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THE TOPOLOGICAL CLASSIFICATION OF THE LENS SPACES

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The n-dimensional lens spaces can be classified under semi-linear equivalence by the torsion invariant of Reidemeister [1]. It seems desirable to prove the topological invariance of this classification without reference to the Hauptvermutung. In this paper we carry out such a proof for $n = 3$, using a method which, intuitively speaking, distinguishes manifolds by the differences in their "knot theories". This method was proposed by Fox [16, p. 247] at the Princeton Bicentennial Conference of 1946, and later applied [17, p. 455] to re-prove the semi-linear classification of the lens spaces.

Associated with every simple closed polygon k in a complex K there is a principal ideal (Δ) of the 1-dimensional Betti group ring of $K - k$; Δ is known as the Alexander polynomial of k. [2, 3, 4, 5]. Let M be an orientable triangulable 3-dimensional manifold, γ an element of $H_1(M)$, k a simple closed curve which represents γ and which is polygonal in some triangulation of M. Let Δ be the Alexander polynomial of k, $T_1(M - k)$ and $B_1(M - k)$ the torsion subgroup and Betti group of $H_1(M - k)$, i: $H_1(M - k) \to H_1(M)$ the injectionhomomorphism, and $*: B_1(M - k) \to H_1(M)/iT_1(M - k)$ the homomorphism induced by i. Then we prove:

(i) $H_1(M - k)$ and $iT_1(M - k)$ depend only upon γ;

(ii) Δ^* depends only upon γ.

Unlike the Reidemeister torsion, the invariant Δ^* is applicable to all orientable 3-dimensional manifolds. However, if one is interested only in the lens spaces, one may restrict oneself to the case $H_1(M - k)$ infinite cyclic, which allows considerable simplification of the proof (cf., §4.). As an additional example we givė the classification of the topological sums of two 3-dimensional lens spaces.

In what follows, *manifold* will mean compact orientable 3-dimensional manifold in the sense of [6]; similarly for *bounded manifold*. The term *polygonal curve* will mean polygonal in *some triangulation* of the given space; it should be emphasized that the assumption that two or more curves are polygonal *does not* necessarily postulate their polygonality in the *same triangulation* unless specifically stated. The reader is also cautioned that all groups, including homology groups, will be written multiplicatively to avoid confusion with group ring notation. Also, homomorphisms will frequently be written as superscripts, that is, $f(x)$ as x^f and $f(x^{-1})$ as x^{-f}.

1. For the basic notions of group presentations, Jacobians, and elementary ideals we refer to [5] and [7]. Let $(X: R)$ be a finitely generated presentation of a group G, φ the associated homomorphism of X onto G, and ψ the natural homomorphism of G onto its Betti group B (i.e., the free part of the commutator quotient group of G). By the *Alexander matrix* of G we mean the Jacobian matrix of $(X: R)$ evaluated at ψ. The Betti group ring JB is essentially a polynomial ring and has the unique factorization property; every subset of JB has a greatest common divisor (g.c.d.) determined up to a unit factor of JB. The *Alexander polynomial* Δ of G is the greatest common divisor of the first elementary ideal $\mathfrak{E}$ of the Alexander matrix.

DEFINITION. A group presentation having m generators and n relators is said to have *deficiency* $m-n$. A finitely presented group has *deficiency* r if it has a presentation of deficiency r, but no presentation of deficiency $r+1$.

LEMMA 1.1. *Let $(X: R)$ be a presentation of deficiency one of a group G, B the Betti group of G, $\Delta^{(j)}$ the determinant of the minor obtained by deleting the j^{th} column of the Alexander matrix A of $(X: R)$. Let $\mathfrak{F}$ be the fundamental ideal*[1] *of JB and let $\delta = \text{g.c.d. } \mathfrak{F}$. Then $\delta \cdot \Delta^{(j)} = (x_j - 1)^{\psi\varphi} \cdot \Delta$, where x_j is the generator of G corresponding to the j^{th} column.*

PROOF. Denote by ξ_j the j^{th} column of A. By the fundamental formula (see [5], 2.3) of the free calculus $\sum_j \xi_j \cdot (x_j - 1)^{\psi\varphi} = 0$; hence

$$\begin{aligned}\Delta^{(j)} \cdot (x_{k-1})^{\psi\varphi} &= |\xi_1, \cdots, \hat{\xi}_j, \cdots, \xi_k \cdot (x_k - 1)^{\psi\varphi}, \cdots, \xi_\nu| \\ &= -|\xi_1, \cdots, \hat{\xi}_j, \cdots, \xi_j(x_j - 1)^{\psi\phi}, \cdots, \xi_\nu| \\ &= (-1)^{j-k} \cdot |\xi_1, \cdots, \xi_j(x_j - 1)^{\psi\varphi}, \cdots, \hat{\xi}_k, \cdots, \xi_\nu| \\ &= (-1)^{j-k} \cdot (x_j - 1)^{\psi\varphi} \cdot \Delta^{(k)}\end{aligned}$$

where $\hat{\xi}_j$ denotes deletion of ξ_j. Therefore

$$\Delta^{(j)} \cdot \delta = \text{g.c.d.}_k \, \Delta^{(j)} \cdot (x_k - 1)^{\psi\varphi} = \text{g.c.d.}_k \, \Delta^{(k)} \cdot (x_j - 1)^{\psi\varphi} = \Delta \cdot (x_j^{\psi\varphi} - 1) .$$

We remark that if $p(B)$ denotes the rank of B, we have $\delta = 0$ for $p(B) = 0$, $\delta = 1$ for $p(B) > 1$, and for $p(B) = 1$, $\delta = b - 1$ where b is a generator of B.

LEMMA 1.2. *Let $\mathfrak{E}$ be the first elementary ideal of the Alexander matrix A of a finitely generated group presentation. Let $\mathfrak{E}^{(j)}$ be the 0^{th} elementary ideal of the submatrix of A obtained by deleting the j^{th}*

[1] See [5]. We recall that $\mathfrak{F}$ is generated by the elements of $b - 1$ when b ranges over any set of generators of B.

column. Then

$$\mathfrak{F} \cdot \mathfrak{E}^{(j)} = (x_j^{\psi\varphi} - 1) \cdot \mathfrak{E}$$

PROOF. If A has n columns, let α be any set of $n-1$ rows. Then, as in the proof of 1.1,

$$(x_k - 1)^{\psi\varphi} \cdot \Delta_\alpha^{(j)} = (x_j - 1)^{\psi\varphi} \cdot \Delta_\alpha^{(k)} .$$

"Summing" over α

$$(x_k - 1)^{\psi\varphi} \cdot \mathfrak{E}^{(j)} = (x_j - 1)^{\psi\varphi} \cdot \mathfrak{E}^{(k)}$$

and the result now follows by summing over k.

THEOREM 1.3. *Let G_0, G_1, G_2 be groups with presentations*

$$G_1 = (x_1, \cdots, x_m, z_1, z_2 \colon r_1(x, z), \cdots, r_m(x, z), w(z)) ,$$

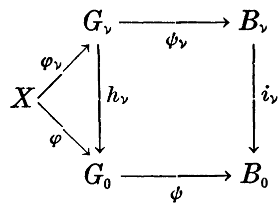
of deficiency one,

$$G_2 = (z_1, z_2, y_1, \cdots, y_n \colon w(z), s_1(y, z), s_2(y, z), \cdots)$$

with an arbitrary number (possibly infinite) of relators, and

$$G_0 = (x_1, \cdots, x_m, z_1, z_2, y_1, \cdots, y_n \colon \\ r_1(x, z), \cdots, r_m(x, z), w(z), s_1(y, z), s_2(y, z), \cdots) .$$

These presentations naturally define homomorphisms $h_\nu \colon G_\nu \to G_0$ and $i_\nu \colon B_\nu \to B_0$ for $\nu = 1, 2$, and the diagrams



$\nu = 1, 2$

are commutative. If $\mathfrak{E}_\mu$, $\mathfrak{F}_\mu$ and Δ_μ denote the first elementary ideals, fundamental ideals, and Alexander polynomials respectively, of G_μ and $\delta_\mu =$ g.c.d. $\mathfrak{F}_\mu$, then

$$\left(\frac{\partial w}{\partial z_2}\right)^{\psi\varphi} \cdot (z_1^{\psi\varphi} - 1) \cdot \delta_1^{t_1} \cdot (\text{g.c.d. } \mathfrak{F}_2^{t_2}) \cdot \Delta_0 \\ = (z_1^{\psi\varphi} - 1)^2 \cdot (\text{g.c.d. } \mathfrak{F}_0) \cdot \Delta_1^{t_1} \cdot (\text{g.c.d. } \mathfrak{S}_2^{t_2})$$

PROOF. Consider the Alexander matrices

$$A_1 = \left\| \begin{matrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial z_1} & \dfrac{\partial r}{\partial z_2} \\ 0 & \dfrac{\partial w}{\partial z_1} & \dfrac{\partial w}{\partial z_2} \end{matrix} \right\|^{\psi_1\varphi_1} , \qquad A_2 = \left\| \begin{matrix} \dfrac{\partial w}{\partial z_1} & \dfrac{\partial w}{\partial z_2} & 0 \\ \dfrac{\partial s}{\partial z_1} & \dfrac{\partial s}{\partial z_2} & \dfrac{\partial s}{\partial y} \end{matrix} \right\|^{\psi_2\varphi_2}$$

and

$$\mathbf{A}_0 = \left\| \begin{matrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial z_1} & \frac{\partial r}{\partial z_2} & 0 \\ 0 & \frac{\partial w}{\partial z_1} & \frac{\partial w}{\partial z_2} & 0 \\ 0 & \frac{\partial s}{\partial z_1} & \frac{\partial s}{\partial z_2} & \frac{\partial s}{\partial y} \end{matrix} \right\|^{\psi\varphi} .$$

Let $\mathfrak{E}_\mu^{(1)}$ denote the 0^{th} elementary ideal of the submatrix of A_μ obtained by deleting the column corresponding to z_1. Then[2]

$$\mathfrak{E}_1^{(1)} = \left(\left(\frac{\partial w}{\partial z_2}\right)^{\psi_1\varphi_1} \cdot \left|\frac{\partial r}{\partial x}\right|^{\psi_1\varphi_1}\right),$$

$$\mathfrak{E}_2^{(1)} = \left(\mathbf{U}_\alpha\left(\left(\frac{\partial w}{\partial z_2}\right)^{\psi_2\varphi_2} \cdot \left|\frac{\partial s}{\partial y}\right|_\alpha^{\psi_2\varphi_2}\right) \cup \mathbf{U}_\beta\left(\left|\frac{\partial s}{\partial z_2} \quad \frac{\partial s}{\partial y}\right|_\beta^{\psi_2\varphi_2}\right)\right)$$

$$\mathfrak{E}_0^{(1)} = \left(\mathbf{U}_\alpha\left(\left(\frac{\partial w}{\partial z_2}\right)^{\psi\varphi} \cdot \left|\frac{\partial r}{\partial x}\right|^{\psi\varphi} \cdot \left|\frac{\partial s}{\partial y}\right|_\alpha^{\psi\varphi}\right) \cup \mathbf{U}_\beta\left(\left|\frac{\partial r}{\partial x}\right|^{\psi\varphi} \cdot \left|\frac{\partial s}{\partial z_2} \quad \frac{\partial s}{\partial y}\right|_\beta^{\psi\varphi}\right)\right)$$

where the index α ranges over all sets of n rows, β over all sets of $n + 1$ rows. We conclude that[3]

$$(1) \qquad \left(\frac{\partial w}{\partial z_2}\right)^{\psi\varphi} \cdot \mathfrak{E}_0^{(1)} = ((\mathfrak{E}_1^{(1)})^{i_1} \cdot (\mathfrak{E}_2^{(1)})^{i_2})$$

By Lemmas 1.1 and 1.2,

$$\delta_1^{i_1} \cdot (\mathfrak{E}_1^{(1)})^{i_1} = (z_1^{\psi\varphi} - 1) \cdot \Delta_1^{i_1},\ \mathfrak{F}_2^{i_2} \cdot (\mathfrak{E}_2^{(1)})^{i_2} = (z_1 - 1)^{\psi\varphi} \cdot \mathfrak{E}_2^{i_2},$$
$$\mathfrak{F}_0 \cdot (\mathfrak{E}_0^{(1)}) = (z_1^{\psi\varphi} - 1) \cdot \mathfrak{E}_0 .$$

Substituting these in equation (1) we have

$$\left(\frac{\partial w}{\partial z}\right)^{\psi\varphi} \cdot (z_1^{\psi\varphi} - 1) \cdot \delta_1^{i_1} \cdot \mathfrak{F}_2^{i_2} \cdot \mathfrak{E}_0 = (z_1^{\psi\varphi} - 1)^2 \cdot \mathfrak{F}_0 \cdot \Delta_1^{i_1} \cdot \mathfrak{E}_2^{i_2}$$

and the result follows by taking the greatest common divisor of both sides.

2. By a *tube* we shall mean the topological image of an anchor ring (Vollring); more generally by *tubular manifold* we mean a Henkelkorper in the sense of [6]. A tubular manifold of genus h in S^3 is representable as a tubular neighborhood of the graph L which we call an *h-leaved rose*. The closed complement R_0 of this tubular neighborhood is also a tubular

[2] The outer parentheses here mean "ideal generated by".

[3] The homomorphic image of an ideal is not necessarily an ideal, but the product of two subsets of JB_0 is defined in the obvious way.

manifold of genus h. Hence any closed 3-dimensional manifold M' may be constructed by welding a tubular manifold R of genus h to the boundary $\dot{R}_0$ of $\overline{S-R_0}$ according to a given Heegard diagram. Any bounded 3-dimensional manifold M may be represented as the complement in a closed manifold M' of a tubular neighborhood $R_1 \cup \cdots \cup R_\mu$ of μ roses $L_1 \cup \cdots \cup L_\mu$. This system of roses may be deformed by an isotopy of M' into $M' - R$. Thus we have

LEMMA 2.1. *An arbitrary 3-dimensional bounded manifold may be obtained from the closed complement in S^3 of a tubular neighborhood $R_0 \cup R_1 \cup \cdots \cup R_\mu$ of a disjoint system $L_0 \cup L_1 \cup \cdots \cup L_\mu$ of roses by welding $\dot{R}_0$ to the boundary $\dot{R}$ of a tubular manifold of the same genus.*

The fundamental group of a surface $\dot{R}_i$ (of genus $h_i \geqq 1$) is presentable as $(x_{i1}, \cdots, x_{ih_i}, \xi_{i1}, \cdots, \xi_{ih_i}: \prod_{k=1}^{h_i} [x_{ik}, \xi_{ik}])$ where $x_{i1}, \cdots, x_{ih_i}, \xi_{i1}, \cdots, \xi_{ih_i}$ are represented by canonical curves on $\dot{R}_i$. Such a presentatio nof $\pi_1(\dot{R}_i)$ is called *canonical*, the generators are called *canonical generators*, and the relator, a *canonical relator*.

THEOREM 2.2. *Let M be an orientable 3-dimensional manifold bounded by surfaces of genus zero and surfaces S_i of genus $h_i \geqq 1 (i=1, \cdots, \mu \geqq 1)$. Then $\pi_1(M)$ has a presentation of deficiency*

$$1 + \sum_{i=1}^{\mu} (h_i - 1)$$

containing canonical presentations of $\pi_1(S_1), \cdots, \pi_1(S_{\mu-1})$.

PROOF. We prove the theorem first for the special case of a manifold M' imbedded in S^3 whose boundary surfaces are the boundaries $\dot{R}_0, \cdots, \dot{R}_\mu$ of tubular neighborhoods of roses $L_0, \cdots, L_\mu$. The group $\pi_1(M') = \pi_1(S - L_0 \cup \cdots \cup L_\mu)$ has the Wirtinger presentation with generators $x_{ijk} (i = 0, \cdots, \mu;\ j = 1, \cdots, h_i;\ k = 1, \cdots, \nu_{ij})$ and relators r_{ijk} $(i = 0, \cdots, \mu;\ j = 1, \cdots, h_i;\ k = 2, \cdots, \nu_{ij})$ and $r_i\ (i = 0, \cdots, \mu)$. The generator x_{ijk} corresponds to the k^{th} branch of the j^{th} leaf of the i^{th} rose L_i and

$$r_{ijk} = w_{ijk} x_{ijk-1} w_{ijk}^{-1} x_{ijk}^{-1}, \qquad r_i = \prod_{j=1}^{h_i} x_{ij1} x_{ij\nu_{ij}}^{-1} .$$

The relators r_{ijk} correspond to crossings, while r_i is read around the node of the i^{th} rose. This presentation has $n = \sum_{i=0}^{\mu} \left(\sum_{j=1}^{h_i} \nu_{ij} \right)$ generators and $m = \mu + 1 + \sum_{i=0}^{\mu} \sum_{j=1}^{h_i} (\nu_{ij} - 1)$ relators. Any of these relators, say r_μ, is a consequence of the others and may be deleted. Furthermore

$$(w_{ij2}^{-1} r_{ij2} w_{ij2})(w_{ij2}^{-1} w_{ij3}^{-1} r_{ij3} w_{ij3} w_{ij2}) \cdots (w_{ij2}^{-1} \cdots w_{ij\nu}^{-1} r_{ij\nu} w_{ij\nu} \cdots w_{ij2})$$
$$= x_{ij1} \xi_{ij} x_{ij\nu}^{-1} \xi_{ij}^{-1}$$

where

$$\xi_{ij} = (w_{ij\nu} w_{ij\nu-1} \cdots w_{ij2})^{-1} .$$

The elements $x_{ij1}, \xi_{ij}(j = 1, \cdots, h_i)$ represent a system of canonical curves on $\dot{R}_i$ and the corresponding canonical relator r'_i is obtained from r_i by replacing $x_{ij\nu}$ by $\xi_{ij}^{-1} x_{ij1} \xi_{ij}$ for $j = 1, \cdots, h_i$. After we have deleted the relator r_μ from the given presentation we may apply Tietze transformations of the first kind to replace r_i by r'_i for $i = 0, 1, \cdots, \mu - 1$. The deficiency of this presentation is

$$\left(\sum_{i=0}^{\mu} \sum_{j=1}^{h_i} \nu_{ij}\right) - \left(\mu + \sum_{i=0}^{\mu} \sum_{j=1}^{h_i} (\nu_{ij} - 1)\right) = 1 + \sum_{i=0}^{\mu} (h_i - 1) .$$

By Tietze operations of the second kind, which do not alter the deficiency of the presentation, we introduce $\xi_{ij} = (w_{ij\nu} \cdots w_{ij2})^{-1}$. The resulting presentation of $\pi_1(M')$ has the required properties.

By Lemma 2.1 an arbitrary bounded orientable 3-manifold M is obtained from such an M' by welding $\dot{R}_0$ with the boundary of a tubular manifold R of genus h_0; $\pi_1(R)$ has a presentation given by the generators $x_{0j1}, \xi_{0j}(j = 1, \cdots, h_0)$ and relators $\rho_{01}, \cdots, \rho_{0h_0}$ corresponding to a system of meridian 2-cells in R. The relator r'_0 is a consequence of $\rho_{01}, \cdots, \rho_{0h_0}$. Hence a presentation of $\pi_1(M)$ is obtained from the above presentation of $\pi_1(M')$ by adjoining $\rho_{01}, \cdots, \rho_{0h_0}$, and deleting r'_0. This presentation has deficiency equal to $1 + \sum_{i=1}^{\mu} (h_i - 1)$ and contains canonical presentations of $\pi_1(S_1), \cdots, \pi_1(S_{\mu-1})$.

THEOREM 2.3. *Let N be a manifold bounded by at least one surface of genus one, M a bounded manifold and T a tube with $N = M \cup T$, $\dot{T} = M \cap T$. Let $\Delta(M)$, $\Delta(N)$ be the Alexander polynomials, $\mathfrak{F}_N$ and $\mathfrak{F}_T$ the fundamental ideals of $JB_1(N)$ and $JB_1(T)$ respectively,*

$$\delta(N) = \text{g.c.d. } \mathfrak{F}_N, \qquad \delta(T) = \text{g.c.d. } \mathfrak{F}_T ,$$

then

$$\delta_N(T) \cdot \Delta(N) = \delta(N) \cdot \Delta_N(M)$$

where the subscript N denotes injection into $JB_1(N)$.

PROOF. By Theorem 2.2, $\pi_1(M)$ has a presentation

$$(x_1, \cdots, x_m, z_1, z_2 : r_1(x, z), \cdots, r_m(x, z), [z_1, z_2])$$

where z_1 and z_2 are canonical generators of $\dot{T}$. The group $\pi_1(T)$ has a presentation $(z_1, z_2 : w(z))$ where $w(z)$ is the relator corresponding to a canonical 2-cell of T. Since $[z_1, z_2]$ is a consequence of $w(z)$, $\pi_1(N)$ has the presentation

$$(x_1, \cdots, x_m, z_1, z_2 : r_1(x, z), \cdots, r_m(x, z), w(z))$$

of deficiency one. We distinguish two cases:

CASE I. Some $z_i^{\psi\varphi} \neq 1$, say z_1. From the Alexander matrix of T, using Lemma 1.1,

$$\delta(T) \cdot \left(\frac{\partial w}{\partial z_2}\right)^{\psi_2\varphi_2} = (z_1 - 1)^{\psi_2\varphi_2} \cdot \Delta(T) = (z_1 - 1)^{\psi_2\varphi_2}$$

hence in N,

$$\left(\frac{\partial w}{\partial z_2}^{\psi\varphi}\right) \cdot \delta_N(T) = (z_1 - 1)^{\psi\varphi}$$

From the Alexander matrix of M,

$$\delta(M) \cdot (z_1 - 1)^{\psi_1\varphi_1} \cdot \left|\frac{\partial r}{\partial x}\right|^{\psi_1\varphi_1} = (z_1 - 1)^{\psi_1\varphi_1} \cdot \Delta(M) ,$$

and $\delta(M) = 1$ since $p_1(M) \geqq 2$. From the matrix of N,

$$\begin{aligned}(z_1 - 1)^{\psi\varphi} \cdot \Delta(N) \cdot \delta_N(T) &= \delta(N) \cdot \delta_N(T) \cdot \left(\frac{\partial w}{\partial z_2}\right)^{\psi\varphi} \cdot \left|\frac{\partial r}{\partial x}\right|^{\psi\varphi} \\ &= \delta(N) \cdot (z_1^{\psi\varphi} - 1) \cdot \left|\frac{\partial r}{\partial x}\right|^{\psi\varphi} \\ &= \delta(N) \cdot (z_1 - 1)^{\psi\varphi} \cdot \Delta_N(M)\end{aligned}$$

hence

$$\delta_N(T) \cdot \Delta(N) = \delta(N) \cdot \Delta_N(M) .$$

CASE II. $z_1^{\psi\varphi} = z_2^{\psi\varphi} = 1$. Then $\delta_N(T) = 0$. Since $p_1(N) \geqq 1$, we must have some $x_i^{\psi\varphi} \neq 1$, say $x_1^{\psi\varphi} \neq 1$. Then from the matrix of M,

$$\begin{aligned}(x_1 - 1)^{\psi_1\varphi_1} \cdot \Delta(M) = (z_1^{\psi_1\varphi_1} - 1) &\cdot \left|\frac{\partial r}{\partial x_2} \cdots \frac{\partial r}{\partial x_m}\ \frac{\partial r}{\partial z_1}\right|^{\psi_1\varphi_1} \\ &- (1 - z_2^{\psi_1\varphi_1}) \cdot \left|\frac{\partial r}{\partial x_2} \cdots \frac{\partial r}{\partial x_m}\ \frac{\partial r}{\partial z_2}\right|^{\psi_1\varphi_1} .\end{aligned}$$

Hence $\Delta_N(M) = 0$. So the conclusion of the theorem is trivially true in this case.

We now state without proof a classical theorem of van Kampen [8] in a simplified form due to R. H. Fox [2].

THEOREM 2.4. *Let X and Y be closed connected subsets of a connected topological space $X \cup Y$ such that $X \cap Y$ is connected and is a neighborhood deformation retract of X and of Y. Let $i_X: \pi_1(X \cap Y) \to \pi_1(X)$ and $i_Y: \pi_1(X \cap Y) \to \pi_1(Y)$ be the injections, P the free product $\pi_1(X) * \pi_1(Y)$ and Q the normal subgroup of P generated by all elements $i_X(z) * i_Y(z^{-1})$ where z ranges over the set of generators of $\pi_1(X \cap Y)$. Then $\pi_1(X \cup Y) \approx P/Q$.*

In particular, using a circle to denote direct product, we have the abelianized form of 2.4:

COROLLARY 2.5. *The section*

$$H_1(X \cap Y) \xrightarrow{\psi} H_1(X) \circ H_1(Y) \xrightarrow{\varphi} H_1(X \cup Y) \longrightarrow 0$$

of the Mayer-Vietoris sequence [9] *of the triad* $(X \cup Y; X, Y)$ *is exact in the singular homology theory* (*if the triad satisfies the hypotheses of Theorem* 2.4).

Consider a tube T imbedded in a complex K. The fundamental cycle τ of the center line of T may be approximated in K by a simplicial mapping σ of a circle S^1 (i.e., simplicial in the affine structure of K) such that σ is homotopic to τ in T. Furthermore, by making small deformations we may guarantee that σ is a homeomorphism. Let T_1 be a tubular neighborhood ([6], p. 225) of the simple closed polygon $\sigma(S^1)$ in the affine structure Σ of K with $T_1 \subset$ interior T. We call T_1 an *approximation* to T in Σ.

LEMMA 2.6. *Let T be a tube in the interior of a bounded manifold M. The image of the injection i: $B_1(\dot{T}) \rightarrow B_1(M - T)$ is non-trivial.*

PROOF. Let T_1 be an approximation to T in the given affine structure of M. It is known [10] that the image of the injection $i_1: B_1(\dot{T}_1) \rightarrow B_1(M - T_1)$ is non-trivial. The injections i': $B_1(\dot{T}) \rightarrow B_1(T - T_1)$ and i_1': $B_1(\dot{T}_1) \rightarrow B_1(T - T_1)$ are isomorphisms and we have $i_1 = i \cdot i'^{-1} \cdot i_1'$.

LEMMA 2.7. *Let T' be a tube in the interior of a bounded manifold M, T'' a tube in T' such that the center line of T'' is homologous in T' to the center line of T'. Then the injection $H_1(M - T') \rightarrow H_1(M - T'')$ is an isomorphism.*

PROOF. Consider the exact sequence

$$H_1(\dot{T}') \xrightarrow{\psi} H_1(M - T') \circ H_1(T - T'') \xrightarrow{\varphi} H_1(M - T'') \longrightarrow 0$$

given by Corollary 2.5.

We have

$$\psi(x) = \psi_1(x) \circ \psi_2(x^{-1}), \qquad \varphi(y) = \varphi_1(y) \cdot \varphi_2(y) ,$$

where

$$\varphi_1: H_1(M - T') \longrightarrow H_1(M - T''), \qquad \varphi_2: H_1(T - T'') \longrightarrow H_1(M - T'') ,$$
$$\psi_1: \quad H_1(\dot{T}') \longrightarrow H_1(M - T'), \qquad \psi_2: H_1(\dot{T}') \longrightarrow H_1(T - T'')$$

are injections; clearly ψ_2 is an isomorphism. Now we have $\varphi(H_1(M - T')) = H_1(M - T'')$. For given $a \in H_1(M - T'')$, let

$$(a_1, a_2) \in H_1(M - T') \circ H_1(T - T'')$$

with $\varphi(a_1, a_2) = a$, let $b \in H_1(\dot{T}')$ with $\psi_2(b) = a_2$. Then $(a_1, a_2) \cdot \psi(b^{-1}) = (a_1', 1) \in H_1(M - T')$ and $\varphi((a_1, a_2) \cdot \psi(b^{-1})) = \varphi(a_1, a_2) = a$. Furthermore kernel $(\varphi \mid H_1(M - T')) = (\text{kernel } \varphi) \cap H_1(M - T') = (\text{image } \psi) \cap H_1(M - T') = 0$ since kernel $\psi_2 = 0$. Thus

$$\varphi_1 \colon H_1(M - T') \longrightarrow H_1(M - T'')$$

is an isomorphism.

LEMMA 2.8. *Assume the hypotheses of Lemma 2.7 and moreover that T' is smooth*[4]. *Then there exists a homomorphism Ω of $\pi_1(M - T'')$ onto $\pi_1(M - T')$ such that the diagram*

$$\begin{array}{ccc} \pi_1(M - T'') & \xrightarrow{\;\Omega\;} & \pi_1(M - T') \\ {\scriptstyle \psi''}\downarrow & & \downarrow{\scriptstyle \psi} \\ H_1(M - T'') & \xleftarrow{\;\varphi\;} & H_1(M - T') \end{array}$$

is commutative, where ψ'' and ψ are the natural homomorphisms and φ is the isomorphism of Lemma 2.7.

PROOF. Since T' is smooth we may assume that T' is a tubular neighborhood of the center line of another smooth tube T. Consider the commutative diagram

$$\begin{array}{ccccccccc} H_1(M - T) & \xleftarrow{\;\psi\;} & \pi_1(M - T) & & & & \pi_1(T - T'') & \xrightarrow{\;\bar{\bar{\psi}}\;} & H_1(T - T'') \\ {\scriptstyle \varphi_1}\uparrow & & \downarrow{\scriptstyle \Omega_1} & \nwarrow{\scriptstyle i_1} & & \nearrow{\scriptstyle i_2} & \downarrow{\scriptstyle \Omega_2} & & \uparrow{\scriptstyle \varphi_2} \\ & & & & \pi_1(\dot{T}) & & & & \\ & & & \swarrow{\scriptstyle i_1} & & \searrow{\scriptstyle j_2} & & & \\ H_1(M - T) & \xleftarrow{\;\psi\;} & \pi_1(M - T) & & & & \pi_1(T - T') & \xrightarrow{\;\bar{\psi}\;} & H_1(T - T') \end{array}$$

where $\varphi_1, \Omega_1, i_1, i_2, j_2$ are injections (in particular, φ_1 and Ω_1 are identities), $\psi, \bar{\psi}, \bar{\bar{\psi}}$ are natural homomorphisms, $\varphi_2 = (\bar{\bar{\psi}} i_2) j_2^{-1} \bar{\psi}^{-1}$ and $\Omega_2 = \bar{\psi}^{-1} \varphi_2^{-1} \bar{\bar{\psi}}$; the last two definitions are permissible because $(\bar{\bar{\psi}} i_2)$, j_2 and $\bar{\psi}$ are isomorphisms. The homomorphisms Ω_1, Ω_2 define a homomorphism $\Omega_1 * \Omega_2$ of $\pi_1(M - T) * \pi_1(T - T'')$ onto $\pi_1(M - T) * \pi_1(T - T')$. If $\tau \in \pi_1(\dot{T})$ then $\Omega_1 * \Omega_2(i_1(\tau) * i_2(\tau^{-1})) = i_1(\tau) * j_2(\tau^{-1})$. Hence we have by Theorem 2.4, $\Omega_1 * \Omega_2$ induces a homomorphism Ω of $\pi_1(M - T'')$ onto $\pi_1(M - T')$. Since the isomorphism $\varphi \colon H_1(M - T') \to H_1(M - T'')$ of Lemma 2.7 is induced by $(\varphi_1, \varphi_2) \colon H_1(M - T) \circ H_1(T - T') \to H_1(M - T) \circ H_1(T - T'')$, the remainder of the lemma follows from Corollary 2.5 and the commutativity

[4] That is, T' is contained in a concentric tube in M. A tubular neighborhood of a polygonal curve in the interior of a bounded manifold is smooth.

of the squares in the above diagram.

LEMMA 2.9. *Let T be a smooth tube in a finite complex K. Then $\pi_1(K - T)$ is finitely generated.*

PROOF. Let $T_1 \subset T_2$ be tubes concentric with and containing T. There exists a polyhedron P in a subdivision of K with $K - T \supset P \supset K - T_1$. The injection $h: \pi_1(K - T_1) \to \pi_1(K - T)$ is an isomorphism, since $K - T_1$ is a deformation retract of $K - T$. But h is the composition of the injections $i: \pi_1(K - T_1) \to \pi_1(P)$ and $j: \pi_1(P) \to \pi_1(K - T)$. Hence j is onto and since $\pi_1(P)$ is finitely generated, $\pi_1(K - T)$ is finitely generated.

Let us remark that in all that follows equations involving the Alexander polynomial are *only true* up to a unit factor of the ring JB.

LEMMA 2.10. *Let M be a bounded 3-dimensional manifold bounded by at least one surface of genus one. Let T be a smooth tube in the interior of M (not necessarily in a triangulation of M), T' a tubular neighborhood of the center line k of T in a triangulation σ of T, T'' a tube in σ whose center line is homologous to k in T. Then $\Delta_M(M - T'') = \Delta_M(M - T')$.*

PROOF. By Theorem 2.2 $\pi_1(T - T'')$ and $\pi_1(T - T')$ have presentations of deficiency one that contain canonical presentations of $\pi_1(\dot{T})$. Hence by Theorems 2.4 and 1.3

$$(z_{M-T'} - 1)^2 \cdot \delta_{M-T'}(T - T') \cdot (\text{g.c.d. } \mathfrak{F}_{M-T'}(M - T)) \cdot \Delta(M - T') =$$
$$(z_{M-T'} - 1)^2 \cdot (\text{g.c.d. } \mathfrak{F}(M - T')) \cdot \Delta_{M-T'}(T - T') \cdot (\text{g.c.d. } \mathfrak{E}_{M-T'}(M - T))$$

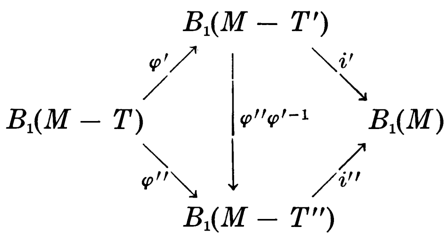
where z could be either of the generators of $B_1(\dot{T})$. By Lemma 2.6, z_{M-T} must be non-trivial for one of the generators, so by Lemma 2.7 the corresponding $z_{M-T'}$ must be non-trivial, and hence the factors $(z_{M-T'} - 1)^2$ may be cancelled. Let T_1 be an approximation to T in a subdivision of M. Again by Lemma 2.7 g.c.d. $\mathfrak{F}_{M-T'}(M - T) =$ g.c.d. $\mathfrak{F}(M - T') = 1$; and obviously $p_1(M - T') = p_1(M - T) = p_1(M - T_1) \geqq 2$ (see [10]), $\delta(T - T') = 1$. From Theorem 2.3 one calculates directly that $\Delta_T(T - T') = \Delta(T) = 1$, hence $\Delta_M(T - T') = 1$. So if we inject the above equation into $JB_1(M)$ we are left with

$$\Delta_M(M - T') = (\text{g.c.d. } \mathfrak{E}_{M-T'}(M - T))_M$$

and similarly

$$\Delta_M(M - T'') = (\text{g.c.d. } \mathfrak{E}_{M-T''}(M - T))_M$$

But by Lemma 2.7 we have a commutative diagram



and from this follows the equality of the terms on the right.

THEOREM 2.11. *Let N be a manifold bounded by at least one surface of genus one, T a smooth tube in the interior of N. Then*

$$\delta_N(T) \cdot \Delta(N) = \delta(N) \cdot \Delta_N(N - T) .$$

PROOF. Let T_1 be an approximation to T in a subdivision of N, and let T'' be an approximation to T_1 in a triangulation of T. Now, it follows from 4.3 and 4.4 of [7] that the elementary ideals of a group G are divided by the corresponding elementary ideals of any homomorph[5] of G. Hence by Lemma 2.8 $\Delta_N(N - T)$ divides $\Delta_N(N - T_1)$ and $\Delta_N(N - T_1)$ divides $\Delta_N(N - T'')$. If T' is a tubular neighborhood of the center line of T in a triangulation of T, then $\Delta_N(N - T) = \Delta_N(N - T')$. By Lemma 2.10 $\Delta_N(N - T') = \Delta_N(N - T'')$. Hence $\Delta_N(N - T) = \Delta_N(N - T_1)$. By Theorem 2.3, $\delta_N(T_1) \cdot \Delta(N) = \delta(N) \cdot \Delta_N(N - T_1)$ and since $\delta_N(T_1) = \delta_N(T)$, we have

$$\delta_N(T) \cdot \Delta(N) = \delta(N) \cdot \Delta_N(N - T) .$$

3. THEOREM 3.1. *Let k, k' be disjoint knots carrying homologous cycles in a 3-dimensional orientable manifold M. Then*

(a) *$B_1(M - k) \approx B_1(M - k')$, in particular, $B_1(M - k - k')$ has a subgroup which injects isomorphically onto both $B_1(M - k)$ and $B_1(M - k')$. If k carries an element of order ρ in M, then $p_1(M - k - k') = 1 + p_1(M - k) = 2 + p_1(M)$ if $\rho > 0$, and $p_1(M - k - k') = 1 + p_1(M - k) = 1 + p_1(M)$ if $\rho = 0$.*

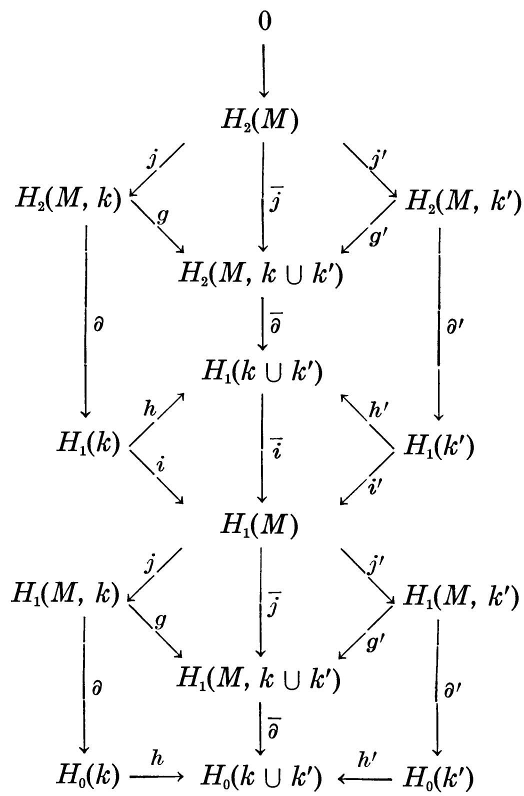
(b) *The injection of the torsion group $T_1(M - k - k') \to T_1(M - k)$ is an isomorphism.*

The proof is preceded by an easy lemma.

LEMMA. *If the sequence $G \xrightarrow{\mu} H \xrightarrow{\nu} J \longrightarrow 0$ is exact and J is free abelian on generators $a_1, \cdots, a_r$, and $a_i = \nu b_i$, then $H = (\mu G) \circ (b_1) \circ \cdots \circ (b_r)$, where (b_i) denotes the subgroup generated by b_i.*

PROOF OF THEOREM 3.1. Consider the commutative diagram

[5] That is, a homomorph the kernel of which is contained in the kernel of $\psi\varphi$.



formed from the exact sequences of the pairs (M, k), $(M, k \cup k')$, (M, k'), and additional injections g and h. We first prove (a) by exhibiting a basis for $B_1(M - k - k')$ having a sub-basis which represents bases for both $B_1(M - k)$ and $B_1(M - k')$. By hypothesis $iH_1(k) = i'H_1(k') = (\alpha)$, a cyclic group of order ρ, $0 \leqq \rho$. We consider explicitly only the case $\rho > 0$; the modifications of the proof for $\rho = 0$ will be obvious. Let $H_1(k) = (\beta)$, $H_1(k') = (\beta')$ with $i\beta = i'\beta' = \alpha$. Let $\gamma = (h'\beta') \cdot (h\beta^{-1})$. Then $\bar{i}\gamma = 0$ and it may be verified by a trivial calculation that

$$\text{kernel } \bar{i} = (h\beta^{\rho}) \circ (\gamma) = (h'(\beta')^{\rho}) \circ (\gamma) .$$

By exactness there exist δ_0, δ_r in $H_2(M, k \cup k')$ with $\bar{\partial}\delta_0 = h\beta^{\rho}$, $\bar{\partial}\delta_r = \gamma$. Let $\delta'_0 = \delta_0 \cdot \delta_r^{\rho}$; then $\bar{\partial}\delta'_0 = h'(\beta')^{\rho}$. By the lemma,

$$H_2(M, k \cup k') = \bar{j}H_2(M) \circ (\delta_0) \circ (\delta_r) = \bar{j}H_2(M) \circ (\delta'_0) \circ (\delta_r) .$$

By exactness there exist $\tilde{\varepsilon}_0$ in $H_2(M, k)$, $\tilde{\varepsilon}'_0$ in $H_2(M, k')$ with $\partial\tilde{\varepsilon}_0 = \beta^{\rho}$, $\partial'\tilde{\varepsilon}'_0 = (\beta')^{\rho}$, then $g\tilde{\varepsilon}_0 = \delta_0 \cdot \bar{j}\eta$, $g'\tilde{\varepsilon}'_0 = \delta'_0 \cdot \bar{j}\eta'$. Let $\varepsilon_0 = \tilde{\varepsilon}_0 \cdot j\eta^{-1}$, $\varepsilon'_0 = \tilde{\varepsilon}'_0 \cdot j'\eta'^{-1}$, then $\partial\varepsilon_0 = \beta^{\rho}$, $\partial'\varepsilon'_0 = (\beta')^{\rho}$, $g\varepsilon_0 = \delta_0$, $g'\varepsilon'_0 = \delta'_0$. Since β^{ρ} and $(\beta')^{\rho}$

generate the images of ∂ and ∂' respectively, we have by the lemma,

$$H_2(M, k) = jH_2(M) \circ (\varepsilon_0),\ H_2(M, k') = j'H_2(M) \circ (\varepsilon'_0) .$$

Using the fact that j, j' and $\bar{j}$ have kernel 0 it is easily verified that

$$g : H_2(M, k) \longrightarrow \bar{j}H_2(M) \circ (\delta_0)$$
$$g' : H_2(M, k') \longrightarrow \bar{j}H_2(M) \circ (\delta'_0)$$

are isomorphisms. Let $\lambda_1, \cdots, \lambda_{r-1}$ be a Betti basis for $H_2(M)$, $\varepsilon_i = j\lambda_i$, $\varepsilon'_i = j'\lambda_i$, $\delta_i = \bar{j}\lambda_i$ for $i = 1, \cdots, r-1$. Then $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{r-1}$ is a Betti basis for $H_2(M, k)$, $\varepsilon'_0, \varepsilon'_1, \cdots, \varepsilon'_{r-1}$ is a Betti basis for $H_2(M, k')$, and $\delta_0, \delta_1, \cdots, \delta_r$ is a Betti basis for $H_2(M, k \cup k')$. By the Lefschetz duality theorem there exists a basis $\check{\delta}_0, \check{\delta}_1, \cdots, \check{\delta}_r$ of $B_1(M - k - k')$ with intersection matrix $||\mathfrak{S}(\check{\delta}_i, \delta_j)|| = ||\delta_{ij}||$ where δ_{ij} is the Kronecker delta. The sub-basis $\check{\delta}_0, \check{\delta}_1, \cdots, \check{\delta}_{r-1}$ represents a basis for $B_1(M - k)$; it also represents a basis for $B_1(M - k')$. For $\mathfrak{S}(\check{\delta}_i, \varepsilon_j) = \mathfrak{S}(\check{\delta}_i, \delta_j) = \delta_{ij}$ for $0 \leqq i, j \leqq r-1$; $\mathfrak{S}(\check{\delta}_i, \varepsilon'_j) = \mathfrak{S}(\check{\delta}_i, \delta_j) = \delta_{ij}$ for $0 \leqq i \leqq r-1, 1 \leqq j \leqq r-1$, and $\mathfrak{S}(\check{\delta}_i, \varepsilon'_0) = \mathfrak{S}(\check{\delta}_i, \delta'_0) = \mathfrak{S}(\check{\delta}_i, \delta_0 \cdot \delta_r^p) = \delta_{i0}$ for $0 \leqq i \leqq r-1$. This proves (a).

Now we prove (b) by exhibiting a basis for $T_1(M - k - k')$ which also represents a basis for $T_1(M - k)$. Since k is a connected set, $\partial H_1(M, k) = 0$, so by exactness j is onto. Now $\bar{\partial} H_1(M, k \cup k')$ is an infinite cyclic group (α). Consider β in $H_1(M, k \cup k')$ with $\bar{\partial}\beta = \alpha$. By the lemma we have

$$H_1(M, k \cup k') = \bar{j}H_1(M) \circ (\beta) = gH_1(M, k) \circ (\beta)$$

Since image i=image $\bar{i}$, we have kernel j = kernel $\bar{j}$; from $gjH_1(M) = \bar{j}H_1(M)$ it then follows that kernel $g=0$. By the Lefschetz duality theorem there exist dual bases $\check{\tau}_1, \cdots, \tau_s$ and $\check{\tau}_1, \cdots, \check{\tau}_s$ of $T_1(M, k \cup k')$ and $T_1(M - k - k')$ respectively with linking matrix $||\mathfrak{V}(\check{\tau}_i, \tau_j)|| = ||\delta_{ij}/\text{order } \tau_i||$. Since β is of order zero, $g^{-1}\tau_1, \cdots, g^{-1}\tau_s$ is a basis for $T_1(M, k)$, and $||\mathfrak{V}(\check{\tau}_i, g^{-1}\tau_j)|| = ||\mathfrak{V}(\check{\tau}_i, \tau_j)||$. So $\check{\tau}_1, \cdots, \check{\tau}_s$ also represents a basis for $T_1(M - k)$. This completes the proof.

We are now in a position to make our basic definition. Let M be an orientable 3-dimensional manifold, α an element of $H_1(M)$, k a polygonal knot carrying α. The injection i: $H_1(M - k) \rightarrow H_1(M)$ induces a homomorphism

$$* : B_1(M - k) \longrightarrow H_1(M)/iT_1(M - k) = D(k)$$

The Alexander polynomial $\Delta(M - k)$ is determined up to a multiplicative factor $\pm\beta$, β in $B_1(M - k)$. So the image $\Delta^*(M - k)$ in $JD(k)$ is determined only up to a multiplicative factor $\pm\beta^*$. We define $\Delta^*(\alpha)$ to be $\Delta^*(M - k)$.

We proceed to justify our definition. Let k' be another polygonal knot carrying α. By Theorem 3.1b, $iT_1(M - k) = i'T_1'(M - k')$, hence $D(k) = D(k') = D(\alpha)$ so $\Delta^*(M - k)$ and $\Delta^{*\prime}(M - k')$ are elements of the same ring. Furthermore we may assume that k and k' are disjoint, for a polygonal knot k'' homologous to and disjoint from both k and k' is easily constructed. It remains to prove:

THEOREM 3.2. *Let k and k' be disjoint polygonal knots carrying homologous cycles in an orientable 3-dimensional manifold M. Then* $\Delta^*(M - k) = \Delta^{*\prime}(M - k')$.

We first prove a lemma which will enable us to assume k, k' in the same triangulation.

LEMMA. *Let Σ, Σ' be affine structures on M, and let k be a knot in Σ. Then there exists a knot k' in Σ' such that k' is homotopic to k and* $\Delta^*(M - k') = \Delta^{*\prime}(M - k)$.

PROOF. Let T be a tubular neighborhood of k and $V \subset U \subset T$ tubes concentric with T. It follows from Lemma 2.6 that there is an element σ of $B_1(\dot{U})$ which injects non-trivially into $B_1(M - V)$ and is represented by a loop m homotopic to k in U, hence in M. Let m be approximated by a knot k' in Σ' with a tubular neighborhood T' disjoint from V. By Theorem 2.11,

$$\delta_{M-V}(T') \cdot \Delta(M - V) = \delta(M - V) \cdot \Delta_{M-V}(M - V - T') \tag{1}$$

$$\delta_{M-T'}(V) \cdot \Delta(M - T') = \delta(M - T') \cdot \Delta_{M-T'}(M - V - T') \tag{1'}$$

The injections induce a commutative diagram

$$\begin{array}{ccccc} & & B_1(M - V) & & \\ & \overset{i_1}{\nearrow} & \downarrow * & \overset{i_2}{\nwarrow} & \\ B_1(M - V - T') & \longrightarrow & D(k) & \longleftarrow & B_1(M - T) \\ & \underset{i_3}{\searrow} & \uparrow *' & \underset{i_4}{\swarrow} & \\ & & B_1(M - T') & & \end{array}$$

and i_2, i_4 are isomorphisms. If we consider a tube T'' in $M - T$, near $\dot{T}$ and congruent to T', and denote by σ the element of $B_1(M - T)$ carried by T'', it is clear that $i_2(\sigma - 1) = \delta_{M-V}(T')$ and $i_4(\sigma - 1) = \delta_{M-T'}(V)$. Thus if we apply i_2^{-1} to (1) and i_4^{-1} to (1$'$) we have

$$(\sigma - 1) \cdot i_2^{-1}\Delta(M - V) = \delta(M - T) \cdot i_2^{-1}i_1\Delta(M - V - T')$$
$$= (\sigma - 1) \cdot i_4^{-1}\Delta(M - T')$$

and since $\sigma \neq 1$, $i_2^{-1}\Delta(M - V) = i_4^{-1}\Delta(M - T')$, hence

$$\Delta^*(M - k) = \Delta^*(M - V) = \Delta^{*\prime}(M - T') = \Delta^{*\prime}(M - k') .$$

PROOF OF THEOREM. By [10] we have $p_1(M - k) \geqq 1$, so we may distinguish two cases.

CASE I: $B_1(M - k)$ has rank one: in the notation of the proof of Theorem 3.1, $B_1(M - k) = (\check{\delta}_i) = B_1(M - k')$, where $i = 0$ (hence $r = 1$) if $\rho > 0$ and $i = 1$ (hence $r = 2$) if $\rho = 0$. Let $\check{\delta}_i$ be represented by a polygonal knot $m \subset M - k - k'$, in the triangulation carrying k and k'. We have $\mathfrak{S}(\check{\delta}_i, \delta_r) = 0$, and we may assume that δ_r is represented by an orientable surface bounded by $k \cup k'$, which is intersected an even number of times by the polygon m. By joining the intersections in pairs by small tubes enclosing segments of m we may construct an orientable surface in $M - m$ bounded by k and k'. Hence we have k homologous to k' in $M - m$. Now suppose k represented an element of $T_1(M - m)$, (which implies $\rho > 0$.) Then there would exist an element ε of $H_2(M, k)$ with $\partial\varepsilon = \beta^{\sigma\rho}$, $\sigma > 0$. But any such element is of the form $\varepsilon_0^\sigma \cdot \prod_{i=1}^{r-1} \varepsilon_i^{a_i}$, so $\mathfrak{S}(\check{\delta}_0, \varepsilon) = \sigma$ hence ε cannot be represented in $M - m$. So k and similarly k' represent elements of order zero of $H_1(M-m)$. By applying Theorem 2.11 to the appropriate tubular neighborhoods we have

$$(2) \qquad \delta_{M-k}(m) \cdot \Delta(M - k) = \delta(M - k) \cdot \Delta_{M-k}(M - k - m)$$
$$(2') \qquad \delta_{M-k'}(m) \cdot \Delta(M - k') = \delta(M - k') \cdot \Delta_{M-k'}(M - k' - m)$$

and since $\delta(M - k - m) = \delta(M - k' - m) = 1$:

$$(3) \qquad \delta_{M-k-m}(k') \cdot \Delta(M - k - m) = \Delta_{M-k-m}(M - k - k' - m)$$
$$(3') \qquad \delta_{M-k'-m}(k) \cdot \Delta(M - k' - m) = \Delta_{M-k'-m}(M - k - k' - m) .$$

Using $\delta_{M-k}(m) = \delta(M - k)$ and $\delta_{M-k'}(m) = \delta(M - k')$, (2) and (2') become

$$(4) \qquad \Delta(M - k) = \Delta_{M-k}(M - k - m)$$
$$(4') \qquad \Delta(M - k') = \Delta_{M-k'}(M - k' - m) .$$

Let $g\colon H_1(M - k - m) \to H_1(M - m)$ and $g'\colon H_1(M - k' - m) \to H_1(M - m)$ be injections, and let G be the subgroup of $H_1(M - m)$ generated by $gT_1(M - k - m)$ and $g'T_1(M - k' - m)$ (it can be shown that these images are identical but this fact is not required). Define $C(M - m) = H_1(M - m)/G$. Because G maps into $iT_1(M - k)$ under the injection $j\colon H_1(M - m) \to H_1(M)$ we have the commutative diagram

$$\begin{array}{ccccccc}
 & & B_1(M-k-m) & \longrightarrow & B_1(M-K) & & \\
 & \nearrow & & \searrow^{\#} & & \searrow^{*} & \\
B_1(M-k-k'-m) & & & & C(M-m) & \xrightarrow{j} & D(\alpha) \\
 & \searrow & & \nearrow_{\#'} & & \nearrow_{*'} & \\
 & & B_1(M-k'-m) & \longrightarrow & B_1(M-k') & &
\end{array}$$

where all homomorphisms are induced by injection. Injecting (3) and (3′) into $C(M-m)$

$$\delta^{\#}(k') \cdot \Delta^{\#}(M-k-m) = \delta^{\#'}(k) \cdot \Delta^{\#'}(M-k'-m)$$

and since k, k', represent elements of order zero of $H_1(M-m)$, $\delta^{\#}(k') = \delta^{\#'}(k)$ cannot be a zero divisor in $JC(M-m)$. Hence $\Delta^{\#}(M-k-m) = \Delta^{\#'}(M-k'-m)$. Using (4), (4′) and the commutativity of our diagram, it follows by applying the homomorphism j that

$$\Delta^*(M-k) = \Delta^{*'}(M-k')$$

CASE II. $B_1(M-k)$ is of the rank > 1. Again we consider two cases: (a) α is an element of order zero of $H_1(M)$. By Theorem 2.11,

$$\delta_{M-k}(k') \cdot \Delta(M-k) = \Delta_{M-k}(M-k-k')$$

hence

$$\delta^*(k') \cdot \Delta^*(M-k) = \Delta^*(M-k-k') = \delta^*(k) \cdot \Delta^*(M-k')$$

and since $\delta^*(k') = \delta^*(k)$ is not a zero divisor of $JD(\alpha)$,

$$\Delta^*(M-k) = \Delta^*(M-k')\ .$$

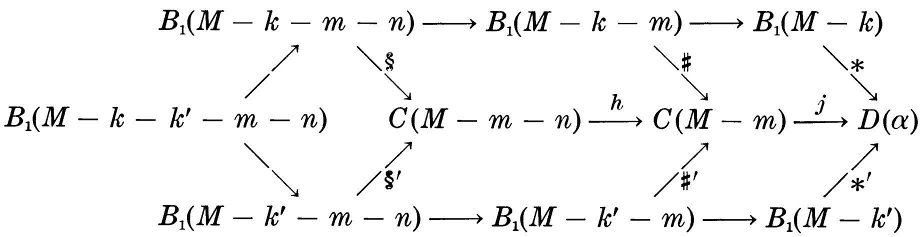
(b) α is in $T_1(M)$, thus, in the notation of Theorem 3.1, $B_1(M-k) = (\check{\delta}_0) \circ (\check{\delta}_1) \circ \cdots \circ (\check{\delta}_{r-1})$, $r > 1$. Let $m, n \subset M-k-k'$ be disjoint polygonal knots in the given triangulation, representing $\check{\delta}_1$ and $\check{\delta}_0 \cdot \check{\delta}_1$ respectively. We verify as in Case I that k and k' represent the same element, of order zero, of $H_1(M-m-n)$; note that n represents an element of order zero of $H_1(M)$, hence of $H_1(M-m)$. By Theorem 2.11,

$$(5) \qquad \delta_{M-k}(m) \cdot \Delta(M-k) = \Delta_{M-k}(M-k-m)$$

$$(6) \qquad \delta_{M-k-m}(n) \cdot \Delta(M-k-m) = \Delta_{M-k-m}(M-k-m-n)$$

$$(7) \qquad \delta_{M-k-m-n}(k') \cdot \Delta(M-k-m-n) = \Delta_{M-k-m-n}(M-k-k'-m-n)$$

and similar equations (5′), (6′), (7′) involving k'. Let $f: H_1(M-k-m-n) \rightarrow H_1(M-m-n)$ and $f': H_1(M-k'-m-n) \rightarrow H_1(M-m-n)$ be the injections, F the subgroup of $H_1(M - m - n)$ generated by $fT_1(M-k-m-n)$ and $f'T_1(M - k' - m - n)$. Define $C(M - m - n) = H_1(M - m - n)/F$, and $C(M - m)$ as in Case I. Then the injections induce a commutative diagram



Applying (§), (§′) to (7), (7′),

$$\delta^{\S}(k') \cdot \Delta^{\S}(M - k - m - n) = \delta^{\S'}(k) \cdot \Delta^{\S'}(M - k' - m - n) .$$

Since $\delta^{\S}(k') = \delta^{\S'}(k)$ is not a zero divisor,

$$\Delta^{\S}(M - k - m - n) = \Delta^{\S'}(M - k' - m - n) . \tag{8}$$

Applying h to (8) and using (6), (6′),

$$\delta^{\#}(n) \cdot \Delta^{\#}(M - k - m) = \delta^{\#'}(n) \cdot \Delta^{\#'}(M - k' - m) .$$

Since $\delta^{\#}(n)$ is not a zero divisor,

$$\Delta^{\#}(M - k - m) = \Delta^{\#'}(M - k' - m) . \tag{9}$$

Applying j to (9) and using (5), (5′),

$$\delta^*(m) \cdot \Delta^*(M - k) = \delta^{*'}(m) \cdot \Delta^{*'}(M - k')$$

Since $\delta^*(m)$ is not a zero divisor,

$$\Delta^*(M - k) = \Delta^{*'}(M - k') .$$

4. EXAMPLE I. THE 3-DIMENSIONAL LENS SPACES. Let T and U be tubes, a and b the meridian and longitude of $\dot{T}$, c and d the meridian and longitude of $\dot{U}$. The lens space $L(p, q)$ is obtained from T and U by identifying $\dot{T}$ with $\dot{U}$ in such a way that $c = a^q \cdot b^p$ and $d = a^r \cdot b^{\bar{q}}$ where $q\bar{q} - rp = 1$ (hence $q\bar{q} \equiv 1 \bmod p$). Thus $\pi_1(L)$ is the cyclic group $(b: b^p)$.

Any polygonal knot k in L may be transformed by an isotopy of L into a knot in U. Let $n = \mathfrak{V}(c, k)$. In order to compute Δ^* we may, by Theorem 3.2, assume that k is in the form of a torus knot of type $(n, 1)$ (see [6, p. 179]). Using Theorem 2.4 to combine T and $U - k$, we have

$$\begin{aligned}\pi_1(L - k) &= (c, d, x: [c, d], cd^n \cdot x^{-n}, a, c^{-1} \cdot a^q \cdot b^p, d^{-1} \cdot a^r \cdot b^{\bar{q}}) \\ &= (b, x: b^{p+n\bar{q}} \cdot x^{-n}) .\end{aligned}$$

We propose to characterize the homology classes γ having the properties:

(1) $H_1(L - \gamma)$ is infinite cyclic.

(2) $\Delta^*(\gamma) = 1$.

From the relation matrix for $H_1(L - k)$

$$\begin{array}{cc} b & x \\ \|p + n\bar{q} & -n\| \end{array}$$

we see that condition (1) is satisfied if and only if $(n, p) = 1$; in that case $H_1(L - k) = (t)$ where $b = t^n$ and $x = t^{p+n\bar{q}}$. Hence the Alexander matrix for the given presentation of $\pi_1(L - k)$ is then

$$\begin{array}{cc} b & x \\ \left\| \dfrac{t^{n(p+n\bar{q})} - 1}{t^n - 1} \right. & \left. \dfrac{t^{n(p+n\bar{q})} - 1}{t^{p+n\bar{q}} - 1} \right\| \end{array}$$

Using Lemma 1.1 we have

$$\Delta(L - k) = \varepsilon \cdot t^\nu \cdot \left(\frac{(t^{n(p+n\bar{q})} - 1) \cdot (t -)1}{(t^n - 1) \cdot (t^{p+n\bar{q}} - 1)}\right) \tag{3}$$

with $\varepsilon = \pm 1$. Multiplying (3) by $t^n - 1/t - 1$ and mapping into $JD(\gamma)$ we find that for condition (2) to be satisfied it is necessary that

$$\varepsilon \cdot t^\nu \cdot \left(\frac{t^{n^2\bar{q}} - 1}{t^{n\bar{q}} - 1}\right) = \frac{t^n - 1}{t - 1} \qquad (\text{mod } t^p - 1)\,.$$

Letting $t = 1$ we have $\varepsilon = 1$. Multiplying (3) by $(t^n - 1)(t^{p+n\bar{q}} - 1)$ and mapping into JD we obtain

$$t^{n\bar{q}+n-\nu} + t^{-\nu} + t^{n^2\bar{q}} + t = t^{n^2\bar{q}+1} + t^{n\bar{q}-\nu} + t^{n-\nu} + 1 \qquad (\text{mod } t^p - 1)\,.$$

For this to be an identity in $JD(\gamma)$, the exponents on the right and on the left must cancel in pairs (mod p). Since n and $\bar{q}$ are relatively prime to p we must have

(i) either $\nu \equiv 0$ or $n\bar{q} + n - \nu \equiv 0$ (mod p), and

(ii) either $n\bar{q} - \nu \equiv 1$ or $n - \nu \equiv 1$ (mod p).

These four possibilities lead to $n = 1, -1, q, -q$ (mod p). But conditions (1) and (2) are satisfied for $n = \pm 1$ by the curves $\pm d$ and for $n = \pm q$ by the curves $\pm b$. Since k represents the homology classes $b^{\pm n\bar{q}}$, the elements γ in $H_1(L)$ satisfying (1) and (2) are precisely

$$b,\ b^{-1},\ b^{\bar{q}},\ b^{-\bar{q}}\,.$$

Under change of generator $b \to (b')^h$, with $(h, p) = 1$, these become

$$(b')^h, (b')^{-h}, (b')^{h\bar{q}}, (b')^{-h\bar{q}} .$$

Thus the ratios of the exponents of the elements of $H_1(L)$ satisfying (1) and (2), expressed as powers of a single generator, are ± 1, $\pm q$, $\pm 1/q$ (mod p), and are invariants of the lens space $L(p, q)$. But one may exhibit [6] a homeomorphism of $L(p, q)$ and $L(p, q')$ if $qq' \equiv \pm 1 \pmod{p}$. Therefore:

THEOREM. *The lens spaces* $L(p, q)$ *and* $L(p, q')$ *are homeomorphic if and only if either* $q \equiv \pm q'$ *or* $qq' \equiv \pm 1 \pmod{p}$.

EXAMPLE II. THE TOPOLOGICAL SUM OF TWO LENS SPACES. Let M_1 and M_2 be oriented 3-dimensional manifolds, E_1 and E_2 open polyhedral 3-cells with $E_1 \subset M_1$ and $E_2 \subset M_2$. The *topological sum* $M = M_1 \oplus M_2$ is the oriented manifold obtained from $M_1 - E_1$ and $M_2 - E_2$ by semi-linearly matching $\dot{E}_1$ with $\dot{E}_2$ in such a manner that $M_1 - E_1$ and $M_2 - E_2$ are coherently oriented in M. By Theorem 2.4, $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$.

Let V_1 and V_2 be disjoint tubular neighborhoods of circles C_1 and C_2 in the plane $z = 0$ of S^3. Let c_i, d_i be longitude and meridian respectively of $\dot{V}_i$, $i = 1, 2$. Let T_1 and T_2 be tubes; let a_i, b_i be meridian and longitude, respectively, of $\dot{T}_i$, $i = 1, 2$. The topological sum

$$M = L(p_1, q_1) \oplus L(p_2, q_2)$$

may be formed from $S^3 - V_1 - V_2 = U$, T_1 and T_2 by identifying $\dot{V}_i$ with $\dot{T}_i$ in such a manner that $c_i = a_i^{q_i} \cdot b_i^{p_i}$ and $d_i = a_i^{r_i} \cdot b_i^{\bar{q}_i}$ where $q_i\bar{q}_i - r_i p_i = 1$, $i = 1, 2$. We have $\pi_1(M) = (b_1, b_2 \colon b_1^{p_1}, b_2^{p_2})$. It is known [12] that the elements of finite order of this group are either transforms of powers of b_1 or transforms of powers of b_2, hence every element of $H_1(M)$ which is an image of an element of finite order of $\pi_1(M)$ is either a power of b_1, or a power of b_2. Thus $H_1(M)$ has two topologically distinguished cyclic summands:

$$H_1^1 = (b_1 \colon b_1^{p_1}) \text{ and } H_1^2 = (b_2 \colon b_2^{p_2}) .$$

Any polygonal knot k in M may be transformed by an isotopy of M into a knot in U. Let $n_i = \mathfrak{V}(c_i, k)$ for $i = 1, 2$. Using Corollary 2.5 we have a relation matrix

$$\begin{array}{ccc} \tilde{b}_1 & e & \tilde{b}_2 \end{array}$$
$$\left\| \begin{array}{ccc} p_1 & -n_1 & 0 \\ 0 & -n_2 & p_2 \end{array} \right\|$$

for $H_1(M-k)$; $\tilde{b}_i$ is represented by the longitude of $\dot{T}_i$ and e by the meridian of a tubular neighborhood of k in U.

Let γ be an element of $H_1(M)$ and let $i: T_1(M-\gamma) \to H_1(M)$ be the injection. We wish to characterize those elements γ having the following properties:

$$(1) \qquad H_1(M-\gamma) = Z \circ Z_{p_2}, \qquad iT_1(M-\gamma) = iZ_{p_2} = H_1^2$$

$$(2) \qquad \Delta^*(\gamma) = p_2 .$$

Since kernel $i = (e)$, $i\tilde{b}_1 = b_1$, $i\tilde{b}_2 = b_2$ we "know" that $i^{-1}H_1^1$ is generated by $\tilde{b}_1$ and e, and that $i^{-1}H_1^2$ is generated by $\tilde{b}_2$ and e.

(i) Since $H_1(M-\gamma) = i^{-1}H_1^1 \circ Z_{p_2}$, $i^{-1}H_1^1$ must be an infinite cyclic group Z'. Since its relation matrix is

$$\begin{matrix} \tilde{b}_1 & e \\ \| p_1 & -n_1 \| \end{matrix}$$

we must have $(n_1, p_1) = 1$.

(ii) $i^{-1}H_1^2$ must be the direct sum of Z_{p_2} and the cyclic group p_1Z'. Since its relation matrix is

$$\begin{matrix} e & \tilde{b}_2 \\ \| -n_2 & p_2 \| \end{matrix}$$

we must have $n_2 \equiv 0 \bmod p_2$.

The latter condition implies that k is homologous in U to a knot whose linking number with the circle C_2 is zero, hence to a knot which is completely contained in a 3-cell F containing V_1 and disjoint from V_2. By Theorem 2.4,

$$\pi_1(M-k) = \pi_1((F-V_1) \cup T_1) * \pi_1((S-F-V_2) \cup T_2) .$$

Now $(F-V_1) \cup T_1$ is the complement in the lens space $L(p_1, q_1)$ of the knot k and a 3-cell, while $(S-F-V_2) \cup T_2$ is the complement of a 3-cell in the lens space $L(p_2, q_2)$. Hence the Alexander matrix for $M-k$ takes the form

$$\left\| \begin{matrix} \mathbf{A}_1 & 0 \\ 0 & p_2 \end{matrix} \right\|$$

where $\mathbf{A}_1$ is an Alexander matrix for $L(p_1, q_1) - k$. Since $B_1(M-k)$ has rank one we have, by deleting an appropriate column of $\mathbf{A}_1$ and applying Lemma 1.1,

$$\Delta(M-k) = p_2 \cdot \Delta_{M-k}(L(p_1, q_1) - k)$$

hence condition (2) requires

$$p_2 \cdot \Delta^*(L(p_1, q_1) - k) = p_2$$

or

$$\Delta^*(L(p_1, q_1) - k) = 1 .$$

This may be written as

$$\varepsilon \cdot t^{\nu} \cdot \frac{(t^{n_1(p_1+n_1\bar{q}_1)} - 1) \cdot (t - 1)}{(t^{n_1} - 1) \cdot (t^{p_1+n_1\bar{q}_1} - 1)} = 1 \bmod (t^{p_1} - 1) .$$

Hence we are left with the condition of Example I, and we conclude that $n_1 = \pm 1, \pm q_1 \pmod{p_1}$. However conditions (1) and (2) are realized by curves $\pm d_1$ for $n_1 = \pm 1$ and by $\pm b_1$ for $n_1 = \pm q_1$. Hence the homology classes satisfying (1) and (2) are

$$b_1, b_1^{-1}, b_1^{\bar{q}_1}, b_1^{-\bar{q}_1} .$$

Reversing the roles of L_1 and L_2 we find that the homology classes satisfying the analogous conditions $(1)_2$ and $(2)_2$ are

$$b_2, b_2^{-1}, b_2^{\bar{q}_2}, b_2^{-\bar{q}_2} .$$

Applying a change of generators $b_1 \to (b_1')^{h_1}$, $b_2 \to (b_2')^{h_2}$ *which preserves the distinguished sub-groups* of $H_1(M)$, we find as in Example I that the ratios $\pm q_1, \pm 1/q_1 \pmod{p_1}$ and $\pm q_2, \pm 1/q_2 \pmod{p_2}$ are invariants of M. Thus we have proved:

THEOREM 4.2. *$L(p_1, q_1) \oplus L(p_2, q_2) \equiv L(p_1, q_1') \oplus L(p_2, q_2')$ if and only if the components of the sum are pairwise homeomorphic.*

5. The author does not know whether the function, which assigns to every γ in $H_1(M)$ the factor group $D(\gamma)$ of $H_1(M)$, is preserved by homotopy equivalence. That the function $\Delta^*: H_1(M) \to \sum_{\gamma \text{ in } H_1(M)} JD(\gamma)$ is not a homotopy invariant is shown by the fact that the homotopy classification of the lens spaces [11] does not coincide with the topological classification.

As possibilities for further application of the Δ^* invariant we mention

(i) the manifolds constructed by Ausbohrung [6] of knots in S^3,

(ii) the "fibre spaces" of Seifert [13], and

(iii) Alexander's theorem [14] that every 3-dimensional manifold is a branched covering [15] of a multiple knot in S^3.

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