. Both partial order relations are compatible with the partial order relations for braid group orbits of bases and diagrams introduced in §3. In fact, one has to use an obvious analogue of these partial order relations for orbits of $(\{\pm1\}^n \times J_n) \rtimes B_n$ and $\{\pm1\}^n \rtimes B_n$. The following result is due to G.N. Tjurina [106] and D. Siersma [103].

THEOREM 4.10. Let $[f]$ and $[g]$ be μ-equivalence classes of germs of functions with k variables. Let n,m be the Milnor numbers and L,M be Milnor lattices of f,g. Let $\delta[f]$ and $\delta[g]$ be the sets of diagrams of $[f]$ and $[g]$. These are orbits of $\{\pm1\}^n \rtimes B_n$ and $\{\pm1\}^m \rtimes B_m$ respectively. Then the relation $[g] \underset{B}{\leq} [f]$ implies that there is a primitive embedding $M \subseteq L$ such that the $(\{\pm1\}^m \times J_m) \rtimes B_m$-orbit of distinguished bases of M is less or equal to the $(\{\pm1\}^n \times J_n) \rtimes B_n$-orbit of distinguished bases of L. In particular the following implication holds:

$$\boxed{[g] \underset{B}{\leq} [f] \Rightarrow \delta[g] \leq \delta[f]}$$

This theorem shows the relevance of the embedding problem for diagrams for the problem of determining the "adjacency"-relations $\underset{B}{\leq}$ for singularities. In [25] I have conjectured the converse of the implication 4.10. However Bauer-Balkenborg-Bilitewski found a counterexample: The diagram S_{239} of Arnold's μ-equivalence class E_{14} is a subdiagram of a diagram with one more vertex, which belongs to Arnold's μ-equivalence class $S^{\#}_{1,1}$. But the relation $E_{14} \underset{B}{\leq} S^{\#}_{1,1}$ does not hold (c.f. [14] p.127).

In the present context the following problems are obviously important:

(A) Determination of μ-equivalence classes of singularities.

(B) Determination of diagrams of these classes.

(C) Determination of adjacency relations between classes of singularities.

An enormous amount of computational work has been done in connection with these problems. By far the greatest contribution to (A) has been made by V.I. Arnold [7],[8]. In particular, he has determined the classes of all 0-modular, 1-modular and 2-modular singularities. The 0-modular singularities are by definition the simple singularities. They are classified as the classes A_k, D_k, E_6, E_7, E_8. The classical Coxeter diagrams of these types are diagrams for these singularities. These singularities have been discovered again and again during the past 100 years. Their equations occur already in a paper of H.A. Schwarz on the hypergeometric series of Gauss [101] and in F. Klein's lectures on the icosahedron. After that they were discovered by P. DuVal [35], by D. Kirby [62], by M. Artin [13] and by V.I. Arnold [6]. In addition to the characterization of the simple singularities found by these authors, there are many others, see A. Durfee [34] for a survey. The simple singularities are basic entities of the mathematical world, like Platonic solids or root systems, simple Liegroups, and they are related to all these objects (see P. Slodowy [104] for a survey). The μ-equivalence classes of 1-modular singularities are of three types: simply elliptic singularities with diagrams T_{333}, T_{244}, T_{236}, cusp singularities with diagrams T_{pqr}, where $1/p+1/q+1/r<1$, and 14 exceptional classes with diagrams S_{pqr}. All these singularities have been thoroughly investigated from many points of view, but this work is beyond the scope of my report.

Concerning problem (B), several methods for computing diagrams have been developped and applied to particular cases by N. A'Campo [3], A.M. Gabrielov [40],[41],[42], S.M. Gusein-Zade [48] and A. Hefez - F. Lazzeri [52]. W. Ebeling found "normal forms" of diagrams for many classes of singularities, which reflect to a certain extent the structure of V.I. Arnold's classification, where certain infinite series of classes form families such as the T_{pqr}-series (c.f. [36],[38]).

Concerning problem (C), many authors have contributed to the determination of at least some adjacency relations. The most comprehensive work on this problem based on an enormous amount of computations is that of Bauer - Balkenborg - Bilitewski [14].

The definitions and results quoted above establish a close link between orbits and invariants of actions of the braid groups B_n on cartesian products $\Delta_\varepsilon(L)^n$ of the automorphic sets $\Delta_\varepsilon(L)$ associated to lattices and invariants of classes of singularities such as geometrically distinguished bases of Milnor lattices, Coxeter diagrams, monodromy groups, classical monodromy, variation operator and linking matrix of the Milnor fibration. Certain of the familiar relations between these invariants of singularities are simply special cases of the corresponding relations in the general context of the automorphic set $\Delta_\varepsilon(L)$. However the invariants associated to singularities do have special properties. I shall finish my report by quoting the most important ones. There may be others which we do not yet know.

<u>THEOREM 4.11.</u> Let f be a germ of a function with an isolated singularity which is different from an ordinary double point with an even number of variables. Let L be a Milnor lattice of f as above and let $\Delta_e \subset \Delta_\varepsilon(L)$ be the automorphic subset belonging to any geometrically distinguished basis $e \in \Delta_\varepsilon(L)^n$. Then this "set of vanishing cycles" $\Delta = \Delta_e$ is independent on the choice of e, and (L,Δ) is a vanishing lattice. In particular Δ is

very homogeneous, i.e. the monodromy group acts transitively on Δ . Therefore Δ is irreducible and the Coxeter diagrams of the singularity are connected.

This theorem is essentially due to F. Lazzeri [73], who noticed that this is a consequence of the irreducibility of the discriminant (c.f. A.M. Gabrielov [43] and E. Looijenga [80] 7.8). Because of 4.11 one can apply theorems 3.21 and 3.22. In particular this leads to the following theorem of W. Ebeling [37]. (It was noticed by E. Looijenga that the last statement (ii) is a consequence of Ebeling's result (i).)

THEOREM 4.12. Let f be a germ of a function of an odd number k of variables. Assume that f is not a cusp singularity with a diagram T_{pqr} , where $1/p+1/q+1/r<1$ and $(p,q,r) \neq (3,3,4),(2,4,5),(2,3,7)$. Let L be a Milnor lattice of f , let $\Gamma \subset O(L)$ be the monodromy group, $\Delta \subset \Delta_\varepsilon(L)$ the set of vanishing cycles and $\varepsilon = k(k-1)$. Then

(i) $\Gamma = O^*_\varepsilon(L)$

(ii) $\Delta = \{a \in \Delta_\varepsilon(L) \mid \langle a,L\rangle = \mathbb{Z}\}$

In (ii) it is assumed that f is not an ordinary double point.

The next theorem, which is sometimes called the monodromy theorem, answers a question of J. Milnor [87], p.72. This theorem and also various global versions were proved by several authors and by widely different methods (c.f. A. Landman [72], A. Grothendieck, SGA 7,I, Exposé I [47], C.H. Clemens [28], E. Brieskorn [21], N. Katz [61], A. Borel, W. Schmid [100], N. A'Campo [4]).

THEOREM 4.13. The classical monodromy of a germ of a function with an isolated singularity is quasiunipotent, i.e. its eigenvalues are roots of unity.

For irreducible singularities of plane curves the monodromy is not only quasiunipotent, but even of finite order (Lê Dũng Tráng [74]). But for reducible singularities of plane curves the monodromy is not semisimple in general (N. A'Campo [2]). The size of the blocks in the Jordan normal form has been estimated by several authors and by different methods. Our last theorem was proved by N. A'Campo [1].

THEOREM 4.14. The classical monodromy of a germ of a function of k variables with an isolated singularity has trace $(-1)^k$.

Let me finish with a few words on complete intersections, i.e. germs of maps $(\mathbb{C}^k,0) \to (\mathbb{C}^q,0)$ with fibres of codimension $q>1$ with isolated singularities. They are treated in an important forthcoming paper of W. Ebeling

[39]. The basic approach of Lefschetz can still be applied in this more general situation. One has the Milnor lattice L, the monodromy group $\Gamma \subset Aut(L)$ and the set of vanishing cycles $\Delta \subset L$. One also constructs geometrically distinguished systems of generators of L, but now these do not form a basis of L. Therefore the situation is more complicated than for $q = 1$. But the braid groups still operate on these systems of generators, and one can define and compute invariants of the orbits such as diagrams and monodromy groups. By developping this general theory and analyzing these data very carefully for a number of special classes of singularities Ebeling finally arrives at a perfect analogue of theorem 4.12 for complete intersections. Let me close this report by expressing my admiration for this beautiful work of Wolfgang Ebeling.

POSTSCRIPTUM. In §1 we introduced the groups $T(F)$. These groups are interesting in the context of singularity theory, because there is a relation between these groups and so called "weakly distinguished bases" of Milnor lattices which is analogous to the relation between the braid groups and geometrically distinguished bases. It is in this context that S.P. Humphries [54] proved that $T(F)$ is generated by the elements $\varkappa_{ij}$. The work of W. Ebeling [36] shows that weakly distinguished bases are really a relatively weak invariant (see also E. Voigt [109]). This is why I did not mention them in this report. On the other hand, the group $T(F)$ is interesting in itself (see S.P. Humphries [54],[55]). Therefore it is also interesting to have a presentation of $T(F)$. Just a few days ago, Aleksandar Lipkovski and Sava Krstič from the university of Beograd kindly sent me a preprint of J. McCool [86], where such a presentation is proved. Here it is:

$$\varkappa_{ij}\varkappa_{km} = \varkappa_{km}\varkappa_{ij} \qquad \text{where} \quad \#(i,j,k,m) = 4 \ ,$$

$$\varkappa_{ij}\varkappa_{ik} = \varkappa_{ik}\varkappa_{ij} \qquad \text{where} \quad \#(i,j,k) = 3 \ ,$$

$$\varkappa_{ij}(\varkappa_{ki}\varkappa_{kj}) = (\varkappa_{ki}\varkappa_{kj})\varkappa_{ij} \qquad \text{where} \quad \#(i,j,k) = 3 \ .$$

BIBLIOGRAPHY

1. N. A'Campo: "Le nombre de Lefschetz d'une monodromie", Nederl. Akad. Wetensch. Indag. Math., 35, No 2 (1973), 113-118.

2. N. A'Campo: "Sur la monodromie des singularités isolées d'hypersurfaces complexes", Invent. Math., 20 (1973), 147-169.

3. N. A'Campo: "Le groupe de monodromie du déploiement des singularités isolées de courbes planes. I", Math. Ann., 213 (1975), 1-32 and II, Actes du Congrès Int. des Math. Vancouver 1974, 395-404.

4. N. A'Campo: "La fonction zêta d'une monodromie", Comment. Math. Helv., 50 (1975), 233-248.

5. V.I. Arnold: "Remark on the branching of hyperelliptic integrals as functions of the parameters", Funktsional. Anal. i Prilozhen. 2, No. 3 (1968), 1-3, translated in Functional Anal. Appl. 2 (1968), 187-189.

6. V.I. Arnold: "Normal forms for functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities", Funktsional. Anal. i Prilozhen. 6, No. 4 (1972), 3-25, translated in Functional Anal. Appl., 6 (1972), 254-272.

7. V.I. Arnold: "Critical points of smooth functions", Proc. Internat. Congress of Mathematicians, Vancouver 1974, 19-39.

8. V.I. Arnold: Singularity theory. Selected papers. London Math. Soc. Lecture Note Ser. 53, Cambridge Univ. Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1981.

9. V.I. Arnold: "On some problems in singularity theory", in: Geometry and Analysis (Papers dedicated to the memory of V.K. Patodi), Bombay 1980. Also in Proc. Indian Acad. Sci. Math. Sci. 9 (1981), 1-9.

10. V.I. Arnold - S.M. Gusein-Zade - A.N. Varchenko: Singularities of differentiable maps, Vol. I, Birkhäuser, Boston, Basel, Stuttgart 1985.

11. E. Artin: "Theorie der Zöpfe", Abh. Math. Sem. Univ. Hamburg, 4 (1925), 101-126.

12. E. Artin: "Theory of braids", Ann. of Math. (2), 48 (1947), 101-126.

13. M. Artin: "On isolated rational singularities of surfaces", Amer. J. Math., 88 (1966), 129-136.

14. D. Balkenborg - R. Bauer - F.J. Bilitewski: "Beiträge zur Hierarchie der bimodularen Singularitäten", Thesis, University of Bonn 1984.

15. J.S. Birman: Braids, links and mapping class groups, Ann. of Math. Stud. 82, Princeton Univ. Press, Princeton, N.J., 1974.

16. J. Birman - B. Wajnryb: "Markov classes in certain finite quotients of Artin's braid group", preprint.

17. N. Bourbaki: Algèbre, Chapitre 4, Polynomes et fractions rationnelles, Chapitre 5, Corps commutatifs, Hermann, Paris, 1959.

18. N. Bourbaki: Algèbre, Chapitre 9, Formes sesquilinéaires et formes quadratiques, Hermann, Paris, 1959.

19. N. Bourbaki: Groupes et algèbres de Lie, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

20. E. Brieskorn: "Beispiele zur Differentialtopologie von Singularitäten", Invent. math. 2 (1966), 1-14.

21. E. Brieskorn: "Die Monodromie der isolierten Singularitäten von Hyperflächen", Manuscripta Math., 2 (1970), 103-161.

22. E. Brieskorn: "Singular elements of semi-simple algebraic groups", Actes du congrès int. des mathématiciens, Nice 1970, Tome 2, 279-284, Gauthier-Villars, Paris 1971.

23. E. Brieskorn: "Sur les groupes de tresses (d'après V.I. Arnold)", Séminaire N. Bourbaki 1971/1972, Exposé 401, Lecture Notes in Math. 317, Springer-Verlag, Berlin - Heidelberg - New York, 1973.

24. E. Brieskorn - K. Saito: "Artin-Gruppen und Coxeter-Gruppen", Invent. math., 17 (1972), 245-271.

25. E. Brieskorn: "Milnor lattices and Dynkin diagrams", Proc. Sympos. Pure Math., 40 (1983), Part 1, 153-165.

26. R.W. Carter: "Conjugacy classes in the Weyl group", Compositio Math., 25 (1972), 1-59.

27. W.L. Chow: "On the algebraic braid group", Ann. of Math. (2), 49 (1948), 654-658.

28. C.H. Clemens: "Picard-Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities", Trans. Amer. Math. Soc., 136 (1969), 93-108.

29. D.B. Cohen: "The Hurwitz monodromy group", J. Algebra, 32 (1974), 501-517.

30. P. Deligne: "Les immeubles des groupes de tresses généralisés", Invent. math., 17 (1972), 273-302.

31. P. Deligne - N. Katz: Séminaire de géométrie algébrique du Bois-Marie, SGA 7 II, "Groupes de Monodromie en géométrie algébrique", Lecture notes in Math. 340, Springer-Verlag, Berlin - Heidelberg - New York, 1973.

32. P. Deligne: Letter to E. Looijenga of March 9, 1974.

33. A. Durfee: "Fibered knots and algebraic singularities", Topology 13 (1974), 47-59.

34. A.H. Durfee: "Fifteen characterizations of rational double points and simple critical points", Enseign. Math., 25 (1979), 131-136.

35. P. DuVal: "On isolated singularities of surfaces which do not affect the conditions of adjunction I,II,III, Proc. of the Cambridge Philosophical Soc., 30 (1934), 453-459, 460-465, 483-491.

36. W. Ebeling: "Milnor lattices and geometric bases of some special singularities", Enseign. Math., 29 (1983), 263-280.

37. W. Ebeling: "An arithmetic characterization of the symmetric monodromy groups of singularities", Invent. math., 77 (1984), 85-99.

38. W. Ebeling - C.T.C. Wall: "Kodaira singularities and an extension of Arnold's strange duality", Compositio Math., 56 (1985), 3-77.

39. W. Ebeling: "Vanishing lattices and monodromy groups of isolated complete intersection singularities", to appear.

40. A.M. Gabrielov: "Intersection matrices for certain singularities", Funktsional. Anal. i Prilozhen., 7 (1973), 18-23, translated in Functional Anal. Appl. 7 (1973), 182-193.

41. A.M. Gabrielov: "Dynkin diagrams of unimodal singularities", Funktsional. Anal. i Prilozhen., 8 (1974), 1-6, translated in Functional Anal. Appl., 8 (1974), 192-196.

42. A.M. Gabrielov: "Polar curves and intersection matrices of singularities", Invent. math., 54 (1979), 15-22.

43. A.M. Gabrielov: "Bifurcations, Dynkin diagrams and modality of isolated singularities", Funktsional. Anal. i Prilozhen., 8, No. 2 (1974), 7-12, translated in Functional Anal. Appl., 8 (1974), 192-196.

44. F.A. Garside: "The braid group and other groups", Quart. J. Math. Oxford Ser. (2), 20 (1969), 235-254.

45. H. Grauert - R. Remmert: Analytische Stellenalgebren, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

46. H. Grauert: "Über die Deformation isolierter Singularitäten analytischer Mengen", Invent. math., 15 (1972), 171-198.

47. A. Grothendieck: Séminaire de géométrie algébrique du Bois-Marie 1967-1969, SGA 7 I "Groupes de monodromie en géométrie algébrique", Lecture Notes in Math. 288, Springer-Verlag, Berlin - Heidelberg - New York, 1972.

48. S.M. Gusein-Zade: "Dynkin diagrams of singularities of functions of two variables", Funktsional. Anal. i Prilozhen., 8, No. 4 (1974), 23-30, translated in Functional Anal. Appl. 8 (1974), 295-300.

49. S.M. Gusein-Zade: "The monodromy groups of isolated singularities of hypersurfaces", Uspekhi Mat. Nauk, 32 (1977), 23-65, translated in Russian Math. Surveys 32 (1977), 23-69.

50. H.A. Hamm: "Lokale topologische Eigenschaften komplexer Räume", Math. Ann., 191 (1971), 235-252.

51. F. Hirzebruch: "Singularities and exotic spheres", Séminaire N. Bourbaki 1966/67, Exposé n° 314.

52. A. Hefez - F. Lazzeri: "The intersection matrix of Brieskorn singularities", Invent. math., 25 (1974), 143-157.

53. R. Horowitz: "Induced automorphisms on Fricke characters of free groups", Trans. Amer. Math. Soc., 208 (1975), 41-50.

54. S.P. Humphries: "On weakly distinguished bases and free generating sets of free groups", Quart. J. Math. Oxford Ser. (2), 36 (1985), 215-219.

55. S.P. Humphries: "Graphs and Nielsen transformations of symmetric, orthogonal and symplectic groups", Quart. J. Math. Oxford Ser. (2), 36 (1985), 297-313.

56. A. Hurwitz: "Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten", Math. Ann., 39 (1891), 1-61.

57. W.A.M. Janssen: "Skew-symmetric vanishing lattices and their monodromy groups", Math. Ann., 266 (1983), 115-133.

58. W.A.M. Janssen: "Skew-symmetric vanishing lattices and their monodromy groups. II, Math. Ann., 272 (1985), 17-22.

59. D. Joyce: "A classifying invariant of knots, the knot quandle", J. Pure Appl. Algebra, 23 (1982), 37-65.

60. A. Kas - M. Schlessinger: "On the versal deformation of a complex space with an isolated singularity", Math. Ann., 196 (1972), 23-29.

61. N. Katz: "Nilpotent connections and the monodromy theorem. Applications of a result of Turritin", Inst. Hautes Études Sci. Publ. Math. 39 (1970), 175-232.

62. D. Kirby: "The structure of an isolated multiple point of a surface I, II, Proc. London Math. Soc. (3), 6 (1956), 597-609 and 7 (1957), 1-28.

63. M. Kneser: "Erzeugung ganzzahliger orthogonaler Gruppen durch Spiegelungen", Math. Ann., 255 (1981), 453-462.

64. F. Klein: "Über Flächen dritter Ordnung", Math. Ann. 6 (1873), 551-581. (also in: F. Klein, Gesammelte Mathematische Abhandlungen, Bd II, 11-44, with additions p.44-62, J. Springer, Berlin 1923, Reprint: Springer 1973)

65. F. Klein: Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, B.G. Teubner, Leipzig 1884.

66. P. Kluitmann: "Hurwitz action and finite quotients of braid groups", to appear in the Proceedings of the Conference on Artin's braid groups held at the University of California in Santa Cruz, 1986.

67. P. Kluitmann: "Ausgezeichnete Basen erweiterter affiner Wurzelgitter", Ph.D.thesis, University of Bonn, 1986.

68. H. Knörrer: "Zum $K(\pi,1)$-Problem für isolierte Singularitäten von vollständigen Durchschnitten", Compositio Math., 45 (1982), 333-340.

69. B. Krüger: "Zur Operation der Artinschen Zopfgruppe auf symmetrischen $(n \times n)$-Matrizen, deren sämtliche Diagonalelemente gleich 2 sind, insbesondere im Fall $n = 3$ ", Thesis, University of Bonn, 1986.

70. K. Lamotke: "Die Homologie isolierter Singularitäten", Math. Z., 143 (1975), 27-44.

71. K. Lamotke: "The topology of complex projective varieties after S. Lefschetz", Topology 20 (1981), 15-51.

72. A. Landman: "On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities", Trans. Amer. Math. Soc., 181 (1973), 89-126. (with an appendix by Ph.A. Griffiths).

73. F. Lazzeri: "A theorem on the monodromy of isolated singularities", in: "Singularités à Cargèse", Astérisque 7 et 8 (1973), 269-275, Soc. Math. de France.

74. Lê Dũng Tráng: "Sur les noeuds algébriques", Compositio Math., 25 (1972), 281-321.

75. Lê Dũng Tráng - C.P. Ramanujam: "The invariance of Milnor's number implies the invariance of the topological type", Amer. J. Math., 98 (1976), 67-78.

76. Lê Dũng Tráng: "The geometry of the monodromy", in: C.P. Ramanujam - A Tribute, Studies in Mathematics, No. 8, Tata Institute of Fundamental Research, Bombay, 1978.

77. S. Lefschetz: L'analysis situs et la géométrie algébrique, Gauthier-Villars et Cie, Paris 1924.

78. J. Levine: "Polynomial invariants of knots of codimension two", Ann. of Math. (2), 84 (1966), 537-554.

79. E. Looijenga: "The complement of the bifurcation variety of a simple singularity", Invent. Math., 23 (1974), 105-116.

80. E.J.N. Looijenga: Isolated singular points on complete intersections, London Math. Soc. Leture Note Ser. 77, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1984.

81. R.C. Lyndon - P.E. Schupp: Combinatorial group theory, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

82. W. Magnus - A. Karrass - D. Solitar: Combinatorial group theory, Second revised edition, Dover Publications. Inc., New York, 1976.

83. W. Magnus - A. Peluso: "On a theorem of V.I. Arnol'd", Comm. Pure Appl. Math., 22 (1969), 683-692.

84. W. Magnus: "Braids and Riemann surfaces", Comm. Pure Appl. Math., 25 (1972), 151-162.

85. W. Magnus: "Rings of Fricke characters and automorphism groups of free groups", Math. Z., 170 (1980), 91-103.

86. J. McCool: "On basis-conjugating automorphisms of free groups", preprint, Dept. of Mathematics, Univ. of Toronto.

87. J. Milnor: Singular points of complex hypersurfaces, Ann. of Math. Stud. 61, Princeton University Press and the University of Tokyo Press, Princeton, N.J., 1968.

88. B. Mitchell: Theory of Categories, Academic Press, New York and London, 1965.

89. L.J. Mordell: Diophantine Equations, Academic Press, New York - London, 1969.

90. V.V. Nikulin: "Integral symmetric bilinear forms and some of their applications", Izv. Akad. Nauk SSSR Ser. Mat., 43 (1979), 111-177, translated in Math. USSR-Izv., 14 (1980), 103-166.

91. P. Orlik (Editor): "Proceedings of the summer institute on singularities held at Humboldt State University, Arcata, California, July 20 - August 7, 1981", Proc. Sympos. Pure Math., 40 (1983).

92. B. Perron: " "μ constant" implique "type topologique constant" en dimension complexe trois", Université de Dijon, preprint, withdrawn.

93. F. Pham: "Formules de Picard-Lefschetz généralisées et ramification des intégrales", Bull. Soc. Math. France, 93 (1965), 333-367.

94. F. Pham: "Tresses des fonctions algébriques d'après V. Arnold" and "Formules de Picard-Lefschetz", Séminaire Leray, Collège de France, Exposés faits le 5 mars et le 12 mars 1969.

95. E. Picard - G. Simart: Théorie des fonctions algébriques de deux variables indépendantes, Tome I, II, Gauthier-Villars, Paris, 1897, 1906.

96. H. Poincaré: "Sur les cycles des surfaces algébriques; quatrième complément à l'analysis situs", Journal de Mathématiques 8 (1902), 169-214 (Oeuvres de Henri Poincaré, Tome VI, 397-434, Gauthier-Villars, Paris, 1953).

97. G. Rosenberger: "Fuchssche Gruppen, die freies Produkt zweier zyklischer Gruppen sind, und die Gleichung $x^2 + y^2 + z^2 = xyz$, (Auszug aus der Dissertation), Math. Ann., 199 (1972), 213-227.

98. K. Saito: "Extended affine root systems I (Coxeter transformations)", Publ. Res. Inst. Math. Sci. Kyoto Univ., 21 (1985), 75-179.

99. J. Scherk: "A propos d'un théorème de Mather et Yau", C.R. Acad. Sci. Paris, Sér. I Math., 296 (1983), 513-515.

100. W. Schmid: "Variation of Hodge structure: the singularities of the period mapping", Invent. Math., 22 (1973), 211-319.

101. H.A. Schwarz: "Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elements darstellt", J. Reine Angew. Math., 75 (1872), 292-335.

102. Ch.A. Scott: "The nature and effect of singularities of plane algebraic curves", Amer. J. Math., 15 (1893), 221-243.

103. D. Siersma: "Classification and deformation of singularities", Thesis, University of Amsterdam, 1974.

104. P. Slodowy: "Platonic solids, Kleinian singularities and Lie groups", in "Algebraic Geometry", Proceedings 1981, edited by I. Dolgachev, Lecture Notes in Math. 1008, Springer-Verlag, Berlin - Heidelberg - New York - Tokyo, 1983.

105. R. Thom: Stabilité structurelle et morphogénèse, W.A. Benjamin Inc., Reading, Mass., 1972.

106. G.N. Tjurina: "The topological properties of isolated singularities of complex spaces of codimension one", Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 605-620, translated in Math. USSR-Izv. (1968), 557-571.

107. G.N. Tjurina: "Locally semi-universal flat deformations of isolated singularities of complex spaces", Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969), 1026-1058, translated in Math. USSR-Izv. 3 (1969), 967-1000.

108. H. Van der Lek: "The homotopy type of complex hyperplane complements", Ph.D.Thesis, University of Nijmwegen, 1983.

109. E. Voigt: "Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten", Abh. Math. Sem. Univ. Hamburg, 55 (1985), 183-190. Details in 109', "Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten", Bonner Mathematische Schriften 160 (1985).

110. G. Wassermann: Stability of unfoldings, Lecture Notes in Math. 393, Springer-Verlag, Berlin - Heidelberg - New York, 1974.

111. K. Wirthmüller: "Singularities determined by their discriminant", Math. Ann., 252 (1980), 237-245.

MATHEMATISCHES INSTITUT
UNIVERSITÄT BONN
D-5300 BONN 1